Journal of Algebra and Its Applications
Vol. 11, No. 3 (2012) 1250046 (7 pages)

DOI: 10.1142/S0219498811005737

RINGS OVER WHICH FLAT COVERS OF SIMPLE MODULES ARE PROJECTIVE

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Received 27 May 2010
Accepted 28 June 2011

Communicated by J. Trlifaj

Let R be a ring with identity. We prove that, the flat cover of any simple right R-module is projective if and only if R is semilocal and $J(R)$ is cotorsion if and only if R is semilocal and any indecomposable flat right R-module with unique maximal submodule is projective.

Keywords: Flat cover; projective module; perfect ring.

Mathematics Subject Classification: 16E30, 16N80

1. Introduction

Throughout, R is a ring with an identity element and all modules are unital right R-modules. Let $\mathcal{F}$ be the class of all flat right R-modules. Following [9], an *$\mathcal{F}$-precover* of an R-module M is a homomorphism $\varphi : F \to M$ with $F \in \mathcal{F}$ such that for any $\varphi' : F' \to M$ with $F' \in \mathcal{F}$, there exists a homomorphism $f : F' \to F$ such that $\varphi' = \varphi f$. An $\mathcal{F}$-precover $\varphi : F \to M$ is said to be an *$\mathcal{F}$-cover* if every endomorphism f of F with $\varphi f = \varphi$ is an isomorphism. For an R-module M, an epimorphism $f : P \to M$ is said to be a *projective cover* of M if P is projective and $\operatorname{Ker} f \ll P$. In [7] Bican *et al.* proved that all modules have flat covers. In contrast, an arbitrary module may not have a projective cover, in general. The rings over which all right modules have projective covers are exactly the right perfect rings (see, [8]). In this case, it is of interest to know when the flat cover of a given module is projective. It is well known that, a ring R is right perfect if and only if flat cover of any right R-module is projective. The rings with the property that flat covers of finitely generated right R-modules are projective are characterized in [1, 2]. In [1, 2], a ring R is called *almost-perfect* (A-perfect) if every flat R-module is R-projective.

They proved that the right A-perfect rings are exactly those rings R over which flat covers of finitely generated modules are projective.

Let R be a ring and $\mathcal{S}$ be the class of all representatives of simple right R-modules (i.e. each element of $\mathcal{S}$ is isomorphic to R/I for some maximal right ideal I of R). We call R *right B-perfect* if for every flat module F and $S \in \mathcal{S}$, and homomorphisms $f : R \to S$, $h : F \to S$ there exists a homomorphism $g : F \to R$ such that $h = fg$. It is clear that any right A-perfect ring is right B-perfect.

The main objective of this paper is to introduce and give several characterizations of the rings over which flat covers of simple modules are projective. We prove that, R is right B-perfect if and only if flat covers of simple right R-modules are projective if and only if R is semiperfect and flat covers of simple modules are finitely generated (Theorem 2.4). We also prove that, R is right B-perfect if and only if R is semilocal and $J(R)$ is cotorsion if and only if R and the maximal right ideals of R are cotorsion (Theorem 2.11).

For a ring R and a right R-module M, $J(R)$ and $J(M)$ will stand for the Jacobson radical of R and the Jacobson radical of M, respectively.

2. Characterizations of Right *B*-Perfect Rings

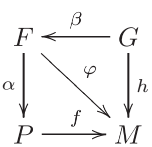
The following lemma characterizes when the flat cover of a given module is projective. The proof is easy and standard, we include it for completeness.

Lemma 2.1. *Let $\varphi : F \to M$ be a flat cover of M. Then, the following are equivalent.*

(1) *F is projective.*
(2) *There exists an epimorphism $f : P \to M$ with P projective such that the induced map $\mathrm{Hom}(F, P) \to \mathrm{Hom}(F, M)$ is surjective.*
(3) *There exists a flat precover $f : P \to M$ with P projective.*

Proof. (1) $\Rightarrow$ (2): Take $P = F$, then the proof is clear.

