QUANTUM CALCULUS OF CLASSICAL HEAT-BURGERS' HIERARCHY AND QUANTUM COHERENT STATES

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ABSTRACT

QUANTUM CALCULUS OF CLASSICAL HEAT-BURGERS' HIERARCHY AND QUANTUM COHERENT STATES

The purpose of this thesis is an application of quantum calculus to classical Heat-Burgers' hierarchy and quantum coherent states. First we construct random walk on q-lattice, corresponding q-heat equation and exact solutions in terms of new family of q-exponential functions. Then we introduce a new type of q-diffusive heat equation and q-viscous Burgers' equation, their polynomial solutions as generalized Kampe-de Feriet polynomials, corresponding dynamical symmetry and description in terms of Bell polynomials. Shock soliton solutions with fusion and fission of shocks are found and studied for different values of q. The q-semiclassical expansion of these equations in terms of Bernoulli polynomials is derived as corrections in power of $\ln q$. A new class of complex valued function of complex argument as q-analytic functions in terms of q-analytic binomials is introduced and shown that these binomials are generalized analytic functions. As an application, we construct a new type of quantum states as q-analytic coherent states and corresponding q-analytic Fock-Bargmann representation. Then, we extend the concept of q-analytic function for two complex arguments, called double q-analytic functions, which has q-Hermite binomial expansion. As hyperbolic extension, we describe the q-analogue of traveling waves and find the D'Alembert solution of q-wave equation. By introducing q-translation operators we obtain q-binomials, q-analytic and q-anti analytic functions, q-travelling waves and non-commutative binomials. New type of quantum states as Hermite coherent states and Kampe-de Feriet coherent states are studied by generalization of the known Mehler formula. We introduce Golden quantum calculus, and as an application we study Golden quantum oscillator and its angular momentum representations.

ÖZET

KLASİK ISI-BURGERS' HİYERARŞİSİNİN VE KUANTUM KOHERENT DURUMLARIN KUANTUM HESAPLAMASI

Bu tezin amacı, kuantum hesaplamanın klasik ısı-Burgers hiyerarşisine ve kuantum coherent durumlara uygulanmasıdır. İlk olarak, q-latis(örgü) üzerinde rassal yürüyüş inşa edip, ilgili q-ısı denklemini ve bunun tam çözümlerini q-üstel fonksiyonların yeni ailesi cinsinden bulduk. Daha sonra yeni bir q-difüzyon ısı ve q-viskoz Burgers denklemlerini tanıtıp, çözümlerini genelleştirilmiş Kampe-de Feriet polinomlar cinsinden yazıp, ilgili dinamik simetri ve Bell polinomları cinsinden açıklamasını yaptık. Füzyon ve fisyon şoklardan oluşan şok soliton çözümleri bulunup, bu çözümler farklı q değerleri için incelendi. Denklemlerin, Bernoulli polinomları cinsinden q-yarı klasik açılımı $\ln q$ nun kuvvetleri cinsinden yazıldı. Kompleks parametreli kompleks değerli yeni bir fonksiyon sınıfı, q-analitik binomlar cinsinden q-analitik fonksiyonlar olarak tanıtılmıştır ve bu binomların genelleştirilmiş analitik fonksi-yonlar olduğu gösterilmiştir. Bunun uygulaması olarak q-analitik koherent durumlar olan yeni bir çeşit kuantum durumlar ve ilgili q-analitik Fock-Bargmann gösterimlerini inşa ettik. Daha sonra q-analitik fonksiyon kavramını, çift q-analitik fonksiyonlar olarak adlandırdığımız iki kompleks parametreli fonksiyonlara genişletip, bunların q-Hermite polinomları cinsinden açılımını bulduk. Bu fonksiyonların hiperbolik genişlemesi olarak, q-hareket eden dalgaları tanımlayıp, q-dalga denkleminin D'Alembert çözümünü bulduk. q-öteleme operatörleri tanıtılarak q-binomlar, q-analitik ve q-anti analitik fonksiyonlar, q-hareket eden dalgalar ve sırabağımlı binomlar elde ettik. Bilinen Mehler formülünü genelleyerek, Hermite koherent ve Kampe-de Feriet koherent durumlar olan yeni kuantum durumlar bulundu. Altın kuantum hesaplamayı tanıttık, ve uygulaması olarak Altın kuantum osilatörü ve açısal momentum gösterimini çalıştık.

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CHAPTER 1

INTRODUCTION

The mathematical concepts of scale transformation and scale invariance have origin in more fundamental concepts of human perception. As described by J. Piaget (Piaget and Inhelder, 1971) in Chapter "Similarity and Proportions" of his paper "The Child's Conception of Space", the origin of the similarity idea is in the real perception of forms, starting from childhood, and possibility to select the forms as an invariant objects of the size variations. Penetrating to philosophical and religious systems, it produced mathematical analysis as possibility to split the world to hierarchy of scales with simple rules. The mathematical calculus of scales and proportions is known as the q-calculus. It takes origin in works of Euler, Gauss, Fermat, Pascale, and was developed at the end of XIX - beginning of XX- centuries by Jackson, Thomae, Heine, Ramanujan and others. At modern times, it has attracted the attention of researchers in quantum theory of exactly solvable models and this is the reason why it is also called also as the quantum calculus. This leaded to discovery of quantum algebras and quantum groups as deformations of the usual Lie algebras and Lie groups with deformation parameter q. As physical applications it includes quantum spin chains, anyons, conformal and Chern-Simons field theory. Nonextensive statistical mechanics, Moyal's quantization and non-commutative geometry, q-special functions and q-difference equations, q-integrable models and q-quantum oscillators and q-deformed Poincare groups - are active field of research now. The present thesis is devoted to study applications of quantum calculus to description of new type of q-diffusive Heat-Burgers' equations hierarchy, new type of complex functions and applications to theory of special functions and quantum states.

1.1. Heat-Burgers' Hierarchy and q-Diffusive Equations

The heat equation and its modifications are the simplest equations in mathematical physics, modelling diffusion, the heat transfer and other phenomena. To model a more reach class of diffusion phenomena, several extensions of the diffusion equation by fractional calculus (Miller and Ross, 1993), quantum or q-calculus (Nalci and Pashaev, 2010), (Pashaev and Nalci, 2012), noncommutative calculus (Martina and Pashaev, 2013), etc. were proposed. Described in terms of relative gradients, the heat equation appears in the form of nonlinear Burgers' equation. Solution of this Burgers' equation as the shock solitons and their interactions play fundamental role in description of soliton phenomena. Extensions of this equation by q-deformations lead to a new type of soliton solutions, like q-shock solitons (Nalci and Pashaev, 2010), (Pashaev and Nalci, 2012), noncommutative shock solitons (Martina and Pashaev, 2013), etc. This is the reason why, any exactly solvable extension of the heat and Burgers' equations play essential role in description of new type of soliton interactions, in exact solvability of corresponding equations and in modelling a new physical phenomena associated with them.

Recently, several extensions of diffusion equation by the q-deformation of partial derivatives were proposed (Nalci and Pashaev, 2010), (Pashaev and Nalci, 2012) and exact solutions in the form of q-shock solitons constructed and represented in terms of q-special functions. By such an approach the q-deformation of classical damped oscillator as the q-deformed oscillator was studied in (Nalci and Pashaev, 2011). The quantum versions of q-oscillator have attracted attentions due to the relations with quantum groups and exact solvability for different realizations of quantum symmetry, such as symmetrical (Biedenharn, 1989), (Macfarlane, 1989), non symmetrical (Arik and Coon, 1976), Fibonacci (Arik et al, 1992) and Golden calculus (Pashaev and Nalci, 2012), etc. In the set of papers by Man'ko and coauthors (Man'ko et all, 1997) a physical approach to q-oscillator as a nonlinear oscillator was proposed. Then, as was shown in (Pashaev, 2015), every integrable system in action-angle variables was described as a set of nonlinear oscillators and appeared in the form of the q- or more generally, the f- oscillator. Motivated by this, in papers (Pashaev, 2015), (Pashaev, 2016) the linear Schrödinger equation with q-modified dispersion was introduced and the Madellung form of this equation as q-dispersive complex nonlinear Burgers' equation was derived.

In the present thesis, following similar ideas we propose a new type of heat equation with modified non-symmetric q-diffusive term. This equation belongs to the heat hierarchy of infinite order diffusive equations. Description of this equation in terms of relative gradients leads to the q-viscous Burgers' equation, which is a specific member of Burgers' hierarchy. We study several classes of exact solutions, polynomial and shock soliton type. The polynomial solutions are generalizations of the Kampe de Feriet polynomials written in terms of Bell polynomials. We derive generating function for these polynomials by using the dynamical symmetry and the Zassenhaus formula. Generating exact solutions and the dynamical symmetry, the generalized boost operator is constructed explicitly. Then we find one, two and multiple shock soliton solutions and study their interactions. We show that the q-deformation modifies the speed of our solitons, so that for $q < 1$ the speed is bounded above and as a result, fission of soliton takes place. Finally, we develop the "q-semiclassical expansion" of our equations in $\lambda = \ln q$ as higher order deformations, written in terms of Bernoulli polynomials.

1.2. Complex Functions and Quantum States

1.2.1. q-Periodic Analytic Functions

Development of infinite dimensional group theory, conformal field theory and quantum integrable systems, has illuminated from new direction the classical subject known as q-calculus (Kac and Cheung, 2002). Besides quantum groups and anyon physics, this calculus found recently applications in old classical problem of hydrodynamics in circular multiple connected domain (Pashaev and Yilmaz, 2008). Here q-periodic analytic functions allowed us to formulate the two circle theorem for irrotational and incompressible flow in double connected domain bounded by two circles (Pashaev and Nalci , 2014), (Pashaev, 2009). Let $f(z)$ be the complex potential of the flow in plane, then with addition of two concentric circular cylinders with cross sections $C_1 : |z| = r_1$ and $C_2 : |z| = r_2$, the flow between cylinders becomes

$$F(z) = \sum_{n=-\infty}^{\infty} f(q^n z) + \sum_{n=-\infty}^{\infty} \bar{f}\left(q^n \frac{r_1^2}{z}\right). \tag{1.1}$$

Here parameter q has simple geometrical meaning $q = r_2^2/r_1^2$ as a unique characteristic of the double connected domain. This solution shows that complex potential is q-periodic analytic function $F(qz) = F(z)$. Corresponding complex velocity $\bar{V}(z) = dF(z)/dz$ is scale-invariant analytic function $\bar{V}(qz) = q^{-1}\bar{V}(z)$ and admits representation $\bar{V}(z) = z^{-1}A_q(z)$, where $A_q(qz) = A_q(z)$ is q-periodic analytic function. The form of scale-invariant function $W(qt) = q^d A_q(t)$ is characteristic of fractal self-similar functions. So the famous Mandelbrot-Weierstrass fractal function is represented in this form. Besides of hydrodynamics, analytical extension of this function in Fock-Bargman or in coherent state representation can be used for construction of quantum wave function analog of everywhere continuous but nowhere differentiable function of Mandelbrot-Weierstrass fractal. This is the reason why naturally to consider it as quantum fractal. The structure of quantum fractal is typical for hierarchical lattices and phase transitions critical phenomena (Erzan, 1997).

1.2.2. q-Analytic Functions

The q-binomial formula as a main tool to develop q-Taylor series expansion, allows us to introduce complex valued function of complex argument which we call q-analytic function. The Glauber coherent states and corresponding Fock-Bargman representation in quantum theory give direct meaning to an entire analytic function as the wave function of quantum states. Solutions of the planar electrons problem in magnetic field (Landau levels) and the Quantum Hall effect (the Laughlin wave function) include an arbitrary analytic wave function, which reflects degeneracy of the ground state.

Motivated by hydrodynamic problem mentioned above with q-periodic and self-similar structure, in the present thesis we study complex functions under finite scaling transformations. This allows us to introduce a new class of complex functions of complex argument, depending on real parameter q and reducible to analytic functions for a particular value of $q = 1$. The construction is based on q-derivative extension of the Riemann holomorphicity equation. As an example of this q-analytic function we treat in details the complex q-binomial. We show that this q-analytic function, being non-analytic in the classical sense for $q \neq 1$, still is the generalized analytic function. This function allows us to construct a new type of quantum coherent states and quantum fractals.

As a hyperbolic version of q-analytic functions here we introduce the q-traveling wave and q-wave equation. Also we derive the analytic Hermite binomial formula and the double q-analytic q-Hermite binomial formulas.

All binomials formulas can be derived by specific translation operators which are equivalent to the first order equations on q-analytic functions. As a next result we derive the Hermite coherent states, the Kampe-de Feriet coherent states and the Bernoulli coherent states. These states are related with squeezed coherent states, which have application in quantum optics and with dynamical symmetry of quantum oscillator.

1.3. Golden Quantum Calculus

One more direction studied in present thesis is related with the so-called Fibonacci or Golden quantum calculus. Fibonacci numbers are known from ancient times and have many applications from human proportions, architecture (Golden section), natural plants (branches of trees, arrangement of leaves) up to financial market (Koshy, 2001).

The Fibonacci numbers satisfy the recursion relation

$$F_1 = F_2 = 1 \quad \text{(Initial Condition)}, \tag{1.2}$$

$$F_n = F_{n-1} + F_{n-2}, \quad \text{for } n \geq 3 \quad \text{(Recursion Formula)}. \tag{1.3}$$

First few Fibonacci numbers are $1, 1, 2, 3, 5, 8, 13, \ldots$ For these numbers, starting from de

Moivre, Lame and Binet, next representation is known as the Binet formula (Koshy, 2001):

$$F_n = \frac{\varphi^n - \varphi'^n}{\varphi - \varphi'}, \tag{1.4}$$

where φ, φ' are positive and negative roots of the equation

$$x^2 - x - 1 = 0,$$

respectively. These roots are given explicitly as

$$\varphi = \frac{1+\sqrt{5}}{2}, \qquad \varphi' = \frac{1-\sqrt{5}}{2} = -\frac{1}{\varphi}. \tag{1.5}$$

The number φ is known as the Golden ratio or the Golden section. There is a huge amount of work devoted to the applications of Golden ratio in many fields from natural phenomena to architecture and music.

Fibonacci numbers can be considered as a particular case of Fibonacci polynomials $F_n(a)$:

$$F_1(a) = 1, \ \ F_2(a) = a, \tag{1.6}$$

$$F_{n+1}(a) = aF_n(a) + F_{n-1}(a), \ \ \text{for } n \geq 2, \tag{1.7}$$

when $a = 1$: $F_n(1) = F_n$. The Binet representation for these polynomials is

$$F_n(a) = \frac{q^n - (-\frac{1}{q})^n}{q - (-\frac{1}{q})}, \tag{1.8}$$

where parameter $a = q - \frac{1}{q}$, so that $q = \frac{a+\sqrt{a^2+4}}{2}$ and $-\frac{1}{q} = \frac{a-\sqrt{a^2+4}}{2}$ are roots of quadratic equation $x^2 = ax + 1$.

Here we notice that Binet formula can be considered as a special case of the so-called q-numbers in q-calculus with two basis q and Q, where $Q = -\frac{1}{q}$. The pair (Q, q) calculus generalizes the q-calculus. In particular cases when $Q = 1$ it becomes non-symmetrical calculus. In case $Q = \frac{1}{q}$ it becomes so-called symmetrical q-calculus. It appears in the study of generalized quantum q-harmonic oscillator (Arik et al, 1992), (Chakrabarti and Jagannathan, 1991) and mentioned as a convenient form for generalization, generalization of the q-calculus in (Kac and Cheung, 2002).

Recently, we found that it appears naturally in construction of q-Binomial formula for noncommutative elements. Noncommutative q-binomials were considered in (Nalci Tumer and Pashaev , in preparation) for description q-Hermite polynomial solutions for q-Heat equation. From another side it appears also in description of AKNS Hierarchy of integrable systems where $Q = R$ is recursion operator of AKNS Hierarchy and q is the spectral parameter (Pashaev and Nalci , 2014).

In the present thesis we would like to explore the possibility to interpret Binet formula for Fibonacci polynomials and Fibonacci numbers as q-numbers, and develop corresponding q-calculus.

1.3.1. Generalized q-Deformed Fermion Algebra

In addition to q-bosonic quantum algebra several attempts were done to construct q-deformed fermionic oscillators (Parthasarathy and Viswanathan, 1991). These fermionic quantum algebras were applied to several problems: for the dynamic mass generation of quarks and nuclear pairing (Tripodi and Lima, 1997), (Timoteo and Lima, 2006), as description of higher order effects in many-body interactions in nuclei (Sviratcheva et al. , 2004), (Ballesteros et al. , 2002).

A non-trivial q-deformation of the fermion oscillator algebra has been proposed in (Parthasarathy and Viswanathan, 1991):

$$f_q f_q^+ + \sqrt{q}\, f_q^+ f_q = q^{-\frac{N}{2}} \tag{1.9}$$

$$[N, f_q^+] = f_q^+, \quad [N, f_q] = -f_q; \quad f_q^2 \neq 0. \tag{1.10}$$

In this q-deformed fermionic oscillator algebra, the Pauli exclusion principle is not valid anymore. The oscillator allows more than two q-fermions in a given quantum state. For such q-fermion algebra the Fock space construction requires to introduce the "fermionic q-numbers"

(Parthasarathy and Viswanathan, 1991),

$$[n]_q^F = \frac{q^{-\frac{n}{2}} - (-1)^n q^{\frac{n}{2}}}{q^{-\frac{1}{2}} + q^{\frac{1}{2}}}. \tag{1.11}$$

For generic q, this representation is infinite-dimensional. Though in the limit $q \to 1$, the Fock space reduces to two states: the vacuum state and one-fermion state, so that the Pauli principle is recovered. Here we note that this fermionic q-number (1.11) under substitution $q \to \frac{1}{\sqrt{q}}$ becomes Binet formula (1.8) for Fibonacci polynomials $F_n(\frac{1}{\sqrt{q}} - \sqrt{q})$, and for Golden Ratio base $q = \frac{1}{\varphi^2}$, it gives Fibonacci numbers (1.4). This relation allows us to connect Fibonacci polynomials and Fibonacci numbers considered as q-numbers, with fermionic q-numbers (Parthasarathy and Viswanathan, 1991). Statistical properties of these q-deformed fermions were investigated in (Chaichian et al. , 1993) for description of fractional statistics. Later it was shown (Narayana , 2005) that the thermodynamics of these generalized fermions should involve the q-calculus with Jackson type q-derivative in the form

$$D_q^x f(x) = \frac{1}{x} \frac{f(q^{-1}x) - f(-qx)}{q + q^{-1}} \tag{1.12}$$

Here we notice that under substitution $q \to \frac{1}{q}$ this derivative becomes the Fibonacci derivative (8.21), and for $q \to \frac{1}{\varphi}$, the Golden Derivative (8.22). The above consideration indicate on emergency of the Fibonacci q-calculus in description of q-deformed fermions and their statistics.

1.3.2. Hecke Condition for R Matrix

Another motivation is related with quantum integrable systems approach to the theory of quantum groups via solution of the Yang-Baxter equation for the R-matrix (Faddeev et al. , 1990). If one introduces the $\hat{R}$ matrix, $\hat{R} = PR$, where P- is permutation matrix, then this invertible $\hat{R}$ matrix obeys a characteristic equation. For two roots, this equation is represented in the form of the Hecke condition

$$(\hat{R} - q\hat{I})(\hat{R} + \frac{1}{q}\hat{I}) = 0 \tag{1.13}$$

or

$$\hat{R}^2 = a\hat{R} + \hat{I}. \tag{1.14}$$

By studying representations of the braid group satisfying this quadratic relation, (Jones , 1987) obtained a polynomial invariant in two variables for oriented links. If calculating higher powers of matrix $\hat{R}$, we repeatedly apply the Hecke condition (1.14) as a result we find

$$\hat{R}^n = F_n(a)\hat{R} + F_{n-1}(a)\hat{I}, \tag{1.15}$$

where $F_n(a) = aF_{n-1}(a) + F_{n-2}(a)$ - are Fibonacci polynomials (1.8) with $a = q - \frac{1}{q}$.

1.3.3. Entangled N Qubit Spin Coherent States

One more motivation is coming from quantum information theory. The unit of quantum information, the qubit, in the spin coherent state representation

$$|\psi\rangle = \frac{1}{\sqrt{1+|\psi|^2}} \begin{pmatrix} 1 \\ \psi \end{pmatrix} \tag{1.16}$$

is parametrized by complex number $\psi \in C$, given by the stereographic projection $\psi = \tan\frac{\theta}{2}e^{i\phi}$ of the Bloch sphere for qubit

$$|\theta, \phi\rangle = \cos\frac{\theta}{2}|0\rangle + \sin\frac{\theta}{2}e^{i\phi}|1\rangle. \tag{1.17}$$

For arbitrary representation j of $SU(2)$, the scalar product of two coherent states is

$$\langle\phi|\psi\rangle = \frac{(1+\bar{\phi}\psi)^{2j}}{(1+|\phi|^2)^j(1+|\psi|^2)^j}. \tag{1.18}$$

The orthogonality condition $\langle\phi|\psi\rangle = 0$ implies $1 + \bar{\phi}\psi = 0$ or a two states at the inverse-symmetric points in the unit circle ψ and $\phi = -\frac{1}{\bar{\psi}}$ (Pashaev and Gurkan , 2011). These points correspond to antipodal points on Bloch sphere, $M(x,y,z)$ and $M*(-x,-y,-z)$. According to these points, recently we have constructed maximally entangled set of orthonormal two qubit coherent states (Pashaev and Gurkan , 2011),

$$|P_{\pm}\rangle = \frac{1}{\sqrt{2}}(|\psi\rangle|\psi\rangle \pm \left|-\frac{1}{\bar{\psi}}\right\rangle\left|-\frac{1}{\bar{\psi}}\right\rangle) \tag{1.19}$$

$$|G_{\pm}\rangle = \frac{1}{\sqrt{2}}(|\psi\rangle - \left|\frac{1}{\bar{\psi}}\right\rangle \pm \left|-\frac{1}{\bar{\psi}}\right\rangle|\psi\rangle) \tag{1.20}$$

with concurrence $C = 1$. These states generalize the Bell states and reduce to the last ones in the limit $\psi \to 0$ and $-\frac{1}{\bar{\psi}} \to \infty$. This construction can be extended to arbitrary N-qubit coherent states. First set of entangled states expanded in computational basis is

$$\frac{|\psi\rangle^N - \left|-\frac{1}{\bar{\psi}}\right\rangle^N}{\psi + \frac{1}{\bar{\psi}}} = F_1(\alpha,\beta)(|10...0\rangle + |01...0\rangle + ... |00...1\rangle) \tag{1.21}$$

$$+F_2(\alpha,\beta)(|110...0\rangle + |101...0\rangle + ... |00...11\rangle) \tag{1.22}$$

$$... + F_N(\alpha,\beta)(|111...1\rangle \tag{1.23}$$

and is characterized by the set of complex Fibonacci polynomials $F_n(a)$, where $a = \psi - \frac{1}{\bar{\psi}}$. Another set of entangled N-qubit coherent states is

$$|\psi\rangle^N + \left|-\frac{1}{\bar{\psi}}\right\rangle^N = |00...0\rangle + L_1(\alpha,\beta)(|10...0\rangle + |01...0\rangle + ... |00...1\rangle) \tag{1.24}$$

$$+L_2(\alpha,\beta)(|110...0\rangle + |101...0\rangle + ... |00...11\rangle) \tag{1.25}$$

$$... + L_N(\alpha,\beta)(|111...1\rangle \tag{1.26}$$

and is characterized by complex Lucas polynomials $L_n(\alpha,\beta) = \psi^n + (-\frac{1}{\bar{\psi}})^n$. The inverse-symmetric points ψ and $-\frac{1}{\bar{\psi}}$ are roots of complex quadratic equation $z^2 = \alpha z + \beta$, where $\alpha = \psi - \frac{1}{\bar{\psi}}$ and $\beta = \frac{\psi}{\bar{\psi}}$. From polar representation of complex numbers $\psi = qe^{i\phi}$ and $z = re^{i\phi}$ we get $r^2 = ar + 1$, where $a = q - \frac{1}{q}$, and $r^n = rF_n(a) + F_{n-1}(a)$ with Fibonacci polynomials $F_n(a)$ (1.8). The interesting point here is that the symmetric points under the unit circle appear in the problem of vortex images in circular domain (Pashaev and Yilmaz, 2008), where these points correspond to the line vortex at ψ and its image in the circle at $\frac{1}{\bar{\psi}}$. Then parameter a, $\alpha = ae^{i\phi}$ in Fibonacci polynomials has simple geometrical meaning as the distance between vortex and its image. In particular when this distance is equal to one, $a = q - \frac{1}{q} = 1$, the position of the vortex is at the Golden Ratio distance from origin $r = \varphi = \frac{1+\sqrt{5}}{2}$ and Fibonacci polynomials turn to Fibonacci numbers. In this case the line interval connecting vortex and the inverse-symmetric point intersects the unit circle at a point which divide this interval on two parts of length ϕ and $\frac{1}{\phi}$.

The above motivations show that Fibonacci q-calculus is interesting subject to develop with fruitful potential applications.

The goal of the present thesis is to study quantum calculus of classical Heat-Burgers' hierarchy and quantum coherent states.

The thesis is organized as follows.

In Chapter 2, we study random walk on q-lattice and q-deformed heat equation. After introduction of the heat and Burgers' equations by Cole Hopf transformation in Section 2.1 we discuss shock soliton solutions and IVP. Random walk on q-lattice as Fermat partition and its relation with q-heat equation with specific q-dependence for time and space variables are discussed in Section 2.2. To describe exact solutions of this equation here we introduce and study a new type of q-exponential functions. Section 2.3 is devoted to solution of q-heat equation in terms of these new q-exponential functions. This solution includes q-oscillator hierarchy and allows extending to a family of q-heat equations. Then we show that specific case of random walk on q-lattice is described by the symmetrical q-calculus.

In Chapter 3, a new type of heat equation with nonsymmetric q-extension of the diffusion term is introduced, which we call q-diffusive heat equation. We find polynomial solutions of this equation as generalized Kampe de Feriet polynomials (Section 3.3), corresponding dynamical symmetry and description in terms of Bell polynomials in Section 3.4. By using the Cole Hopf transformation the q-viscous Burgers' equation is derived in Section 3.5. Its solutions as shock solitons and their interactions are constructed and analyzed for different q values. Due to specific dependence of the group velocity on wave number, in addition to fusion of the solitons as in usual Burgers equation, a new process of fission of shock solitons with higher amplitude is shown. In Section 3.6 the semiclassical expansion of these equations is obtained in terms of Bernoulli polynomials as corrections in power of $\ln q$. The Bäcklund transformations are subject of Section 3.7

In Chapter 4, we introduce a new class of complex valued function of complex argument which we call q-analytic functions (Sections 4.1-4.5), satisfying q-Cauchy-Riemann and q-Laplace equations. We show that q-analytic functions are not the analytic functions in the usual sense. However some of these complex functions fall to the class of the generalized analytic functions (Section 4.6). A new type of quantum states as q-analytic coherent states and corresponding q-analytic Fock-Bargmann representation are constructed in Sections 4.8-4.9.

In Chapter 5, the concept of q-analytic function is extended to expansion of q-binomial in terms of q-Hermite polynomials (Sections 5.1-5.3), analytic in two complex arguments. Based on this representation, we introduce a new class of complex functions of two complex arguments, which we call the double q-analytic functions (Section 5.4). As another hyperbolic extension, in Section 5.5 we describe the q-analogue of traveling waves, which are not preserving the shape during evolution. Then IVP for q-wave equation is solved in the q-Hermite polynomial form.

In Chapter 6, we introduce q-translation operator acting on monomials, which produces q-binomials, q-analytic and q-anti analytic functions, and q-travelling waves. Another type q-translation operator as q-commutative (non-commutative) translation operator is introduced. This produces non-commutative binomials, functions for non-commutative coordinates. All these translations can be described by first order q-difference equations.

