APOLLONIUS REPRESENTATION AND COMPLEX GEOMETRY OF ENTANGLED QUBIT STATES

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ABSTRACT

APOLLONIUS REPRESENTATION AND COMPLEX GEOMETRY OF ENTANGLED QUBIT STATES

In present thesis, a representation of one qubit state by points in complex plane is proposed, such that the computational basis corresponds to two fixed points at a finite distance in the plane. These points represent common symmetric states for the set of quantum states on Apollonius circles. It is shown that, the Shannon entropy of one qubit state depends on ratio of probabilities and is a constant along Apollonius circles. For two qubit state and for three qubit state in Apollonius representation, the concurrence for entanglement and the Cayley hyperdeterminant for tritanglement correspondingly, are constant along Apollonius circles. Similar results are obtained also for n- tangle hyperdeterminant with even number of qubit states. It turns out that, for arbitrary multiple qubit state in Apollonius representation, fidelity between symmetric qubit states is also constant along Apollonius circles. According to these, the Apollonius circles are interpreted as integral curves for entanglement characteristics. For generic two qubit state in Apollonius representation, we formulated the reflection principle relating concurrence of the state, with fidelity between symmetric states.

The Möbius transformations, corresponding to universal quantum gates are derived and Apollonius representation for multi-qubit states is generated by circuits of quantum gates. The bipolar and the Cassini representations for qubit states are introduced, and their relations with qubit coherent states are established. We proposed the differential geometry for qubit states in Apollonius representation, defined by the metric on a surface in conformal coordinates, as square of the concurrence. The surfaces of the concurrence, as surfaces of revolution in Euclidean and Minkowski (Pseudo-Euclidean) spaces are constructed. It is shown that, curves on these surfaces with constant Gaussian curvature becomes Cassini curves. The hydrodynamic interpretation of integral curves for concurrence as a flow in the plane is given and the spin operators in multiqubit $|PP...P\rangle$ states are discussed.

ÖZET

DOLAŞIK KÜBİT DURUMLARININ APOLLONIUS TEMSİLİ VE KOMPLEKS GEOMETRİSİ

Bu tezde, kompleks düzlemde bir noktaya göre bir kübit durumunun temsili, hesaplama tabanı düzlemde sonlu bir mesafede iki sabit noktaya karşılık gelecek şekilde önerilmiştir. Bu noktalar, Apollonius çemberlerinde bulunan kuantum durumları için ortak simetrik durumları temsil eder. Bir kübit durumu için Shannon entropisinin, olasılıkların oranına bağlı olup ve Apollonius çemberleri boyunca sabit olduğu gösterilmiştir. Apollonius temsilinde dolaşıklık, iki kübit durumu için dolaşıklık derecesi (concurrence) ve üç kübit durumu için Cayley hiperderminant hesaplanmıştır. Bu özellikler Apollonius çemberleri boyunca sabittir. Benzer sonuçlar, çift sayıdaki n- kübit durumları için hiperdeterminant hesaplanarak, n- dolaşık (n- tangle) olarak elde edilir. Apollonius temsilinde keyfi seçilmiş birden fazla kübit durumu için, simetrik kübit durumları arasındaki bağlılığın(fidelity) da Apollonius çemberleri boyunca sabit olduğu ortaya çıkmaktadır. Buna göre, Apollonius çemberleri dolaşıklık özelliklerine göre integral eğrileri olarak yorumlanır. Apollonius temsilinde genel iki kübit durumu için kübitlerin yansıma ilkesini dolaşıklık derecesi (concurrence) ile ilişkilendirilerek formüle ettik.

Evrensel kuantum kapılarına karşılık gelen Möbius dönüşümleri türetilmiş ve çok-kübitli durumlar için kuantum kapılarının devreleri tarafından Apollonius temsili üretilmiştir. Kübit durumları için bipolar ve Cassini temsilleri tanıtıldı ve eş uyumlu kübit durumlar ile ilişkilendirildi. Apollonius temsilindeki kübit durumları için, diferensiyel geometride konformal koordinatlardaki bir yüzey üzerinde tanımlanan metriği dolaşıklık derecesinin (concurrence) karesi olarak önerdik. Öklidyen ve Minkowski (Sözde - Öklidyen) uzaylarında concurrence yüzeyi dönel yüzey olarak inşa edilmiştir. Bu yüzeylerdeki sabit Gauss eğriliğine sahip eğrilerin Cassini eğrileri olduğu gösterilmiştir. Dolaşıklık derecesi (concurrence) düzlemde bir akış olarak, integral eğrilerinin hidrodinamik yorumu olarak verilmiş ve çok-kübitli $|PP...P\rangle$ durumları için spin operatörleri tartışılmıştır.

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CHAPTER 1

INTRODUCTION

"Complex variable theory is so beautiful that I feel that nature must have made good use of it, and , very likely, we need to make stronger use of it than we've done up to the present. " P. A. M. Dirac

Quantum computation and quantum information become very hot topics at recent time. Process of miniaturization of classical computers inevitably leads to quantum laws and necessity of creation quantum computer with quantum algorithms, taking into account quantum mechanical laws. (Benenti, Casati and Strini, 2004) , (Chuang and Nielsen, 2011)

In contrast to classical unit of information as a bit, with 0 and 1 classical states, in quantum information, a qubit as unit of information is a vector in Hilbert space, characterized by infinite number of values on unit sphere (Bloch sphere). Multiple qubit states, representing input and output in quantum computers, belong to multidimensional Hilbert space and have special properties. One of them is the entanglement, which is non-classical and non-local property of two qubit quantum states. Entanglement plays fundamental role in processing of quantum computer, this is why characterisation of entanglement is crucial in understanding quantum information. One of the qualifications of entanglement for two qubit state is the concurrence (Wootters, 1998). For three qubit states, similar qualification is given by Cayley hyper-determinant (Cayley, 1889), (Coffman, Kundu and Wootters, 2000), but different kinds of entanglement in this case could appear (as partial and total entanglement). Going to multiqubit states, variety of entanglements is growing (Wong and Christensen, 2001). This is the reason, why proper representation of qubit and entanglement by simple geometrical structures becomes very actual problem.

The present thesis is devoted to geometrical characterisation of entanglement for special class of multiple qubit states. The main idea is related with representation of Bloch sphere by points in complex plane, which is known as coherent qubit state representation (Pashaev and Gurkan, 2012). This representation allows one to use algebra and analysis of complex variables for description of qubit states. Disadvantage of coherent state representation is that, one of the computational basis states is at infinity (state $|0\rangle$ is at the origin, state $|1\rangle$ is at infinity). This makes difficult to construct simple geometrical characteristics related with distance between states in the plane. In this thesis, new representation of qubit state by a point in complex plane is proposed, so that $|0\rangle$ and $|1\rangle$ states are placed at two finite points in the plane. This allows one to describe characteristics of qubit states in terms of simple geometrical objects like distances, areas and volumes. The proposed representation is based on definition of circle as a ratio of distances from two fixed points a and b: $\frac{|z-a|}{|z-b|} = r$, first discovered by Apollonius of Perga in ancient times (Brannan, Esplen and Gray, 2012). The meaning of this representation becomes clear with introduction of Möbius transformation $w = \frac{z-a}{z-b}$ of concentric circles at origin $|w| = r$ to the set of circles $\frac{|z-a|}{|z-b|} = r$, with common symmetric points a and b (Ahlfors, 1966). This representation allows one to consider $|0\rangle$ and $|1\rangle$ states of qubit as common symmetric states for the set of Apollonius circle states. Calculation of the Shannon entropy for one qubit state in this representation, shows that it is a constant along every Apollonius circle. An extension of Apollonius representation to two qubit states and to multiple qubit states, show that the concurrence for two qubit states, the 3-tangle for three qubit states, the n-tangle for even n-qubit states, and fidelity for symmetric multiqubit states are constant along Apollonius circles. Apollonius circles, supplemented by the set of orthogonal circles passing from points a and b, describe the bipolar coordinate system in the plane. This orthogonal coordinate system has several applications in hydrodynamics and electrostatics. In our case to describe states with variable concurrence, we introduce bipolar representation of qubit states. So that Apollonius circles becomes equi-concurrent curves and the concurrence is changing along the orthogonal set of curves. This orthogonal set of curves could be related with some new constant characteristics of qubit states. To generate our Apollonius qubit states, we propose several circuits of unitary universal gates and represent them in the form of universal Möbius transformations.

By considering the set of Apollonius circles as integral curves of some vector field, we can interpret concurrence as the stream function of two dimensional rotational flow. In addition to this hydrodynamic representation, we developed differential geometrical description of qubit states in coherent and Apollonius representation. For these, we choose conformal metric on the surface as $g(x, y) = C^2(x, y)$, where $C(x, y)$ is the concurrence. The surfaces of the concurrence, as surfaces of revolution in Euclidean and Minkowski (Pseudo-Euclidean) spaces are constructed. By calculating Gaussian curvature, we show that it is a constant along the set of Cassini curves. This curves where proposed by Cassini for description of planet motion, but dismissed by Newton's description of elliptic orbits or conic sections (Sivardiere , 1994). Cassini curves have beautiful geometrical property complimentary to Apollonius circles, as $|z-a||z-b| = r^2$. In our thesis, we established relation between Cassini curves and Apollonius circles, which allow us to introduce Cassini representation of qubit. This Cassini representation is an example of multivalued representation of qubit, when two points in the plane represent one qubit state. This multivalued property implies that, some points in plane reflected in y- axis should be identified.

The thesis is organized in the following form.

In Chapter 2, we introduce bit and qubit as units of classical and quantum information, Section 2.1. In Section 2.2, we describe geometry of one qubit state, Bloch sphere and probability distribution. In Section 2.3, we discuss unitary one qubit gates and universality of one qubit computations.

In Chapter 3, we derive representation of qubit in complex plane. Section 3.1 is devoted to stereographic projection of Bloch sphere to complex plane. This is known as qubit coherent states, Section 3.2. Qubit gates as Möbius transformation are discussed in Section 3.3. Universality of one qubit computations in special form is subject of Section 3.4 and in Section 3.5, fidelity between symmetric states is discussed.

Chapter 4 is devoted to description of multiple qubit states. We start from Section 4.1, by describing separability criterium for two qubit states in terms of concurrence and area. Then, in Section 4.2 we introduce concurrence as a determinant. The concurrence and fidelity relation discussed in Section 4.3. In Section 4.4, we generated symmetric states by using antipodal points. In Section 4.5, as geometrical characteristics we found inner product metric relation with concurrence. To physical meaning of concurrence as reduced density matrix devoted Section 4.6. Relation between entanglement and Von Neumann entropy is studied in Section 4.7. To complete this chapter, we discussed Reimannian metric and concurrence in Section 4.8.

In Chapter 5, we introduce Apollonius representation for qubit states. In Section 5.1, we introduced Apollonius circles and related Möbius transformations. The Hadamard gate and generation of Apollonius representation of qubit state is considered in Section 5.2. Section 5.3. is devoted to Apollonius representation for one and two qubit states and corresponding entanglement characteristics. In Section 5.3.1, we show that entropy and fidelity for non-symmetric states are constant on Apollonius circles. In a similar way in Section 5.3.2, we treat concurrence and entropy for two qubit states in non-symmetric and symmetric cases. In Section 5.3.2.4 we find geometrical meaning of concurrence in terms of areas and angles. Relation of concurrence with reflection principle is subject of Section 5.3.2.5. In Section 5.4, we introduce multiple qubits in Apollonius representation. This representation for generic two qubit states is derived in Section 5.5. Relation between concurrence and fidelity of reflected qubits for the generic case is discussed in Section

5.6.

In Chapter 6, characteristics of entanglement for three and even n qubit states are studied. The Cayley hyperdeterminant and 3-tangle for three qubit states are subject of Sections 6.1. Generalization to arbitrary even n number of qubit states in form of n-tangle hyperdeterminant is obtained in Section 6.2.

Chapter 7 is devoted to Cassini multivalued representation of qubit states. In Section 7.1, we introduce Cassini curves in cartesian and polar form. In Section 7.2, we relate Cassini curves with Apollonius circles by set of conformal transformations. The Cassini representation of one qubit state and corresponding entropy are calculated in Section 7.3. Two qubit Cassini states, fidelity for these states (7.4.1), and inversion in leminiscate and symmetric Cassini states are found in Section 7.4. In Section 7.5. 3-tangle for three qubit Cassini state and in Section 7.6., n-tangle for the even number of Cassini qubit states are derived. Transformation between Cassini and Apollonius states are discussed in Section 7.7.

Bipolar representation of qubit states is subject of Chapter 8. We represent one and two qubit states in bipolar coordinates with corresponding entropy and concurrence for non-symmetric case in Section 8.1. The symmetric case is treated in Section 8.2.

In Chapter 9, we developed conformal differential geometry description of qubit states. In Section 9.1, we identified equi-concurrent Apollonius circles with stream lines of two dimensional hydrodynamic flow. In Section 9.2, we introduce conformal metric in terms of concurrence for symmetric and non-symmetric Apollonius representation. It leads us to consider nonlinear Laplace equation in Section 9.3. In Section 9.4, we related constant Gaussian curvature concurrence surfaces with Cassini curves. In Section 9.5, we describe the concurrence surface as the surface of revolution in Euclidean and Pseudo-Euclidean spaces. In Section 9.6. generic conformal transformation of coherent states is derived. The Liouville equation for concurrence is discussed in Section 9.7.

In Chapter 10, relation between spin operators and qubit states are discussed. The average of spin $\frac{1}{2}$ operators on qubit states and related characteristics of maximally random states are subject of Section 10.1. In Section 10.2., we introduce n qubit $|PP..P\rangle$ state and calculate averages of spin operators in these states. For maximally entangled states, as maximally random states, the averages of spin operators are zero.

Our conclusion are presented in Chapter 11.

CHAPTER 2

QUBIT QUANTUM STATES

In this chapter, the basic definitions and concepts of quantum information theory are introduced. For more details see (Chuang and Nielsen, 2011)

2.1. Bit and Qubit

The bit is the fundamental concept of classical computation and classical information. The classical bit is a unit of information, which has two states, either 0 or 1. The quantum computation and quantum information are built upon the quantum bit, for shortly it is called the qubit (Figure 2.1). The qubit has two basis states, as vectors in Hilbert space denoted as $|0\rangle$ and $|1\rangle$. They are correspond to classical bits 0 and 1 respectively. The standard notations for states in quantum mechanics is Dirac notation "$|\ \rangle$" that is called *the ket state* and "$\langle\ |$" that is called *the bra state*. In these notations a qubit can be represented as a *superposition* of two states

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle, \tag{2.1}$$

where α and β are complex numbers. This is why, the state of the qubit is a vector in *Hilbert Space ($\mathcal{H}$)*. Here, the Hilbert space is a two-dimensional complex vector space $\mathbb{C}^2$ and states $|0\rangle$ and $|1\rangle$ are known as *computational basis*, which is the orthonormal basis for this vector space.

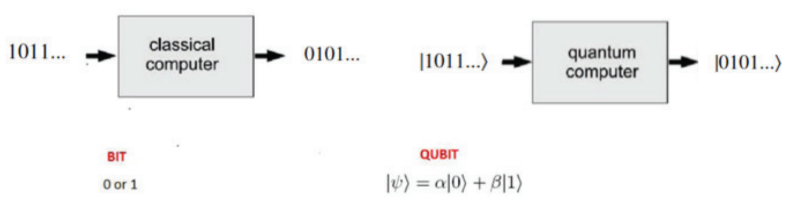


Figure 2.1. Classical and Quantum Computer

There is a big difference between bit and qubit, the bit can be examined exactly to determine whether it is in the position 0 or 1. In contrast to this, the qubit cannot be examined exactly to determine its quantum state. Quantum mechanics gives only restricted information about it. For the state (2.1), the measurement result is the state $|0\rangle$ with probability p_0 or it is the state $|1\rangle$ with probability p_1. Between these probabilities exist a constraint, which is called the *normalization condition*,

$$|\alpha|^2 + |\beta|^2 = p_0 + p_1 = 1,$$

where probabilities are determined by modulus of complex numbers

$$p_0 = |\alpha|^2 = \alpha \cdot \bar{\alpha} \quad p_1 = |\beta|^2 = \beta \cdot \bar{\beta}.$$

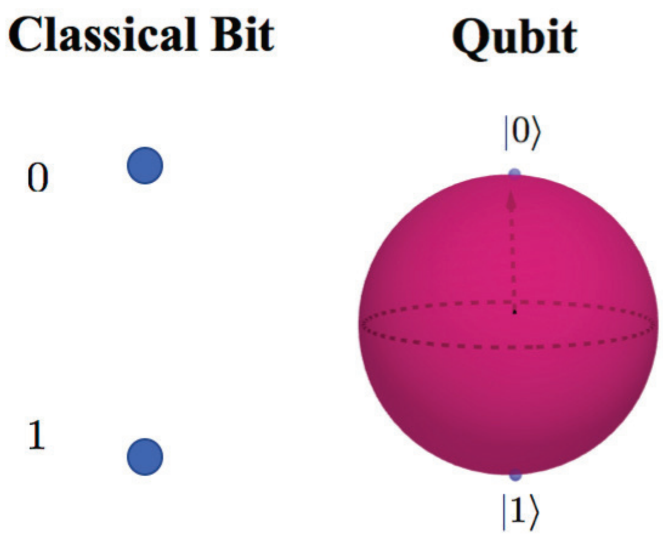


Figure 2.2. Bit and Qubit Representation

2.2. Geometry of Qubit States

The one qubit quantum computation takes place in vector space $\mathbb{C}^2$, where every vector represents qubit as a unit of quantum information. Every vector in $\mathbb{C}^2$ can be written as a linear combination of two vectors,

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle = \alpha\begin{pmatrix}1\\0\end{pmatrix} + \beta\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}\alpha\\\beta\end{pmatrix}$$

corresponding to computational basis in this space

$$|0\rangle = \begin{pmatrix}1\\0\end{pmatrix}, \qquad |1\rangle = \begin{pmatrix}0\\1\end{pmatrix}.$$

These basis vectors are orthonormal, which means that they are normalized and orthogonal:

$$\langle 0|0\rangle = 1 = \langle 1|1\rangle, \ \langle 0|1\rangle = 0 = \langle 1|0\rangle.$$

The geometrical meaning of one qubit state can be understood by applying normalization condition,

$$\langle\psi|\psi\rangle = 1$$

giving

$$|\alpha|^2 + |\beta|^2 = 1,$$

where coefficients $\alpha = \alpha_1 + i\alpha_2$ and $\beta = \beta_1 + i\beta_2$ are complex numbers. This shows that the normalization condition represents the unit sphere

$$\alpha_1^2 + \alpha_2^2 + \beta_1^2 + \beta_2^2 = 1,$$

in four dimensional real space $\mathbf{S}^3 \in \mathbb{R}^4$, where $(\alpha_1, \alpha_2, \beta_1, \beta_2) \in \mathbb{R}^4$. This unit sphere $\mathbf{S}^3$ is reduced to unit sphere $\mathbf{S}^2 \in \mathbb{R}^2$, due to global phase identification.

2.2.1. Global Phase

In Quantum mechanics the states are defined up to the global phase and are the rays in the Hilbert space.

Definition 2.1 *The ray is an equivalence class of vectors, that are differ by multiplication by a non-zero complex scalar, called the global phase* $e^{i\gamma}$. *Ray in Hilbert space is the set*

$$\{e^{i\gamma}|\psi\rangle : \ \forall\gamma \in \mathbb{R}\}.$$

Quantum state $|\psi\rangle$ *is determined up to global phase, such that* $|\psi\rangle \equiv e^{i\gamma}|\psi\rangle$ *and represents a ray in* $\mathcal{H}$.

Probability density for every state $|\psi\rangle$ is the same for every ray in Hilbert space $\{e^{i\gamma}|\psi\rangle\}$, since

$$\langle\psi|\psi\rangle = \langle\psi|e^{-i\gamma}e^{i\gamma}|\psi\rangle.$$

2.2.2. Bloch Sphere -S^2

Due to this global phase identification, the qubit state takes values on the unit sphere S^2, which is called the Bloch sphere. This sphere follows from the representation,

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle, \quad |\alpha|^2 + |\beta|^2 = 1.$$

Let's solve normalization constraint as $\alpha = \cos\frac{\theta}{2}e^{i\chi_1}, \beta = \sin\frac{\theta}{2}e^{i\chi_2}$ and substitute into the state

$$|\psi\rangle = \cos\frac{\theta}{2}e^{i\chi_1}|0\rangle + \sin\frac{\theta}{2}e^{i\chi_2}|1\rangle$$

By extracting the global phase $e^{i\chi_1}$ and denoting $\chi_2 - \chi_1 \equiv \varphi$ the qubit becomes

$$|\psi\rangle = e^{i\chi_1}(\cos\frac{\theta}{2}|0\rangle + \sin\frac{\theta}{2}e^{i\varphi}|1\rangle),$$

where χ_1, θ and φ are real numbers. Identifying the qubit states with different global phases χ_1, one gets the Bloch sphere representation of qubit.

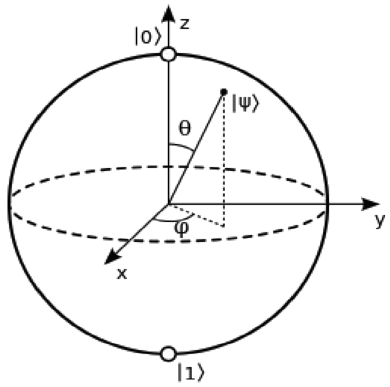


Figure 2.3. Bloch Sphere Representation

Definition 2.2 *One qubit in Bloch sphere representation is*

$$|\psi\rangle = \cos\frac{\theta}{2}|0\rangle + \sin\frac{\theta}{2}e^{i\varphi}|1\rangle, \tag{2.2}$$

where

$$0 \leq \theta \leq \pi, \qquad 0 \leq \varphi \leq 2\pi.$$

According to this definition, every state of qubit is represented by a point on unit sphere with coordinates (θ, φ) as altitude and latitude respectively (Figure 2.3). So that, $|0\rangle$ state corresponds to the north pole of the sphere, while $|1\rangle$ state is represented by the south pole of the sphere. By measuring generic qubit state, the unit vector on Bloch sphere jumps to the north or the south poles and corresponding qubit state collapses to $|0\rangle$ or $|1\rangle$ state with corresponding probabilities (Figure 2.4). These probabilities are completely determined by angle θ:

- probability to get state $|0\rangle$ is $p_0 = |\langle 0|\psi\rangle|^2 = \cos^2\frac{\theta}{2}$
- probability to get state $|1\rangle$ is $p_1 = |\langle 1|\psi\rangle|^2 = \sin^2\frac{\theta}{2}$.

Addition of these probabilities is one : $p_0 + p_1 = \cos^2\frac{\theta}{2} + \sin^2\frac{\theta}{2} = 1$.

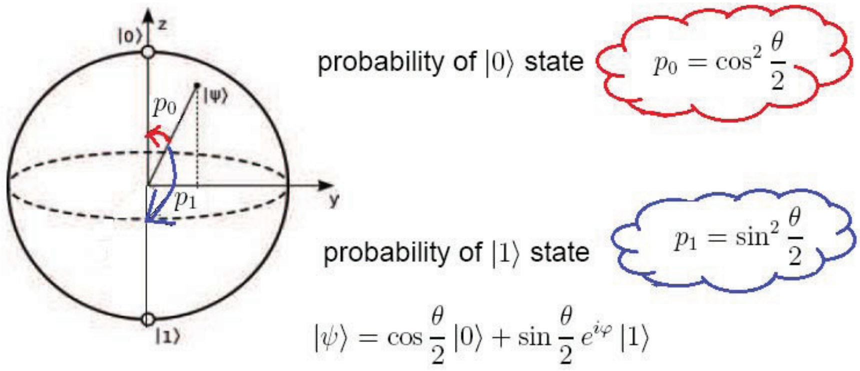


Figure 2.4. Probabilities of state $|\psi\rangle$

2.2.3. $\mathbf{S^2}$ Identification and Surface of Revolution

As it has seen above, the qubit state without global phase identification is described by a point on unit sphere $\mathbf{S^3}$ in $\mathbb{R}^4$, while with global phase identification it corresponds to the point on Bloch sphere $\mathbf{S^2}$ in $\mathbb{R}^3$. Then, the global phase identification can be described as identification of points on $\mathbf{S^3}$ sphere. To describe this explicitly, firstly elementary example would be considered.

Let $\mathbf{S^2}$ is a unit sphere. This sphere is a surface of revolution for the unit circle $\mathbf{S^1}$. Applying identification of points on this surface with different φ, the surface of revolution reduces to the generating curve. For sphere $\mathbf{S^2}$ this curve becomes $\mathbf{S^1}$ circle. It can be seen also from parametric representation of sphere in the form

$$x = \sin\theta\cos\varphi, \ \ y = \sin\theta\sin\varphi, \ \ z = \cos\theta.$$

Identification of points with different angles φ with the ones in the plane where $\varphi = 0$, gives

$$x = \sin\theta, \ \ y = 0, \ \ z = \cos\theta,$$

representing $\mathbf{S^1}$ - circle in xz-plane with equation

$$x^2 + z^2 = 1, \ \ y = 0.$$

2.2.4. $\mathbf{S^3}$ Identification and Bloch Sphere

In a similar way, for $\mathbf{S^3}$ - sphere with angles (θ, φ, χ) $0 \le \theta \le \pi,\ \ 0 \le \varphi \le 2\pi,\ \ 0 \le \chi \le 2\pi$, identification of points with different χ leads to $\mathbf{S^2}$ - sphere, which is the Bloch sphere. $\mathbf{S^3}$ - sphere can be represented as a hyper surface of revolution in the parametric form:

$$x = \sin\theta\cos\varphi\cos\chi, y = \sin\theta\cos\varphi\sin\chi, z = \sin\theta\sin\varphi, t = \cos\theta,$$

such that

$$x^2 + y^2 + z^2 + t^2 = 1.$$

Identification of points on $\mathbf{S^3}$ with different angles χ with the ones on the hyper-plane where $\chi = 0$, gives $\mathbf{S^2}$ - sphere in (x, z, t) coordinates, with parametric equation

$$x = \sin\theta\cos\varphi,\ \ y = 0,\ \ z = \sin\theta\sin\varphi,\ \ t = \cos\theta,$$

such that

$$x^2 + z^2 + t^2 = 1.$$

2.3. One Qubit Quantum Gates

One qubit quantum gate is a device, which performs a fixed unitary operation acting on the selected qubit in a fixed period of time. (Ekert, Hayden and Inamori, 2000)

2.3.1. Unitary Transformations and Quantum Gates

Quantum mechanics postulates, that the time evolution of the quantum system is necessarily unitary. This constraint is unique also for quantum gates.

- What is an unitary transformation?
  The unitary transformation is complex analogue of rotation in complex space $\mathbb{C}^2$

and preserving the lengths of vectors. It is described by complex matrices:

$$U = \begin{pmatrix} a & c \\ b & d \end{pmatrix},$$

satisfying unitarity condition

$$UU^\dagger = U^\dagger U = I,$$

where $U^\dagger$ as Hermitian conjugate of this matrix,

$$U^\dagger = \begin{pmatrix} \bar{a} & \bar{b} \\ \bar{c} & \bar{d} \end{pmatrix}.$$

As a result, the unitary matrix in $\mathbb{C}^2$ space in $U(2)$ form

$$U = \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix}, \qquad |a|^2 + |b|^2 \neq 0.$$

If

$$\det U = |a|^2 + |b|^2 = 1$$

then $U \in SU(2)$.

Definition 2.3 *(Loceff, 2015), (Stillwell, 1992) Unitary quantum gate is a linear transformation of Hilbert space, that maps the normalized (unit) vectors to other unit vectors. Since Hilbert space for one qubit is two dimensional, a unitary quantum operator can be represented by a* 2×2 *matrix.*

- Unitary transformation in the Hilbert space of one qubit maps the basis states $|0\rangle$ and $|1\rangle$ to orthonormal states $|v_0\rangle = a|0\rangle - \bar{b}|1\rangle$ and $|v_1\rangle = b|0\rangle + \bar{a}|1\rangle$.

2.3.2. Unitary Gates and Rotation of Bloch Sphere

2.3.3. Pauli Gates

Pauli gates are defined according to (Loceff, 2015).

- **X - Gate**

Definition 2.4 *Pauli X gate is denoted as quantum NOT gate (QNOT) and is defined as*

$$X \equiv \sigma_x \equiv \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

$X\dagger = X$ *and* $X^2 = I$.

Applying the gate to basis states gives

$$X|0\rangle = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} = |1\rangle, \qquad X|1\rangle = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = |0\rangle,$$

or

$$X|0\rangle = |1\rangle, \qquad X|1\rangle = |0\rangle,$$

Since this gate interchanges the basis states, X gate usually called as *the bit flip* gate. If the quantum state $|\psi\rangle$ is written in matrix form

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle = \begin{pmatrix} \alpha \\ \beta \end{pmatrix},$$

application of this gate to the state gives

$$X|\psi\rangle = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} \beta \\ \alpha \end{pmatrix} = \beta|0\rangle + \alpha|1\rangle.$$

It shows that X operator swaps the amplitudes of any state vector. The circuit diagram for this gate is

$$\alpha|0\rangle + \beta|1\rangle \quad \text{———} \boxed{\mathrm{X}} \text{———} \quad \beta|0\rangle + \alpha|1\rangle$$

- **Y - Gate**

Definition 2.5 *The Y gate is defined as*

$$Y \equiv \sigma_y \equiv \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}.$$

Applying it to computational basis states

$$Y|0\rangle = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ i \end{pmatrix} = i|1\rangle, \qquad Y|1\rangle = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -i \\ 0 \end{pmatrix} = -i|0\rangle,$$

gives

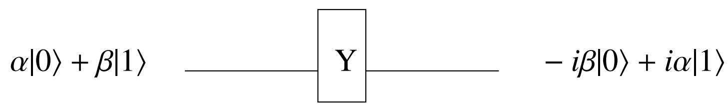
$$Y|0\rangle = i|1\rangle, \qquad Y|1\rangle = -i|0\rangle.$$

Since it flips both, the bits and relative phases, Y gate is called *the bit and phase flip* gate.

Applying this gate to arbitrary state $|\psi\rangle$ gives

$$Y|\psi\rangle = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = -i\begin{pmatrix} \beta \\ -\alpha \end{pmatrix} = -i\beta|0\rangle + i\alpha|1\rangle.$$

The circuit diagram for Y gate is



- **Z - Gate**

Definition 2.6 *The Z gate is defined by the following matrix*

$$Z \equiv \sigma_z \equiv \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Applying Z gate to basis states gives

$$Z|0\rangle = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = |0\rangle, \qquad Z|1\rangle = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = -|1\rangle,$$

or

$$Z|0\rangle = |0\rangle, \qquad Z|1\rangle = -|1\rangle.$$

This gate is known as the *phase flip* gate, since it is changing only the sign of state $|1\rangle$. Application of it to state $|\psi\rangle$ gives,

$$Z|\psi\rangle = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} \alpha \\ -\beta \end{pmatrix} = \alpha|0\rangle - \beta|1\rangle.$$

As can be seen, it changes the relative phase of two amplitudes in $|\psi\rangle$ and describes 180 degree rotation of β in complex plane $(-1 = e^{i\pi})$. The circuit diagram for Z gate is

$$\alpha|0\rangle + \beta|1\rangle \quad \text{———} \boxed{\mathrm{Z}} \text{———} \quad \alpha|0\rangle - \beta|1\rangle$$

2.3.4. Hadamard Gate and Phase Gate

Here the Hadamard and Phase gate are defined according to (Benenti, Casati and Strini, 2004) and (Loceff, 2015)

Definition 2.7 *The Hadamard gate is defined as*

$$H = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}.$$

This single qubit gate corresponds to rotations and reflections of Bloch sphere. Rotation around $\frac{\pi}{4}$ followed by a reflection. In addition to this, $H^\dagger = H$ since H is real and symmetric, and $H^2 = I$.

Applying Hadamard gate to computational basis states gives

$$H|0\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{|0\rangle + |1\rangle}{\sqrt{2}},$$

$$H|1\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{|0\rangle - |1\rangle}{\sqrt{2}},$$

or

$$H|0\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}} \equiv |+\rangle, \qquad H|1\rangle = \frac{|0\rangle - |1\rangle}{\sqrt{2}} \equiv |-\rangle.$$

States $|+\rangle$ and $|-\rangle$ are orthonormal Hadamard basis states. Applying H to generic state $|\psi\rangle$ gives,

$$H|\psi\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}\begin{pmatrix}\alpha\\ \beta\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix}\alpha+\beta\\ \alpha-\beta\end{pmatrix},$$

which implies

$$H|\psi\rangle = \frac{\alpha+\beta}{\sqrt{2}}|0\rangle + \frac{\alpha-\beta}{\sqrt{2}}|1\rangle = \alpha|+\rangle + \beta|-\rangle.$$

The circuit diagram for this gate is

$$\alpha|0\rangle + \beta|1\rangle \quad \text{———}\boxed{\text{H}}\text{———} \quad \frac{\alpha+\beta}{\sqrt{2}}|0\rangle + \frac{\alpha-\beta}{\sqrt{2}}|1\rangle$$

Definition 2.8 *The Phase gate is defined as*

$$R_z(\theta) = \begin{pmatrix}1 & 0\\ 0 & e^{i\theta}\end{pmatrix},$$

where θ is any real number. This gate generates a counter-clockwise rotation through an angle θ about z-axis of the Bloch sphere.

Applying phase gate to basis states gives

$$R_z(\theta)|0\rangle = \begin{pmatrix}1 & 0\\ 0 & e^{i\theta}\end{pmatrix}\begin{pmatrix}1\\ 0\end{pmatrix} = |0\rangle,$$

$$R_z(\theta)|1\rangle = \begin{pmatrix}1 & 0\\ 0 & e^{i\theta}\end{pmatrix}\begin{pmatrix}0\\ 1\end{pmatrix} = e^{i\theta}|1\rangle,$$

or

$$R_z(\theta)|0\rangle = |0\rangle, \qquad R_z(\theta)|1\rangle = e^{i\theta}|1\rangle,$$

and to generic state $|\psi\rangle$ is,

$$R_z(\theta)|\psi\rangle = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\theta} \end{pmatrix}\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} \alpha \\ e^{i\theta}\beta \end{pmatrix}.$$

It implies

$$R_z(\theta)|\psi\rangle = \alpha|0\rangle + e^{i\theta}\beta|1\rangle.$$

The circuit diagram for this gate is

$$\alpha|0\rangle + \beta|1\rangle \quad ——— \boxed{R_z(\theta)} ——— \quad \alpha|0\rangle + e^{i\theta}\beta|1\rangle$$

It is important to realize that any unitary operation on a single qubit can be constructed by using only Hadamard and phase gates. It means that, the unitary transformation moves the qubit state from one point to another point of the Bloch sphere by using only Hadamard and phase gates.

2.3.5. Universality of One Qubit Computations

Two gates introduced in previous section represent the so called universal one qubit quantum gates. It turns out that arbitrary one qubit gate can be implemented by sequence of Hadamard and phase gates. This property is called universality of gates and computations on one qubit then become universal quantum computations.This means that by the set of universal gates, arbitrary one qubit state can be transformed to another arbitrary state. To show this, first one applies Hadamard and phase gates to basis state $|0\rangle$, giving the generic state $|\psi\rangle$ (2.2),

$$R_z\left(\frac{\pi}{2}+\phi\right) H\, R_z(\theta)\, H\, |0\rangle = e^{i\frac{\theta}{2}}\left(\cos\frac{\theta}{2}|0\rangle + \sin\frac{\theta}{2}e^{i\varphi}|1\rangle\right)$$

or up to global phase

$$R_z\left(\frac{\pi}{2}+\phi\right) H\, R_z(\theta)\, H\, |0\rangle = |\psi\rangle.$$

For two arbitrary qubits

$$|\psi_1\rangle = \cos\frac{\theta_1}{2}|0\rangle + \sin\frac{\theta_1}{2}e^{i\phi_1}|1\rangle, \qquad |\psi_2\rangle = \cos\frac{\theta_2}{2}|0\rangle + \sin\frac{\theta_2}{2}e^{i\phi_2}|1\rangle,$$

applying circuit

$$R_z\left(\frac{\pi}{2}+\phi_2\right) H\, R_z(\theta_2-\theta_1)\, H\, R_z\left(-\frac{\pi}{2}-\phi_1\right) |\psi_1\rangle = e^{i\left(\frac{\theta_2}{2}-\frac{\theta_1}{2}\right)}\left(\cos\frac{\theta_2}{2}|0\rangle + \sin\frac{\theta_2}{2}e^{i\phi_2}|1\rangle\right)$$

up to global phase gives relation

$$R_z\left(\frac{\pi}{2}+\phi_2\right) H\, R_z(\theta_2-\theta_1)\, H\, R_z\left(-\frac{\pi}{2}-\phi_1\right) |\psi_1\rangle = |\psi_2\rangle$$

This transformation allows one to generate arbitrary qubit $|\psi_2\rangle$ from arbitrary qubit $|\psi_1\rangle$.

In addition, X, Y and Z gates can be represented by using only phase and Hadamard gates as follows,

$$X = H\, R_z(\pi)\, H, \qquad Y = R_z\left(\frac{\pi}{2}\right) H\, R_z(\pi)\, H\, R_z\left(-\frac{\pi}{2}\right), \qquad Z = R_z(\pi).$$

CHAPTER 3

QUBIT IN COMPLEX PLANE

The Bloch sphere representation of one qubit states suggests to use complex numbers to parametrize these states. If one considers the Bloch sphere as a Riemann sphere for complex plane $\mathbb{C}$, then every point in this plane corresponds to some point on the sphere and represents a qubit state. On the Bloch sphere, as the extended complex plane $\mathbb{C} \cup \{\infty\}$, two basis states $|0\rangle$ and $|1\rangle$, representing the north and the south poles of the sphere, corresponds to 0 and ∞ points in this extended plane.

3.1. Bloch Sphere in Complex Plane Representation

An arbitrary qubit state is represented as a point (θ, φ) on the Bloch sphere

$$|\psi\rangle = |\theta, \varphi\rangle = \cos\frac{\theta}{2}\,|0\rangle + \sin\frac{\theta}{2}\,e^{i\varphi}\,|1\rangle, \tag{3.1}$$

where $0 \leq \theta \leq \pi$, $0 \leq \varphi \leq 2\pi$. The stereographic projection from the south pole $(0, 0, -1)$ of this unit sphere to complex plane $\mathbb{C}$, the reflection plane between the north and the south poles as symmetric points, is determined by formula (Figure 3.1)

$$z = \tan\frac{\theta}{2}\,e^{i\varphi} = |z|e^{i\varphi}, \tag{3.2}$$

where $z = x + iy \in \mathbb{C}$ and $|z| = \tan\dfrac{\theta}{2}$.

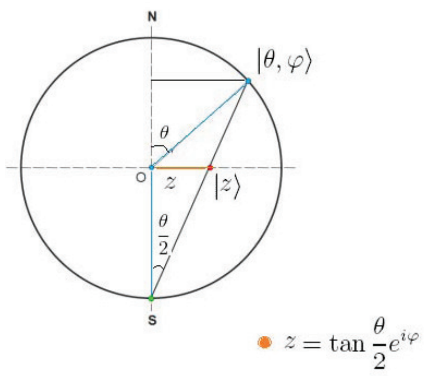


Figure 3.1. Stereographic projection of Bloch sphere

By extracting $\cos\frac{\theta}{2}$ from (3.1),

$$|\psi\rangle = \cos\frac{\theta}{2}\left(|0\rangle + \tan\frac{\theta}{2}\, e^{i\varphi}\,|1\rangle\right),$$

and using

$$\frac{1}{\sqrt{1+|z|^2}} = \frac{1}{\sqrt{1+\tan^2\frac{\theta}{2}}} = \cos\frac{\theta}{2},$$

the one qubit state (3.1)

$$|\psi\rangle = \cos\frac{\theta}{2}\left(|0\rangle + \tan\frac{\theta}{2}\, e^{i\varphi}\,|1\rangle\right) = \frac{1}{\sqrt{1+|z|^2}}\left(|0\rangle + z\,|1\rangle\right)$$

becomes

$$|z\rangle = \frac{|0\rangle + z|1\rangle}{\sqrt{1+|z|^2}}. \tag{3.3}$$

This parametrization of qubit by complex number z is called the qubit coherent state. For every complex number $z \in \mathbb{C} \cup \{\infty\}$ it represents the qubit state $|z\rangle$. Points inside of unit circle $|z| < 1$ correspond to states $|z\rangle$ in upper Bloch hemisphere and points with $|z| > 1$ correspond to states in lower Bloch hemisphere. Point $z = 0$ represents state $|0\rangle$ and point $z = \infty$ corresponds to state $|1\rangle$. Representation of qubit by complex numbers allows one to apply techniques of complex algebra and analysis to study qubit states, which would be discussed in this section. Disadvantage of qubit coherent state representation is that basis state $|1\rangle$ doesn't correspond to finite point in the plane. This prevents proper visualisation of geometrical characteristics of qubits. To correct this disadvantage, in Chapter 5 the Apollonius representation of qubit would be introduced.

3.2. Qubit Coherent States

Generic one qubit state

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle = \begin{pmatrix} \alpha \\ \beta \end{pmatrix},$$

where $|\alpha|^2 + |\beta|^2 = 1$ can be represented in terms of homogenous coordinates $z = \dfrac{\beta}{\alpha}$. By extracting β,

$$|\psi\rangle = \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \beta \begin{pmatrix} 1 \\ z \end{pmatrix},$$

and fixing β by normalization condition $\langle\psi|\psi\rangle = 1$, up to the global phase the qubit state can be written as

$$|z\rangle = \frac{1}{\sqrt{1 + |z|^2}} \begin{pmatrix} 1 \\ z \end{pmatrix}$$

or

$$|z\rangle = \frac{|0\rangle + z|1\rangle}{\sqrt{1 + |z|^2}}.$$

This state is called $SU(2)$ or the spin coherent state, and would be referred as the "coherent qubit state". Every point z in extended complex plane determines the qubit state in this representation.

Definition 3.1 *(Ahlfors, 1966) Two points z and z^* are called symmetric with respect to the circle through z_1, z_2, z_3 if and only if $(z^*, z_1, z_2, z_3) = \overline{(z, z_1, z_2, z_3)}$, where the cross ratio of four points is*

$$(z, z_1, z_2, z_3) = \frac{(z - z_2)(z_1 - z_3)}{(z - z_3)(z_1 - z_2)}.$$

The circle in this definition is the generalized circle, that includes also a line, regarded as a circle with an infinite radius. On the Riemann or the Bloch sphere, all generalized circles are coming from the intersection of the sphere with a plane, so that if the plane passes through the projection pole, the corresponding projection would be a line.

According to definition for given point z exist symmetric points as :

1. Reflection in x- axis: $z^* = \bar{z}$

2. Reflection in y- axis: $z^* = -\bar{z}$

3. Inversion in the unit circle: $z^* = \frac{1}{\bar{z}}$
   Combination of first two reflections give point $-z$, which after inversion in unit circle gives the antipodal point: $z^* = -\frac{1}{\bar{z}}$.

