DOI 10.1007/s11253-017-1411-x
Ukrainian Mathematical Journal, Vol. 69, No. 7, December, 2017 (Ukrainian Original Vol. 69, No. 7, July, 2017)

CO-COATOMICALLY SUPPLEMENTED MODULES

R. Alizade[1] **and S. Güngör**[2] UDC 512.5

It is shown that if a submodule N of M is co-coatomically supplemented and M/N has no maximal submodule, then M is a co-coatomically supplemented module. If a module M is co-coatomically supplemented, then every finitely M-generated module is a co-coatomically supplemented module. Every left R-module is co-coatomically supplemented if and only if the ring R is left perfect. Over a discrete valuation ring, a module M is co-coatomically supplemented if and only if the basic submodule of M is coatomic. Over a nonlocal Dedekind domain, if the torsion part $T(M)$ of a reduced module M has a weak supplement in M, then M is co-coatomically supplemented if and only if $M/T(M)$ is divisible and $T_P(M)$ is bounded for each maximal ideal P. Over a nonlocal Dedekind domain, if a reduced module M is co-coatomically amply supplemented, then $M/T(M)$ is divisible and $T_P(M)$ is bounded for each maximal ideal P. Conversely, if $M/T(M)$ is divisible and $T_P(M)$ is bounded for each maximal ideal P, then M is a co-coatomically supplemented module.

1. Introduction

Throughout the paper, R denotes an associative ring with identity and all modules are *left* unitary R-modules ($_RM$), unless otherwise stated. Let U be a submodule of M. A submodule V of M is called a *supplement* of U in M if V is a minimal element in the set of submodules $L \leq M$ with $U + L = M$. The submodule V is a supplement of U in M if and only if $U + V = M$ and $U \cap V \ll V$. A module M is called *supplemented* if every submodule of M has a supplement in M (see [9], Section 41, or [5], Chapter 4). Semisimple, artinian, and hollow (in particular local) modules are supplemented. A module M is called *coatomic* if every proper submodule of M is contained in a maximal submodule (see [12]).

Let N be a submodule of a module M. We say that N is a *co-coatomic* submodule in M if M/N is coatomic. Semisimple, finitely generated, and local modules are coatomic modules. Since every factor module of a coatomic module is coatomic, every submodule of semisimple finitely generated and local modules is co-coatomic. A module M is said to be a *co-coatomically supplemented* module if every co-coatomic submodule of M has a supplement in M. A submodule N of M is called *cofinite* if M/N is finitely generated. M is called a *cofinitely supplemented* module if every cofinite submodule of M has a supplement in M (see [1]). Clearly, a co-coatomically supplemented module is cofinitely supplemented and a coatomic module is co-coatomically supplemented if and only if it is a supplemented module. A module M is called *co-coatomically weak supplemented* if every co-coatomic submodule N of M has a weak supplement in M, i.e., $N + K = M$ and $N \cap K \ll M$ for some submodule K of M. It is clear that a co-coatomically supplemented module is co-coatomically weak supplemented. A submodule U of an R-module M has *ample supplements* in M if, for every submodule V of M with $U + V = M$, there exists a supplement V' of U with $V' \leq V$ (see [5, p. 237]). A module M is called *co-coatomically amply supplemented* if every co-coatomic submodule of M has ample supplements in M. Clearly, a co-coatomically amply supplemented module is co-coatomically supplemented.

[1] Yaşar University, Izmir, Turkey.
[2] Izmir Institute of Technology, Izmir, Turkey.

Published in Ukrains'kyi Matematychnyi Zhurnal, Vol. 69, No. 7, pp. 867–876, July, 2017. Original article submitted April 30, 2012, revision submitted April 3, 2017.

In Section 2, we show that if a submodule N of M is co-coatomically supplemented and M/N has no maximal submodule, then M is co-coatomically supplemented. Every left R-module is co-coatomically supplemented if and only if the ring R is left perfect.

In Section 3, we study co-coatomically supplemented modules over a discrete valuation ring. It is shown that a module M is co-coatomically supplemented if and only if the basic submodule of M is coatomic if and only if $M = T(M) \oplus X$, where the reduced part of $T(M)$ is bounded and $X/\operatorname{Rad}(X)$ is finitely generated.

In Section 4, we study co-coatomically supplemented modules over nonlocal Dedekind domains. A torsion module M is co-coatomically weak supplemented if and only if it is co-coatomically supplemented. We show that, for a reduced module M, if the torsion part $T(M)$ of M has a weak supplement in M, then M is co-coatomically supplemented if and only if $M/T(M)$ is divisible and $T_P(M)$ is bounded for each maximal ideal P. For a reduced module M, if M is co-coatomically amply supplemented, then $M/T(M)$ is divisible and $T_P(M)$ is bounded for each maximal ideal P of R. Conversely, if $M/T(M)$ is divisible and $T_P(M)$ is bounded for each maximal ideal P of R, then M is a co-coatomically supplemented module.