In Chapter 7, applying evolution operator to Glauber coherent states, we introduce a new type of quantum states as Hermite coherent states (Section 7.2) and Kampe-de Feriet coherent states (Section 7.4), characterized by Hermite polynomials and Kampe-de Feriet polynomials correspondingly. In Section 7.5, we find a generalization of the Mehler formula. By this formula we normalize these Coherent states and construct corresponding Fock Bargmann representation. In section 7.6, we introduce Bernoulli coherent states are related Fock-Bargmann representation. By q-translation operator in Section 7.7, we discuss q-coherent states.

In chapter 8, we introduce Golden quantum calculus. By Fibonacci and Golden derivatives we derive main ingredients of these calculus as Golden Leibnitz rule, Taylor expansion, Golden binomial and Golden integral. In Section 8.3 we study Golden quantum oscillator and its angular momentum representations.

Conclusions of this thesis are given in Chapter 9.

CHAPTER 2

RANDOM WALK ON Q-LATTICE AND Q-HEAT EQUATIONS

Here by considering random walk on q-lattice as Fermat partition we introduce a new type of q-heat equation with specific q-dependence for time and space variables. In order to find exact solutions of this equation a new type of q-exponential functions are introduced and some properties are studied. These q-exponential functions are generalizations of Jackson's q-exponential functions. We obtain a solution of q-heat equation in terms of these q-exponential functions. This solution includes q-oscillator hierarchy and allows extending to a family of q-heat equations. Then we show that specific case of random walk on q-lattice is described by symmetrical q-calculus.

2.1. Random Walk on Equidistant h-Lattice and Heat Equation

Consider a symmetric random walk on one dimensional lattice. Starting from the origin, a particle moves one step to the right or the left with equal probabilities $\frac{1}{2}$.

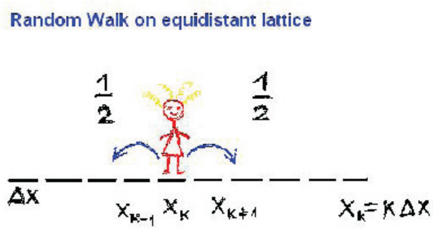

Figure 2.1. One dimensional random walk on equidistant lattice with step-size Δx

The consecutive steps are independent and taken at times

$$t_k = k\,\Delta t, \quad k = 1, 2, 3, ...$$

with the step size (lattice distance) Δx, so that the set of possible positions of particle is

$$x_k = k\,\Delta x.$$

The function $X(t, x)$ indicates whether the side x is occupied ($X = 1$) or unoccupied ($X = 0$) at time t and

$$u(t_k, x_k) = P\{X(t_k, x_k) = 1\}$$

shows the probability that at time t_k the particle is at site x_k. Then we can write

$$u(t_{k+1},\ x_k) = \frac{1}{2}u(t_k, x_{k-1}) + \frac{1}{2}u(t_k, x_{k+1}). \tag{2.1}$$

This equation can be rewritten as follows:

$$u(t_{k+1}, x_k) - u(t_k, x_k) = \frac{1}{2}\left(u(t_k, x_{k-1}) + u(t_k, x_{k+1}) - 2u(t_k, x_k)\right). \tag{2.2}$$

Let $u(t_k,\ x_k) \equiv u(t, x)$ and expand both sides in Taylor at x, t :

$$\begin{aligned} u(t_{k+1}, x_k) &= u(t + \Delta t, x) = u(t, x) + \frac{\partial u}{\partial t}\,\Delta t + O((\Delta t)^2), &(2.3)\\ u(t_k, x_{k\pm1}) &= u(t, x \pm \Delta x) = u(t, x) \pm \frac{\partial u}{\partial x}\,\Delta x + \frac{1}{2}\frac{\partial^2 u}{\partial x^2}\,(\Delta x)^2 + O((\Delta x)^3). &(2.4)\end{aligned}$$

Substituting (8.33) and (8.34) into (2.2) we get

$$\frac{\partial u}{\partial t} = \frac{1}{2}\frac{(\Delta x)^2}{\Delta t}\frac{\partial^2 u}{\partial x^2}.$$

We see that for ($\Delta t \to 0$) and ($\Delta x \to 0$) nontrivial dynamics only happen when $\frac{(\Delta x)^2}{\Delta t} = a$ is a constant.

Thus we obtain the linear heat equation as

$$\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial x^2}, \tag{2.5}$$

where $\nu \equiv \frac{a}{2}$.

2.1.1. Burger's Equation and Cole-Hopf Transformation

Nonlinear heat equation which is also known as Burger's equation is

$$u_t + uu_x = \nu u_{xx}. \tag{2.6}$$

By using the Cole-Hopf transformation

$$u(x,t) = -2\nu \frac{\phi_x(x,t)}{\phi(x,t)}, \tag{2.7}$$

the equation (2.6) reduces to the linear heat equation

$$\phi_t = \nu \phi_{xx}. \tag{2.8}$$

Shock soliton solutions are particular solutions of this equation.

2.1.2. IVP for Burgers' Equation

IVP for the Burgers' equation defined in the following form

$$\begin{aligned} u_t + uu_x &= \nu u_{xx}, \quad t > 0, \\ u(x,0) &= F(x), \quad -\infty < x < \infty, \end{aligned}$$

can be transformed to IVP for the Heat equation

$$\begin{aligned}\phi_t &= \nu\phi_{xx},\\ \phi(x,0) &= e^{\frac{-1}{2\nu}\int^x F(\eta)d\eta}.\end{aligned}$$

Solution of the IVP for heat equation

$$\phi(x,t) = \frac{1}{\sqrt{4\pi\nu t}}\int_{-\infty}^{\infty}\phi(\eta,0)e^{-\frac{(x-\eta)^2}{4\nu t}}d\eta$$

implies the solution of IVP for Burgers' equation in the following form

$$u(x,t) = -2\nu\frac{\phi_x}{\phi} = \frac{\int_{-\infty}^{\infty}\frac{x-\eta}{t}e^{-\frac{G}{2\nu}}d\eta}{\int_{-\infty}^{\infty}e^{-\frac{G}{2\nu}}d\eta},$$

where

$$G(\eta;x,t) \equiv \int^{\eta}F(\eta')d\eta' + \frac{(x-\eta)^2}{2t}.$$

2.1.3. Shock Soliton Solutions of Burgers' Equation

One of the simplest solution of the heat equation

$$\phi_t = \nu\phi_{xx} \tag{2.9}$$

is

$$\phi = e^{\eta_1}, \qquad \eta_1 = k_1x + \omega_1t + \eta_1^0 \tag{2.10}$$

with $\omega_1 = \nu k_1^2$. Parameterizing $k_1 = a_1/2\nu$, we have

$$\phi = e^{-\frac{a_1}{2\nu}x + \frac{a_1^2}{4\nu}t + \eta_1^0}.$$

The corresponding solution of Burgers' equation is

$$u(x,t) = -2\nu\frac{\phi_x}{\phi} = a_1. \tag{2.11}$$

Since the heat equation is linear, any superposition of solutions is also a solution (superposition principle). Then for

$$\phi = e^{\eta_1} + e^{\eta_2}, \quad \eta_i = -\frac{a_i}{2\nu}x + \frac{a_i^2}{4\nu}t + \eta_i^0, \quad (i = 1, 2),$$

the corresponding solution of Burgers' equation is in the form of shock solitons

$$u(x,t) = \frac{a_1 e^{\eta_1} + a_2 e^{\eta_2}}{e^{\eta_1} + e^{\eta_2}}, \tag{2.12}$$

with asymptotics: $u \to a_1$ if $x \to -\infty$, $u \to a_2$ if $x \to +\infty$, where $0 < k_1 < k_2$.

2.1.4. Initial Step Function to Shock

Importance of shock solitons follows from initial value problem with step function:

$$u(x,0) = F(x) = \begin{cases} a_1, & x > 0 \\ a_2 > a_1, & x < 0. \end{cases}$$

Then at time $t > 0$:

$$u(x,t) = a_1 + \frac{a_2 - a_1}{1 + h(x,t)e^{\frac{(a_2-a_1)}{2\nu}(x - vt - x_0)}} \tag{2.13}$$

where $v = \frac{a_1+a_2}{2}$.

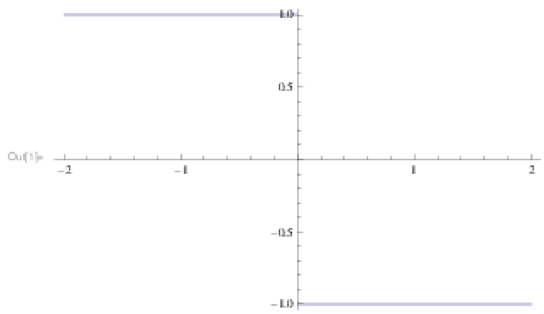

Figure 2.2. Initial step function

When $x \to \infty, t \to \infty$, so that x/t is fixed and we obtain shock soliton solution

$$h(x,t) = \frac{\int_{-\frac{(x-a_1 t)}{\sqrt{4\nu t}}}^{\infty} e^{-\zeta^2} d\zeta}{\int_{\frac{(x-a_2 t)}{\sqrt{4\nu t}}}^{\infty} e^{-\zeta^2} d\zeta} \to 1. \tag{2.14}$$

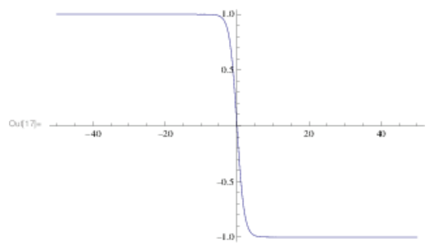

Figure 2.3. Shock soliton solution

2.2. Random Walk on q-Lattice and q-Heat equation

2.2.1. Fermat Partition and q-Lattice

Pierre de Fermat (1601-1665), along with Descartes (1596-1650), invented $xy-$ co-ordinate system and analytic geometry. Besides developing analytic geometry, Fermat and Descartes were also early researchers in the subject that we now call the q calculus.

Fermat evaluated the area under the graph of a power function $y = x^{\alpha}$, where α is a rational number except -1, that is, how he determined what we now write as

$$\int_0^a x^{\alpha}\, dx = \frac{a^{\alpha+1}}{\alpha + 1}.$$

He approached the area by rectangular estimates.

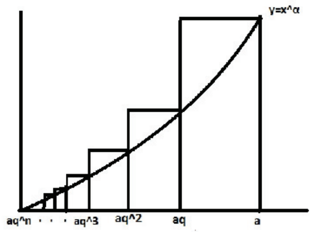

Figure 2.4. Area on q-lattice

Fermat partitioned the interval $[0, a]$ into subintervals which are not the same size, but ordered in geometric progression.

Let us calculate the area under the curve of $y = x^{\alpha}$ in interval [0,a] by Fermat's upper rectangular estimates. For $0 < q < 1$, we divide the closed interval $[0, a]$ into n-subintervals of different lengths at the points $q^0 a, q^1\, a, q^2\, a, q^3\, a, ..., q^n\, a$, then the partition is

$$P\{a_0, a_1, a_2, ..., a_n\} = \{a, qa, q^2 a, ...q^n a\}.$$

The width of the leftmost subinterval is $\Delta a_1 = a_0 - a_1 = a - qa = a(1-q)$, the next one $\Delta a_2 = a_1 - a_2 = qa(1-q)$, and the last one is $\Delta a_n = a_{n-1} - a_n = q^{n-1}a(1-q)$. Then, the sum of rectangular areas from right to the left is expressed in terms of the Riemann Sum S_q^n as

$$\begin{aligned} S_q^n &= \sum_{k=1}^{n} f(c_k)\Delta a_k \\ &= a^{\alpha+1}(1-q)\sum_{k=1}^{n}\left(q^{\alpha+1}\right)^{k-1} \\ &= a^{\alpha+1}(1-q)\sum_{k=1}^{n} Q^{k-1} \\ &= a^{\alpha+1}(1-q)\left(1 + Q^2 + Q^3 + ... + Q^{n-1}\right), \end{aligned} \tag{2.15}$$

where $q^{\alpha+1} \equiv Q$.

The exact area is given by the limit

$$\lim_{n\to\infty} S_q^n = a^{\alpha+1}(1-q)\frac{1}{1-q^{\alpha+1}},$$

where $Q = q^{\alpha+1} < 1$. Now if $q \to 1$, then $S_q^\infty \to \frac{a^{\alpha+1}}{\alpha+1}$, which is exact value of the integral.

2.2.2. From Fermat's Approach to Jackson's Integral

Let us evaluate the area under the curve $y = f(x)$ in the interval $[0, x]$ by using Fermat partition

$$P\{x_0, x_1, x_2, ..., x_n\} = \{x, qx, q^2x, ..., q^n x = 0\},$$

where $0 < q < 1$. The Riemann sum is obtained as

$$\begin{aligned} S_q^n &= x(1-q)\sum_{k=1}^{n} f\left(q^{k-1}x\right)q^{k-1} \\ &= x(1-q)\sum_{j=0}^{n} f\left(q^j x\right)q^j. \end{aligned} \tag{2.16}$$

When $n \to \infty$, we obtain the Jackson's integral

$$\begin{aligned}\lim_{n\to\infty} S_q^n &= x(1-q)\sum_{j=0}^{\infty} f\left(q^j x\right) q^j \\ &= \int f(x) d_q x = F(x). \end{aligned} \tag{2.17}$$

2.2.3. q-Lattice and Symmetric Points in Two Concentric Circles

Inversion of point a in circle with radius R is $\frac{R^2}{a}$

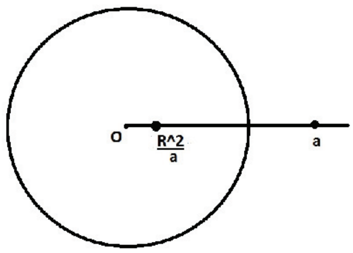

Figure 2.5. Inversion of point a in circle R

Inversion of point a in 2 circles with radiuses R_1 and R_2 is determined by infinite set of points

$$\begin{aligned}&..., \frac{1}{q^2}a, \frac{1}{q}a, a, aq, aq^2, ... \\ &..., \frac{1}{q}\frac{R_1^2}{a}, \frac{R_1^2}{a}, q\frac{R_1^2}{a}, q^2\frac{R_1^2}{a}, ...\end{aligned}$$

Boundary value problem corresponds infinite set of vortex images arranged as two q-lattices with $q = \frac{R_2^2}{R_1^2}$.

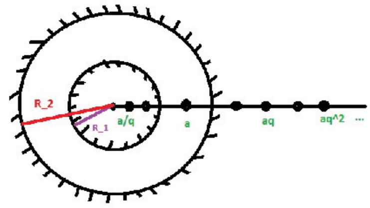

Figure 2.6. q-Lattice of hydrodynamic vortex images in annular domain, $q = \frac{R_2^2}{R_1^2}$

2.2.4. Random Walk on q-Lattice

Now let us consider a non-equidistant random walk on the one dimensional lattice. Starting from the origin, a particle moves one step to the right or the left with different probabilities, which are inversely proportional to the distances. Here, the lattice constructed by geometrical progression rule $x_k = q^k x_0$, where $q > 1$ and $k = 0, \pm 1, \pm 2, ...$, and corresponding probability moves one step to the right is $\frac{1}{1+q}$ and one step to the left is $\frac{q}{1+q}$.

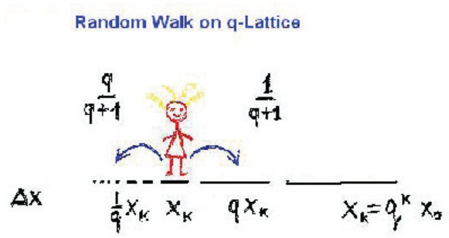

Figure 2.7. One dimensional random walk on q-lattice

The consecutive steps are independent and taken at times

$$t_k = Q^k t_0, \quad k = 1, 2, 3, ...$$

with the step size (lattice distance) Δx, so that the set of possible positions of particle is

$$x_k = q^k x_0.$$

The function $X(t, x)$ indicates whether the side x is occupied ($X = 1$) or unoccupied ($X = 0$) at time t, so that

$$u(t_k, x_k) = P\{X(t_k, x_k) = 1\}$$

shows the probability that at time t_k the particle is at site x_k. Then we can write

$$u(t_{k+1}, x_k) = \frac{q}{q+1} u(t_k, x_{k-1}) + \frac{1}{q+1} u(t_k, x_{k+1}).$$

This equation can be rewritten as follows

$$u(t_{k+1}, x_k) - u(t_k, x_k) = \frac{q}{q+1} u(t_k, x_{k-1}) + \frac{1}{q+1} u(t_k, x_{k+1}) - u(t_k, x_k). \tag{2.18}$$

We denote $u(t_k, x_k) \equiv u(t, x)$, $\quad t \equiv t_k = Q^k t_0$, $x \equiv x_k = q^k x_0$. Then we can write

$$\begin{aligned} u(t_{k+1}, x_k) &= u(Qt, x), \\ u(t_k, x_{k+1}) &= u(t, qx), \\ u(t_k, x_{k-1}) &= u(t, \frac{x}{q}), \end{aligned}$$

and the equation (2.18) can be rewritten in the following form

$$\begin{aligned} u(Qt, x) - u(t, x) &= \frac{q}{q+1} u(t, \frac{x}{q}) + \frac{1}{q+1} u(t, qx) - u(t, x) \\ (Q-1) t D_Q^t u(t, x) &= \frac{q}{q+1} u(t, \frac{x}{q}) + \frac{1}{q+1} u(t, qx) - \frac{q}{q+1} u(t, x) - \frac{1}{q+1} u(t, x) \\ (Q-1) t D_Q^t u(t, x) &= \frac{1}{q+1} \left(q u(t, \frac{x}{q}) + u(t, qx) - (q+1) u(t, x) \right) \end{aligned}$$

$$
\begin{aligned}
(Q-1)tD^t_Q u(t,x) &= \frac{1}{q+1} M^x_{\frac{1}{q}} \left(q(q-1)^2 x^2 (D^x_q)^2 u(t,x) \right) \\
Q^k (Q-1) t_0 D^t_Q u(t,x) &= \frac{1}{q+1} M^x_{\frac{1}{q}} \left(q(q-1)^2 q^{2k} x_0^2 (D^x_q)^2 u(t,x) \right),
\end{aligned}
\tag{2.19}
$$

and we get

$$
M^x_q \, D^t_Q \, u(t,x) = \frac{(q-1)^2 \, q \, q^{2k}}{(q+1)(Q-1)Q^k} \frac{x_0}{t_0} (D^x_q)^2 u(t,x) \tag{2.20}
$$

where the partial $q-$derivatives are defined as

$$
D^x_q u(t,x) = \frac{u(t,qx) - u(t,x)}{(q-1)x}, \qquad D^t_Q u(t,x) = \frac{u(Qt,x) - u(t,x)}{(Q-1)t}, \tag{2.21}
$$

and dilatation operator is $M^x_q u(t,x) = u(t,qx)$.

By proper choice $Q \equiv q^2$ in order to make equation (2.20) independent of steps number k, we obtain the $(q,q^2)-$ heat equation

$$
D^t_{q^2} u(t,x) = \nu M^x_{\frac{1}{q}} (D^x_q)^2 u(t,x) \tag{2.22}
$$

or it can be written as

$$
D^t_{q^2} u(t,x) = \nu D^x_{\frac{1}{q}} D^x_q u(t,x),
$$

where

$$
\frac{x_0^2}{t_0} \equiv a, \qquad \frac{q(q-1)a}{q+1} \equiv \nu. \tag{2.23}
$$

Random walk on q-lattice produces different kind of Q-space difference q-time difference heat equations. As a special cases we mention :
In (Nalci and Pashaev, 2010), (Nalci and Pashaev, 2014) we studied the case $Q = q$: q-Space-Time Difference Heat equation, its polynomial solutions and corresponding q-Burgers' equation with q-shock soliton solutions.
Furthermore in (Pashaev and Nalci, 2012), (Nalci and Pashaev, 2014) the case $Q = 1$:

q-Space Difference and Time Differential Heat equation is studied with details.

In order to solve (q, q^2)- heat equation (2.22) we need to introduce new type of q-exponential functions.

2.2.5. New Family of q-Exponential Functions

Definition 2.1 *N-Weighted q-exponential function is defined as*

$$_Ne_q(x) \equiv \sum_{n=0}^{\infty} q^{N\frac{n(n-1)}{2}} \frac{x^n}{[n]_q!} \tag{2.24}$$

The q-exponential function $_Ne_q(x)$ is entire function for $N > 0$ and $q < 1$, and $N < 0$ or $0 < N < 1$ and $q > 1$.

Particular cases of this exponential function are:

1. For $N = 0$ it reduces to Jackson's q-exponential function

$$_0e_q(x) \equiv e_q(x) = \sum_{n=0}^{\infty} \frac{x^n}{[n]_q!}, \tag{2.25}$$

which is an entire function of x for $q > 1$, and for $q < 1$ it converges for $|x| < \frac{1}{|q-1|}$.

2. For $N = 1$ it is the second Jackson's q-exponential function

$$_1e_q(x) \equiv E_q(x) = \sum_{n=0}^{\infty} q^{\frac{n(n-1)}{2}} \frac{x^n}{[n]_q!}. \tag{2.26}$$

It is easy to see that

$$e_{\frac{1}{q}}(x) = E_q(x). \tag{2.27}$$

Lemma 2.1 *q-Derivative of the exponential function $_Ne_q(ax)$ is found as*

$$D_q^x \; {}_Ne_q(ax) = a \; {}_Ne_q(aq^N x). \tag{2.28}$$

Proof By direct application of q-derivative we find

$$\begin{aligned} D_q^x \ {}_N e_q(ax) &= \sum_{n=0}^{\infty} q^{N\frac{n(n-1)}{2}} D_q^x \frac{ax^n}{[n]_q!} = \sum_{n=1}^{\infty} q^{N\frac{n(n-1)}{2}} \frac{a^n x^{n-1}}{[n-1]_q!} = \sum_{n=0}^{\infty} q^{N\frac{n(n-1)}{2}} q^{Nn} a^{n+1} \frac{x^n}{[n]_q!} \\ &= a \ {}_N e_q(aq^N x). \end{aligned}$$

□

Lemma 2.2 *Due to the symmetry between q and $\frac{1}{q}$, the q-exponential function ${}_N e_q(x)$ satisfies the following reciprocity relation*

$$_N e_{\frac{1}{q}}(x) = {}_{1-N}\, e_q(x). \tag{2.29}$$

Proof By using the definition of q-numbers, the relation between q and $\frac{1}{q}$-numbers is found as

$$[n]_q = q^{n-1}[n]_{\frac{1}{q}}, \tag{2.30}$$

and the relation between q and $\frac{1}{q}$-factorials is obtained in the form

$$[n]_q! = q^{\frac{n(n-1)}{2}} [n]_{\frac{1}{q}}!. \tag{2.31}$$

Then, by using the definition of N-weighted q-exponential function and the relation (2.31) we get the desired result

$$\begin{aligned} {}_N e_{\frac{1}{q}}(x) &= \sum_{n=0}^{\infty} \left(\frac{1}{q}\right)^{N\frac{n(n-1)}{2}} \frac{x^n}{[n]_{\frac{1}{q}}!} = \sum_{n=0}^{\infty} q^{-N\frac{n(n-1)}{2}} q^{\frac{n(n-1)}{2}} \frac{x^n}{[n]_q!} = \sum_{n=0}^{\infty} q^{(1-N)\frac{n(n-1)}{2}} \frac{x^n}{[n]_q!}{}_{1-N} e_q(x) \\ &= {}_{1-N} e_q(x). \end{aligned}$$

□

For $N = 0$, we obtain the known relation between the first and the second Jackson's q-exponential functions

$$_0 e_{\frac{1}{q}}(x) \equiv e_{\frac{1}{q}}(x) = {}_1\, e_q(x) \equiv E_q(x). \tag{2.32}$$

This reciprocity relation gives transition between two Jackson's q-exponential functions or gives transition between symmetric points in the unit circle, q and $\frac{1}{q}$.

2.3. q-Heat Equation

After introducing a new type of q-exponential function ${}_Ne_q(x)$, in order to solve the (q, q^2)-heat equation

$$M_q^x D_{q^2}^t u(t,x) = \nu (D_q^x)^2 u(t,x), \tag{2.33}$$

we consider the separation of variable method $u(t,x) = T(t)X(x)$. By taking corresponding derivatives

$$\frac{D_{q^2}^t T(t)}{\nu T(t)} = \frac{(D_q^x)^2 X(x)}{X(qx)} = -\kappa^2$$

we obtain two equations which only depends on t and x,

$$D_{q^2}^t T(t) = -\nu\kappa^2 T(t), \tag{2.34}$$
$$(D_q^x)^2 X(x) = -\kappa^2 X(qx). \tag{2.35}$$

The solution for time dependent part (2.34) is found in terms of Jackson's q-exponential function $e_q(x)$ with base q^2 as

$$T(t) = T(0) e_{q^2}(-\nu\kappa^2 t),$$

where $T(0)$ is the initial condition. The solution of space dependent part (2.35) is obtained in terms of new q-exponential function ${}_Ne_q(x)$ with $N = \frac{1}{2}$ in the following form

$$X(x) = {}_{\frac{1}{2}} e_q\left(\pm i \frac{\kappa x}{q^{\frac{1}{4}}}\right). \tag{2.36}$$

As a result, particular solution is obtained as

$$u_{\pm}(x,t) = X(x)T(t) = {}_{\frac{1}{2}}\, e_q\left(\pm i\frac{\kappa x}{q^{\frac{1}{4}}}\right) e_{q^2}(-\nu\kappa^2 t). \tag{2.37}$$

Definition 2.2 *New family of q-trigonometric functions are defined in the following form*

$$\begin{aligned} {}_N\cos_q(x) &\equiv \frac{{}_N e_q(ix) + {}_N e_q(-ix)}{2}, \\ {}_N\sin_q(x) &\equiv \frac{{}_N e_q(ix) - {}_N e_q(-ix)}{2i}, \end{aligned} \tag{2.38}$$

which satisfy the relations

$$ {}_N\cos_{\frac{1}{q}}(x) = {}_{1-N}\cos_q(x) \tag{2.39}$$

$$ {}_N\sin_{\frac{1}{q}}(x) = {}_{1-N}\sin_q(x). \tag{2.40}$$

Their derivatives are found as

$$D_q^x\, {}_N\sin_q(ax) = a\; {}_N\cos_q(q^N ax),$$

$$D_q^x\, {}_N\cos_q(ax) = -a\; {}_N\sin_q(q^N ax).$$

Then, a general solution is written in terms of new family of q-trigonometric functions in the form

$$u(x,t) = \left(A_{\frac{1}{2}}\cos_q\left(\frac{\kappa x}{q^{1/4}}\right) + B_{\frac{1}{2}}\sin_q\left(\frac{\kappa x}{q^{1/4}}\right)\right) e_{q^2}(-\nu\kappa^2 t), \tag{2.41}$$

in which we introduce Euler type formula

$$ {}_N e_q(ix) = {}_N\cos_q(x) + i\, {}_N\sin_q(x). \tag{2.42}$$

Here, A and B could be fixed by initial functions, but in order to fix κ we need to use boundary conditions, which requires to study zeros of ${}_{\frac{1}{2}}\cos_q(x)$ and ${}_{\frac{1}{2}}\sin_q(x)$ functions.

In Figures 2.8 and 2.9, we compare Jackson's q-exponential function and $\frac{1}{2}$-weighted q-exponential function ${}_{\frac{1}{2}}e_q(x)$ with the standard exponential function e^x for different values of q.