(2) $\Rightarrow$ (3): Let G be a flat module and $h \in \mathrm{Hom}(G, M)$. By (2) there exist $\alpha \in \mathrm{Hom}(F, P)$ such that $\varphi = f\alpha$. On the other hand, since $\varphi : F \to M$ is a flat cover, $h = \varphi\beta$ for some $\beta \in \mathrm{Hom}(G, F)$. Hence we obtain the following diagram:



In this case, from $h = \varphi\beta$ and $\varphi = f\alpha$, we get $h = f(\alpha\beta)$ with $\alpha\beta \in \mathrm{Hom}(G, P)$. This proves (3).

(3) $\Rightarrow$ (1): By [9, Theorem 1.2.7], M has a flat cover that is a direct summand of P. Therefore F is projective, because flat covers (of M) are isomorphic (see, [9, Theorem 1.2.6]). $\square$

In [1, Theorem 3.7], it is shown that a ring R is right A-perfect if and only if the flat cover of any finitely generated right R-module is projective. From Lemma 2.1 we see that, an arbitrary module has a projective flat cover if and only if it has a projective flat precover. Hence the following corollary is clear.

Corollary 2.2. *A ring R is right A-perfect if and only if any finitely generated right R-module has a projective flat precover.*

Recall that a ring R is right (or left) semiperfect if R is semilocal (i.e. $R/J(R)$ is semisimple artinian) and the idempotents of $R/J(R)$ lift to R. A ring R is semiperfect if and only if every simple right (left) R-module has a projective cover [8, 42.6]. Over a semiperfect ring any finitely generated flat right R-module is projective. As we have mentioned, a flat cover $\varphi : F \to M$ of a module M need not be a projective cover. But, in case, F is projective then $\operatorname{Ker}\varphi \ll F$ and so $\varphi : F \to M$ is a projective cover (see, [9, Theorem 1.2.12]). We shall use these facts in the sequel. Recall that, if R is a semilocal ring then $J(M) = M.J(R)$ for any right R-module M (see, [3, Corollary 15.18]).

Lemma 2.3. *Let R be a semilocal ring and X_R be a simple right R-module. If $\phi : F \to X$ is the flat cover of X, then the kernel $\operatorname{Ker}\phi = F.J(R)$. Moreover, F is indecomposable.*

Proof. See the proof of [6, Corollary 23]. □

Theorem 2.4. *For a ring R the following statements are equivalent:*

(1) *R is right B-perfect.*
(2) *Flat covers of simple modules are projective.*
(3) *R is semiperfect and flat covers of simple modules are cyclic.*
(4) *R is semiperfect and flat covers of simple modules are local.*
(5) *Flat covers of finitely generated semisimple modules are projective.*
(6) *R is semiperfect and flat covers of finitely generated semisimple modules are finitely generated.*

Proof. (1) $\Rightarrow$ (2) This is clear from the definition of B-perfect rings.

(2) $\Rightarrow$ (3) Let $f : F \to X$ be the flat cover of the simple right R-module X. By (2) F is projective and so $\operatorname{Ker} f \ll F$. Now we have $F/\operatorname{Ker} f \simeq X$ is cyclic and $\operatorname{Ker} f \ll F$. Hence F is cyclic. On the other hand $f : F \to X$ is a projective cover of the simple R-module X. Therefore R is semiperfect.

(3) $\Rightarrow$ (4) Let $f : F \to X$ be the flat cover of the simple right R-module X. We only need to prove that F is local. By Lemma 2.3 $\operatorname{Ker} f = F.J$ where $J = J(R)$. Since $F/F.J \simeq X$ is simple and $F.J \subseteq J(F)$, we get $F.J$ is the unique maximal submodule of F. On the other hand F is finitely generated. Therefore $F.J$ is the largest submodule of F, and so F is a local module by [8, p. 351].

(4) ⇒ (5) Finitely generated flat modules are projective over semiperfect rings. Let $M = \bigoplus_{i=1}^{n} S_i$ be a semisimple module and $f_i : F_i \to S_i$ be the flat cover of S_i for each $i = 1, \ldots, n$. Then $\bigoplus_{i=1}^{n} f_i : \bigoplus_{i=1}^{n} F_i \to \bigoplus_{i=1}^{n} S_i$ is a flat cover of $\bigoplus_{i=1}^{n} S_i$ by [9, Theorem 1.2.10]. By hypothesis F_i is projective for each $i = i, \ldots, n$, hence $\bigoplus_{i=1}^{n} F_i$ is projective.

