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ON PSEUDO SEMISIMPLE RINGS

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A necessary and sufficient condition is obtained for a right pseudo semisimple ring to be left pseudo semisimple. It is proved that a right pseudo semisimple ring is an internal exchange ring. It is also proved that a right and left pseudo semisimple ring is an SSP ring.

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1. Introduction

Throughout, R will denote an associative ring with unity, and J will denote the Jacobson radical of R. S and Z will stand for the right socle and the right singular ideal of R, respectively. The left socle of R will be denoted by S' and the left singular ideal will be denoted by Z'. R is local if R/J is a division ring. For the purposes of the paper, $J \neq 0$ will be assumed throughout for a local ring. R is *regular* (in the sense of Von Neumann) if for every $a \in R$, there exists $x \in R$ such that $axa = a$.

A right ideal P of R is called *right pseudo maximal* if P is maximal in the set of right ideals which are not isomorphic to R_R (left pseudo maximal ideals are defined symmetrically). If I is an ideal of R, then I is a maximal right ideal if and only if R/I is a division ring if and only if I is a maximal left ideal. For a subset X of R, ${}^0X$ and X^0 will stand for the left and right annihilator of X, respectively. For an element $x \in R$ and a right ideal L of R, the set $\{r \in R : xr \in L\}$ will be denoted by $(L : x)$.

Let M be a right module. For submodules X and Y of M, $X \leq Y$ $(X < Y)$ will mean that X is a submodule (proper submodule) of Y. By a summand of M we will always mean a direct summand. The notation $N \leq^{\oplus} M$ will indicate that N is a summand of M.

M is said to have the *n-exchange property* if whenever $M \leq^{\oplus} A = \bigoplus_{i=1}^{n} A_i$, then $A = (\bigoplus_{i=1}^{n} A_i') \oplus M$ with $A_i' \leq A_i$. M has the *finite exchange property* if M has the n-exchange property for every positive integer n. A decomposition $M = \bigoplus_{i=1}^{n} M_i$ is *exchangeable* if for any summand N of M, $M = \bigoplus_{i=1}^{n} M_i' \oplus N$ with $M_i' \leq M_i$ (this generalizes the notion of decompositions that complement direct summands, see [2]). If every finite decomposition of M is exchangeable, then M is said to have the *finite internal exchange property*. Clearly the exchange property implies the internal exchange property. Also the 2-exchange (internal exchange) property implies the finite exchange (internal exchange) property (see, [8, Proposition 16; 9, Proposition 1.11]). A ring R is said to be an *exchange* (*internal exchange*) *ring* if R_R, equivalently ${}_RR$ has the exchange (internal exchange) property.

A ring R is *right hereditary* (*respectively*, *right PP*) if every right ideal (respectively, cyclic right ideal) is projective (see, [10]).

The split extension of a ring R by an R-R bimodule M, denoted by $R \rtimes M$, is the ring of all matrices of the form $\begin{pmatrix} r & m \\ 0 & r \end{pmatrix}$, with $r \in R$ and $m \in M$.

A ring R is called *right pseudo semisimple* if any right ideal of R is either semisimple or isomorphic to R_R. Trivial examples of such rings are semisimple rings ($S = R$) or principal right ideal domains ($S = 0$). So it is only interesting to study pseudo semisimple rings in which $0 < S < R$. In this paper the term right (respectively, left) pseudo semisimple ring will mean one in which $0 < S < R$ (respectively, $0 < S' < R$). A number of examples of such rings is given in [7]. Yet an example of a right pseudo semisimple ring with $S^2 \neq 0$ and S is not a maximal right ideal, is not known.

In this paper we discuss conditions for a right pseudo semisimple ring with $S^2 \neq 0$ to be left pseudo semisimple (Theorems 3.8 and 3.10). In addition we investigate the relation of pseudo semisimple rings with other classes of rings such as SSP rings, and (internal) exchange rings.