2. Co-Coatomically Supplemented Modules

For any module M, $\operatorname{Soc}(M)$ denotes the socle of M and $\operatorname{Rad}(M)$ denotes the radical of M. The Jacobson radical of ${}_RR$ is denoted by $\operatorname{Jac}(R)$.

Let $\{M_\lambda\}_{\lambda\in\Lambda}$ be the family of simple submodules of M that are direct summands of M. By $\operatorname{Soc}^{\oplus}(M)$ we denote the sum of M_λs for all $\lambda \in \Lambda$, i.e.,

$$\operatorname{Soc}^{\oplus}(M) = \sum_{\lambda\in\Lambda} M_\lambda.$$

Clearly,

$$\operatorname{Soc}^{\oplus}(M) \leq \operatorname{Soc}(M).$$

Theorem 2.1. *Let R be a ring. The following assertions are equivalent for an R-module M:*

1. *Every co-coatomic submodule of M is a direct summand of M.*
2. *Every cofinite submodule of M is a direct summand of M.*
3. *Every maximal submodule of M is a direct summand of M.*
4. *$M/\operatorname{Soc}^{\oplus}(M)$ does not contain a maximal submodule.*
5. *$M/\operatorname{Soc}(M)$ does not contain a maximal submodule.*

Proof. (1) $\Rightarrow$ (2) is clear since every cofinite submodule is co-coatomic.

(2) $\Rightarrow$ (3). Clear.

(3) $\Rightarrow$ (4). Suppose that $M/\operatorname{Soc}^{\oplus}(M)$ contains a maximal submodule $K/\operatorname{Soc}^{\oplus}(M)$. Thus, K is a maximal submodule of M. By the hypothesis, $M = K \oplus K'$ and K' is simple. Hence, we get

$$K' \leq \operatorname{Soc}^{\oplus}(M) \leq K.$$

A contradiction.

(4) $\Rightarrow$ (5). This is clear because $\operatorname{Soc}^{\oplus}(M) \leq \operatorname{Soc}(M)$.

(5) $\Rightarrow$ (1). Let N be a co-coatomic submodule of M. Since

$$M/(N + \operatorname{Soc}(M)) \cong (M/N)/\big((N + \operatorname{Soc}(M))/N\big)$$

and M/N is coatomic, we conclude that $M/(N+\operatorname{Soc}(M))$ is also coatomic. Since $M/\operatorname{Soc}(M)$ has no maximal submodule, $M/(N + \operatorname{Soc}(M))$ also has no maximal submodule. Therefore, $M = N + \operatorname{Soc}(M)$. It follows that $M = N \oplus N'$ for any submodule N' such that

$$\operatorname{Soc}(M) = \big(N \cap \operatorname{Soc}(M)\big) \oplus N'.$$

A supplemented module is co-coatomically supplemented but co-coatomically supplemented modules need not be supplemented as shown in the following example:

Example 2.1. The $\mathbb{Z}$-module $\mathbb{Q}$ is co-coatomically supplemented since the only co-coatomic submodule is $\mathbb{Q}$ itself. At the same time, the $\mathbb{Z}$-module $\mathbb{Q}$ is not supplemented because $\mathbb{Q}$ is not torsion (see [10], Theorem 3.1).

Proposition 2.1. *Let M be a semilocal module with small radical* $\operatorname{Rad}(M)$. *Then M is co-coatomically supplemented if and only if M is supplemented.*

Proof. Let N be a submodule of M. Since M is semilocal, $M/\operatorname{Rad}(M)$ is semisimple, i.e., coatomic. Consider the following statement:

$$M/\big(N + \operatorname{Rad}(M)\big) \cong \big(M/\operatorname{Rad}(M)\big)/\big(\big(N + \operatorname{Rad}(M)\big)/\operatorname{Rad}(M)\big).$$

Since $M/\operatorname{Rad}(M)$ is coatomic, $M/\big(N+\operatorname{Rad}(M)\big)$ is also coatomic. Therefore, $N+\operatorname{Rad}(M)$ has a supplement in M, say, K. Then

$$M = N + \operatorname{Rad}(M) + K \quad \text{and} \quad \big(N + \operatorname{Rad}(M)\big) \cap K \ll K.$$

Since $\operatorname{Rad}(M) \ll M$, we conclude that $M = N + K$ and

$$N \cap K \leq \big(N + \operatorname{Rad}(M)\big) \cap K \ll K.$$

Thus, M is supplemented.