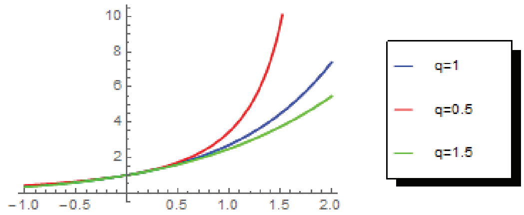

Figure 2.8. q-Exponential function $e_q(x)$

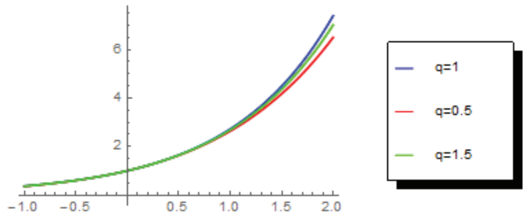

Figure 2.9. $\frac{1}{2}$-Weighted exponential function ${}_{\frac{1}{2}}e_q(x)$

If we consider the Dirichlet boundary conditions $X(0) = X(L) = 0$ for (q, q^2)-heat equation we obtain the solution as

$$u(x,t) = {}_{\frac{1}{2}}\sin_q\left(\frac{x}{q^{1/4}}\right)e_{q^2}(-t), \tag{2.43}$$

where for simplicity we choose $B = 1,\ \kappa = 1,\ \nu = 1$ and $L = q^{\frac{1}{4}}x_0,\ x_0 \cong 4.6$ is a zero of ${}_{\frac{1}{2}}\sin_q(x)$

In Figure 2.10 we see the solution of (q, q^2)-heat equation at time $t = 0.1$ for $q = 0.5$, $q = 1.2$, and $q = 1$ with interval depending on q.

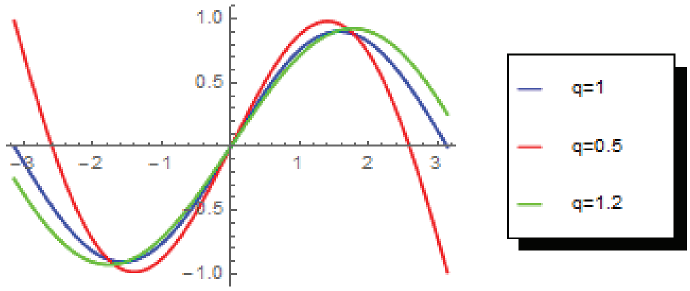

Figure 2.10. Solution of (q, q^2)-heat equation at time $t = 0.1$

2.3.1. q-Oscillators Hierarchy

The q-oscillator equation is defined as

$$\left(D_q^t\right)^2 y(t) + \omega^2 y(t) = 0, \tag{2.44}$$

with the q-exponential solution as

$$y(t) = Ae_q(i\omega t) + Be_q(-i\omega t) = a\cos_q(\omega t) + b\sin_q(\omega t). \tag{2.45}$$

But previous consideration implies the q-oscillator equation in the following time delateted form

$$\left(D_q^t\right)^2 y(t) + \omega^2 y(qt) = 0. \tag{2.46}$$

It has a solution as N-weighted q-exponential function with $N = \frac{1}{2}$

$$y(t) = A\ {}_{\frac{1}{2}}e_q\left(\frac{i\omega}{q^{1/4}}t\right) + B\ {}_{\frac{1}{2}}e_q\left(-\frac{i\omega}{q^{1/4}}t\right) = a\ {}_{\frac{1}{2}}\cos_q(\frac{\omega}{q^{1/4}}t) + b\ {}_{\frac{1}{2}}\sin_q(\frac{\omega}{q^{1/4}}t). \tag{2.47}$$

For arbitrary N-weighted q-exponential function we introduce the q-oscillator hierarchy

$$\left(D_q^t\right)^2 y(t) + \omega^2 y(q^M t) = 0. \tag{2.48}$$

The general solution for M-th member of family of q-oscillators is

$$y(t) = A \ {}_{\frac{M}{2}}e_q\left(\frac{i\omega}{q^{M/4}}t\right) + B \ {}_{\frac{M}{2}}e_q\left(-\frac{i\omega}{q^{M/4}}t\right) = a \ {}_{\frac{M}{2}}\cos_q\left(\frac{\omega}{q^{M/4}}t\right) + b \ {}_{\frac{M}{2}}\sin_q\left(\frac{\omega}{q^{M/4}}t\right). \tag{2.49}$$

2.3.2. Family of q-Heat Equations

The above hierarchy of q-oscillators suggests to introduce a family of q-heat equations in the form

$$D_Q^t u(q^M x, t) = (D_q^x)^2 u(x, t) \tag{2.50}$$

whose general solution is given by

$$\begin{aligned} u(x,t) &= e_Q(-\kappa^2 t)\left(A \ {}_{\frac{M}{2}}e_q(\frac{i\kappa}{q^{M/4}})x + B \ {}_{\frac{M}{2}}e_q(-\frac{i\kappa}{q^{M/4}}x)\right) \\ &= e_Q(-\kappa^2 t)\left(a \ {}_{\frac{M}{2}}\cos_q(\frac{\omega}{q^{M/4}}x) + b \ {}_{\frac{M}{2}}\sin_q(\frac{\omega}{q^{M/4}}x)\right). \end{aligned} \tag{2.51}$$

Property: The eigenvalue problem for q-weighted q-exponential function is

$$\left((M_{\frac{1}{q}})^N D_q^x\right) \ {}_N e_q(ax) = a \ {}_N e_q(ax). \tag{2.52}$$

Property: The higher order equation

$$(D_q^x)^k f(x) = \lambda f(q^M x) \tag{2.53}$$

has solution in the following form

$$f(x) =_{\frac{M}{k}} e_q\left(\frac{\lambda^{1/k}}{q^{N\frac{k-1}{2}}}x\right). \tag{2.54}$$

2.3.3. Multiple q_1, q_2 Numbers

Definition 2.3 *The q-number with two basis q_1 and q_2 is defined (Nalci and Pashaev, 2014) as*

$$[n]_{q_1,q_2} = \frac{q_1^n - q_2^n}{q_1 - q_2}. \tag{2.55}$$

By choosing $q_1 = q^N$ and $q_2 = q^{N-1}$, the q-number with two basis (q_1, q_2) is written as

$$[n]_{q^N,q^{N-1}} = \frac{(q^N)^n - (q^{N-1})^n}{q^N - q^{N-1}} = \frac{q^{Nn}(1 - q^{-n})}{q^N(1 - q^{-1})} = q^{N(n-1)}[n]_{\frac{1}{q}}, \tag{2.56}$$

and the q-factorial with two basis (q^N, q^{N-1}) becomes

$$[n]_{q^N,q^{N-1}}! = q^{N\frac{n(n-1)}{2}}[n]_{\frac{1}{q}}!. \tag{2.57}$$

Definition 2.4 *(q_1, q_2)-Exponential functions are defined (Nalci and Pashaev, 2014) in the following form*

$$e_{q_1,q_2}(x) \equiv \sum_{n=0}^{\infty} \frac{1}{[n]_{q_1,q_2}!}x^n \tag{2.58}$$

$$E_{q_1,q_2}(x) \equiv \sum_{n=0}^{\infty} \frac{1}{[n]_{q_1,q_2}!}(q_1\, q_2)^{\frac{n(n-1)}{2}} x^n. \tag{2.59}$$

Proposition 2.1 *N-weighted q-exponential function is written in terms of q-exponential func-*

tion with two basis (q^N, q^{N-1}) *as follows*

$$ {}_N e_q(x) = E_{q^N, q^{N-1}}(x). \tag{2.60} $$

Proof Using definition of (q_1, q_2)-exponential function (2.59) with bases $q_1 = q^N$ and $q_2 = q^{N-1}$ we have $E_{q^N,q^{N-1}}(x)$ as follows

$$
\begin{aligned}
E_{q^N,q^{N-1}}(x) &= \sum_{N=0}^{\infty} \frac{1}{[n]_{q^N,q^{N-1}}!}(q^N q^{N-1})^{\frac{n(n-1)}{2}} x^n = \sum_{n=0}^{\infty} \frac{1}{q^{N\frac{n(n-1)}{2}}[n]_{\frac{1}{q}}!}(q^N q^{N-1})^{\frac{n(n-1)}{2}} x^n \\
&= \sum_{N=0}^{\infty} \frac{q^{\frac{n(n-1)}{2}}}{q^{N\frac{n(n-1)}{2}}[n]_q!}(q^N q^{N-1})^{\frac{n(n-1)}{2}} x^n = \sum_{n=0}^{\infty} \frac{q^{N\frac{n(n-1)}{2}}}{[n]_q!} x^n =_N e_q(x).
\end{aligned} \tag{2.61}
$$

□

Definition 2.5 *The* (q_1, q_2)*-analogue of* $(x-a)^n$ *is the polynomial (Nalci and Pashaev, 2014)*

$$
(x-a)^n_{q_1,q_2} = \begin{cases} 1 & \text{if } n = 0, \\ (x - q_1^{n-1}a)(x - q_1^{n-2}q_2 a)...(x - q_1 q_2^{n-2}a)(x - q_2^{n-1}a) & \text{if } n \geq 1 \end{cases}
$$

Proposition 2.2 *The N-weighted q-exponential function has factorization formula*

$$ {}_N e_q(x+y)_{q^N,q^{N-1}} =_N e_{\frac{1}{q}}(x)\, {}_N e_q(y). \tag{2.62} $$

Proof By using the factorization formula (Nalci and Pashaev, 2014)

$$ e_{q_i,q_j}(x+y)_{q_i,q_j} = e_{q_i,q_j}(x) E_{q_i,q_j}(y) \tag{2.63} $$

for $q_i = q^N$ and $q_j = q^{N-1}$ we get

$$
\begin{aligned}
e_{q^N,q^{N-1}}(x+y)_{q^N,q^{N-1}} &= \underbrace{e_{q^N,q^{N-1}}(x)}_{E_{q^{-N},q^{1-N}}} E_{q^N,q^{N-1}}(y) \\
{}_N e_q(x+y)_{q^N,q^{N-1}} &= {}_N e_{\frac{1}{q}}(x)\, {}_N e_q(y).
\end{aligned} \tag{2.64}
$$

□

For special choice $N = \frac{1}{2}$, the N-weighted q-exponential function with $N = \frac{1}{2}$ is written in terms of exponential function (2.59) with base $(\sqrt{q}, \frac{1}{\sqrt{q}})$

$$ {}_{\frac{1}{2}}e_q(x) = E_{\sqrt{q},\frac{1}{\sqrt{q}}}(x). \tag{2.65} $$

For base $(\sqrt{q}, \frac{1}{\sqrt{q}})$, it is easy to see from definition (2.4) that both exponential functions are the same

$$ {}_{\frac{1}{2}}e_q(x) = E_{\sqrt{q},\frac{1}{\sqrt{q}}}(x) = e_{\sqrt{q},\frac{1}{\sqrt{q}}}(x) \equiv \widetilde{e}_{\sqrt{q}}(x), \tag{2.66} $$

which means that for symmetric basis, the exponential functions coincide. Here, $\widetilde{e}_{\sqrt{q}}(x)$ is the symmetric q-exponential function which is defined as

$$ \widetilde{e}_{\sqrt{q}}(x) = \widetilde{E}_{\sqrt{q}}(x) = \sum_{n=0}^{\infty} \frac{x^n}{\widetilde{[n]}_{\sqrt{q}}!}, \tag{2.67} $$

where the symmetric $\widetilde{[n]}_{\sqrt{q}}$-number is

$$ \widetilde{[n]}_{\sqrt{q}} = \frac{(\sqrt{q})^n - (\frac{1}{\sqrt{q}})^n}{\sqrt{q} - \frac{1}{\sqrt{q}}}. \tag{2.68} $$

The factorization formula for $N = \frac{1}{2}$ is

$$ {}_{\frac{1}{2}}e_q(x+y)_{\sqrt{q},\frac{1}{\sqrt{q}}} = {}_{\frac{1}{2}}\, e_{\frac{1}{q}}(x)\, {}_{\frac{1}{2}}e_{\frac{1}{q}}(y) = \widetilde{e}_{\sqrt{q}}(x)\widetilde{e}_{\sqrt{q}}(y), \tag{2.69} $$

where

$$ {}_{\frac{1}{2}}e_q(x+y)_{\sqrt{q},\frac{1}{\sqrt{q}}} = \sum_{n=0}^{\infty} q^{\frac{n(n-1)}{4}} \frac{(x+y)^n_{\sqrt{q},\frac{1}{\sqrt{q}}}}{[n]_q!}, \tag{2.70} $$

and symmetrical binomial is

$$ \widetilde{(x+y)}^n_{\sqrt{q}} = (x + (\sqrt{q})^{n-1}y)(x + (\sqrt{q})^{n-3}y)...(x + (\sqrt{1})^{3-n}y)(x + (\sqrt{q})^{1-n}y). \tag{2.71} $$

As a result, the q-difference equation (2.35) has a general solution in terms of symmetric exponential functions as

$$\begin{aligned} X(x) &= D_{\frac{1}{2}} e_q\left(\pm i \frac{\kappa x}{q^{\frac{1}{4}}}\right) = D \widetilde{e}_{\sqrt{q}}\left(\pm i \frac{\kappa x}{q^{\frac{1}{4}}}\right) \\ &= = A \widetilde{\cos}_{\sqrt{q}}\left(\frac{\kappa x}{q^{\frac{1}{4}}}\right) + B \widetilde{\sin}_{\sqrt{q}}\left(\frac{\kappa x}{q^{\frac{1}{4}}}\right), \end{aligned} \tag{2.72}$$

where symmetric $\sqrt{q}$-trigonometric functions are

$$\widetilde{\cos}_{\sqrt{q}}(x) = \frac{\widetilde{e}_{\sqrt{q}}(ix) + \widetilde{e}_{\sqrt{q}}(-ix)}{2}, \tag{2.73}$$

$$\widetilde{\sin}_{\sqrt{q}}(x) = \frac{\widetilde{e}_{\sqrt{q}}(ix) - \widetilde{e}_{\sqrt{q}}(-ix)}{2i}. \tag{2.74}$$

Applying to the random walk on q-lattice, equation (2.20) can be written in the following form

$$\begin{aligned} (Q-1) t D_Q^t u(t,x) &= \frac{1}{q+1}\left(q u(t, \frac{x}{q}) + u(t, qx) - (q+1) u(t,x)\right) \\ &= \frac{1}{q+1}\left(\sqrt{q} - \frac{1}{\sqrt{q}}\right)^2 \sqrt{q} x^2 \left(\widetilde{D^x_{\sqrt{q}}}\right)^2 u(t,x), \end{aligned} \tag{2.75}$$

where symmetric $\sqrt{q}$-derivative is defined as

$$\widetilde{D^x_{\sqrt{q}}} f(x) = \frac{f(\sqrt{q}x) - f\left(\frac{x}{\sqrt{q}}\right)}{(\sqrt{q} - \frac{1}{\sqrt{q}})x}. \tag{2.76}$$

We denote $t = t_0 Q^k$, $x = x_0 q^k$. To get an equation independent of steps numbers k, we choose $Q = q^2$, and hence we find the following q-heat equation

$$D^t_{q^2} u(t,x) = \nu (\widetilde{D^x_{\sqrt{q}}})^2 u(t,x), \tag{2.77}$$

where

$$\nu \equiv \frac{(q-1)a}{\sqrt{q}(q+1)^2}, \quad \frac{x_0^2}{t_0} \equiv a.$$

By the method of separation of variables $u(t,x) = T(t)X(x)$, particular solution is found

$$u(t,x) = e_{q^2}(\nu k^2 t)\widetilde{e}_{\sqrt{q}}(kx), \tag{2.78}$$

which is the generating function of new type of Kampe de Feriet polynomials

$$e_{q^2}(\nu k^2 t)\widetilde{e}_{\sqrt{q}}(kx) = \sum_{N=0}^{\infty} \frac{k^N}{N!} K_N(x,t;q), \tag{2.79}$$

where

$$K_N(x,t;q) = \sum_{n=0}^{[\frac{N}{2}]} \frac{N! x^{N-2n}(\nu t)^n}{[n]_{q^2}! \widetilde{[N-2n]}_{\sqrt{q}}!}. \tag{2.80}$$

CHAPTER 3

Q-DIFFUSIVE HEAT EQUATION AND Q-VISCOUS BURGERS' EQUATION

Here we propose a q-diffusive heat equation with nonsymmetric q-extension of the diffusion term. Written in relative gradient variables, this system appears as the q-viscous Burgers' equation. Exact solutions of this equation in polynomial form of generalized Kampe de Feriet polynomials, corresponding dynamical symmetry and description in terms of Bell polynomials are derived. We find the generating function for these polynomials by application of dynamical symmetry and the Zassenhaus formula. Shock soliton solutions and their interactions are constructed and analyzed for different q. For $q < 1$ the soliton speed becomes bounded from above and as a result, in addition to usual Burgers soliton process of fusion, we found a new phenomena, when soliton with higher amplitude but smaller velocity is fissing to two solitons. In terms of Bernoulli polynomials we develop the semiclassical expansion of these equations. Finally, we obtain the Bäcklund transformation which relates two solutions of q-viscous Burgers equation.

3.1. Q-Diffusive Heat Equation

We introduce q-diffusive deformation of the heat equation in the following form

$$\frac{\partial}{\partial t}\phi(x,t) = \left[\nu\frac{\partial^2}{\partial x^2}\right]_q \phi(x,t), \tag{3.1}$$

where ν is diffusion constant and the q-operator

$$\left[\nu\frac{\partial^2}{\partial x^2}\right]_q = \frac{q^{\nu\frac{\partial^2}{\partial x^2}} - 1}{q-1} \tag{3.2}$$

is defined as a formal power series. In the limiting case $q \to 1$, equation (3.1) reduces to the standard heat equation.

By the method of separation of variables we search solution of this equation in the form

$$\phi(x,t) = X(x)T(t).$$

Substituting this into (3.1) we get

$$\frac{T^{'}(t)}{T(t)} = \frac{\left[\nu\frac{\partial^2}{\partial x^2}\right]_q X(x)}{X(x)} = -\lambda.$$

As a result, we obtain two ordinary differential equations

$$T^{'}(t) + \lambda T(t) = 0, \tag{3.3}$$
$$\left[\nu\frac{\partial^2}{\partial x^2}\right]_q X(x) + \lambda X(x) = 0. \tag{3.4}$$

Solution of the first equation in t is

$$T(t) = e^{-\lambda t}T(0),$$

where $T(0)$ is a constant.

3.1.1. Finite Interval Case

For the space part we consider the following eigenvalue problem on finite interval with the Dirichlet boundary conditions

$$\left[\nu\frac{\partial^2}{\partial x^2}\right]_q X(x) = -\lambda X(x), \tag{3.5}$$
$$X(0) = X(l) = 0. \tag{3.6}$$

In order to solve this problem, we use the following boundary value problem

$$\begin{aligned} -X''(x) &= \mu X(x), \\ X(0) &= X(l) = 0, \end{aligned} \tag{3.7}$$

with eigenvalues

$$\mu_n = \left(\frac{n\pi}{l}\right)^2, \tag{3.8}$$

and the corresponding eigenfunctions

$$X_n(x) = \sqrt{\frac{2}{l}} \sin \frac{n\pi}{l} x. \tag{3.9}$$

This set of eigenfunctions is orthonormal and complete in L^2. Then, substituting the last equation to equation (3.5), and by using definition of the q-operator we obtain

$$\begin{aligned} \left[\nu \frac{\partial^2}{\partial x^2}\right]_q X(x) &= \frac{q^{\nu \frac{\partial^2}{\partial x^2}} X(x) - X(x)}{q-1} \\ &= \frac{q^{-\mu\nu} - 1}{q-1} X(x) = [-\mu\nu]_q \, X(x), \end{aligned} \tag{3.10}$$

which gives the following relation between the eigenvalues of q-equation (3.5) and equation (3.7)

$$\lambda = -\left[-\mu\nu\right]_q. \tag{3.11}$$

Therefore, solution of the q-deformed initial value problem (3.5) is obtained in terms of solution of standard Sturm-Liouville problem (3.7) with eigenvalues as q-numbers

$$\lambda_n = -[-\mu_n \nu]_q = -\left[-\left(\frac{n\pi}{l}\right)^2 \nu\right]_q,$$

and the corresponding eigenfunctions as in (3.9).

Then, we find a particular solution of q-diffusive heat equation (3.1) in the form

$$\phi_n(x,t) = X_n(x)T_n(t) = e^{-\lambda_n t}T(0)\ \sqrt{\frac{2}{l}}\ \sin\frac{n\pi}{l}x,$$

where $\lambda_n = -[-(\frac{n\pi}{l})^2\nu]_q$.

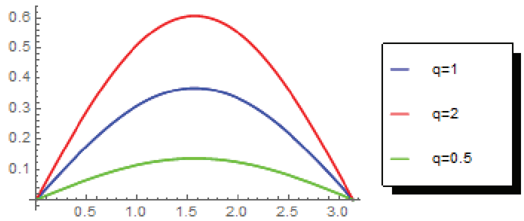

Figure 3.1. Evolution of $n = 1$ solution at time $t = 1$

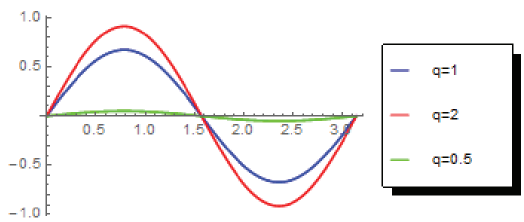

Figure 3.2. Evolution of $n = 2$ solution at time $t = 0.1$

In Figures 3.1 and 3.2 we show particular solutions for $n = 1$ and $n = 2$ modes correspondingly, in $q > 1$, $q < 1$ and $q = 1$ cases. As we can see, comparing with the usual heat equation with $q = 1$, depending on q the decaying process is going faster for $q < 1$, or slower for $q > 1$.

The general solution is a proper superposition of these solutions

$$\phi(x,t) = \sum_{n=1}^{\infty} A_n e^{-\lambda_n t} \sin\frac{n\pi}{l}x = \sum_{n=1}^{\infty} A_n e^{t[-(\frac{n\pi}{l})^2\nu]_q} \sin\frac{n\pi}{l}x. \tag{3.12}$$

To fix the Fourier coefficients A_n we pose the following IVP

$$\phi(x,0) = f(x),$$

so that we get

$$\phi(x,0) = f(x) = \sum_{n=0}^{\infty} A_n \sin\frac{n\pi}{l}x.$$

Then the coefficients are found as

$$A_m = \frac{2}{l}\int_0^l f(x)\sin(\frac{m\pi}{l}x)\,dx,$$

and solution is obtained in the form

$$\phi(x,t) = \frac{2}{l}\sum_{n=0}^{\infty}\int_0^l dy\, f(y)\sin(\frac{n\pi}{l}x)\ \sin(\frac{n\pi}{l}y)\,e^{-\lambda_n t}. \tag{3.13}$$

We define the Green function for equation (3.1) as

$$G(x,y;t) = \frac{2}{l}\sum_{n=0}^{\infty}\sin(\frac{n\pi}{l}x)\ \sin(\frac{n\pi}{l}y)\ e^{t[-(\frac{n\pi}{l})^2\nu]_q}, \tag{3.14}$$

so that solution of IBVP is

$$\phi(x,t) = \int_0^l G(x,y;t)f(y)dy. \tag{3.15}$$

The Green function (3.14) as evident, satisfies $G(x,y;t) = G(y,x;t)$ and at initial time $t = 0$ it is just the Dirac delta function

$$G(x,y;0) = \frac{2}{l}\sum_{n=0}^{\infty}\sin(\frac{n\pi}{l}x)\sin(\frac{n\pi}{l}y) = \delta(x-y). \tag{3.16}$$

Due to relation

$$F\left(\frac{d}{dx}\right)e^{ikx} = F(ik)e^{ikx}, \tag{3.17}$$

where F is an analytic function, which implies

$$e^{t[\nu\frac{d^2}{dx^2}]_q}e^{\pm i\frac{n\pi}{l}x} = e^{t[\nu(\frac{in\pi}{l})^2]_q}e^{\pm i\frac{n\pi}{l}x},$$

we can rewrite (3.14) in an operator form by using the evolution operator

$$\begin{aligned} G(x-y\,;t) &= e^{t[\nu\frac{d^2}{dx^2}]_q}\delta(x-y) = \frac{2}{l}\sum_{n=0}^{\infty} e^{t[\nu\frac{d^2}{dx^2}]_q}\sin(\frac{n\pi}{l}x)\sin(\frac{n\pi}{l}y) \\ &= \frac{2}{l}\sum_{n=0}^{\infty} e^{t[-\nu(\frac{n\pi}{l})^2]_q}\sin(\frac{n\pi}{l}x)\sin(\frac{n\pi}{l}y). \end{aligned} \tag{3.18}$$

3.1.2. Infinite Interval Case

Now we consider the initial value problem for q-diffusive heat equation in infinite interval:

$$\begin{aligned} \frac{\partial}{\partial t}\phi(x,t) &= \left[\nu\frac{\partial^2}{\partial x^2}\right]_q \phi(x,t), \\ \phi(x,0) &= f(x), \\ -\infty < &\ x < \infty. \end{aligned} \tag{3.19}$$

By using the Fourier transform

$$\phi(x,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{ikx}\ \tilde{\phi}(k,t)dk, \tag{3.20}$$

and substituting into (3.19) we have

$$\int_{-\infty}^{\infty} e^{ikx}\ \tilde{\phi}_t(k,t)dk = \int_{-\infty}^{\infty} \tilde{\phi}(k,t)\left[\nu\frac{\partial^2}{\partial x^2}\right]_q e^{ikx}dk. \tag{3.21}$$

Due to property (3.17)

$$\left[\nu\frac{d^2}{dx^2}\right]_q e^{ikx} = \left[\nu(ik)^2\right]_q e^{ikx} = \left[-\nu k^2\right]_q e^{ikx}. \tag{3.22}$$

the integral (3.21) becomes

$$\int_{-\infty}^{\infty} e^{ikx}\ \tilde{\phi}_t(k,t)dk = \int_{-\infty}^{\infty} \tilde{\phi}(k,t)\left[-\nu k^2\right]_q e^{ikx}dk \tag{3.23}$$

and

$$\int_{-\infty}^{\infty}(\tilde{\phi}_t(k,t) - \left[-\nu k^2\right]_q \tilde{\phi}(k,t))e^{ikx}dk = 0, \tag{3.24}$$

which implies

$$\tilde{\phi}_t(k,t) = \left[-\nu k^2\right]_q \tilde{\phi}(k,t).$$

The general solution of the last equation is found in the form

$$\tilde{\phi}(k,t) = \tilde{\phi}(k,0)\ e^{t[-\nu k^2]_q}. \tag{3.25}$$

Substituting (3.25) into Fourier transform (3.20) we get solution

$$\phi(x,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{ikx+t[-\nu k^2]_q}\tilde{\phi}(k,0)dk.$$

By using the inverse Fourier transform, we can fix $\tilde{\phi}(k,0)$ by the initial function

$$\tilde{\phi}(k,0) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\phi(y,0)e^{-iky}dy.$$

Then solution of the initial value problem for infinite interval is

$$\phi(x,t) = \int_{-\infty}^{\infty} G(x,y;t)\phi(y,0)dy\,, \tag{3.26}$$

where the Green function is defined as

$$G(x,y;t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ik(x-y)+t[-\nu k^2]_q}dk. \tag{3.27}$$

Using property (3.22)

$$e^{t[-\nu k^2]_q}e^{ik(x-y)} = e^{t[\nu(ik)^2]_q}e^{ik(x-y)} = e^{t[\nu\frac{d^2}{dx^2}]_q}e^{ik(x-y)},$$

the Green function becomes

$$G(x,y;t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{t[\nu\frac{d^2}{dx^2}]_q}e^{ik(x-y)}dk = e^{t[\nu\frac{d^2}{dx^2}]_q}\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ik(x-y)}dk.$$

As a result, the Green function for q-diffusive heat equation (3.19) can be expressed as time evolution of the Dirac delta function

$$G(x-y\,;t) = e^{t[\nu\frac{d^2}{dx^2}]_q}\delta(x-y). \tag{3.28}$$

Definition 3.1 *The evolution operator is defined in terms of q-deformed operator as*

$$U(t) = e^{t[\nu\frac{d^2}{dx^2}]_q}, \tag{3.29}$$

and gives evolution of the initial function $\phi(x,0)$

$$\phi(x,t) = e^{t[\nu\frac{d^2}{dx^2}]_q}\phi(x,0). \tag{3.30}$$

As an example, we consider the q-diffusive heat equation with initial value as the Dirac Delta function:

$$\begin{aligned}\frac{\partial}{\partial t}\phi(x,t) &= \left[\nu\frac{\partial^2}{\partial x^2}\right]_q \phi(x,t),\\ \phi(x,0) &= \delta(x).\end{aligned} \tag{3.31}$$

Then the solution is

$$G(x,t) = e^{t[\nu \frac{d^2}{dx^2}]_q}\delta(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx+t[-\nu k^2]_q}\,dk. \tag{3.32}$$

3.2. The Generalized Kampe-De Feriet Polynomials

Here we are going to construct polynomial solutions of equation (3.1). For this reason we consider the plane wave solution of (3.1) as the generating function for the Kampe de Feriet type polynomials,

$$\phi(x,t) = e^{kx+\omega(k)t},$$

which implies the q-deformed dispersion $\omega(k) = [\nu k^2]_q$ and

$$\phi(x,t) = e^{kx+t[\nu k^2]_q}. \tag{3.33}$$

The phase velocity of this plane wave solution is characterized by q :

$$v_{ph} = \frac{\omega(k)}{k} = \frac{[\nu k^2]_q}{k} = \frac{e^{\nu k^2 \ln q} - 1}{k(q-1)}.$$

In Figure 3.3 we show the phase velocity for different values of q. In contrast to the linear dependence for $q = 1$ case, for $q < 1$ the velocity is bounded from above and reaches the maximum value and then it starts to decline fast. As we show in Section 5, this leads to a new process of soliton fissions. However, for $q > 1$ case the phase velocity is growing infinitely, that is, it has no upper limit.