2. Preliminaries

For the reader's convenience, we state here [5, Proposition 2.1] as it includes most of the basic properties of nontrivial right pseudo semisimple rings.

Proposition 2.1. *The following hold in a right pseudo semisimple ring R.*

(1) *If $R = A \oplus B$ for nonzero right ideals A and B of R, then exactly one of them is semisimple and the other one is isomorphic to R; in particular none of them is an ideal, and so any nontrivial idempotent of R is not central.*
(2) *S is the smallest essential right ideal of R and is right pseudo maximal.*
(3) $^0S = Z \leq S \cap J$.
(4) *$S =^0 x$ for every $0 \neq x \in J$; in particular if $J \neq 0$, then $S =^0 J$.*
(5) *$Z \leq A$ for any right ideal A not contained in S.*
(6) *If $b^0 = 0$, then $(Z : b) = Z$.*
(7) *If a is not in S, then $(S : a) = S$ and $aZ = Z$.*
(8) *R/S is a principal right ideal domain.*
(9) *$SZ = 0$ and Z is torsion free divisible as a left R/S module.*

Let g be an idempotent in the right socle S of an arbitrary ring R. It is known that $(1-g)R \cong R$ if and only if $R \oplus gR \cong R$ if and only if there exist t and t^* in R such that $t^*t = 1$ and $tt^* = 1 - g$. We call t a *shift* for g. We say R *has enough shifts* if for every isomorphism type of indecomposable idempotents in S there is a representative f which has a shift.

Corollary 2.2. *Let R be a right pseudo semisimple ring. If e is an idempotent in S, then $(1-e)R \cong R_R$ and $R(1-e) \cong_R R$.*

Proof. $(1-e)R \cong R_R$ follows by Proposition 2.1(1). Then $R(1-e) \cong {}_RR$ by [4, p. 63]. □

Corollary 2.3. *Assume that R has enough shifts, and let $R = A \oplus B$ for some left ideals A and B. If $A \leq S$, then $B \cong {}_RR$.*

We also record here the following proposition for easy reference.

Proposition 2.4 ([5, Proposition 2.2]). *A ring R is right pseudo semisimple if and only if S is right pseudo maximal and R has enough shifts.*

By [5, Lemma 2.6] and its right–left symmetry, a right (left) pseudo semisimple ring, has $S' \leq S$ ($S \leq S'$). This fact will be used frequently in this paper without any further reference.

Lemma 2.5. *For a right pseudo semisimple ring R, we have*:

(1) *If R/S is a division ring, then $J \leq S'$,*
(2) *Either $J \cap S' = 0$ or R/S is a division ring and $0 < J \leq S' \leq S$.*

Proof. (1) For a nonzero $x \in J$, we have $^0x = S$ by Proposition 2.1(4). Hence Rx is a minimal left ideal of R. This implies that $x \in S'$.

(2) Assume $J \cap S' \neq 0$ and consider a minimal left ideal $Rx \leq J$. Then ${}^0x$ is a maximal left ideal. As ${}^0x = S$, we have R/S is a division ring. The result now follows by (1). □

Proposition 2.6. *Let R be a right pseudo semisimple ring with $S^2 = 0$. Then either $S' = 0$ or $S' = J = S$ and R is a local ring with $J^2 = 0$.*

Proof. As $S^2 = 0$, we get $S \leq J$ and so $S' \leq J$. Hence $J \cap S' = S'$. It then follows by Lemma 2.5 that $S' = 0$ or S is a maximal right ideal and $0 < J \leq S' \leq S$. As $S \leq J$, we get $S' = J = S$. Then R/J is a division ring, and so R is local. Also $J^2 = SJ = 0$. □

Remark 2.7. A local ring with $J^2 = 0$ is left and right pseudo semisimple. We record here [5, Example 2.8], which is an example of a right pseudo semisimple ring R with $S' = 0$. Such a ring cannot be left pseudo semisimple (see, Proposition 3.5). Also R is an example of a ring having S as a right pseudo maximal ideal which is not a maximal right ideal.