The qubit state $|z\rangle$, for $z = 0$ is state $|0\rangle$ and for $z = \infty$ is state $|1\rangle$. But $z = 0$ and $z = \infty$ are symmetric points with respect to the unit circle. Therefore, corresponding quantum states $|0\rangle$ and $|1\rangle$ is called symmetric states in unit circle.

Above definition suggests for given generic qubit state $|z\rangle$ to define the corresponding symmetric state $|z^*\rangle$ with respect to given circle $\mathbb{S}$. These states can be called symmetric qubits :

1. Symmetric qubit state with respect to the x-axis

$$|\bar{z}\rangle = \frac{|0\rangle + \bar{z}|1\rangle}{\sqrt{1 + |z|^2}}$$

2. Symmetric qubit state with respect to the y-axis

$$|-\bar{z}\rangle = \frac{|0\rangle - \bar{z}|1\rangle}{\sqrt{1+|z|^2}}$$

3. Symmetric qubit state with respect to the unit circle

$$\left|\frac{1}{\bar{z}}\right\rangle = \frac{\bar{z}|0\rangle + |1\rangle}{\sqrt{1+|z|^2}}$$

To antipodal point $z^* = -\frac{1}{\bar{z}}$, corresponds the antipodal qubit state,

$$\left|-\frac{1}{\bar{z}}\right\rangle = \frac{-\bar{z}|0\rangle + |1\rangle}{\sqrt{1+|z|^2}}.$$

The computational basis states $|0\rangle$ and $|1\rangle$ are also antipodal states and orthogonal to each other. It turns out that every pair of antipodal states is orthogonal :

$$\left\langle -\frac{1}{\bar{z}} \middle| z \right\rangle = 0.$$

3.3. Möbius Transformations and Qubit Gates

Definition 3.2 *(Ahlfors, 1966) The linear fractional transformation of the form*

$$w = M(z) = \frac{az+b}{cz+d}, \qquad ad - bc \neq 0 \tag{3.4}$$

is called the Möbius transformation.

It is known from complex analysis that,

- The Möbius transformations transform every generalized circle to another generalized circle.

- The cross ratio defined in section 3.2 is invariant under the Möbius transformation M.
- Symmetric points with respect to a circle transforms by M to another pair of symmetric points for transformed circle.

The Möbius transformations are related with linear transformations in $\mathbb{C}^2$. For two vectors $|z\rangle$ and $|w\rangle$ related by

$$\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix},$$

corresponds the Möbius transformation

$$M(z) = \frac{az+b}{cz+d}$$

acting on the homogenous coordinates $z = \dfrac{z_1}{z_2}$ and $w = \dfrac{w_1}{w_2}$.

This can be applied to qubit coherent states. Transformation between two states $|z\rangle$ and $|w\rangle$

$$|w\rangle = U|z\rangle$$

or in matrix form

$$\begin{pmatrix} w_0 \\ w_1 \end{pmatrix} = \begin{pmatrix} d & c \\ b & a \end{pmatrix} \begin{pmatrix} z_0 \\ z_1 \end{pmatrix},$$

implies the Möbius transformation $M(z)$ (7.4) for homogenous coordinates $w = \dfrac{w_1}{w_0}$ and $z = \dfrac{z_1}{z_0}$. This way every $2x2$ matrix transformation of qubit states is related with Möbius transformation in complex plane :

$$U = \begin{pmatrix} d & c \\ b & a \end{pmatrix} \iff M(z) = \frac{az + b}{cz + d} \tag{3.5}$$

Möbius matrices in (3.5) and (3.6) are related by X gate linear transformation (flipped Mobius transformation)

$$\begin{pmatrix} d & c \\ b & a \end{pmatrix} = X \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix} X .$$

3.3.1. Unitary Möbius Transformations

From all linear transformations for qubit the important is the one class of transformations, preserving norm of the qubit states. If $|w\rangle = U|z\rangle$ then

$$\langle w|w\rangle = \langle z|U\dagger U|z\rangle = \langle z|z\rangle$$

and

$$U^{\dagger} U = I,$$

so that matrix U should be unitary. General form of 2×2 unitary matrix is

$$U = \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix} \tag{3.6}$$

where $|a|^2 + |b|^2 = 1$. This transformation determines generic one qubit gate. Corresponding Möbius transformation from (3.5) is

$$w = M(z) = \frac{\bar{a}z - \bar{b}}{bz + a}$$

Theorem 3.1 *(Stillwell, 1992)(Gauss) The maps of the z plane $\mathbb{C}$ induced by rotations of S^2 are precisely the functions*

$$w = M(z) = \frac{az + b}{-\bar{b}z + \bar{a}},$$

where $a, b \in \mathbb{C}$ and $|a|^2 + |b|^2 = 1$.

This theorem implies that every rotation of the Bloch sphere is determined by unitary matrix, described by above Möbius transformation.

Definition of Möbius transformation (3.5), due to the special choice for homogenous coordinate z is different from the standard notation in complex analysis, which corresponds to $z \to \frac{1}{z}$. These two transformations are connected by simple change of notations $a \to \bar{a}$ and $b \to \bar{b}$. In addition, the general form of unitary transformation U (3.6) is defined up to phase,

$$U \to \pm iU.$$

Indeed, the Möbius transformation is invariant under rescaling :

$$U \to \gamma U,$$

where $\gamma \in \mathbb{C}$. Due to unitarity, $|\gamma| = 1$ and $\gamma = e^{i\lambda}$. The special case $\lambda = \pm\frac{\pi}{2}$ gives identification of U and $\pm iU$.

3.3.2. Anti-Unitary Möbius Transformations

Symmetric point with respect to generalized circles are not Möbius transformed points. Since it includes reflection or inversion in a line or in a circle with operation of complex conjugation. To consider these points as Möbius transformed ones, an extension of Möbius transformation is required.

Definition 3.3 *(Stillwell, 1992) The anti - Möbius transformation or an anti- homography transformations are*

$$w = M(\bar{z}) = \frac{a\bar{z} + b}{c\bar{z} + d},$$

with $a, b, c, d \in \mathbb{C}$ and $ad - bc \neq 0$.

Theorem 3.2 *(Blair, 2000) Anti - Homographies (anti - Möbius) map lines and circles to lines and circles and are conformal.*

Definition 3.4 *(Blair, 2000) The set of all Möbius and anti - Möbius transformations forms the group called the extended Möbius transformations.*

Reflections in axis and inversions in the circles considered in previous section corresponds to special cases of anti - Möbius transformations.

1. Reflection in x- axis: $z^* = \bar{z} \Longleftrightarrow a = d = 1$ and $b = c = 0$
2. Reflection in y- axis: $z^* = -\bar{z} \Longleftrightarrow a = 1, d = -1$ and $b = c = 0$
3. Inversion in the unit circle: $z^* = \frac{1}{\bar{z}} \Longleftrightarrow a = 0, d = 0$ and $b = 1, c = 1$

   The antipodal point is also result of anti- Möbius transformations : $z^* = -\frac{1}{\bar{z}} \Longleftrightarrow a = 0, d = 0$ and $b = 1, c = -1$

As easy to see, every anti - Möbius transformation is composition of the usual Möbius transformation M and special transformation $K : z \to \bar{z}$. According to this, anti- unitary Möbius transformations and anti- unitary transformations can be derived.

Definition 3.5 *Anti- unitary transformation U_A is defined as $U_A = U \cdot K$ where U is unitary matrix and K is anti - unitary map $K : z \to \bar{z}$.*

This implies the following definition :

Definition 3.6 *(Stillwell, 1992) Anti- unitary Möbius transformation is*

$$w = M(\bar{z}) = \frac{a\bar{z} + b}{-\bar{b}\bar{z} + \bar{a}},$$

with $a, b, c, d \in \mathbb{C}$ and $|a|^2 + |b|^2 = 1$.

Theorem 3.3 *(Stillwell, 1992) The maps of the z plane $\mathbb{C}$ induced by orientation - reversing isometries of S^2 are precisely the functions.*

$$w = M(\bar{z}) = \frac{a\bar{z} + b}{-\bar{b}\bar{z} + \bar{a}}$$

with $a, b, c, d \in \mathbb{C}$ and $|a|^2 + |b|^2 = 1$.

This theorem implies that every isometry of the Bloch sphere is determined by unitary and anti- unitary matrices described by Möbius and anti - Möbius transformation.

3.3.3. Möbius Qubit Gates

According to previous consideration, every one qubit gate can be represented by corresponding Möbius transformation acting on point in complex plane, representing qubit state. For Pauli gates Möbius transformations are following :

1. **Möbius X - Gate**

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \longrightarrow w = M_X(z) = \frac{1}{z}$$

2. **Möbius Y - Gate**

$$Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \longrightarrow w = M_Y(z) = -\frac{1}{z}$$

3. **Möbius Z - Gate**

$$Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \longrightarrow w = M_Z(z) = -z$$

Universal one qubit gates, the Hadamard H and the phase gate $R_z(\theta)$, correspond to Möbius gates :

1. **Möbius Hadamard Gate**

$$H = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \longrightarrow w = M_H(z) = \frac{1-z}{1+z} \tag{3.7}$$

2. **Möbius Phase Gate**

$$R_z(\theta) = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\theta} \end{pmatrix} \longrightarrow w = M_\theta(z) = z\, e^{i\theta} \tag{3.8}$$

Since Hadamard H and phase gate $R_z(\theta)$ are universal one qubit gates, $M_H(z)$ and $M_\theta(z)$ are universal Möbius transformations, such that every unitary Möbius transformation is a combination of these two gates.

3.3.3.1. Qubit States Generated by Möbius Gates

The above Möbius gates generates the following qubit states.

1. **By Möbius X - Gate**

$$M_X \,:\, z \rightarrow \frac{1}{z} \Longleftrightarrow \left|\frac{1}{z}\right\rangle = \frac{z|0\rangle + |1\rangle}{\sqrt{1+|z|^2}}$$

2. **By Möbius Y - Gate**

$$M_Y \,:\, z \rightarrow -\frac{1}{z} \Longleftrightarrow \left|-\frac{1}{z}\right\rangle = \frac{-z|0\rangle + |1\rangle}{\sqrt{1+|z|^2}}$$

3. **By Möbius Z - Gate**

$$M_Z\ : z \to -z \Longleftrightarrow |-z\rangle = \frac{|0\rangle - z|1\rangle}{\sqrt{1+|z|^2}}$$

The qubit states generated by universal Möbius gates are

1. **By Möbius Hadamard Gate**

$$M_H(z) = \frac{1-z}{1+z} \Longleftrightarrow \left|\frac{1-z}{1+z}\right\rangle = \frac{(1+z)\,|0\rangle + (1-z)\,|1\rangle}{\sqrt{|1+z|^2 + |1-z|^2}}$$

This state is called "Apollonius one gubit state" and it is studied in Chapter 5.

2. **By Möbius Phase Gate**

$$M_\theta(z) = z\,e^{i\theta} \Longleftrightarrow |z\,e^{i\theta}\rangle = \frac{|0\rangle + z\,e^{i\theta}\,|1\rangle}{\sqrt{1+|z|^2}}.$$

3.3.4. Anti - Möbius Qubit Gates

Since every anti - unitary transformation U_A is composition of unitary transformation U and anti - unitary K, one can describe corresponding qubit gates and anti - unitary Möbius transformations by such decomposition. Anti - unitary Möbius transformation $K : z \to \bar{z}$ is acting on one qubit coherent state as $|\bar{z}\rangle = K|z\rangle$. Combination of this transformation with Möbius gates gives generic anti - Möbius transformation.

Definition 3.7 *The set of Möbius and anti - Möbius qubit gates are called extended Möbius gates.*

Since every one qubit gate is composition of Hadamard and phase gates which are universal gates, an addition of anti - unitary gate K allows one to describe also symmetric and antipodal states.

Theorem 3.4 *The Hadamard gate H, the phase gate $R_\theta(z)$ and anti-unitary transformation K represent extended universal one qubit quantum gates.*

3.3.4.1. Qubit States Generated by Anti - Möbius Gates

1. **By K - Gate**

$$K\ : z \to \bar{z} \Longleftrightarrow |\bar{z}\rangle = \frac{|0\rangle + \bar{z}\,|1\rangle}{\sqrt{1+|z|^2}}$$

2. **By Anti - Möbius Z - Gate**

$$M_Z\,K\ : z \to -\bar{z} \Longleftrightarrow |-\bar{z}\rangle = \frac{|0\rangle - \bar{z}\,|1\rangle}{\sqrt{1+|z|^2}}$$

3. **By Anti - Möbius X - Gate**

$$M_X\,K : z \to \frac{1}{\bar{z}} \Longleftrightarrow \left|\frac{1}{\bar{z}}\right\rangle = \frac{\bar{z}|0\rangle + |1\rangle}{\sqrt{1+|z|^2}}$$

This state is symmetric qubit state with respect to the unit circle.

4. **By Anti - Möbius Y - Gate**

$$M_Y\,K : z \to -\frac{1}{\bar{z}} \Longleftrightarrow \left|-\frac{1}{\bar{z}}\right\rangle = \frac{-\bar{z}|0\rangle + |1\rangle}{\sqrt{1+|z|^2}}$$

This state is antipodal qubit state.

3.3.5. Gate Action on Qubit States

Extended Mobius transformations acting on given state $|z\rangle$ produce the set of related states as reflections in coordinate axis and in the unit circle :

$$|z\rangle = \frac{|0\rangle + z\,|1\rangle}{\sqrt{1+|z|^2}}, \qquad |\bar{z}\rangle = \frac{|0\rangle + \bar{z}\,|1\rangle}{\sqrt{1+|z|^2}},$$

$$|-z\rangle = \frac{|0\rangle - z\,|1\rangle}{\sqrt{1+|z|^2}}, \qquad |-\bar{z}\rangle = \frac{|0\rangle - \bar{z}\,|1\rangle}{\sqrt{1+|z|^2}},$$

$$\left|\frac{1}{z}\right\rangle = \frac{z|0\rangle + |1\rangle}{\sqrt{1+|z|^2}}, \qquad \left|\frac{1}{\bar{z}}\right\rangle = \frac{\bar{z}|0\rangle + |1\rangle}{\sqrt{1+|z|^2}},$$

$$\left|-\frac{1}{z}\right\rangle = \frac{-z|0\rangle + |1\rangle}{\sqrt{1+|z|^2}}, \qquad \left|-\frac{1}{\bar{z}}\right\rangle = \frac{-\bar{z}|0\rangle + |1\rangle}{\sqrt{1+|z|^2}}.$$

In this list, the states with complex conjugation $z \to \bar{z}$ are connected by anti - unitary transformation K.

Pauli gates are acting on the above states in the following way :

$$X\,|z\rangle = \left|\frac{1}{z}\right\rangle, \qquad Y\,|z\rangle = i\left|-\frac{1}{z}\right\rangle, \qquad Z\,|z\rangle = |-z\rangle,$$

$$X\left|-\frac{1}{\bar{z}}\right\rangle = |-\bar{z}\rangle, \qquad Y\left|-\frac{1}{\bar{z}}\right\rangle = -i\,|\bar{z}\rangle, \qquad Z\left|-\frac{1}{\bar{z}}\right\rangle = -\left|\frac{1}{\bar{z}}\right\rangle.$$

Transition amplitudes between the states are

$$\langle z | -z \rangle = \frac{1 - |z|^2}{1 + |z|^2} = \left\langle -\frac{1}{\bar{z}} \middle| \frac{1}{\bar{z}} \right\rangle,$$

$$\left\langle z \middle| \frac{1}{z} \right\rangle = \frac{z + \bar{z}}{1 + |z|^2} = -\left\langle -\frac{1}{\bar{z}} \middle| -\bar{z} \right\rangle,$$

$$\left\langle z \middle| -\frac{1}{z} \right\rangle = \frac{\bar{z} - z}{1 + |z|^2} = \left\langle -\frac{1}{\bar{z}} \middle| \bar{z} \right\rangle.$$

3.4. Universality of One Qubit Computations

In Section 2.3.5 universality of one qubit computations was shown for qubits on the Bloch sphere. Below universality of one qubit computations in coherent state representation is derived. An arbitrary coherent state $|z\rangle$ can be generated from basis state $|0\rangle$:

$$|0\rangle \xrightarrow{A_0(z)} |z\rangle,$$

where

$$A_0(z) = \frac{1}{\sqrt{1 + |z|^2}} \begin{pmatrix} 1 & -\bar{z} \\ z & 1 \end{pmatrix}, \tag{3.9}$$

and

$$A_0^\dagger(z) = \frac{1}{\sqrt{1 + |z|^2}} \begin{pmatrix} 1 & \bar{z} \\ -z & 1 \end{pmatrix}.$$

For this matrix $\det A_0(z) = 1$, and it is unitary $A_0\dagger(z)A_0(z) = I$, so that

$$A_0^\dagger(z)A_0(z) = \frac{1}{1+|z|^2}\begin{pmatrix} 1+|z|^2 & 0 \\ 0 & 1+|z|^2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I.$$

In a similar way, the state $|z\rangle$ can be generated from state $|1\rangle$:

$$|1\rangle \xrightarrow{A_1(z)} |z\rangle,$$

where

$$A_1(z) = \frac{1}{\sqrt{1+|z|^2}}\begin{pmatrix} \bar{z} & 1 \\ -1 & z \end{pmatrix}, \tag{3.10}$$

and

$$A_1^\dagger(z) = \frac{1}{\sqrt{1+|z|^2}}\begin{pmatrix} z & -1 \\ 1 & \bar{z} \end{pmatrix}.$$

For this matrix $\det A_1(z) = 1$, and it is unitary $A_1^\dagger(z)A_1(z) = I$, so that

$$A_1^\dagger(z)A_1(z) = \frac{1}{1+|z|^2}\begin{pmatrix} 1+|z|^2 & 0 \\ 0 & 1+|z|^2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I.$$

By using $A_0(z)$ (3.9), it is possible to relate arbitrary state $|z_1\rangle$ with another arbitrary state $|z_2\rangle$, $|z_1\rangle \to |z_2\rangle$:

$$\begin{aligned} |z_1\rangle &= A_0(z_1)\,|0\rangle, \qquad & |z_2\rangle = A_0(z_2)\,|0\rangle, \\ (A_0^\dagger(z_1))\,|z_1\rangle &= |0\rangle, \qquad & (A_0^\dagger(z_2))\,|z_2\rangle = |0\rangle. \end{aligned}$$

This implies that

$$(A_0^\dagger(z_1))\,|z_1\rangle = (A_0^\dagger(z_2))\,|z_2\rangle \rightarrow |z_2\rangle = A_0(z_2)(A_0^\dagger(z_1))\,|z_1\rangle. \tag{3.11}$$

Universality of one qubit computations in coherent state form means that, transformations $A_0(z)$ and $A_1(z)$ can be written as decomposition of the Hadamard and the phase gate. It can be seen explicitly from following formulas

$$A_0(\theta,\phi) = e^{-i\frac{\theta}{2}}\,R\left(\phi+\frac{\pi}{2}\right)H\,R(\theta)\,H\,R\left(-\phi-\frac{\pi}{2}\right), \tag{3.12}$$

and

$$A_1 = A_0\,i\,X$$

giving

$$A_1(\theta,\phi) = e^{-i\left(\frac{\theta}{2}-\frac{\pi}{2}\right)}R\left(\phi+\frac{\pi}{2}\right)H\,R(\theta)\,H\,R(-\phi)\,H\,R(\pi)\,H\,R\left(-\frac{\pi}{2}\right). \tag{3.13}$$

The circuits for A_0 (3.12) and for A_1 (3.13) generate one qubit coherent state from $|0\rangle$ and $|1\rangle$ states, respectively,

$$|0\rangle \xrightarrow{A_0(z)} |z\rangle = \frac{|0\rangle + z\,|1\rangle}{\sqrt{1+|z|^2}},$$

$$|1\rangle \xrightarrow{A_1(z)} |z\rangle = \frac{|0\rangle + z\,|1\rangle}{\sqrt{1+|z|^2}}.$$

In addition to this, applying A_0 to state $|1\rangle$ gives the antipodal state,

$$|1\rangle \xrightarrow{A_0(z)} \left|-\frac{1}{\bar{z}}\right\rangle = |z^*\rangle = \frac{-\bar{z}\,|0\rangle + |1\rangle}{\sqrt{1+|z|^2}}.$$

Since Hadamard and phase gates are universal Möbius gates, transformation $A_0^\dagger(z_2)A_0(z_1)$ (3.11) between arbitrary states $|z_1\rangle$ and $|z_2\rangle$, can be implemented by the set of these Möbius gates.

3.4.1. Universality of Möbius Gates

The relation between 2×2 matrix gate

$$U = \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix},$$

and the Möbius transformation

$$w = M(\xi) = \frac{\bar{a}\xi - \bar{b}}{b\xi + a},$$

which has been discussed in Section 3.3. According to this relation, the Möbius gates corresponding to $A_0(z)$ and $A_1(z)$ can be derived as

$$A_0(z) = \begin{pmatrix} 1 & -\bar{z} \\ z & 1 \end{pmatrix} \xrightarrow{a=1, b=-\bar{z}} M_0^z(\xi) = \frac{\xi + z}{-\bar{z}\xi + 1}, \tag{3.14}$$

and

$$A_1(z) = \begin{pmatrix} \bar{z} & 1 \\ -1 & z \end{pmatrix} \xrightarrow{a=\bar{z}, b=1} M_1^z(\xi) = \frac{z\xi - 1}{\xi + \bar{z}}. \tag{3.15}$$

The matrices $A_0(z)$ and $A_1(z)$ are acting on states $\begin{pmatrix}\xi_0\\ \xi_1\end{pmatrix}$ and corresponding Möbius transformations act on projective coordinates $\xi = \frac{\xi_1}{\xi_0}$. The Möbius transformation $M_0(z)$ is determined by zero at $\xi = -z$ and pole at $\xi = \frac{1}{\bar{z}}$, while $M_1(z)$ is determined by zero at $\xi = \frac{1}{z}$ and pole at $\xi = -\bar{z}$. These transformations act on computational basis $|0\rangle$ and $|1\rangle$, corresponding to $z = 0$ and $z = \infty$, as

$$|0\rangle \xrightarrow{A_0(z)} |z\rangle, \qquad M_0^z(0) = z,$$

$$|1\rangle \xrightarrow{A_0(z)} \left|-\frac{1}{\bar{z}}\right\rangle, \qquad M_0^z(\infty) = -\frac{1}{\bar{z}},$$

$$|1\rangle \xrightarrow{A_1(z)} |z\rangle \qquad M_1^z(\infty) = z$$

$$|0\rangle \xrightarrow{A_1(z)} \left|-\frac{1}{\bar{z}}\right\rangle \qquad M_1^z(0) = -\frac{1}{\bar{z}}$$

Combining these transformations like in (3.11), it is possible to derive the Möbius transformation, relating arbitrary one qubit states $|z_1\rangle \to |z_2\rangle$,

$$A_0(z_2)A_0^\dagger(z_1) \longleftrightarrow M^{z_1 z_2}(\xi) = \frac{(1 + \bar{z}_1 z_2)\xi + z_2 - z_1}{(\bar{z}_1 - \bar{z}_2)\xi + 1 + z_1\bar{z}_2},$$

$$|z_1\rangle \xrightarrow{A_0(z_2)A_0^\dagger(z_1)} |z_2\rangle, \qquad z_2 = M^{z_1 z_2}(z_1).$$

Universality decomposition of one qubit gates (3.12) and (3.13) in terms of the Hadamard and the phase gates, implies that corresponding Möbius transformations (3.14) and (3.15) can be decomposed to universal Möbius gates, namely the Hadamard Möbius gate (3.7), $w = M_H(z) = \dfrac{1-z}{1+z}$, and the phase Möbius gate (3.8), $w = M_\theta(z) = z\, e^{i\theta}$:

$$M_0^z(\xi) = M_{\phi+\frac{\pi}{2}}\left(M_H\left(M_\theta\left(M_H\left(M_{-\phi-\frac{\pi}{2}}(\xi)\right)\right)\right)\right),$$

$$M_1^z(\xi) = M_{\phi+\frac{\pi}{2}}\left(M_H\left(M_\theta\left(M_H\left(M_{-\phi}\left(M_H\left(M_\pi\left(M_H\left(M_{-\frac{\pi}{2}}(\xi)\right)\right)\right)\right)\right)\right)\right)\right),$$

3.5. Fidelity Between Symmetric States

Definition 3.8 *Fidelity between symmetric states is*

$$F = |\langle z^*|z\rangle| \tag{3.16}$$

where $|z\rangle$ and $|z^*\rangle$ are symmetric qubit states, corresponding to symmetric points z and z^* with respect to a generalized circle.

From definition, it is evident that

$$0 \le F \le 1.$$

Indeed, on the generalized circle the symmetric states coincide and $F = 1$. This characteristic of one qubit state is important due to several reasons. For two qubit states it gives concurrence characteristics of entanglement. As it will be discuss in Chapter 5, the concurrence in this form is constant along Apollonius circles. It is also can characterize multi-qubit states of the special form.

It would be shown below that fidelity defined in this way is invariant under unitary Möbius transformations. As is well known the generic Möbius transformation transforms symmetric points z, z^* to symmetric points w, w^*. This implies that corresponding unitary transformation maps symmetric states $|z\rangle$, $|z^*\rangle$ with respect to circle S_1, to symmetric states $|w\rangle$, $|w^*\rangle$ with respect to another circle S_2 where circle S_2 is Möbius image of circle S_1. Since our Möbius transformations are unitary, the inner product between symmetric states is preserved.

$$\langle z^*|z\rangle = \langle w^*|w\rangle.$$

Then, the corresponding fidelities coincide:

$$F_z = |\langle z^*|z\rangle| = |\langle w^*|w\rangle| = F_w. \tag{3.17}$$

This is why if

$$w = M(z) \qquad w^* = M(z^*)$$

is Möbius transformation of symmetric points, then fidelity is not changing.

$$F = |\langle z^*|z\rangle| = |\langle w^*|w\rangle| = |\langle M(z^*)|M(z)\rangle|.$$

CHAPTER 4

TWO QUBIT STATES

Here some notions for tensor product of qubit states is introduced. For more details see (Benenti, Casati and Strini, 2004) and (McMahon, 2008).

4.1. Two Qubit States

Definition 4.1 *Tensor Product*
Consider two Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$ of dimensions m and n, respectively. The Hilbert space $\mathcal{H}$ is the tensor product of these two spaces, such that $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$. The state $|\psi\rangle \in \mathcal{H}$, associated with each pair of vectors $|\alpha\rangle \in \mathcal{H}_1$ and $|\beta\rangle \in \mathcal{H}_2$ is denoted as

$$|\psi\rangle = |\alpha\rangle \otimes |\beta\rangle,$$

and is called the tensor product of states $|\alpha\rangle$ and $|\beta\rangle$. The shorthand notation for tensor product is

$$|\alpha\rangle \otimes |\beta\rangle \equiv |\alpha\beta\rangle \equiv |\alpha\rangle\,|\beta\rangle.$$

Definition 4.2 *The matrix representation for tensor product of one qubit states*

$$|\alpha\rangle = \begin{pmatrix} \alpha_0 \\ \alpha_1 \end{pmatrix}$$

and

$$|\beta\rangle = \begin{pmatrix} \beta_0 \\ \beta_1 \end{pmatrix}$$

is

$$|\alpha\rangle \otimes |\beta\rangle = \begin{pmatrix} \alpha_0 \\ \alpha_1 \end{pmatrix} \otimes \begin{pmatrix} \beta_0 \\ \beta_1 \end{pmatrix} = \begin{pmatrix} \alpha_0 \cdot \begin{pmatrix} \beta_0 \\ \beta_1 \end{pmatrix} \\ \alpha_1 \cdot \begin{pmatrix} \beta_0 \\ \beta_1 \end{pmatrix} \end{pmatrix} = \begin{pmatrix} \alpha_0\beta_0 \\ \alpha_0\beta_1 \\ \alpha_1\beta_0 \\ \alpha_1\beta_1 \end{pmatrix}.$$

Definition 4.3 *If $\mathcal{H}_1$ and $\mathcal{H}_2$ are two dimensional vector spaces* $(m = n = 2)$*, with basis vectors $|0\rangle$ and $|1\rangle$, then $\mathcal{H}$ has dimension $m \cdot n = 4$ with following basis vectors,*

$$|0\rangle \otimes |0\rangle = |00\rangle, \quad |0\rangle \otimes |1\rangle = |01\rangle, \quad |1\rangle \otimes |0\rangle = |10\rangle, \quad |1\rangle \otimes |1\rangle = |11\rangle$$

called the computational basis. The matrix representation for the computational basis states is

$$|0\rangle \otimes |0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad |0\rangle \otimes |1\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix},$$

$$|1\rangle \otimes |0\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad |1\rangle \otimes |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}.$$

Definition 4.4 *The generic two qubit state $|\psi\rangle \in \mathcal{H}$ is defined as*

$$|\psi\rangle = \sum_{i,j=0,1} c_{ij}|i\rangle \otimes |j\rangle = c_{00}|00\rangle + c_{01}|01\rangle + c_{10}|10\rangle + c_{11}|11\rangle, \tag{4.1}$$

where complex valued coefficients $c_{ij} = \langle ij|\psi\rangle$ determine probabilities $p_{ij} = |\langle ij|\psi\rangle|^2$,

$i, j = 0, 1$. The total probability is

$$\sum_{ij} p_{ij} = p_{00} + p_{01} + p_{10} + p_{11} = 1,$$

implying normalization condition for state $|\psi\rangle$,

$$\langle\psi|\psi\rangle = |c_{00}|^2 + |c_{01}|^2 + |c_{10}|^2 + |c_{11}|^2 = 1.$$

4.1.1. Classification of Two Qubit States

This section introduces, classification of two qubit states as separable and entangled states .

Definition 4.5 *If the generic two qubit state $|\psi\rangle$ in (4.1) can be represented as a tensor product of one qubit states, then state $|\psi\rangle$ is called the* ***separable state***

$$\boxed{|\psi\rangle = |\alpha\rangle \otimes |\beta\rangle \equiv |\alpha\beta\rangle}$$

Proposition 4.1 *The generic separable two qubit state as the tensor product of one qubit states, $|\alpha\rangle = \alpha_0|0\rangle + \alpha_1|1\rangle$ and $|\beta\rangle = \beta_0|0\rangle + \beta_1|1\rangle$ has the form*

$$|\alpha\beta\rangle = \alpha_0\beta_0|00\rangle + \alpha_0\beta_1|01\rangle + \alpha_1\beta_0|10\rangle + \alpha_1\beta_1|11\rangle.$$

Proof Tensor product of states $|\alpha\rangle = \alpha_0|0\rangle + \alpha_1|1\rangle$ and $|\beta\rangle = \beta_0|0\rangle + \beta_1|1\rangle$ gives

$$\begin{aligned} |\alpha\beta\rangle = |\alpha\rangle|\beta\rangle &= (\alpha_0|0\rangle + \alpha_1|1\rangle)(\beta_0|0\rangle + \beta_1|1\rangle) \\ &= \alpha_0|0\rangle(\beta_0|0\rangle + \beta_1|1\rangle) + \alpha_1|1\rangle(\beta_0|0\rangle + \beta_1|1\rangle) \\ &= \alpha_0\beta_0|00\rangle + \alpha_0\beta_1|01\rangle + \alpha_1\beta_0|10\rangle + \alpha_1\beta_1|11\rangle. \end{aligned}$$

□

Definition 4.6 *If the generic two qubit state $|\psi\rangle$ in (4.1), cannot be represented as a tensor product of one qubit states, then state $|\psi\rangle$ is called the **entangled state***

$$\boxed{|\psi\rangle \neq |\alpha\rangle \otimes |\beta\rangle}$$

4.1.1.1. Generic Separable States

In this section, criterium of separability for generic two qubit state is derived.

Lemma 4.1 *An arbitrary generic two qubit state*

$$|\psi\rangle = c_{00}|00\rangle + c_{01}|01\rangle + c_{10}|10\rangle + c_{11}|11\rangle$$

can be represented as

$$|\psi\rangle = |0\rangle \otimes |\psi_1\rangle + |1\rangle \otimes |\psi_2\rangle, \tag{4.2}$$

where the one qubit states are

$$|\psi_1\rangle = c_{00}|0\rangle + c_{01}|1\rangle, \qquad |\psi_2\rangle = c_{10}|0\rangle + c_{11}|1\rangle.$$

Proof It follows from decomposition

$$\begin{aligned} |\psi\rangle &= c_{00}|00\rangle + c_{01}|01\rangle + c_{10}|10\rangle + c_{11}|11\rangle \\ &= |0\rangle \otimes \underbrace{(c_{00}|0\rangle + c_{01}|1\rangle)}_{|\psi_1\rangle} + |1\rangle \otimes \underbrace{(c_{10}|0\rangle + c_{11}|1\rangle)}_{|\psi_2\rangle}, \end{aligned}$$

thus

$$|\psi\rangle = |0\rangle \otimes |\psi_1\rangle + |1\rangle \otimes |\psi_2\rangle.$$

$\square$

From this lemma the criterium of separability follows.

Theorem 4.1 ***Separability and Linear Dependence***
The state

$$|\psi\rangle = |0\rangle \otimes |\psi_1\rangle + |1\rangle \otimes |\psi_2\rangle$$

is separable if and only if the states $|\psi_1\rangle$ *and* $|\psi_2\rangle$ *are linearly dependent*

$$|\psi_1\rangle = \lambda|\psi_2\rangle.$$

Proof

- **($\Longrightarrow$)** Assume $|\psi\rangle$ is a separable two qubit state, such that

  $$|\psi\rangle = |\alpha\rangle \otimes |\beta\rangle,$$

  where $|\alpha\rangle = \alpha_0|0\rangle + \alpha_1|1\rangle$. Then

  $$|\psi\rangle = |\alpha\rangle \otimes |\beta\rangle = (\alpha_0|0\rangle + \alpha_1|1\rangle) \otimes |\beta\rangle = \alpha_0|0\rangle \otimes |\beta\rangle + \alpha_1|1\rangle \otimes |\beta\rangle,$$

  and the state $|\psi\rangle$ can be represented as

  $$|\psi\rangle = \alpha_0|0\rangle \otimes |\beta\rangle + \alpha_1|1\rangle \otimes |\beta\rangle = |0\rangle \otimes \underbrace{\alpha_0|\beta\rangle}_{|\psi_1\rangle} + |1\rangle \otimes \underbrace{\alpha_1|\beta\rangle}_{|\psi_2\rangle}.$$

  It is clear that, states $|\psi_1\rangle$ and $|\psi_2\rangle$ are linearly dependent,

  $$|\psi_1\rangle = \lambda|\psi_2\rangle.$$

- **($\Longleftarrow$)** Assume that the state $|\psi\rangle$ in (4.2) is such that $|\psi_1\rangle = \lambda|\psi_2\rangle$ are linearly dependent states. Substitution of state $|\psi_1\rangle$ into (4.2) gives

  $$|\psi\rangle = |0\rangle \otimes \lambda|\psi_2\rangle + |1\rangle \otimes |\psi_2\rangle = (\lambda|0\rangle + |1\rangle) \otimes |\psi_2\rangle.$$

This shows that the state $|\psi\rangle$ is separable.

$\square$

It should be noticed that instead of representation (4.2), another similar form of decomposition is possible to use

$$|\chi\rangle = \underbrace{(c_{00}|0\rangle + c_{10}|1\rangle)}_{|\chi_1\rangle} \otimes|0\rangle + \underbrace{(c_{10}|0\rangle + c_{11}|1\rangle)}_{|\chi_2\rangle} \otimes|1\rangle. \tag{4.3}$$

In this case, separability of state $|\chi\rangle$ is related with linear dependence of states $|\chi_1\rangle$ and $|\chi_2\rangle$: $|\chi_1\rangle = \lambda|\chi_2\rangle$.

4.1.1.2. Separability and Determinant

As it is well known, for the linear dependent vectors, the determinant of corresponding coefficients vanishes. This is why, separability of two qubit states can be connected with values of the determinant.

Theorem 4.2 *The generic state $|\psi\rangle$ in* (4.1) *is separable if and only if determinant of the coefficients vanishes*

$$D = \begin{vmatrix} c_{00} & c_{01} \\ c_{10} & c_{11} \end{vmatrix} = 0. \tag{4.4}$$

Proof As it is mentioned above, the state $|\psi\rangle$, represented in the form (4.2) is separable, if and only if one qubit states are linearly dependent

$$|\psi_1\rangle = \lambda|\psi_2\rangle. \tag{4.5}$$

- ($\Longrightarrow$) Suppose that one qubit states $|\psi_1\rangle$ and $|\psi_2\rangle$ are expended as $|\psi_1\rangle = c_{00}|0\rangle + c_{01}|1\rangle$ and $|\psi_2\rangle = c_{10}|0\rangle + c_{11}|1\rangle$. Substitution into (4.5) gives

$$|\psi_1\rangle = c_{00}|0\rangle + c_{01}|1\rangle = \lambda(c_{10}|0\rangle + c_{11}|1\rangle) = \lambda|\psi_2\rangle.$$

This implies

$$c_{00} = \lambda c_{10}, \qquad c_{01} = \lambda c_{11},$$

and corresponding determinant

$$D = \begin{vmatrix} c_{00} & c_{01} \\ c_{10} & c_{11} \end{vmatrix} = \begin{vmatrix} \lambda c_{10} & \lambda c_{11} \\ c_{10} & c_{11} \end{vmatrix} = 0.$$

- ($\Longleftarrow$) Let for states $|\psi_1\rangle = c_{00}|0\rangle + c_{01}|1\rangle$ and $|\psi_2\rangle = c_{10}|0\rangle + c_{11}|1\rangle$ the determinant vanishes

$$D = \begin{vmatrix} c_{00} & c_{01} \\ c_{10} & c_{11} \end{vmatrix} = 0.$$

Then

$$c_{00} \cdot c_{11} = c_{10} \cdot c_{01}$$

or

$$\frac{c_{00}}{c_{10}} = \frac{c_{01}}{c_{11}} \equiv \lambda,$$

and as follows, the states $|\psi_1\rangle$ and $|\psi_2\rangle$ are linearly dependent

$$|\psi_1\rangle = \lambda|\psi_2\rangle.$$

□

As a result, this theorem establishes separability criterium for two qubit state in terms of vanishing determinant.

4.1.1.3. Separability, Determinant and Area Relation

Since determinant corresponding to two real vectors in plane has geometrical meaning of the parallelogram area, it allows one to relate separability condition with that area. In the special case of two qubit states with real coefficients, expansion (4.2) relates this qubit state with pair of real vectors $\vec{r_0} \equiv (r_{00}, r_{01})$ and $\vec{r_1} \equiv (r_{10}, r_{11})$,

$$\begin{aligned} |\psi\rangle &= r_{00}|00\rangle + r_{01}|01\rangle + r_{10}|10\rangle + r_{11}|11\rangle \\ &= |0\rangle(r_{00}|0\rangle + r_{01}|1\rangle) + |1\rangle(r_{10}|0\rangle + r_{11}|1\rangle). \end{aligned} \tag{4.6}$$

Definition 4.7 *The one qubit state*

$$|r\rangle = r_0|0\rangle + r_1|1\rangle, \quad r_0^2 + r_1^2 = 1$$

with real coefficients r_0 and r_1 is called the rebit.

This definition implies that generic two qubit rebit state (4.6) can be represented by two one rebit states. Then, separability condition for two rebit state is related with linear dependence of two real vectors $\vec{r_0} \equiv (r_{00}, r_{01})$ and $\vec{r_1} \equiv (r_{10}, r_{11})$ corresponding to rebits

$$|r_0\rangle = r_{00}|0\rangle + r_{01}|1\rangle,$$

$$|r_1\rangle = r_{10}|0\rangle + r_{11}|1\rangle.$$

These vectors determine the parallelogram in plane with area

$$A = |\vec{r_0} \times \vec{r_1}| = \begin{vmatrix} r_{00} & r_{01} \\ r_{10} & r_{11} \end{vmatrix} = |\vec{r_0}||\vec{r_1}| \sin\theta. \tag{4.7}$$

For separable two rebit states, vectors $\vec{r}_0$ and $\vec{r}_1$ are linearly dependent

$$\vec{r}_0 = \lambda \vec{r}_1$$

and corresponding determinant and area are vanishing.The vanishing area condition $A = 0$ means that angle between these two vectors is $\theta = 0$ (parallel) or $\theta = \pi$ (anti-parallel).

- **Maximum and Minimum Area**

If $A \neq 0$, the corresponding determinant $D \neq 0$ and related two rebit state is entangled. Since $0 \leq A \leq A_{max}$, this area can be considered as a measure of entanglement for two rebit states.

To find maximum value of area A_{max}, the following optimization problem can be formulated.

Optimization Problem (Real Case)

Find maximal area of parallelogram $(\vec{r_0}, \vec{r_1})$ with constraint $|\vec{r_0}|^2 + |\vec{r_1}|^2 = 1$.

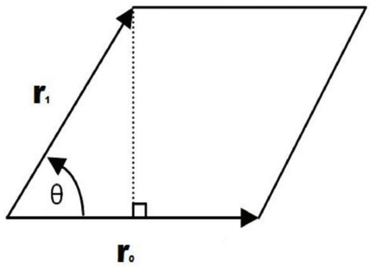


Figure 4.1. Parallelogram

Solution: The maximal area of parallelogram (4.7) corresponds to maximal value of $\sin\theta$, which is 1 for $\theta = \frac{\pi}{2}$. Then one needs to find maximal value of the area

$$A = |r_0||r_1| \sin\frac{\pi}{2} = |r_0||r_1|,$$

with constraint $|r_0|^2 + |r_1|^2 = 1$. To find this value two approaches can be proposed.

1. In the first approach by parametrization $|r_0| = \cos\beta$, $|r_1| = \sin\beta$ the area formula becomes
$$A_{max} = (\cos\beta \sin\beta)_{max} = \frac{1}{2}(\sin 2\beta)_{max} = \frac{1}{2}.$$

2. In the second approach, by denoting $r_0 \equiv \mu$ the constraint becomes

$$r_0^2 + r_1^2 = \mu^2 + r_1^2 = 1.$$

It gives $r_1 = \sqrt{1-\mu^2}$ and the area

$$A(\mu) = \mu\sqrt{1-\mu^2}.$$

The maximal value of this area corresponds to extremum point :

$$\frac{dA}{d\mu} = \frac{1-2\mu^2}{\sqrt{1+\mu^2}} = 0, \tag{4.8}$$

implying that $\mu^2 = \frac{1}{2}$. Then, substituting this into $A(\mu)$ gives the maximal area

$$A(\mu)_{max} = \mu\sqrt{1-\mu^2} = \sqrt{\frac{1}{2}}\sqrt{\frac{1}{2}} = \frac{1}{2}.$$

According to above optimization problem, the area of parallelogram corresponding to arbitrary two rebit states is bounded: $0 \leq A \leq \frac{1}{2}$. This suggests to introduce positive number

$$C = 2A = \begin{vmatrix} r_{00} & r_{01} \\ r_{10} & r_{11} \end{vmatrix},$$

bounded between $0 \leq C \leq 1$. This number can characterize the level of entanglement for arbitrary two rebit state. If the state is separable then $C = 0$, if the state is maximally entangled, then $C = 1$.