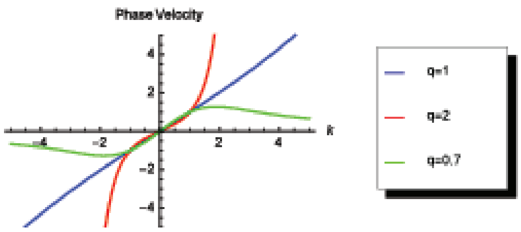

Figure 3.3. Phase velocity

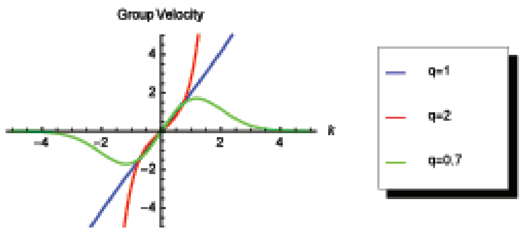

Figure 3.4. Group velocity

The group velocity of this solution also depends on q and is given by

$$v_g = \frac{d\omega(k)}{dk} = \frac{2\nu k \ln q}{q-1} e^{\nu k^2 \ln q}.$$

In Figure 3.4 we show the group velocity for three different values of q. For $q < 1$ the group velocity is bounded from the above function, taking maximal value

$$|v_{max}| = \sqrt{\frac{2\nu}{e} \ln \frac{1}{q} \frac{1}{1-q}} \tag{3.34}$$

for $k = \pm 1/\sqrt{2\nu \ln 1/q}$.

Definition 3.2 *The generalized Kampe-de Feriet polynomials $K_n(x,t)$ are defined as*

$$e^{kx + t[\nu k^2]_q} = \sum_{n=0}^{\infty} \frac{k^n}{n!} K_n(x,t). \tag{3.35}$$

Property of these polynomials can be studied in a similar way as the usual Hermite and Kampe de Feriet polynomials. But in contrast to $q = 1$ case, our generating function contains all powers of k^2 and requires introduction of the Bell polynomials. Before to proceed with this approach, in the next section we follow a more direct way by using dynamical symmetry of q-diffusive heat equation.

3.3. Dynamical Symmetry for q-Diffusive Heat Equation

For given differential equation $\hat{S}\phi = 0$ with

$$\hat{S} = \frac{\partial}{\partial t} - H(P_1) \tag{3.36}$$

exists the commuting operator $\hat{K}$ in the following form (Pashaev, 2009),

$$\hat{K} = x + tH^{'}(P_1), \tag{3.37}$$

where $P_1 = \frac{d}{dx}$, such that $[\hat{S}, \hat{K}] = 0$. This $\hat{K}$ operator generates the dynamical symmetry for differential equation $\hat{S}\phi = 0$: from given solution ϕ of the equation it creates another solution $\psi = \hat{K}\phi$ of the same equation $\hat{S}\psi = 0$. The $\hat{K}$ operator in this form is linear in x and t and it represents the generalized Boost operator.

For our q-diffusive heat equation

$$\frac{\partial}{\partial t}\phi = \left[\nu\frac{\partial^2}{\partial x^2}\right]_q \phi,$$

we have $H(P_1) = [\nu P_1^2]_q$, and by taking derivative of $H(P_1)$ according to P_1 we get

$$H^{'}(P_1) = \frac{d}{dP_1}[\nu P_1^2]_q = \frac{d}{dP_1}\frac{e^{\nu \ln q P_1^2} - 1}{q-1} = \frac{2\nu \ln q P_1}{q-1} e^{\nu \ln q P_1^2}.$$

Substituting the result into definitions (3.36) and (3.37) we obtain the q-diffusive heat operator and the q-boost operator in the following form

$$\hat{S} = \frac{\partial}{\partial t} - \left[\nu\frac{\partial^2}{\partial x^2}\right]_q, \tag{3.38}$$

$$\hat{K} = x + \frac{2\nu \ln q}{q-1} t \frac{d}{dx} e^{\nu \ln q \frac{d^2}{dx^2}}. \tag{3.39}$$

Algebra of symmetry operators is

$$[\hat{P}_0, \hat{P}_1] = 0, \qquad [\hat{P}_0, \hat{K}] = \frac{2\nu \ln q}{q-1} \frac{\partial}{\partial x} e^{\nu \ln q \frac{\partial^2}{\partial x^2}}, \qquad [\hat{P}_1, \hat{K}] = 1,$$

where $\hat{P}_0 = \frac{\partial}{\partial t}$ and $\hat{P}_1 = \frac{\partial}{\partial x}$.

Proposition 3.1 *The q-diffusive heat operator (3.38) and the q-boost operator (3.39) are commutative* $[\hat{S}, \hat{K}] = 0$.

Proof

$$\begin{aligned} [\hat{S}, \hat{K}] &= \left[\frac{\partial}{\partial t} - [\nu \frac{\partial^2}{\partial x^2}]_q,\ x + \frac{2\nu \ln q}{q-1} t \frac{d}{dx} e^{\nu \ln q \frac{\partial^2}{\partial x^2}}\right] \\ &= \frac{2\nu \ln q}{q-1} \frac{d}{dx} e^{\nu \ln q \frac{\partial^2}{\partial x^2}} \underbrace{\left[\frac{\partial}{\partial t}, t\right]}_{1} - \underbrace{\left[[\nu \frac{\partial^2}{\partial x^2}]_q, x\right]}_{*}. \end{aligned} \tag{3.40}$$

In order to find the commutator $(*)$, we use the following property:

For any real analytic function $f(x)$ we have

$$\left[f(\frac{d}{dx}), x\right] = f'(\frac{d}{dx}),$$

which implies that the commutator $(*)$ can be written in the form:

$$\left[\left[\nu \frac{\partial^2}{\partial x^2}\right]_q, x\right] = \left([\nu \frac{\partial^2}{\partial x^2}]_q\right)'. \tag{3.41}$$

Calculating derivative of the operator

$$\left([\nu \frac{\partial^2}{\partial x^2}]_q\right)' = \left(\frac{q^{\nu \frac{\partial^2}{\partial x^2}} - 1}{q-1}\right)', \tag{3.42}$$

and denoting $P \equiv \frac{\partial}{\partial x}$ we get

$$\frac{d}{dP}\left(\frac{e^{\nu \ln q P^2} - 1}{q-1}\right) = \frac{2\nu P \ln q}{q-1} e^{\nu \ln q P^2} = \frac{2\nu \ln q}{q-1} \frac{\partial}{\partial x} q^{\nu \frac{\partial^2}{\partial x^2}}. \tag{3.43}$$

Substituting the result into (3.40), finally we proved that $[\hat{S}, \hat{K}] = 0$. $\square$

Proposition 3.2 *If $\phi(x,t)$ is a solution of q-diffusive heat equation (3.1) and $[\hat{S}, \hat{K}] = 0$, then $\psi(x,t) = \hat{K}\phi(x,t)$ is also solution of this equation, where $\hat{S}$ is the q-dispersive heat operator (3.38) and $\hat{K}$ is the q-boost operator (3.39).*

According to this proposition: if $\phi(x,t)$ is a solution of the q-diffusive heat equation $\frac{\partial}{\partial t}\phi(x,t) = [\nu\frac{\partial^2}{\partial x^2}]_q\phi(x,t)$, then

$$\psi = \left(x + t\frac{2\nu\ln q}{q-1}\frac{\partial}{\partial x}e^{\nu\ln q\frac{\partial^2}{\partial x^2}}\right)\phi(x,t) \tag{3.44}$$

is also solution.

3.3.1. Bell Polynomials

The generating function of Bell polynomials with n-variables denoted by $B_n(g_1, ..., g_n)$ is defined as (Comtet, 1974)

$$\exp\sum_{n=1}^{\infty}\frac{g_n z^n}{n!} = \sum_{n=0}^{\infty}B_n(g_1, q_2, ..., g_n)\frac{z^n}{n!}. \tag{3.45}$$

And a few Bell polynomials are given below

$$\begin{aligned}
&B_0 = 1,\\
&B_1(g_1) = g_1,\\
&B_2(g_1, g_2) = g_2 + g_1^2,\\
&B_3(g_1, g_2, g_3) = g_3 + 3g_1g_2 + g_1^3.
\end{aligned}$$

In particular case, when all independent variables are equal $g_1 = g_2 = ... = g_n = x$, the corresponding generating function (3.45) reduces to the generating function for Bell polynomials of one variable x defined in (Knuth et al, 1994) as

$$e^{x(e^z-1)} = \sum_{n=0}^{\infty} B_n(x)\frac{z^n}{n!}. \tag{3.46}$$

A few Bell polynomials then are

$$B_0(x) = 1, \quad B_1(x) = x, \quad B_2(x) = x + x^2, \quad B_3(x) = x + 3x^2 + x^3, ... \tag{3.47}$$

Proposition 3.3 *The plane wave solution of equation (3.1) determines the q-Kampe-de Feriet type polynomials $K_N(x, t; q)$*

$$e^{kx}e^{[\nu k^2]_q t} = \sum_{N=0}^{\infty} \frac{k^N}{N!} K_N(x, t; q), \tag{3.48}$$

which can be represented in terms of the Bell polynomials $B_n(t)$:

$$K_N(x, t; q) = \sum_{n=0}^{[\frac{N}{2}]} \frac{x^{N-2n} N!}{(N-2n)!n!} B_n(\frac{t}{q-1})(\nu \ln q)^n.$$

Proof By expanding the plane wave solution in k, we have

$$e^{kx}e^{[\nu k^2]_q t} = \left(\sum_{m=0}^{\infty} \frac{k^m}{m!} x^m\right) e^{[\nu k^2]_q t}. \tag{3.49}$$

Then, $e^{[\nu k^2]_q t}$ can be expanded in terms of Bell polynomials as

$$e^{[\nu k^2]_q t} = e^{\frac{q^{\nu k^2}-1}{q-1} t} = e^{\frac{t}{q-1}(e^{\nu \ln q k^2}-1)} = \sum_{n=0}^{\infty} B_n\left(\frac{t}{q-1}\right)\frac{(\nu \ln q k^2)^n}{n!}, \tag{3.50}$$

and the plane wave solution is written in the following form

$$e^{kx}e^{[\nu k^2]_q t} = \sum_{m,n=0}^{\infty} \frac{k^{m+2n}}{m!n!} x^m B_n(\frac{t}{q-1})(\nu \ln q)^n. \tag{3.51}$$

By changing order of summation $m + 2n = N$,

$$e^{kx}e^{[\nu k^2]_q t} = \sum_{N=0}^{\infty}\sum_{n=0}^{[\frac{N}{2}]} \frac{k^N}{(N-2n)!n!} x^{N-2n} B_n\left(\frac{t}{q-1}\right)(\nu \ln q)^n$$

we obtain the plane wave solution in the form of q-Kampe de Feriet type polynomials

$$e^{kx}e^{[\nu k^2]_q t} = \sum_{N=0}^{\infty} \frac{k^N}{N!} K_n(x,t;q), \tag{3.52}$$

where

$$K_N(x,t;q) = \sum_{n=0}^{[\frac{N}{2}]} \frac{x^{N-2n} N!}{(N-2n)!n!} B_n\left(\frac{t}{q-1}\right)(\nu \ln q)^n.$$

□

Using the first few Bell Polynomials (3.47) we can calculate q-Kampe de Feriet Polynomials

$$\begin{aligned}
K_0(x,t;q) &= 1 \\
K_1(x,t;q) &= x \\
K_2(x,t;q) &= x^2 + \frac{2t}{q-1}\nu_q \\
K_3(x,t;q) &= x^3 + \frac{6t\nu_q}{q-1} \\
K_4(x,t;q) &= x^4 + 12t\frac{\nu_q}{q-1}x^2 + 12(\frac{\nu_q t}{q-1})^2 + 12\frac{t}{q-1}(\nu_q)^2,
\end{aligned}$$

where $\nu_q \equiv \nu \ln q$. In the limit $q \to 1$, these polynomials reduce to the standard Kampe-de Feriet polynomials.

We can find the time evolution of zeros for these polynomials. For $n = 2$, we have two zeros evolving as

$$x_{1,2} = \pm\sqrt{\frac{2t\nu_q}{1-q}}.$$

In Figures 3.5 and 3.6 we show the evolution of zeros, depending of values of q. For $q < 1$, zeros are moving faster than $q = 1$ case, and for $q > 1$, the motion slow down.

In order to find the general form of these Kampe-de Feriet polynomials for arbitrary n, we apply relation (3.44) and the boost operator (3.39). Starting from $K_0(x,t;q) = 1$ by

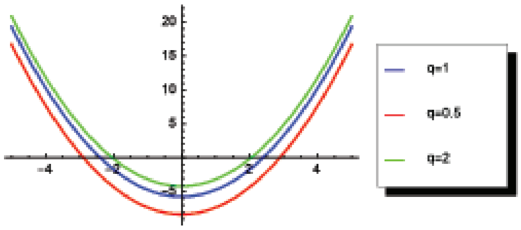

Figure 3.5. Motion of zeros at $t = -3$

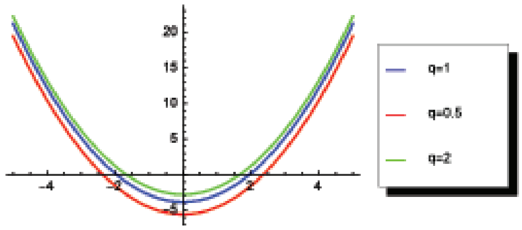

Figure 3.6. Motion of zeros at $t = -2$

successive application of this formula we obtain

$$K_n(x,t;q) = \left(x + \frac{2t\nu_q}{q-1}\frac{\partial}{\partial x}e^{\nu_q \frac{\partial^2}{\partial x^2}}\right)^n \cdot 1. \tag{3.53}$$

The polynomials result from evolution in time of monomials

$$K_n(x,0;q) = x^n$$

applying the evolution operator (3.30)

$$K_n(x,t;q) = e^{t[\nu \frac{d^2}{dx^2}]_q} x^n. \tag{3.54}$$

3.3.2. Dynamical Symmetry and Generating Function

Here we would like to find the generating function for our q-Kampe de Feriet polynomials (3.35) by application of the boost operator

$$\sum_{n=0}^{\infty} \frac{k^n}{n!} K_n(x,t;q) = \sum_{n=0}^{\infty} \frac{k^n}{n!}\left(x + \frac{2t\nu_q}{q-1}\frac{d}{dx}e^{\nu_q \frac{d^2}{dx^2}}\right)^n \cdot 1 = e^{k(x+\frac{2t\nu_q}{q-1}\frac{d}{dx}e^{\nu_q \frac{d^2}{dx^2}})} \cdot 1 \tag{3.55}$$

and show that it gives the plane wave solution (3.33).

Proposition 3.4 *We have the following factorization formula*

$$e^{k(x+\frac{2t\nu_q}{q-1}\frac{d}{dx}e^{\nu_q\frac{d^2}{dx^2}})}\cdot 1 = e^{kx}\ e^{[\nu k^2]_q t} \tag{3.56}$$

To show this we need to use the Zassenhaus formula.

Proposition 3.5 *The Zassenhaus formula (Magnus, 1954) for two operators X and Y is given by*

$$e^{\xi(X+Y)} = e^{\xi X}e^{\xi Y}e^{-\frac{\xi^2}{2}[X,Y]}e^{\frac{\xi^3}{6}(2[Y,[X,Y]]+[X,[X,Y]])}\ \cdots \tag{3.57}$$

where ξ is an arbitrary constant parameter.

In order to apply the Zassenhaus formula to our case, we denote $P \equiv \frac{d}{dx}$, which satisfies the following commutation relations:

$$[P,x] = 1,\quad [P^2,x] = 2P,\ \cdots [P^n,x] = nP^{n-1}$$

and can be generalized in the following form

$$[f(P),x] = \frac{d}{dP}f(P), \tag{3.58}$$

for $\forall$ analytic function $f(P)$.

In our formula (3.56) by changing variables

$$Y \equiv \frac{2t}{q-1}\nu_q\frac{d}{dx}e^{\nu_q\frac{d^2}{dx^2}},\qquad Z \equiv e^{\nu_q\frac{d^2}{dx^2}},\quad X \equiv x$$

we obtain

$$e^{k(x+\frac{2t\nu_q}{q-1}\frac{\partial}{\partial x}e^{\nu_q\frac{d^2}{dx^2}})}\cdot 1 = e^{k(X+Y)}\cdot 1. \tag{3.59}$$

In order to factorize the exponential function we need to calculate the commutator relations:

$$[X,Y],\quad [Y,[X,Y]],\quad [X,[X,Y]],...$$

As easy to see all commutators are vanishing

$$[[X, Y], Y] = [[[X, Y], Y], Y] = ... = 0 \tag{3.60}$$

and therefore the following commutator is zero

$$[P, Z] = [P, e^{e^{\nu_q P^2}}] = 0.$$

Explicit calculation of Z and X commutator is obtained in terms of the commutators of X and Y, which we need for the Zassenhaus formula

$$[Z, X] = [e^{\nu_q P^2}, X] = 2\nu_q P e^{\nu_q P^2} = \frac{q-1}{t} Y,$$
$$[[Z, X], X] = \frac{q-1}{t}[Y, X] = -\frac{q-1}{t}[X, Y],$$
$$[[[Z, X], X], X] = \frac{q-1}{t}[[Y, X], X] = (-1)^2 \frac{q-1}{t}[X, [X, Y]],$$
$$\cdot$$
$$\cdot$$
$$\cdot$$

$$[[Z, \underbrace{X], X, ..., X}_{\text{n-times X}}] = \frac{q-1}{t}[[Y, \underbrace{X], X, ..., X}_{\text{n-1 -times X}}] = \frac{q-1}{t}[\underbrace{X, [X, ..., [X}_{\text{n-1-times X}}, Y]. \tag{3.61}$$

Now let us find commutator of operators Z and X.

Calculation of the following commutators give us derivatives

$$[Z, X] = [e^{\nu_q P^2}, X] = \frac{d}{dP} e^{\nu_q P^2} = 2P\nu_q e^{\nu_q P^2} = 2P\nu_q Z = \frac{d}{dP} Z$$

$$[[Z, X], X] = [2P\nu_q Z, X] = 2\nu_q(P[Z, X] + [P, X]Z) = 2\nu_q(P\frac{d}{dP} + 1)Z = 2\nu_q(2P^2\nu_q + 1)Z = \frac{d^2}{dP^2} Z$$

$$\begin{aligned} [[[Z, X], X], X] &= 2\nu_q([2P^2\nu_q Z + Z, X]) = 2\nu_q(2\nu_q[P^2 Z, X] + [Z, X]) \\ &= 2\nu_q(2\nu_q(P^2[Z, X] + [P^2, X]Z) + [Z, X]) \\ &= 2\nu_q(4\nu_q^2 P^3 Z + 4\nu_q PZ + 2\nu_q PZ) = \frac{d^3}{dP^3} Z, \end{aligned} \tag{3.62}$$

which can be generalized in the following proposition:

Proposition 3.6 *We present the following identity for commutators*

$$[[[[Z,\underbrace{X],X],...],X}_{n\text{-times } X}] = \frac{d^n}{dP^n}Z = \frac{d^n}{dP^n}e^{\nu_q P^2} \tag{3.63}$$

Proposition 3.7 *The commutation relation (3.63) can be expressed in terms of Hermite polynomials with operator argument*

$$[[Z,\underbrace{X],X,...X}_{n\text{-times } X}] = (-i)^n(\nu_q)^{\frac{n}{2}}H_n\left(i\sqrt{\nu_q}\frac{d}{dx}\right)e^{\nu_q\frac{d^2}{dx^2}}. \tag{3.64}$$

Proof From definition of Hermite polynomials

$$H_n(\xi) = (-1)^n e^{\xi^2}\frac{d^n}{d\xi^n}e^{-\xi^2}, \tag{3.65}$$

we have

$$H_n(\xi)e^{-\xi^2} = (-1)^n\frac{d^n}{d\xi^n}e^{-\xi^2}. \tag{3.66}$$

By considering commutation relation (3.63)

$$[[Z,\underbrace{X],X,...,X}_{\text{n-times}}] = \frac{d^n}{dP^n}Z = \frac{d^n}{dP^n}e^{\nu_q P^2},$$

and by changing variables $\nu_q P^2 \equiv -\xi^2 \Rightarrow \xi = i\sqrt{\nu_q}P \Rightarrow \frac{dP}{d\xi} = \frac{-i}{\sqrt{\nu_q}}$,

$$\begin{aligned} H_n(\xi)e^{-\xi^2} &= (-1)^n\frac{d^n}{d\xi^n}e^{-\xi^2} \\ H_n(i\sqrt{\nu_q}P)e^{\nu_q P^2} &= (-1)^n(\frac{dP}{d\xi}\frac{d}{dP})^n e^{\nu_q P^2} = \frac{i^n}{(\nu_q)^{\frac{n}{2}}}\frac{d^n}{dP^n}e^{\nu_q P^2} \end{aligned} \tag{3.67}$$

$$(-i)^n(\nu_q)^{\frac{n}{2}}H_n(i\sqrt{\nu_q}P)e^{\nu_q P^2} = \frac{d^n}{dP^n}e^{\nu_q P^2},$$

we can express commutation relation in terms of Hermite polynomials of the operator argument

$$\begin{aligned}[][[Z,X],X,...X] &= \frac{d^n}{dP^n}e^{\nu_q P^2} = (-i)^n(\nu_q)^{\frac{n}{2}}H_n(i\sqrt{\nu_q}P)e^{\nu_q P^2} \\ &= (-i)^n(\nu_q)^{\frac{n}{2}}H_n(i\sqrt{\nu_q}\frac{d}{dx})e^{\nu_q\frac{d^2}{dx^2}}. \end{aligned} \tag{3.68}$$

□

Using the Zassenhaus formula

$$e^{\xi(X+Y)}\cdot 1 = e^{\xi X}e^{\xi Y}e^{-\frac{\xi^2}{2}[X,Y]}e^{\frac{\xi^3}{6}(2[Y,[X,Y]]+[X,[X,Y]])}...e^{(-1)^{n+1}\frac{\xi^n}{n!}[X,[X,...,[X,Y]]]}...\cdot 1 \tag{3.69}$$

and (3.61) we can factorize the following exponential function as

$$\begin{aligned} e^{\xi(x+2t\frac{\nu_q}{q-1}\frac{d}{dx}e^{\nu_q\frac{d^2}{dx^2}})}\cdot 1 &= e^{\xi x}\,e^{\xi 2t\frac{\nu_q}{q-1}\frac{d}{dx}e^{\nu_q\frac{d^2}{dx^2}}}\,e^{\sum_{n=2}^{\infty}(-1)^{n+1}\frac{\xi^n}{n!}[X,[X,...,[X,Y]]]}\cdot 1 \\ &= e^{\xi x}\,e^{\xi 2t\frac{\nu_q}{q-1}\frac{d}{dx}e^{\nu_q\frac{d^2}{dx^2}}}\,e^{\sum_{n=2}^{\infty}\frac{\xi^n}{n!}\frac{t}{q-1}(-i)^n\nu_q^{\frac{n}{2}}H_n(i\sqrt{\nu_q}\frac{d}{dx})e^{\nu_q\frac{d^2}{dx^2}}}\cdot 1 \\ &= e^{\xi x}\,e^{\xi 2t\frac{\nu_q}{q-1}\frac{d}{dx}e^{\nu_q\frac{d^2}{dx^2}}}\prod_{n=2}^{\infty}e^{\frac{\xi^n}{n!}\frac{t}{q-1}(-i)^n\nu_q^{\frac{n}{2}}H_n(i\sqrt{\nu_q}\frac{d}{dx})e^{\nu_q\frac{d^2}{dx^2}}}\cdot 1 \\ &= e^{\xi x}\prod_{n=2}^{\infty}e^{\frac{\xi^n}{n!}\frac{t}{q-1}(-i)^n\nu_q^{\frac{n}{2}}H_n(0)}. \end{aligned} \tag{3.70}$$

Due to relations for Hermite polynomials

$$\begin{aligned} H_{2n}(0) &= (-1)^n\frac{(2n)!}{n!}, \\ H_{2n+1}(0) &= 0, \end{aligned}$$

we find that only the terms with even numbers survive

$$e^{\xi(x+2t\frac{\nu_q}{q-1}\frac{d}{dx}e^{\nu_q\frac{d^2}{dx^2}})}\cdot 1 = e^{\xi x}\prod_{k=1}^{\infty}e^{\frac{\xi^{2k}}{k!}\frac{t}{q-1}\nu_q^k}. \tag{3.71}$$

Replacing ξ by k and using $\nu_q = \nu \ln q$ we obtain

$$e^{k(x+2t\frac{\nu_q}{q-1}\frac{d}{dx}e^{\nu_q \frac{d^2}{dx^2}})} \cdot 1 = e^{kx} e^{\frac{t}{q-1}\sum_{l=1}^{\infty}\frac{k^{2l}}{l!}\nu^l (\ln q)^l}. \tag{3.72}$$

Finally we can factorize this expression in the form of the plane wave solution

$$e^{k(x+2t\frac{\nu_q}{q-1}\frac{d}{dx}e^{\nu_q \frac{d^2}{dx^2}})} \cdot 1 = e^{kx}\ e^{[\nu k^2]_q t}. \tag{3.73}$$

3.4. q-Viscous Burgers' Equation

We can relate our q-diffusive heat equation with nonlinear q-viscous Burgers' equation. By dividing equation (3.1) with $\phi(x,t)$ we obtain

$$(\ln \phi(x,t))_t = \frac{1}{\phi(x,t)}\left[\nu \frac{\partial^2}{\partial x^2}\right]_q \phi(x,t) \tag{3.74}$$

and taking the x derivative of both sides and denoting

$$(\ln \phi(x,t))_x = \frac{\phi_x}{\phi} \equiv u, \tag{3.75}$$

we get

$$u_t = \left(\frac{1}{\phi}\left[\nu \frac{\partial^2}{\partial x^2}\right]_q \phi(x,t)\right)_x . \tag{3.76}$$

Proposition 3.8 *We present the following relation*

$$\frac{1}{\phi}\left[\nu\frac{\partial^2}{\partial x^2}\right]_q \phi = \left[\nu(\frac{d}{dx}+u)^2\right]_q \cdot 1, \tag{3.77}$$

where $u = \frac{\phi_x}{\phi}$

Proof Using definition of the q-operator number

$$\begin{aligned} \frac{1}{\phi}\left[\nu\frac{\partial^2}{\partial x^2}\right]_q \phi &= \frac{1}{\phi}\frac{q^{\nu\frac{\partial^2}{\partial x^2}}-1}{q-1}\phi = \frac{1}{\phi}\frac{1}{q-1}(e^{\nu\ln q\frac{\partial^2}{\partial x^2}}-1)\phi \\ &= \frac{1}{q-1}\frac{1}{\phi}\sum_{n=1}^{\infty}\frac{(\nu\ln q)^n}{n!}\frac{\partial^{2n}}{\partial x^{2n}}\phi \end{aligned} \tag{3.78}$$

and denoting $\phi \equiv e^f$, which implies

$$f = \ln\phi, \quad f_x = (\ln\phi)_x = \frac{\phi_x}{\phi} \equiv u,$$

we find

$$\begin{aligned} \frac{1}{\phi}\left[\nu\frac{\partial^2}{\partial x^2}\right]_q \phi &= \frac{1}{q-1}\sum_{n=1}^{\infty}\frac{(\nu\ln q)^n}{n!}(\frac{d}{dx}+u)^{2n}\cdot 1 = \frac{1}{q-1}(e^{\nu\ln q(\frac{d}{dx}+u)^2}-1)\cdot 1 \\ &= \frac{q^{\nu(\frac{d}{dx}+u)^2}-1}{q-1}\cdot 1 = \left[\nu\left(\frac{d}{dx}+u\right)^2\right]_q \cdot 1 \end{aligned} \tag{3.79}$$

□

Substituting (3.77) into equation (3.76) we obtain the q-viscous Burgers' equation as

$$u_t = \left(\left[\nu(\frac{d}{dx}+u)^2\right]_q \cdot 1\right)_x. \tag{3.80}$$

By using solution of q-diffusive heat equation we can find the solution of q-viscous Burgers' equation. As a first particular solution of q-diffusive heat equation (3.1) we choose the traveling plane wave solution

$$\phi(x,t) = e^{kx+[\nu k^2]_q t}.$$

This plane wave is the generating function for Kampe de Feriet polynomials (3.48), being polynomial solution of q-diffusive equation. Moving zeros of Kampe-de Feriet Polynomials then correspond to moving poles of q-viscous Burgers' equation (3.80).