Example 2.8. Let $A = F[X]$ be the ring polynomials over a field F and $M = F(X)$, the quotient field of A. For $m \in M$ and $r = a_0 + a_1X + \cdots + a_nX^n \in A$, we define $r \cdot m$ as the natural multiplication in M and $m \cdot r := ma_0$. This makes M an A-bimodule. Define $R = A \ltimes M$. One can check that

$$S = J = \begin{pmatrix} 0 & M \\ 0 & 0 \end{pmatrix}, \quad S^0 = \begin{pmatrix} XA & M \\ 0 & XA \end{pmatrix}, \quad S' = 0.$$

Clearly $S^2 = 0$, $R/S \cong A$ and S is torsion-free divisible as a left R/S-module. Then by [7, Theorem 1.7], R is right pseudo semisimple.

As $S' = 0 \neq S$, R is not left pseudo semisimple. Also $S < S^0 < R$, and so S is not a maximal right ideal.

3. Right–Left Pseudo Semisimple Rings

A module M has (C_2) if whenever a submodule N of M is isomorphic to a summand of M, then $N \leq^{\oplus} M$. (C_2) implies the weaker condition (C_3): If X and Y are summands of M with $X \cap Y = 0$, then $X \oplus Y \leq^{\oplus} M$ (see [6]). Clearly every regular ring has (C_2). A stronger version of (C_3) states that if X and Y are summands of M, then $X + Y \leq^{\oplus} M$. This condition is called SSP (see [1, 3, 11]). The relation between (C_2) and SSP is not obvious.

We say that a module M is (C_4) if every submodule of M that contains an isomorphic copy of M, is itself isomorphic to M. A ring R is right (respectively, left) (C_4) if R_R (respectively, ${}_RR$) is (C_4). The condition (C_4) is not left–right symmetric for a ring R. For an example consider a principal right ideal domain which is not left principal ideal ring.

Lemma 3.1. *A ring R with S maximal right ideal is an exchange ring.*

Proof. Let $R = A + B$ for right ideals A and B of R. As S is a maximal right ideal we may assume $A \not\leq S$. Let C be a complement of A. Then maximality of S implies $R = A \oplus C$. Hence $R = A + B$ with $A \leq^{\oplus} R$. The result now follows by [9, Proposition 2.9]. □

Theorem 3.2. *A right pseudo semisimple ring R is an internal exchange ring.*

Proof. We only need to show that R_R has the 2-internal exchange property (see [8, Proposition 16]). Let $R = A \oplus B$, for right ideals A and B, and let C be a summand of R. By Proposition 2.1(1), we may assume that B is semisimple. Hence $B = (A + C) \cap B \oplus B'$, for some $B' \leq B$, and therefore $R = (A + C) \oplus B'$.

Let $f : A \oplus B \to B$ be the natural projection, and let f' denote the restriction of f to C. Again B is semisimple implies that $f'(C)$ is a summand of B, hence projective. It follows that $C = \operatorname{Ker} f' \oplus D$ with $D \cong f'(C)$. Hence $A \cap C = \operatorname{Ker} f' \leq^{\oplus} C \leq^{\oplus} R$. Therefore $A = A' \oplus A \cap C$ for some $A' \leq A$. Hence $A + C = A' \oplus C$. Consequently, we obtain that $R = A' \oplus C \oplus B'$. □

Theorem 3.3. *A right pseudo semisimple ring R with $Z = J$ has SSP.*

Proof. Let A and B be summands of R. We consider two cases.