A co-coatomically supplemented module is cofinitely supplemented but the example presented in what follows shows that a cofinitely supplemented module is not necessarily co-coatomically supplemented.

A ring R is called semiperfect if $R/\operatorname{Jac}(R)$ is semisimple and the idempotents in $R/\operatorname{Jac}(R)$ can be lifted to R (see [9], 42.6).

A ring is called left perfect if $R/\operatorname{Jac}(R)$ is left semisimple and $\operatorname{Jac}(R)$ is right t-nilpotent (see [9], 43.9).

By ${}_RR^{(\mathbb{N})}$ we denote the direct sum of R-module R by the index set $\mathbb{N}$. Note that $\mathbb{N}$ denotes the set of all positive integers.

Any direct sum of cofinitely supplemented modules is cofinitely supplemented [1] (Corollary 2.4).

Example 2.2. Let p be a prime integer. We consider the following ring:

$$R = \mathbb{Z}_{(p)} = \left\{ \frac{a}{b} \mid a,\, b \in \mathbb{Z},\, b \neq 0,\, (b, p) = 1 \right\},$$

which is the localization of $\mathbb{Z}$ at (p). In this case, the R-module R is supplemented. Then the R-module $R^{(\mathbb{N})}$ is cofinitely supplemented by [1] (Corollary 2.4). Furthermore, R is a semiperfect ring and, therefore, $R/\operatorname{Jac}(R)$ is semisimple (see [9], 42.6). Hence, R is semilocal. However, R is not a perfect ring because its Jacobson radical is not t-nilpotent by [9] (43.9). Note that $\operatorname{Rad}\big({}_R R^{(\mathbb{N})}\big)$ is a co-coatomic submodule of ${}_R R^{(\mathbb{N})}$ but $\operatorname{Rad}\big({}_R R^{(\mathbb{N})}\big)$ does not have a supplement in ${}_R R^{(\mathbb{N})}$ because R is not a perfect ring (see [3], Theorem 1). Hence, ${}_R R^{(\mathbb{N})}$ is not co-coatomically supplemented.

Example 2.2 shows that the cofinitely supplemented modules and co-coatomically supplemented modules not necessarily coincide over semiperfect rings and discrete valuation rings.

Proposition 2.2. *A factor module of a co-coatomically supplemented module is co-coatomically supplemented.*

Proof. Let M be a co-coatomically supplemented module and let N be a submodule of M. Then any co-coatomic submodule of M/N is a submodule of the form L/N, where L is co-coatomic submodule of M. By the hypothesis, L has a supplement in M, say, K. This implies that $(K+N)/N$ is a supplement of L/N in M/N by [9] (41.1(7)).

Proposition 2.3. *Let M be a co-coatomically supplemented module. Then every co-coatomic submodule of the module $M/\operatorname{Rad}(M)$ is a direct summand.*

Proof. Any co-coatomic submodule of $M/\operatorname{Rad}(M)$ has the form $N/\operatorname{Rad}(M)$, where N is a co-coatomic submodule of M. Since M is co-coatomically supplemented, there exists a submodule K of M such that $M = N+K$ and $N\cap K \ll K$. This yields $N \cap K \leq \operatorname{Rad}(M)$. Thus,

$$M/\operatorname{Rad}(M) = \big(N/\operatorname{Rad}(M)\big) + \big((K+\operatorname{Rad}(M))/\operatorname{Rad}(M)\big),$$

$$\big(N/\operatorname{Rad}(M)\big) \cap \big((K+\operatorname{Rad}(M))/\operatorname{Rad}(M)\big) = \big(N\cap K+\operatorname{Rad}(M)\big)/\operatorname{Rad}(M) = 0.$$

Hence,

$$M/\operatorname{Rad}(M) = \big(N/\operatorname{Rad}(M)\big) \oplus \big((K+\operatorname{Rad}(M))/\operatorname{Rad}(M)\big).$$

To prove that a finite sum of co-coatomically supplemented modules is a co-coatomically supplemented module, we use the following standard lemma (see [9], 41.2):

Lemma 2.1. *Let N and L be submodules of an R-module M such that N is co-coatomic, L is co-coatomically supplemented, and $N+L$ has a supplement in M. Then N has a supplement in M.*

Proof. Let K be a supplement of $N+L$ in M. Note that

$$L/\big(L\cap(N+K)\big) \cong (N+K+L)/(N+K) = M/(N+K).$$

This module is coatomic and, therefore, there is a supplement H of $L\cap(N+K)$ in L, i.e.,

$$L = H + L\cap(N+K) \qquad \text{and} \qquad H\cap L\cap(N+K) \ll H.$$

Hence,

$$M = N+L+K = N+K+H+L\cap(N+K) = N+K+H,$$

$$N \cap (H + K) \leq H \cap (N + K) + K \cap (N + H)$$

$$\leq H \cap (N + K) + K \cap (N + L) \ll H + K.$$

Therefore, $H + K$ is a supplement of N in M.