By using the Cole-Hopf transformation, the plane wave solution gives the constant solution of the q-viscous Burgers' equation

$$u(x,t) = \frac{\phi_x}{\phi} = k.$$

By considering the superposition of two plane waves with different wave numbers k_1, k_2,

$$\phi(x,t) = e^{k_1 x+[\nu k_1^2]_q t} + e^{k_2 x+[\nu k_2^2]_q t}, \tag{3.81}$$

we get shock soliton solution in the following form

$$u(x,t) = \frac{\phi_x}{\phi} = \frac{k_1 e^{k_1 x+[\nu k_1^2]_q t} + k_2 e^{k_2 x+[\nu k_2^2]_q t}}{e^{k_1 x+[\nu k_1^2]_q t} + e^{k_2 x+[\nu k_2^2]_q t}}. \tag{3.82}$$

In Figure 3.7 we show one shock soliton for different values of q. Depending on value of q the soliton is moving faster ($q < 1$) or slower ($q > 1$) than in usual $q = 1$ case. By fixing constants $k_2 > k_1 > 0$, at fixed time we have asymptotic

$$\begin{aligned} x \to +\infty \quad &\Rightarrow \quad u \to k_2 \\ x \to -\infty \quad &\Rightarrow \quad u \to k_1. \end{aligned}$$

Then our q-shock soliton solution can be written as

$$u(x,t) = \left(k_1 + \frac{k_2 - k_1}{1 + e^{(k_2-k_1)(x-vt)}}\right), \tag{3.83}$$

where the velocity of shock is

$$v = -\frac{[k_1^2 \nu]_q - [k_2^2 \nu]_q}{k_1 - k_2}.$$

To analyze this expression we choose $k_1 = 0$ and denote $k_2 \equiv k$, so that the soliton velocity is

$$v = \frac{2\nu k \ln q}{q-1} e^{\nu k^2 \ln q}. \tag{3.84}$$

For $q < 1$ this velocity is bounded from the above, and takes maximal value

$$|v_{max}| = \sqrt{\frac{2\nu}{e} \ln \frac{1}{q}} \frac{1}{1-q} \tag{3.85}$$

for $k = \pm 1/\sqrt{2\nu \ln 1/q}$.

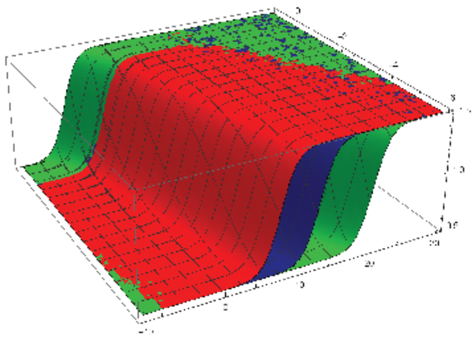

Figure 3.7. One shock soliton for $q = 1$(blue), $q = 0.5$(red), $q = 2$(green)

We show graph of this velocity in Figure 3.8. This dependence creates a new property of the shock soliton. Namely, for values of k bigger than the extremum point, and corresponding amplitudes, the velocity is not growing, but decaying. It produces new type of shock interaction. To see this we look for two shock soliton solutions.

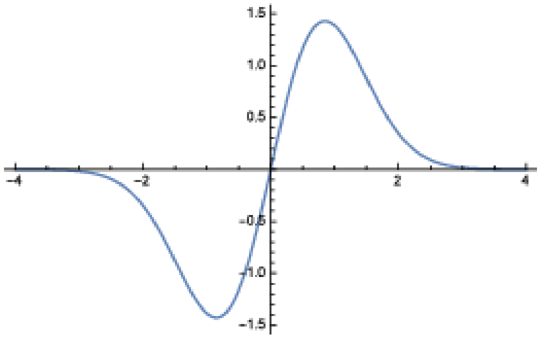

Figure 3.8. Soliton velocity for $q = 0.5$

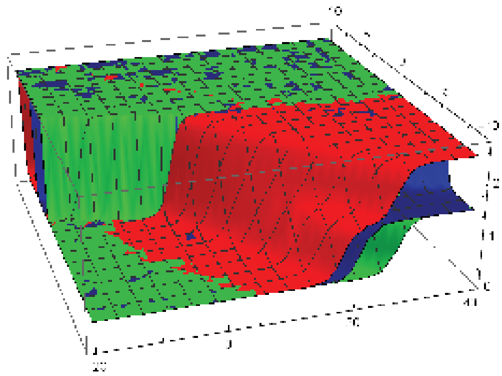

Figure 3.9. Two shock solitons for $q = 1$(blue), $q = 0.5$(red), $q = 2$(green)

By taking superposition of 3-plane waves

$$\phi(x,t) = e^{k_1 x + [\nu k_1^2]_q t} + e^{k_2 x + [\nu k_2^2]_q t} + e^{k_3 x + [\nu k_3^2]_q t}, \tag{3.86}$$

we find two shock soliton solution in the form

$$u(x,t) = \frac{\phi_x}{\phi} = \frac{k_1 e^{k_1 x + [\nu k_1^2]_q t} + k_2 e^{k_2 x + [\nu k_2^2]_q t} + k_3 e^{k_3 x + [\nu k_3^2]_q t}}{e^{k_1 x + [\nu k_1^2]_q t} + e^{k_2 x + [\nu k_2^2]_q t} + e^{k_3 x + [\nu k_3^2]_q t}}. \tag{3.87}$$

In Figure 3.9 we show fusion of two shock solitons moving with speeds, depending on values of q. For $q > 1$ they move slower and for $q < 1$ the speed of shocks collision is going faster than in $q = 1$ case. In addition to this, for $q < 1$ case here we have a new type of phenomena. By choosing parameters $k_1 = 0$, $k_2 < k_0$ and $k_3 > k_0$, where $k_0 > 0$ is extremum point with maximal speed, we find that the soliton with higher amplitude is moving slowly and splits to two solitons, one of which with smaller amplitude is moving faster. We illustrate this behavior as soliton fission in Figures 3.10 and 3.11.

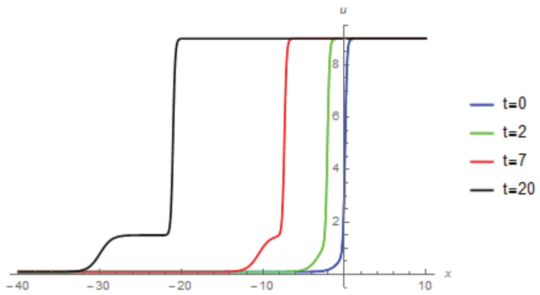

Figure 3.10. Shock fission

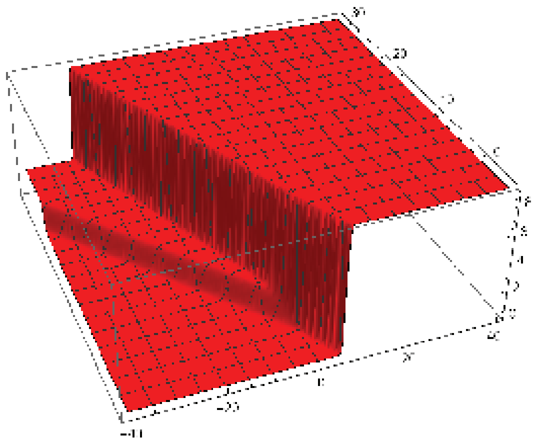

Figure 3.11. Soliton fission for $q = 0.5$

Superposition of $n+1$ plane waves with wave numbers $k_1, k_2, ..., k_{n+1}$ and constants $\eta_1, ..., \eta_{k+1}$ gives n-shock soliton solution in the form

$$u(x,t) = \frac{\sum_{i=1}^{n+1} k_i e^{k_i x + [\nu k_i^2]_q t + \eta_i}}{\sum_{i=1}^{n+1} e^{k_i x + [\nu k_i^2]_q t + \eta_i}}. \tag{3.88}$$

3.5. q-Semiclassical Expansion of q-Diffusive Heat Equation

If in q-diffusive heat equation (3.1) we expand the right hand side according to ν, then we get infinite order equation with even order derivative in x,

$$\begin{aligned}
\frac{\partial \phi}{\partial t} = \left[\nu \frac{\partial^2}{\partial x^2}\right]_q \phi &= \frac{q^{\nu \frac{\partial^2}{\partial x^2}-1}}{q-1}\phi \\
&= \frac{1}{q-1}(e^{\ln q \nu \frac{\partial^2}{\partial x^2}} - 1)\phi \\
&= \frac{1}{q-1}\sum_{n=1}^{\infty}(\ln q)^n \nu^n (\frac{\partial^2}{\partial x^2})^n \phi \\
&= \frac{1}{q-1}(\nu \ln q \frac{\partial^2}{\partial x^2} + \frac{(\nu \ln q)^2}{2!}\frac{\partial^4}{\partial x^4} + ...)\phi, \qquad (3.89)
\end{aligned}$$

where the first order equation for $\nu \ll 1$, is the standard heat equation, but with deformed diffusion coefficient $\nu_q = \nu \frac{\ln q}{q-1}$,

$$\frac{\partial \phi}{\partial t} = \nu_q \frac{\partial^2}{\partial x^2}\phi. \qquad (3.90)$$

As $q \to 1$, this gives standard heat equation with diffusion coefficient ν.

From another side, if we like to consider deformations of Heat equation for every power of $\ln q$, which we called the "q-semiclassical" expansion (since $q = 1$ case corresponds to "classical case") we need to use the Bernoulli polynomials.

Proposition 3.9 *Generating function for Bernoulli polynomials is defined as (Knuth et al, 1994)*

$$\frac{te^{xt}}{e^t - 1} = \sum_{n=0}^{\infty} B_n(x)\frac{t^n}{n!}. \qquad (3.91)$$

For $x = 0$ we have the generating function of Bernoulli numbers

$$\frac{t}{e^t - 1} = \sum_{n=0}^{\infty} B_n(0)\frac{t^n}{n!}, \qquad (3.92)$$

where $B_n(0) = B_n$-Bernoulli numbers.

In the generating function of Bernoulli numbers (8.33) by choosing $e^t \equiv q$ we have

$$\frac{\ln q}{q-1} = \sum_{n=0}^{\infty} B_n \frac{(\ln q)^n}{n!}. \tag{3.93}$$

This gives the modified diffusion coefficient as expansion with Bernoulli numbers in powers of $\ln q$

$$\nu_q = \nu \sum_{n=0}^{\infty} \frac{B_n}{n!} (\ln q)^n.$$

Proposition 3.10 $[n]_q$ *number can be expressed in terms of Bernoulli polynomials as*

$$[n]_q = n + \sum_{m=1}^{\infty} (B_{m+1}(n) - B_{m+1}(0)) \frac{(\ln q)^m}{(m+1)!} \tag{3.94}$$

Proof Using the definition of q-numbers

$$[n]_q = \frac{q^n - 1}{q-1} = \frac{e^{n \ln q}}{q-1} - \frac{1}{q-1} = \frac{1}{\ln q}\left(\frac{\ln q \; e^{n \ln q}}{q-1}\right) - \frac{1}{\ln q}\left(\frac{\ln q}{q-1}\right) \tag{3.95}$$

and denoting $\ln q = t$, we obtain

$$[n]_q = \frac{1}{t}\left(\frac{te^{nt}}{e^t - 1}\right) - \frac{1}{t}\left(\frac{t}{e^t - 1}\right). \tag{3.96}$$

The generating function for Bernoulli polynomials (3.91) and (3.92) allow us to get

$$[n]_q = \frac{1}{t}\underbrace{(B_0(n) - B_0(0))}_{*} + \sum_{m=0}^{\infty} (B_{m+1}(n) - B_{m+1}(0)) \frac{t^m}{(m+1)!}. \tag{3.97}$$

The term $*$ vanishes due to $B_0(x) = 1$. And we can write

$$[n]_q = \sum_{m=0}^{\infty} (B_{m+1}(n) - B_{m+1}(0)) \frac{t^m}{(m+1)!} = \underbrace{B_1(n) - B_1(0)}_{**} + \sum_{m=1}^{\infty} (B_{m+1}(n) - B_{m+1}(0)) \frac{(\ln q)^m}{(m+1)!}.$$

The term $**$ becomes n since $B_1(x) = x - \frac{1}{2}$ and the desired result is obtained. $\square$

The q-number operator for an operator A written as a formal power series in terms of Bernoulli polynomials is given as

$$[A]_q = A + \sum_{m=1}^{\infty} (B_{m+1}(A) - B_{m+1}(0)) \frac{(\ln q)^m}{(m+1)!}. \tag{3.98}$$

By expansion of q-diffusive heat equation (3.1) in powers of $\ln q$, we get higher derivative corrections to the Heat equation

$$\frac{\partial \phi}{\partial t} = \nu \frac{\partial^2}{\partial x^2}\phi + \sum_{m=1}^{\infty} \left(B_{m+1} \left(\nu \frac{\partial^2}{\partial x^2} \right) - B_{m+1} \right) \frac{(\ln q)^m}{(m+1)!} \phi. \tag{3.99}$$

For the q-Galilean boost operator we obtain

$$K = x + 2\nu t \frac{d}{dx} + 2\nu t \sum_{m=1}^{\infty} B_m \left(\nu \frac{d^2}{dx^2} \right) \frac{(\ln q)^m}{m!}. \tag{3.100}$$

The particular solution of q-diffusive heat equation for finite interval case can be expanded in the following form

$$\begin{aligned} \phi_n(x,t) &= e^{t[-\nu(\frac{\pi n}{l})^2]_q} \sin \frac{n\pi}{l} x \\ &= e^{-\nu(\frac{n\pi}{l}^2)x} \sin \frac{n\pi}{l} x \prod_{m=1}^{\infty} e^{(B_{m+1}(-\nu(\frac{n\pi}{t})^2 t) - B_{m+1}) \frac{(\ln q)^m}{(m+1)!}}, \end{aligned} \tag{3.101}$$

which shows how the solution of q-diffusive heat equation is modified by q-diffusivity.

We can expand the Green function of q-diffusive heat equation for infinite interval case as

$$\begin{aligned} G(x,y;t) &= \frac{2}{l} \sum_{n=0}^{\infty} \sin(\frac{n\pi}{l} x) \sin(\frac{n\pi}{l} y) e^{t[-\nu(\frac{n\pi}{l})^2]_q} \\ &= \frac{2}{l} \sum_{n=0}^{\infty} \sin(\frac{n\pi}{l} x) \sin(\frac{n\pi}{l} y) e^{-\nu(\frac{n\pi}{l}^2)t} \prod_{m=1}^{\infty} e^{(B_{m+1}(-\nu(\frac{n\pi}{t})^2 t) - B_{m+1}) \frac{(\ln q)^m}{(m+1)!}} \end{aligned}$$

showing modification due to q-diffusivity.

The q-viscous Burgers' equation is also expandable in terms of higher order derivatives as an arguments of Bernoulli polynomials

$$\begin{aligned} u_t &= \left(\left[\nu(\frac{d}{dx}+u)^2\right]_q \cdot 1\right)_x \\ &= \nu u_{xx} + 2\nu u u_x + \left(\sum_{m=1}^{\infty}(B_{m+1}\left(\nu\left(\frac{d}{dx}+u\right)^2\right) - B_{m+1})\frac{(\ln q)^m}{(m+1)!} \cdot 1\right)_x \end{aligned}$$

First two terms of this expansion give the standard Burgers' equation.

In a similar way we get expansion of the plane wave solution of q-diffusive heat equation in terms of powers of $\ln q$,

$$\phi(x,t) = e^{kx+[\nu k^2]_q t} = e^{kx+\nu k^2 t}\prod_{m=1}^{\infty} e^{(B_{m+1}(\nu k^2 t)-B_{m+1})\frac{(\ln q)^m}{(m+1)!}}, \tag{3.102}$$

showing modification of the standard plane wave solution.

And by using the superposition of two travelling waves (3.82) with different wave numbers k_1, k_2 as a solution of q-diffusive heat equation we obtain shock soliton solution of q-viscous Burgers' equation as q- modification of standard shock soliton solution

$$\begin{aligned} u(x,t) &= \frac{\phi_x}{\phi} = \frac{k_1 e^{k_1 x+[\nu k_1^2]_q t} + k_2 e^{k_2 x+[\nu k_2^2]_q t}}{e^{k_1 x+[\nu k_1^2]_q t} + e^{k_2 x+[\nu k_2^2]_q t}} \\ &= \frac{k_1 + e^{(k_2-k_1)x+\nu(k_2^2-k_1^2)t}\prod_{m=1}^{\infty} e^{(B_{m+1}(\nu k_2^2 t)-B_{m+1}(\nu k_1^2 t))\frac{(\ln q)^m}{(m+1)!}}}{1 + e^{(k_2-k_1)x+\nu(k_2^2-k_1^2)t}\prod_{m=1}^{\infty} e^{(B_{m+1}(\nu k_2^2 t)-B_{m+1}(\nu k_1^2 t))\frac{(\ln q)^m}{(m+1)!}}} \end{aligned}$$

For the speed of q-shock soliton,

$$v = -\frac{[k_1^2\nu]_q - [k_2^2\nu]_q}{k_1 - k_2},$$

we have expansion

$$v = -\left(\nu(k_1 + k_2) + \sum_{m=1}^{\infty} \frac{B_{m+1}(k_1^2\nu) - B_{m+1}(k_2^2\nu)}{k_1 - k_2} \frac{(\ln q)^m}{(m+1)!} \right). \tag{3.103}$$

By using explicit formula for Bernoulli polynomials for $n \geq 0$

$$B_n(x) = \sum_{j=0}^{n} \binom{n}{j} b_j x^{n-j}, \tag{3.104}$$

where b_j are Bernoulli numbers, the modified velocity is written as

$$v = -\nu(k_1 + k_2) \left(1 + \sum_{m=1}^{\infty} \frac{(\ln q)^m}{(m+1)!} S_{m+1}(k_1, k_2) \right), \tag{3.105}$$

where

$$S_{m+1} \equiv \sum_{j=0}^{m+1} \binom{m+1}{j} b_{m+1-j} [j]_{\nu k_1^2, \nu k_2^2} \tag{3.106}$$

and the q-number with q_1, q_2 basis are defined as

$$[n]_{q_1,q_2} = \frac{q_1^n - q_2^n}{q_1 - q_2}. \tag{3.107}$$

This shows the modification of the standard velocity of shock soliton.

Now we consider finite ν, but expand in terms of $\epsilon = q - 1 \ll 1$ in (3.89). Then for $q \to 1,\ \epsilon \ll 1, q = 1 + \epsilon$ and ν-arbitrary finite, we obtain higher order derivative corrections to heat equation

$$\begin{aligned} \frac{\partial \phi}{\partial t} &= \nu \frac{\partial^2}{\partial x^2}\phi + \epsilon(-\frac{\nu}{2}\frac{\partial^2}{\partial x^2} + \frac{\nu^2}{2}\frac{\partial^4}{\partial x^4})\phi + O(\epsilon^2) \\ &= \nu\left(1 - \frac{\epsilon}{2}\right)\frac{\partial^2}{\partial x^2}\phi + \epsilon\frac{\nu^2}{2}\frac{\partial^4}{\partial x^4}\phi + O(\epsilon^2), \end{aligned} \tag{3.108}$$

In the first term, for small q we have diffusion coefficient which is modified by ϵ.

In the next section we are going to construct q-viscous Burgers' equation, related to our q-diffusive heat equation. For this we need next proposition:

Proposition 3.11 *For any $f \in C^{\infty}$ we have the identity*

$$e^{-f}\frac{d^n}{dx^n}e^f = \left(\frac{d}{dx} + f_x\right)^n. \tag{3.109}$$

Proof

$$e^{-f}\frac{d}{dx}e^f\psi = e^{-f}\frac{d}{dx}(e^f\psi) = e^{-f}(f_x e^f\psi + e^f\frac{d}{dx}\psi) = (f_x + \frac{d}{dx})\psi, \tag{3.110}$$

so that

$$e^{-f}\frac{d}{dx}e^f = f_x + \frac{d}{dx} \tag{3.111}$$

and then we can generalize it as follows

$$\begin{aligned} e^{-f}\frac{d^n}{dx^n}e^f &= e^{-f}\frac{d}{dx}\frac{d}{dx}...\frac{d}{dx}e^f \\ &= e^{-f}\frac{d}{dx}e^f e^{-f}\frac{d}{dx}e^f e^{-f}...e^f e^{-f}\frac{d}{dx}e^f \\ &= \left(\frac{d}{dx} + f_x\right)^n. \end{aligned} \tag{3.112}$$

□

3.5.1. Corrections to Burgers' Equation

Expansion in $\epsilon = q - 1$ provides higher derivative order corrections to the Burgers' equation.

$$\begin{aligned}
u_t &= \left([\nu(\frac{d}{dx} + u)^2]_q\right)_x = \left(\frac{1}{q-1}(e^{\nu \ln q(\frac{d}{dx}+u)^2} - 1)\cdot 1\right)_x = \left(\frac{1}{q-1}\sum_{n=1}^{\infty}\frac{(\nu \ln q)^n}{n!}(\frac{d}{dx}+u)^{2n}\cdot 1\right)_x \\
&= \frac{1}{q-1}\left(\frac{\nu \ln q}{1!}(\frac{d}{dx}+u)^2\cdot 1 + \frac{(\nu \ln q)^2}{2!}(\frac{d}{dx}+u)^4\cdot 1 + ...\right)_x \\
&= \frac{1}{q-1}\left(\nu \ln q\ (u_x + u^2) + \frac{(\nu \ln q)^2}{2!}(u_{xxx} + 4uu_{xx} + 3u_x^2 + 6u^2u_x + u^4) + ...\right)_x \\
&= \frac{\nu \ln q}{q-1}(u_{xx} + 2uu_x) + \frac{(\nu \ln q)^2}{(q-1)2!}(...) + ... \qquad (3.113)
\end{aligned}$$

This gives the deformation of Burgers' equation with parameter ν. In the limit $q \to 1$ it reduces to standard Burgers' equation

$$u_t = \nu u_{xx} + 2\nu uu_x.$$

For small $q = 1 + \epsilon$, $\epsilon \ll 1$ we write

$$\ln q = \ln(1+\epsilon) = \epsilon - \frac{\epsilon^2}{2} + \frac{\epsilon^3}{3} - ... = \epsilon(1 - \frac{\epsilon}{2} + \frac{\epsilon^2}{3} - ...)$$

and after substitution into (3.113) we get

$$\begin{aligned}
u_t &= \nu u_{xx} + 2\nu uu_x - \frac{\nu\epsilon}{2}(u_{xx} + 2uu_x) + \frac{\nu^2}{2!}\epsilon(...) + O(\epsilon^2) \\
&= (1 - \frac{\epsilon}{2})\nu u_{xx} + 2(1 - \frac{\epsilon}{2})\nu uu_x + ... \qquad (3.114)
\end{aligned}$$

This shows lower order corrections to Burgers equation from q-deformed viscosity.

3.6. Bäcklund Transformation for q-Viscous Burgers' Equation

In this section we find the Bäcklund transformation which relates two solutions of q-viscous Burgers equation (3.80).