(5) ⇒ (6) Flat cover of any simple module is projective by (5), hence R is semiperfect. Now, if M is a finitely generated semisimple module and $\phi : F \to M$ is a flat cover, then F is projective by (5). Then $\operatorname{Ker}\phi \ll F$ and this implies that F is finitely generated as $F/\operatorname{Ker}\phi \simeq M$ is finitely generated.

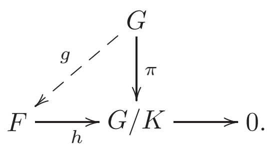
(6) ⇒ (1) Let X be a simple module and $f : R \to X$ be an epimorphism. Let G be a flat module and $g : G \to X$ be a homomorphism. If $\phi : F \to X$ is a flat cover, by (6) F is finitely generated. Then F is projective as R is semiperfect. Therefore there is a homomorphism $h : F \to R$ such that $f = \phi h$. Then by Lemma 2.1 there is a homomorphism $t : G \to R$ such that $g = ft$. Hence R is B-perfect. □

As we have mentioned, the class of A-perfect rings is contained in the class of B-perfect rings. An example of a B-perfect ring which is not A-perfect is constructed in [1] as follows.

Example 2.5 ([1, Examples 2.17, 2.22]). Let K be a field and $S = K[y_1, y_2, \ldots]$ be the polynomial ring in indeterminates $y_1, y_2, \ldots$ over K. Let L be the ideal of S generated by $\{y_i y_j \mid i, j = 1, 2, \ldots\}$. Consider the ring $R = S/L$. Then the ring $R[[x]]$ is not A-perfect by [1, Example 2.17]. On the other hand, flat cover of any simple $R[[x]]$-module is projective by [1, Example 2.22]. Hence $R[[x]]$ is a B-perfect ring by Theorem 2.4(1) ⇔ (2).

Theorem 2.6. *Let R be any ring. Then R is right B-perfect if and only if R is semilocal and any indecomposable flat right R-module with a unique maximal submodule is projective.*

Proof. Necessity: Suppose R is right B-perfect. Then R is semiperfect by Theorem 2.4. Let G be an indecomposable flat module with a unique maximal submodule K. Let $h : F \to G/K$ be the flat cover of the simple module G/K. Then F is projective by Theorem 2.4, and so $\operatorname{Ker} h \ll F$. If $\pi : G \to G/K$ is an epimorphism, then, by definition of flat cover, there exists $g : G \to F$ such that the following diagram is commutative.



From the proof of [9, Theorem 1.2.12], g is an epimorphism, and so the map $g : G \to F$ splits. But G is indecomposable, so that g must be an isomorphism. Hence G is projective.

Sufficiency: Let X_R be a simple module and $f: F \to X$ be the flat cover of X. Then $\operatorname{Ker} f = F.J$ is the unique maximal submodule of F. Moreover F is indecomposable by [5, Theorem 15]. So that F is projective by the hypothesis. Hence R is right B-perfect by Theorem 2.4. □

Definition 2.7. A right R-module C is said to be *cotorsion* if $\operatorname{Ext}^1_R(F, K) = 0$ for any flat right R-module F.

Lemma 2.8 ([9, Lemma 2.1.1]). *Let $f: F \to M$ be a flat cover of the R-module M and $K = \operatorname{Ker}(f)$. Then for any flat R-module G, $\operatorname{Ext}^1_R(G, K) = 0$, i.e. K is cotorsion.*

Lemma 2.9 ([9, Proposition 3.3.3]). *Let I be an ideal of R with $I \neq R$. If C is cotorsion as an R/I-module, then it is cotorsion as an R-module.*

The following lemma is an easy consequence of Lemma 2.9.

Lemma 2.10. *Let R be a semilocal ring. Then any semisimple R-module is cotorsion.*

Proof. Suppose M is a semisimple module. Then M is an $R/J(R)$-module as $M.J(R) = 0$. Since R is semilocal, $R/J(R)$ is a semisimple ring. So that every $R/J(R)$-module is cotorsion. In particular M is a cotorsion $R/J(R)$-module. Therefore M is a cotorsion R-module by Lemma 2.9. □

It is well known that, the right perfect rings are exactly those rings R over which every flat right R-module is projective. This implies that, R is right perfect if and only if every right R-module is cotorsion. In [1, Theorem 2.13], the authors prove that a ring R is right A-perfect if and only if any right ideal of R is cotorsion.