A (direct) sum of infinitely many co-coatomically supplemented modules need not be co-coatomically supplemented by Example 2.2 but a finite sum of co-coatomically supplemented modules is always co-coatomically supplemented.

Theorem 2.2. *A finite sum of co-coatomically supplemented modules is co-coatomically supplemented.*

Proof. Clearly, it is sufficient to prove that the sum $M = M_1 + M_2$ of two co-coatomically supplemented modules M_1 and M_2 is a co-coatomically supplemented. Let U be a co-coatomic submodule of M. Then $M = M_1 + M_2 + U$. Since $M_2 + U$ is a co-coatomic submodule of M and M_1 is co-coatomically supplemented, M_2+U has a supplement in M by Lemma 2.1. Since M_2 is co-coatomically supplemented and U is co-coatomic, by Lemma 2.1, U has a supplement in M. Thus, M is co-coatomically supplemented.

Let M and N be R-modules. If there is an epimorphism $f\colon M^{(\Lambda)} \to N$ for some finite set Λ, then N is called a *finitely M-generated* module.

The following assertion is a corollary of Proposition 2.2 and Theorem 2.2:

Corollary 2.1. *If M is co-coatomically supplemented module, then any finitely M-generated module is a co-coatomically supplemented module.*

A ring R is called a left *V-ring* if every simple R-module is injective (see [9, p. 192]). A commutative ring R is a V-ring if and only if R is a von Neumann regular ring (see [9], 23.5).

Proposition 2.4. *A module M over a V-ring R is co-coatomically supplemented if and only if M is semisimple.*

Proof. $(\Leftarrow)$ Clear.

$(\Rightarrow)$ Since M is a co-coatomically supplemented module, $M/\operatorname{Soc}(M)$ has no maximal submodule by Theorem 2.1. It follows from [9] (23.1) that

$$M/\operatorname{Soc}(M) = \operatorname{Rad}\big(M/\operatorname{Soc}(M)\big) = 0$$

because R is a V-ring. Thus, M is semisimple.

Corollary 2.2. *Any direct sum of co-coatomically supplemented modules is co-coatomically supplemented over a left V-ring.*

Proof. By Proposition 2.4, co-coatomically supplemented and semisimple modules coincide over left V-rings.

Theorem 2.3. *Let N be a co-coatomically supplemented submodule of an R-module M such that M/N has no maximal submodule. Then M is a co-coatomically supplemented module.*

Proof. Let L be a submodule of M such that M/L is coatomic. Clearly, $M/(N + L)$ is also coatomic. Since M/N has no maximal submodule, $M/(N + L)$ also has no maximal submodule. Therefore, $M = N + L$. By Lemma 2.1, L has a supplement in M. Thus, M is a co-coatomically supplemented module.

The following corollary is a direct result of Theorem 2.3:

Corollary 2.3. *Let M be a module and let $M/\operatorname{Soc}(M)$ have no maximal submodule. Then M is co-coatomically supplemented.*

Proposition 2.5. *Let M be a co-coatomically supplemented R-module. If M contains a maximal submodule, then M contains a local submodule.*

Proof. Let L be a maximal submodule of M. Then L is a co-coatomic submodule of M. Since M is a co-coatomically supplemented module, there exists a submodule K of M such that K is a supplement of L in M, i.e., $M = K + L$ and $K \cap L \ll K$. It follows from [9] (41.1(3)) that K is local.

A module M is called *linearly compact* if, for any family of cosets $\{x_i + M_i\}_\triangle,\ x_i \in M$, and submodules $M_i \leq M$ (with finitely cogenerated M/M_i), the intersection of any group of finitely many cosets from this family is nonempty, then the intersection of the entire family of cosets is also nonempty (see [9], 29.7(c)).

The following proposition gives a characterization of a co-coatomically supplemented module by a linearly compact submodule:

Proposition 2.6. *Let K be a linearly compact submodule of an R-module M. Then M is co-coatomically supplemented if and only if M/K is co-coatomically supplemented.*

Proof. $(\Rightarrow)$ By Proposition 2.2.

$(\Leftarrow)$ Let N be a co-coatomic submodule of M. Then $(N+K)/K$ is co-coatomic submodule of M/K because $N+K$ is co-coatomic submodule of M. Since M/K is co-coatomically supplemented, $(N+K)/K$ has a supplement in M/K. The submodule K has a supplement in every submodule L of M with $K \leq L$ because K is linearly compact (see [8], Lemma 2.3). Moreover, K is supplemented by [9] (29.8(2)) and [8] (Lemma 2.3). Therefore, N has a supplement in M by [8] (Corollary 2.7). Thus, M is co-coatomically supplemented.