Proposition 3.12 *The Bäcklund transfomation relating a solution u to an another solution v of q-viscous Burgers' equation is written as*

$$v = \frac{\psi_x}{\psi} = \frac{1 + xu + \frac{2\nu \ln q}{q-1} t\, q^{\nu(\frac{d}{dx}+u)^2}(u_x + u^2)}{x + \frac{2\nu \ln q}{q-1} t\, q^{\nu(\frac{d}{dx}+u)^2} u}. \tag{3.115}$$

By taking the logarithmic derivative of (3.44)

$$v \equiv (\ln\psi)_x = \frac{\psi_x}{\psi} = \frac{\phi + x\phi_x + \frac{2\nu \ln q}{q-1} t \frac{d^2}{dx^2} e^{\nu \ln q \frac{d^2}{dx^2}} \phi}{x\phi + \frac{2\nu \ln q}{q-1} t \frac{d}{dx} e^{\nu \ln q \frac{d^2}{dx^2}} \phi}, \tag{3.116}$$

and taking the ϕ parenthesis in RHS we have

$$\frac{\psi_x}{\psi} = \frac{1 + x\frac{\phi_x}{\phi} + \frac{2\nu \ln q}{q-1} t \frac{1}{\phi} \frac{d^2}{dx^2} e^{\nu \ln q \frac{d^2}{dx^2}} \phi}{x + \frac{2\nu \ln q}{q-1} t \frac{1}{\phi} \frac{d}{dx} e^{\nu \ln q \frac{d^2}{dx^2}} \phi}. \tag{3.117}$$

In order to write the above transformation in a proper form we need to find the following expressions

$$I \equiv \frac{1}{\phi}\frac{d}{dx} e^{\nu \ln q \frac{d^2}{dx^2}} \phi \tag{3.118}$$

$$II \equiv \frac{1}{\phi}\frac{d^2}{dx^2} e^{\nu \ln q \frac{d^2}{dx^2}} \phi. \tag{3.119}$$

These expressions can be written in terms of Bell polynomials

Definition 3.3 *The Bell polynomials are defined by the exponential generating function*

$$Y_n(\vec{y}) \equiv Y_n(y_1, y_2, ..., y_n) = e^{-y(x)} \frac{d^n}{dx^n} e^{y(x)}, \tag{3.120}$$

where

$$y_x \equiv y_1,\ y_{xx} = y_2,\ y_{xxx} = y_3, ...$$

Proposition 3.13 *The recursion formula for Bell polynomials is given as*

$$Y_n(y_1, ..., y_n) = e^{-y(x)} \frac{d^n}{dx^n} e^{y(x)} = \left(\frac{d}{dx} + y_1 \right)^n \cdot 1. \tag{3.121}$$

Proof It can be proved by using mathematical induction:

For $n = 1$ we have

$$Y_1(y_1) = e^{-y(x)} \frac{d}{dx} e^{y(x)} = \left(\frac{d}{dx} + y_1 \right) \cdot 1 = y_1,$$

for $n = 2$

$$Y_2(y_1, y_2) = e^{-y(x)} \frac{d^2}{dx^2} e^{y(x)} = \left(\frac{d}{dx} + y_1 \right)^2 \cdot 1 = y_1^2 + y_2,$$

and we suppose that it is true for n :

$$Y_n(y_1, ..., y_n) = e^{-y(x)} \frac{d^n}{dx^n} e^{y(x)} = \left(\frac{d}{dx} + y_1 \right)^n \cdot 1.$$

We need to prove it for $n + 1$ case:

$$\begin{aligned} Y_{n+1}(y_1, ..., y_n, y_{n+1}) &= e^{-y(x)} \frac{d^{n+1}}{dx^{n+1}} e^{y(x)} = e^{-y(x)} \frac{d}{dx} \frac{d^n}{dx^n} e^{y(x)} \\ &= e^{-y(x)} \frac{d}{dx} e^{y(x)} \underbrace{e^{-y(x)} \frac{d^n}{dx^n} e^{y(x)}}_{Y_n(y_1, ..., y_n)} = \left(\frac{d}{dx} + y_1 \right) Y_n(y_1, ..., y_n) \\ &= \left(\frac{d}{dx} + y_1 \right) \left(\frac{d}{dx} + y_1 \right)^n \cdot 1 = \left(\frac{d}{dx} + y_1 \right)^{n+1} \cdot 1, \end{aligned} \tag{3.122}$$

where $y_1 = y_x$. □

By using the above definition for $y = \ln \phi$, the first identity can be rewritten as infinitive series of Bell polynomials

$$\begin{aligned} I &= \frac{1}{\phi} \frac{d}{dx} e^{\nu \ln q \frac{d^2}{dx^2}} \phi = \frac{1}{\phi} \sum_{n=0}^{\infty} \frac{(\nu \ln q)^n}{n!} \frac{d^{2n+1}}{dx^{2n+1}} \phi = e^{-\ln \phi} \sum_{n=0}^{\infty} \frac{(\nu \ln q)^n}{n!} \frac{d^{2n+1}}{dx^{2n+1}} e^{\ln \phi} \\ &= e^{-\ln \phi} \sum_{n=0}^{\infty} \frac{(\nu \ln q)^n}{n!} e^{\ln \phi}\ Y_{2n+1} \left((\ln \phi)_x,\ (\ln \phi)_{xx}, ..., (\ln \phi)_{2n+1} \right) \end{aligned}$$

$$I = \sum_{n=0}^{\infty} \frac{(\nu \ln q)^n}{n!} Y_{2n+1}\left((\ln\phi)_x,\ (\ln\phi)_{xx}, ..., (\ln\phi)_{2n+1}\right) \tag{3.123}$$

$$\frac{d^{2n+1}}{dx^{2n+1}} e^{\ln\phi} = e^{\ln\phi}\ Y_{2n+1}\left((\ln\phi)_x,\ (\ln\phi)_{xx}, ..., (\ln\phi)_{2n+1}\right). \tag{3.124}$$

Similarly, the second identity is also expressed as infinite sum of Bell polynomials

$$II = \frac{1}{\phi}\frac{d^2}{dx^2} e^{\nu \ln q \frac{d^2}{dx^2}} \phi = \sum_{n=0}^{\infty} \frac{(\nu \ln q)^n}{n!} Y_{2n+2}\left((\ln\phi)_x,\ (\ln\phi)_{xx}, ..., (\ln\phi)_{2n+2}\right). \tag{3.125}$$

By using Cole-Hopf transformation and its derivatives

$$(\ln\phi)_x = \frac{\phi_x}{\phi} = u, \quad (\ln\phi)_{xx} = u_x \equiv u_1, \quad (\ln\phi)_{xxx} = u_{xx} \equiv u_2,\ ...,\ (\ln\phi)_n = u_{n-1},$$

the identities I and II can be written in terms of Bell polynomials with arguments of u and its derivatives

$$\begin{aligned} I &= \sum_{n=0}^{\infty} \frac{(\nu \ln q)^n}{n!} Y_{2n+1}(u, u_1, u_2, ..., u_{2n}), &(3.126)\\ II &= \sum_{n=0}^{\infty} \frac{(\nu \ln q)^n}{n!} Y_{2n+2}(u, u_1, u_2, ..., u_{2n+1}). &(3.127)\end{aligned}$$

By using the above proposition, we can write I in terms of covariant momentum in the following form

$$\begin{aligned} I &= \sum_{n=0}^{\infty} \frac{(\nu \ln q)^n}{n!} \left(\frac{d}{dx} + u\right)^{2n+1} \cdot 1 = \left(\frac{d}{dx} + u\right) \sum_{n=0}^{\infty} \frac{(\nu \ln q)^n}{n!} \left(\frac{d}{dx} + u\right)^{2n} \cdot 1 \\ &= \left(\frac{d}{dx} + u\right) e^{\nu \ln q \left(\frac{d}{dx} + u\right)^2} \cdot 1 = e^{\nu \ln q \left(\frac{d}{dx} + u\right)^2} \left(\frac{d}{dx} + u\right) \cdot 1 = e^{\nu \ln q \left(\frac{d}{dx} + u\right)^2} u \\ &= q^{\nu \left(\frac{d}{dx} + u\right)^2} u. \end{aligned} \tag{3.128}$$

Similarly we can write

$$
\begin{aligned}
II &= \sum_{n=0}^{\infty} \frac{(\nu \ln q)^n}{n!} (\frac{d}{dx} + u)^{2n+2} \cdot 1 = (\frac{d}{dx} + u)^2 \sum_{n=0}^{\infty} \frac{(\nu \ln q)^n}{n!} (\frac{d}{dx} + u)^{2n} \cdot 1 \\
&= (\frac{d}{dx} + u)^2 e^{\nu \ln q (\frac{d}{dx}+u)^2} \cdot 1 = (\frac{d}{dx} + u) e^{\nu \ln q (\frac{d}{dx}+u)^2} (\frac{d}{dx} + u) \cdot 1 \\
&= e^{\nu \ln q (\frac{d}{dx}+u)^2} (\frac{d}{dx} + u) u = q^{\nu (\frac{d}{dx}+u)^2} (u_x + u^2). \qquad (3.129)
\end{aligned}
$$

Finally, putting the results (3.128) and (3.129) into the (3.117) we get the Bäcklund transformation between two solutions of q-viscous Burgers' equation

$$
v = \frac{\psi_x}{\psi} = \frac{1 + xu + \frac{2\nu \ln q}{q-1} t\, q^{\nu (\frac{d}{dx}+u)^2} (u_x + u^2)}{x + \frac{2\nu \ln q}{q-1} t\, q^{\nu (\frac{d}{dx}+u)^2} u}.
$$

As an example, we consider the constant solution $\phi(x, t) = C$ of q-diffusive heat equation (3.1), and the Cole Hopf transformation gives zero solution $u = 0$ for q-viscous Burgers equation. By using the Bäcklund transformation (3.115) for the solution $u = 0$ we find rational solution of the q-viscous Burgers equation in the form

$$
v = \frac{\psi_x}{\psi} = \frac{1}{x}. \qquad (3.130)
$$

It is instructive to prove this result by direct substitution to q-viscous Burgers' equation

$$
v_t = \left([\nu (\frac{d}{dx} + v)^2]_q \cdot 1 \right)_x .
$$

Using the definition of q-operator (3.2) the above equation is written as

$$
v_t = \frac{1}{q-1} \left(q^{\nu (\frac{d}{dx}+v)^2} \cdot 1 \right)_x . \qquad (3.131)
$$

Proposition 3.14 *We present the following relation*

$$
q^{\nu (\frac{d}{dx}+y_1)^2} \cdot 1 = e^{-y} q^{\nu \frac{d^2}{dx^2}} e^{y}, \qquad (3.132)
$$

where $v = y_1 = y_x$.

Proof Proof is easy by using the recursion operator for Bell polynomials (3.121),

$$\begin{aligned} q^{\nu(\frac{d}{dx}+y_1)^2}\cdot 1 &= e^{\nu\ln q(\frac{d}{dx}+y_1)^2}\cdot 1 = \sum_{n=0}^{\infty}\frac{(\nu\ln q)^n}{n!}\underbrace{\left(\frac{d}{dx}+y_1\right)^{2n}\cdot 1}_{e^{-y}\frac{d^{2n}}{dx^{2n}}e^y} \\ &= e^{-y}q^{\nu\frac{d^2}{dx^2}}e^y. \end{aligned} \tag{3.133}$$

□

For $v = y_1 = y_x = \frac{1}{x}$ we have $y = \ln x$, and the above proposition gives

$$q^{\nu(\frac{d}{dx}+\frac{1}{x})^2}\cdot 1 = e^{-\ln x}q^{\nu\frac{d^2}{dx^2}}e^{\ln x} = \frac{1}{x}q^{\nu\frac{d^2}{dx^2}}x.$$

As a result the equation (3.131) is written in the following form:

$$\begin{aligned} v_t &= \frac{1}{q-1}\left(\frac{1}{x}q^{\nu\frac{d^2}{dx^2}}x\right)_x = \frac{1}{q-1}\left(-\frac{1}{x^2}q^{\nu\frac{d^2}{dx^2}}x + \frac{1}{x}\left(q^{\nu\frac{d^2}{dx^2}}x\right)_x\right) \\ &= \frac{1}{q-1}\left(-\frac{1}{x^2}\underbrace{\sum_{n=0}^{\infty}\frac{(\nu\ln q)^n}{n!}\frac{d^{2n}}{dx^{2n}}x}_{x} + \frac{1}{x}\underbrace{\left(\sum_{n=0}^{\infty}\frac{(\nu\ln q)^n}{n!}\frac{d^{2n}}{dx^{2n}}x\right)_x}_{1}\right) \\ 0 = v_t &= \frac{1}{q-1}\left(-\frac{1}{x^2}x + \frac{1}{x}1\right) = 0, \end{aligned} \tag{3.134}$$

which proves that the rational solution $v = 1/x$ is solution for q-viscous Burgers' equation.

As a next example, we consider the travelling wave solution $\phi(x,t) = e^{kx+[\nu k^2]_q t}$ of q-diffusive heat equation (3.1) which gives constant solution $u(x,t) = \frac{\phi_x}{\phi} = k$ of q-viscous Burgers equation.

In order to find new solution for q-viscous Burgers' equation we use Bäcklund transformation (3.115) for $u = k$:

$$v = (\ln\psi)_x = \frac{1 + xk + \frac{2\nu\ln q}{q-1}tq^{\nu(\frac{d}{dx}+k)^2}k^2}{x + \frac{2\nu\ln q}{q-1}tq^{\nu(\frac{d}{dx}+k)^2}k} \tag{3.135}$$

and $y_1 = y_x = k$ implies $y = kx$. By using the proposition (6.1) we can calculate

$$\begin{aligned}
q^{\nu(\frac{d}{dx}+k)^2}k &= kq^{\nu(\frac{d}{dx}+k)^2}\cdot 1 = ke^{-kx}q^{\nu\frac{d^2}{dx^2}}e^{kx} = kq^{\nu k^2} \\
q^{\nu(\frac{d}{dx}+k)^2}k^2 &= k^2q^{\nu(\frac{d}{dx}+k)^2}\cdot 1 = k^2e^{-kx}q^{\nu\frac{d^2}{dx^2}}e^{kx} = k^2q^{\nu k^2}
\end{aligned} \tag{3.136}$$

and putting the results into the equation (3.135) we find the rational solution of q-viscous Burgers' equation as

$$v = k + \frac{1}{x + \frac{2\nu\ln q}{q-1}kte^{\nu\ln qk^2}}. \tag{3.137}$$

This solution has pole singularity moving with constant speed, equal to the group velocity (3.4).

Now we show in explicit form that the rational solution (3.137) satisfies the q-viscous Burgers' equation:

$$v_t = \left(\left[\nu(\frac{d}{dx} + v)^2\right]_q \cdot 1\right)_x.$$

First we write the RHS of the equation in the following form:

$$\begin{aligned}
\left(\left[\nu(\frac{d}{dx} + v)^2\right]_q \cdot 1\right)_x &= \frac{1}{q-1}\frac{d}{dx}\left(\sum_{n=1}^{\infty}\frac{(\nu\ln q)^n}{n!}(\frac{d}{dx} + v)^{2n}\cdot 1\right) \\
&= \frac{1}{q-1}\frac{d}{dx}\left(\sum_{n=1}^{\infty}\frac{(\nu\ln q)^n}{n!}e^{-y}\frac{d^{2n}}{dx^{2n}}e^{y}\right) \\
&= \frac{d}{dx}\left(e^{-y}e^{\nu\ln q\frac{d^2}{dx^2}}e^{y} - 1\right) \\
&= \frac{d}{dx}\left(\frac{1}{\psi}e^{\nu\ln q\frac{d^2}{dx^2}}\psi\right),
\end{aligned} \tag{3.138}$$

where in the last line we substituted $y = \ln\psi$. By using the Boost operator $\hat{K}$ we can rewrite ψ in the form

$$\psi = \hat{K}\phi = \left(x + \frac{2\nu\ln q}{q-1}t\frac{d}{dx}e^{\nu\ln q\frac{d^2}{dx^2}}\right)e^{kx+[\nu k^2]_q t},$$

and substituting into the equation and using the following property:

$$\begin{aligned} e^{\nu \ln q \frac{d^2}{dx^2}} x e^{kx} &= e^{\nu \ln q \frac{d^2}{dx^2}} \frac{d}{dk} e^{kx} = \frac{d}{dk} e^{\nu \ln q \frac{d^2}{dx^2}} e^{kx} = \frac{d}{dk} e^{\nu \ln q k^2} e^{kx} \\ &= (2\nu \ln q k + x) e^{kx + \nu \ln q k^2}, \end{aligned} \tag{3.139}$$

we get

$$\begin{aligned} RHS &= \frac{1}{q-1} \frac{d}{dx} \frac{(2\nu k \ln q + x + \frac{2\nu \ln q}{q-1} k t e^{\nu \ln q k^2}) e^{\nu \ln q k^2}}{x + \frac{2\nu \ln q}{q-1} k t e^{\nu \ln q k^2}} \\ &= \frac{1}{q-1} \frac{-e^{\nu \ln q k^2} 2\nu k \ln q}{(x + \frac{2\nu \ln q}{q-1} k t e^{\nu \ln q k^2})^2} \\ &= v_t. \end{aligned} \tag{3.140}$$

The last relation can be easily seen by taking derivative of (3.137) in t.

CHAPTER 4

Q-ANALYTIC FUNCTIONS

We introduce a new class of complex functions of complex argument which we call q-analytic functions. These functions satisfy q-Cauchy-Riemann equations and have real and imaginary parts as q-harmonic functions. We show that q-analytic functions are not the analytic functions in the usual sense. q-deformation here shows deviation from analyticity. Some of these complex functions, like q-analytic binomials, fall to the class of the generalized analytic functions. As a main example we study the complex q-binomial functions and their integral representation as a solution of the D-bar problem. In terms of these functions the complex q-analytic fractal, satisfying the self-similar q-difference equation is derived. A new type of quantum states as q-analytic coherent states and corresponding q-analytic Fock-Bargmann representation are constructed. As an application, we solve quantum q-oscillator problem in this representation, and show that the wave functions of quantum states are given by complex q-binomials.

4.1. q-Analytic Function

The q-differential of finite scale transformation for real function of one variable is defined as (Kac and Cheung, 2002)

$$d_q f(x) = f(qx) - f(x) = (D_q^x f(x)) d_q x, \tag{4.1}$$

where $d_q x = (q-1)x$, and q-derivative is

$$D_q^x f(x) = \frac{f(qx) - f(x)}{(q-1)x}. \tag{4.2}$$

For a complex-valued function $f(x, y)$ of two real variables x and y, the q-differential of f

$$d_q f(x, y) = f(qx, qy) - f(x, y), \tag{4.3}$$

can be rewritten as

$$d_q f(x,y) = (M_q^y D_x f(x,y))\, d_q x + (D_y f(x,y))\, d_q y, \tag{4.4}$$

where $d_q x = (q-1)x$, $d_q y = (q-1)y$, D_q^x and D_q^y are partial q-derivatives in x and y variables. Here M_q^y is the dilatation operator in y variable: $M_q^y F(x,y) = F(x,qy)$. In operator form we have $M_q^y = q^{y\frac{d}{dy}}$, and $D_q^y = \frac{q^{y\frac{d}{dy}}-1}{(q-1)y}$. In terms of complex coordinates $z = x+iy$, $\bar{z} = x-iy$ we can rewrite complex q-differentials $d_q z = d_q x + i d_q y$, $d_q \bar{z} = d_q x - i d_q y$ as $d_q z = (q-1)z$ and $d_q \bar{z} = (q-1)\bar{z}$. For q-differential of an arbitrary complex-valued function $f(x,y)$ then we get

$$d_q f(x,y) = (M_q^y D_z f)\, d_q z + (M_q^y D_{\bar{z}} f)\, d_q \bar{z}, \tag{4.5}$$

where we have introduced two linear operators of complex q-derivatives

$$D_z \equiv \frac{1}{2}(D_q^x - i D_{\frac{1}{q}}^y), \quad D_{\bar{z}} \equiv \frac{1}{2}(D_q^x + i D_{\frac{1}{q}}^y). \tag{4.6}$$

In the limiting $q \to 1$, q-differential formula (4.5) is reduced the known differential formula for a complex valued function $f(x,y)$

$$df(x,y) = \frac{\partial f}{\partial z} dz + \frac{\partial f}{\partial \bar{z}} d\bar{z}. \tag{4.7}$$

Definition 4.1 *A complex-valued function $f(x,y)$ of two real variables is called q-analytic in a region if the following identity holds*

$$D_{\bar{z}} f = \frac{1}{2}(D_x + i D_{\frac{1}{q}}^y) f = 0, \tag{4.8}$$

in the region.

The q-differential of q-analytic function then is given by

$$d_q f = (M_q^y D_q^z f)\, d_q z. \tag{4.9}$$

In the limit $q \to 1$, this definition reduces to the standard analyticity condition

$$\frac{\partial}{\partial \bar{z}} f = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right) f = 0 \tag{4.10}$$

leading to independence of $\bar{z}$: $df = \frac{\partial f}{\partial z} dz$.

In a similar way, we define q-anti-analytic function f as the one satisfying

$$D_z f = \frac{1}{2}(D_q^x - iD_{\frac{1}{q}}^y) f = 0, \tag{4.11}$$

and

$$d_q f = (M_q^y D_{\bar{z}} f)\, d_q \bar{z}. \tag{4.12}$$

Notice that analytic function $f(z)$, as a function of z, can depend on several constants. In the case of q-holomorphic function (4.8) these constants could be arbitrary q-periodic functions of z. For example $D_{\bar{z}} f(z) = 0$ determines $f(z)$ not uniquely but up to $f(z) + A_q(\bar{z})$, where $D_{\bar{z}} A_q(\bar{z}) = 0$, and $A_q(\bar{z})$-is q-periodic function $A_q(q\bar{z}) = A_q(\bar{z})$.

4.1.1. q-Analytic Binomial

The simplest and most important set of q-analytic functions is given by complex q-binomials

$$(x + iy)_q^n \equiv (x + iy)(x + iqy)(x + iq^2 y)...(x + iq^{n-1} y) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q q^{k(k-1)/2} i^k x^{n-k} y^k$$

expandable according to Gauss' binomial formula. Here, we follow notations for real q-binomial introduced in (Kac and Cheung, 2002). By direct calculation we have

$$D_{\bar{z}} (x + iy)_q^n = 0 \tag{4.13}$$

and

$$D_z(x+iy)_q^n = [n]_q(x+iy)_q^{n-1}. \tag{4.14}$$

Then for q-differential we get

$$d_q(x+iy)_q^n = (M_q^y D_z(x+iy)_q^n)\, d_q z = [n]_q(x+iqy)_q^{n-1}\, d_q z. \tag{4.15}$$

In a similar way, it is easy to show that complex conjugate q-binomial $(x-iy)_q^n$ is q-anti-analytic.

Here we notice an interesting limit of this binomial. For $q<1$ and $x=1$ the limit $n\to\infty$ exists and is given by the q-analogue of the Euler Formula

$$(1+iy)_q^\infty = E_q^{iy/(1-q)} = Cos_q\frac{y}{1-q} + iSin_q\frac{y}{1-q},$$

where E_q^x is the second Jackson's q-exponential function.

4.1.2. Negative Power q-Analytic Binomial

For $n\in N$, we define complex q-binomial of negative power as

$$(x+iy)_q^{-n} = \frac{1}{(x+iq^{-n}y)_q^n}. \tag{4.16}$$

For $z\neq 0$, it is an q-analytic function since

$$D_{\bar z}(x+iy)_q^{-n} = 0 \tag{4.17}$$

and

$$D_z(x+iy)_q^{-n} = [-n]_q(x+iy)_q^{-(n+1)}. \tag{4.18}$$

For its q-differential we have

$$d_q(x+iy)_q^{-n} = (M_q^y D_z(x+iy)_q^{-n})\, d_q z = [-n]_q(x+iqy)_q^{-(n+1)}\, d_q z. \tag{4.19}$$

4.2. q-Taylor Formula for q-Analytic Polynomial

By taking linear combination of complex q-binomials, we get q-analytic polynomials. Conversely, any complex-valued q-analytic polynomial function $P(z;q)$ of degree N has the following q-Taylor expansion

$$P(z;q) = \sum_{k=0}^{N} (D_z^k P)(0) \frac{(x+iy)_q^k}{[k]!}. \tag{4.20}$$

It follows from the expansion

$$P(z;q) = \sum_{k=0}^{N} a_k (x+iy)_q^k \tag{4.21}$$

where polynomials $\{(x+iy), (x+iy)_q^2, ..., (x+iy)_q^N\}$ are linearly independent. They constitute a basis for the space of complex q-analytic polynomials degree of N. Due to q-analyticity condition, the above expansion includes only $(x+iy)_q^k$ polynomials, and not the complex conjugate ones. Then differentiating this expression k-times in z, and putting $z = 0$ we find coefficients $a_k = (D_z^k P)(0)/[k]!$.

4.3. q-Taylor Representation for q-Analytic Functions

In the limit $N \to \infty$, the above Taylor formula for convergent series, represents q-analytic function

$$f(z;q) = \sum_{k=0}^{\infty} a_k (x+iy)_q^k = \sum_{k=0}^{\infty} (D_z^k f)(0) \frac{(x+iy)_q^k}{[k]!}. \tag{4.22}$$

It is clear that this q-analytic function satisfies the equation (4.8): $D_{\bar{z}} f(z) = 0$. If we fix base $|q| < 1$, and $n = 0, 1, 2, ...$ then we get the inequality

$$x^2 + q^{2n} y^2 \leq x^2 + y^2,$$

which implies

$$|(x+iy)_q^n| \leq |(x+iy)^n| \Rightarrow |\sum_{n=0}^{\infty} a_n (x+iy)_q^n| \leq \sum_{n=0}^{\infty} |a_n||(x+iy)^n|.$$

Proposition 4.1 *For a given complex-valued function $f(z)$ analytic inside the disk of radius R, $C_R : |z| < R$, $\partial f(z)/\partial \bar{z} = 0$, with the Taylor expansion*

$$f(z) = \sum_{n=0}^{\infty} a_n z^n, \tag{4.23}$$

there exists a q-analytic function $f(z;q)$, $D_{\bar{z}} f(z;q) = 0$, $|q| < 1$, convergent in the same disk C_R with the q-Taylor expansion

$$f(z;q) = \sum_{n=0}^{\infty} a_n (x+iy)_q^n. \tag{4.24}$$

According to this, every analytic function corresponds to a q-analytic function. For $q = 1$ a q-analytic function becomes analytic as $f(z; q = 1) = f(z)$ and parameter q shows deviation from this analyticity.

4.3.1. q-Analytic Function Examples

From standard exponential and trigonometric functions we have the following entire q-analytic functions with $|q| < 1$:

$$e(z;q) = \sum_{n=0}^{\infty} \frac{(x+iy)_q^n}{n!}, \tag{4.25}$$

$$\sin(z;q) = \sum_{n=0}^{\infty}(-1)^n \frac{(x+iy)_q^{2n+1}}{(2n+1)!}, \tag{4.26}$$

$$\cos(z;q) = \sum_{n=0}^{\infty}(-1)^n \frac{(x+iy)_q^{2n}}{(2n)!}. \tag{4.27}$$

From the definition of Jackson's q-exponential function,

$$e_q(z) \equiv \sum_{n=0}^{\infty} \frac{(x+iy)^n}{[n]!_q},$$

which is an entire function for $q > 1$, we have q-analytic q-exponential function

$$e_q(z;q) \equiv \sum_{n=0}^{\infty} \frac{(x+iy)_q^n}{[n]!_q},$$

or in terms of $z \equiv x + iy, \quad z_q \equiv x + iqy, \quad ... \quad z_{q^n} \equiv x + iq^n y, ...,$

$$e_q(z;q) = \sum_{n=0}^{\infty} \frac{z\, z_q \ldots z_{q^{n-1}}}{[n]_q!}.$$

This function $e_q(x+iy;q)$ is q-analytic since $D_{\bar{z}} e_q(z;q) = 0$ for $q > 1$ in the strip $-\infty < x < \infty$, $|y| < q/(q-1)$, and can be factorized in terms of Jackson's q-exponential functions as

$$e_q(x+iy;q) = e_q(x)E_q(iy) = e_q(x)\left(Cos_q(y) + iSin_q(y)\right). \tag{4.28}$$

This formula is q-analogue of Euler formula for analytic function $e^z = e^x e^{iy} = e^x(\cos y + i \sin y)$.

Here we like to emphasize that q-analytic functions as complex valued functions are not analytic functions in the usual sense, because arguments

$$z_{q^n} = x + iq^n y = \frac{(1+q^n)}{2} z + \frac{(1-q^n)}{2} \bar{z},$$

include both z and $\bar{z}$, so that $\frac{\partial}{\partial \bar{z}} e_q(x+iy;q) \neq 0$. The only exception for $q \neq 1$ is a linear function $f = az + b$.

Geometrically, we can represent every complex variable $z_{q^n} = x + iq^n y, n = 0, \pm 1, \pm 2, ...$ in a complex plane with coordinates $(x, q^n y)$ whose y coordinate is re-scaled. All these planes are intersecting along the real axis x. Then, the q-analytic function depends on infinite set of complex variables on these planes $z, z_{q^{\pm 1}}, z_{q^{\pm 2}}, ...$ and not on $\bar{z}, \bar{z}_{q^{\pm 1}}, \bar{z}_{q^{\pm 2}}, ...$ In the limiting case $q \to 1$, all planes coincide with the complex plane z, and q-analytic function becomes the standard analytic function.

4.4. q-Laurent Expansion for q-Analytic Functions

The Laurent formula for an analytic function in annular domain allows us to introduce corresponding q-analytic function.