4.2. Concurrence and Determinant

For genetic two qubit state with complex coefficients, the separability and entanglement are related with linear dependence of qubits $|\psi_1\rangle$ and $|\psi_2\rangle$ in (4.2). For separable states, the complex vectors $\vec{c_0} \equiv (c_{00}, c_{01})$ and $\vec{c_1} \equiv (c_{10}, c_{11})$ are linearly dependent and determinant (4.4) vanishes. For entangled states, these vectors are linearly independent and determinant is non zero. In the real case of rebit states, the determinant as a real number was bounded and related with area of parallelogram. In a similar way, in complex case, by taking modulus of complex determinant as a complex area the real number is defined

$$D = \left|\det\begin{pmatrix} c_{00} & c_{01} \\ c_{10} & c_{11} \end{pmatrix}\right| = \left\|\begin{matrix} c_{00} & c_{01} \\ c_{10} & c_{11} \end{matrix}\right\|.$$

This determinant is bounded and can be considered as a measure of entanglement

$$0 \leq D \leq D_{max}.$$

When $D = 0$ the state is separable and for $D = D_{max}$ it is maximally entangled. To find maximal value of this determinant D_{max}, the following lemma would be used.

Lemma 4.2 *Module of determinant D satisfies inequality*

$$D = \left|\det\begin{pmatrix} c_{00} & c_{01} \\ c_{10} & c_{11} \end{pmatrix}\right| \leq max\,(|c_{00}||c_{11}| + |c_{01}||c_{10}|)$$

Proof

$$D^2 = |c_{00}c_{11} - c_{01}c_{10}|^2 = |c_{00}|^2|c_{11}|^2 + |c_{01}|^2|c_{10}|^2 - 2|c_{00}||c_{11}||c_{01}||c_{10}|\cos\alpha$$

where $\alpha = \arg c_{00} + \arg c_{11} - \arg c_{01} - \arg c_{10}$. Since $|\cos\alpha| \leq 1$, maximal value of determinant is

$$D^2_{max} = max(|c_{00}|^2|c_{11}|^2 + |c_{01}|^2|c_{10}|^2 + 2|c_{00}||c_{11}||c_{01}||c_{10}|) = max((|c_{00}||c_{11}| + |c_{01}||c_{10}|)^2)$$

and as follows

$$D_{max} = max\,(|c_{00}||c_{11}| + |c_{01}||c_{10}|).$$

□

To find maximal value of the determinant, the following optimization problem arises. By denoting $x \equiv |c_{00}|$, $y \equiv |c_{11}|$, $z \equiv |c_{01}|$, $t \equiv |c_{10}|$ the problem is to find maximal value of

$$D(x, y, z, t) = xy + zt,$$

where x, y, z, t are non negative numbers, satisfying constraint

$$x^2 + y^2 + z^2 + t^2 = 1.$$

Proposition 4.2 *Module of determinant D is bounded*

$$0 \leq D \leq \frac{1}{2}$$

Proof By solving constraint for $t = \sqrt{1 - x^2 - y^2 - z^2}$, the function becomes

$$D(x, y, z) = xy + z\sqrt{1 - x^2 - y^2 - z^2}.$$

For critical points it satisfying

$$\frac{\partial D}{\partial x} = \frac{\partial D}{\partial y} = \frac{\partial D}{\partial z} = 0,$$

and gives relations

$$1 - x^2 - y^2 - z^2 = z^2, \qquad x = y.$$

This implies the circle equation

$$x^2 + z^2 = \frac{1}{2}.$$

Parametrization

$$x = \frac{1}{\sqrt{2}}\cos\mu, \quad z = \frac{1}{\sqrt{2}}\sin\mu \Longrightarrow y = x = \frac{1}{\sqrt{2}}\cos\mu$$

and

$$t = \sqrt{1 - x^2 - y^2 - z^2} = \sqrt{z^2} = z = \frac{1}{\sqrt{2}}\sin\mu.$$

In this parametrization,

$$D_{max} = xy + zt = x^2 + z^2 = \frac{1}{2}.$$

For separable states $D_{min} = 0$, this completes the proof. □

Due to the last proposition, it is convenient, instead of D to introduce $C = 2D$ so that $0 \leq C \leq 1$.

Definition 4.8 *For the generic two qubit state*

$$|\psi\rangle = c_{00}|00\rangle + c_{01}|01\rangle + c_{10}|10\rangle + c_{11}|11\rangle,$$

with normalization condition

$$|c_{00}|^2 + |c_{01}|^2 + |c_{10}|^2 + |c_{11}|^2 = 1,$$

the concurrence as a degree of entanglement is given by the determinant formula

$$C = \left| 2 \begin{vmatrix} c_{00} & c_{01} \\ c_{10} & c_{11} \end{vmatrix} \right|, \qquad 0 \leq C \leq 1. \tag{4.9}$$

- If $\mathbf{C = 0} \Longrightarrow |\psi\rangle$ is *seperable state*
- If $\mathbf{C = 1} \Longrightarrow |\psi\rangle$ is *maximally entangled state*
- If $\mathbf{0 < C \leq 1} \Longrightarrow |\psi\rangle$ is *entangled state*

For particular case of two rebit states, the concurrence coincides with double area:

$$C = 2A,$$

and in generic case is double of complex area.

The Bell States

$$|\alpha_\pm\rangle = \frac{|00\rangle \pm |11\rangle}{\sqrt{2}} \qquad |\beta_\pm\rangle = \frac{|01\rangle \pm |10\rangle}{\sqrt{2}}$$

are maximally entangled states with $C = 1$, as easy to see from the determinant formula (4.9).

Generalized Bell States The above Bell states are the particular cases of the generalized Bell states. For the first pair of states it is

$$|\psi\rangle = \cos\frac{\theta}{2}|00\rangle + \sin\frac{\theta}{2}e^{i\varphi}|11\rangle.$$

This state is normalized $\langle\psi|\psi\rangle = \cos^2\frac{\theta}{2} + \sin^2\frac{\theta}{2} = 1$ and the concurrence is

$$C = \left|2\left|\begin{array}{cc} c_{00} & c_{01} \\ c_{10} & c_{11} \end{array}\right|\right| = \left|2\left|\begin{array}{cc} \cos\frac{\theta}{2} & 0 \\ 0 & \sin\frac{\theta}{2}e^{i\varphi} \end{array}\right|\right| = |\sin\theta|.$$

These states can be represented by points (θ, φ) on unit sphere $\mathbf{S^2}$. Values of θ characterizes of degree of entanglement.

- If $\theta = 0 \Rightarrow C = 0$ then $|\psi\rangle = |00\rangle$ represents separable state- *the north pole*
- If $\theta = \pi \Rightarrow C = 0$ then $|\psi\rangle = |11\rangle$ is separable state- *the south pole*
- If $\theta = \frac{\pi}{2}$ and $\theta = \frac{3\pi}{2} \Rightarrow C = 1$ for two maximally entangled states

  $$|\psi_1\rangle = \frac{|00\rangle + e^{i\varphi}|11\rangle}{\sqrt{2}}, \quad |\psi_2\rangle = \frac{|00\rangle - e^{i\varphi}|11\rangle}{\sqrt{2}},$$

  on the equator.

  It is possible to restrict value of θ between

  $$0 < \theta < \pi,$$

  then both of these states can be described by the same formula.

- These states are orthogonal; $\langle\psi_1|\psi_2\rangle = 0$ for any φ. For $\varphi = 0$ they reduce to the first pair of Bell states. Similar consideration can be done for the second pair of Bell states.

4.2.1. Rotation Invariance of Concurrence

Determinant formula for concurrence, as well as the area formula shows invariance of concurrence under rotation of two parallelogram vectors. Indeed, for two rebit state, the concurrence as an area is

$$C = 2A = 2|\vec{r_0} \times \vec{r_1}| = 2 \begin{vmatrix} r_{00} & r_{10} \\ r_{01} & r_{11} \end{vmatrix}. \tag{4.10}$$

For rotation of vectors

$$\vec{r'_0} = R\vec{r_0}, \qquad \vec{r'_1} = R\vec{r_1},$$

on angle α the matrix representation is

$$\begin{pmatrix} r_{00} \\ r_{01} \end{pmatrix} = \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} r'_{00} \\ r'_{01} \end{pmatrix}, \qquad \begin{pmatrix} r_{10} \\ r_{11} \end{pmatrix} = \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} r'_{10} \\ r'_{11} \end{pmatrix}.$$

From these matrices

$$\begin{aligned} r_{00} &= \cos\alpha\, r'_{00} + \sin\alpha\, r'_{01}, \;\; r_{10} = \cos\alpha\, r'_{10} + \sin\alpha\, r'_{11}, \\ r_{01} &= -\sin\alpha\, r'_{00} + \cos\alpha\, r'_{01}, \;\; r_{11} = -\sin\alpha\, r'_{10} + \cos\alpha\, r'_{11}. \end{aligned}$$

Then the concurrence is

$$\begin{aligned} C = 2\begin{vmatrix} r_{00} & r_{10} \\ r_{01} & r_{11} \end{vmatrix} &= 2\begin{vmatrix} \cos\alpha\, r'_{00} + \sin\alpha\, r'_{01} & \cos\alpha\, r'_{10} + \sin\alpha\, r'_{11} \\ -\sin\alpha\, r'_{00} + \cos\alpha\, r'_{01} & -\sin\alpha\, r'_{10} + \cos\alpha\, r'_{11} \end{vmatrix} \\ &= 2(\cos^2\alpha + \sin^2\alpha) r'_{00}\, r'_{11} - 2(\cos^2\alpha + \sin^2\alpha) r'_{01}\, r'_{10} \\ &= 2\begin{vmatrix} r'_{00} & r'_{10} \\ r'_{01} & r'_{11} \end{vmatrix} = 2|\vec{r'_0} \times \vec{r'_1}|. \end{aligned}$$

In the complex form, denoting

$$z_0 \equiv r_{00} + ir_{01}, \;\; \bar{z}_0 \equiv r_{00} - ir_{01},$$

$$z_1 \equiv r_{10} + ir_{11}, \;\; \bar{z}_1 \equiv r_{10} - ir_{11}.$$

and solving for real vectors

$$\begin{aligned} r_{00} &= \frac{z_0 + \bar{z}_0}{2}, \quad r_{01} = \frac{z_0 - \bar{z}_0}{2i}, \\ r_{10} &= \frac{z_1 + \bar{z}_1}{2}, \quad r_{11} = \frac{z_1 + \bar{z}_1}{2i}. \end{aligned}$$

the concurrence becomes

$$C = |\bar{z}_0 z_1 - z_0 \bar{z}_1|.$$

Rotation of complex vectors $z_0 = z'_0 e^{-i\alpha}$, $z_1 = z'_1 e^{-i\alpha}$ preserves this concurrence formula. (Similar invariance relation can be derived for concurrence in generic two qubit state) As it was shown in Section 3, fidelity between symmetric states is invariant under Möbius transformation, and as the concurrence it is bounded $0 \leq F \leq 1$. This suggests on possible relation between concurrence and fidelity for symmetric states.

4.3. Concurrence and Fidelity

In Section 3, fidelity between symmetric qubit states was defined and it was shown that this fidelity (3.16) is invariant under Möbius transformation. In previous section, it was shown that, concurrence for two qubit states is invariant under rotations. This is why, the next problem arises:

"For given generic two qubit state $|\psi\rangle$, find the symmetric two qubit state $|\tilde{\psi}\rangle$, such that fidelity between these states

$$F = |\langle\tilde{\psi}|\psi\rangle| = C,$$

gives the concurrence ?"

For solving this problem the generic two qubit state

$$|\psi\rangle = c_{00}|00\rangle + c_{01}|01\rangle + c_{10}|10\rangle + c_{11}|11\rangle = \begin{pmatrix} c_{00} \\ c_{01} \\ c_{10} \\ c_{11} \end{pmatrix},$$

can be represented in the form

$$|\psi\rangle = |0\rangle\,(c_{00}|0\rangle + c_{01}|1\rangle) + |1\rangle\,(c_{10}|0\rangle + c_{11}|1\rangle),$$

with one qubit states

$$|c_0\rangle = c_{00}|0\rangle + c_{01}|1\rangle, \qquad |c_1\rangle = c_{10}|0\rangle + c_{11}|1\rangle,$$

giving

$$|\psi\rangle = |0\rangle \otimes |c_0\rangle + |1\rangle \otimes |c_1\rangle. \tag{4.11}$$

Suppose that, the symmetric state is in the generic form

$$|\tilde{\psi}\rangle = \tilde{c}_{00}|00\rangle + \tilde{c}_{01}|01\rangle + \tilde{c}_{10}|10\rangle + \tilde{c}_{11}|11\rangle,$$

and can be represented as

$$|\tilde{\psi}\rangle = |0\rangle \otimes |\tilde{c}_0\rangle + |1\rangle \otimes |\tilde{c}_1\rangle,$$

where the pair of one qubits is

$$|\tilde{c}_0\rangle = \tilde{c}_{00}|0\rangle + \tilde{c}_{01}|1\rangle, \qquad |\tilde{c}_1\rangle = \tilde{c}_{10}|0\rangle + \tilde{c}_{11}|1\rangle.$$

Fidelity between these states is

$$\begin{aligned} F = |\langle\tilde{\psi}|\psi\rangle| &= |\langle\tilde{c}_0|c_0\rangle + \langle\tilde{c}_1|c_1\rangle| \\ &= |\overline{\tilde{c}_{00}}c_{00} + \overline{\tilde{c}_{01}}c_{01} + \overline{\tilde{c}_{10}}c_{10} + \overline{\tilde{c}_{11}}c_{11}|. \end{aligned}$$

Comparison with the concurrence C, calculated according to the determinant formula

$$C = \left| 2 \left| \begin{array}{cc} c_{00} & c_{01} \\ c_{10} & c_{11} \end{array} \right| \right| = |2\,|c_{00}c_{11} - c_{01}c_{10}|| = |c_{11}c_{00} - c_{10}c_{01} - c_{01}c_{10} + c_{00}c_{11}|$$

gives a solution, relating coefficients between symmetric states

$$\tilde{c}_{00} = e^{-i\gamma}\bar{c}_{11}, \quad \tilde{c}_{11} = e^{-i\gamma}\bar{c}_{00}, \quad \tilde{c}_{01} = -e^{-i\gamma}\bar{c}_{10}, \quad \tilde{c}_{10} = -e^{-i\gamma}\bar{c}_{01}, \tag{4.12}$$

where γ is an arbitrary phase. Then, desired one qubit states are

$$|\tilde{c}_0\rangle = e^{-i\gamma}\,(\bar{c}_{11}|0\rangle - \bar{c}_{10}|1\rangle), \qquad |\tilde{c}_1\rangle = e^{-i\gamma}\,(-\bar{c}_{01}|0\rangle + \bar{c}_{00}|1\rangle).$$

By choosing phase $\gamma = \pi$ one gets in particular

$$|\tilde{c}_0\rangle = -\bar{c}_{11}|0\rangle + \bar{c}_{10}|1\rangle, \qquad |\tilde{c}_1\rangle = \bar{c}_{01}|0\rangle - \bar{c}_{00}|1\rangle.$$

Then, the symmetric state $|\tilde{\psi}\rangle$ is represented as

$$|\tilde{\psi}\rangle = |0\rangle\,(-\bar{c}_{11}|0\rangle + \bar{c}_{10}|1\rangle) + |1\rangle\,(\bar{c}_{01}|0\rangle - \bar{c}_{00}|1\rangle)$$

or

$$|\tilde{\psi}\rangle = -\bar{c}_{11}|00\rangle + \bar{c}_{01}|01\rangle + \bar{c}_{10}|10\rangle - \bar{c}_{00}|11\rangle. \tag{4.13}$$

The symmetric state obtained above, can be generated by anti - unitary two qubit gates.

Proposition 4.3 *Symmetric state $|\tilde{\psi}\rangle$ results from application of unitary $Y \otimes Y$ gate and anti-unitary K gate*

$$|\tilde{\psi}\rangle = Y \otimes Y\,|\bar{\psi}\rangle = (Y \otimes Y)\,K|\psi\rangle,$$

where $|\bar{\psi}\rangle = K|\psi\rangle$.

Proof Application $Y|0\rangle = i|1\rangle$ and $Y|1\rangle = -i|0\rangle$ gives

$$\begin{aligned} Y \otimes Y\,|00\rangle &= Y|0\rangle \otimes Y|0\rangle = -|11\rangle, \quad Y \otimes Y\,|11\rangle = Y|1\rangle \otimes Y|1\rangle = -|00\rangle, \\ Y \otimes Y\,|01\rangle &= Y|0\rangle \otimes Y|1\rangle = |10\rangle, \quad Y \otimes Y\,|10\rangle = Y|1\rangle \otimes Y|0\rangle = |01\rangle. \end{aligned}$$

Then

$$Y \otimes Y|\bar{\psi}\rangle = -\bar{c}_{11}|00\rangle + \bar{c}_{10}|01\rangle + \bar{c}_{01}|10\rangle - \bar{c}_{00}|11\rangle$$

and comparison with (4.13) shows that it is the symmetric state $|\tilde{\psi}\rangle$. □

Combining the above results together, solution of the problem posed at the beginning of this section is given by following proposition,

Proposition 4.4 *Concurrence for generic two qubit state is equal to fidelity between symmetric states $|\psi\rangle$ and $|\tilde{\psi}\rangle = Y \otimes Y|\bar{\psi}\rangle$,*

$$C = 2D = |2\,|c_{00}c_{11} - c_{01}c_{10}|| = |\langle\tilde{\psi}|\psi\rangle| = F.$$

4.4. Antipodal Points Generating Symmetric States

The generic two qubit state has 6 real or 3 complex parameters. Then, transition from given state $|\psi\rangle$ to the symmetric state $|\tilde{\psi}\rangle$ can be implemented by a proper transformation of these complex numbers in complex planes. In present section it would be shown for that generic state $|\psi\rangle$, characterized by three complex numbers z, w, η the corresponding symmetric state $|\tilde{\psi}\rangle$ appears as combined antipodal transformation of these points. For state $|\psi\rangle$ written in the form

$$\begin{aligned} |\psi\rangle &= c_{00}|00\rangle + c_{01}|01\rangle + c_{10}|10\rangle + c_{11}|11\rangle \\ &= |0\rangle\,(c_{00}|0\rangle + c_{01}|1\rangle) + |1\rangle\,(c_{10}|0\rangle + c_{11}|1\rangle) \\ &= |0\rangle\, c_{00}\,(|0\rangle + z\,|1\rangle) + |1\rangle\, c_{10}\,(|0\rangle + w\,|1\rangle), \end{aligned} \tag{4.14}$$

where

$$z \equiv \frac{c_{01}}{c_{00}}, \qquad w \equiv \frac{c_{11}}{c_{10}},$$

the symmetric state $|\tilde{\psi}\rangle$ is

$$\begin{aligned} |\tilde{\psi}\rangle &= \tilde{c}_{00}|00\rangle + \tilde{c}_{01}|01\rangle + \tilde{c}_{10}|10\rangle + \tilde{c}_{11}|11\rangle \\ &= |0\rangle\,(\tilde{c}_{00}|0\rangle + \tilde{c}_{01}|1\rangle) + |1\rangle\,(\tilde{c}_{10}|0\rangle + \tilde{c}_{11}|1\rangle) \\ &= |0\rangle\, \tilde{c}_{00}\,(|0\rangle + \tilde{z}\,|1\rangle) + |1\rangle\, \tilde{c}_{10}\,(|0\rangle + \tilde{w}\,|1\rangle), \end{aligned}$$

and

$$\tilde{z} \equiv \frac{\tilde{c}_{01}}{\tilde{c}_{00}}, \qquad \tilde{w} \equiv \frac{\tilde{c}_{11}}{\tilde{c}_{10}}.$$

Due to relations between symmetric states (4.12),

$$\tilde{z} = \frac{\tilde{c}_{01}}{\tilde{c}_{00}} = \frac{-e^{-i\gamma}\bar{c}_{10}}{e^{-i\gamma}\bar{c}_{11}} = \frac{1}{-\frac{\bar{c}_{11}}{\bar{c}_{10}}} = -\frac{1}{\bar{w}},$$

and

$$\tilde{w} = \frac{\tilde{c}_{11}}{\tilde{c}_{10}} = \frac{e^{-i\gamma}\bar{c}_{00}}{-e^{-i\gamma}\bar{c}_{01}} = \frac{1}{-\frac{\bar{c}_{01}}{\bar{c}_{00}}} = -\frac{1}{\bar{z}}.$$

This shows that the symmetric state is related with transformation of points z and w to the antipodal mutual points

$$z \longrightarrow -\frac{1}{\bar{w}}, \qquad w \longrightarrow -\frac{1}{\bar{z}}.$$

By choosing

$$c_{00} = \frac{\gamma_{00}}{\sqrt{1+|z|^2}}, \qquad c_{10} = \frac{\gamma_{10}}{\sqrt{1+|w|^2}},$$

and substituting these into (4.14), one gets

$$|\psi\rangle = \gamma_{00}\,|0\rangle|z\rangle + \gamma_{10}\,|1\rangle|w\rangle, \tag{4.15}$$

where one qubit states are written in coherent state form

$$|z\rangle = \frac{|0\rangle + z\,|1\rangle}{\sqrt{1+|z|^2}}, \qquad |w\rangle = \frac{|0\rangle + w\,|1\rangle}{\sqrt{1+|w|^2}}. \tag{4.16}$$

The state (4.15) can be rewritten as

$$|\psi\rangle = \gamma_{00}\,(|0\rangle|z\rangle + \frac{\gamma_{10}}{\gamma_{00}}\,|1\rangle|w\rangle),$$

where

$$\langle\psi|\psi\rangle = |\gamma_{00}|^2 + |\gamma_{10}|^2 = 1.$$

By denoting

$$\eta \equiv \frac{\gamma_{10}}{\gamma_{00}}$$

and solving the constraint, the state $|\psi\rangle$ up to global phase acquires the form

$$|\psi\rangle = \frac{|0\rangle|z\rangle + \eta\,|1\rangle|w\rangle}{\sqrt{1+|\eta|^2}}. \tag{4.17}$$

4.4.1. Coherent Like Two Qubit States

The generic two qubit state (4.17), in particular cases depends on one complex parameter and formally looks like the one qubit coherent state (4.16):

1. In the limit $z \to 0$, $w \to \infty$ states $|z\rangle \to |0\rangle$, $|w\rangle \to |1\rangle$ and state

$$|\psi\rangle = \frac{|00\rangle + \eta\,|11\rangle}{\sqrt{1+|\eta|^2}}. \tag{4.18}$$

2. In the limit $z \to \infty$, $w \to 0$ states $|z\rangle \to |1\rangle$, $|w\rangle \to |0\rangle$ and state

$$|\psi\rangle = \frac{|01\rangle + \eta\,|10\rangle}{\sqrt{1+|\eta|^2}} \equiv |\chi\rangle. \tag{4.19}$$

This one qubit like states can be generated from $|00\rangle$ and $|11\rangle$ states by one qubit gates $A_0(\eta)$ (3.9) and $A_1(\eta)$ (3.10), and generalized CNOT gates.

The concurrence for both states is the same and equal

$$C = \frac{2|\eta|}{1+|\eta|^2}. \tag{4.20}$$

In the limits $\eta \to 0$ and $\eta \to \infty$ it is zero $C = 0$. The concurrence is constant along concentric circles $|\eta| = r$

$$C = \frac{2r}{1+r^2}$$

in complex η plane. For the circle $|\eta| = 1$, the concurrence reaches maximal value $C = 1$. Comparison of these two, particular two qubit states with one qubit coherent state shows formal similarity. Moreover, in both cases Shannon entropy for one qubit and the concurrence for two qubits, are constant along the concentric circles. This suggest that concurrence is related with level of randomness for two qubit states. And it measures how close is the state and the symmetric one. The symmetric states with respect to (4.18) and (4.19) are

$$|\tilde{\psi}\rangle = \frac{\bar{\eta}\,|00\rangle + |11\rangle}{\sqrt{1+|\eta|^2}}, \qquad |\tilde{\chi}\rangle = \frac{\bar{\eta}\,|01\rangle + |10\rangle}{\sqrt{1+|\eta|^2}}.$$

Like in one qubit coherent state representation, these states correspond to inversion of point η in the unit circle: $\eta \to \frac{1}{\bar{\eta}}$. According to proposition (4.4), the concurrence for states (4.18) and (4.19) in (4.20) can be calculated by fidelity between symmetric states. Indeed,

$$\begin{aligned} F_\psi &= |\langle\tilde{\psi}|\psi\rangle| = \frac{2|\eta|}{1+|\eta|^2} = C \\ F_\chi &= |\langle\tilde{\chi}|\chi\rangle| = \frac{2|\eta|}{1+|\eta|^2} = C \end{aligned}$$

coincide with the formula (4.20) .

4.4.2. Concurrence for the Generic Case

Returning back to the generic state (4.17) and calculating the concurrence for this state according to the determinant formula gives

$$C = 2 \begin{vmatrix} \frac{1}{\sqrt{(1+|z|^2)(1+|\eta|^2)}} & \frac{z}{\sqrt{(1+|z|^2)(1+|\eta|^2)}} \\ \frac{\eta}{\sqrt{(1+|z|^2)(1+|\eta|^2)}} & \frac{\eta\, w}{\sqrt{(1+|z|^2)(1+|\eta|^2)}} \end{vmatrix} = \frac{2\,|\eta|}{(1+|\eta|^2)\sqrt{(1+|z|^2)(1+|w|^2)}} \begin{vmatrix} 1 & z \\ 1 & w \end{vmatrix}$$

or

$$C = 2\,\frac{|\eta|\,|w - z|}{(1 + |\eta|^2)\,\sqrt{(1 + |z|^2)(1 + |w|^2)}}. \tag{4.21}$$

This formula in particular cases

1. in the limit $z \to 0$, $w \to \infty$
2. in the limit $z \to \infty$, $w \to 0$

reduces the concurrence (4.20.

Complex numbers z and w, determining states $|z\rangle$ and $|w\rangle$ are involved to concurrence formula (4.21) in a symmetric way : it is invariant under exchange $z \leftrightarrow w$. If two points z and w coincide: $z = w$, then $C = 0$ and states are separable. For entangled states $z \neq w$ and $|z - w| \neq 0$. The Euclidean distance between these two points determines the level of entanglement. It is noted that in contrast to separability condition for two qubit state (4.11), where two one qubit states are linearly dependent, now separability corresponds just to equality of two states $|z\rangle$ and $|w\rangle$: $|z\rangle = |w\rangle$ and points $z = w$. This is due to that z and w are homogenous coordinates, given by ratio of two complex numbers. And this ratio is not changing under scaling transformation like

$$z = \frac{c_{01}}{c_{00}} = \frac{\lambda c_{01}}{\lambda c_{00}}.$$

According to proposition (4.4), the generic concurrence for (4.17) is given by fidelity between symmetric states. For symmetric state $|\tilde{\psi}\rangle = Y \otimes Y|\bar{\psi}\rangle$, first calculate

$$|\bar{\psi}\rangle = \frac{|0\rangle|\bar{z}\rangle + \bar{\eta}\,|1\rangle|\bar{w}\rangle}{\sqrt{1 + |\eta|^2}},$$

where

$$|\bar{z}\rangle = \frac{|0\rangle + \bar{z}\,|1\rangle}{\sqrt{1 + |z|^2}}, \qquad |\bar{w}\rangle = \frac{|0\rangle + \bar{w}\,|1\rangle}{\sqrt{1 + |w|^2}}.$$

Then

$$Y \otimes Y|\bar{\psi}\rangle = \frac{|1\rangle\frac{\bar{z}|0\rangle - |1\rangle}{\sqrt{1+|z|^2}} + \eta|0\rangle\frac{-\bar{w}|0\rangle + |1\rangle}{\sqrt{1+|w|^2}}}{\sqrt{1+|\eta|^2}},$$

and

$$|\tilde{\psi}\rangle = \frac{1}{\sqrt{1+|\eta|^2}}\left(\bar{\eta}|0\rangle\frac{-\bar{w}|0\rangle + |1\rangle}{\sqrt{1+|w|^2}} - |1\rangle\frac{-\bar{z}|0\rangle + |1\rangle}{\sqrt{1+|z|^2}}\right).$$

Since

$$\left|-\frac{1}{\bar{z}}\right\rangle = \frac{-\bar{z}|0\rangle + |1\rangle}{\sqrt{1+|z|^2}}, \qquad \left|-\frac{1}{\bar{w}}\right\rangle = \frac{-\bar{w}|0\rangle + |1\rangle}{\sqrt{1+|w|^2}},$$

then up to global phase

$$|\tilde{\psi}\rangle = -\frac{-\bar{\eta}\,|0\rangle\left|\frac{1}{\bar{w}}\right\rangle + |1\rangle\left|-\frac{1}{\bar{z}}\right\rangle}{\sqrt{1+|\eta|^2}}. \tag{4.22}$$

The concurrence as a fidelity is

$$C = F = |\langle\tilde{\psi}|\psi\rangle| = \frac{|\eta|}{1+|\eta|^2}\left|\left\langle -\frac{1}{\bar{w}}\middle| z\right\rangle - \left\langle -\frac{1}{\bar{z}}\middle| w\right\rangle\right|, \tag{4.23}$$

where

$$\left\langle -\frac{1}{\bar{w}}\middle| z\right\rangle = \frac{-w+z}{\sqrt{(1+|w|^2)(1+|z|^2)}}, \qquad \left\langle -\frac{1}{\bar{z}}\middle| w\right\rangle = \frac{w-z}{\sqrt{(1+|w|^2)(1+|z|^2)}}.$$

Substituting these into (4.23) gives the same form of the concurrence (4.21). Therefore the following proposition holds.

Proposition 4.5 *The symmetric state*

$$|\tilde{\psi}\rangle = -\frac{-\bar{\eta}\,|0\rangle\left|\frac{1}{\bar{w}}\right\rangle + |1\rangle\left|-\frac{1}{\bar{z}}\right\rangle}{\sqrt{1+|\eta|^2}}$$

for the generic state

$$|\psi\rangle = \frac{|0\rangle|z\rangle + \eta\,|1\rangle|w\rangle}{\sqrt{1+|\eta|^2}}$$

results from combined antipodal transformations

$$z \longrightarrow -\frac{1}{\bar{w}}, \qquad w \longrightarrow -\frac{1}{\bar{z}}, \qquad \eta \longrightarrow -\frac{1}{\bar{\eta}}.$$

4.5. Concurrence and Inner Product Metric

It was shown in Subsection 4.2.1 that concurrence for two rebit state is given by the area formula (4.10). Since area of parallelogram is related with the inner product metric and with Riemannian metric, here the concurrence would be connected with this metric.

It is known that (Dubrovin, Fomenko and Novikov, 1984), if $\vec{r_0}$ and $\vec{r_1}$ are vectors in the Euclidean plane, then they determine the parallelogram consisting of all vectors

$$\lambda\vec{r_0} + \mu\vec{r_1}, \qquad 0 \leq \lambda, \mu \leq 1.$$

The area of this parallelogram is given by

$$a = |\det A|,$$

where

$$A = \begin{pmatrix} r_{00} & r_{01} \\ r_{10} & r_{11} \end{pmatrix}$$

and $\vec{r_0} = (r_{00}, r_{01})$, $\vec{r_1} = (r_{10}, r_{11})$ are components of vectors $\vec{r_0}$ and $\vec{r_1}$, relative to an orthonormal basis $\vec{e_0}$, $\vec{e_1}$:

$$\vec{r_0} = r_{00}\, \vec{e_0} + r_{01}\, \vec{e_1}, \qquad \vec{r_1} = r_{01}\, \vec{e_0} + r_{11}\, \vec{e_1}.$$

Indeed,

$$a = |\vec{r_0} \times \vec{r_1}| = |\vec{r_0}|\, |\vec{r_1}| \sin \alpha = |\det A|.$$

This area formula can be related with the Riemannian metric (Dubrovin, Fomenko and Novikov, 1984). Let in 2- dimensional inner product space over the reals, as a 2- dimensional vector space equipped with an inner (scalar) product, the orthonormal basis is $\vec{e_0}$, $\vec{e_1}$. Then, it is natural to define the area of parallelogram by analogy with the Euclidean case.

Definition 4.9 *The metric in two dimensional vector space is*

$$\langle \vec{r_0}, \vec{r_0} \rangle = g_{00}, \qquad \langle \vec{r_1}, \vec{r_1} \rangle = g_{11}, \qquad \langle \vec{r_0}, \vec{r_1} \rangle = g_{01} = g_{10}.$$

By using components of vectors $\vec{r_0}$ and $\vec{r_1}$ it gives

$$g_{00} = r_{00}^2 + r_{01}^2, \qquad g_{11} = r_{10}^2 + r_{11}^2, \qquad g_{01} = g_{10} = r_{00}\, r_{10} + r_{01}\, r_{11}.$$

The inner product matrix can be factorized as

$$g_{ij} = \begin{pmatrix} g_{00} & g_{01} \\ g_{10} & g_{11} \end{pmatrix} = \begin{pmatrix} r_{00}^2 + r_{01}^2 & r_{00}\, r_{10} + r_{01}\, r_{11} \\ r_{00}\, r_{10} + r_{01}\, r_{11} & r_{10}^2 + r_{11}^2 \end{pmatrix} = \begin{pmatrix} r_{00} & r_{01} \\ r_{10} & r_{11} \end{pmatrix} \begin{pmatrix} r_{00} & r_{10} \\ r_{01} & r_{11} \end{pmatrix} = A\, A^{\intercal}$$

or

$$G = A\, A^{\intercal}.$$

Then

$$\det G = \det A\; \det A^{\intercal} = (\det A)^2,$$

and the area of parallelogram is

$$a = |\det A| = \sqrt{\det G}. \tag{4.24}$$

This result constitutes the following lemma.

Lemma 4.3 *(Dubrovin, Fomenko and Novikov, 1984) The area of the parallelogram, determined by the vectors $\vec{r_0}$, $\vec{r_1}$ of the inner product space is $\sqrt{\det G}$, where*

$$\det G = g_{00}\, g_{11} - g_{01}^2.$$

4.5.1. Determinant Formula For Two Rebit Concurrence

- **Right Decomposition**

  Let arbitrary two rebit state

  $$|\psi\rangle = r_{00}\,|00\rangle + r_{01}\,|01\rangle + r_{10}\,|10\rangle + r_{11}\,|11\rangle,$$

  where $r_{ij} \in \mathbb{R}$ is represented as

  $$|\psi\rangle = |0\rangle\,|r_0\rangle + |1\rangle\,|r_1\rangle,$$

  where one rebits are

  $$|r_0\rangle = r_{00}\,|0\rangle + r_{01}\,|1\rangle, \qquad |r_1\rangle = r_{10}\,|0\rangle + r_{11}\,|1\rangle.$$

  Two vectors $\vec{r_0} = (r_{00}, r_{10})$, $\vec{r_1} = (r_{01}, r_{11})$ determines the parallelogram of states

  $$\lambda\vec{r_0} + \mu\vec{r_1}, \qquad 0 \le \lambda, \mu \le 1.$$

The area of the parallelogram

$$|\det A| = \begin{vmatrix} r_{00} & r_{10} \\ r_{01} & r_{11} \end{vmatrix}$$

is half of the concurrence C,

$$C = 2|\det A| = 2 \begin{vmatrix} r_{00} & r_{10} \\ r_{01} & r_{11} \end{vmatrix}$$

if the vectors are normalized as

$$r_{00}^2 + r_{01}^2 + r_{10}^2 + r_{11}^2 = 1.$$

The Hilbert space of rebits is the real inner product space, so that components of the inner products are

$$\langle r_0 | r_0 \rangle = g_{00}, \qquad \langle r_1 | r_1 \rangle = g_{11}, \qquad \langle r_0 | r_1 \rangle = g_{01} = g_{10},$$

or in component form

$$\langle r_0 | r_0 \rangle = r_{00}^2 + r_{01}^2, \qquad \langle r_1 | r_1 \rangle = r_{10}^2 + r_{11}^2, \qquad \langle r_0 | r_1 \rangle = r_{00}\, r_{10} + r_{01}\, r_{11},$$

and corresponding matrix is factorized as

$$g_{ij} = \begin{pmatrix} g_{00} & g_{01} \\ g_{10} & g_{11} \end{pmatrix} = \begin{pmatrix} r_{00}^2 + r_{01}^2 & r_{00}\, r_{10} + r_{01}\, r_{11} \\ r_{00}\, r_{10} + r_{01}\, r_{11} & r_{10}^2 + r_{11}^2 \end{pmatrix} = \begin{pmatrix} r_{00} & r_{01} \\ r_{10} & r_{11} \end{pmatrix} \begin{pmatrix} r_{00} & r_{10} \\ r_{01} & r_{11} \end{pmatrix} = A\, A^{\intercal}$$

or

$$G = A\, A^{\intercal}.$$

Then

$$\det G = (\det A)^2.$$

Normalization condition $\langle\psi|\psi\rangle = 1$ or

$$r_{00}^2 + r_{01}^2 + r_{10}^2 + r_{11}^2 = 1$$

implies that

$$\mathrm{Tr}\, G = 1.$$

In this case, the concurrence is

$$C = 2|\det A| = 2\sqrt{\det G}. \tag{4.25}$$

But $\mathrm{Tr}\, G$ and $\det G$ are invariants of matrix G. The characteristic equation for this matrix is

$$\det|G - \lambda I| = 0 \Longrightarrow \lambda^2 - (\mathrm{Tr}\, G)\,\lambda + \det G = 0 \tag{4.26}$$

with eigenvalues

$$\lambda_{1,2} = \frac{1}{2}\,\mathrm{Tr}\, G \pm \sqrt{\frac{1}{4}(\mathrm{Tr}\, G)^2 - \det G}.$$

For rebit states, the characteristic equation becomes

$$\mathrm{Tr}\, G = 1 \Longrightarrow \lambda^2 - \lambda + \frac{C^2}{4} = 0,$$

with eigenvalues

$$\lambda_{1,2} = \frac{1 \pm \sqrt{1 - C^2}}{2}.$$

Then, the concurrence is the product of eigenvalues of the inner product metric

$$C^2 = 4\lambda_1\lambda_2 \Longrightarrow C = 2\sqrt{\lambda_1\lambda_2}.$$

- **Left Decomposition**
  As it was noticed in (4.3), another decomposition of two rebit state is possible with

$$|\psi\rangle = |l_0\rangle|0\rangle + |l_1\rangle|1\rangle,$$

where one qubit rebits are

$$|l_0\rangle = r_{00}\,|0\rangle + r_{10}\,|1\rangle, \qquad |l_1\rangle = r_{10}\,|0\rangle + r_{11}\,|1\rangle.$$

In this case, the area of the parallelogram will be the same

$$|\det A| = \begin{vmatrix} r_{00} & r_{10} \\ r_{01} & r_{11} \end{vmatrix} = \begin{vmatrix} r_{00} & r_{01} \\ r_{10} & r_{11} \end{vmatrix},$$

since transposition in matrix A such that $A \to A^\intercal$ doesn't change the determinant. As a result, the concurrence formula will not change as well. However, the inner product matrix will change. Indeed, instead of

$$\langle r_i|r_j\rangle = g_{ij} \qquad i, j = 0, 1$$

appears the matrix

$$h_{ij} = \langle l_i|l_j\rangle, \qquad i, j = 0, 1\ ,$$

where

$$\begin{aligned}
H &= \begin{pmatrix} h_{00} & h_{01} \\ h_{10} & h_{11} \end{pmatrix} = \begin{pmatrix} \langle l_0|l_0\rangle & \langle l_0|l_1\rangle \\ \langle l_1|l_0\rangle & \langle l_1|l_1\rangle \end{pmatrix} \\
&= \begin{pmatrix} r_{00}^2 + r_{10}^2 & r_{00}\,r_{01} + r_{10}\,r_{11} \\ r_{00}\,r_{01} + r_{10}\,r_{11} & r_{01}^2 + r_{11}^2 \end{pmatrix}. \\
&= \begin{pmatrix} r_{00} & r_{10} \\ r_{01} & r_{11} \end{pmatrix} \begin{pmatrix} r_{00} & r_{01} \\ r_{10} & r_{11} \end{pmatrix} \\
&= A^{\intercal} A.
\end{aligned}$$

Then

$$H = A^{\intercal} A \Longrightarrow \det H = (\det A)^2.$$

4.5.2. Determinant Formula For Two Qubit Concurrence

In this section extension of previous result to generic two qubit state is obtained.

- **Right Decomposition**
  The generic two qubit state

$$|\psi\rangle = c_{00}\,|00\rangle + c_{01}\,|01\rangle + c_{10}\,|10\rangle + c_{11}\,|11\rangle,$$

where $c_{i,j} \in \mathbb{C}$, $i, j = 0, 1$, can be represented as

$$|\psi\rangle = |0\rangle\,|c_0\rangle + |1\rangle\,|c_1\rangle, \tag{4.27}$$

where one qubit states are

$$|c_0\rangle = c_{00}\,|0\rangle + c_{10}\,|1\rangle, \qquad |c_1\rangle = c_{10}\,|0\rangle + c_{11}\,|1\rangle.$$

The Hilbert space of qubits is complex Hermitian space with the Hermitian inner product.