(i) $B \leq S$: Then $B = \bigoplus_{i=1}^{n} B_i$ where B_i is a minimal right ideal and $B_i \leq^{\oplus} R$. Using induction we may assume that B is minimal. If $B \leq A$, we have nothing to prove. So assume that $B \not\leq A$. Then $A + B = A \oplus B$. Also $R = A \oplus C$ for some right ideal C of R and so $A \oplus B = A \oplus X$ with $(A \oplus B) \cap C = X \cong B$. This implies $X \cap Z = 0$ and consequently $X \cap J = 0$. It follows that $X^2 \neq 0$ and so $X \leq^{\oplus} R$. As $X \leq C$, we get $A + B = A \oplus X \leq^{\oplus} R$.

(ii) $B \not\leq S$: Write $R = B \oplus D$. Then $D \leq S$ by Proposition 2.1(1). Now

$$A + B = B \oplus (A + B) \cap D.$$

As D is semisimple, $(A + B) \cap D \leq^{\oplus} D$. Hence $A + B \leq^{\oplus} R$. □

Remark 3.4. The above theorem shows that R_R has SSP. By [11, Theorem 2.4], ${}_RR$ also has SSP.

Proposition 3.5. *Let R be a right and left pseudo semisimple ring. Then the following hold*:

(1) $S' = S$,
(2) $Z = J = Z'$,
(3) R/S *is a division ring or* $J = 0$.

Proof. (1) is obvious by [5, Lemma 2.6], and its left–right symmetry.
(2) $JS' = 0$ and so $JS = 0$. Hence $J \leq Z$. However $Z \leq J$, hence $Z = J$. Similarly $Z' = J$.
(3) Follows by Lemma 2.5. □

$Z = J$ is a necessary condition for a right pseudo semisimple ring to be left pseudo semisimple. In [5, Example 3.3], R is a right pseudo semisimple ring with $Z = 0$ and $J \neq 0$, so R is not left pseudo semisimple.

Corollary 3.6. *A right and left pseudo semisimple ring is an* SSP *ring.*

Proof. The result follows by Theorem 3.3 and Proposition 3.5. □

In a right and left pseudo semisimple ring either S is a maximal right ideal or $J = 0$. The result [7, Corollary 2.3] deals with the case S maximal right ideal and $J = 0$. In the following we will separate the two cases. First we note that this corollary may be rephrased as follows.

Theorem 3.7. *The following are equivalent for a ring R with $0 < S < R$.*

(1) *R is right pseudo semisimple and regular,*
(2) *R is semiprime, right and left pseudo semisimple with R/S a division ring,*
(3) *R is left pseudo semisimple and regular.*

Note that in a right pseudo semisimple ring, $J = 0$ if and only if R is semiprime. Indeed, R is semiprime implies $S^2 \neq 0$, and hence $J < S$ by [5, Lemma 2.4]. So $J^2 \leq SJ = 0$, and consequently $J = 0$.

We generalize Theorem 3.7 by dropping the semiprimeness condition in (2) and replacing regularity by the weaker condition (C_2).

Theorem 3.8. *The following are equivalent for a ring R with $0 < S < R$.*

(1) *R is right pseudo semisimple with (C_2),*
(2) *R is right and left pseudo semisimple with R/S a division ring,*
(3) *R is left pseudo semisimple with (C_2).*

Proof. (1) ⇒ (2) Clearly (C_2) implies S is a maximal right ideal, and so R/S is a division ring. Then $J \leq S'$ by Lemma 2.5. Also $S' \leq S$. Write $S = J \oplus K$. We prove that $K \leq S'$. Consider a minimal right ideal $E \leq K$. As $E \cap J = 0$, $E = eR$ for some $e^2 = e \in R$. We prove that Re is a minimal left ideal. Consider a nonzero element $re \in Re$. As $reR \cong eR$, we get by (C_2) that $reR \leq^{\oplus} R$. Hence $reRreR \neq 0$ and therefore $eRre \neq 0$. Since eRe is a division ring, $eRre = eRe$. Then

$$Re = ReRe = ReRre \leq Rre \leq Re.$$

So that Re is a minimal left ideal of R. It follows that $e \in Re \leq S'$. As S' is an ideal, we get $eR \leq S'$. This proves that $K \leq S'$. Hence $S \leq S'$ and so $S = S'$.