Remark 2.1. A module M is called Σ-*selfprojective* if, for each index set I, the module $M^{(I)}$ is selfprojective. For an R-module M, if M is Σ-selfprojective and $U \leq \operatorname{Rad}(M)$, then the following assertion is true: U has a supplement in M and, hence, U is small in M [11] (Satz 4.1). Clearly, ${}_RR^{(\mathbb{N})}$ is Σ-selfprojective and

$$\operatorname{Rad}\big({}_RR^{(\mathbb{N})}\big) \leq \operatorname{Rad}\big({}_RR^{(\mathbb{N})}\big).$$

Therefore, if $\operatorname{Rad}\big({}_RR^{(\mathbb{N})}\big)$ has a supplement in ${}_RR^{(\mathbb{N})}$, then

$$\operatorname{Rad}\big({}_RR^{(\mathbb{N})}\big) \ll {}_RR^{(\mathbb{N})}.$$

Theorem 2.4. *Every left R-module is co-coatomically supplemented if and only if the ring R is left perfect.*

Proof. $(\Leftarrow)$ Clear.

$(\Rightarrow)$ By the hypothesis, every left R-module is co-coatomically supplemented and, hence, every left R-module is cofinitely supplemented. Then R is semiperfect by [1] (Theorem 2.13). Thus, $R/\operatorname{Jac}(R)$ is semisimple by [9] (42.6). This means that ${}_RR^{(\mathbb{N})}/\operatorname{Rad}\big({}_RR^{(\mathbb{N})}\big)$ is semisimple. Therefore, $\operatorname{Rad}\big({}_RR^{(\mathbb{N})}\big)$ is co-coatomic in ${}_RR^{(\mathbb{N})}$. By the hypothesis, $\operatorname{Rad}\big({}_RR^{(\mathbb{N})}\big)$ has a supplement in ${}_RR^{(\mathbb{N})}$. By Remark 2.1,

$$\operatorname{Rad}\big({}_RR^{(\mathbb{N})}\big) \ll {}_RR^{(\mathbb{N})}.$$

Since $R/\operatorname{Jac}(R)$ is semisimple and $\operatorname{Rad}\big({}_RR^{(\mathbb{N})}\big) \ll {}_RR^{(\mathbb{N})}$, ${}_RR$ is perfect by [9] (43.9). Thus, the ring R is left perfect.

3. Co-Coatomically Supplemented Modules Over Discrete Valuation Rings

Throughout this section R is a discrete valuation ring. An R-module M is called radical-supplemented if $\operatorname{Rad}(M)$ has a supplement in M (see [11]). A module M is radical supplemented if and only if the basic submodule of M is coatomic (see [11], Satz 3.1). A module M is coatomic if and only if M is reduced and supplemented (see [10], Lemma 2.1).

Proposition 3.1. *Let M be an R-module. Then M is a co-coatomically supplemented module if and only if the basic submodule of M is coatomic.*

Proof. $(\Rightarrow)$ $M/\operatorname{Rad}(M) = M/pM$ is semisimple and, therefore, coatomic. Since M is a co-coatomically supplemented module, pM has a supplement. Thus, M is a radical-supplemented module. Then the basic submodule of M is coatomic by [11] (Satz 3.1).

$(\Leftarrow)$ Let X be a submodule of M such that M/X is coatomic and let B be the basic submodule of M. Then $M/(X+B)$ is also coatomic. Furthermore, $M/(X+B)$ is reduced by [10] (Lemma 2.1). On the other hand, $M/(X+B)$ is divisible because M/B is divisible. Therefore, $M/(X+B) = 0$, i.e., $M = X + B$. By the hypothesis, B is coatomic and, hence, supplemented by [10] (Lemma 2.1). Therefore, X has a supplement in M by Lemma 2.1. Hence, M is a co-coatomically supplemented module.