Definition 4.10 *(Dubrovin, Fomenko and Novikov, 1984) The inner product of an ordered pair of vectors $|x\rangle, |y\rangle \in \mathcal{H}$ is a complex number, denoted as $\langle x \,|\, y\rangle$, with the following requirements:*

1. *Skew- symmetry:* $\langle x \,|\, y\rangle = \overline{\langle y \,|\, x\rangle}$
2. *Linearity:* $\langle x \,|\, cy + dz\rangle = c\langle x \,|\, y\rangle + d\langle x \,|\, z\rangle$ *with* $|x\rangle, |y\rangle, |z\rangle \in \mathbb{C}^2$*,where for any complex numbers* $c, d \in \mathbb{C}$*,*
3. *Positivity:* $\langle x \,|\, x\rangle \geq 0$ *for any state* $|x\rangle \in \mathcal{H}$*, with equality if and only if* $|x\rangle$ *is the zero vector.*
4. $\langle \lambda x \,|\, y\rangle = \bar{\lambda}\langle x \,|\, y\rangle$*, where* $\lambda \in \mathbb{C}$*,*
5. $\langle x \,|\, \lambda y\rangle = \lambda\langle x \,|\, y\rangle$*.*

Any inner product on $\mathcal{H} = \mathbb{C}^n$ *with the above properties is called an Hermitian inner product.*

For two vectors $|c_0\rangle$ and $|c_1\rangle$, it gives complex inner product matrix with elements

$$\langle c_0|c_0\rangle = g_{00}, \qquad \langle c_1|c_1\rangle = g_{11}, \qquad \langle c_0|c_1\rangle = g_{01}, \tag{4.28}$$

$$\langle c_1|c_0\rangle = g_{10} = \bar{g}_{01} = \overline{\langle c_0|c_1\rangle}.$$

In components of vectors it gives

$$\begin{aligned} g_{00} &= |c_{00}|^2 + |c_{01}|^2, \qquad g_{11} = |c_{10}|^2 + |c_{11}|^2, \\ g_{01} &= \bar{c}_{00}\, c_{10} + \bar{c}_{01}\, c_{11}, \quad g_{10} = c_{00}\, \bar{c}_{10} + c_{01}\, \bar{c}_{11} = \bar{g}_{01}, \end{aligned} \tag{4.29}$$

and in the matrix form is

$$G = \begin{pmatrix} g_{00} & g_{01} \\ g_{10} & g_{11} \end{pmatrix} = \begin{pmatrix} \langle c_0|c_0\rangle & \langle c_0|c_1\rangle \\ \overline{\langle c_0|c_1\rangle} & \langle c_1|c_1\rangle \end{pmatrix}$$

$$= \begin{pmatrix} |c_{00}|^2 + |c_{01}|^2 & \bar{c}_{00}\, c_{10} + \bar{c}_{01}\, c_{11} \\ c_{00}\, \bar{c}_{10} + c_{01}\, \bar{c}_{11} & |c_{10}|^2 + |c_{11}|^2 \end{pmatrix}. \tag{4.30}$$

Let

$$A \equiv \begin{pmatrix} c_{00} & c_{01} \\ c_{10} & c_{11} \end{pmatrix}, \qquad B \equiv \begin{pmatrix} c_{00} & c_{10} \\ c_{01} & c_{11} \end{pmatrix},$$

and corresponding Hermitian conjugate matrices are

$$A^\dagger \equiv \begin{pmatrix} \bar{c}_{00} & \bar{c}_{10} \\ \bar{c}_{01} & \bar{c}_{11} \end{pmatrix}, \qquad B^\dagger \equiv \begin{pmatrix} \bar{c}_{00} & \bar{c}_{01} \\ \bar{c}_{10} & \bar{c}_{11} \end{pmatrix}.$$

Matrices A and B are just transpose of each other

$$A^\intercal = B, \qquad A = B^\intercal.$$

Then, matrix G can be written as product

$$G = B^\dagger B, \qquad G^\intercal = AA^\dagger.$$

As easy to see G is Hermitian matrix:

$$G^\dagger = (B^\dagger B)^\dagger = B^\dagger B = G,$$

and as follows, corresponding eigenvalues are real. By taking trace and using normalization condition $\langle\psi|\psi\rangle = 1$ implies

$$\operatorname{Tr} G = |c_{00}|^2 + |c_{01}|^2 + |c_{10}|^2 + |c_{11}|^2 = 1.$$

The determinant of the matrix is

$$\begin{aligned}\det G &= \det B^{\dagger}\ \det B = \overline{\det B}\ \det B \\ &= |\det B|^2.\end{aligned}$$

In a similar way

$$\det G = |\det A|^2.$$

But $|\det A|$ is the half of the concurrence for two qubit state

$$C = 2|\det A| = 2\begin{vmatrix} c_{00} & c_{10} \\ c_{01} & c_{11} \end{vmatrix}.$$

Therefore, the concurrence C is given by formula similar to (4.25) ,

$$C = 2\sqrt{\det G}.$$

The characteristic equation for this matrix is

$$\lambda^2 - \lambda + \frac{C^2}{4} = 0,$$

with the eigenvalues

$$\lambda_{1,2} = \frac{1 \pm \sqrt{1 - C^2}}{2}.$$

From the above consideration follows that the norm and the concurrence of two qubit state are two invariants, $\mathrm{Tr}\, G$ and $\det G$ respectively, of the inner product matrix G.

Proposition 4.6 *The norm and the concurrence of generic two qubit state are in-*

variants, $\mathrm{tr}\, G$ *and* $\det G$ *respectively, of the inner product matrix* G *:*

$$G_{ij} = \langle c_i | c_j \rangle, \quad i, j = 0, 1.$$

- **Left Decomposition**

In addition to expansion (4.27), possible to have

$$|\psi\rangle = |d_0\rangle\, |0\rangle + |d_1\rangle\, |1\rangle,$$

where one qubit states are

$$|d_0\rangle = c_{00}\, |0\rangle + c_{10}\, |1\rangle, \qquad |d_1\rangle = c_{10}\, |0\rangle + c_{11}\, |1\rangle.$$

In this case the Hermitian inner product matrix becomes

$$h_{ij} = \langle d_i | d_j \rangle, \qquad i, j = 0, 1,$$

where

$$\begin{aligned} H &= \begin{pmatrix} h_{00} & h_{01} \\ h_{10} & h_{11} \end{pmatrix} = \begin{pmatrix} \langle d_0 | d_0 \rangle & \langle d_0 | d_1 \rangle \\ \overline{\langle d_0 | d_1 \rangle} & \langle d_1 | d_1 \rangle \end{pmatrix} \\ &= \begin{pmatrix} |c_{00}|^2 + |c_{10}|^2 & \bar{c}_{00}\, c_{01} + \bar{c}_{10}\, c_{11} \\ c_{00}\, \bar{c}_{01} + c_{10}\, \bar{c}_{11} & |c_{01}|^2 + |c_{11}|^2 \end{pmatrix}. \end{aligned} \tag{4.31}$$

It can be decomposed to matrices

$$A^\dagger = \begin{pmatrix} \bar{c}_{00} & \bar{c}_{10} \\ \bar{c}_{01} & \bar{c}_{11} \end{pmatrix}, \qquad B^\dagger = \begin{pmatrix} \bar{c}_{00} & \bar{c}_{01} \\ \bar{c}_{10} & \bar{c}_{11} \end{pmatrix}.$$

In terms of matrix

$$B = \begin{pmatrix} c_{00} & c_{10} \\ c_{01} & c_{11} \end{pmatrix},$$

the matrices G and H can be written as

$$G = B^{\dagger}B, \qquad H^{\top} = BB^{\dagger}. \tag{4.32}$$

By matrix

$$A = \begin{pmatrix} c_{00} & c_{01} \\ c_{10} & c_{11} \end{pmatrix},$$

matrices G and H become

$$G^{\top} = AA^{\dagger}, \qquad H = A^{\dagger}A. \tag{4.33}$$

From (4.32) and (4.33) follow

$$\det H = |\det A|^2 = \det G.$$

This means that concurrence determined by these matrices is the same

$$C = 2\sqrt{\det G} = 2\sqrt{\det H}.$$

4.6. Concurrence and Reduced Density Matrix

Geometrical characteristics of entangled states, obtained in previous chapters are related with physical characteristics as density matrix of quantum states.

Definition 4.11 *The density matrix or density operator ρ for pure state $|\psi\rangle$ is defined as*

$$\rho = |\psi\rangle\langle\psi|.$$

If $|\psi\rangle$ *is given in the form* $|\psi\rangle = \alpha|u_1\rangle + \beta|u_2\rangle$ *, then in matrix form it is represented as*

$$\rho \equiv \begin{pmatrix} \langle u_1|\rho|u_1\rangle & \langle u_1|\rho|u_2\rangle \\ \langle u_2|\rho|u_1\rangle & \langle u_2|\rho|u_2\rangle \end{pmatrix}. \tag{4.34}$$

Main properties of density operator are following:

- The density operator is Hermitian, $\rho = \rho^\dagger$
- $\mathrm{tr}(\rho) = 1$
- ρ is a positive operator, $\langle\varphi|\rho|\varphi\rangle \geq 0$, for any state $|\varphi\rangle$.

Definition 4.12 *Suppose A and B are physical systems, whose states is described by a density operator* ρ_{AB}*. Then the reduced density operator for system A is defined as*

$$\rho_A \equiv \mathrm{tr}_B(\rho_{AB}),$$

where tr_B *is partial trace over the system B. The partial trace is defined as*

$$\mathrm{tr}_B(|a_1\rangle\langle a_2| \otimes |b_1\rangle\langle b_2|) = |a_1\rangle\langle a_2|\,\mathrm{tr}(|b_1\rangle\langle b_2|).$$

The reduced density operator for system B is defined as

$$\rho_B \equiv \mathrm{tr}_A(\rho_{AB}).$$

For pure states, the density operator is a projection operator:

$$\rho^2 = |\psi\rangle\langle\psi||\psi\rangle\langle\psi| = |\psi\rangle\langle\psi| = \rho.$$

Since $\mathrm{tr}(\rho) = 1$ for pure states, than clearly

$$\mathrm{tr}(\rho^2) = 1.$$

This gives following criterium of mixed and pure states:

- **Pure State:** $\mathrm{tr}(\rho^2) = 1$
- **Mixed State:** $\mathrm{tr}(\rho^2) < 1.$

It is instructive to rewrite generic two qubit state in density matrix form, to establish link between the inner product metric and reduced density matrix. The level of mixture for this reduced state is determined by criterium for trace of square of reduced density matrix.

- **Right Decomposition**
  For arbitrary two qubit pure state in the form

$$|\psi\rangle = |0\rangle|c_0\rangle + |1\rangle|c_1\rangle,$$

where

$$|c_0\rangle = c_{00}\,|0\rangle + c_{10}\,|1\rangle, \qquad |c_1\rangle = c_{10}\,|0\rangle + c_{11}\,|1\rangle, \tag{4.35}$$

the density matrix is

$$\begin{aligned} \rho = |\psi\rangle\langle\psi| &= (|0\rangle|c_0\rangle + |1\rangle|c_1\rangle)(\langle 0|\langle c_0| + \langle 1|\langle c_1|) \\ &= |0\rangle\langle 0|\,|c_0\rangle\langle c_0| + |1\rangle\langle 1|\,|c_1\rangle\langle c_1| + |0\rangle\langle 1|\,|c_0\rangle\langle c_1| + |1\rangle\langle 0|\,|c_1\rangle\langle c_0|, \end{aligned}$$

and reduced density matrix appears as

$$\rho_A = \mathrm{tr}_A\,\rho = |c_0\rangle\langle c_0| + |c_1\rangle\langle c_1|.$$

Substituting one qubit states (4.35) into reduced density matrix gives

$$\begin{gathered} \rho_A = (|c_{00}|^2 + |c_{10}|^2)\,|0\rangle\langle 0| + (|c_{01}|^2 + |c_{11}|^2)\,|1\rangle\langle 1| \\ +(c_{00}\bar{c}_{01} + c_{10}\bar{c}_{11})\,|0\rangle\langle 1| + (c_{01}\bar{c}_{00} + c_{11}\bar{c}_{10})\,|1\rangle\langle 0|. \end{gathered}$$

Therefore, the reduced density matrix in matrix form is

$$\rho_A = \begin{pmatrix} |c_{00}|^2 + |c_{10}|^2 & c_{00}\bar{c}_{01} + c_{10}\bar{c}_{11} \\ c_{01}\bar{c}_{00} + c_{11}\bar{c}_{10} & |c_{01}|^2 + |c_{11}|^2 \end{pmatrix}. \tag{4.36}$$

- **Left Decomposition**

  The left decomposition of arbitrary two qubit state is

$$|\psi\rangle = |d_0\rangle|0\rangle + |d_1\rangle|1\rangle,$$

where

$$|d_0\rangle = c_{00}\,|0\rangle + c_{10}\,|1\rangle, \qquad |d_1\rangle = c_{01}\,|0\rangle + c_{11}\,|1\rangle. \tag{4.37}$$

Then from the density matrix

$$\begin{aligned} \rho = |\psi\rangle\langle\psi| &= (|0\rangle|d_0\rangle + |1\rangle|d_1\rangle)(\langle 0|\langle d_0| + \langle 1|\langle d_1|) \\ &= |0\rangle\langle 0|\,|d_0\rangle\langle d_0| + |1\rangle\langle 1|\,|d_1\rangle\langle d_1| + |0\rangle\langle 1|\,|d_0\rangle\langle d_1| + |1\rangle\langle 0|\,|d_1\rangle\langle d_0|, \end{aligned}$$

the following reduced density matrix appears

$$\rho_B = \mathrm{tr}_B\,\rho = |d_0\rangle\langle d_0| + |d_1\rangle\langle d_1|.$$

Substituting one qubit states (4.37) into this matrix gives

$$\begin{aligned} \rho_B = (|c_{00}|^2 + |c_{01}|^2)\,|0\rangle\langle 0| + (|c_{10}|^2 + |c_{11}|^2)\,|1\rangle\langle 1| \\ +(c_{00}\bar{c}_{10} + c_{01}\bar{c}_{11})\,|0\rangle\langle 1| + (c_{10}\bar{c}_{00} + c_{11}\bar{c}_{01})\,|1\rangle\langle 0|. \end{aligned}$$

Then, the matrix form for this reduced density matrix is

$$\rho_B = \begin{pmatrix} |c_{00}|^2 + |c_{01}|^2 & c_{00}\bar{c}_{10} + c_{01}\bar{c}_{11} \\ c_{10}\bar{c}_{00} + c_{11}\bar{c}_{01} & |c_{10}|^2 + |c_{11}|^2 \end{pmatrix}. \tag{4.38}$$

Comparison of reduced density matrices (4.36) and (4.38) with the inner space metric G (4.30) and H (4.31) shows that they coincide,

$$\rho_A = H^\intercal = BB^\dagger, \qquad \rho_B = G^\intercal = AA^\dagger.$$

Therefore, determinants of the reduced density matrices are equal

$$\det\rho_A = \det H = |\det B|^2 = |\det A|^2, \tag{4.39}$$

$$\det\rho_B = \det G = |\det A|^2 = |\det B|^2, \tag{4.40}$$

implying that concurrence is

$$C = 2|\det A| = 2\sqrt{\det\rho_A} = 2\sqrt{\det\rho_B}. \tag{4.41}$$

This formula expresses concurrence by determinants of reduced density matrices. Since concurrence characterizes entanglement of two qubit state, it is possible now to compare the entanglement with the mixed character of the reduced quantum states.

For reduced density matrices (4.36) and (4.38) following relations are valid:

$$1) \qquad \mathrm{tr}\,\rho_A = 1, \qquad \mathrm{tr}\,\rho_B = 1 \tag{4.42}$$

$$2) \qquad \rho_A^2 = (H^\intercal)^2, \qquad \rho_B^2 = (G^\intercal)^2 \tag{4.43}$$

$$3) \qquad \operatorname{tr}\rho_A^2 = 1 - 2|c_{00}c_{11} - c_{01}c_{10}|^2, \qquad \operatorname{tr}\rho_B^2 = 1 - 2|c_{00}c_{11} - c_{01}c_{10}|^2. \qquad (4.44)$$

Since concurrence $C = 2|c_{00}c_{11} - c_{01}c_{10}|$, it gives

$$\operatorname{tr}\rho_A^2 = 1 - \frac{1}{2}C^2, \qquad \operatorname{tr}\rho_B^2 = 1 - \frac{1}{2}C^2.$$

These formulas represent Pythagoras theorem for concurrence and reduced density matrices:

$$\operatorname{tr}\rho_A^2 + \frac{1}{2}C^2 = 1, \qquad \operatorname{tr}\rho_B^2 + \frac{1}{2}C^2 = 1.$$

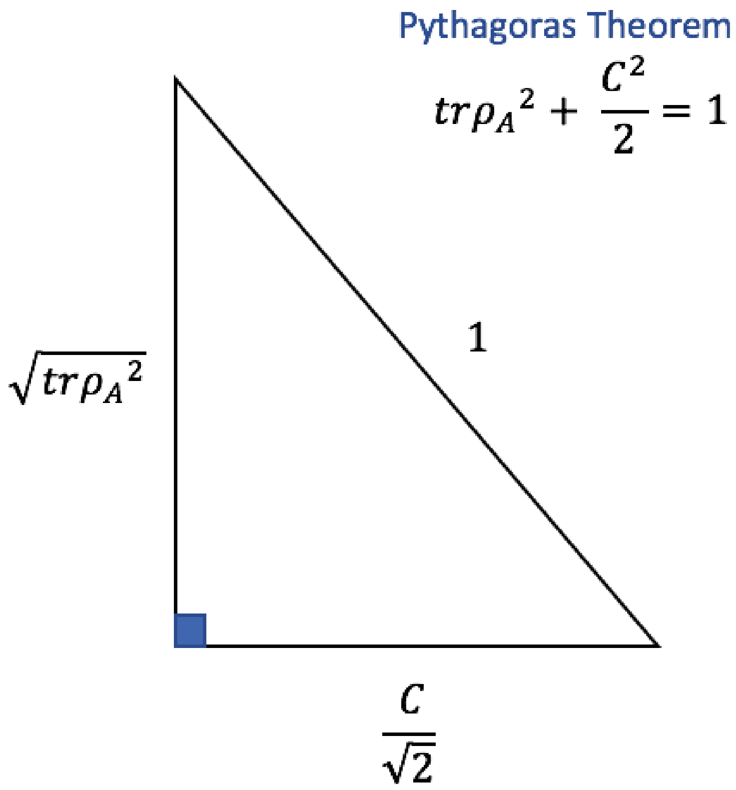


Figure 4.2. Relation between concurrence and reduced density matrix

- **Seperable states:**
  If $C = 0$, then $\mathrm{tr}\,\rho_A^2 = 1$ and the state is pure

- **Entangled states:**

  1. If $C = 1$, then $\mathrm{tr}\,\rho_A^2 = \frac{1}{2} < 1$ and the state is maximally mixed
  2. If $0 < C < 1$, then $\mathrm{tr}\,\rho_A^2 = 1 - \frac{1}{2}C^2 < 1$ - and the state is mixed .

  The same analysis is valid also for matrix ρ_B.

In addition to above formulas, possible to express the concurrence as a function of reduced density matrix. Indeed from

$$\mathrm{tr}\,\rho_A^2 = \mathrm{tr}\,\rho_B^2 = 1 - \frac{1}{2}C^2,$$

follows that

$$C = \sqrt{2(1 - \mathrm{tr}\,\rho_A^2)} = \sqrt{2(1 - \mathrm{tr}\,\rho_B^2)}.$$

4.7. Entanglement and Von Neumann Entropy

In classical information theory the measure of randomness or measure of unpredictability is determined by the Shannon entropy.

Definition 4.13 *Let X is random variable taking values $x_1, x_2, ..., x_n$ characterized by probability distribution $p_1, p_2, ..., p_n$ where*

$$\sum_i^n p_i = 1 \qquad 0 \leq p_i \leq 1.$$

Then, the Shannon entropy of X is defined as expected value

$$S(X) = -\sum_{i}^{n} p_i \log_2 p_i.$$

The Shannon entropy satisfies the following bounds

$$0 \le S(X) \le \log_2 n.$$

For $n = 2$ elements, the bound is

$$0 \le S(X) \le 1.$$

The quantum mechanical analogue of the Shannon entropy is the Von Neumann entropy.

Definition 4.14 *Von Neumann entropy of a quantum state ρ is defined by formula*

$$S(\rho) = -\operatorname{tr}(\rho \log_2 \rho),$$

where ρ is the density matrix of the quantum state, satisfying constraint

$$\operatorname{tr}\rho = 1.$$

Proposition 4.7 *If ρ is diagonalized in a basis |i⟩,*

$$\rho = \sum_{i} p_i |i\rangle\langle i|,$$

then the Von Neumann entropy takes the form of the Shannon entropy

$$S = -\sum_i p_i \log_2 p_i.$$

Proof Evaluating the trace in the basis states $|n\rangle$

$$S = -k \sum_{n,m} \langle n|\rho|m\rangle \langle m| \log_2 \rho |n\rangle$$

and transforming basis $|n\rangle$ to $|i\rangle$ with diagonal ρ

$$\rho = \sum_i p_i |i\rangle\langle i|, \qquad \log_2 \rho = \sum_i \log_2 p_i |i\rangle\langle i|$$

the entropy becomes

$$S = -\sum_i p_i \log_2 p_i$$

where $p_i = \langle i|\rho|i\rangle$.

□

The Von Neumann entropy plays essential role in definition of entanglement for qubit states. The basic idea is that, by taking partial trace of a state one can decide, if the reduced state is pure state or the mixed state, if it is random or not.

Definition 4.15 *(Wootters, 1998) The entanglement E for a pure two qubit state $|\psi\rangle$ is defined as the entropy in the form of the Von Neumann entropy*

$$\begin{aligned} E(\psi) &= -\operatorname{tr}(\rho_A \log_2 \rho_A) \\ &= -\operatorname{tr}(\rho_B \log_2 \rho_B) \end{aligned}$$

where reduced density matrices are

$$\rho_A = \operatorname{tr}_B \rho = \operatorname{tr}_B |\psi\rangle\langle\psi|, \qquad \rho_B = \operatorname{tr}_A \rho = \operatorname{tr}_A |\psi\rangle\langle\psi|$$

and

$$\rho = |\psi\rangle\langle\psi|.$$

Characteristic equations for these two matrices are equal. Indeed,

$$\lambda^2 - \lambda \operatorname{tr} \rho_A + \det \rho_A = 0,$$

and

$$\lambda^2 - \lambda \operatorname{tr} \rho_B + \det \rho_B = 0,$$

for matrices (4.36) and (4.38), due to

$$\operatorname{tr} \rho_A = \operatorname{tr} \rho_B = 1,$$

and

$$\det \rho_A = \det \rho_B = \det G = \det H,$$

which follows from (4.39), (4.40), are the same. Since the determinants can be expressed by the concurrence (4.41),

$$\det \rho_A = \det \rho_B = \frac{C^2}{4},$$

the characteristic equation becomes

$$\lambda^2 - \lambda + \frac{C^2}{4} = 0.$$

This equation has two real solutions as eigenvalues of reduced density matrices,

$$\lambda_{\pm} = \frac{1 \pm \sqrt{1 - C^2}}{2},$$

with eigenstates $|\lambda_+\rangle$ and $|\lambda_-\rangle$. In terms of these eigenstates the density matrices ρ_A and ρ_B are diagonal and the entanglement E takes form of the Shannon entropy

$$E = -\lambda_+ \log_2 \lambda_+ - \lambda_- \log_2 \lambda_-.$$

It gives the following expression for the entanglement E as a function of concurrence C:

$$E(C) = -\frac{1 + \sqrt{1 - C^2}}{2} \log_2 \left(\frac{1 + \sqrt{1 - C^2}}{2} \right) - \frac{1 - \sqrt{1 - C^2}}{2} \log_2 \left(\frac{1 - \sqrt{1 - C^2}}{2} \right). \quad (4.45)$$

Plot of this function is shown in Figure 4.1, and the function is monotonically increasing from 0 to 1 on the interval $0 \leq C \leq 1$. This means that concurrence C can characterizes entanglement E as the level of entropy or the randomness. The maximal concurrence $C = 1$ corresponds to maximally entangled states, which are maximally random states. The separable states with $C = 0$ give minimum of randomness with $E = 0$.

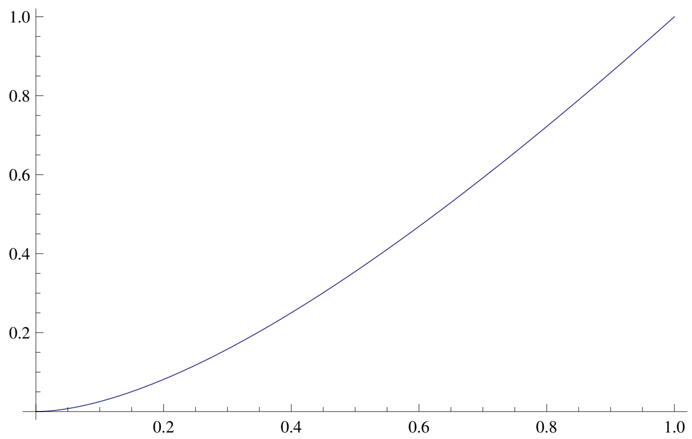


Figure 4.3. Entanglement Function of Concurence

4.7.1. Entanglement of Two Qubit Coherent Like State

For two qubit coherent like state

$$|z\rangle = \frac{|00\rangle + z|11\rangle}{\sqrt{1+|z|^2}}$$

the concurrence is

$$C = \frac{2|z|}{1+|z|^2}.$$

Calculation of entanglement $E(C)$, (4.45), in terms of z gives

$$1 - C^2 = 1 - \frac{4|z|^2}{(1+|z|^2)^2} = \frac{(1-|z|^2)^2}{(1+|z|^2)^2},$$

$$\sqrt{1-C^2} = \left|\frac{(1-|z|^2)}{(1+|z|^2)}\right| = \frac{|1-|z|^2|}{1+|z|^2},$$

then

$$1 + \sqrt{1-C^2} = \frac{1+|z|^2+|1-|z|^2|}{1+|z|^2} \tag{4.46}$$

or

$$1 + \sqrt{1-C^2} = \begin{cases} \dfrac{2|z|^2}{1+|z|^2} & \text{for} \quad |z|^2 > 1 \\ 1 & \text{for} \quad |z|^2 = 1 \\ \dfrac{2}{1+|z|^2} & \text{for} \quad |z|^2 < 1 \end{cases} \tag{4.47}$$

Similar calculation

$$1 - \sqrt{1 - C^2} = \frac{1 + |z|^2 - |1 - |z|^2|}{1 + |z|^2} \tag{4.48}$$

gives

$$1 - \sqrt{1 - C^2} = \begin{cases} \dfrac{2}{1 + |z|^2} & \text{for} \quad |z|^2 > 1 \\ 1 & \text{for} \quad |z|^2 = 1 \\ \dfrac{2|z|^2}{1 + |z|^2} & \text{for} \quad |z|^2 < 1 \end{cases} \tag{4.49}$$

Due to these results the entanglement is

$$E = \begin{cases} -\dfrac{|z|^2}{1 + |z|^2} \log_2\left(\dfrac{|z|^2}{1 + |z|^2}\right) - \dfrac{1}{1 + |z|^2} \log_2\left(\dfrac{1}{1 + |z|^2}\right) & \text{for} \quad |z|^2 > 1 \\ -\dfrac{1}{2} \log_2\left(\dfrac{1}{2}\right) - -\dfrac{1}{2} \log_2\left(\dfrac{1}{2}\right) = 1 & \text{for} \quad |z|^2 = 1 \\ -\dfrac{1}{1 + |z|^2} \log_2\left(\dfrac{1}{1 + |z|^2}\right) - \dfrac{|z|^2}{1 + |z|^2} \log_2\left(\dfrac{|z|^2}{1 + |z|^2}\right) & \text{for} \quad |z|^2 < 1 \end{cases}$$

For arbitrary $|z|^2$,

$$\begin{aligned} E(|z|^2) &= -\frac{1}{1 + |z|^2} \log_2\left(\frac{1}{1 + |z|^2}\right) - \frac{|z|^2}{1 + |z|^2} \log_2\left(\frac{|z|^2}{1 + |z|^2}\right) \\ &= \frac{1}{1 + |z|^2} \log_2 (1 + |z|^2) + \frac{|z|^2}{1 + |z|^2}(\log_2 (1 + |z|^2) - \log_2 |z|^2), \end{aligned}$$

giving entanglement

$$E(|z|^2) = \log_2 (1 + |z|^2) - \frac{|z|^2 \log_2 |z|^2}{1 + |z|^2}.$$

As easy to see, on every circle $|z|^2 = r^2$, the entanglement is a constant

$$E(r^2) = \log_2(1 + r^2) - \frac{r^2 \log_2 r^2}{1 + r^2}, \tag{4.50}$$

coinciding with the Shannon entropy (5.3).

4.8. Concurrence and Riemannian Metric

The inner product metric (4.28), for the generic two qubit state (4.17), depends on three complex parameters η, z, w or six real parameters.

Comparision the generic state representation (4.27), with (4.17) gives

$$|c_0\rangle = \frac{|z\rangle}{\sqrt{1 + |\eta|^2}}, \qquad |c_1\rangle = \frac{\eta\,|w\rangle}{\sqrt{1 + |\eta|^2}}.$$

Then, elements of the inner product matrix can be found as

$$g_{00} = \langle c_0|c_0\rangle = \frac{1}{1 + |\eta|^2}, \quad g_{01} = \langle c_0|c_1\rangle = \frac{\eta}{1 + |\eta|^2}\,\frac{1 + \bar{z}w}{\sqrt{(1 + |z|^2)(1 + |w|^2)}},$$

$$g_{11} = \langle c_1|c_1\rangle = \frac{|\eta|^2}{1 + |\eta|^2}, \quad g_{10} = \overline{\langle c_0|c_1\rangle} = \frac{\bar{\eta}}{1 + |\eta|^2}\,\frac{1 + z\bar{w}}{\sqrt{(1 + |z|^2)(1 + |w|^2)}}.$$

This metric has the matrix form

$$G = \frac{1}{1 + |\eta|^2}\begin{pmatrix} 1 & \frac{\eta\,(1 + \bar{z}w)}{\sqrt{(1 + |z|^2)(1 + |w|^2)}} \\ \frac{\bar{\eta}\,(1 + z\bar{w})}{\sqrt{(1 + |z|^2)(1 + |w|^2)}} & |\eta|^2 \end{pmatrix}$$

Trace of this matrix is one, $\mathrm{Tr}\, G = 1$, and determinant with concurrence are

$$\det G = |\det A|^2 = \frac{C^2}{4},$$

where the concurrence is given by (4.21).

In particular cases, when this metric depends on only one complex variable, or two real variables, this metric becomes the Riemannian metric on a surface. Depending on the reduction, several possibilities exist.

Particular Cases -"Entangled Metric":

1. By taking limits in (4.16) as $z \to 0 \Longrightarrow |z\rangle \to |0\rangle$, $w \to \infty \Longrightarrow |w\rangle \to |1\rangle$ one qubit states are represented as

$$|c_0\rangle = \frac{|0\rangle}{\sqrt{1+|\eta|^2}}, \qquad |c_1\rangle = \frac{\eta\,|1\rangle}{\sqrt{1+|\eta|^2}},$$

and the state (4.17) becomes

$$|\psi\rangle = \frac{|00\rangle + \eta\,|11\rangle}{\sqrt{1+|\eta|^2}}.$$

The matrix elements for the metric are

$$g_{00} = \langle c_0|c_0\rangle = \frac{1}{1+|\eta|^2}, \quad g_{01} = \langle c_0|c_1\rangle = 0,$$

$$g_{11} = \langle c_1|c_1\rangle = \frac{|\eta|^2}{1+|\eta|^2}, \quad g_{10} = \overline{\langle c_0|c_1\rangle} = 0,$$

and the matrix is

$$G = \frac{1}{1+|\eta|^2}\begin{pmatrix} 1 & 0 \\ 0 & |\eta|^2 \end{pmatrix}.$$

2. By taking limits in (4.16) as $z \to \infty \Longrightarrow |z\rangle \to |1\rangle$, $w \to 0 \Longrightarrow |w\rangle \to |0\rangle$, one qubit states are represented as

$$|c_0\rangle = \frac{|1\rangle}{\sqrt{1+|\eta|^2}}, \qquad |c_1\rangle = \frac{\eta\,|0\rangle}{\sqrt{1+|\eta|^2}},$$

the state (4.17) becomes

$$|\psi\rangle = \frac{|01\rangle + \eta\,|10\rangle}{\sqrt{1+|\eta|^2}}.$$

The matrix elements for the metric are

$$g_{00} = \langle c_0|c_0\rangle = \frac{1}{1+|\eta|^2}, \quad g_{01} = \langle c_0|c_1\rangle = 0,$$

$$g_{11} = \langle c_1|c_1\rangle = \frac{|\eta|^2}{1+|\eta|^2}, \quad g_{10} = \overline{\langle c_0|c_1\rangle} = 0,$$

and corresponding matrix is

$$G = \frac{1}{1+|\eta|^2}\begin{pmatrix} 1 & 0 \\ 0 & |\eta|^2 \end{pmatrix}.$$

In both cases 1 and 2, the metric is the same. Invariants of this metric $\mathrm{Tr}\, G = 1$, and

$$\det G = |\det A|^2 = \frac{|\eta|^2}{(1+|\eta|^2)^2} = \frac{C^2}{4},$$

imply the concurrence

$$C = \frac{2|\eta|^2}{\sqrt{1+|\eta|^2}}.$$

3. By taking limits in (4.16) as $w \to \infty \implies |w\rangle \to |1\rangle$ and $\eta = 1$, one qubit states are represented as

$$|c_0\rangle = \frac{|z\rangle}{\sqrt{2}}, \qquad |c_1\rangle = \frac{|1\rangle}{\sqrt{2}},$$

the state (4.17) becomes

$$|\psi\rangle = \frac{|0\rangle|z\rangle + |1\rangle|1\rangle}{\sqrt{2}}.$$

Then matrix elements for the metric are

$$g_{00} = \langle c_0|c_0\rangle = \frac{1}{2}, \quad g_{01} = \langle c_0|c_1\rangle = \frac{\bar{z}}{\sqrt{1+|z|^2}},$$

$$g_{11} = \langle c_1|c_1\rangle = \frac{1}{2}, \quad g_{10} = \overline{\langle c_0|c_1\rangle} = \frac{z}{\sqrt{1+|z|^2}},$$

and corresponding matrix is

$$G = \frac{1}{2}\begin{pmatrix} 1 & \frac{\bar{z}}{\sqrt{1+|z|^2}} \\ \frac{z}{\sqrt{1+|z|^2}} & 1 \end{pmatrix}.$$

Then $\mathrm{Tr}\, G = 1$ with

$$\det G = |\det A|^2 = \frac{1}{4(1+|z|^2)} = \frac{C^2}{4},$$

which implies that concurrence

$$C = \frac{1}{\sqrt{1+|z|^2}}.$$

It reaches maximal value for $z = 0$, giving the Bell state.

4. By taking limits in (4.16) as $z \to 0 \implies |z\rangle \to |0\rangle$ and $\eta = 1$, one qubit states are represented as

$$|c_0\rangle = \frac{|0\rangle}{\sqrt{2}}, \qquad |c_1\rangle = \frac{|w\rangle}{\sqrt{2}},$$

the state (4.17) becomes

$$|\psi\rangle = \frac{|0\rangle|0\rangle + |1\rangle|w\rangle}{\sqrt{2}}.$$

The matrix elements for the metric

$$g_{00} = \langle c_0|c_0\rangle = \frac{1}{2}, \quad g_{01} = \langle c_0|c_1\rangle = \frac{w}{\sqrt{1+|w|^2}},$$

$$g_{11} = \langle c_1|c_1\rangle = \frac{1}{2}, \quad g_{10} = \overline{\langle c_0|c_1\rangle} = \frac{\bar{w}}{\sqrt{1+|w|^2}},$$

give

$$G = \frac{1}{2}\begin{pmatrix} 1 & \dfrac{w}{\sqrt{1+|w|^2}} \\ \dfrac{\bar{w}}{\sqrt{1+|w|^2}} & 1 \end{pmatrix}.$$

Then, $\operatorname{Tr} G = 1$ and

$$\det G = |\det A|^2 = \frac{1}{4(1+|w|^2)} = \frac{C^2}{4},$$

which implies that

$$C = \frac{1}{\sqrt{1+|w|^2}}.$$

5. By taking $z = \eta$ and $w = 0$, one qubit states are represented as

$$|c_0\rangle = \frac{|\eta\rangle}{\sqrt{1+|\eta|^2}}, \qquad |c_1\rangle = \frac{\eta|0\rangle}{\sqrt{1+|\eta|^2}},$$

and the state (4.17) becomes

$$|\psi\rangle = \frac{|0\rangle|\eta\rangle + \eta\,|1\rangle|0\rangle}{\sqrt{1+|\eta|^2}}.$$

Then matrix elements for the metric are

$$g_{00} = \langle c_0|c_0\rangle = \frac{1}{1+|\eta|^2}, \quad g_{01} = \langle c_0|c_1\rangle = \frac{\eta}{(1+|\eta|^2)^{3/2}},$$

$$g_{11} = \langle c_1|c_1\rangle = \frac{|\eta|^2}{1+|\eta|^2}, \quad g_{10} = \overline{\langle c_0|c_1\rangle} = \frac{\bar{\eta}}{(1+|\eta|^2)^{3/2}},$$

and corresponding matrix is

$$G = \frac{1}{1+|\eta|^2}\begin{pmatrix} 1 & \dfrac{\eta}{\sqrt{1+|\eta|^2}} \\ \dfrac{\bar{\eta}}{\sqrt{1+|\eta|^2}} & |\eta|^2 \end{pmatrix}.$$

For this metric $\mathrm{Tr}\,G = 1$ and

$$\det G = |\det A|^2 = \frac{|\eta|^4}{(1+|\eta|^2)^2} = \frac{C^2}{4},$$

which implies that

$$C = \frac{2|\eta|^2}{1+|\eta|^2}.$$

Particular Case -"Separable Metric":

By taking $z = w$, the state (4.17) becomes separable

$$|\psi\rangle = \frac{|0\rangle|z\rangle + \eta\,|1\rangle|z\rangle}{\sqrt{1+|\eta|^2}} = \frac{|0\rangle + \eta|1\rangle}{\sqrt{1+|\eta|^2}}\,|z\rangle = |\eta\rangle\,|z\rangle,$$

and is a direct product of one qubit coherent states. With respect to these, one qubit states are represented as

$$|c_0\rangle = \frac{|z\rangle}{\sqrt{1+|\eta|^2}}, \qquad |c_1\rangle = \frac{\eta|z\rangle}{\sqrt{1+|\eta|^2}}.$$

The matrix elements for corresponding metric are

$$g_{00} = \langle c_0|c_0\rangle = \frac{1}{1+|\eta|^2}, \quad g_{01} = \langle c_0|c_1\rangle = \frac{\eta}{1+|\eta|^2},$$

$$g_{11} = \langle c_1|c_1\rangle = \frac{|\eta|^2}{1+|\eta|^2}, \quad g_{10} = \overline{\langle c_0|c_1\rangle} = \frac{\bar{\eta}}{1+|\eta|^2},$$

and the matrix is

$$G = \frac{1}{1+|\eta|^2}\begin{pmatrix} 1 & \eta \\ \bar{\eta} & |\eta|^2 \end{pmatrix}.$$

Since the state $|\psi\rangle$ is separable:

$$\det G = |\det A|^2 = 0 = \frac{C^2}{4},$$

which implies that

$$C = 0.$$

This means that for separable states the metric is degenerate, $\det G = 0$.

CHAPTER 5

APOLLONIUS QUBIT STATES

5.1. Apollonius Circles and Möbius Transformations

Definition 5.1 *(Brannan, Esplen and Gray, 2012) Apollonius Circle: A circle can be defined as the set of points $z = x+iy$ in complex plane that have specified ratio of distances from two fixed points. The ratio is*

$$\frac{|z-a|}{|z-b|} = r, \tag{5.1}$$

where a and b are common symmetric points playing role of the fixed points. (Figure 5.1*)*

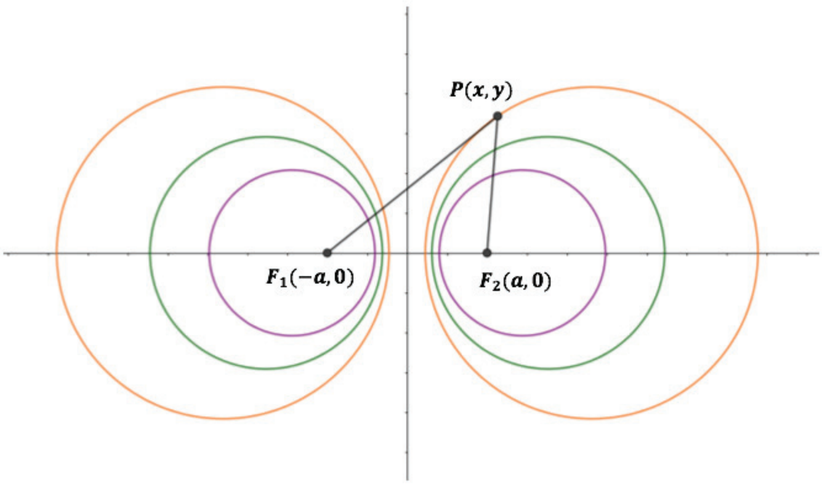


Figure 5.1. Apollonius circles with $z = a$ and $z = -a$ fixed points

5.2. Hadamard Gate and Apollonius Representation

In Section 3.2, one qubit in coherent state representation was defined as

$$|z\rangle = \frac{|0\rangle + z|1\rangle}{\sqrt{1+|z|^2}},$$

where complex number $z = \tan\frac{\theta}{2}e^{i\varphi}$ denotes the stereographic projection of Bloch sphere.

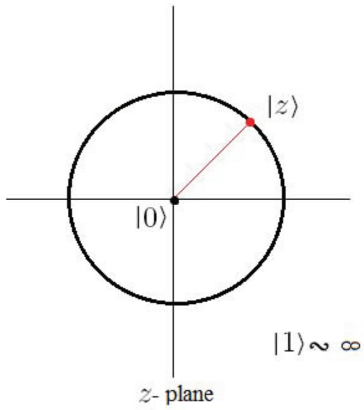
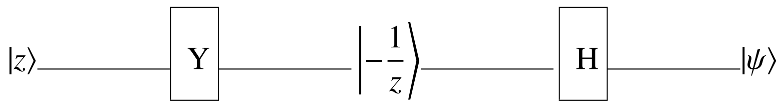


Figure 5.2. Coherent states plane

In this representation $|0\rangle$ state corresponds to the origin $z = 0$, and the state $|1\rangle$ is going to infinity, and belongs to the extended complex plane(Figure 5.2). This creates some disadvantages for visualization of geometrical characteristics of qubits. For this reason, more convenient to use a new parametrization of qubit state with $|0\rangle$ and $|1\rangle$ states located at two finite points in complex plane. This parametrization is related with Möbius transformations and Apollonius circles. By using Hadamard gate, one can move 0 and ∞ points to a finite points 1 and -1 in the plane. As a result, it gives a new representation of qubit, with state $|0\rangle$ at point 1 and state $|1\rangle$ at point -1:

$$H|z\rangle = |b\rangle = \frac{(1+z)|0\rangle + (1-z)|1\rangle}{\sqrt{2}\sqrt{1+|z|^2}}.$$

To get ordered basis qubits $|0\rangle$ and $|1\rangle$ at positions -1 and 1, correspondingly, the following circuit diagram:



can be used, so that the following state appears

$$|\psi\rangle = \frac{(z-1)|0\rangle + (z+1)|1\rangle}{\sqrt{2}\sqrt{1+|z|^2}}.$$

Definition 5.2 *The one qubit state*

$$|\psi\rangle = \frac{(z-1)|0\rangle + (z+1)|1\rangle}{\sqrt{2}\sqrt{1+|z|^2}} = \frac{(z-1)|0\rangle + (z+1)|1\rangle}{\sqrt{|z-1|^2 + |z+1|^2}},$$

is called the symmetric Apollonius qubit state.

When $z = 1$ it gives state $|1\rangle$ located on real axis at point 1,and when $z = -1$ it gives state $|0\rangle$ located on real axis at point -1.