We have the following corresponding result for right B-perfect rings.

Theorem 2.11. *For a ring R the following statements are equivalent*:

(1) *R is right B-perfect.*
(2) *every right ideal of R containing $J(R)$ is cotorsion.*
(3) *R and the maximal right ideals of R are cotorsion.*
(4) *R is semilocal and $J(R)$ is cotorsion.*

Proof. (1) $\Rightarrow$ (2) By Theorem 2.4, R is semiperfect and flat covers of finitely generated semisimple modules are projective. Therefore flat covers and projective covers of finitely generated semisimple modules coincide. Since $J(R) \ll R$, the usual map $\pi: R \to R/J(R)$ is a projective (flat) cover. Then $\operatorname{Ker} \pi = J(R)$ is cotorsion by Lemma 2.8. Let I be a right ideal containing $J(R)$. Since R is semilocal and $I/J(R)$ is semisimple and finitely generated, $I/J(R)$ is a cotorsion R-module by Lemma 2.10. Since cotorsion modules are closed under extension and $J(R)$, $I/J(R)$ are cotorsion, I is cotorsion.

(2) $\Rightarrow$ (3) clear.

(3) $\Rightarrow$ (1) Let F be a flat module and I_R be a maximal right ideal of R. By hypothesis $\mathrm{Ext}^1_R(F, I) = 0$. Therefore by applying the Hom functor to the exact sequence

$$0 \to I \to R \to R/I \to 0$$

we get the epimorphism

$$\mathrm{Hom}(F, R) \to \mathrm{Hom}(F, R/I) \to 0.$$

This implies that R is B-perfect.

(4) $\Rightarrow$ (2) Let I be a right ideal containing $J(R)$. Then Since R is semilocal, $R/J(R)$ is semisimple, and so $I/J(R)$ is finitely generated and semisimple. Then $I/J(R)$ is a cotorsion R-module by Lemma 2.10. By hypothesis $J(R)$ is also cotorsion. Hence I is cotorsion, because cotorsion modules are closed under extension.

(1) $\Rightarrow$ (4) By Theorem 2.4, R is semiperfect, and so it is semilocal. By repeating the arguments in the proof of (1) $\Rightarrow$ (2) we obtain that $J(R)$ is cotorsion. □

Example 2.12. Let $\mathbb{Z}$ be the ring of integers and p be a prime integer. Consider the ring $R = \{\frac{a}{b} \mid a\, b \in \mathbb{Z}, (b, p) = 1\}$. Then R is a local ring with the unique maximal ideal pR. The flat cover F of R/pR is isomorphic to the set of p-adic integers (see, [9, Theorem 1.3.8] and the example after [9, Theorem 1.3.8]). It is known that F is not a projective R-module. Hence R is not B-perfect. On the other hand, since R is a local ring it is semiperfect.

Actually, we have the following for local rings.

Proposition 2.13. *Let R be a local ring. Then R is B-perfect if and only if the usual epimorphism $\pi : R \to R/J(R)$ is a flat cover of $R/J(R)$.*

Proof. Suppose R is B-perfect. Then $J(R)$ is cotorsion by Theorem 2.11. So that $\pi : R \to R/J(R)$ is a flat precover by [9, Proposition 2.1.3]. Let $F \to R/J(R)$ be a flat cover of $R/J(R)$. Then F is a direct summand of R by [9, Theorem 1.2.7]. But R is indecomposable, so that $R \simeq F$. Hence $\pi : R \to R/J(R)$ is a flat cover of $R/J(R)$.

Conversely, suppose $\pi : R \to R/J(R)$ is flat cover. Then $\mathrm{Ker}\, \pi = J(R)$ is cotorsion, hence R is B-perfect by Theorem 2.11. □

Acknowledgment

The author is grateful to the referee for the valuable suggestions and comments which improved the presentation of the paper.

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