In (4.16) for the negative power q-binomial

$$(x+iy)_q^{-n} = \frac{1}{(x+iq^{-n}y)_q^n}, \tag{4.29}$$

we found that for $z \neq 0$ it is q-analytic function, $D_{\bar{z}}(x+iy)_q^{-n} = 0$. If we fix the base $|q| < 1$, then we have inequality

$$\frac{1}{|(x+iq^{-n}y)_q^n|} \leq \frac{1}{|(x+iy)^n|} \tag{4.30}$$

and hence

$$\left| \sum_{n=1}^{\infty} \frac{b_n}{(x+iq^{-n}y)_q^n} \right| \leq \sum_{n=1}^{\infty} \frac{|b_n|}{|(x+iy)^n|}. \tag{4.31}$$

According to this relation we can extend class of q-analytic functions.

Proposition 4.2 *For a given complex-valued function $f(z)$ analytic inside the annular domain $D: r < |z| < R$, $\partial f(z)/\partial \bar{z} = 0$, and with the Laurent expansion*

$$f(z) = \sum_{n=-\infty}^{\infty} b_n z^n. \tag{4.32}$$

Then there exists a q-analytic function $f(z;q)$, $D_{\bar{z}}f(z;q) = 0$, convergent in the same domain D, with the q-Laurent expansion

$$f(z;q) = \sum_{n=-\infty}^{\infty} b_n (x+iy)_q^n. \tag{4.33}$$

As an example we have

$$e\left(\frac{1}{z};q\right) = \sum_{n=0}^{\infty} \frac{(x+iy)_q^{-n}}{n!} = \sum_{n=0}^{\infty} \frac{1}{n!(x+iq^{-n}y)_q^n} \tag{4.34}$$

which is q-analytic everywhere except $z = 0$.

4.5. The q-Cauchy-Riemann Equations

Expanding a q-holomorphic function to real and imaginary parts $f(x+iy;q) = u(x,y;q) + iv(x,y;q)$ due to (4.8), $(D_q^x + iD_{\frac{1}{q}}^y)(u+iv) = 0$, and we get the system of q-Cauchy-Riemann Equations

$$D_q^x u = D_{\frac{1}{q}}^y v, \quad D_q^x v = -D_{\frac{1}{q}}^y u. \tag{4.35}$$

The **q-Laplace operator** is defined in terms of q-holomorphic derivatives (4.6) as

$$\Delta_q \equiv 4D_z D_{\bar{z}} = \left(D_q^x\right)^2 + \left(D_{\frac{1}{q}}^y\right)^2 = (D_q^x)^2 + \frac{1}{q}(D_{\frac{1}{q}}^y)^2, \tag{4.36}$$

where the order of M_q^y and D_q^y operators are interchanged according to Q-commutative formula $(D_q^y M_Q^y) = Q(M_Q^y D_q^y)$.

Due to (4.8), the operator D_z acts on q-holomorphic function $f(z;q)$ just as D_q^x derivative:

$$D_z f(z;q) = \frac{1}{2}(D_q^x - iD_{\frac{1}{q}}^y) f(z;q) = D_q^x f(z;q). \tag{4.37}$$

Definition 4.2 *The real function $\phi(x,y)$ is a **q-harmonic function** if it satisfies the q-Laplace* *equation*

$$\Delta_q \phi(x,y) = 0. \quad (4.38)$$

Due to factorization $\Delta_q f = 4D_z D_{\bar{z}} f = 0$, the real and imaginary parts of a q-analytic function are conjugate q-harmonic functions

$$\Delta_q u(x,y) = 0, \qquad \Delta_q v(x,y) = 0.$$

These functions can be used for solution of two-dimensional q-heat and q-Schrödinger equations. Recently we have studied the q-heat equations in a line (Nalci and Pashaev, 2010), (Pashaev and Nalci, 2012). Different forms of these equations can be derived in the problems of random walk on quantum group (Protogenov, 2015) and gauge theory of self-similar systems (Olemskoi, 2000). Two dimensional version of stationary heat distribution in such systems is described by the q-Laplace equation $\Delta_q u = 0$ with general solution in terms of q-harmonic functions.

4.5.1. Examples of q-Harmonic Functions

From q-binomial for $n = 2$

$$(x+iy)_q^2 = (x+iy)(x+iqy) = x^2 - qy^2 + (1+q)ixy$$

we have q-harmonically conjugate functions $u(x,y) = x^2 - qy^2$, $v(x,y) = (1+q)xy$.

For arbitrary $n = 1, 2, 3...$, polynomial q-harmonic functions are

$$u(x,y) = \frac{1}{2}[(x+iy)_q^n + (x-iy)_q^n], \quad v(x,y) = \frac{1}{2i}[(x+iy)_q^n - (x-iy)_q^n]. \quad (4.39)$$

Simplest non-polynomial q-harmonic functions follow from (4.28) as

$$u(x,y) = e_q(x) Cos_q(y), \quad v(x,y) = e_q(x) Sin_q(y). \quad (4.40)$$

4.6. q-Analytic Function as Generalized Analytic Function

In previous sections we have seen that q-analytic functions depend on both z and $\bar{z}$ variables and are not analytic. Nevertheless, here we are going to show that some of q-analytic functions are generalized analytic functions (Vekua, 1962). This class of functions is related with the $\bar{\partial}$-problem (D-Bar Problem). The scalar equation

$$\frac{\partial \Phi(z, \bar{z})}{\partial \bar{z}} = f(z, \bar{z}) \tag{4.41}$$

for simple connected domain in complex z-plane called $\bar{\partial}$-problem (Ablowitz and Fokas, 1997). For complex functions

$$\Phi = u + iv, \quad f = \frac{g + ih}{2}, \quad z = x + iy$$

it is equivalent to the system of a generalized Cauchy-Riemann equations

$$\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} = g(x, y), \quad \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = h(x, y). \tag{4.42}$$

In case of analytic functions, $g(x, y) = h(x, y) = 0 \rightarrow f(x, y) = 0$ it recovers the Cauchy-Riemann equations.

Definition 4.3 *Complex function* $\Phi(z, \bar{z})$ *in a region R, satisfying equation*

$$\frac{\partial \Phi}{\partial \bar{z}} = A(z, \bar{z})\Phi + B(z, \bar{z})\bar{\Phi} \tag{4.43}$$

is called generalized analytic function.

The particular case $B = 0$, the last equation reduces to D-Bar equation

$$\frac{\partial \Phi}{\partial \bar{z}} = A(z, \bar{z})\Phi \tag{4.44}$$

which can be solved in closed form (Vekua, 1962), (Ablowitz and Fokas, 1997) as:

$$\Phi(z,\bar{z}) = \omega(z) e^{\frac{1}{2\pi i}\int\int_D \frac{A(\zeta,\bar{\zeta})}{\zeta - z}\, d\zeta\wedge d\bar{\zeta}}, \tag{4.45}$$

where $\omega(z)$ is an arbitrary analytic function. This solution was first obtained by N. Theodoresco in 1931, (Theodoresco, 1936).

4.6.1. Complex q-Binomial

Here we intend to show that complex q-binomials $\Phi(z,\bar{z}) = (x+iy)^n_q$ are generalized analytic functions. Calculating the partial derivatives

$$\frac{\frac{\partial}{\partial x}(x+iy)^n_q}{(x+iy)^n_q} = \frac{\partial}{\partial x}\sum_{k=0}^{n-1}\ln(x+iq^k y) = \sum_{k=0}^{n-1}\frac{1}{x+iq^k y} \tag{4.46}$$

we get

$$\begin{aligned}\frac{\partial}{\partial x}(x+iy)^n_q &= (x+iy)^n_q \sum_{k=0}^{n-1}\frac{1}{x+iq^k y},\\ \frac{\partial}{\partial y}(x+iy)^n_q &= (x+iy)^n_q \sum_{k=0}^{n-1}\frac{iq^k}{x+iq^k y},\end{aligned} \tag{4.47}$$

and

$$\frac{\partial}{\partial\bar{z}}(x+iy)^n_q = \frac{1-q}{2}(x+iy)^n_q \sum_{k=0}^{n-1}\frac{[k]_q}{x+iq^k y} \tag{4.48}$$

Therefore $\Phi(z,\bar{z}) = (x+iy)^n_q$ is the generalized analytic function satisfying $\bar{\partial}$-equation (4.44)

$$\frac{\partial}{\partial\bar{z}}\Phi(z,\bar{z}) = \Phi(z,\bar{z})(1-q)\sum_{k=0}^{n-1}\frac{[k]_q}{(1+q^k)z + (1-q^k)\bar{z}}, \tag{4.49}$$

where

$$A(z,\bar{z}) = (1-q)\sum_{k=1}^{n-1}\frac{[k]_q}{(1+q^k)z+(1-q^k)\bar{z}}.$$

Here parameter q expresses deviation from analyticity, and for $q = 1$ we have $A(z,\bar{z}) \equiv 0$ and D-Bar Equation ($\bar{\partial}$-problem) reduces to the holomorphicity condition $\frac{\partial}{\partial\bar{z}} z^n = 0$. By using (4.44) and (4.45) we find a new representation for q-Binomial:

$$(x+iy)_q^n = \omega(z)\exp\left[\frac{1}{2\pi i}\int\int_D \frac{1-q}{\zeta - z}\sum_{k=1}^{n-1}\frac{[k]_q}{(1+q^k)\zeta+(1-q^k)\bar{\zeta}}d\zeta\wedge d\bar{\zeta}\right], \tag{4.50}$$

where

$$\omega(z) = \left(\frac{z}{2}\right)^n\prod_{k=0}^{n-1}(1+q^k). \tag{4.51}$$

Details of these calculations are given in Appendix. This representation shows explicit relation between complex q-binomial $(x+iy)_q^n$ and complex binomial $(x+iy)^n = z^n$.

4.7. Complex q-Analytic Fractals

In this section we are going to construct self-similar fractal surface as a q-analytic function. In papers (Erzan, 1997), (Erzan and Eckmann, 1997) it was shown how generators of fractal and multi-fractal sets with discrete dilatation symmetries can be related to q-derivative operator. It was applied then to free energy of spin systems on hierarchical lattices (Erzan, 1997), (Erzan and Eckmann, 1997) and irreversible dynamics on such lattices (Erzan and Gorbon, 1999). Key point is that singular part of critical spin systems on hierarchical lattices possesses discrete dilatation symmetry and satisfies the homogeneity relation. Following similar arguments here we consider complex q-derivative and q-analytic functions to obtain new type of fractal sets.

We introduce complex valued function $f(x,y)$, as homogeneous function of degree d:

$$f(qx,qy) = q^d f(x,y). \tag{4.52}$$

The q-differential of this function is

$$d_q f = f(qx, qy) - f(x, y) = (q^d - 1) f(x, y) \tag{4.53}$$

and from (4.5), it can be rewritten as

$$(q^d - 1) f(x, y) = (M_q^y D_z f)\, d_q z + (M_q^y D_{\bar{z}} f)\, d_q \bar{z}. \tag{4.54}$$

For q-analytic function $D_{\bar{z}} f = 0$ the last term vanishes and we have the homogeneous q-difference equation

$$z M_q^y D_z f = \frac{q^d - 1}{q - 1} f. \tag{4.55}$$

Below we consider only the case $d = n$ as a positive integer. To find a solution of this equation, first we notice that complex q-binomial $(x + iy)_q^n$ is a homogenous function of degree n

$$(\lambda x + i\lambda y)_q^n = \lambda^n (x + iy)_q^n. \tag{4.56}$$

Combining this condition for $\lambda = q$ with q-analyticity condition $D_{\bar{z}}(x + iy)_q^n = 0$, we find that it satisfies the equation (4.55). That is

$$z M_q^y D_z (x + iy)_q^n = [n]_q (x + iy)_q^n. \tag{4.57}$$

Then the general q-analytic fractal solution is

$$f(x, y) = (x + iy)_q^n\, A_q(x, y), \tag{4.58}$$

where $A_q(qx, qy) = A_q(qx, y) = A_q(x, qy) = A_q(x, y)$ is complex valued q-periodic function in both x and y.

By choosing $A_q(x,y)$ as a real q-periodic function, we get the q-harmonic fractals as

$$U(x,y) = u(x,y)\; A_q(x,y), \qquad V(x,y) = v(x,y)\; A_q(x,y), \tag{4.59}$$

where u and v are q-harmonic functions (4.39). Specific form of these fractals depend on structure of $A_q(x,y)$. To fix it we have next Proposition.

Proposition 4.3 *A q-periodic function can be represented in the general form*

$$A_q(x,y) = (xy)^{-s} \sum_{k=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} q^{-s(k+l)}\; G(q^k x, q^l y) \tag{4.60}$$

Proof Consider

$$A_q(qx,y) = (qxy)^{-s} \sum_{k=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} q^{-s(k+l)}\; G(q^{k+1} x, q^l y)$$

By replacing k by $k-1$, it is obvious that $A_q(qx,y) = A_q(x,y)$. Similarly, it is easy to see that $A_q(x,y)$ is q-periodic in y argument as well. □

According to the above proposition, the general q-analytic fractal solution of q-difference self-similarity equation (4.57) is

$$f(x,y) = (xy)^{-s}(x+iy)_q^n \sum_{k=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} q^{-s(k+l)}\; G(q^k x, q^l y). \tag{4.61}$$

4.7.1. Examples of q-Periodic Functions

For $G(x,y) = \sin x\, \sin y$, from (4.60) we find

$$A_q(x,y) = (xy)^{-s} \sum_{k=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} \frac{\sin(q^k x)\; \sin(q^l y)}{q^{s(k+l)}}. \tag{4.62}$$

With another choice $G(x,y) = (1-e^{ix})\,(1-e^{iy})$, we get

$$A_q(x, y) = (xy)^{-s} \sum_{k=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} \frac{(1 - e^{iq^k x})(1 - e^{iq^l y})}{q^{s(k+l)}}.$$

This function can be written as a product of one dimensional q-periodic functions $A(x)$ and $B(y)$,

$$A_q(x, y) = x^{-s} \sum_{k=-\infty}^{\infty} \frac{1 - e^{iq^k x}}{q^{sk}} y^{-s} \sum_{l=-\infty}^{\infty} \frac{1 - e^{iq^l y}}{q^{sl}} = A(x)\, B(y),$$

representing the q-periodic parts of the Weierstrass-Mandelbrot function, and a canonical example of a fractal curve.

4.7.2. Double Mellin Series Expansion

Below we restrict our consideration to $A_q(x, y) = A_q(x)\, B_q(y)$, where $A_q(qx) = A_q(x)$, $B_q(qx) = B_q(x)$ are q-periodic functions. Without loss of generality we consider $A_q(x)$ case only in details. By changing argument $\ln x = t$ and $\ln q = T$ we have

$$A_q(qx) = A_q(x) \Rightarrow A_q(e^T e^t) = A_q(e^t).$$

Denoting $A_q(e^t) \equiv F(t)$ we find that it is T-periodic, $F(t + T) = F(T)$, and can be expanded to Fourier series

$$F(t) = \sum_{n=-\infty}^{\infty} c_n e^{\frac{i2\pi nt}{T}},$$

with Fourier coefficients

$$c_n = \frac{1}{T} \int_0^T F(t) e^{\frac{-i2\pi nt}{T}} dt.$$

According to this, q-periodic function $A_q(x)$ can be represented by complex series (the Mellin series)

$$A_q(x) = F(\ln x) = \sum_{n=-\infty}^{\infty} c_n x^{\frac{i2\pi n}{\ln q}}, \tag{4.63}$$

where

$$c_n = \frac{1}{\ln q} \int_1^q A_q(x)\, x^{\frac{-i2\pi n}{\ln q}} \frac{dx}{x}.$$

In a similar way for $B_q(y)$ we have

$$B_q(y) = \sum_{n=-\infty}^{\infty} d_n y^{\frac{i2\pi n}{\ln q}}, \tag{4.64}$$

with coefficients

$$d_n = \frac{1}{\ln q} \int_1^q B_q(y)\, y^{\frac{-i2\pi n}{\ln q}} \frac{dy}{y}.$$

Combining together we get the double-Mellin series representation of q-periodic function in the following form

$$A_q(x, y) = \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} c_n\, d_m\, x^{\frac{i2\pi n}{\ln q}}\, y^{\frac{i2\pi n}{\ln q}}. \tag{4.65}$$

By substituting to (4.58) and expanding q-binomial according to Gauss's Binomial formula, we obtain expansion of self-similar q-analytic function (q-analytic fractal) to double-Mellin series

$$f(x, y) = \sum_{k=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} c_k\, d_m \sum_{l=0}^{n} \begin{bmatrix} n \\ l \end{bmatrix}_q i^l\, q^{\frac{l(l-1)}{2}}\, x^{n-l+\frac{i2\pi k}{\ln q}}\, y^{l+\frac{i2\pi m}{\ln q}}. \tag{4.66}$$

In case of fractal (4.62) the expansion is

$$\begin{aligned} f(x,y) &= A_q(x,y)(x+iy)_q^n \\ &= \sum_{k=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \sum_{l=0}^{n} \begin{bmatrix} n \\ l \end{bmatrix}_q i^l \, q^{\frac{l(l-1)}{2}} \, x^{n-l-s} \, y^{l-s} \frac{\sin(q^k x) \, \sin(q^m y)}{q^{s(k+m)}}. \end{aligned}$$

4.7.3. Examples of q-Analytic Fractals

By choosing function

$$A_q(x,y) = \sin\left(\frac{2\pi}{\ln q} \ln|x|\right) \cdot \sin\left(\frac{2\pi}{\ln q} \ln|y|\right), \tag{4.67}$$

as q-periodic in x and y, we obtain the following set of homogenous self-similar q-analytic fractals of degree n,

$$f_n(x,y) = \sin\left(\frac{2\pi}{\ln q} \ln|x|\right) \cdot \sin\left(\frac{2\pi}{\ln q} \ln|y|\right) \cdot (x+iy)_q^n, \tag{4.68}$$

and for $Ref_n(x,y) \equiv u_n$ and $Imf_n(x,y) \equiv v_n$ the set of self-similar q-harmonic functions.

For $n = 0$, the simplest q-harmonic and q-periodic function is $f_0(x,y) = A_q(x,y)$ from (4.67). In Figure 4.1 and Figure 4.2, we plot $f_0(x,y)$ for $q = 2$ and $-0.5 \le x \le 0.5$, $-0.5 \le y \le 0.5$. By changing scale $(x,y) \to (q^n x, q^n y)$, or in our example magnifying our figure in scales $..., \frac{1}{4}, \frac{1}{2}, 2, 4, 8, ...,$ etc. we find that the figure shows the self-similar character remaining in the same form.

For $n = 2$, we have

$$\begin{aligned} u_2(x,y) &= (x^2 - qy^2) \cdot \sin\left(\frac{2\pi}{\ln q} \ln|x|\right) \cdot \sin\left(\frac{2\pi}{\ln q} \ln|y|\right), \\ v_2(x,y) &= [2]_q(xy) \cdot \sin\left(\frac{2\pi}{\ln q} \ln|x|\right) \cdot \sin\left(\frac{2\pi}{\ln q} \ln|y|\right). \end{aligned} \tag{4.69}$$

In Figure 4.3 we show 3D plot of $u_2(x,y)$ at $q = 2$ and $-10 \le x \le 10$, $-10 \le y \le 10$. By re-scaling coordinates in 2^n scale we get the same figures, showing self-similar structure of our q-harmonic function.

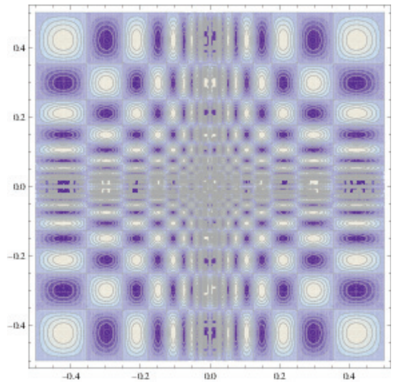

Figure 4.1. Contour plot of q-periodic q-harmonic function

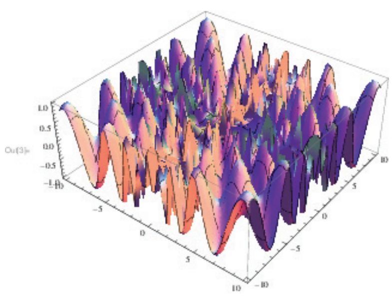

Figure 4.2. 3D plot of q-periodic q-harmonic function

For $n = 3$, we get

$$\begin{aligned} u(x,y) &= x(x^2 - qy^2 - [2]_q q^2 y^2) \cdot \sin\left(\frac{2\pi}{\ln q} \ln|x|\right) \cdot \sin\left(\frac{2\pi}{\ln q} \ln|y|\right), \\ v(x,y) &= y([2]_q x^2 + q^2(x^2 - qy^2)) \cdot \sin\left(\frac{2\pi}{\ln q} \ln|x|\right) \cdot \sin\left(\frac{2\pi}{\ln q} \ln|y|\right). \end{aligned} \tag{4.70}$$

In Figure 4.4 we show 3D plot of q-harmonic fractal $u_3(x,y)$ at $q = 2$ and $-1 \leq x \leq 1$, $-1 \leq y \leq 1$. This figure also shows self-similar structure at 2^n scale.

For $n = -1$, we have

$$f_{-1}(x,y) = \sin\left(\frac{2\pi}{\ln q} \ln|x|\right) \cdot \sin\left(\frac{2\pi}{\ln q} \ln|y|\right) \cdot (x + iy)_q^{-1}. \tag{4.71}$$

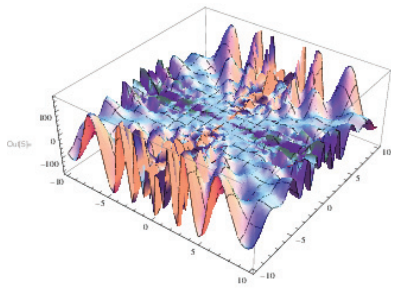

Figure 4.3. 3D plot of $n = 2$ q-harmonic function

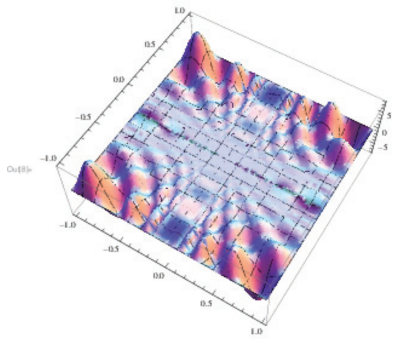

Figure 4.4. 3D plot of $n = 3$ q-harmonic function

and corresponding self-similar q-harmonic functions for $(x + iy \neq 0)$ are

$$\begin{aligned} u(x, y) &= \frac{q^2 x}{x^2 q^2 + y^2} \cdot \sin\left(\frac{2\pi}{\ln q} \ln |x|\right) \cdot \sin\left(\frac{2\pi}{\ln q} \ln |y|\right) \\ v(x, y) &= \frac{-qy}{x^2 q^2 + y^2} \cdot \sin\left(\frac{2\pi}{\ln q} \ln |x|\right) \cdot \sin\left(\frac{2\pi}{\ln q} \ln |y|\right) \end{aligned} \tag{4.72}$$

In Figure 4.5 and 4.6 we show contour plot and 3D plot of this q-harmonic fractal $u(x, y)$ at $q = 2$.

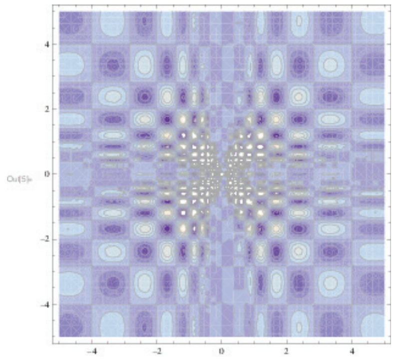

Figure 4.5. Contour plot of $n = -1$ q-harmonic function

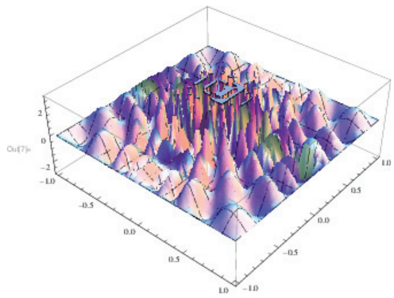

Figure 4.6. 3D plot of $n = -1$ q-harmonic function

4.8. q-Analytic Coherent States

In this section, we apply our q-analytic functions to construct quantum states of harmonic oscillator. We consider bosonic operators

$$[a, a^{+}] = I, \ \ [a, I] = 0, \ \ [a^{+}, I] = 0 \tag{4.73}$$

and the vacuum state $|0\rangle$:

$$a\,|0\rangle = 0, \langle 0|0\rangle = 1. \tag{4.74}$$

The orthonormal set of n-particle states, $n = 0, 1, 2, ...,$

$$|n\rangle = \frac{(a^+)^n}{\sqrt{n!}}\,|0\rangle\,,\ \ \langle n|m\rangle = \delta_{nm}, \tag{4.75}$$

generates the normalized Glauber coherent states, with complex α (Perelomov, 1986),

$$|\alpha\rangle = e^{-\frac{1}{2}|\alpha|^2}\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}\,|n\rangle\,. \tag{4.76}$$

By analogy with these coherent states we introduce a new set of q-analytic coherent states, parameterized by complex number $\alpha = \alpha_1 + i\alpha_2$:

$$|\alpha; q\rangle = C\sum_{n=0}^{\infty}\frac{(\alpha_1 + i\alpha_2)_q^n}{\sqrt{[n]_q!}}\,|n\rangle\,. \tag{4.77}$$

Normalization condition gives

$$1 = \langle\alpha; q|\alpha; q\rangle = |C|^2\sum_{n=0}^{\infty}\frac{(\alpha_1^2 + \alpha_2^2)_{q^2}^n}{[n]_q!} = |C|^2 e_q(\alpha_1^2 + \alpha_2^2)_{q^2},$$

where we denoted

$$e_q(\alpha_1^2 + \alpha_2^2)_{q^2} = \sum_{n=0}^{\infty}\frac{(\alpha_1^2 + \alpha_2^2)_{q^2}^n}{[n]_q!} \tag{4.78}$$

in Hahn's notations (Hahn, 1949), see also (Ernst, 2001). Then the normalized q-analytic coherent states are given by

$$|\alpha; q\rangle = \left(e_q(\alpha_1^2 + \alpha_2^2))_{q^2}\right)^{-\frac{1}{2}}\sum_{n=0}^{\infty}\frac{(\alpha_1 + i\alpha_2)_q^n}{\sqrt{[n]_q!}}\,|n\rangle\,. \tag{4.79}$$

For $|q| < 1$, due to evident relation $(\alpha_1^2 + \alpha_2^2)_{q^2}^n \leq (\alpha_1^2 + \alpha_2^2)^n$, we get inequality

$$e_q(\alpha_1^2 + \alpha_2^2)_{q^2} \leq e_q(\alpha_1^2 + \alpha_2^2), \tag{4.80}$$

where on the r.h.s we have the Jackson q-exponential function. From infinite product representation (Kac and Cheung, 2002) of the last function

$$e_q(|\alpha|^2) = \frac{1}{(1-(1-q)|\alpha|^2)_q^{\infty}}, \tag{4.81}$$

we can see that singularities of this function are located on the set of concentric circles with radiuses given by growing geometric progression $r_n = r_0 q^{-n/2}$, $r_0 = 1/\sqrt{1-q}$. Then both functions convergent in the disc $D : |\alpha|^2 \leq 1/(1-q)$. This is the region where normalization of our q-analytic coherent states is defined.

When $q \to 1$ these states reduce to the Glauber coherent states (4.76) and radius of convergency $r_0 \to \infty$. Here we emphasize that our q-analytic coherent states are also different from the q-coherent states appearing in representation of q-deformed Heisenberg-Weyl algebra (Vitiello, 2012), (Vitiello, 2009), (Vitiello, 2008). The last ones are analytic in α, while our states are not analytic but the q-analytic.