In principle, one can fix $|0\rangle$ and $|1\rangle$ states at arbitrary points in the plane. For illustration reasons the natural choice is to consider the special case, when $|0\rangle$ state is located at the origin 0, and $|1\rangle$ state is located at point 1.

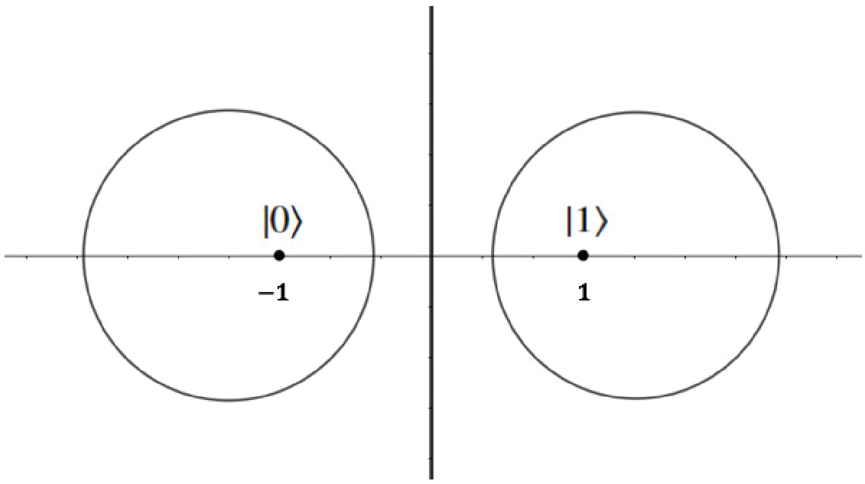


Figure 5.3. Symmetric Apollonius One Qubit State

Definition 5.3 *The one qubit state*

$$|\psi\rangle = \frac{(z-1)|0\rangle + z|1\rangle}{\sqrt{2}\sqrt{1+|z|^2}} = \frac{(z-1)|0\rangle + z|1\rangle}{\sqrt{|z-1|^2+|z|^2}}$$

is called the non- symmetric Apollonius qubit state.

It is useful for comparison of bits and qubits. Indeed, one bit corresponds just to two points 0 and 1 in plane, while the qubit is determined by an arbitrary point in the plane.

5.3. Non-Symmetric Apollonius Qubit States

5.3.1. One Qubit State

To fix position of states $|0\rangle$ and $|1\rangle$ at points 0 and 1 correspondingly, one replaces z to $2z-1$ (scaling and translation) and gets one qubit state $|\psi\rangle$ in the form

$$|\psi\rangle = \frac{(z-1)|0\rangle + z|1\rangle}{\sqrt{|z-1|^2+|z|^2}},$$

which is the non-symmetric Apollonius qubit representation. Probabilities to measure states $|0\rangle$ or $|1\rangle$ are:

$$p_0 = \frac{|z-1|^2}{|z-1|^2+|z|^2},$$

$$p_1 = \frac{|z|^2}{|z-1|^2+|z|^2},$$

where $p_0 + p_1 = 1$ and the ratio of probabilities is

$$\frac{p_1}{p_0} = \frac{|z|^2}{|z-1|^2} \equiv r^2 \,. \tag{5.2}$$

As easy to see, for fixed ratio of probabilities r^2, the set of points in complex plane z belongs to the Apollonius circles. This is why, the set of qubit states in Apollonius representation, with fixed ratio of probabilities, is located on an Apollonius circle. This representation splits the set of all qubit states to the states on different Apollonius circles.

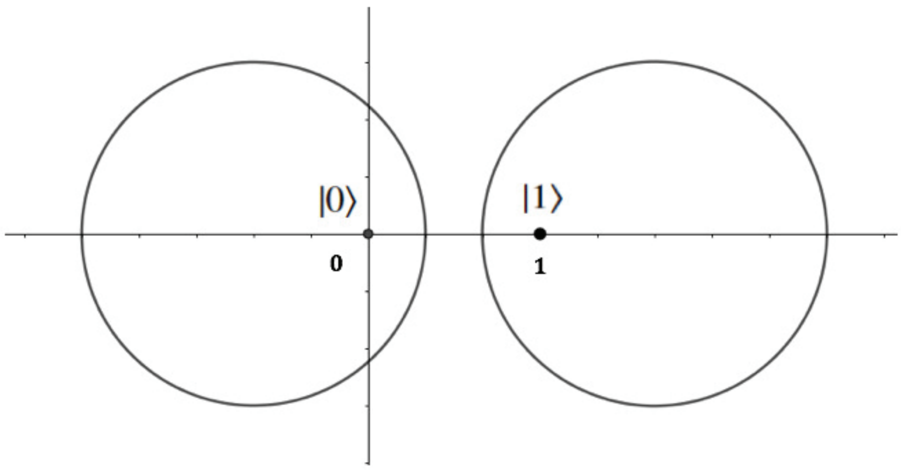


Figure 5.4. Non - Symmetric Apollonius One Qubit State

5.3.1.1. Entropy of One Qubit State

For Apollonius state $|\psi\rangle$ probabilities to measure states $|0\rangle$ and $|1\rangle$ are

$$p_0 = |\langle 0|z\rangle|^2 = \frac{|z-1|^2}{|z-1|^2 + |z|^2} = \frac{1}{1+r^2} \,,$$

$$p_1 = |\langle 1|z\rangle|^2 = \frac{|z|^2}{|z-1|^2 + |z|^2} = \frac{r^2}{1+r^2} \,,$$

where r^2 as ratio of probabilities is defined in (5.2). The level of randomness for this state $|\psi\rangle$ then could be characterized by the Shannon entropy

$$S = -p_0 \log_2 p_0 - p_1 \log_2 p_1$$

represented as

$$S(r^2) = \log_2(1 + r^2) - \frac{r^2}{1 + r^2} \log_2 r^2. \tag{5.3}$$

This formula shows that the Shannon entropy or the level of randomness for Apollonius qubit states is constant along Apollonius circles.

Maximally Random States: For maximally random state, derivative of entropy with respect to r^2 vanishes

$$\frac{dS}{dr^2} = -\frac{1}{(1 + r^2)^2} \log_2 r^2 = 0 \quad \Rightarrow \quad r = 1.$$

The second derivative gives

$$\frac{d^2S}{(dr^2)^2} = -\frac{2}{(1 + r^2)^3} \log_2 r^2 - \frac{1}{(1 + r^2)^2} \frac{1}{r^2 \ln 2}$$

and

$$S''|_{r=1} = -\frac{1}{4 \ln 2} < 0,$$

which implies that $r = 1$ is the local maximum. Therefore, Apollonius circles are level curves of the same randomness (constant entropy S along these level curves). The maximally random states with $S = 1$ are located at vertical line $\operatorname{Re} z = \frac{1}{2}$(Figure 5.3). In contrast, the computational basis states with $r = 0$ and $r = \infty$ have zero entropy: $S(0) = 0$ and $S(\infty) = 0$.

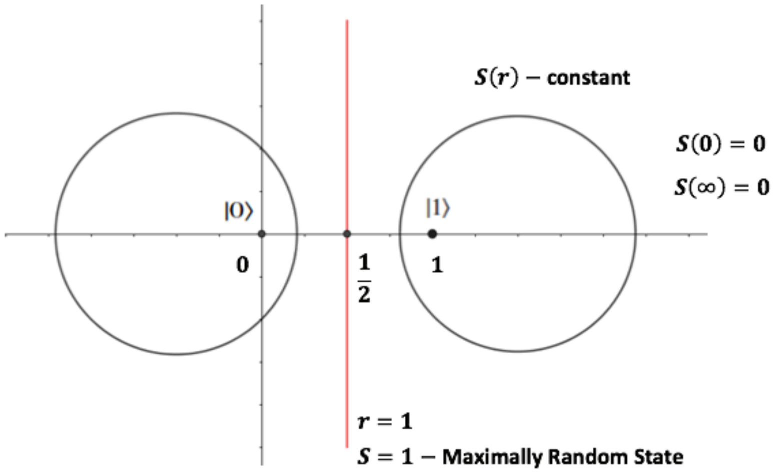


Figure 5.5. Entropy on Apollonius circles

5.3.1.2. Fidelity and Distance

Definition 5.4 *Fidelity of two quantum states $|\psi_1\rangle$ and $|\psi_2\rangle$ is defined by $F \equiv |\langle\psi_1|\psi_2\rangle|$. It is a measure of the distance between two quantum states. Fidelity is bounded $0 \leq F \leq 1$; it is $F = 1$, when $|\psi_1\rangle$ coincides with $|\psi_2\rangle$ and if $F = 0$, when $|\psi_1\rangle$ and $|\psi_2\rangle$ are orthogonal.*

Another characteristics, which is constant along Apollonius circles is the fidelity between symmetric states, reflected in vertical axis $\operatorname{Re} z = \frac{1}{2}$. This reflection corresponds to substitution $z \to 1 - \bar{z}$ and gives the symmetric Apollonius state $|\psi\rangle$

$$|\psi_s\rangle = \frac{-\bar{z}|0\rangle + (1-\bar{z})|1\rangle}{\sqrt{|z-1|^2 + |z|^2}},$$

with fidelity

$$F = |\langle\psi_s|\psi\rangle| = \frac{2|z||z-1|}{|z-1|^2 + |z|^2}.$$

This fidelity depends only on the ratio $r = \dfrac{|z|}{|z-1|}$ and is constant along the Apollonius circles. This constant is bounded $0 \leq F \leq 1$ and vanishes for $|0\rangle$ and $|1\rangle$ states as orthogonal symmetric states.

Figure 5.4 shows the entropy and the fidelity versus r. Both curves in this figure reach maximal value at $r = 1$ and vanish at $r = 0$ and $r = \infty$. Comparison of these curves show that maximally random state corresponds to maximal fidelity between symmetric states and it happens, when these states belong to the line $\mathrm{Re}\, z = \frac{1}{2}$. Increasing the geometrical distance between symmetric states will decrease the level of randomness. So that $|0\rangle$ and $|1\rangle$ states as maximally far symmetric states are orthogonal and as a result $F = 0$.

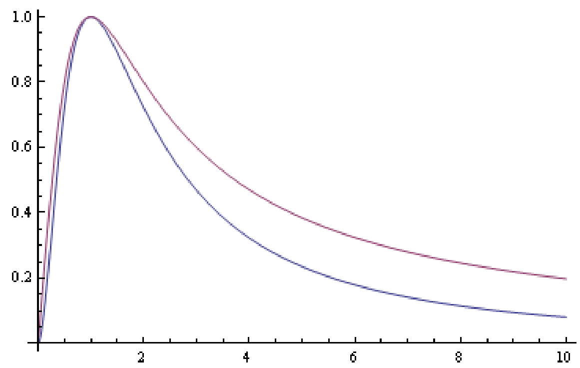


Figure 5.6. Entropy (blue line) and fidelity (pink line) between symmetric states versus r

The standard distance between symmetric states in the Hilbert space is given by formula

$$|||\psi\rangle - |\psi_s\rangle|| = 2\frac{\left|\mathrm{Re}\, z - \frac{1}{2}\right|}{\sqrt{|z-1|^2 + |z|^2}}.$$

It shows that the distance reaches maximal value for orthogonal states at $z = 0$ and $z = 1$ and it vanishes on the vertical line $\mathrm{Re}\, z = \frac{1}{2}$. Due to this property, one can introduce another distance characteristics between states in terms of fidelity

$$d = \sqrt{1 - F^2}.$$

For Apollonius qubit state $|\psi\rangle$ and the symmetric one $|\psi_s\rangle$ due to $r = \frac{|z|}{|z-1|}$, the distance is

$$d = \frac{||z-1|^2 - |z|^2|}{|z-1|^2 + |z|^2} = \frac{|1-r^2|}{1+r^2}. \tag{5.4}$$

This formula shows that distance between symmetric states depends only on Apollonius circle and is determined by its parameter r. It is invariant under substitution $r \to \frac{1}{r}$, corresponding to the pair of symmetric circles as reflections in axis $\mathrm{Re}\, z = \frac{1}{2}$. For symmetric states on the line with $r = 1$, the distance is minimal $d = 0$. For $r = 0$ and $r = \infty$, corresponding to states $|0\rangle$ and $|1\rangle$ respectively, which are orthogonal states, the distance takes maximal value $d = 1$. This value coincides with geometrical (Euclidean) distance between corresponding points 0 and 1 in complex plane. The distance (5.4) is the same for symmetric (reflected in vertical line $\mathrm{Re}\, z = \frac{1}{2}$) states on reflected Apollonius circles with values r and $\frac{1}{r}$. It is given just by Euclidean distance between two points of intersection of Apollonius circles with real line interval $[0, 1]$.

5.3.2. Apollonius Two Qubit State

Application of the $CNOT$ gate to the product of one qubit states:

$$|a\rangle \otimes |0\rangle \;\text{———}\; \boxed{\text{CNOT}} \;\text{———}\; |A\rangle$$

where $|a\rangle$ is the Apollonius one qubit, generates the Apollonius two qubit state:

$$|A\rangle = \frac{(z-1)|00\rangle + z|11\rangle}{\sqrt{|z-1|^2 + |z|^2}}. \tag{5.5}$$

In this representation state $|00\rangle$ is located at $z = 0$ and state $|11\rangle$ at $z = 1$(Figure 5.5).

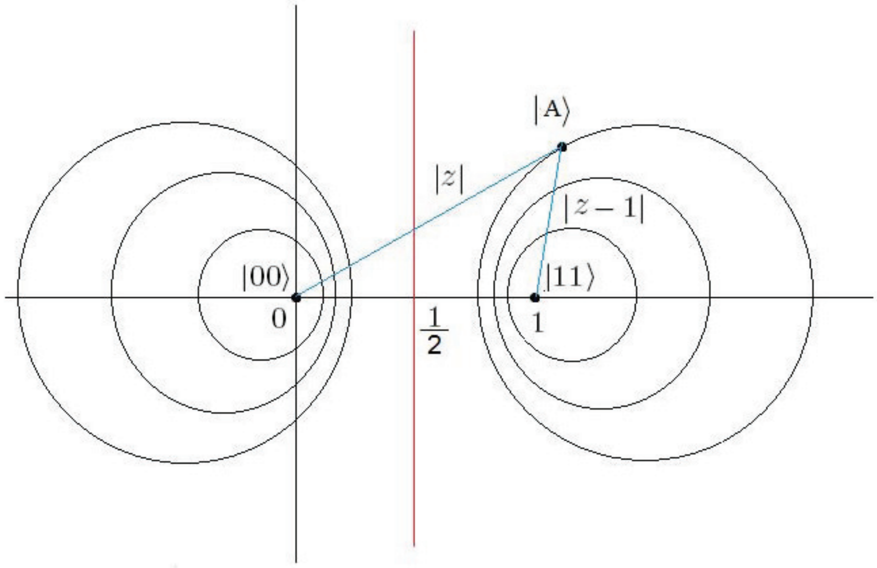


Figure 5.7. Apollonius Two Qubit State

5.3.2.1. Concurrence and Entropy for Two Qubit States

The concurrence for this state $|A\rangle$, by the determinant formula from Section 4.2 is

$$C = \frac{2|z||z-1|}{|z-1|^2 + |z|^2} = \frac{2r}{1+r^2},$$

where $r = \frac{|z|}{|z-1|}$. 3D plot of this concurrence is shown in Figure 5.6 and the contour plot in Figure 5.7. The concurrence depends on r and as follows it depends on Apollonius circle. Therefore, the concurrence and Apollonius circles are related; the concurrence is a constant along Apollonius circle for given r(Figure 5.7). The qubit states with $r = 1$ belong the line $Re(z) = \frac{1}{2}$ and are maximally entangled with $C_{max} = 1$. While states $|00\rangle$ and $|11\rangle$ with $C_{min} = 0$ are separable and correspond to common symmetric points for Apollonius circles.

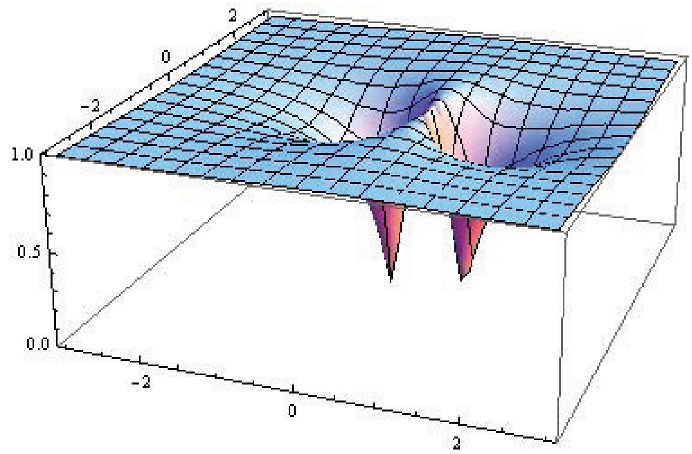


Figure 5.8. Concurence 3D

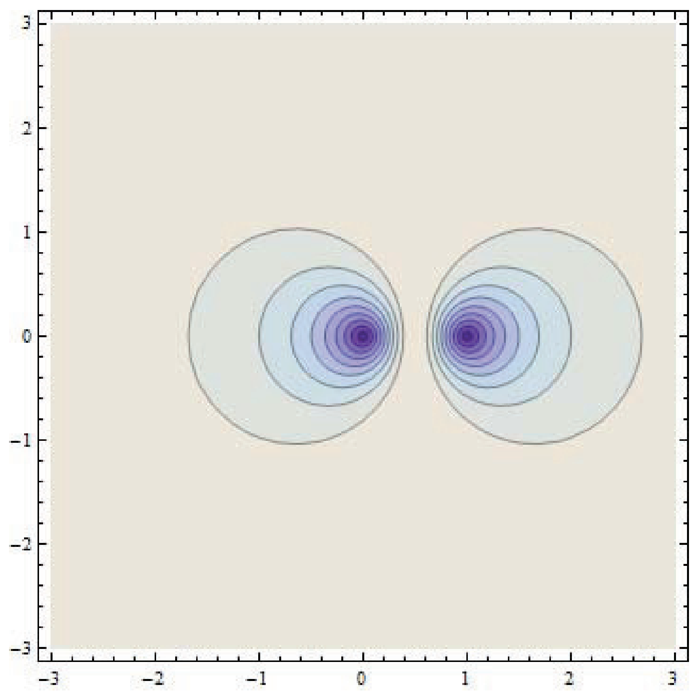


Figure 5.9. Concurence Contour Plot

By calculating the Shannon entropy (S) for two qubit Apollonius state (5.5) one gets the same expression as in (5.3), (calculations are identical to the one qubit case). Since the concurrence $C = C(r)$ is function of r only the entropy can be rewritten in the form

$S = S(C)$: S is a function of C, where

$$S(C) = 1 + \left(3 - \frac{C}{r(C)}\right) \log_2 r(C) - \log_2 C$$

and

$$r(C) = \frac{1 \pm \sqrt{1 - C^2}}{C}.$$

As it was noticed before that, both the entropy S and concurrence $C(r)$ are constant along Apollonius circles. Now, this formula shows explicitly how the level of randomness S depends on concurrence C.

5.3.2.2. Entanglement for Non- Symmetric Apollonius States

For two qubit non-symmetric Apollonius state

$$|z\rangle = \frac{(z-1)\,|00\rangle + z\,|11\rangle}{\sqrt{|z-1|^2 + |z|^2}}$$

the concurrence is

$$C = \frac{2|z|\,|z-1|}{|z-1|^2 + |z|^2}.$$

By calculating entanglement $E(C)$, (4.45):

$$1 - C^2 = 1 - \frac{4|z|^2|z-1|^2}{(|z-1|^2 + |z|^2)^2} = \frac{(|z-1|^2 - |z|^2)^2}{(|z-1|^2 + |z|^2)^2},$$

$$\sqrt{1 - C^2} = \left|\frac{|z-1|^2 - |z|^2}{|z-1|^2 + |z|^2}\right| = \frac{\left||z-1|^2 - |z|^2\right|}{|z-1|^2 + |z|^2},$$

then

$$\sqrt{1-C^2} = \begin{cases} \dfrac{|z-1|^2 - |z|^2}{|z-1|^2 + |z|^2} & \text{for} \quad |z-1|^2 > |z|^2 \\[2ex] \dfrac{|z|^2 - |z-1|^2}{|z-1|^2 + |z|^2} & \text{for} \quad |z-1|^2 < |z|^2 \end{cases}$$

It gives

$$\frac{1+\sqrt{1-C^2}}{2} = \begin{cases} \dfrac{|z-1|^2}{|z-1|^2 + |z|^2} & \text{for} \quad |z-1|^2 > |z|^2 \\[2ex] \dfrac{|z|^2}{|z-1|^2 + |z|^2} & \text{for} \quad |z-1|^2 < |z|^2 \end{cases}$$

and

$$\frac{1-\sqrt{1-C^2}}{2} = \begin{cases} \dfrac{|z|^2}{|z-1|^2 + |z|^2} & \text{for} \quad |z-1|^2 > |z|^2 \\[2ex] \dfrac{|z-1|^2}{|z-1|^2 + |z|^2} & \text{for} \quad |z-1|^2 < |z|^2 \end{cases}$$

Therefore, the entanglement $E(z)$ as a function of z is a constant along non-symmetric Apollonius circles $\left|\dfrac{z}{z-1}\right| = r$. Indeed, from

$$E(z) = \log_2(|z-1|^2 + |z|^2) - \frac{|z|^2 \log_2 |z|^2 + |z-1|^2 \log_2 |z-1|^2}{|z-1|^2 + |z|^2} \tag{5.6}$$

follows

$$E(r^2) = \log_2(1+r^2) - \frac{r^2}{1+r^2}\log_2 r^2$$

5.3.2.3. Entanglement for Symmetric Apollonius States

For two qubit symmetric Apollonius state

$$|z\rangle = \frac{(z-1)\,|00\rangle + (z+1)\,|11\rangle}{\sqrt{|z-1|^2 + |z+1|^2}}$$

the concurrence is

$$C = \frac{2|z^2-1|}{|z-1|^2 + |z+1|^2}.$$

By calculating the entanglement $E(C)$, (4.45):

$$1 - C^2 = 1 - \frac{4|z^2-1|^2}{(|z-1|^2 + |z+1|^2)^2} = \frac{(|z-1|^2 - |z+1|^2)^2}{(|z-1|^2 + |z+1|^2)^2}$$

$$\sqrt{1-C^2} = \left|\frac{|z-1|^2 - |z+1|^2}{|z-1|^2 + |z+1|^2}\right| = \frac{||z-1|^2 - |z+1|^2|}{|z-1|^2 + |z+1|^2},$$

then

$$\sqrt{1-C^2} = \begin{cases} \dfrac{|z-1|^2 - |z+1|^2}{|z-1|^2 + |z+1|^2} & \text{for} \quad |z-1|^2 > |z+1|^2 \\[2ex] \dfrac{|z+1|^2 - |z-1|^2}{|z-1|^2 + |z+1|^2} & \text{for} \quad |z-1|^2 < |z+1|^2 \end{cases}$$

It gives

$$\frac{1 + \sqrt{1-C^2}}{2} = \begin{cases} \dfrac{|z-1|^2}{|z-1|^2 + |z+1|^2} & \text{for} \quad |z-1|^2 > |z+1|^2 \\[2ex] \dfrac{|z+1|^2}{|z-1|^2 + |z+1|^2} & \text{for} \quad |z-1|^2 < |z+1|^2 \end{cases}$$

and

$$\frac{1-\sqrt{1-C^2}}{2} = \begin{cases} \dfrac{|z+1|^2}{|z-1|^2+|z+1|^2} & \text{for} \quad |z-1|^2 > |z+1|^2 \\ \\ \dfrac{|z-1|^2}{|z-1|^2+|z+1|^2} & \text{for} \quad |z-1|^2 < |z+1|^2 \end{cases}$$

As a result, entanglement $E(z)$ is a constant along symmetric Apollonius circles $\left|\frac{z+1}{z-1}\right| = r$. Indeed, from

$$E(z) = \log_2 (|z-1|^2 + |z+1|^2) - \frac{|z-1|^2 \log_2 |z-1|^2 + |z+1|^2 \log_2 |z+1|^2}{|z-1|^2 + |z+1|^2}.$$

follows

$$E(r^2) = \log_2(1+r^2) - \frac{r^2}{1+r^2} \log_2 r^2.$$

Entanglement contour plot is shown in Figure 5.10 and 3D plot in Figure 5.11.

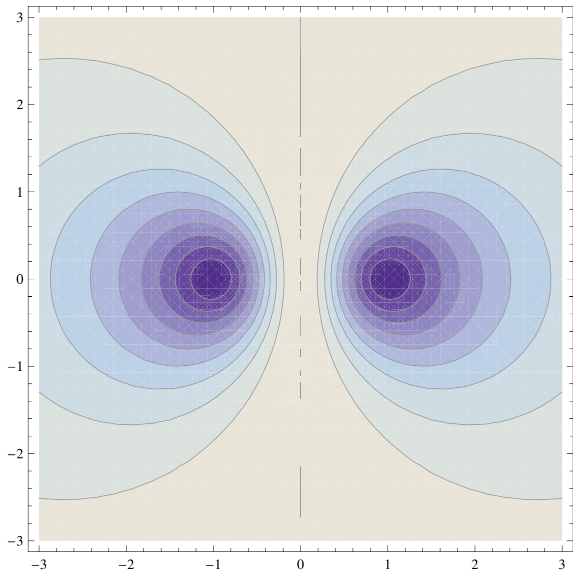


Figure 5.10. Contour Plot of Entanglement for Apollonius Symmetric States

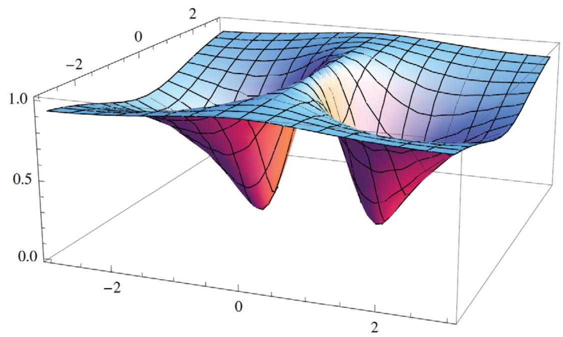


Figure 5.11. 3D Plot of Entanglement for Apollonius Symmetric States

5.3.2.4. Geometrical Meaning of Concurrence

The concurrence for two qubit Apollonius state has simple geometrical meaning. Since concurrence has the same value for arbitrary point on the given Apollonius circle, the intersection of this circle with the orthogonal circle

$$\left|z - \frac{1}{2}\right| = \frac{1}{4}$$

can be considered. The intersection points in Figure5.12 shows that the concurrence is determined as the double area of the shaded rectangle. In Figure 5.10 it is a distance between two intersection points.

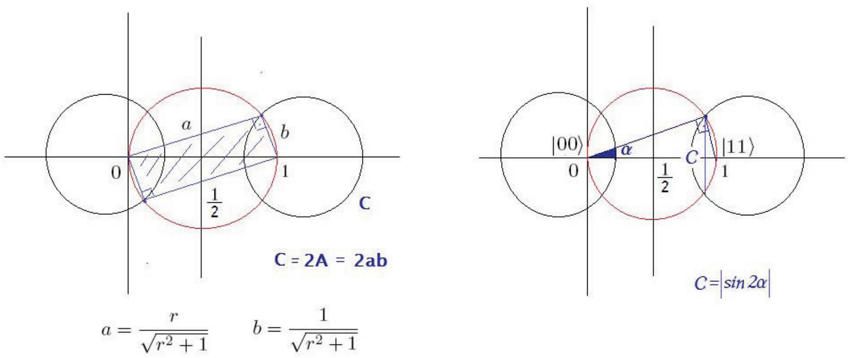


Figure 5.12. a) Concurrence as an area, b) Concurrence as a distance

5.3.2.5. Concurrence and Reflection Principle

Reflecting Apollonius two qubit state $|A\rangle$ with respect to the line $Re(z) = \frac{1}{2}$ gives the symmetric two qubit state (Figure 5.13)

$$|A_s\rangle = \frac{-\bar{z}|00\rangle + (1-\bar{z})|11\rangle}{\sqrt{|z-1|^2 + |z|^2}} \, .$$

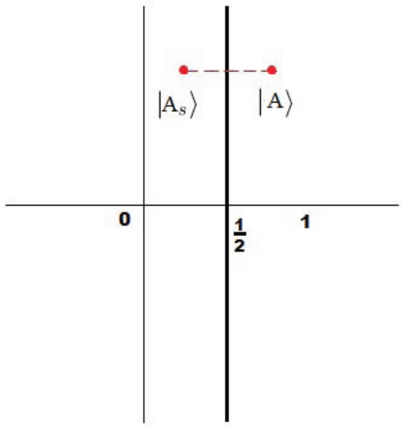


Figure 5.13. Symmetric qubit states

Fidelity between these two symmetric states coincides with the concurrence

$$F = |\langle A_s|A\rangle| = \frac{2|z||z-1|}{|z-1|^2 + |z|^2} = C,$$

and is constant for the symmetric states on reflected Apollonius circles.

5.4. Multiple Qubits in Apollonius Representation

By applying the following circuit

$$|a\rangle \otimes |0\rangle ... |0\rangle \otimes |0\rangle \,\rule{1em}{0.4pt}\, \boxed{\text{CNOT}} \otimes ... I \otimes I \,\rule{1em}{0.4pt}\, ... \,\rule{1em}{0.4pt}\, I \otimes I ... \otimes \boxed{\text{CNOT}} \,\rule{1em}{0.4pt}\, |A\rangle$$

the n-qubit Apollonius state can be generated in the form

$$|A\rangle = \frac{(z-1)|00...0\rangle + z|11...1\rangle}{\sqrt{|z-1|^2 + |z|^2}} .$$

The corresponding symmetric state is

$$|A_s\rangle = \frac{-\bar{z}|00...0\rangle + (1-\bar{z})|11...1\rangle}{\sqrt{|z-1|^2 + |z|^2}}$$

and fidelity between these states

$$F = |\langle A_s|A\rangle| = \frac{2|z||z-1|}{|z-1|^2 + |z|^2} = \frac{2r}{1+r^2}$$

is constant on Apollonius circle with fixed r.

5.5. Apollonius Representation for Generic Two Qubit State

The Apollonius states, as introduced in Section 5.3.2 are characterized by one complex parameter z. For the one qubit case it represents the generic state. However, for multiple generic qubit states, more parameters are required. Below, the Apollonius representation for the generic two qubit state

$$|\psi\rangle = c_{00}|00\rangle + c_{01}|01\rangle + c_{10}|10\rangle + c_{11}|11\rangle , \tag{5.7}$$

with normalization

$$|c_{00}|^2 + |c_{01}|^2 + |c_{10}|^2 + |c_{11}|^2 = 1\,.$$

would be derived. First of all, instead of four complex variables c_{ij}, $i, j = 0, 1$, another set of four complex variables η, ζ, a and b, according to formulas

$$c_{00} = (\eta - 1)\, a, \quad c_{11} = \eta\, a\,,$$
$$c_{01} = (\zeta - 1)\, b, \quad c_{10} = \zeta\, b\,,$$

is introduced, where complex a and b could be expressed in terms of complex α and β as:

$$a = \frac{\alpha}{\sqrt{|\eta - 1|^2 + |\eta|^2}}, \quad b = \frac{\beta}{\sqrt{|\zeta - 1|^2 + |\zeta|^2}}\,.$$

By introducing Apollonius two qubit states in the form

$$|\eta\rangle = \frac{(\eta - 1)|00\rangle + \eta|11\rangle}{\sqrt{|\eta - 1|^2 + |\eta|^2}}, \quad |\zeta\rangle = \frac{(\zeta - 1)|01\rangle + \zeta|10\rangle}{\sqrt{|\zeta - 1|^2 + |\zeta|^2}}\,,$$

the generic state (5.7) appears as superposition of these states

$$|\psi\rangle = \alpha|\eta\rangle + \beta|\zeta\rangle\,.$$

Parameters α and β can be fixed by normalization condition. Since Apollonius states $|\eta\rangle$ and $|\zeta\rangle$ are orthogonal and normalized:

$$\langle\eta|\eta\rangle = \langle\zeta|\zeta\rangle = 1, \quad \langle\eta|\zeta\rangle = \langle\zeta|\eta\rangle = 0,$$

it implies $|\alpha|^2 + |\beta|^2 = 1$. By choosing

$$\alpha = (\xi - 1)\,\lambda, \quad \beta = \xi\,\lambda\,,$$

where ξ is an arbitrary complex number, the following condition holds

$$|\lambda| = \frac{1}{\sqrt{|\xi - 1|^2 + |\xi|^2}}\,.$$

Then, by neglecting an arbitrary global phase factor, the normalized generic two qubit state in Apollonius representation can be characterized by three arbitrary complex numbers η, ζ and ξ :

$$|\psi\rangle = \frac{(\xi - 1)|\eta\rangle + \xi|\zeta\rangle}{\sqrt{|\xi - 1|^2 + |\xi|^2}}\,.$$

The concurrence of this state, calculated by the determinant formula is

$$C = \frac{2}{\sqrt{|\xi - 1|^2 + |\xi|^2}} \left| (\xi - 1)^2 \frac{\eta(\eta - 1)}{\sqrt{|\eta - 1|^2 + |\eta|^2}} - \xi^2 \frac{\zeta(\zeta - 1)}{\sqrt{|\zeta - 1|^2 + |\zeta|^2}} \right| \tag{5.8}$$

In particular cases, this states and the concurrence are reduced to the previous results

$$\xi = 0 \Rightarrow C = \frac{2|\eta||\eta - 1|}{\sqrt{|\eta - 1|^2 + |\eta|^2}}\,,$$
$$\xi = 1 \Rightarrow C = \frac{2|\zeta||\zeta - 1|}{\sqrt{|\zeta - 1|^2 + |\zeta|^2}}\,.$$

5.6. Reflected Qubits and Concurrence

The concurrence formula (5.8) can be derived from the reflection principle for Apollonius generic two qubit state as

$$C = |\langle\psi_s|\psi\rangle|\,,$$

where the symmetric qubit state $|\psi_s\rangle$ is coming from reflection of input qubits in three steps.

1) Reflection in complex plane η, in the vertical line $\mathrm{Re}\,\eta = \frac{1}{2}$ (Figure 5.14):
$\eta_s \equiv \eta^* = 1 - \bar{\eta}$

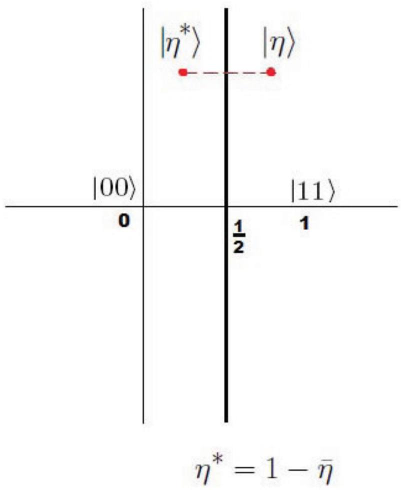


Figure 5.14. Symmetric qubits $|\eta\rangle$ and $|\eta^*\rangle$

2) Reflection in complex plane ζ in the vertical line $\mathrm{Re}\,\zeta = \frac{1}{2}$ (Figure 5.15):
$\zeta_s \equiv \zeta^* = 1 - \bar{\zeta}$

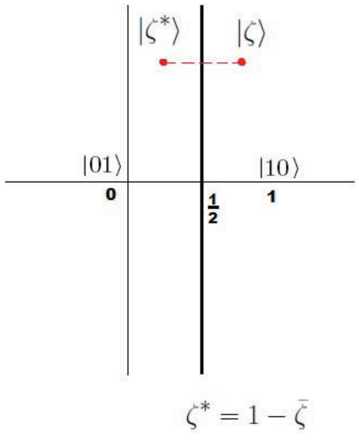


Figure 5.15. Symmetric qubits $|\zeta\rangle$ and $|\zeta^*\rangle$

3) Inversion in complex plane ξ in circle $\left|\xi - \frac{1}{2}\right| = \frac{1}{4}$ (Figure 5.16) :

$$\xi_s \equiv \xi^* = \frac{1}{2} + \frac{\frac{1}{4}}{\bar{\xi} - \frac{1}{2}}$$

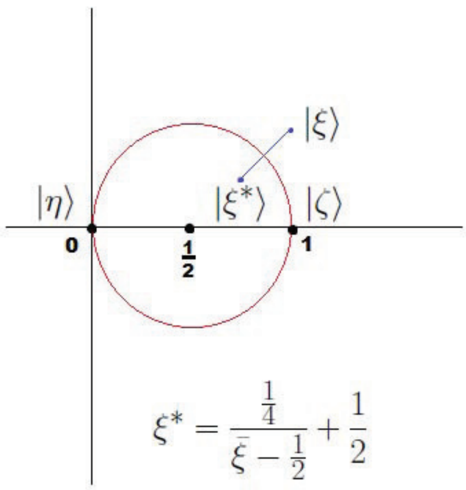


Figure 5.16. Symmetric qubits $|\xi\rangle$ and $|\xi^*\rangle$ by inversion in circle

The resulting state is

$$|\psi_s\rangle = \frac{(\xi^* - 1)|\eta^*\rangle + \xi^*|\zeta^*\rangle}{\sqrt{|\xi^* - 1|^2 + |\xi^*|^2}},$$

or up to global phase

$$|\psi_s\rangle = \frac{(\bar{\xi} - 1)|\eta^*\rangle - \bar{\xi}|\zeta^*\rangle}{\sqrt{|\xi - 1|^2 + |\xi|^2}}, \tag{5.9}$$

where symmetric qubit states are

$$|\eta^*\rangle = -\frac{\bar{\eta}|00\rangle + (\bar{\eta} - 1)|11\rangle}{\sqrt{|\eta - 1|^2 + |\eta|^2}},$$

$$|\zeta^*\rangle = -\frac{\bar{\zeta}|01\rangle + (\bar{\zeta} - 1)|10\rangle}{\sqrt{|\zeta - 1|^2 + |\zeta|^2}}.$$

Calculating the concurrence $C = |\langle\psi_s|\psi\rangle|$, the same result as by determinant formula (5.8) is obtained.

It is instructive to see how the phase flipping gate action as in Section 4.3 is related with reflection of Apollonius qubits. Applying the gate to anti-unitary transformed states

$$K|\eta\rangle = |\bar{\eta}\rangle, \quad K|\zeta\rangle = |\bar{\zeta}\rangle,$$

the reflected states appear

$$Y \otimes Y|\bar{\eta}\rangle = |\eta^*\rangle,$$

$$Y \otimes Y|\bar{\zeta}\rangle = -|\zeta^*\rangle,$$

and

$$Y \otimes Y|\bar{\psi}\rangle = \frac{(\bar{\xi}-1)Y \otimes Y|\bar{\eta}\rangle + \bar{\xi}\, Y \otimes Y|\bar{\zeta}\rangle}{\sqrt{|\xi-1|^2+|\xi|^2}} = \frac{(\bar{\xi}-1)|\eta^*\rangle - \bar{\xi}|\zeta^*\rangle}{\sqrt{|\xi-1|^2+|\xi|^2}} = |\psi_s\rangle\,.$$

CHAPTER 6

ENTANGLEMENT FOR MULTIPLE QUBIT STATES

Entanglement for two qubit system is related with bipartite expansion of two qubits on product of one qubits. For three and more qubits, several partitions are possible. For example, for a three qubit state $|\psi\rangle$, one partition is $|\psi\rangle = |a\rangle |b\rangle |c\rangle$ and the another one is $|\psi\rangle = |a\rangle |\psi_2\rangle$, where $|\psi_2\rangle$ is a two qubit state. This is why, entanglement characteristics for multiple qubits are more complicated. In this Chapter the n-tangle of n qubit state, which is a pure state, is studied.

6.1. Hyperdeterminant and 3-tangle for Three Qubit State

As it was shown in Chapter 4, the concurrence as a measure of entanglement can be derived from several geometrical and physical ideas. To generalize it to three qubit state, the determinant formula is instructive,

$$C = 2 \begin{vmatrix} c_{00} & c_{01} \\ c_{10} & c_{11} \end{vmatrix} = 2|c_{00}c_{11} - c_{01}c_{10}|.$$

Since determinant is skew-symmetric, it can be represented by absolute skew -symmetric Levi- Civita tensors. Since coefficients in qubit states are taking only two value 0 and 1, the Levi- Civita tensor is the second rank tensor, $\epsilon_{ij} = -\epsilon_{ji}$:

$$\epsilon_{ij} = \begin{cases} 1 & \text{for} \quad i, j = 0, 1 \\ \\ -1 & \text{for} \quad i, j = 1, 0 \end{cases}$$

with only non zero components $\epsilon_{01} = 1, \epsilon_{10} = -1$. Then for two complex vectors $\vec{a}_0 = (a_{00}, a_{01})$ and $\vec{a}_1 = (a_{10}, a_{11})$ (we denote $a_{ij} \equiv c_{ij},\ i, j = 0, 1$) the vector product is

$$\begin{aligned}\vec{a}_0 \times \vec{a}_1 &= \epsilon_{ij}\,(\vec{a}_0)_i\,(\vec{a}_1)_j \\ &= (\vec{a}_0)_0\,(\vec{a}_1)_1 - (\vec{a}_0)_1\,(\vec{a}_1)_0 = a_{00}a_{11} - a_{01}a_{10},\end{aligned}$$

so that

$$C = 2\,|\vec{a}_0 \times \vec{a}_1| = 2\,|\,\epsilon_{ij}\,(\vec{a}_0)_i\,(\vec{a}_1)_j\,|.$$

Since

$$a_{00}a_{11} - a_{01}a_{10} = \frac{1}{2}\,\epsilon_{ij}\,\epsilon_{kl}\,a_{ik}\,a_{jl},$$

this equation can be rewritten as

$$C = 2\,\left|\frac{1}{2}\,\epsilon_{ij}\,\epsilon_{kl}\,a_{ik}\,a_{jl}\right|, \tag{6.1}$$

(the Einstein convention for sum in repeated indices is implied). This tensor form for the concurrence is convenient way for generalization to three qubits. One notices that, the concurrence (6.1) is zero rank tensor (scalar), obtained by contraction of bilinear form of coefficients $a_{ik}\,a_{jl}$ which is the fourth rank tensor, with two second rank Levi-Civita tensors. If three qubit state is

$$|\psi\rangle = \sum_{i,j,k} a_{ijk}|ijk\rangle,$$

then bilinear form as a tensor $a_{i_1 j_1 k_1}\,a_{i_2 j_2 k_2}$ has rank 6 and can be contracted with three Levi-Civita tensors, giving identically zero,

$$\epsilon_{i_1 i_2}\,\epsilon_{j_1 j_2}\,\epsilon_{k_1 k_2}\,a_{i_1 j_1 k_1}\,a_{i_2 j_2 k_2} = 0.$$

This is why, it cannot be proper characteristic of entanglement. As a next generalization the trilinear form $a_{i_1 j_1 k_1}\, a_{i_2 j_2 k_2}\, a_{i_3 j_3 k_3}$ is considered as a tensor of rank 9, which is not even and cannot be contracted with Levi-Civita tensors. Then the next candidate is the quartic form in the coefficients, with rank 12 and it can be contracted to a scalar by 6 Levi-Civita tensors. This is the generalization of the determinant formula (6.1) introduced by A. Cayley, and known as the Cayley's hyperdeterminant, (Cayley, 1889)

$$\det\psi = -\frac{1}{2}\,\epsilon_{i_1 i_2}\,\epsilon_{j_1 j_2}\,\epsilon_{i_3 i_4}\,\epsilon_{j_3 j_4}\,\epsilon_{k_1 k_3}\,\epsilon_{k_2 k_4}\,a_{i_1 j_1 k_1}\,a_{i_2 j_2 k_2}\,a_{i_3 j_3 k_3}\,a_{i_4 j_4 k_4}.$$

As a three dimensional generalization of a two dimensional determinant, it has explicit form

$$\det\psi = d_1 - 2d_2 + 4d_3, \tag{6.2}$$

where

$$d_1 = a_{000}^2 a_{111}^2 + a_{001}^2 a_{110}^2 + a_{010}^2 a_{101}^2 + a_{100}^2 a_{011}^2,$$

$$\begin{aligned} d_2 &= a_{000}\,a_{111}\,a_{011}\,a_{100} + a_{000}\,a_{111}\,a_{101}\,a_{010} \\ &+ a_{000}\,a_{111}\,a_{110}\,a_{001} + a_{011}\,a_{100}\,a_{101}\,a_{010} \\ &+ a_{011}\,a_{100}\,a_{110}\,a_{001} + a_{101}\,a_{010}\,a_{110}\,a_{001}, \end{aligned}$$

$$d_3 = a_{000}\,a_{110}\,a_{101}\,a_{011} + a_{111}\,a_{001}\,a_{010}\,a_{100}.$$

In above formulas the following identity is used

$$\epsilon_{ij}\,\epsilon_{i'j'} = \delta_{ii'}\,\delta_{jj'} - \delta_{ij'}\,\delta_{ji'}. \tag{6.3}$$

This hyperdeterminant can be considered as characteristics of entanglement for three qubit state and it is known as the "residual entanglement " or "3-tangle". (Coffman, Kundu and Wootters, 2000)

Definition 6.1 *The* 3*-tangle of three qubit* $|ABC\rangle$ *state*

$$|\psi\rangle = \sum_{i,j,k} a_{ijk}|ijk\rangle$$

is defined as

$$\tau_{ABC} = 4\,|\det\psi|.$$

Explicitly it is

$$\tau_{ABC}(|\psi\rangle) = 2\left|\sum_{0,1} a_{\alpha_1\alpha_2\alpha_3}\, a_{\beta_1\beta_2\beta_3}\, a_{\gamma_1\gamma_2\gamma_3}\, a_{\delta_1\delta_2\delta_3}\, \epsilon_{\alpha_1\beta_1}\, \epsilon_{\alpha_2\beta_2}\, \epsilon_{\gamma_1\delta_1}\, \epsilon_{\gamma_2\delta_2}\, \epsilon_{\alpha_3\gamma_3}\, \epsilon_{\beta_3\delta_3}\right|,$$

or

$$\tau_{ABC} = 4|d_1 - 2d_2 + 4d_3|. \tag{6.4}$$

6.1.1. Determinant Decomposition of 3-tangle

In Chapter 4, the concurrence characteristics of entanglement was represented by determinant formula. Here, the determinant representation for 3-tangle is derived.