Corollary 3.1. *Co-coatomically supplemented modules and radical supplemented modules coincide.*

The following corollary is a consequence of [11] (Satz 3.1) and Corollary 3.1:

Corollary 3.2. *A module M is co-coatomically supplemented if and only if $M = T(M) \oplus X$, where the reduced part of $T(M)$ is bounded and $X/\operatorname{Rad}(X)$ is finitely generated.*

The following properties were presented in [11] (Lemma 3.2) for the radical-supplemented modules over a discrete valuation ring. Since co-coatomically supplemented modules coincide with radical-supplemented modules, these properties clearly hold for the co-coatomically supplemented modules:

Corollary 3.3. *For an R-module M the following assertions are true:*

1. *The class of co-coatomically supplemented modules is closed under pure submodules and extensions.*
2. *If M is co-coatomically supplemented and M/U is reduced, then U is also co-coatomically supplemented.*
3. *Every submodule of M is co-coatomically supplemented if and only if $T(M)$ is supplemented and $M/T(M)$ has a finite rank.*

4. Co-Coatomically Supplemented Modules over Nonlocal Dedekind Domains

Throughout this section, R is a nonlocal Dedekind domain, unless otherwise stated.

Theorem 4.1. *Let R be a Dedekind domain and let M be an R-module. Then M is a module whose co-coatomic submodules are direct summands if and only if*

1) *$T(M) = M_1 \oplus M_2$, where M_1 is semisimple and M_2 is divisible,*

2) *$M/T(M)$ is divisible.*

Proof. By Theorem 2.1 and [4] (Theorem 6.11).

A submodule N of a module M has (is) a weak supplement in M if $M = N + K$ and $N \cap K \ll M$ for some submodule K of M. Clearly, every supplement is a weak supplement.

Recall that, over an arbitrary ring R, a module M is called co-coatomically weak supplemented if every co-coatomic submodule has a weak supplement in M.

Proposition 4.1. *Over an arbitrary ring, a small cover of a co-coatomically weak supplemented module is co-coatomically weak supplemented.*

Proof. Let M be a small cover of a co-coatomically weak supplemented module N. Then $N \cong M/K$ for some $K \ll M$. We take a co-coatomic submodule L of M. Thus, $(L + K)/K$ is a co-coatomic submodule of M/K because $L + K$ is a co-coatomic submodule of M. By the hypothesis, M/K is co-coatomically weak supplemented and, hence, $(L + K)/K$ has a weak supplement in M/K, say, X/K. Since $K \ll M$, we get

$$(X \cap L) + K = X \cap (L + K) \ll M$$

(see [5], 2.2(3)). Therefore,

$$M = L + X \quad \text{and} \quad L \cap X \ll M,$$

i.e., X is a weak supplement of L in M. Thus, M is co-coatomically weak supplemented.

Proposition 4.2. *Over an arbitrary ring, a factor module of a co-coatomically weak supplemented module is co-coatomically weak supplemented.*

Proof. Let M be a co-coatomically weak supplemented module and let N be a submodule of M. Then any co-coatomic submodule of M/N is a submodule of the form L/N, where L is a co-coatomic submodule of M. By the hypothesis, L has a weak supplement in M, say, K. Thus, $(K + N)/N$ is a weak supplement of L/N in M/N by [5] (2.2(5)).

Let M be a module and let K be a submodule of M. A submodule L of M is called a complement of K in M if it is maximal in the set of all submodules N of M with $K \cap N = 0$. A submodule L of M is called a complement submodule if it is a complement of some submodule of M (see [5], 1.9). A submodule of M is a complement if and only if it is closed (see [5], 1.10). A submodule L of M is called coclosed in M if L has no proper submodules K for which $L/K \ll M/K$ (see [5], 3.6). Over a Dedekind domain, a submodule N of M is closed if and only if N is coclosed (see [10], Lemma 3.3). Over a domain R, a torsion submodule $T(M)$ of a module M is a closed submodule of M (see [7], Example 6.34). Therefore, over a Dedekind domain, a torsion submodule $T(M)$ of a module M is a coclosed submodule of M.

Proposition 4.3. *Let M be a torsion R-module. Then M is co-coatomically weak supplemented if and only if it is co-coatomically supplemented.*

Proof. $(\Leftarrow)$ Clear.

$(\Rightarrow)$ Let K be a submodule of M such that M/K is coatomic. Since M is co-coatomically weak supplemented, K has a weak supplement in M, say, N. Then

$$M = K + N \quad \text{and} \quad K \cap N \ll M.$$

Since M is a torsion, N is also a torsion and, hence, it is coclosed. Therefore, $K \cap N \ll N$ by [5] (3.7(3)). Thus, M is co-coatomically supplemented.

Let R be a Dedekind domain and let $\mathcal{P}$ be the set of all maximal ideals of R. For some $P \in \mathcal{P}$, the submodule

$$\{m \in M \mid P^n m = 0 \text{ for some integer } n \geq 1\}$$

is said to be the P-primary component of M. This submodule is denoted by $T_P(M)$.

Over a discrete valuation ring, if a module M is torsion and reduced and the radical of M has a supplement in M, then M is bounded (see [10, p. 48], 2nd Folgerung).