As R contains enough shifts, we get R is left pseudo semisimple by the left-handed version of Proposition 2.4.

(2) $\Rightarrow$ (1) Let A be a right ideal of R such that $A \cong eR$ for some $e^2 = e \in R$. Let B be a complement of A. Since S is a maximal right ideal, $A \oplus B = S$ or $A \oplus B = R$. In the second case, we have nothing to prove. In the first case, we have $A \leq S$. Since eR is semisimple $Z \cap eR = 0$. Now $A \cong eR$ implies $Z \cap A = 0$. Since $Z = J$ by Proposition 3.5, $J \cap A = 0$, and so each simple right ideal contained in A is a summand of R. Using induction, we get $A = gR$ for some $g^2 = g \in R$.

(3) $\Leftrightarrow$ (2) Follows by symmetry. □

Corollary 3.9. *If R is a right pseudo semisimple ring with (C_2) then R/J is a regular right and left pseudo semisimple ring.*

Proof. By Theorem 3.8, R is right and left pseudo semisimple with R/S a division ring. Then $S' = S$ and $Z = J$ by Proposition 3.5. If $S^2 = 0$, then R is local by Proposition 2.6. Hence R/J is a division ring. On the other hand, assume $S^2 \neq 0$. Then by the right–left symmetry of [5, Theorem 2.11], R/J is right and left pseudo semisimple with $\mathrm{Soc}(R/J) = S/J$. Thus R/J is a semiprime right and left pseudo semisimple with maximal socle. Hence R/J is regular by Theorem 3.7. □

Next we deal with the case $J = 0$.

Theorem 3.10. *The following are equivalent for a ring R with $0 < S < R$.*

(1) *R is right pseudo semisimple, left PP and left (C_4),*
(2) *R is right and left pseudo semisimple with $J = 0$,*
(3) *R is left pseudo semisimple, right PP and right (C_4).*

Proof. (2) $\Rightarrow$ (1) It is clear that any left pseudo semisimple ring is left (C_4). Thus, we only need to show that R is left PP. We have $S' = S$. Also $J = 0$ implies ${}_RS$ is projective. Now let A be a left ideal of R. If $A \leq S$, then $A \leq^{\oplus} S$, and hence projective. On the other hand $A \nleq S$ implies $A \cong {}_RR$, and hence free. (This proves that R is left hereditary.)

(1) $\Rightarrow$ (2) Consider an element $a \in R$ such that a is not in S. As R/S is a domain, by Proposition 2.1(8), ${}^0a \leq S$. Now R is left PP implies Ra is projective, and hence $R = {}^0a \oplus B$, with $B \cong Ra$. As R has enough shifts, we get by Corollary 2.3 that $Ra \cong B \cong {}_RR$. Now applying (C_4), we get $A \cong {}_RR$ for any left ideal A that is not contained in S.

Also R is left PP implies ${}^0x \leq^{\oplus} R$, for every $x \in R$, and so $Z' = 0$. As $J = Z'$ by Proposition 3.5(2), we conclude $J = 0$.

(3) $\Leftrightarrow$ (2) follows by symmetry. □

Summing up Propositions 2.4, 2.6 and Theorems 3.8, 3.10, we get the following corollary.

Corollary 3.11. *Let R be any ring with $0 < S < R$. Then R is right and left pseudo semisimple if and only if*

(1) *R is a local ring with $J^2 = 0$, or*
(2) *R has enough shifts, $S' = S$ and R/S is a division ring, or*
(3) *R has enough shifts, $J = 0$, R/S is a domain, R is hereditary with (C_4).*

One of the open problems in [5] is to find a right pseudo semisimple ring with $S^2 \neq 0$, and S is not a maximal right ideal. If such an example exists for a right and left pseudo semisimple ring, then J should be 0. So, we are asking for a ring of type (3) in Corollary 3.11 in which S is not a maximal right ideal.

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