Proposition 6.1 *The* 3*-tangle formula* (6.4) *can be represented in terms of* 2×2 *determinants. These determinants correspond to areas of parallelograms constructed on vectors,* *combined from coefficients a_{ijk},*

$$\frac{\tau_{ABC}}{4}=\begin{vmatrix} a_{000} & a_{001} \\ a_{110} & a_{111} \end{vmatrix}^2+\begin{vmatrix} a_{010} & a_{011} \\ a_{100} & a_{101} \end{vmatrix}^2+2\begin{vmatrix} a_{000} & a_{001} \\ a_{010} & a_{011} \end{vmatrix}\begin{vmatrix} a_{101} & a_{100} \\ a_{111} & a_{110} \end{vmatrix}+2\begin{vmatrix} a_{000} & a_{001} \\ a_{100} & a_{101} \end{vmatrix}\begin{vmatrix} a_{011} & a_{010} \\ a_{111} & a_{110} \end{vmatrix}$$

or

$$\frac{\tau_{ABC}}{4}=\begin{vmatrix} a_{000} & a_{001} \\ a_{110} & a_{111} \end{vmatrix}^2+\begin{vmatrix} a_{010} & a_{011} \\ a_{100} & a_{101} \end{vmatrix}^2-2\begin{vmatrix} a_{000} & a_{001} \\ a_{010} & a_{011} \end{vmatrix}\begin{vmatrix} a_{101} & a_{100} \\ a_{111} & a_{110} \end{vmatrix}-2\begin{vmatrix} a_{000} & a_{001} \\ a_{100} & a_{101} \end{vmatrix}\begin{vmatrix} a_{011} & a_{010} \\ a_{111} & a_{110} \end{vmatrix}$$

Proof The 3-tangle formula is

$$\tau_{ABC}=4|d_1-2d_2+4d_3|,$$

where

$$d_1 = a_{000}^2a_{111}^2+a_{001}^2a_{110}^2+a_{010}^2a_{101}^2+a_{100}^2a_{011}^2,$$

$$\begin{aligned} d_2 &= a_{000}\,a_{111}\,a_{011}\,a_{100}+a_{000}\,a_{111}\,a_{101}\,a_{010} \\ &+ a_{000}\,a_{111}\,a_{110}\,a_{001}+a_{011}\,a_{100}\,a_{101}\,a_{010} \\ &+ a_{011}\,a_{100}\,a_{110}\,a_{001}+a_{101}\,a_{010}\,a_{110}\,a_{001}, \end{aligned}$$

$$d_3=a_{000}\,a_{110}\,a_{101}\,a_{011}+a_{111}\,a_{001}\,a_{010}\,a_{100}.$$

Substitution and regrouping the terms gives

$$\begin{aligned}
\frac{\tau_{ABC}}{4} &= (a_{000}\,a_{111} - a_{001}\,a_{110})^2 + (a_{010}\,a_{101} - a_{100}\,a_{011})^2 + 2a_{000}\,a_{011}\,(a_{110}\,a_{101} - a_{100}\,a_{111}) \\
&+ 2a_{111}\,a_{010}\,(a_{001}\,a_{100} - a_{000}\,a_{101}) + 2a_{110}\,a_{011}\,(a_{000}\,a_{101} - a_{100}\,a_{001}) \\
&+ 2a_{001}\,a_{010}\,(a_{111}\,a_{100} - a_{101}\,a_{110}) \\
&= \begin{vmatrix} a_{000} & a_{001} \\ a_{110} & a_{111} \end{vmatrix}^2 + \begin{vmatrix} a_{010} & a_{011} \\ a_{100} & a_{101} \end{vmatrix}^2 + 2a_{000}\,a_{011} \begin{vmatrix} a_{101} & a_{100} \\ a_{111} & a_{110} \end{vmatrix} + 2a_{111}\,a_{010} \begin{vmatrix} a_{001} & a_{000} \\ a_{101} & a_{100} \end{vmatrix} \\
&+ 2a_{110}\,a_{011} \begin{vmatrix} a_{000} & a_{001} \\ a_{100} & a_{101} \end{vmatrix} + 2a_{001}\,a_{010} \begin{vmatrix} a_{100} & a_{110} \\ a_{101} & a_{111} \end{vmatrix}.
\end{aligned}$$

By changing sign in the 4^{th} and 6^{th} terms

$$\begin{aligned}
\frac{\tau_{ABC}}{4} &= \begin{vmatrix} a_{000} & a_{001} \\ a_{110} & a_{111} \end{vmatrix}^2 + \begin{vmatrix} a_{010} & a_{011} \\ a_{100} & a_{101} \end{vmatrix}^2 + 2a_{000}\,a_{011} \begin{vmatrix} a_{101} & a_{100} \\ a_{111} & a_{110} \end{vmatrix} - 2a_{111}\,a_{010} \begin{vmatrix} a_{000} & a_{001} \\ a_{100} & a_{101} \end{vmatrix} \\
&+ 2a_{110}\,a_{011} \begin{vmatrix} a_{000} & a_{001} \\ a_{100} & a_{101} \end{vmatrix} - 2a_{001}\,a_{010} \begin{vmatrix} a_{101} & a_{100} \\ a_{111} & a_{110} \end{vmatrix} \\
&= \begin{vmatrix} a_{000} & a_{001} \\ a_{110} & a_{111} \end{vmatrix}^2 + \begin{vmatrix} a_{010} & a_{011} \\ a_{100} & a_{101} \end{vmatrix}^2 + 2(a_{000}\,a_{011} - a_{001}\,a_{010}) \begin{vmatrix} a_{101} & a_{100} \\ a_{111} & a_{110} \end{vmatrix} \\
&+ 2(a_{110}\,a_{011} - a_{010}\,a_{111}) \begin{vmatrix} a_{000} & a_{001} \\ a_{100} & a_{101} \end{vmatrix} \\
&= \begin{vmatrix} a_{000} & a_{001} \\ a_{110} & a_{111} \end{vmatrix}^2 + \begin{vmatrix} a_{010} & a_{011} \\ a_{100} & a_{101} \end{vmatrix}^2 + 2 \begin{vmatrix} a_{000} & a_{001} \\ a_{010} & a_{011} \end{vmatrix} \begin{vmatrix} a_{101} & a_{100} \\ a_{111} & a_{110} \end{vmatrix} \\
&+ 2 \begin{vmatrix} a_{011} & a_{010} \\ a_{111} & a_{110} \end{vmatrix} \begin{vmatrix} a_{000} & a_{001} \\ a_{100} & a_{101} \end{vmatrix}
\end{aligned}$$

then finally equation is found in the form

$$\frac{\tau_{ABC}}{4} = \begin{vmatrix} a_{000} & a_{001} \\ a_{110} & a_{111} \end{vmatrix}^2 + \begin{vmatrix} a_{010} & a_{011} \\ a_{100} & a_{101} \end{vmatrix}^2 + 2\begin{vmatrix} a_{000} & a_{001} \\ a_{010} & a_{011} \end{vmatrix}\begin{vmatrix} a_{101} & a_{100} \\ a_{111} & a_{110} \end{vmatrix} + 2\begin{vmatrix} a_{000} & a_{001} \\ a_{100} & a_{101} \end{vmatrix}\begin{vmatrix} a_{011} & a_{010} \\ a_{111} & a_{110} \end{vmatrix}$$

□

It is known that, a 2×2 determinant can be interpreted as an area of parallelogram, determined by two vectors with real components. In a similar way, 2×2 determinant for complex vectors, appears as complex area in $\mathbb{C}^2$. Due to this, possible to represent the Cayley hyperdeterminant and corresponding 3-tangle in terms of areas and the vector product of complex (real) vectors.

Proposition 6.2 *Let*

$$\vec{r}_{00} = (a_{000}, a_{001}), \ \ \vec{r}_{01} = (a_{010}, a_{011}), \ \ \vec{r}_{10} = (a_{100}, a_{101}), \ \ \vec{r}_{11} = (a_{110}, a_{111})$$

are real vectors in $\mathbb{R}^2$ *satisfying constraint*

$$\vec{r}^{\,2}_{00} + \vec{r}^{\,2}_{01} + \vec{r}^{\,2}_{10} + \vec{r}^{\,2}_{11} = 1.$$

Then, Cayley's hyperdeterminant is

$$\det\psi = (\vec{r}_{00} \times \vec{r}_{11})^2 + (\vec{r}_{01} \times \vec{r}_{10})^2 - 2(\vec{r}_{00} \times \vec{r}_{01})(\vec{r}_{10} \times \vec{r}_{11}) - 2(\vec{r}_{00} \times \vec{r}_{10})(\vec{r}_{01} \times \vec{r}_{11})$$

and 3*-tangle for three qubit (rebit) state is*

$$\tau = 4|\det\psi|.$$

Proof It is evident by identification of determinants with signed areas and with the vector products in the form

$$\vec{r}_{ij} \times \vec{r}_{kl} = \begin{vmatrix} a_{ij0} & a_{ij1} \\ a_{kl0} & a_{kl1} \end{vmatrix},$$

where $i, j, k, l = 0, 1$. $\square$

The vectors and areas are shown in figure 6.1.

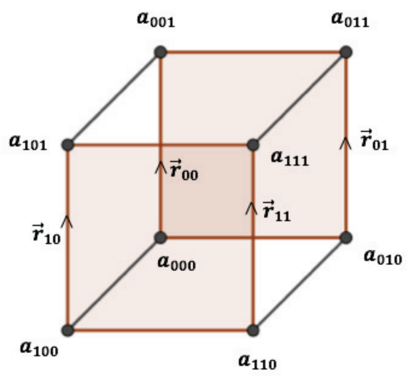


Figure 6.1. Area representation of vectors

6.1.2. Apollonius and Coherent Like Three Qubit States

Here, the 3-tangle of several three qubit states is calculated.

1. **Coherent Like Three Qubit States** For three qubit state

$$|\psi\rangle = \frac{|000\rangle + z|111\rangle}{\sqrt{1 + |z|^2}},$$

the 3-tangle (6.4) is

$$\tau = 4|a_{000}^2 a_{111}^2| = 4\frac{|z|^2}{(1 + |z|^2)^2}.$$

Fidelity between symmetric states then is

$$F = |\langle\tilde{\psi}|\psi\rangle| = 2\frac{|z|}{1 + |z|^2},$$

where

$$|\tilde{\psi}\rangle = \frac{\bar{z}|000\rangle + |111\rangle}{\sqrt{1+|z|^2}}.$$

This is why, the 3- tangle is

$$\tau = F^2,$$

and 3-tangle for such states is a constant along concentric circles $|z| = r$. It reaches maximum value for circle $|z| = 1$. (See 3D plot of 3-tangle in Figure 6.2 and the contour plot in Figure 6.3.)

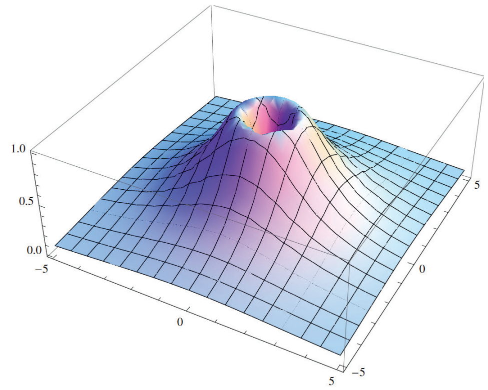


Figure 6.2. 3-tangle Coherent State 3D

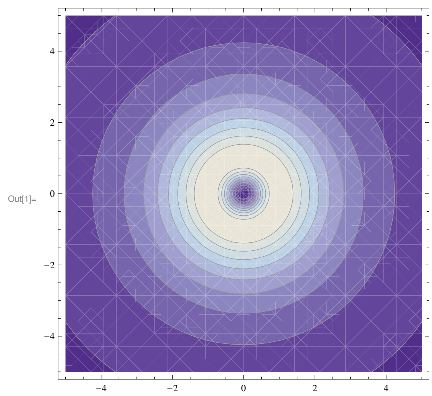


Figure 6.3. 3-tangle Coherent State Contour Plot

2. **Maximally Tritangled States**

   The states have $\tau = 1$ and are maximally tritangled states

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle) \to a_{000} = a_{111} = \frac{1}{\sqrt{2}},$$

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|101\rangle + |010\rangle) \to a_{101} = a_{010} = \frac{1}{\sqrt{2}},$$

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|001\rangle + |110\rangle) \to a_{001} = a_{110} = \frac{1}{\sqrt{2}},$$

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|011\rangle + |100\rangle) \to a_{011} = a_{100} = \frac{1}{\sqrt{2}}.$$

3. **Apollonius Three Qubit States** For state

$$|\psi\rangle = \frac{(z-1)|000\rangle + (z+1)|111\rangle}{\sqrt{|z-1|^2 + |z+1|^2}},$$

   the 3-tangle (6.4) is

$$\tau = 4|a_{000}^2\, a_{111}^2| = 4\frac{|z^2-1|^2}{(|z-1|^2 + |z+1|^2)^2},$$

   and it is a constant along Apollonius circles $\frac{|z+1|}{|z-1|} = r$, and reaches maximal value $\tau = 1$ for vertical line $Re\, z = 0$, and minimal value $\tau = 0$ for $z = \pm 1$.

6.2. n - Tangle of n - Qubit State

In previous Section, the 3-tangle for three qubit state or the "residual entanglement" (Coffman, Kundu and Wootters, 2000) was determined by the Cayley hyperdeterminant formula,

$$\tau_{ABC}(|\psi\rangle) = 2\left|\sum_{0,1} a_{\alpha_1\alpha_2\alpha_3}\, a_{\beta_1\beta_2\beta_3}\, a_{\gamma_1\gamma_2\gamma_3}\, a_{\delta_1\delta_2\delta_3}\, \epsilon_{\alpha_1\beta_1}\, \epsilon_{\alpha_2\beta_2}\, \epsilon_{\gamma_1\delta_1}\, \epsilon_{\gamma_2\delta_2}\, \epsilon_{\alpha_3\gamma_3}\, \epsilon_{\beta_3\delta_3}\right|,$$

where a_{i_1,i_2,i_3} are coefficients of pure three qubit state.

$$|\psi\rangle = \sum_{i_1,i_2,i_3} a_{i_1i_2i_3}|i_1i_2i_3\rangle, \tag{6.5}$$

and $\epsilon_{01} = -\epsilon_{10} = 1$, $\epsilon_{00} = -\epsilon_{11} = 0$.

This formula can be generalized to multiple qubit states with even number $n = 2k$ of qubits. (Wong and Christensen, 2001)

Definition 6.2 *For even n-qubit state*

$$|\psi\rangle = \sum_{i_1i_2...i_n} a_{i_1i_2...i_n}|i_1i_2...i_n\rangle$$

the n-tangle is defined as

$$\tau_{12...n} = 2\left|\sum_{0,1} a_{\alpha_1\alpha_2...\alpha_n}\, a_{\beta_1\beta_2...\beta_n}\, a_{\gamma_1\gamma_2...\gamma_n}\, a_{\delta_1\delta_2...\delta_n}\, \epsilon_{\alpha_1\beta_1}\, \epsilon_{\alpha_2\beta_2} ...\epsilon_{\alpha_n\beta_n}\, \epsilon_{\gamma_1\delta_1}\, \epsilon_{\gamma_2\delta_2} ...\epsilon_{\gamma_{n-1}\delta_{n-1}}\, \epsilon_{\alpha_n\gamma_n}\, \epsilon_{\beta_n\delta_n}\right| \tag{6.6}$$

It was shown that it is invariant under permutations of the qubits (Wong and Christensen, 2001) . In addition, it can be related with fidelity between symmetric states. In Section 4.3, the pure state concurrence for two qubit states was defined as fidelity

$$C(\psi) = |\langle\tilde{\psi}|\psi\rangle|.$$

This formula was generalized to arbitrary even n, (Wong and Christensen, 2001) as

$$C_{12...n}(\psi) = |\langle\tilde{\psi}|\psi\rangle|,$$

where

$$|\tilde{\psi}\rangle = \underbrace{(Y \otimes Y \otimes ... \otimes Y)}_{n \text{ times}} |\bar{\psi}\rangle.$$

For two qubit states, the 2-tangle (6.6) with $n = 2$ is square of the concurrence

$$\tau_{12} = C^2.$$

Generalization of this formula for arbitrary even n is (Wong and Christensen, 2001)

$$\tau_{12...n} = C^2_{12...n}.$$

6.2.1. Apollonius and Coherent Like $n = 2k$ Qubit States

Here, the n-tangle for specific even $n = 2k$ qubit states is calculated.

1. **Coherent Like States**

   For n- qubit coherent state in the form

$$|z\rangle = \frac{|00...0\rangle + z|11...1\rangle}{\sqrt{1 + |z|^2}}, \tag{6.7}$$

   with even number $n = 2k$, the above formula (6.6) gives

$$\tau_{12...n} = 2\,|(-1)^n\, 2\, a^2_{00...0}\, a^2_{11...1}| = 2\left|(-1)^n \frac{2z^2}{(1 + |z|^2)^2}\right|$$

or

$$\tau_{12...n} = \frac{4|z|^2}{(1+|z|^2)^2}.$$

By using the symmetric state

$$\begin{aligned} |\tilde{z}\rangle = \underbrace{(Y \otimes Y \otimes ... \otimes Y)}_{n \text{ times}} |\bar{z}\rangle &= \frac{i^{2k}\,|11...1\rangle + \bar{z}\,(-i)^{2k}\,|00...0\rangle}{\sqrt{1+|z|^2}} \\ &= (-1)^k\,\frac{\bar{z}|00...0\rangle + |11...1\rangle}{\sqrt{1+|z|^2}} \\ &= |\tilde{z}\rangle, \end{aligned}$$

which corresponds to the symmetric point $z^* = \dfrac{1}{\bar{z}}$ of z in the unit circle, it is easy to see that

$$\tau_{12...n} = C^2_{12...n},$$

where

$$C_{12...n} = |\langle\tilde{z}|z\rangle| = \frac{2|z|}{1+|z|^2}.$$

This shows that, n-tangle of state (6.7) is constant $\tau_{12...n} = constant$ along concentric circles $|z| = r$ and it is maximal $\tau_{12...n} = 1$ on the unit circle $|z| = 1$.

2. **Apollonius States**

For n-qubit Apollonius state

$$|z\rangle = \frac{(z-1)\,|00...0\rangle + (z+1)\,|11...1\rangle}{\sqrt{|z-1|^2 + |z+1|^2}}, \tag{6.8}$$

with even $n = 2k$, the n-tangle is

$$\tau_{12...n} = 2|(-1)^n\, 2\, a_{00...0}^2\, a_{11...1}^2| = 2\left|(-1)^n\, 2\, \frac{(z-1)^2(z+1)^2}{(|z-1|^2+|z+1|^2)^2}\right|$$

or

$$\tau_{12...n} = \frac{4\,|z^2-1|^2}{(|z-1|^2+|z+1|^2)^2}.$$

It is vanishing for $z = \pm 1$ states and takes maximal value on the line $Re\, z = 0$. Along every Apollonius circle $\dfrac{|z+1|}{|z-1|} = r$ the n- tangle is a constant. (See for contour plot of n-tangle Figure 6.4 and for 3D plot of n-tangle Figure 6.5) It means that Apollonius circles are "iso-tangle" curves for specific (6.8) even n-qubit states.

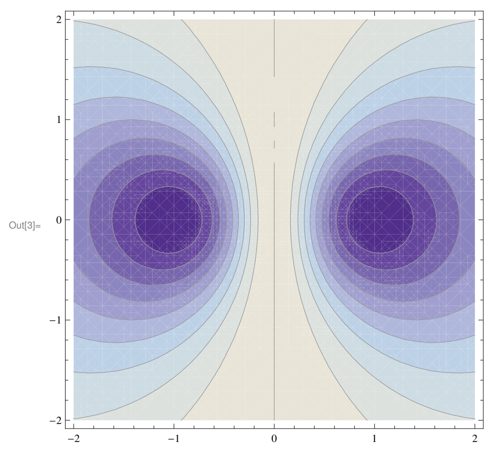


Figure 6.4. n-tangle Apollonius Symmetric Contour Plot

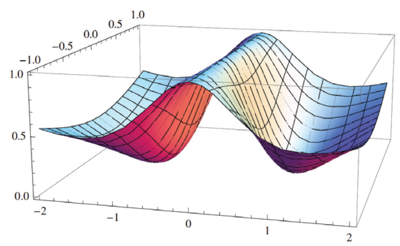


Figure 6.5. n-tangle Apollonius Symmetric 3D

CHAPTER 7

CASSINI QUBIT STATES

In Chapter 5 Apollonius representation of qubit states was introduced and classification of the sates with constant entropy along Apollonius circles was described. It turns out that Apollonius circles can be related with Cassini curves. This is why, it is natural to find representation of qubits on Cassini curves.

7.1. Cassini Curves

In present section Cassini curve in cartesian and in polar form are derived.

Definition 7.1 *(Sivardiere , 1994) A Cassini curve (oval) is a quartic plane curve defined as the set (or locus) of points in the plane, such that the product of the distances to two fixed points is constant. For any point* $P(x, y)$ *in coordinate plane at* $|PF_1|$ *and* $|PF_2|$ *distances to two fixed points* $F_1(-c, 0)$ *,* $F_2(c, 0)$ *the curve is defined as*

$$|PF_1||PF_2| = a^2,$$

where a is a constant (Figure 7.1).

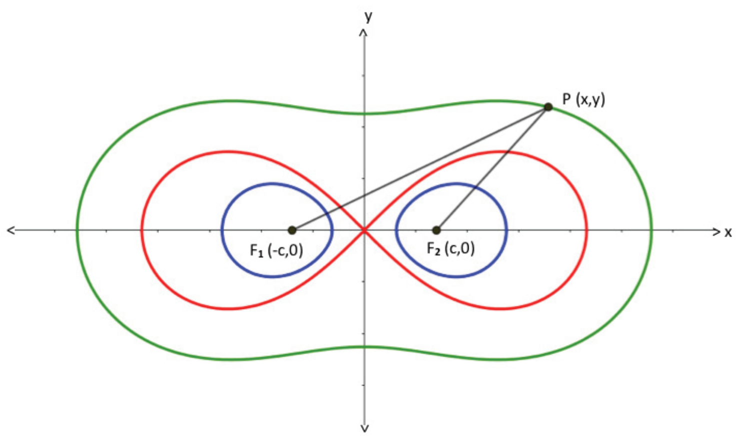


Figure 7.1. Cassini Ovals

7.1.1. Cartesian Form of Cassini Curve

By substituting the distance formulas

$$|PF_1| = \sqrt{(x-c)^2 + (y-0)^2}, \qquad |PF_2| = \sqrt{(x+c)^2 + (y-0)^2}$$

into the product formula, the cartesian form of Cassini curve can be derived

$$|PF_1||PF_2| = \sqrt{(x-c)^2 + (y-0)^2}\,\sqrt{(x+c)^2 + (y-0)^2} = a^2.$$

From this equation, the 1^{st} and 2^{nd} forms of Cassini curve are

$$1)\ (x^2 + y^2 + c^2)^2 - 4x^2c^2 = a^4,$$

$$2)\ (x^2 + y^2)^2 + 2c^2(y^2 - x^2) = a^4 - c^4. \tag{7.1}$$

In this thesis only the 2^{nd} form of Cassini equation (7.1) will be used. Depending on values of a and c, three different cases appear:

- **a = c :** equation (7.1) is represented as

$$(x^2 + y^2)^2 + 2c^2(y^2 - x^2) = 0.$$

By solving this equation the following, roots can be found

$$(0, 0), (\sqrt{2}c, 0), (-\sqrt{2}c, 0).$$

The corresponding curve is represented in Figure 7.2 and is called *"The Bernoulli Leminiscate"*.

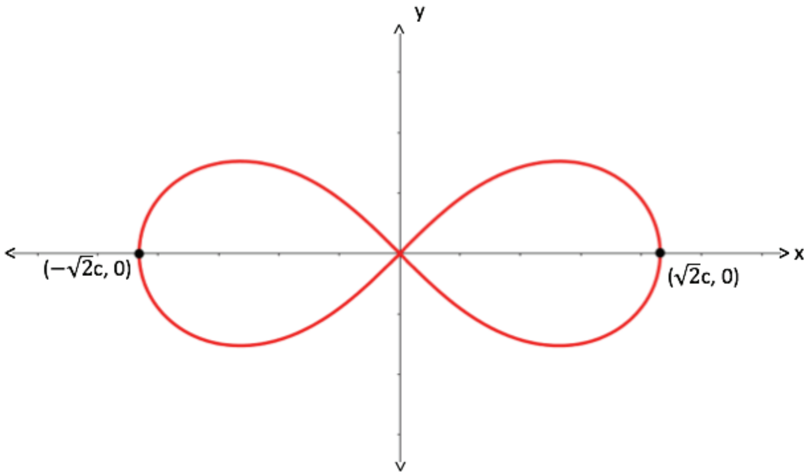


Figure 7.2. Bernoulli Leminiscate

- **a < c :** equation (7.1) has no solution for $x = 0$, but solved for $y = 0$ it has four roots
$$(\sqrt{c^2 + a^2}, 0), (-\sqrt{c^2 + a^2}, 0), (\sqrt{c^2 - a^2}, 0), (-\sqrt{c^2 - a^2}, 0).$$
This curve is represented by two closed ovals, symmetrical with respect to x and y axis and it is shown in Figure 7.3

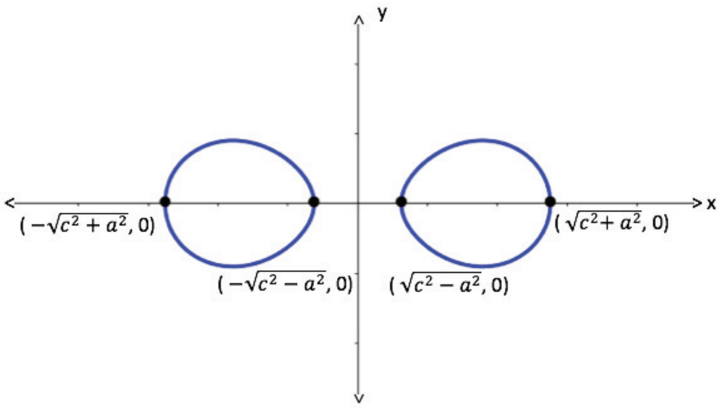


Figure 7.3. Cassini Oval $a < c$

- **a > c :** by taking $x = 0$ two roots appear in the form

$$(0, \sqrt{a^2 - c^2}), (0, -\sqrt{a^2 - c^2})$$

and by choosing $y = 0$, another pair of roots can be found

$$(\sqrt{a^2 + c^2}, 0), (-\sqrt{a^2 + c^2}, 0).$$

Then, the corresponding curve represents closed oval and it is shown in Figure 7.4.

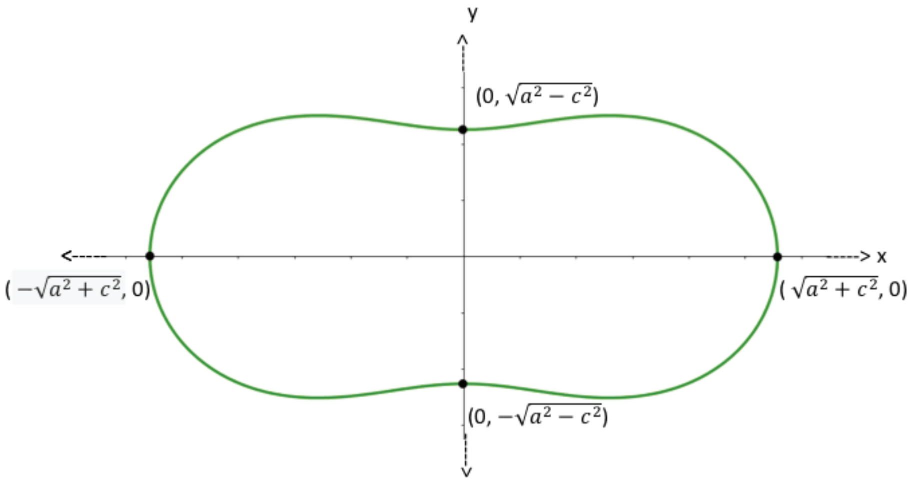


Figure 7.4. Cassini Oval $a > c$

7.1.2. Polar Form of Cassini Curve

By using polar coordinates $x = r\cos\theta$, $y = r\sin\theta$, equation (7.1) can be represented in polar form

$$r^4 - 2c^2r^2\cos 2\theta - a^4 + c^4 = 0. \tag{7.2}$$

Solving this equation, $r(\theta)$ can be find as

$$r(\theta) = \sqrt{c^2 \cos 2\theta \mp \sqrt{a^4 - c^4 + c^4 \cos^2 2\theta}}.$$

Depending on relation between a and c it reduces to the following cases:

- **a = c :** Bernoulli Leminisacate: $r(\theta) = \sqrt{c^2 \cos 2\theta}$
- **a <c :** Two closed ovals: $r(\theta) = \sqrt{c^2 \cos 2\theta \mp \sqrt{a^4 - c^4 + c^4 \cos^2 2\theta}}$
- **a > c :** One closed oval: $r(\theta) = \sqrt{c^2 \cos 2\theta + \sqrt{a^4 - c^4 + c^4 \cos^2 2\theta}}$

7.2. From Cassini Curves to Apollonius Circles

Comparing definition of Cassini curve with definition of Apollonius circles, one can notice complimentary character of their definitions. In the first case, the curve is defined by constant product of distances

$$|PF_1||PF_2| = a^2,$$

while in the second case by ratio of the distances

$$\frac{|PF_1|}{|PF_2|} = a^2$$

from two fixed points.

The natural question appears, if these two curves can be related with each others? Despite that the Apollonius circle curve is quadratic and Cassini curve is quartic, exists transformation between these two curves. This transformation is combination of conformal transformations. To describe it, the Cassini curve can be rewritten in complex form as

$$|z - c||z + c| = |z^2 - c^2| = a^2, \tag{7.3}$$

where $z = x + iy$.

7.2.1. From Cassini Curves to Concentric Circles

Via conformal map $w = z^2$, where $w = u + iv$, the Cassini equation becomes equation of the circle

$$|w - c^2| = a^2,$$

or

$$|w|^2 - c^2(w + \bar{w}) + c^4 = a^4,$$

and

$$(u - c^2)^2 + v^2 = (a^2)^2.$$

This represents the circle in w plane with center $C(c^2, 0)$ and radius $r = a^2$(Figure 7.5). For fixed c and different a, the set of concentric circles around point C occur.

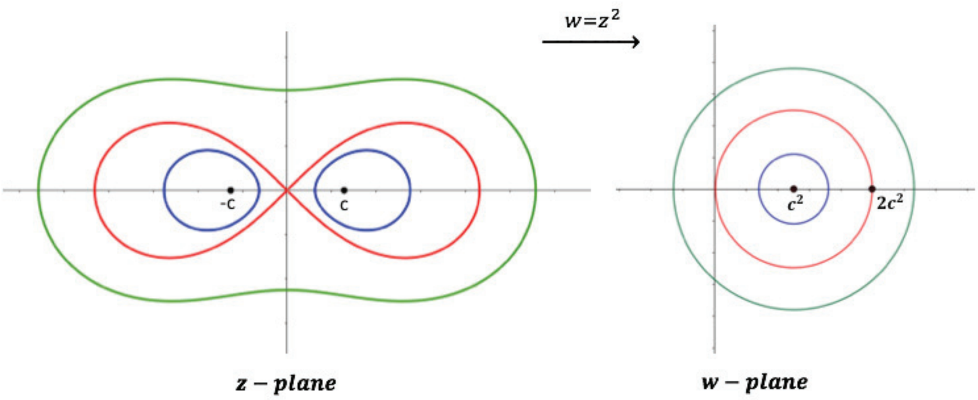


Figure 7.5. Cassini to Concentric circles

7.2.2. Translating Concentric Circles to the Origin

Translating the origin $\xi = w - c^2$, the equation becomes $|w - c^2| = |\xi| = a^2$ and it represents concentric circles in ξ plane with center $C(0, 0)$ and radius $r = a^2$ (Figure 7.6).

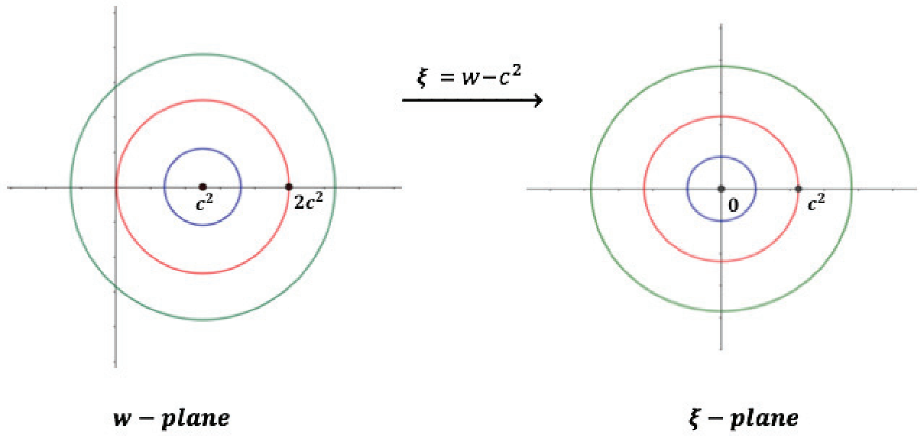


Figure 7.6. Translation of Concentric Circles

7.2.3. Möbius Mapping of Concentric Circles to Apollonius Circles

In ξ plane, 0 and ∞ are symmetric points with respect to concentric circles around origin. These symmetric points determine the Möbius transformation in the form

$$\eta = -c\frac{\xi + c^2}{\xi - c^2}. \tag{7.4}$$

Proposition 7.1 *Transformation* (7.4) *maps concentric circles in* ξ *plane to Apollonius circles in* η *plane.*

Proof Equation (7.4) rewritten in the form

$$\xi = c^2\frac{\eta - c}{\eta + c},$$

implies

$$|\xi| = c^2\frac{|\eta - c|}{|\eta + c|}.$$

For the circle $|\xi| = a^2$, the equation can be written as

$$\frac{|\eta - c|}{|\eta + c|} = \frac{a^2}{c^2}.$$

This is equation of Apollonius circles. Depending on value of constant $\frac{a^2}{c^2}$ three different cases occur (Figure 7.7).

- **a = c :** Equation $|\eta - c| = |\eta + c|$ shows that the circle with center $C(0,0)$ and radius $r = c^2$ in ξ plane is mapped to imaginary axis $Re(\eta) = 0$ in η plane.
- **a <c :** Equation $|\eta - c| > |\eta + c|$ shows that the circle with center $C(0,0)$ and radius $a^2 < c^2$ in ξ plane is mapped to the circle in the right half of η plane.
- **a > c :** Equation $|\eta - c| < |\eta + c|$ shows that the circle with center $C(0,0)$ and radius $a^2 > c^2$ in ξ plane is mapped to the circle in the left half of η plane.

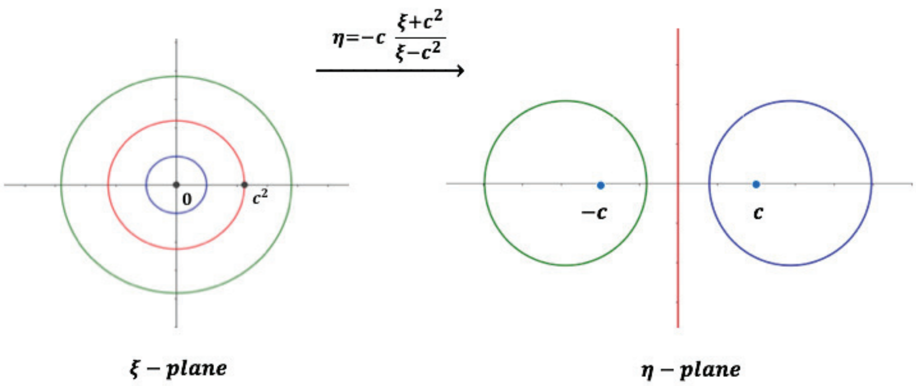


Figure 7.7. Concentric Circles to Apollonius

□

Combining these conformal transformations together

z- plane $\xrightarrow{w = z^2}$ w-plane

w- plane $\xrightarrow{\xi = w - c^2}$ ξ-plane

$$\xi\text{- plane} \xrightarrow{\eta = -c\frac{\xi + c^2}{\xi - c^2}} \eta\text{-plane}$$

the following proposition holds.

Proposition 7.2 *Conformal transformation*

$$\eta = -c\frac{z^2}{z^2 - 2c^2} \tag{7.5}$$

maps Cassini curves in z plane into Apollonius circles in η plane (Figure 7.8*).*

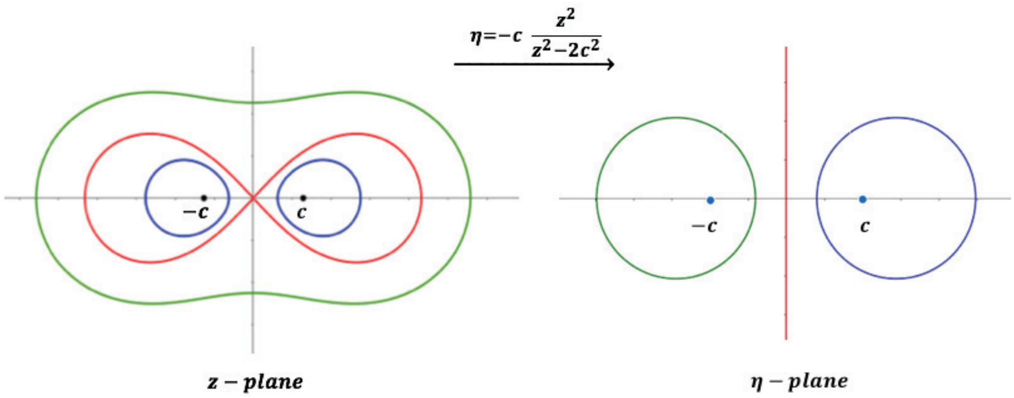


Figure 7.8. Cassini Ovals to Apollonius Circles

7.3. Cassini Representation of One Qubit State

The relation between Cassini curves and Apollonius circles implies representation of one qubit state as Cassini qubit state. In chapter 5, Apollonius one qubit state was represented by the set of Apollonius circles with respect to symmetric points -1 and 1. For points $-c$ and c the state is

$$|\eta\rangle = \frac{(\eta - c)|0\rangle + (\eta + c)|1\rangle}{\sqrt{|\eta - c|^2 + |\eta + c|^2}} \tag{7.6}$$

It is evident that, in η-plane the point $\eta = -c$ represents the state $|0\rangle$, and the point $\eta = c$ represents the state $|1\rangle$. Probabilities to measure these states are

$$p_0 = \frac{|\eta - c|^2}{|\eta - c|^2 + |\eta + c|^2}, \qquad p_1 = \frac{|\eta + c|^2}{|\eta - c|^2 + |\eta + c|^2}.$$

Due to (7.5),

$$\eta - c = -2c\frac{z^2 - c^2}{z^2 - 2c^2}, \qquad \eta + c = -2c\frac{c^2}{z^2 - 2c^2}, \tag{7.7}$$

and in terms of z, one can define Cassini representation of one qubit state,

$$|z\rangle = \frac{(z^2 - c^2)|0\rangle + c^2|1\rangle}{\sqrt{|z^2 - c^2|^2 + c^4}}. \tag{7.8}$$

In this representation, every value of complex number z determines one qubit state, so that $|0\rangle$ state corresponds to $z = \infty$, and $|1\rangle$ state corresponds to points $z = \pm c$. This implies that points $z = -c$ and $z = c$ should be identified. As it is clear from (7.8), the Cassini state is invariant under replacement z to $-z$, which means rotation to angle π around the origin. This implies that Cassini states are uniquely determined by points in the right half plane z: $-\frac{\pi}{2} \leq \arg(z) \leq \frac{\pi}{2}$.