Theorem 4.2. *Let M be a reduced R-module. If $T(M)$ has a weak supplement in M, then M is co-coatomically supplemented if and only if $M/T(M)$ is divisible and $T_P(M)$ is bounded for each maximal ideal P.*

Proof. $(\Rightarrow)$ Let M be a co-coatomically supplemented reduced R-module. Then the module $M/T(M)$ is radical: Suppose K is a maximal submodule of M with $T(M) \subseteq K$. Since M is co-coatomically supplemented, K has a supplement, say, V. Since K is maximal, V is local and, therefore, V is cyclic, i.e., $V \cong R/I$ (see [9], 41.1(3)). On the other hand, R is nonlocal and, thus, $I \neq 0$, i.e., V is torsion. Hence, $V \subseteq T(M)$; a contradiction. Therefore, $M/T(M)$ has no maximal submodule and, thus, $M/T(M)$ is divisible (see [1], Lemma 4.4). By [7] (Example 6.34), $T(M)$ is closed, i.e., it is coclosed by [10] (Lemma 3.3). Since $T(M)$ has a weak supplement, it is a supplement by [5] (20.2). Hence, there is a submodule N in M such that

$$T(M) + N = M \quad \text{and} \quad T(M) \cap N \ll T(M).$$

Then

$$T(M)/T(M) \cap N \cong (T(M) + N)/N = M/N.$$

Since M is co-coatomically supplemented, it is co-coatomically weak supplemented and, thus,

$$T(M)/T(M) \cap N$$

is co-coatomically weak supplemented. By Proposition 4.1, $T(M)$ is co-coatomically weak supplemented. By Proposition 4.2, $T_P(M)$ is also co-coatomically weak supplemented for each P as it is a direct summand of $T(M)$. Moreover, $T_P(M)$ is a co-coatomically supplemented module by Proposition 4.3. Thus, $T_P(M)$ is bounded for each maximal ideal P (see [10, p. 48], 2nd Folgerung).

$(\Leftarrow)$ Each $T_P(M)$ is bounded and, hence, it is supplemented by [10] (Lemma 2.1). Therefore, $T(M)$ is supplemented by [10] (Theorem 3.1). Now let K be a submodule of M such that M/K is coatomic. Then $M/(K + T(M))$ is also coatomic. By the hypothesis, $M/T(M)$ is divisible, i.e., it has no maximal submodules (see [1], Lemma 4.4). Therefore, $M = K + T(M)$. By Lemma 2.1, K has a supplement in M. Hence, M is co-coatomically supplemented.

Remark 4.1. We see that the "if" part of the theorem is true without the condition that "$T(M)$ has a weak supplement in M." We do not know whether this condition is necessary for the "only if" part.

Corollary 4.1. *Let R be a nonlocal Dedekind domain and let M be a reduced R-module. If $\mathrm{Rad}(T(M)) \ll T(M)$, then M is co-coatomically supplemented if and only if $M/T(M)$ is divisible.*

Proof. $(\Rightarrow)$ Clear by the proof of Theorem 4.2.

($\Leftarrow$) By [2] (Corollary 4.1.2.), $T(M)/\operatorname{Rad}(T(M))$ is semisimple and, thus, it is co-coatomically weak supplemented. Then $T(M)$ is co-coatomically weak supplemented because

$$\operatorname{Rad}(T(M)) \ll T(M)$$

by Proposition 4.1. Therefore, $T(M)$ is co-coatomically supplemented by Proposition 4.3. Since $M/T(M)$ is divisible, $M/T(M)$ has no maximal submodule. Hence, M is co-coatomically supplemented by Theorem 2.3.

Theorem 4.3. *Let R be a nonlocal Dedekind domain and let M be a reduced R-module. If M is co-coatomically amply supplemented, then $M/T(M)$ is divisible and $T_P(M)$ is bounded for each $P \in \mathcal{P}$.*

Conversely, if $M/T(M)$ is divisible and $T_P(M)$ is bounded for each maximal ideal P of R, then M is co-coatomically supplemented.