Probabilities to measure $|0\rangle$ and $|1\rangle$ state are

$$p_0 = \frac{|z^2 - c^2|^2}{|z^2 - c^2|^2 + c^4} \qquad p_1 = \frac{c^4}{|z^2 - c^2|^2 + c^4}.$$

and the ratio of these probabilities

$$\frac{p_0}{p_1} = \frac{|z^2 - c^2|^2}{c^4}$$

is constant along Cassini curves:

$$|z^2 - c^2|^2 = a^4, \tag{7.9}$$

where

$$a^4 = c^4 \frac{p_0}{p_1} \Longrightarrow \frac{p_0}{p_1} = \frac{a^4}{c^4}.$$

Depending on ratio of probabilities, the qubit states are classified according following Cassini curves:

- **a = c :** Bernoulli Leminiscate: $\frac{p_0}{p_1} = 1 \rightarrow p_0 = p_1$
- **a < c :** Two closed ovals: $\frac{p_0}{p_1} < 1 \rightarrow p_0 < p_1$
- **a > c :** One closed oval: $\frac{p_0}{p_1} > 1 \rightarrow p_0 > p_1$

It should be noted that, due to uniqueness of Cassini states mentioned above, only half of these curves in the right half plane should be taken into account.

7.3.1. Shannon Entropy For One Qubit Cassini State

Along Cassini curve (7.9), probabilities for one qubit state (7.8) are constant and can be represented as;

$$p_0 = \frac{a^4}{a^4 + c^4} = \frac{\frac{a^4}{a^4}}{1 + \frac{c^4}{a^4}} = \frac{1}{1 + r^2},$$

and

$$p_1 = \frac{c^4}{a^4 + c^4} = \frac{\frac{c^4}{a^4}}{1 + \frac{c^4}{a^4}} = \frac{r^2}{1 + r^2},$$

where r^2 is the ratio of probabilities $r^2 = \frac{c^4}{a^4} = \frac{p_1}{p_0}$. By substituting these probabilities into the Shannon entropy formula,

$$S = -p_0 \log_2 p_0 - p_1 \log_2 p_1,$$

the entropy of Cassini state can be found as

$$S = \log_2 (a^4 + c^4) - \frac{1}{a^4 + c^4}(a^4 \log_2 a^4 + c^4 \log_2 c^4)$$

or

$$S(r^2) = \log_2(1 + r^2) - \frac{r^2}{1 + r^2}(\log_2 r^2).$$

Maximally random state corresponds to maximal entropy, and can be find at $r = 1$ or $a = c$, which corresponds to Bernoulli leminiscate. In this case probabilities $p_0 = p_1 = \frac{1}{2}$ and entropy $S = 1$. For $r^2 \neq 1$, there exists symmetry for entropy values between Cassini curves. For every closed Cassini oval with $r^2 > 1$, exists two Cassini ovals with the same entropy, determined by $\frac{1}{r^2} < 1$.

7.4. Cassini Representation of Two Qubit States

The two qubit analogy of the Cassini state, (7.8) are given in the following form,

$$|z\rangle = \frac{(z^2 - c^2)|00\rangle + c^2|11\rangle}{\sqrt{|z^2 - c^2|^2 + c^4}}, \tag{7.10}$$

and

$$|z\rangle = \frac{(z^2 - c^2)|01\rangle + c^2|10\rangle}{\sqrt{|z^2 - c^2|^2 + c^4}}.$$

Level of entanglement for these two qubit states can be calculated by the determinant formula for concurrence. For Cassini state (7.10), the concurrence is

$$C = 2\begin{vmatrix} \frac{z^2 - c^2}{\sqrt{|z^2 - c^2|^2 + c^4}} & 0 \\ 0 & \frac{c^2}{\sqrt{|z^2 - c^2|^2 + c^4}} \end{vmatrix} = \frac{2|z^2 - c^2|c^2}{|z^2 - c^2|^2 + c^4}. \tag{7.11}$$

This concurrence

$$C(a, c) = \frac{2a^2c^2}{a^4 + c^4} \tag{7.12}$$

is constant along every Cassini curve (7.9). For maximally entangled state:

$$\frac{dC}{da^2} = 0 \rightarrow a = c.$$

This is why, maximally entangled states with $C = 1$ are located on the Bernoulli leminiscate. For states with $C < 1$, due to symmetry (7.12) in a and c ,the Cassini curves with $\frac{a}{c} < 1$ and with $\frac{c}{a} > 1$ gives the same concurrence.

7.4.1. Fidelity for Two Qubit Cassini State

As it is seen in Chapter 5, concurrence for two qubit states can be rewritten as fidelity $F = |\langle\tilde{\psi}|\psi\rangle|$ between the qubit state $|\psi\rangle$ and the bit-phase flipped state $|\tilde{\psi}\rangle$ such that

$$|\tilde{\psi}\rangle = Y \otimes Y|\bar{\psi}\rangle,$$

where the complex conjugate state $|\bar{\psi}\rangle$ results from application of anti -unitary operator K,

$$|\bar{\psi}\rangle = K|\psi\rangle.$$

For two qubit state in Cassini representation (7.10), the bit-flipped state is

$$|\tilde{z}\rangle = Y \otimes Y|\bar{z}\rangle = -\frac{c^2|00\rangle + (\bar{z}^2 - c^2)|11\rangle}{\sqrt{|z^2 - c^2|^2 + c^4}}. \tag{7.13}$$

The corresponding fidelity coincides with concurrence

$$F = |\langle\tilde{z}|z\rangle| = \frac{2|z^2 - c^2|c^2}{|z^2 - c^2|^2 + c^4} = C$$

and it is a constant along every Cassini curve $|z^2 - c^2| = a^2$.

7.4.2. Inversion in Leminiscate and Symmetric Cassini States

As it is mentioned in previous section, for Cassini curves with parameters $a_1^2 = \dfrac{c^2}{r}$ and $a_2^2 = c^2 r$, the Shannon entropy and concurrence coincides. Here it is shown that, these two curves represent inversions of each others in Bernoulli leminiscate, corresponding to $r = 1$ and $a_1^2 = c^2 = a_2^2$.

Definition 7.2 *(Hurwitz and Courant, 1964) Inversion of point $z = x + iy$ in analytical curve $F(x, y) = 0$ gives the point z^*. Explicit formula to find this symmetric point z^* is*

$$F\left(\frac{z + \bar{z}^*}{2}, \frac{z - \bar{z}^*}{2i}\right) = 0.$$

For leminiscate curve (7.1),

$$(x^2 + y^2)^2 - 2c^2(x^2 - y^2) = 0$$

it gives relation

$$(z^*)^2 = \frac{\bar{z}^2 c^2}{\bar{z}^2 - c^2}$$

or

$$(z^2 - c^2)((\bar{z}^*)^2 - c^2) = c^4 \tag{7.14}$$

For given Cassini curve $|z^2 - c^2| = a_1^2$, inversion (7.14) gives symmetric curve

$$|(z^*)^2 - c^2| = a_2^2,$$

which is also Cassini curve, so that

$$a_1^2 \cdot a_2^2 = c^4.$$

For these symmetric curve the concurrence for two qubit coincides.(Similar result is valid for Shannon entropy for one qubit state)

Inversion formula implies also that, for given Cassini state $|z\rangle$ (7.9) exists the symmetric state

$$|z^*\rangle = \frac{c^2|00\rangle + (\bar{z}^2 - c^2)|11\rangle}{\sqrt{|z^2 - c^2|^2 + c^4}}$$

and as easy to see, up to global phase this state coincides with $|\tilde{z}\rangle$ (7.13). This shows that, the concurrence in Cassini representation

$$C = |\langle z^*|z\rangle| = F$$

is just fidelity between symmetric Cassini states, reflected in the leminiscate.

7.5. Tritangle for Three Qubit Cassini State

The three qubit Cassini state

$$|z\rangle = \frac{(z^2 - c^2)|000\rangle + c^2|111\rangle}{\sqrt{|z^2 - c^2|^2 + c^4}} \tag{7.15}$$

is determined by complex parameter $z = x + iy$ and real parameter c, so that in extended complex plane $\mathbb{C}$

- state $|111\rangle$ corresponds to two points $z = \pm c$
- state $|000\rangle$ corresponds to point $z = \infty$.

By identifying points $+z$ and $-z$, the plane $\mathbb{C}$ is reduced to the right half plane

$$Re\, z = x \geq 0,$$

and the state $|111\rangle$ is located at just $z = c$ (Figure 7.9).

Tritangle for Cassini three qubit state (7.15) is

$$\tau = \frac{4|z^2 - c^2|^2 c^4}{(|z^2 - c^2|^2 + c^4)^2} \tag{7.16}$$

and as a function of z, it is related with the concurrence for Cassini two qubit state (7.11) by formula

$$\tau = C^2.$$

Then, along arbitrary Cassini curve

$$|z^2 - c^2| = a^2,$$

determined by real number a, the tritangle (7.16) is constant

$$\tau = \frac{4a^4 c^4}{(a^4 + c^4)^2}.$$

It means that Cassini curves in plane z are *iso-tritangle* curves, and tritangle is a constant along the curve, with $0 \leq \tau \leq 1$. Particular values of tritangle are following

- For $a = 0$, two points $z = \pm c$, corresponding to $|111\rangle$ state give $\tau = 0$.
- For $a = \infty$, solution is $z = \infty$, corresponding to the state $|000\rangle$ has $\tau = 0$.
- For $a = \pm c$ maximal value $\tau = 1$, corresponds to the "Bernoulli Leminiscate".

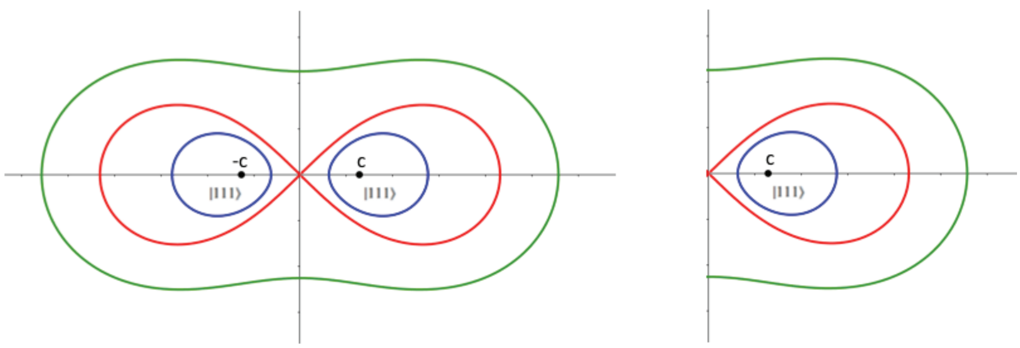


Figure 7.9. Half Cassini Ovals

7.6. n - tangle for Even Cassini Qubit States

The $n = 2k$ - even number Cassini qubit state is defined as

$$|z\rangle = \frac{(z^2 - c^2)|00...0\rangle + c^2|11...1\rangle}{\sqrt{|z^2 - c^2|^2 + c^4}},$$

where $z = x + iy$, and c is real. The state $|11...1\rangle$ corresponds to points $z = \pm c$ and the state $|00...0\rangle$ corresponds to point $z = \infty$. The n - tangle for this state is determined by (6.6) as

$$\tau_{12...n} = 2|(-1)^n\, 2\, a_{00...0}^2\, a_{11...1}^2| = 2\left|(-1)^n\, 2\, \frac{(z^2 - c^2)^2 c^4}{(|z^2 - c^2|^2 + c^4)^2}\right|$$

or

$$\tau_{12...n} = \frac{4|z^2 - c^2|^2 c^4}{(|z^2 - c^2|^2 + c^4)^2}.$$

Along Cassini cures

$$|z^2 - c^2|^2 = a^4$$

the n-tangle becomes a constant

$$\tau_{12...n} = \frac{4a^4 c^4}{(a^4 + c^4)^2} = C^2(a, c).$$

The symmetric state

$$\begin{aligned} |\tilde{z}\rangle = \underbrace{(Y \otimes Y \otimes ... \otimes Y)}_{n \text{ times}} |\bar{z}\rangle \quad &= \quad \frac{(\bar{z}^2 - c^2) i^{2k}\, |11...1\rangle + \bar{z}\, (-i)^{2k}\, |00...0\rangle}{\sqrt{|z^2 - c^2|^2 + c^4}} \\ &= \quad (-1)^k\, \frac{c^2|00...0\rangle + (\bar{z}^2 - c^2)|11...1\rangle}{\sqrt{|z^2 - c^2|^2 + c^4}} \\ &= \quad |\tilde{z}\rangle, \end{aligned}$$

is reflected state in leminiscate, like in Section 7.4.2. Then fidelity becomes

$$F = |\langle\tilde{z}|z\rangle| = \frac{2|z^2 - c^2|c^2|}{|z^2 - c^2|^2 + c^4}$$

and as easy to see, it is related with n - tangle formula

$$\tau_{12...n} = F^2.$$

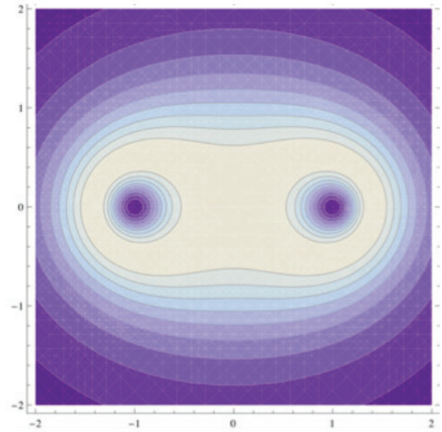


Figure 7.10. n - tangle for Cassini Qubit Contour Plot

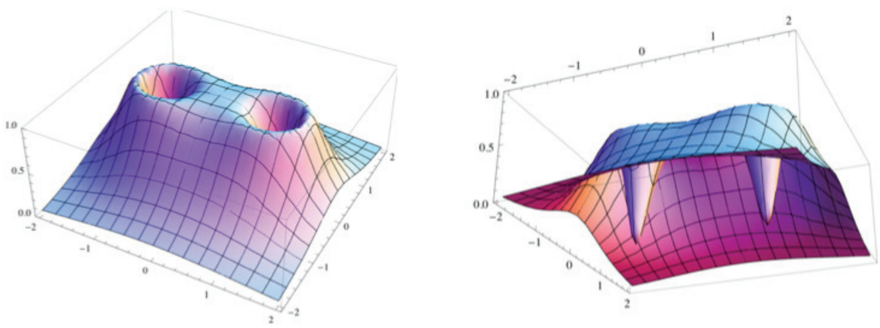


Figure 7.11. n - tangle for Cassini Qubit 3D Plots

For very small range Bernoulli Leminiscate occurs.

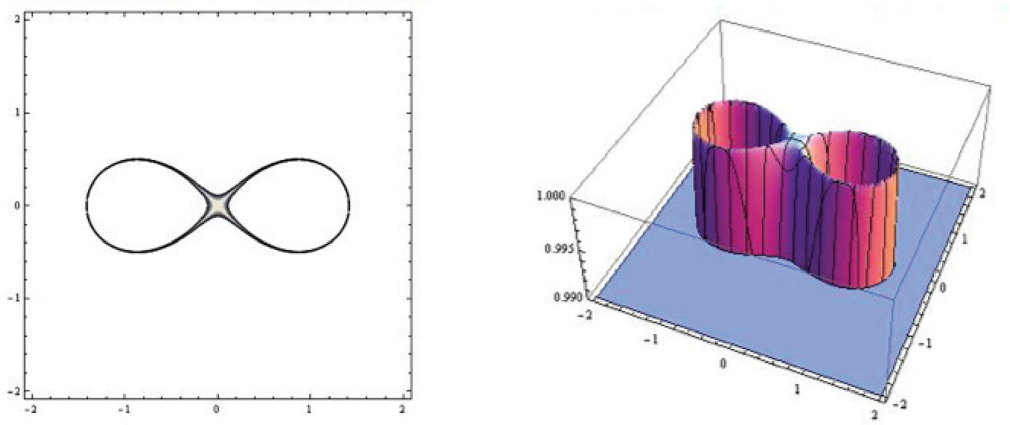


Figure 7.12. n - tangle for Cassini Qubit Bernoulli Leminiscate Contour and 3D Plot

7.7. Transforming Cassini State to Apollonius State

As it is known in Section 7.2.3, the conformal map

$$\eta = -c \, \frac{z^2}{z^2 - 2c^2} \tag{7.17}$$

transforms Cassini curves to Apollonius circles. Here, it will introduced by using qubit states. Cassini qubit state is

$$|z\rangle = \frac{(z^2 - c^2)|0\rangle + c^2|1\rangle}{\sqrt{|z^2 - c^2|^2 + c^4}}$$

can be transform into Apollonius states by using same conformal mapping. To find this, (7.17) can be rewritten as

$$z^2 - c^2 = c^2 \frac{\eta - c}{\eta + c},$$

substituting into state $|z\rangle$ gives symmetric Apollonius state

$$|\eta\rangle = \frac{(\eta - c)\,|0\rangle + (\eta + c)\,|1\rangle}{\sqrt{|\eta - c|^2 + |\eta + c|^2}}.$$

Scaling $\eta \to 2\eta - c$ gives non - symmetric Apollonius state

$$|\eta\rangle = \frac{(\eta - c)\,|0\rangle + \eta\,|1\rangle}{\sqrt{|\eta - c|^2 + |\eta|^2}}.$$

This result can be generalized for two, three and n- qubits.

CHAPTER 8

BIPOLAR REPRESENTATION OF QUBIT STATES

In Chapter 5, Apollonius circles were defined as a ratio of distances between two symmetric points. These symmetric points determine the symmetric one - qubit states. As an example, computational basis states $|0\rangle$ and $|1\rangle$ are symmetric states. In Section 5.3, it was shown that, entropy of one qubit state along these Apollonius circles is a constant. It is known that, Apollonius circles represent part of the so called bipolar coordinates, determined by two fixed points in the plane. These coordinates have applications in electro-magnetic theory, determining the electric and magnetic field of two infinitely long parallel cylindrical conductors. This is why, it is natural to use bipolar coordinates to parametrize qubit states. In the present Chapter, bipolar coordinates for one and two qubit states are derived for two different choices of symmetric states.

8.1. Bipolar Coordinates: Non-Symmetric Case (0,1)

Here the bipolar coordinates are introduced for fixed points 0 and 1. For given complex number $z = x + iy$, two real variables are defined as τ and σ,

$$z = \frac{e^{\tau}}{e^{\tau} - e^{i\sigma}} \,,$$

where

$$-\infty < \tau < \infty, \qquad -\pi < \sigma < \pi,$$

and

$$\frac{|z|}{|z-1|} = r = e^{\tau} \,.$$

Relation of these variables with cartesian coordinates is

$$x = \frac{1}{2} + \frac{1}{2}\frac{\sinh\tau}{\cosh\tau - \cos\sigma}, \qquad y = \frac{1}{2}\frac{\sin\sigma}{\cosh\tau - \cos\sigma} \,,$$

so that

$$z = x + iy = \frac{1}{2} + \frac{1}{2}\frac{\sinh\tau + i\sin\sigma}{\cosh\tau - \cos\sigma}.$$

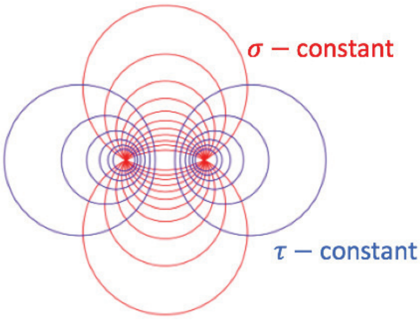


Figure 8.1. Bipolar Coordinates for Apollonius circle

8.1.1. One Qubit State in Bipolar Coordinates (0,1)

The bipolar representation for one qubit state is,

$$|A\rangle = \frac{1}{2}\frac{(e^{i\sigma} - e^{-\tau})|0\rangle + (e^{\tau} - e^{-i\sigma})|1\rangle}{\sqrt{\cosh\tau(\cosh\tau - \cos\sigma)}}.$$

This state, up to the global phase can be rewritten as

$$|\tau, \sigma\rangle = \frac{e^{i\sigma}|0\rangle + e^{\tau}|1\rangle}{\sqrt{1 + e^{2\tau}}},$$

where probabilities in bipolar form are

$$p_0 = \frac{1}{1 + e^{2\tau}}, \quad p_1 = \frac{e^{2\tau}}{1 + e^{2\tau}},$$

with ratio

$$\frac{p_1}{p_0} = e^{2\tau} \equiv r^2.$$

The Shannon entropy represented in bipolar form is

$$S(\tau) = \log_2(1 + e^{2\tau}) - \frac{e^{2\tau}}{1 + e^{2\tau}} \log_2 e^{2\tau}.$$

By simplifying this equation it becomes

$$S(\tau) = 1 + \frac{\ln(\cosh \tau) - \tau \tanh \tau}{\ln 2}.$$

Maximally and Minimally Random States:

- If $\tau = 0$, then $r = e^{\tau} = e^0 = 1$,which shows $S(0) = 1$, that gives the maximally random state.
- If $\tau = \infty$, then $r = e^{\tau} = e^{\infty} = \infty$, which shows $S(\infty) = 0$, that gives the minimally random state $|1\rangle$.
- If $\tau = -\infty$ then $r = e^{\tau} = e^{-\infty} = 0$, which shows $S(-\infty) = 0$, that gives minimally random state. $|0\rangle$.

8.1.2. Two Qubit state in Bipolar Coordinates (0,1)

Definition 8.1 *The Apollonius two qubit state in bipolar coordinates is defined as*

$$|\tau, \sigma\rangle = \frac{e^{i\sigma}|00\rangle + e^{\tau}|11\rangle}{\sqrt{1 + e^{2\tau}}}.$$

where (Figure 8.1*)*

$$-\infty < \tau < \infty, \qquad -\pi < \sigma < \pi.$$

The determinant formula for concurrence of this two qubit state gives expression

$$C = \frac{1}{\cosh \tau} = sech\, \tau. \tag{8.1}$$

It shows that, concurrence is not depending on angle σ, and is constant along the Apollonius circle with fixed coordinate τ. This formula suggests to consider complex valued transition amplitude between symmetric states in bipolar coordinates.

For two qubit state it gives the complex fidelity

$$\mathcal{F} = \langle A_s|A\rangle = Fe^{-i\sigma} = \frac{e^{-i\sigma}}{\cosh\tau},$$

which describes complex version of the concurrence

$$C = \langle A_s|A\rangle = Ce^{-i\sigma} = \frac{e^{-i\sigma}}{\cosh\tau}. \tag{8.2}$$

The modulus of this complex concurrence is just the usual concurrence (8.1)

$$|C| = |\langle A_s|A\rangle| = C = \frac{1}{\cosh\tau}.$$

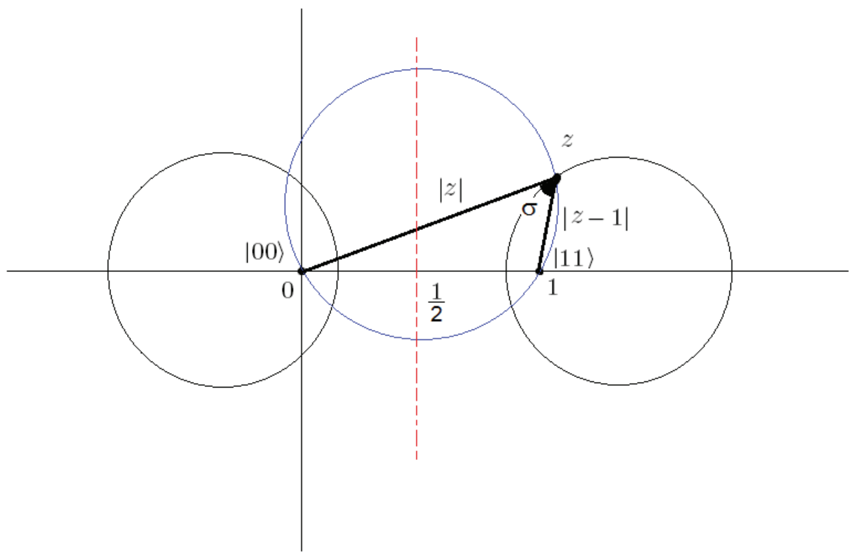


Figure 8.2. Bipolar Coordinates for Two Qubit (0,1)

8.2. Bipolar Coordinates: Symmetric Case (-1,1)

For two fixed points -1 and 1, the distance ratio becomes

$$\left|\frac{z+1}{z-1}\right| = r = e^{\tau}.$$

where $z = x+iy$. This complex variable z is represented in bipolar coordinates by equation

$$z = \frac{e^{\tau-i\sigma}+1}{e^{\tau-i\sigma}-1}.$$

For x and y variables it is

$$x = \frac{z+\bar{z}}{2} = \frac{\sinh\tau}{\cosh\tau-\cos\sigma}, \qquad y = \frac{z-\bar{z}}{2i} = \frac{\sinh\sigma}{\cosh\tau-\cos\sigma},$$

$$z = x+iy = \frac{\sinh\tau+\sinh\sigma}{\cosh\tau-\cos\sigma}.$$

8.2.1. One Qubit State in Bipolar Coordinates (-1,1)

Starting from Apollonius state

$$|\psi\rangle = \frac{(z-1)|0\rangle+(z+1)|1\rangle}{\sqrt{|z-1|^2+|z+1|^2}},$$

by using $e^{\tau} = \dfrac{|z+1|}{|z-1|}$, and

$$z-1 = \frac{2}{e^{\tau-i\sigma}-1}, \qquad z+1 = \frac{2e^{\tau-i\sigma}}{e^{\tau-i\sigma}-1},$$

$$|z-1|^2 = \frac{2e^{\tau}}{\cosh\tau - \cos\sigma}, \qquad |z+1|^2 = \frac{2e^{-\tau}}{\cosh\tau - \cos\sigma},$$

$$|z-1|^2 + |z+1|^2 = \frac{4\cosh\tau}{\cosh\tau - \cos\sigma},$$

the one qubit state up to the global phase is obtained as

$$|\tau,\sigma\rangle = \frac{|0\rangle + e^{\tau - i\sigma}|1\rangle}{\sqrt{1+e^{2\tau}}}.$$

The probabilities are

$$p_0 = \frac{1}{1+e^{2\tau}}, \qquad p_1 = \frac{e^{2\tau}}{1+e^{2\tau}},$$

and the ratio of these probabilities is the same as in non-symmetric case,

$$\frac{p_1}{p_0} = e^{2\tau}.$$

This implies that the Shannon entropy, which depends on this ratio only, takes the same form.

8.2.2. Two Qubit State in Bipolar Coordinates (-1,1)

The Apollonius two qubit state in symmetric form is

$$|\psi\rangle = \frac{(z-1)|00\rangle + (z+1)|11\rangle}{\sqrt{|z-1|^2 + |z+1|^2}},$$

with concurrence

$$C = \frac{2|z^2-1|}{|z-1|^2 + |z+1|^2}.$$

This two qubit state can be represented in bipolar coordinates as

$$|\psi\rangle = \frac{|00\rangle + e^{\tau - i\sigma}|11\rangle}{\sqrt{1 + e^{2\tau}}}.$$

The concurrence for this state in bipolar form is in the same, as in non-symmetric case

$$C = \frac{|e^{\tau - i\sigma}|}{1 + e^{2\tau}} = \frac{1}{\cosh \tau}.$$

CHAPTER 9

CONCURRENCE AS CONFORMAL METRIC

In this Chapter, hydrodynamic and geometric interpretations of entanglement characteristics, as the concurrence and the 3-tangle are given. In the first case, this is the stream function of the hydrodynamic flow, and in the second one, the conformal metric on a surface.

9.1. Hydrodynamic Flow and Concurrence

Definition 9.1 *(Milne-Thomson, 1968) Complex Potential:*
Let $\varphi(x, y)$ is velocity potential and $\psi(x, y)$ is stream function of the irrotational two dimensional motion. The complex potential is defined as

$$F(x, y) = \varphi(x, y) + i\psi(x, y).$$

For irrotational and incompressible flow, these functions satisfy the Cauchy-Riemann equations:

$$\frac{\partial\varphi}{\partial x} = \frac{\partial\psi}{\partial y}, \qquad \frac{\partial\varphi}{\partial y} = -\frac{\partial\psi}{\partial x}.$$

These equations imply that complex valued function $F(z)$ of complex argument $z = x + iy$ is analytic function. Singularities of this function describes vortices, sources, etc.

Definition 9.2 *Point Vortex:*
The point vortex, corresponds to complex potential with logarithmic singularity, which is located at z_0,

$$F(z) = \frac{\gamma}{2\pi i}\log(z - z_0).$$

If $\gamma > 0$, then circulation is counterclockwise, if $\gamma < 0$, then circulation is clockwise.

Definition 9.3 *Vortex Pair:*
A pair of vortices, each of strength γ, but in opposite rotations called the vortex pair.

Considering such pair of vortices with strengths γ at point $A(0, -ia)$ and $-\gamma$ at point $B(0, ia)$, where $|AB| = 2a$, the complex potential becomes

$$F(z) = i\gamma \log \frac{z - ia}{z + ia}.$$

9.1.1. Hydrodynamic Flow: Apollonius and Cassini

Here, two point vortex configurations are considered. According to the sign of vortex strengths, two cases with opposite and the same directions of rotation are possible. This leads to the flow along Apollonius circles and Cassini ovals, respectively.

- **Apollonius Vortex Circles:**
  For two point vortices in x- axis, at $(-a, 0)$, with $\gamma < 0$, and at $(a, 0)$, with $\gamma > 0$, the complex potential is

$$\begin{aligned} F(z) &= \frac{\gamma}{2\pi i} \log(z - a) + \frac{-\gamma}{2\pi i} \log(z - (-a)) \\ &= \frac{\gamma}{2\pi i} \log \frac{z - a}{z + a}. \end{aligned}$$

  The stream function of this flow is

$$\begin{aligned} \psi(x, y) = \frac{F(z) - \overline{F(z)}}{2i} &= -\frac{\gamma}{4\pi} \left(\log \frac{z - a}{z + a} + \log \frac{\bar{z} - a}{\bar{z} + a} \right) \\ &= -\frac{\gamma}{4\pi} \log \left(\frac{z - a}{z + a} \cdot \frac{\bar{z} - a}{\bar{z} + a} \right) \\ &= -\frac{\gamma}{4\pi} \log \frac{|z - a|^2}{|z + a|^2}. \end{aligned}$$

  Then, along the stream lines, the stream function is a constant: $\psi(x, y) = \psi_0$

$$\psi_0 = -\frac{\gamma}{4\pi} \log \frac{|z - a|^2}{|z + a|^2}.$$

Solution of this equation gives

$$\frac{|z-a|}{|z+a|} = e^{\frac{-2\pi\psi_0}{\gamma}}.$$

By denoting $e^{\frac{-2\pi\psi_0}{\gamma}} \equiv k$, where k is a constant, the Apollonius circles appear as the stream lines of vortex motion

$$\frac{|z-a|}{|z+a|} = k.$$

- **Cassini Vortex Ovals:**

  For two point vortices in x- axis, at $(-a, 0)$, and at $(a, 0)$ with the same strength $\gamma > 0$, the complex potential is

$$\begin{aligned} F(z) &= \frac{\gamma}{2\pi i}\log(z-a) + \frac{\gamma}{2\pi i}\log(z-(-a)) \\ &= \frac{\gamma}{2\pi i}\log((z-a)\cdot(z+a)) \\ &= \frac{\gamma}{2\pi i}\log(z^2-a^2). \end{aligned} \tag{9.1}$$

  The stream function of this flow is

$$\begin{aligned} \psi(x,y) = \frac{F(z)+\overline{F(z)}}{2i} &= -\frac{\gamma}{4\pi}\Big(\log(z^2-a^2)+\log(\bar{z}^2-a^2)\Big) \\ &= -\frac{\gamma}{4\pi}\log\Big(\log(z^2-a^2)\cdot(\bar{z}^2-a^2)\Big) \\ &= -\frac{\gamma}{4\pi}\log|z^2-a^2|^2. \end{aligned}$$

  Then, along the stream lines, the stream function is a constant: $\psi(x,y) = \psi_0$ and

$$\psi_0 = -\frac{\gamma}{4\pi}\log|z^2-a^2|^2.$$

Solving this equation gives

$$|z^2 - a^2| = e^{\frac{-2\pi\psi_0}{\gamma}}.$$

By denoting $e^{\frac{-2\pi\psi_0}{\gamma}} \equiv t$, where t is a constant, the Cassini ovals becomes the stream lines of the flow

$$|z - a||z + a| = t.$$

9.1.2. Concurrence Flow for Two Qubit States

The curves along which the concurrence is a constant, can be considered as a stream lines of the some planar flow, which would be called as "the concurrence flow". To describe this flow, the stream function is introduced according to equation

$$\psi(x, y) = \ln C^2(x, y),$$

where $C(x, y)$ is the concurrence, as a function of $z = x + iy$. The concurrence flow is a vector field $\vec{v} = (v_x, v_y)$, where

$$v_x(x, y) = \frac{\partial\psi}{\partial y}, \quad v_y(x, y) = \frac{\partial\psi}{\partial x},$$

which is the tangent to the stream lines of the flow. Vorticity of the flow is determined by formula

$$\omega = \frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}$$

and can be expressed by stream function as

$$\omega = -\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\psi = -\Delta\psi,$$

or

$$\omega = -2\Delta \ln C(x, y).$$

- **Two Qubit Coherent State**

  For the state

  $$|z\rangle = \frac{|00\rangle + z|11\rangle}{\sqrt{1 + |z|^2}},$$

  the concurrence is

  $$C(z) = \frac{2|z|}{1 + |z|^2}.$$

  It determines the stream function

  $$\psi(x, y) = \ln\left(\frac{4|z|^2}{(1 + |z|^2)^2}\right) = \ln\left(\frac{4|(x^2 + y^2)}{(1 + x^2 + y^2)^2}\right).$$

  The concurrence flow is given by

  $$v_x = g(r)\, y, \qquad v_y = g(r)\, x,$$

  where $r^2 = x^2 + y^2$, and

  $$g(r) = \frac{2}{r}\frac{1 - r^2}{1 + r^2}.$$

This vector field is tangent to concentric circles around origin, due to

$$\vec{r} \cdot \vec{v} = xv_x + yv_y = xg(r)y - yg(r)x = 0.$$

The flow with $r > 1 \implies g(r) < 0$ is in counter-clockwise direction, while for $r < 1 \implies g(r) > 0$, it is in clockwise direction. At unit circle $r = 1 \implies g(r) = 0$ and velocity of the flow is zero. This circle corresponds to maximally entangled states and at this circle, the flow change the direction of rotation. Vorticity of the flow

$$\omega = \frac{8}{(1 + r^2)^2}$$

is constant along every circle with fixed radius.

For $r \to 0$

$$v_x \sim \frac{2y}{r^2}, \quad v_y \sim -\frac{2x}{r^2}$$

so that, for $v = \sqrt{v_x^2 + v_y^2}$ it gives $v \sim \frac{2}{r}$.

For $r \to \infty$ it is

$$v_x \sim -\frac{2y}{r^2}, \quad v_y \sim \frac{2x}{r^2}$$

and again $v \sim \frac{2}{r}$. This asymptotic behaviour of the velocity field is similar to point vortex with complex potential

$$f(z) = \frac{\gamma}{2\pi i} \ln z$$

and velocity field

$$v_x = \frac{\gamma}{2\pi}\frac{-y}{r^2}, \qquad v_y = \frac{\gamma}{2\pi}\frac{x}{r^2},$$

located at the origin with strength $\Gamma = +4\pi$ for $r \to \infty$, $\Gamma = -4\pi$ for $r \to 0$.

The total vorticity on the plane is equal

$$\iint \omega(x,y)\,dx\,dy = 8\iint \frac{dx\,dy}{(1+r^2)^2} = 8\int_0^\infty \int_0^{2\pi} \frac{r\,dr\,d\varphi}{(1+r^2)^2} = 8\pi,$$

which according to the Green formula, is equal to difference of circulations around ∞ and 0: $+4\pi - (-4\pi) = 8\pi$.

- **Apollonius Flow**

  For Apollonius two qubit state

$$|z\rangle = \frac{(z-1)|00\rangle + (z+1)|11\rangle}{\sqrt{|z-1|^2 + |z+1|^2}}$$

has concurrence is

$$C(z) = \frac{2|z^2-1|}{|z-1|^2 + |z+1|^2}$$

and it determines the stream function

$$\psi(x,y) = \ln\left(\frac{(x^2-y^2-1)^2 + 4x^2y^2}{(1+x^2+y^2)^2}\right).$$

The flow

$$v_x(x,y) = 4\,y\,h(x,y), \qquad v_y(x,y) = -4\,x\,h(x,y),$$

where

$$h(x,y) = \frac{4x^2}{[(x^2-y^2-1)^2+4x^2y^2][1+x^2+y^2]}.$$

The velocity function

$$v(x,y) = \frac{16x^2\, r}{\sqrt{1+r^2}((1+r^2)^2-4x^2)}$$

is shown in Figure 9.1 , and it is singular around points $z = 1$ and $z = -1$, on the plane, as centers of circulation.

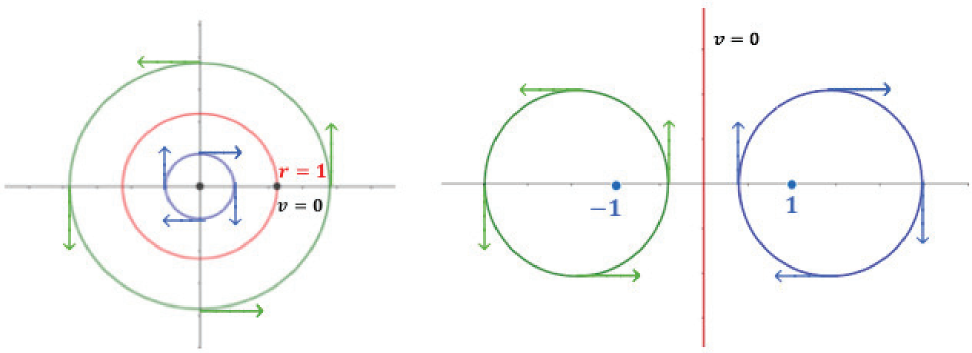


Figure 9.1. Concurrence Flow

9.2. Concurrence as Conformal Metric

It was shown in Chapter 6 that the square of concurrence gives 3-tangle $\tau = C^2$ or n-tangle for even multiqubit states in coherent state representation. In this Section, the square of concurrence $C^2(x,y)$ is treated as a conformal metric on a surface.

9.2.1. Qubit and Conformal Metric on Sphere

Here, a relation between qubit and conformal metric is established. The one qubit state

$$|\psi\rangle = c_0|0\rangle + c_1|1\rangle$$

is determined by two complex numbers c_0, c_1, allowing to define Hermitian metric in $\mathbb{C}^2$ (Dubrovin, Fomenko and Novikov, 1984)

$$dl^2 = dc_0 d\bar{c}_0 + dc_1 d\bar{c}_1.$$

A surface in real space can be considered then as a complex curve in $\mathbb{C}^2$, given by implicit analytic function

$$f(c_0, c_1) = 0.$$

Induced metric on this surface can be calculated as

$$dl^2 = dc_0 d\bar{c}_0 + \frac{dc_1}{dc_0} dc_0 \frac{d\bar{c}_1}{d\bar{c}_0} d\bar{c}_0$$

or

$$dl^2 = \left(1 + \left|\frac{dc_1}{dc_0}\right|^2\right) dc_0 d\bar{c}_0.$$

If $c_0 = x + iy$, then induced metric on surface $f(c_0, c_1) = 0$ is

$$dl^2 = g(c_0, \bar{c}_0)\, dc_0 d\bar{c}_0 = g(x, y)(dx^2 + dy^2)$$

and

$$g(x, y) = g(c_0, \bar{c}_0) = 1 + \left|\frac{dc_1}{dc_0}\right|^2.$$

This metric is called the "conformal metric".

Definition 9.4 *If the metric on a real two dimensional surface has the form*

$$dl^2 = g(x,y)(dx^2 + dy^2) \tag{9.2}$$

in terms of real coordinates x and y on the surface, then these coordinates called "conformal".

But coordinates c_0 and c_1 in addition have to satisfy normalization constraint

$$|c_0|^2 + |c_1|^2 = 1.$$

Parametrizing $c_0 = r_0\, e^{i\varphi_0}, c_1 = r_1\, e^{i(\varphi_0+\varphi_1)}$, this constraint gives the unit circle

$$r_0^2 + r_1^2 = 1. \tag{9.3}$$

So that the metric can be calculated by taking into account the global phase (φ_0) identification $d\varphi_0 = 0$:

$$\begin{aligned} dc_0 &= dr_0 e^{i\varphi_0} + r_0 e^{i\varphi_0} i d\varphi_0, \\ d\bar{c}_0 &= dr_0 e^{-i\varphi_0} - r_0 e^{-i\varphi_0} i d\varphi_0, \end{aligned}$$

$$\begin{aligned} dc_0 d\bar{c}_0 &= dr_0^2 + r_0^2 d\varphi_0^2 \stackrel{\varphi_0=0}{=} dr_0^2 \\ dc_1 d\bar{c}_1 &= dr_1^2 + r_1^2 (d\varphi_0 d\varphi_1)^2 \stackrel{\varphi_0=0}{=} dr_1^2 + r_1^2 d\varphi_1^2. \end{aligned}$$

The circle equation (9.3) gives

$$r_0 dr_0 = -r_1 dr_1 \;\;\rightarrow\;\; dr_0 = -\frac{r_1}{r_0} dr_1$$

and

$$dr_0^2 = \frac{r_1^2}{r_0^2} dr_1^2. \tag{9.4}$$

Then

$$\begin{aligned}
dl^2 &= dc_0 d\bar{c}_0 + dc_1 d\bar{c}_1 \\
&= dr_0^2 + dr_1^2 + r_1^2 d\varphi_1^2 \\
&\overset{(9.4)}{=} \frac{r_0^2 + r_1^2}{r_0^2} dr_1^2 + r_1^2 d\varphi_1^2 \\
&\overset{(9.3)}{=} \frac{1}{1 - r_1^2} dr_1^2 + r_1^2 d\varphi_1^2,
\end{aligned}$$

and the metric becomes

$$dl^2 = \frac{dr_1^2}{1 - r_1^2} + r_1^2\, d\varphi_1^2. \tag{9.5}$$

This is the metric for generic one qubit state

$$|\psi\rangle = c_0|0\rangle + c_1|1\rangle \quad = \quad e^{i\varphi_0}(r_0|0\rangle + r_1 e^{i\varphi_1}|1\rangle),$$

and due to the circle equation (9.3),

$$|\psi\rangle = \sqrt{1 - r_1^2}\,|0\rangle + r_1 e^{i\varphi_1}|1\rangle. \tag{9.6}$$

- **Example 1 :** For $r_1 = \sin\frac{\theta}{2}, \varphi_1 \equiv \varphi$ this gives the Bloch representation of qubit (9.6),

$$|\psi\rangle = |\theta, \varphi\rangle = \cos\frac{\theta}{2}|0\rangle + \sin\frac{\theta}{2}\, e^{i\varphi}|1\rangle,$$

and metric (9.5) is the " spherical metric "

$$dl^2 = \frac{1}{4}d\theta^2 + \sin^2\frac{\theta}{2}d\varphi^2, \tag{9.7}$$

on the Bloch sphere.

- **Example 2 :** Rewriting the state (9.6) as

$$|\psi\rangle = \sqrt{1-r_1^2}\left(|0\rangle + \frac{r_1 e^{i\varphi_1}}{\sqrt{1-r_1^2}}|1\rangle\right)$$

and denoting

$$z \equiv \frac{r_1 e^{i\varphi_1}}{\sqrt{1-r_1^2}},$$

the state (9.6) becomes the coherent qubit state

$$|\psi\rangle = |z\rangle = \frac{|0\rangle + z|1\rangle}{\sqrt{1+|z|^2}}.$$

The corresponding metric is the conformal metric on the unit sphere

$$dl^2 = \frac{dz\,d\bar{z}}{(1+|z|^2)^2}. \tag{9.8}$$

A complex analytical coordinate changes preserve the conformal form of the metric. Indeed, by taking $z = x + iy$ the metric (9.2) is

$$dl^2 = g(z,\bar{z})dz\,d\bar{z}.$$

Let $z = z(w)$ define a complex analytic coordinate change, so that $\frac{\partial z}{\partial \bar{w}} = 0$ and $\bar{z} = \bar{z}(\bar{w})$,

$$dz = \frac{dz}{dw}dw, \qquad d\bar{z} = \frac{d\bar{z}}{d\bar{w}}d\bar{w}.$$

Multiplying them, gives

$$dz\, d\bar{z} = \left|\frac{dz}{dw}\right|^2 dw d\bar{w}.$$

Then, the metric (9.2) in new coordinates remains conformal

$$\begin{aligned} dl^2 = g(z,\bar{z})(dz + d\bar{z}) &= g(z(w),\bar{z}(\bar{w}))\left|\frac{dz}{dw}\right|^2 dw d\bar{w} \\ &= h(w,\bar{w})\, dw d\bar{w}. \end{aligned}$$

In above consideration, a relation between parametrization of one qubit state (9.6) and metric on sphere (9.5) was established. Representation of two qubit state in terms of one complex parameter z implies interpretation of the concurrence $C(z,\bar{z})$, or $C^2(z,\bar{z})$ (the 3-tangle of three qubit state is $\tau = C^2$) as conformal metric on a surface.

9.2.2. Apollonius Metric: Non-symmetric Case

The Apollonius two qubit state in z-plane

$$|\psi\rangle = \frac{(z-1)|00\rangle + z|11\rangle}{\sqrt{|z-1|^2 + |z|^2}} \tag{9.9}$$

has concurrence

$$C = \frac{2|z||z-1|}{|z-1|^2 + |z|^2}. \tag{9.10}$$

This gives conformal metric

$$dl^2 = g(z,\bar{z})\; dz d\bar{z} = C^2(z,\bar{z})\; dz d\bar{z}.$$

The metric can be represented in bipolar coordinates, defined as $z = \dfrac{e^{\tau}}{e^{\tau} - e^{i\sigma}}$. For this, dz and $d\bar{z}$ are calculated,

$$dz = \frac{e^{\tau+i\sigma}}{(e^{\tau} - e^{i\sigma})^2}(-d\tau + id\sigma), \qquad d\bar{z} = \frac{e^{\tau-i\sigma}}{(e^{\tau} - e^{-i\sigma})^2}(-d\tau - id\sigma),$$

$$dzd\bar{z} = \frac{e^{2\tau}}{(e^{\tau} - e^{i\sigma})^2(e^{\tau} - e^{-i\sigma})^2}(d\tau^2 + d\sigma^2),$$

giving the conformal factor,

$$g(z,\bar{z}) = \frac{4|z|^2|z-1|^2}{(|z-1|^2 + |z|^2)^2} = \frac{4e^{2\tau}}{1 + e^{2\tau}} = \frac{4}{4\cosh^2\tau} = \frac{1}{\cosh^2\tau}.$$

Then, the metric is represented in bipolar form as

$$dl^2 = g(z,\bar{z})\ dzd\bar{z} = \frac{d\tau^2 + d\sigma^2}{4\cosh^2\tau(\cosh\tau - \cos\sigma)^2} = g(\tau,\sigma)(d\tau^2 + d\sigma^2),$$

where the conformal factor is

$$g(\tau,\sigma) = \frac{1}{4\cosh^2\tau(\cosh\tau - \cos\sigma)^2}. \tag{9.11}$$

9.2.3. Apollonius Metric: Symmetric Case

Two qubit state in symmetric form,

$$|\psi\rangle = \frac{(z-1)|00\rangle + (z+1)|11\rangle}{\sqrt{|z-1|^2 + |z+1|^2}} \tag{9.12}$$

has concurrence

$$C = \frac{2|z^2 - 1|}{|z-1|^2 + |z+1|^2}. \tag{9.13}$$

The metric is represented as

$$C^2(z, \bar{z}) = \frac{4|z^2 - 1|^2}{(|z-1|^2 + |z+1|^2)^2} = g(z, \bar{z}).$$

In terms of bipolar coordinates

$$z = \frac{e^{\tau - i\sigma} + 1}{e^{\tau - i\sigma} + 1}$$

this metric takes the form

$$dl^2 = \frac{d\tau^2 + d\sigma^2}{\cosh^2 \tau} = g(\tau, \sigma)(d\tau^2 + d\sigma^2),$$

where conformal factor is

$$g(\tau, \sigma) = \frac{1}{\cosh^2 \tau}. \tag{9.14}$$

9.3. Constant Gaussian Curvature Surfaces and Cassini Curves

In Section 9.2, conformal metrics for two dimensional surfaces corresponding to Apollonius qubits were derived. Here, the Gaussian curvature for these surfaces is calculated.

Proposition 9.1 *(Dubrovin, Fomenko and Novikov, 1984) If u and v are conformal coordinates on a surface in* 3 *dimensional Euclidean space, in terms of which induced metric* *has the form*

$$dl^2 = g(u, v)(du^2 + dv^2),$$

then Gaussian curvature of the surface is

$$K = -\frac{1}{2g(u, v)}\Delta \ln g(u, v), \tag{9.15}$$

where

$$\Delta = \frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2}$$

is the Laplace operator.

9.4. Apollonius and Bipolar Representation of Gaussian Curvature

The Apollonius two qubit state metrics in bipolar coordinates are conformal, this is why the Gaussian curvature of these metrics can be calculated according to formula

$$K = -\frac{1}{2g(u, v)}\Delta \ln g(u, v) = -\frac{1}{2g(\tau, \sigma)}\left(\frac{\partial^2}{\partial \tau^2} + \frac{\partial^2}{\partial \sigma^2}\right)\ln(g(\tau, \sigma)). \tag{9.16}$$

Below, two cases of bipolar coordinates, non-symmetric and symmetric one will be considered.

9.4.1. Non-symmetric Case

The Apollonius two qubit state in z-plane (9.9) has concurrence (9.10). Conformal metric, calculated in bipolar coordinate (9.11) is

$$g(\tau, \sigma) = \frac{1}{4\cosh^2 \tau(\cosh \tau - \cos \sigma)^2}.$$

The Gaussian curvature K, with respect to this metric can be calculated by

$$\begin{aligned}
\ln g(\tau,\sigma) &= \ln\frac{1}{4} - 2\ln\cosh\tau - 2\ln(\cosh\tau - \cos\sigma),\\
\left(\frac{\partial^2}{\partial\tau^2} + \frac{\partial^2}{\partial\sigma^2}\right)\ln(g(\tau,\sigma)) &= -2\frac{1}{\cosh^2\tau}.
\end{aligned}$$

Substituting into (9.16), the Gaussian curvature in bipolar form is represented as

$$K = 4(\cosh\tau - \cos\sigma)^2.$$

Calculating

$$\begin{aligned}
K &= 4(\cosh\tau - \cos\sigma)^2\\
&= (e^{\tau} + e^{-\tau} - (e^{i\sigma} + e^{-i\sigma})^2) = \frac{(e^{\tau} - e^{i\sigma})^2(e^{\tau} - e^{-i\sigma})^2}{e^{2\tau}}
\end{aligned}$$

in complex $z = \dfrac{e^{\tau}}{e^{\tau} - e^{i\sigma}}$, $|z|^2 = \dfrac{e^{2\tau}}{(e^{\tau} - e^{i\sigma})(e^{\tau} - e^{-i\sigma})}$ and $|z-1|^2 = \dfrac{1}{(e^{\tau} - e^{i\sigma})(e^{\tau} - e^{-i\sigma})}$, it gives

$$K = \frac{1}{|z|^2|z-1|^2}.$$

According to this formula, the Gaussian curvature is a constant along the curve on the surface, satisfying equation

$$|z||z-1| = \frac{1}{\sqrt{K}}.$$

This is equation for the Cassini curves and the Gaussian curvature is constant along these curves, with fixed points 0 and 1.

9.4.2. Symmetric Case

The symmetric Apollonius two qubit state (9.12) has concurrence (9.13) and corresponding conformal metric (9.14),

$$g(\tau, \sigma) = \frac{1}{\cosh^2 \tau}.$$

Substituting into (9.16), the bipolar form of Gaussian curvature appears

$$K = (\cosh \tau - \cos \sigma)^2.$$

This curvature is constant along curves,

$$|z + 1||z - 1| = \frac{2}{\sqrt{K}},$$

which are Cassini curves, with fixed points -1 and 1.

9.5. Concurrence Surface as Surface of Revolution

In Section 9.2, the square of concurrence $C^2(x, y)$ as conformal metric on a surface was studied. The question is "how to recover this surface globally from local characteristics as the metric ?" This surface would be called as the *concurrence surface*. Intersection of this surface with parallel planes, corresponds to level curves, representing integral curves of constant concurrence. Here, this surface is recovered as a surface of revolution, partially in Euclidean space and partially in Minkowski space.

9.5.1. Surface As a Graph

For a surface given by a graph $z = f(x, y)$ the parametric form in $3D$ space is $\vec{r}(x, y) = (x, y, z = f(x, y))$. If

$$z = f(x, y) = f(x^2 + y^2) \equiv \phi(\sqrt{x^2 + y^2}) = \phi(r),$$

then this surface is "the surface of revolution", with

$$x = r\cos v, \quad y = r\sin v, \quad z = \phi(r), \tag{9.17}$$

where $x^2 + y^2 = r^2$, $0 \leq v \leq 2\pi$. The surface of revolution is invariant under rotations around z - axis.

9.5.2. Induced Metric on Surface

Distance between points on a surface is determined by the induced metric. Depending on metric of $3D$ space, two cases occur.

1. $\mathbb{R}^3$: Euclidean space

   $$dl^2 = dx^2 + dy^2 + dz^2.$$

   Since $z = f(x, y)$, then

   $$dz = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y} = f_x dx + f_y dy,$$

   and

   $$dl^2 = dx^2 + dy^2 + (f_x dx + f_y dy)^2 = (1 + f_x^2)dx^2 + 2f_x f_y dxdy + (1 + f_y^2)dy^2.$$

This gives induced metric

$$dl^2 = (1 + f_x^2)dx^2 + 2f_x f_y dxdy + (1 + f_y^2)dy^2. \qquad (9.18)$$

2. $\mathbb{R}^{2,1}$:Minkowski (Pseudo-Euclidean)space

$$dl^2 = dx^2 + dy^2 - dz^2$$

Similar substitution

$$dl^2 = dx^2 + dy^2 - (f_x dx + f_y dy)^2 = (1 - f_x^2)dx^2 - 2f_x f_y dxdy + (1 - f_y^2)dy^2.$$

gives induced metric

$$dl^2 = (1 - f_x^2)dx^2 - 2f_x f_y dxdy + (1 - f_y^2)dy^2. \qquad (9.19)$$

9.5.3. Induced Metric on Surface of Revolution

The induced metric on surface of revolution can be obtained from representation (9.17).

- **Euclidean Space**

  By

$$dx = dr\cos v - r\sin v dv$$
$$dy = dr\sin v + r\cos v dv$$

and

$$f_x = \phi_x = \frac{d\phi}{dr}\frac{dr}{dx} = \frac{d\phi}{dr}\frac{x}{r}$$
$$f_y = \phi_y = \frac{d\phi}{dr}\frac{dr}{dy} = \frac{d\phi}{dr}\frac{y}{r}$$

substitution into (9.18) gives

$$dl^2 = (1 + (\phi'(r))^2)\, dr^2 + r^2 dv^2. \tag{9.20}$$

This is induced metric on a surface of revolution in Euclidean space.

- **Minkowski Space**
  By

$$dx = dr\cos v - r\sin v dv$$
$$dy = dr\sin v + r\cos v dv$$

and

$$f_x = \phi_x = \frac{d\phi}{dr}\frac{dr}{dx} = \frac{d\phi}{dr}\frac{x}{r}$$
$$f_y = \phi_y = \frac{d\phi}{dr}\frac{dr}{dy} = \frac{d\phi}{dr}\frac{y}{r}$$

substitution into (9.19) gives

$$dl^2 = (1 - (\phi'(r))^2)\, dr^2 + r^2 dv^2. \tag{9.21}$$

This is induced metric on the surface of revolution in Minkowski space.

9.5.4. Conformal Metric on Surface of Revolution

To apply the metric surface of revolution to the concurrence surface, it is required transform the metric to conformal form.

1. **Euclidean Space**

By rewriting metric (9.20) in conformal coordinates $(r, \phi) \to (u, v)$, so that

$$dl^2 = (1 + (\phi'(r))^2)\, dr^2 + r^2 dv^2 = g(u, v)(du^2 + dv^2),$$

identification $g(u, v) = r^2$, due to equality

$$g(u, v)du^2 = r^2\, du^2 = (1 + (\phi'(r))^2)\, dr^2,$$

and

$$\begin{aligned} du^2 &= \frac{1 + (\phi'(r))^2}{r^2} dr^2, \\ du &= \pm \frac{\sqrt{1 + (\phi'(r))^2}}{r} dr, \end{aligned}$$

gives

$$u = \pm \int \frac{\sqrt{1 + (\phi'(r))^2}}{r} dr. \tag{9.22}$$

This shows that u is a function of r only $u = u(r)$ and as inverse function $r = r(u)$. Therefore, the metric

$$g(u, v) = r^2(u)$$

is only function of u, and coordinates u and v are conformal

$$dl^2 = g(u)(du^2 + dv^2). \tag{9.23}$$

2. **Minkowski Space**

- **Elliptic Case:**
  By rewriting metric (9.21) in conformal coordinates $(r, \phi) \to (u, v)$, so that

$$dl^2 = (1 - (\phi'(r))^2)\, dr^2 + r^2 dv^2 = g(u, v)(du^2 + dv^2),$$

identification $g(u, v) = r^2$, due to equality

$$g(u, v)du^2 = r^2\, du^2 = (1 - (\phi'(r))^2)\, dr^2,$$

(if $1 - (\phi'(r))^2 > 0$, then $(\phi'(r))^2 < 1$), and

$$\begin{aligned} du^2 &= \frac{1 - (\phi'(r))^2}{r^2} dr^2, \\ du &= \pm \frac{\sqrt{1 - (\phi'(r))^2}}{r} dr, \end{aligned} \tag{9.24}$$

gives

$$u = \pm \int \frac{\sqrt{1 - (\phi'(r))^2}}{r} dr. \tag{9.25}$$

Here

$$u = u(r), \quad r = r(u),$$

so that

$$g(u, v) = r^2(u)$$

is function of u only. This conformal metric

$$dl^2 = g(u)(du^2 + dv^2) \tag{9.26}$$

is similar to (9.23).

- **Hyperbolic Case:**

  By taking conformal metric in hyperbolic form

$$dl^2 = (1 - (\phi'(r))^2)\,dr^2 + r^2 dv^2 = g(u,v)(-du^2 + dv^2),$$

and using identification $g(u,v) = r^2$, gives

$$-g(u,v)du^2 = r^2\,du^2 = (1 - (\phi'(r))^2)\,dr^2$$

If $(\phi'(r))^2 - 1 > 0$, then $(\phi'(r))^2 > 1$, and as follows

$$du^2 = \frac{(\phi'(r))^2 - 1}{r^2} dr^2,$$
$$du = \frac{\sqrt{(\phi'(r))^2 - 1}}{r} dr,$$

which gives

$$u = \int \frac{\sqrt{(\phi'(r))^2 - 1}}{r} dr. \tag{9.27}$$

It means that $u = u(r)$ and $r = r(u)$, so that

$$g(u,v) = r^2(u),$$

is function of u only. Then the corresponding conformal metric is

$$dl^2 = g(u)(-du^2 + dv^2). \tag{9.28}$$

9.5.5. Gaussian Curvature for Conformal Metric on Surface of Revolution

For conformal metric

$$dl^2 = g(u,v)(du^2 + dv^2),$$

the Gaussian curvature is

$$K = -\frac{1}{2g(u)}\left(\frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2}\right)\ln g. \tag{9.29}$$

For the metric (9.23), $g(u,v) = g(u)$ then

$$K = -\frac{1}{2g(u)}\frac{\partial^2}{\partial u^2}\ln g(u). \tag{9.30}$$

Therefore, for surface of revolution : $g(u) = r^2(u)$

$$K = -\frac{1}{2r^2}\frac{d^2}{du^2}\ln r^2 = -\frac{1}{r^2}\frac{d^2}{du^2}\ln r(u).$$

This formula allows one for given curvature K, to find $r(u)$ and corresponding surface of revolution.

Example 1 : Metric on Sphere

Starting from conformal metric on unit sphere (9.8)

$$dl^2 = \frac{4\,dzd\bar{z}}{(1+|z|^2)^2},$$

and transforming

$$z = e^w \to dz = e^w dw,\; d\bar{z} = e^{\bar{w}}d\bar{w}$$

the metric becomes

$$dl^2 = \frac{4\,e^{w+\bar{w}}}{(1+e^{w+\bar{w}})^2} dw d\bar{w}.$$

Since $w = u + iv \rightarrow |z|^2 = e^{2u}$, then

$$dl^2 = \frac{4\,e^{2u}}{(1+e^{2u})^2}(du^2+dv^2) = \frac{4\,e^{2u}}{(e^u(e^{-u}+e^u))^2}(du^2+dv^2),$$

and due to $\cosh u = \dfrac{e^u+e^{-u}}{2}$, it gives

$$dl^2 = \frac{du^2+dv^2}{\cosh^2 u}.$$

For conformal metric $g(u) = \dfrac{1}{\cosh^2 u}$, the curvature (9.30) is calculated as

$$\begin{aligned} K &= -\frac{\cosh^2 u}{2}\frac{d^2}{du^2}\underbrace{\ln\left(\frac{1}{\cosh^2 u}\right)} \\ &= -\frac{\cosh^2 u}{2}\frac{d^2}{du^2}(-2\ln\cosh u) \\ &= \frac{\cosh^2 u}{\cosh^2 u} = 1. \end{aligned}$$

It is a constant curvature surface with $K = 1$ and the unit radius. To recover the surface, following transformation is used

$$g(u) = r^2(u) = \frac{1}{\cosh^2 u} \rightarrow r = \frac{1}{\cosh u},$$

$$dr = -\frac{1}{\cosh^2 u}\sinh u du = -\frac{\cosh^2 u}{\sinh u} dr.$$

Rewriting this in terms of r, related equation (9.24) gives

$$du = -\frac{dr}{r\sqrt{1-r^2}} = \pm\frac{\sqrt{1+\phi'^2}}{r}dr.$$

Then, by integration

$$\phi^2 + r^2 = 1,$$

appears as the circle equation, and the surface of revolution is the unit sphere.

9.5.6. Concurrence Metric

From conformal metric

$$dl^2 = \frac{4|z|^2}{(1+|z|^2)^2}dzd\bar{z},$$

by taking $z = e^w$ gives

$$dl^2 = \frac{4e^{4u}}{(1+e^{2u})^2}(du^2+dv^2) = \frac{e^{2u}}{\cosh^2 u}(du^2+dv^2),$$

where the conformal factor is

$$g(u) = \frac{e^{2u}}{\cosh^2 u}.$$

Then, the curvature calculated by (9.30) is

$$K = -\frac{1}{2}\frac{1}{\left(\frac{e^{2u}}{\cosh^2 u}\right)}\frac{d^2}{du^2}(2u - 2\ln\cosh u) = -\frac{1}{2}\frac{\cosh^2 u}{e^{2u}}\left(-\frac{2}{\cosh^2 u}\right),$$

or

$$K = \frac{1}{e^{2u}} = \frac{1}{|z|^2}.$$

This curvature is positive for $z = 0$ it becomes infinite. To recover the surface following identification is applied

$$g(u) = r^2(u) = \frac{e^{2u}}{\cosh^2 u} \rightarrow r(u) = \frac{e^u}{\cosh u},$$

where $r(u)$ is bounded $0 \leq r \leq 2$. By taking derivative of $r(u)$,

$$dr = \left(\frac{e^u}{\cosh u} - \frac{e^u \sinh u}{\cosh^2 u} \right) du = (r - r \tanh u)\, du.$$

Expressing $\tanh u$ by r

$$r = \frac{e^u}{\cosh u} = \frac{2e^u}{e^u + e^{-u}} \rightarrow r - 1 = \frac{e^u - e^{-u}}{e^u + e^{-u}} = \tanh u,$$

and

$$r - 1 = \tanh u,$$

gives

$$\frac{du}{dr} = \frac{1}{r(2 - r)}.$$

- **Case 1 : "Concurrence Surface in Euclidean Space"**
  For surface of revolution in Euclidean space

$$\frac{du}{dr} = \pm \frac{\sqrt{1 + (\phi'(r))^2}}{r} = \frac{1}{r(2 - r)} > 0,$$

so that

$$\sqrt{1+(\phi'(r))^2} = \frac{1}{2-r} \geq 1,$$

and

$$|2-r| \leq 1 \quad \rightarrow r \geq 1.$$

The surface can be found for $1 \leq r \leq 2$. To find ϕ, the identification is implied

$$(1+(\phi'(r))^2)^2 = \frac{1}{(2-r)^2} \quad \rightarrow \phi' = \frac{d\phi}{dr} = \pm\frac{\sqrt{(3-r)(r-1)}}{2-r}.$$

By denoting $2-r \equiv x$ it gives

$$-\frac{d\phi}{dx} = \pm\frac{\sqrt{1-x^2}}{x}.$$

Taking integral

$$\begin{aligned}
\phi &= \mp\int\frac{\sqrt{1-x^2}}{x}dx \overset{(x=\cos y)}{=} \mp\int\frac{\sin y}{\cos y}(-\sin y)dy \\
&= \pm\left(\int\frac{1}{\cos y}dy - \int\cos y dy\right) \\
&= \pm(\ln(\sec y + \tan y) - \sin y + C) \\
&= \pm\left(\ln\left(\frac{1}{x} + \frac{\sqrt{1-x^2}}{x}\right) - \sqrt{1-x^2} + C\right)
\end{aligned}$$

leads to

$$\phi(r) = \pm\left(\ln\left(\frac{1+\sqrt{(3-r)(r-1)}}{2-r}\right) - \sqrt{(3-r)(r-1)} + C\right).$$

Then, parametric form of the surface of revolution is

$$x = r\cos v, \quad y = r\sin v, \quad z = \phi(r)$$

where

$$z(r) = z(1) \pm \left(\ln\frac{1 + \sqrt{(3-r)(r-1)}}{2-r} - \sqrt{(3-r)(r-1)}\right)$$

and

$$1 \leq r \leq 2.$$

Corresponding generating curve of revolution is

$$z(x) = z(1) \pm \left(\ln\frac{1 + \sqrt{(3-x)(x-1)}}{2-x} - \sqrt{(3-x)(x-1)}\right)$$

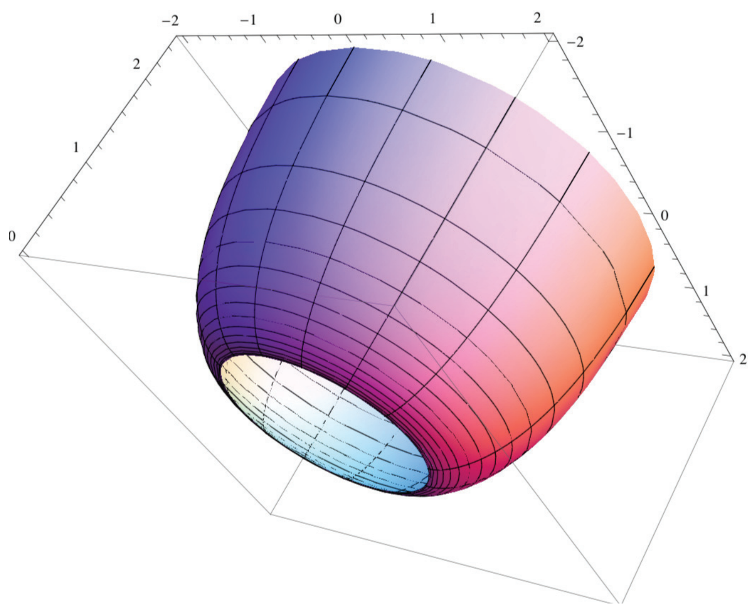


Figure 9.2. Euclidean Surface

- **Case 2: Concurrence Surface in Minkowski Space**

  To recover the surface of revolution for $0 \le r \le 1$ the Euclidean space is not suitable. This is why the Minkowski (Pseudo-Euclidean) space is used, so that

$$\frac{du}{dr} = \pm\frac{\sqrt{1-(\phi'(r))^2}}{r} = \frac{1}{r(2-r)},$$

  or

$$\sqrt{1-(\phi'(r))^2} = \frac{1}{2-r},$$

  and

$$(2-r)^2 \ge 1 \quad \rightarrow r \le 1.$$

  This surface can be found for $0 \le r \le 1$. To find ϕ, by applying the same procedure one gets parametric surface

$$x = r\cos v, \quad y = r\sin v, \quad z = \phi(r),$$

  where

$$\phi(r) = \pm(\sqrt{(3-r)(1-r)} - \arctan\sqrt{(3-r)(1-r)} + C).$$

  Corresponding generating curve of revolution is

$$z(x) = z(1) \mp (\sqrt{(3-x)(1-x)} - \arctan\sqrt{(3-x)(1-x)}),$$

  where $0 \le x \le 1$.

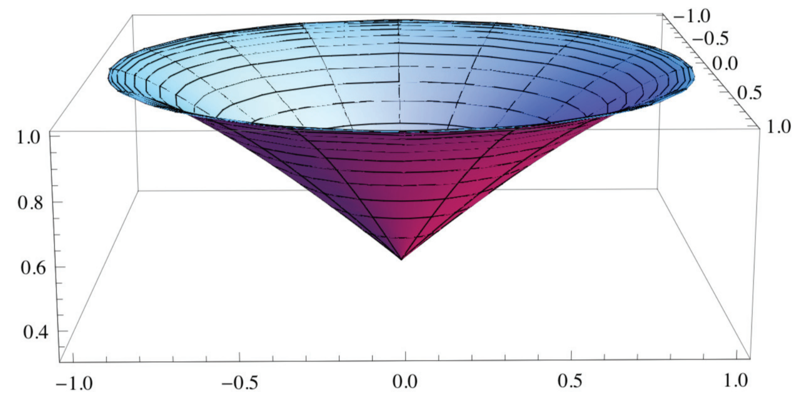


Figure 9.3. Minkowski Surface

9.6. Conformal Transformation of Coherent States

Let

$$|z\rangle = \frac{|0\rangle + z\,|1\rangle}{\sqrt{1+|z|^2}} \tag{9.31}$$

is one qubit coherent state, determined by one complex variable z. In Section 5.3, action of Möbius transformations on this state was derived. Here, more general conformal transformations acting on state (9.31) are considered.

Let $\mu = \mu(z)$ is entire function, so that

$$\frac{\partial}{\partial\bar{z}}\mu(z) = 0.$$

Then, conformal transformation $z \to \mu(z)$ implies transformation to new state

$$|\mu(z)\rangle = \frac{|0\rangle + \mu(z)\,|1\rangle}{\sqrt{1+|\mu(z)|^2}}. \tag{9.32}$$

- **Example 1 :** Hadamard gate as conformal transformation. For

$$\mu(z) = \frac{1-z}{1+z}$$

it transforms the concentric circles in complex plane to the Apollonius circles.

- **Example 2 :** Conformal transformation

$$\mu(z) = \frac{z^2 - c^2}{c^2}$$

splits the plane to the set of Cassini ovals, corresponding to state

$$|\mu(z)\rangle = \frac{c^2\,|0\rangle + (z^2 - c^2)\,|1\rangle}{\sqrt{c^4 + |z^2 - c^2|^2}}$$

like in Section 7.3 and 7.8.

9.7. Concurrence and Liouville Equation

Two qubit analogy of state (9.32) is

$$|\mu(z)\rangle = \frac{|00\rangle + \mu(z)\,|11\rangle}{\sqrt{1 + |\mu(z)|^2}}$$

and the determinant formula for it gives the concurrence

$$C = \frac{2\,|\mu(z)|}{1 + |\mu(z)|^2}.$$

This concurrence determines the conformal metric on a surface

$$dl^2 = g(z, \bar{z})dz\,d\bar{z},$$

where

$$g(z,\bar{z}) = C^2 = \frac{4\,\mu(z)\,\overline{\mu(z)}}{(1+\mu(z)\,\overline{\mu(z)})^2}.$$

The Gaussian curvature of this surface can be calculated by

$$K = -\frac{1}{2g}\Delta \ln g = -\frac{2}{g}\frac{\partial^2 \mu}{\partial z\,\partial \bar{z}} \ln g.$$

The result is

$$K(z,\bar{z}) = \frac{\mu_z(z)}{\mu(z)}\,\overline{\frac{\mu_z(z)}{\mu(z)}} = \left|\frac{\mu_z(z)}{\mu(z)}\right|^2. \tag{9.33}$$

Then, the concurrence square satisfies the variable Liouville equation

$$\Delta \ln C^2 = -2K(z,\bar{z})C^2.$$

For the stream function of the concurrence (entanglement) flow

$$\psi = \ln g = \ln C^2$$

it gives the Liouville equation for vorticity

$$\Delta\psi = -2K(z,\bar{z})\,e^{\psi}.$$

- **Example 1 :** In Apollonius case

  $$\mu(z) = \frac{z+1}{z-1},$$

  the Gaussian curvature is

  $$K(z,\bar{z}) = \frac{4}{|z^2-1|^2},$$

and the Liouville equation becomes

$$\Delta\psi = -\frac{8}{|z^2-1|^2}e^{\psi}.$$

- **Example 2:** Lines on the surface as a set of points with positive constant Gaussian curvature K, which is a constant, are Cassini curves

$$|z^2-1|^2 = \frac{4}{K} \equiv a^2$$

and the flow along these curves is a solution of the Liouville equation

$$\Delta\psi = -\frac{8}{a^2}e^{\psi}.$$

- **Example 3:** For coherent state $\mu(z) = z$, and the curvature is

$$K(z,\bar{z}) = \frac{1}{|z|^2}$$

- **Example 4:** Conformal mapping to constant curvature surface, according to (9.33) is given by

$$\frac{\mu_z(z)}{\mu(z)} = \lambda, \tag{9.34}$$

where λ is a complex constant, so that,

$$K = |\lambda|^2.$$

By integrating of (9.34)

$$\frac{d\mu}{\mu} = \lambda dz$$

it is

$$\mu(z) = e^{\lambda z}.$$

CHAPTER 10

SPIN OPERATORS AND QUBIT STATES

The Pauli gates studied in Chapter 3 are describing spin $\frac{1}{2}$ physical system. This system is simplest two level quantum system, describing qubit. In present Chapter several spin averages of qubit states are considered.

10.1. Spin $\frac{1}{2}$ and Qubit

Spin $\frac{1}{2}$ is described by spin operator

$$\vec{S} = \frac{\hbar}{2}(\sigma_x, \sigma_y, \sigma_z).$$

It is acting on computational states as

$$\begin{aligned}
S_x|0\rangle &= \frac{\hbar}{2}\sigma_x|0\rangle = \frac{\hbar}{2}|1\rangle, \quad S_x|1\rangle = \frac{\hbar}{2}\sigma_x|1\rangle = \frac{\hbar}{2}|0\rangle, \\
S_y|0\rangle &= \frac{\hbar}{2}\sigma_y|0\rangle = i\frac{\hbar}{2}|1\rangle, \quad S_y|1\rangle = \frac{\hbar}{2}\sigma_y|1\rangle = -i\frac{\hbar}{2}|0\rangle, \\
S_z|0\rangle &= \frac{\hbar}{2}\sigma_z|0\rangle = \frac{\hbar}{2}|0\rangle, \quad S_z|1\rangle = \frac{\hbar}{2}\sigma_z|1\rangle = -\frac{\hbar}{2}|1\rangle,
\end{aligned}$$

showing that $|0\rangle$ and $|1\rangle$ states are eigenstates of $S_z = \frac{\hbar}{2}\sigma_z$ with eigenvalues $\frac{\hbar}{2}$ and $-\frac{\hbar}{2}$, correspondingly. The average values are

$$\begin{aligned}
\langle 0|S_x|0\rangle &= \langle 1|S_x|1\rangle = 0, \\
\langle 0|S_y|0\rangle &= \langle 1|S_y|1\rangle = 0, \\
\langle 0|S_z|0\rangle &= \frac{\hbar}{2}, \quad \langle 1|S_z|1\rangle = -\frac{\hbar}{2}.
\end{aligned}$$

For one qubit coherent state

$$|z\rangle = \frac{|0\rangle + z\,|1\rangle}{\sqrt{1+|z|^2}},$$

then

$$\begin{aligned} S_x|z\rangle &= \frac{\hbar}{2}\frac{z\,|0\rangle + |1\rangle}{\sqrt{1+|z|^2}} = \frac{\hbar}{2}\left|\frac{1}{z}\right\rangle, \\ S_y|z\rangle &= \frac{i\hbar}{2}\frac{-z\,|0\rangle + |1\rangle}{\sqrt{1+|z|^2}} = \frac{i\hbar}{2}\left|-\frac{1}{z}\right\rangle, \\ S_z|z\rangle &= \frac{\hbar}{2}\frac{|0\rangle - z\,|1\rangle}{\sqrt{1+|z|^2}} = \frac{\hbar}{2}|z\rangle. \end{aligned}$$

Average values of spin in coherent qubit state are

$$\langle z\,|S_x|\,z\rangle = \frac{\hbar}{2}\,\frac{z+\bar z}{1+|z|^2}, \quad \langle z\,|S_y|\,z\rangle = i\frac{\hbar}{2}\,\frac{-z+\bar z}{1+|z|^2}, \quad \langle z\,|S_z|\,z\rangle = \frac{\hbar}{2}\,\frac{1-|z|^2}{1+|z|^2}.$$

In addition

$$\begin{aligned} \langle z|(S_x + iS_y)|z\rangle &= \langle z|S_+|z\rangle = \frac{\hbar}{2}\,\frac{2z}{1+|z|^2}, \\ \langle z|(S_x - iS_y)|z\rangle &= \langle z|S_-|z\rangle = \frac{\hbar}{2}\,\frac{2\bar z}{1+|z|^2}. \end{aligned}$$

These formulas have simple meaning, that the average values of spin operators

$$\langle z|\vec S|z\rangle = \frac{\hbar}{2}\vec n,$$

give a vector $\vec n$ on the unit sphere

$$\vec n^2 = n_1^2 + n_2^2 + n_3^2 = 1,$$

and complex coordinate z is just the stereographic projection of the corresponding point on the sphere,

$$(\langle z|S_x|z\rangle)^2 + (\langle z|S_y|z\rangle)^2 + (\langle z|S_z|z\rangle)^2 = \frac{\hbar^2}{4}. \tag{10.1}$$

The Shannon entropy for state $|z\rangle$,

$$S(|z|^2) = \log_2(1+|z|^2) - \frac{|z|^2}{1+|z|^2}\log_2|z|^2$$

is constant along concentric circles $|z| = r^2$. The entropy takes maximal value $S = 1$ for $|z| = 1$ circle. In this case

$$\langle z|S_z|z\rangle = 0$$

and $n_3 = 0$, that means the vector $\vec{n}$ is located on equator of the Bloch sphere. Therefore, states on equator are maximally random states. On the contrary, for $z = 0$ and $z = \infty$ states, the entropy $S = 0$ and north and south pole states are not random at all.

10.2. Spin Operators and n-qubit $|PP...P\rangle$ States

The n-qubit coherent like state

$$|z\rangle = \frac{|00...0\rangle + z\,|11...1\rangle}{\sqrt{1+|z|^2}} \tag{10.2}$$

can be generalized to the n-qubit $|PP\rangle$ state

$$|PP...P\rangle = \frac{|\psi\psi...\psi\rangle + z\left|-\frac{1}{\bar{\psi}} - \frac{1}{\bar{\psi}}... - \frac{1}{\bar{\psi}}\right\rangle}{\sqrt{1+|z|^2}}, \tag{10.3}$$

where $|\psi\rangle$ and $\left|-\frac{1}{\bar{\psi}}\right\rangle$ are one qubit antipodal coherent states. The state is normalized

$$\langle PP...P|PP...P\rangle = 1.$$

For $\psi = 0$ this state reduces to (10.2), and for $z = 1$ to states, introduced by (Pashaev and Gurkan, 2012) .

Spin operators for n- qubit state are defined by the tensor products

$$S_x = \frac{\hbar}{2}(\sigma_x \otimes I ... \otimes I + ... + \otimes I \otimes ... \otimes \sigma_x),$$
$$S_y = \frac{\hbar}{2}(\sigma_y \otimes I ... \otimes I + ... + \otimes I \otimes ... \otimes \sigma_y),$$
$$S_z = \frac{\hbar}{2}(\sigma_z \otimes I ... \otimes I + ... + \otimes I \otimes ... \otimes \sigma_z).$$

The average values of these spin operators in $|PP...P\rangle$ state (10.3) are

$$\langle PP...P|\vec{S}|PP...P\rangle \equiv \langle \vec{S}\rangle,$$

and

$$\begin{aligned} \langle S_x\rangle &= \frac{\hbar}{2}n\frac{\psi + \bar{\psi}}{\sqrt{1+|\psi|^2}}\frac{1-|z|^2}{1+|z|^2}, \\ \langle S_y\rangle &= \frac{\hbar}{2}n\frac{\bar{\psi} - \psi}{\sqrt{1+|\psi|^2}}\frac{1-|z|^2}{1+|z|^2}, \\ \langle S_z\rangle &= \frac{\hbar}{2}n\frac{1-|\psi|^2}{\sqrt{1+|\psi|^2}}\frac{1-|z|^2}{1+|z|^2}. \end{aligned}$$

They belong to the sphere

$$\langle S_x\rangle^2 + \langle S_y\rangle^2 + \langle S_z\rangle^2 = \frac{\hbar^2}{4}n^2\left(\frac{1-|z|^2}{1+|z|^2}\right)^2$$

with radius

$$R = \frac{\hbar}{2}n\sqrt{1 - C_z^2},$$

where

$$C_z = \frac{2|z|}{1+|z|^2},$$

is concurrence of coherent state $|z\rangle$. Here, following identity was used

$$1 - C_z^2 = \left(\frac{1-|z|^2}{1+|z|^2}\right)^2.$$

In two qubit $|PP\rangle$ state with $C_z = 1$, $|z| = 1$, and the radius of the sphere $R = 0$, which implies

$$\langle S_x\rangle^2 + \langle S_y\rangle^2 + \langle S_z\rangle^2 = 0$$

or

$$\langle S_x\rangle = \langle S_y\rangle = \langle S_z\rangle = 0.$$

This means that maximally entangled state is maximally random state, so that average of observables, like components of spin, are vanishing.

For separable state $C = 0$ gives radius of sphere $R = 2\frac{\hbar}{2} = \hbar \neq 0$,

$$\langle S_x\rangle^2 + \langle S_y\rangle^2 + \langle S_z\rangle^2 = \left(2\frac{\hbar}{2}\right)^2,$$

which is twice of one qubit coherent state radius in (10.1). More general than two qubit $|PP\rangle$ state

$$|PP\rangle = \frac{|\psi\psi\rangle + z\left|-\frac{1}{\bar{\psi}} - \frac{1}{\bar{\psi}}\right\rangle}{\sqrt{1+|z|^2}}$$

is the state

$$|PT\rangle = \frac{|\psi\chi\rangle + z\left|-\frac{1}{\bar{\psi}}\chi'\right\rangle}{\sqrt{1+|z|^2}}.$$

The concurrence of this state is proportional to geometric distance between complex numbers χ and χ':

$$C = \frac{2|z|}{1+|z|^2}\,\frac{|\chi - \chi'|}{\sqrt{(1+|\chi|^2)(1+|\chi'|^2}}.$$

Since a generic two qubit state is determined by 6 real or 3 complex parameters in $|PT\rangle$ state, one can choose

$$\chi' = -\frac{1}{\bar{\chi}}.$$

Then, most general form of this state is

$$|OT\rangle = \frac{|\psi\chi\rangle + z\left|-\frac{1}{\bar{\psi}} - \frac{1}{\bar{\chi}}\right\rangle}{\sqrt{1+|z|^2}}$$

with concurrence

$$C = \frac{2|z|}{1+|z|^2}.$$

CHAPTER 11

CONCLUSION

In the present thesis, the Apollonius representation for qubit states by symmetric points in complex plane and the set of Apollonius circles was proposed. By using this representation, the randomness characteristics of qubit states, as the Shannon entropy, the concurrence and fidelity, entanglement, 3-tangle and n-tangle were calculated and it was shown that the randomness is constant along Apollonius circles.

By using stereographic projection of Bloch sphere, the qubit was represented by a point in complex plane. It was shown that, unitary gates are acting on this qubit state as Möbius transformations. For two qubit states, entanglement characteristics as determinant, area, concurrence, inner product metric, reduced density matrix and Von Neumann entropy were introduced and their relations with geometrical and physical characteristics of entangled qubit states were studied.

To represent computational basis by finite points in complex plane, the Apollonius representation was introduced. In this representation entropy, fidelity, concurrence, 3-tangle and n- tangle characteristics of multiple qubit states are constant along Apollonius circles.

In addition to Apollonius representation, Cassini and bipolar representation of qubit states were derived. The Cassini representation was connected with Apollonius representation by conformal transformation.

The concurrence flow as vector field for Apollonius integral curves was introduced. By conformal metric, the concurrence surface as the surface of revolution was reconstructed in Euclidean and Minkowski spaces. Conformal transformation of coherent states and Liouville equation for concurrence metric C^2 were obtained.

Average values of observables in coherent and $|PP...P\rangle$ states were calculated and it was shown that for maximally entangled, as maximally random states, they are vanishing.

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