Proof. Let R be a nonlocal Dedekind domain and let M be a co-coatomically amply supplemented reduced R-module. Then, by the proof of Theorem 4.2, $M/T(M)$ is divisible. We now suppose that $T_P(M)$ is not bounded for some $P \in \mathcal{P}$. If a basic submodule $B_p(M)$ is bounded, then, by [6] (Theorem 5), we get

$$T_P(M) = B_P(M) \oplus D,$$

where D is divisible. Therefore, M is not reduced and we arrive at a contradiction. Hence, $B_p(M)$ is not bounded. We now prove that $B_P(M)$ is co-coatomically supplemented. Let K be a co-coatomic submodule of $B_P(M)$, i.e., $B_P(M)/K$ is coatomic. Thus, $B_P(M)/K$ is bounded by [10, p. 48] (2nd Folgerung). We get the following commutative diagram with exact rows and columns:

$$\begin{array}{ccccccccc}
 & & K & & & & & & \\
 & & \downarrow{\scriptstyle i} & & & & & & \\
E: 0 & \longrightarrow & B_P(M) & \xrightarrow{\text{pure}} & M & \longrightarrow & X & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \sigma} & & \downarrow & & \| & & \\
E': 0 & \longrightarrow & B_P(M)/K & \xrightarrow{\text{pure}} & M' & \longrightarrow & X & \longrightarrow & 0
\end{array}$$

Since E is pure E' is also pure. Hence, E' is splitting because $B_P(M)/K$ is bounded (see [6], Theorem 5). By applying Ext, we obtain the exact sequence

$$\to \operatorname{Ext}_R(X, K) \overset{i_*}{\to} \operatorname{Ext}_R\big(X, B_P(M)\big) \overset{\sigma_*}{\to} \operatorname{Ext}_R\big(X, B_P(M)/K\big) \to .$$

Since

$$\operatorname{Ext}\big(X, B_P(M)/K\big) = 0,$$

we find $\sigma_*(E) = 0$ and, therefore, $E \in \operatorname{Ker}\sigma_* = \operatorname{Im} i_*$. Thus, there exists a short exact sequence

$$E'': \ 0 \to K \to N \to X \to 0$$

such that $i_*(E'') = E$. Hence, we arrive at the following diagram:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & K & \longrightarrow & N & \longrightarrow & X & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & B_P(M) & \longrightarrow & M & \longrightarrow & X & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & B_P(M)/K & = & B_P(M)/K & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

Without loss of generality, we can assume that K, $B_P(M)$, and N are submodules of M. In this diagram,

$$B_P(M) \cap N = K \quad \text{and} \quad B_P(M) + N = M$$

(see [9]; the Noether isomorphism theorem). Moreover, M/N is coatomic. Since M is co-coatomically amply supplemented, there exists a submodule L of $B_P(M)$ such that

$$N + L = M \quad \text{and} \quad N \cap L \ll L.$$

Therefore,

$$B_P(M) = B_P(M) \cap (N + L) = L + (B_P(M) \cap N) = L + K$$

and

$$L \cap K \leq L \cap N \ll L.$$

Thus, K has a supplement in $B_P(M)$ and, hence, $B_P(M)$ is co-coatomically supplemented. Therefore, $B_P(M)$ is bounded by [10, p. 48] (2nd Folgerung). This is a contradiction. This means that $T_P(M)$ is bounded for each $P \in \mathcal{P}$.

The converse assertion is clear by Theorem 4.2.

Acknowledgements

The authors would like to express their deep gratitude to E. Büyükaşık for his support during the preparation of the present paper.

REFERENCES

1. R. Aizade, G. Bilhan, and P. F. Smith, "Modules whose maximal submodules have supplements," *Comm. Algebra*, **29**, 2389–2405 (2001).
2. E. Büyükaşık, *Weakly and Cofinitely Weak Supplemented Modules over Dedekind Ring: PhD Thesis*, Dokuz Eylül University (2005).
3. E. Büyükaşık and C. Lomp, "Rings whose modules are weakly supplemented are perfect. Applications to certain ring extensions," *Math. Scand.*, **105**, 25–30 (2009).
4. E. Büyükaşık and D. Pusat-Yılmaz, "Modules whose maximal submodules are supplements," *Hacet. J. Math. Stat.*, **39**, No. 4, 477–487 (2010).
5. J. Clark, C. Lomp, N. Vanaja, and R. Wisbauer, *Lifting Modules*, Birkhäuser-Verlag (2006).
6. I. Kaplansky, "Modules over Dedekind rings and valuation rings," *Trans. Amer. Math. Soc.*, **72**, 327–340 (1952).
7. T. Y. Lam, *Lectures on Modules and Rings*, Springer (1999).
8. P. F. Smith, "Finitely generated supplemented modules are amply supplemented," *Arab. J. Sci. Eng.*, **25**, 69–80 (2000).
9. R. Wisbauer, *Foundations of Modules and Rings*, Gordon & Breach (1991).
10. H. Zöschinger, "Komplementierte moduln über Dedekindringen," *J. Algebra*, **29**, 42–56 (1974).
11. H. Zöschinger, "Moduln die in jeder Erweiterung ein Komplement haben," *Math. Scand.*, **35**, 267–287 (1974).
12. H. Zöschinger and F. Rosenberg, "Koatomare moduln," *Math. Z.*, **170**, 221–232 (1980).