4.9. q-Analytic Fock-Bargmann Representation

The standard Fock-Bargman representation of an arbitrary state

$$|\psi\rangle = \sum_{n=0}^{\infty} c_n |n\rangle\,, \; <\psi|\psi> = \sum_{n=0}^{\infty} |c_n|^2 = 1,$$

is given by the scalar product of this state with Glauber's coherent state (4.76):

$$\langle\alpha|\psi\rangle = e^{-\frac{1}{2}|\alpha|^2}\psi(\bar{\alpha}), \tag{4.82}$$

where the wave function

$$\psi(\alpha) = \sum_{n=0}^{\infty} c_n \frac{\alpha^n}{\sqrt{n!}} \tag{4.83}$$

is an entire analytic function (Perelomov, 1986).

Analytic q-coherent state is defined (Arik and Coon, 1976) as

$$|\alpha\rangle = e^{-\frac{|\alpha|^2}{2}} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{[n]_q!}} |n\rangle\,, \tag{4.84}$$

and corresponding analytic q-Fock-Bargmann representation of an arbitrary state $|\psi\rangle$ is

$$\langle\alpha|\psi\rangle = e^{-\frac{|\alpha|^2}{2}} \psi(\bar{\alpha}), \tag{4.85}$$

where the wave function

$$\psi(\alpha) = \sum_{n=0}^{\infty} c_n \frac{\alpha^n}{\sqrt{[n]_q!}} \tag{4.86}$$

is an analytic function.

As an example, our q-analytic coherent state (4.79) in Fock-Bargman representation $< z|\alpha; q >$ is characterized by analytic function in z:

$$\psi_\alpha(z) = \left(e_q(\alpha_1^2 + \alpha_2^2)_{q^2}\right)^{-\frac{1}{2}} \sum_{n=0}^{\infty} \frac{(\alpha_1 + i\alpha_2)_q^n\, z^n}{\sqrt{[n]_q!\; n!}}. \tag{4.87}$$

By using our q-analytic coherent states (4.79), now we introduce new representation of these states which we call q-analytic Fock-Bargman representation. By taking the scalar product of $|\psi >$ with (4.79) we get

$$\langle\alpha; q|\psi\rangle = \left(e_q(\alpha_1^2 + \alpha_2^2)_{q^2}\right)^{-\frac{1}{2}} \sum_{n=0}^{\infty} c_n \frac{(\alpha_1 - i\alpha_2)_q^n}{\sqrt{[n]_q!}} \tag{4.88}$$

$$= \left(e_q(\alpha_1^2 + \alpha_2^2)_{q^2}\right)^{-\frac{1}{2}} \psi(\bar{\alpha}; q), \tag{4.89}$$

where the wave function

$$\psi(\alpha; q) = \sum_{n=0}^{\infty} c_n \frac{(\alpha_1 + i\alpha_2)_q^n}{\sqrt{[n]_q!}}, \tag{4.90}$$

is complex q-analytic function. Therefore, every complex q-analytic function, $D_{\bar{z}}\psi(x+iy)_q = 0$, determines quantum state in our q-analytic Fock-Bargmann representation.

Proposition 4.1 allows us to compare two wave functions in Fock-Bargman representation (4.83) and in q-analytic Fock-Bargman representation (4.90). Entire character of the first one implies existence of the second one for $|q| < 1$.

As the simplest example we find representation of the orthonormal basis $\{|n\rangle\}$, which is given just by complex q-analytic binomial

$$|n\rangle \;\rightarrow\; \psi_n(\alpha; q) = \frac{(\alpha_1 + i\alpha_2)^n_q}{\sqrt{[n]_q!}}. \tag{4.91}$$

It is not analytic, but as we have seen in Section 4.6.1, it represents the generalized analytic function.

As a next example, we find the Glauber coherent state $|\alpha\rangle$ (4.76) in our q-analytic Fock-Bargmann representation $< z; q|\alpha >$:

$$\psi_\alpha(z; q) = e^{-\frac{1}{2}|\alpha|^2} \sum_{n=0}^{\infty} \frac{(x+iy)^n_q}{\sqrt{[n]_q!}} \frac{\alpha^n}{\sqrt{n!}} = e^{-\frac{1}{2}|\alpha|^2} \sum_{n=0}^{\infty} \frac{(\alpha x + i\alpha y)^n_q}{\sqrt{[n]_q!\, n!}}, \tag{4.92}$$

which is q-analytic in $z = x + iy$.

4.10. Quantum q-Oscillator

We consider q-bosons with creation and annihilation operators

$$b^+ = a^+ \sqrt{\frac{[N+I]_q}{N+I}} = \sqrt{\frac{[N]_q}{N}}\, a^+, \tag{4.93}$$

$$b = \sqrt{\frac{[N+I]_q}{N+I}}\, a = a \sqrt{\frac{[N]_q}{N}}, \tag{4.94}$$

where operators a^+, a are given by (4.73), $N = a^+ a$, $[N]_q = \frac{q^N - 1}{q-1}$. The commutation relations are

$$b\, b^+ - b^+\, b = q^N, \tag{4.95}$$

$$b\, b^+ - q\, b^+\, b = I, \tag{4.96}$$

and for q-number operators we have

$$b^+\, b = [N]_q, \quad b\, b^+ = [N + I]_q. \tag{4.97}$$

As easy to see, n-particle states for b and a operators are the same

$$|n\rangle = \frac{(a^+)^n}{\sqrt{n!}}\, |0\rangle = \frac{(b^+)^n}{\sqrt{[n]_q!}}\, |0\rangle\,, \tag{4.98}$$

where vacuum state is $a\,|0\rangle = b\,|0\rangle = 0$. Then for b, b^+ operators we have

$$b\,|n\rangle = \sqrt{[n]_q}\,|n-1\rangle\,, \quad b^+\,|n\rangle = \sqrt{[n+1]_q}\,|n+1\rangle\,. \tag{4.99}$$

By using last relations we find action of these operators in q-analytic Fock-Bargman representation:

$$b \;\rightarrow\; D_z, \quad b^+ \;\rightarrow\; zM_q^y, \tag{4.100}$$

where D_z is complex derivative operator defined in (4.6). For q-number operator we get representation

$$[N]_q \;\rightarrow\; zM_q^y\, D_z. \tag{4.101}$$

This representation shows interesting connection with self-similarity condition discussed in Section 4.7.3. The eigenvalue problem

$$[N]_q \left|n\right\rangle = [n]_q \left|n\right\rangle, \tag{4.102}$$

in q-analytic Fock-Bargman representation

$$zM_q^y D_z \frac{(x+iy)_q^n}{\sqrt{[n]_q!}} = [n]_q \frac{(x+iy)_q^n}{\sqrt{[n]_q!}}, \tag{4.103}$$

is equivalent to the self-similarity q-difference equation (4.57).

Quantum q-oscillator is described by Hamiltonian operator

$$H = \hbar\omega(b\, b^+ + b^+\, b). \tag{4.104}$$

The Hamiltonian in q-analytic Fock Bargmann representation becomes the operator as in

$$H = \hbar\omega(D_z\, z\, M_q^y + z\, M_q^y\, D_z), \tag{4.105}$$

and the Schrödinger equation

$$H \left|n\right\rangle = E_n \left|n\right\rangle, \tag{4.106}$$

takes the form of q-difference equation

$$\hbar\omega(D_z\, z\, M_q^y + z\, M_q^y\, D_z)\psi_n(z;q) = E_n \psi_n(z;q), \tag{4.107}$$

with q-analytic solution

$$\psi_n(z;q) = \frac{(x+iy)_q^n}{\sqrt{[n]_q!}}, \quad E_n = \hbar\omega([n]_q + [n+1]_q). \tag{4.108}$$

The above consideration shows that our q-analytic functions even non-analytic functions could describe quantum states. Moreover, fractal q-analytic functions discussed in Section 4.7.3 describe quantum states with fractal properties.

CHAPTER 5

Q-ANALYTIC HERMITE BINOMIAL FORMULA

We extend the concept of q-analytic function in two different directions. First motivated by derivation of the Dirac type δ-function for quantum states in Fock-Bargmann representation, we find expansion of q-binomial in terms of q-Hermite polynomials, analytic in two complex arguments. Based on this representation, we introduce a new class of complex functions of two complex arguments, which we call the double q-analytic functions. As another direction, by the hyperbolic version of q-analytic functions, we describe q-analogue of traveling waves, which does not preserve the shape during evolution. The IVP for corresponding q-wave equation is solved in the q-D'Alembert form.

5.1. Different Type of Analiticity

Definition 5.1 *Complex function $f(z)$ of one complex variable z (or two real variables x and y) is analytic if it satisfies*

$$\frac{\partial}{\partial \bar{z}} f(z) = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right) f(z) = 0. \tag{5.1}$$

We now define its q-analogue.

Definition 5.2 *A complex function $f(z;q)$ of one complex variable(two real variables) is q-analytic if*

$$D_q^{\bar{z}} f(z;q) = \frac{1}{2}\left(D_q^x + iD_{\frac{1}{q}}^y\right) f(z;q) = 0. \tag{5.2}$$

Example: Complex q-binomial $z = (x+iy)_q^2 = (x+iy)(x+iqy) = z.\left(\frac{1}{2}(1+q)z + (1-q)\bar{z}\right)$ is not analytic since, $(\frac{\partial}{\partial \bar{z}}(x+iy)_q^2 \neq 0)$ but it is q-analytic $D_q^{\bar{z}}(x+iy)_q^2 = 0$.

Definition 5.3 *Complex function $f(z,w)$ of two complex variables z,w is analytic if*

$$\frac{\partial}{\partial\bar{z}}f(z,w)=\frac{\partial}{\partial\bar{w}}f(z,w)=0. \tag{5.3}$$

Definition 5.4 *Complex function $f(z,w)$ of two complex variables z and w is double analytic if it satisfies*

$$\overline{\partial_{z,w}}f(z,w)=\frac{1}{2}\left(\frac{\partial}{\partial z}+i\frac{\partial}{\partial w}\right)f(z,w)=0. \tag{5.4}$$

Definition 5.5 *Complex function $f(z,w;q)$ of two complex variables z and w is double q-analytic if it satisfies*

$$\overline{D_{z,w}}f(z,w)=\frac{1}{2}\left(D^z_q+iD^w_{\frac{1}{q}}\right)f(z,w)=0. \tag{5.5}$$

Example: For $z=x+iy$ and $w=u+iv$ complex q-binomial $(z+iw)^2_q=z^2+[2]_qiwz-qw^2$ is analytic in z,w ($\frac{\partial}{\partial\bar{z}}(z+iw)^2_q=\frac{\partial}{\partial\bar{w}}(z+iw)^2_q=0$) and double q-analytic ($\bar{D}_{z,w}(z+iw)^2_q=0$).

As is well known, states of a quantum system in Fock-Bargmann representation are described by complex analytic function $f(z)$ and visa versa (Perelomov, 1986). In this representation , due to the formula

$$\int d\mu(z)e^{\xi\bar{z}}f(z)=f(\xi) \tag{5.6}$$

where a measure $d\mu(z)=dzd\bar{z}e^{-z\bar{z}}$, the exponential function plays the role of Dirac type δ function (Floratos , 1991). Proof of this formula is based on following identity

$$\int d\mu(z)e^{\xi\bar{z}}z^n=\xi^n. \tag{5.7}$$

Motivated by derivation of Dirac type δ- function for quantum states, in Fock-Bargmann representation we find holomorphic Newton binomial of two complex variables z and w expanded in terms of holomorphic Hermite polynomials (Nalci and Pashaev, 2010)

$$(z + iw)^n = \frac{1}{2^n} \sum_{k=0}^{n} \binom{n}{k} i^k H_{n-k}(z) H_k(w). \tag{5.8}$$

In this chapter, q-analogue of this formula for complex q-binomial is obtained in terms of q-Hermite polynomials

$$(z + iw)_q^n = \frac{1}{[2]_q^n} \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q i^k q^{\frac{k(k-1)}{2}} H_{n-k}(z; q) H_k(qw, \frac{1}{q}), \tag{5.9}$$

which is double q-analytic function function $D_{\bar{z},\bar{w}}(z + iw)_q^n = \frac{1}{2}(D_q^z + iD_{\frac{1}{q}}^w)(z + iw)_q^n = 0$.

It shows expansion of double q-analytic function of two complex variables in z and w, in terms of standard analytic functions as q-Hermite polynomials. This formula can be used for description of double q-analytic functions and corresponding Fock-Bargmann representation.

This representation allow us to introduce q-analogue of travelling waves $(x \pm ct)_q^n$ which can be expressed in terms of q-Hermite polynomials and corresponding q-travelling wave equation

$$\left((D_{\frac{1}{q}}^t)^2 - c^2 (D_q^x)^2\right) u(x, t) = 0$$

and its general solution in q-D'Alembert form.

5.2. Analytic Hermite Binomial Formula

We start with the following Lemma

Lemma 5.1 *For all ξ, η-complex numbers, we have*

$$\sum_{k=0}^{n} \binom{n}{k} (-1)^k \frac{d^{n-k}}{d\xi^{n-k}} \frac{d^k}{d\eta^k} e^{\xi^2/4 - \eta^2/4} = \left(\frac{\xi + \eta}{2}\right)^n e^{\xi^2/4 - \eta^2/4}. \tag{5.10}$$

Proof Consider left hand side of (5.10)

$$\sum_{k=0}^{n} \binom{n}{k} (-1)^k \frac{d^{n-k}}{d\xi^{n-k}} \frac{d^k}{d\eta^k} e^{\xi^2/4 - \eta^2/4} = \left(\frac{d}{d\xi} - \frac{d}{d\eta}\right)^n e^{\xi^2/4 - \eta^2/4} \tag{5.11}$$

By changing variables ξ and η to λ and μ respectively according to

$$\lambda + \mu = \xi, \qquad \lambda - \mu = \eta$$

we have

$$\lambda\mu = \frac{\xi^2 - \eta^2}{4} = (\frac{\xi - \eta}{2})(\frac{\xi + \eta}{2})$$

and

$$\left(\frac{d}{d\xi} - \frac{d}{d\eta}\right)^n e^{\lambda\mu} = \left(\frac{d}{d\mu}\right)^n e^{\lambda\mu} = \lambda^n e^{\lambda\mu} = \left(\frac{\xi + \eta}{2}\right)^n e^{\xi^2/4 - \eta^2/4}. \tag{5.12}$$

□

Corollary 5.1 *We have*

$$\sum_{k=0}^{n} \binom{n}{k} (-1)^k \frac{d^{n-k}}{d\xi^{n-k}} e^{\xi^2/4} \frac{d^k}{d\xi^k} e^{-\xi^2/4} = \xi^n. \tag{5.13}$$

Proof By taking the limit of expression (5.10) as $\eta \to \xi \Rightarrow \mu \to 0, \lambda \to \xi$, we get

$$\lim_{\eta\to\xi} \sum_{k=0}^{n} \binom{n}{k} (-1)^k \frac{d^{n-k}}{d\xi^{n-k}} \frac{d^k}{d\eta^k} e^{\xi^2/4 - \eta^2/4} = \lim_{\mu\to 0, \lambda\to\xi} \lambda^n e^{\lambda\mu} = \xi^n. \tag{5.14}$$

□

Lemma 5.2 *For $n = 1, 2, \ldots$ $f(z) = z^n$, the below equation holds*

$$\int dz d\bar{z} e^{-z\bar{z}} e^{\xi\bar{z}} z^n = \xi^n. \tag{5.15}$$

Proof By changing complex coordinates to cartesian coordinates the integral is expressed in terms of summation formula

$$\begin{aligned} \int dz d\bar{z} e^{-z\bar{z}} e^{\xi\bar{z}} z^n &= \frac{1}{\pi} \int dx dy e^{\xi(x-iy)} e^{-(x^2+y^2)} (x+iy)^n \\ &= \frac{1}{\pi} \sum_{k=0}^{n} \binom{n}{k} i^k \int dx \ \ x^{n-k} e^{-x^2} e^{\xi x} \int dy \ \ y^k e^{-y^2} e^{-i\xi y} \end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{\pi}\sum_{k=0}^{n}\binom{n}{k} i^k \frac{d^{n-k}}{d\xi^{n-k}}\int dx\ \ e^{-x^2+\xi x}\frac{d^k}{d(-i\xi)^k}\int dy\ \ e^{-y^2-i\xi y} \\
&= \sum_{k=0}^{n}\binom{n}{k}(-1)^k \frac{d^{n-k}}{d\xi^{n-k}} e^{\xi^2/4}\frac{d^k}{d\xi^k}e^{-\xi^2/4}, \qquad (5.16)
\end{aligned}
$$

where we have used the results of Gaussian integrals

$$
\int e^{-x^2+ax}dx = \sqrt{\pi}e^{a^2/4}
$$

and

$$
\int e^{-x^2+ibx}dx = \sqrt{\pi}e^{-b^2/4}.
$$

By using Corollary 5.1 we find the desired result

$$
\int dz d\bar{z} e^{-z\bar{z}} e^{\xi\bar{z}} z^n = \xi^n. \qquad (5.17)
$$

□

Now as evident we can generalize this result for any analytic functions $f(z) = \sum_{n=0}^{\infty} a_n z^n$ as

$$
\int d\mu(z) e^{\xi\bar{z}} f(z) = f(\xi).
$$

The above proof implies some interesting binomial identity formula for Hermite polynomials. For this, we need Rodrigues formula for Hermite polynomials:

Definition 5.6 *Rodrigues formula for Hermite polynomials of complex argument is defined by*

$$
H_n(z) = (-1)^n e^{z^2}\frac{d^n}{dz^n}e^{-z^2} \qquad (5.18)
$$

and replacing $z \to iz$ we get

$$H_n(iz) = i^n e^{-z^2} \frac{d^n}{dz^n} e^{z^2}. \tag{5.19}$$

Identity 5.1 *The following identity*

$$\frac{1}{2^n} \sum_{k=0}^{n} \binom{n}{k} (-i)^{n-k} H_{n-k}(\frac{i}{2}\xi) H_k(\frac{\xi}{2}) = \xi^n \tag{5.20}$$

holds.

Proof According to the previous proof we have

$$\xi^n = \int dz d\bar{z} e^{-z\bar{z}} e^{\xi \bar{z}} z^n = \sum_{k=0}^{n} \binom{n}{k} (-1)^k \frac{d^{n-k}}{d\xi^{n-k}} e^{\xi^2/4} \frac{d^k}{d\xi^k} e^{-\xi^2/4} \tag{5.21}$$

In order to use the Rodrigues formula we multiply the above expression by $e^{\xi^2/4} e^{-\xi^2/4}$

$$\begin{aligned} & \sum_{k=0}^{n} \binom{n}{k} (-1)^k e^{\xi^2/4} \left(e^{-\xi^2/4} \frac{d^{n-k}}{d\xi^{n-k}} e^{\xi^2/4} \right) \left(\frac{d^k}{d\xi^k} e^{-\xi^2/4} \right) \\ = \quad & \sum_{k=0}^{n} \binom{n}{k} (-1)^k \left(\frac{1}{2i} \right)^{n-k} H_{n-k}(\frac{i}{2}\xi) \left(e^{\xi^2/4} \frac{d^k}{d\xi^k} e^{-\xi^2/4} \right) \\ = \quad & \sum_{k=0}^{n} \binom{n}{k} (-1)^k \left(\frac{1}{2i} \right)^{n-k} H_{n-k}(i\frac{\xi}{2}) \left(-\frac{1}{2} \right)^k H_k(\frac{\xi}{2}) \\ = \quad & \frac{1}{2^n} \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} i^{n-k} H_{n-k}(i\frac{\xi}{2}) H_k(\frac{\xi}{2}) \\ = \quad & \xi^n. \end{aligned} \tag{5.22}$$

□

Particular case: By simple change of variable $\xi \to -2iz$ in (5.20), we obtain

$$\frac{1}{2^{2n}} \sum_{k=0}^{n} \binom{n}{k} i^k H_{n-k}(z) H_k(-iz) = z^n, \tag{5.23}$$

and by reductions $\xi = x$ and $\xi = iy$ we get

$$\frac{1}{2^n}\sum_{k=0}^{n}\binom{n}{k} i^{n-k} H_{n-k}(\frac{ix}{2})H_k(\frac{x}{2}) = x^n, \tag{5.24}$$

$$\frac{1}{2^n}\sum_{k=0}^{n}\binom{n}{k} i^{n-k} H_{n-k}(\frac{-y}{2})H_k(i\frac{y}{2}) = i^n y^n. \tag{5.25}$$

Identity 5.2 *More general identity is given as*

$$(z + iw)^n = \frac{1}{2^n}\sum_{k=0}^{n}\binom{n}{k} i^k H_{n-k}(z)H_k(w). \tag{5.26}$$

The proof can be done by generating functions for Hermite polynomials.

Proof Generating function for Hermite polynomials is defined as follows

$$g(z, t) = e^{-t^2+2tz} = \sum_{n=0}^{\infty} H_n(z)\frac{t^n}{n!} \tag{5.27}$$

and

$$g(w, \tau) = e^{-\tau^2+2\tau w} = \sum_{k=0}^{\infty} H_k(w)\frac{\tau^k}{k!} \tag{5.28}$$

by changing variable $\tau = it$ and multiplying (5.27) and (5.28) we have

$$g(z, t)\, g(w, it) = e^{2t(z+iw)} = \sum_{l=0}^{\infty}\sum_{k=0}^{\infty} \frac{H_l(z)}{l!}\frac{H_k(w)i^k}{k!} t^{l+k}. \tag{5.29}$$

In order to change the order of double sum we choose $l + k = n$, and by expanding the left hand side in t we get

$$g(z, t)\, g(w, it) = \sum_{n=0}^{\infty} \frac{2^n t^n (z + iw)}{n!} = \sum_{n=0}^{\infty} \frac{t^n}{n!} \sum_{k=0}^{n}\binom{n}{k} H_{n-k}(z)H_k(w)i^k. \tag{5.30}$$

By equating the term of t^n we obtain the desired result (5.26):

$$(z+iw)^n = \frac{1}{2^n}\sum_{k=0}^{n}\binom{n}{k} i^k H_{n-k}(z)H_k(w).$$

□

Another proof can be done by using the complex Laplace equation.

Proof $\zeta^n = (z+iw)^n$ is analytic function of two complex variables z and w. Therefore it satisfies complex Laplace equation $\Delta\zeta^n = 0$, where $\Delta \equiv \frac{d^2}{dz^2} + \frac{d^2}{dw^2}$ which implies $\Delta^k\zeta^n = 0$ for $k = 0, 1, 2...$ As evident,

$$e^{-\frac{1}{4}\Delta}(z+iw)^n = \left(1 - \frac{1}{4}\Delta + \frac{(-\frac{1}{4}\Delta)^2}{2!} + ... + \frac{(-\frac{1}{4}\Delta)^n}{n!} + ...\right)(z+iw)^n = (z+iw)^n \tag{5.31}$$

or explicitly

$$\begin{aligned} e^{-\frac{1}{4}\Delta}(z+iw)^n &= e^{-\frac{1}{4}\frac{d^2}{dz^2}} e^{-\frac{1}{4}\frac{d^2}{dw^2}} \sum_{k=0}^{n}\binom{n}{k} z^{n-k} i^k w^k \\ &= \sum_{k=0}^{n}\binom{n}{k} i^k \left(e^{-\frac{1}{4}\frac{d^2}{dz^2}} z^{n-k}\right)\left(e^{-\frac{1}{4}\frac{d^2}{dw^2}} w^k\right). \end{aligned} \tag{5.32}$$

By using the identity for Hermite Polynomials:

$$H_n(x) = 2^n e^{-\frac{1}{4}\frac{d^2}{dx^2}} x^n$$

we get

$$(z+iw)^n = \frac{1}{2^n}\sum_{k=0}^{n}\binom{n}{k} H_{n-k}(z)H_k(w) i^k. \tag{5.33}$$

□

5.3. q-Hermite Polynomials

In paper (Nalci and Pashaev, 2010), we define $q-$Hermite polynomials according to generating function

$$e_q(-t^2)e_q([2]_q tx) = \sum_{n=0}^{\infty} H_n(x;q)\frac{t^n}{[n]_q!}, \tag{5.34}$$

where

$$e_q(x) = \sum_{n=0}^{\infty} \frac{x^n}{[n]_q!}, \qquad E_q(x) = \sum_{n=0}^{\infty} q^{\frac{n(n-1)}{2}} \frac{x^n}{[n]_q!}$$

are Jackson's q-exponential functions and q-numbers and q-factorials are defined as follows:

$$[n]_q = \frac{q^n - 1}{q - 1}, \quad [n]_q! = [1]_q[2]_q...[n]_q.$$

From this generating function we have the special values

$$H_{2n}(0;q) = (-1)^n \frac{[2n]_q!}{[n]_q!}, \tag{5.35}$$

$$H_{2n+1}(0;q) = 0, \tag{5.36}$$

and the parity relation

$$H_n(-x;q) = (-1)^n H_n(x;q). \tag{5.37}$$

By q-differentiating the generating function (5.34) according to x and t we have the recurrence relations correspondingly

$$D_x H_n(x;q) = [2]_q[n]_q H_{n-1}(x;q), \tag{5.38}$$

$$
\begin{aligned}
H_{n+1}(x;q) = [2]_q \, xH_n(x;q) - [n]_q \, H_{n-1}(qx;q) \\
-[n]_q \, q^{\frac{n+1}{2}} H_{n-1}(\sqrt{q}x;q).
\end{aligned} \tag{5.39}
$$

We notice that the generating function and the form of our q-Hermite polynomials are different from the known ones in the literature.

First few polynomials are

$$
\begin{aligned}
H_0(x;q) &= 1 \\
H_1(x;q) &= [2]_q \, x \\
H_2(x;q) &= [2]_q^2 \, x^2 - [2]_q \\
H_3(x;q) &= [2]_q^3 \, x^3 - [2]_q^2[3]_q \, x \\
H_4(x;q) &= [2]_q^4 \, x^4 - [2]_q^2[3]_q[4]_q \, x^2 + [2]_q[3]_q[2]_{q^2}.
\end{aligned}
$$

When $q \to 1$ these polynomials reduce to the standard Hermite polynomials.

In generating function (5.34) for $t = 1$ it gives expansion of q-exponential function in terms of q-Hermite polynomials

$$
\sum_{n=0}^{\infty} \frac{H_n(x;q)}{[n]_q!} = \frac{e_q([2]_q x)}{E_q(1)}. \tag{5.40}
$$

In the limit $q \to 1$

$$
e^{2x-1} = \sum_{n=0}^{\infty} \frac{H_n(x)}{n!}
$$

For $x = 1$ and $x = 0$ case:

$$
e = \sum_{n=0}^{\infty} \frac{H_n(1)}{n!}, \quad \frac{1}{e} = \sum_{n=0}^{\infty} \frac{H_n(0)}{n!} = \sum_{n=0}^{\infty} \frac{H_{2n}(0)}{(2n)!} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!}.
$$

Identity 5.3 *q-Analogue of identity (5.26) is as follows*

$$(z+iw)_q^n = \frac{1}{[2]_q^n}\sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}_q i^k q^{\frac{k(k-1)}{2}} H_{n-k}(z;q)H_k(qw,\frac{1}{q}), \tag{5.41}$$

where H_i stands for q-Hermite polynomials.

Proof By using the generating function for q-Hermite polynomials (5.34) by replacing $x \to Z$ we obtain

$$e_q(-t^2)e_q([2]_qZt) = \sum_{n=0}^{\infty} H_n(Z;q)\frac{t^n}{[n]_q!} \tag{5.42}$$

and then replacing $t \to it$, $Z \to W$ and $q \to 1/q$ we have

$$e_{\frac{1}{q}}(t^2)e_{\frac{1}{q}}([2]_{\frac{1}{q}}iWt) = \sum_{n=0}^{\infty} H_n(W;\frac{1}{q})i^n\frac{t^n}{[n]_{\frac{1}{q}}!}. \tag{5.43}$$

Multiplying (5.42), (5.43) and using the factorization of q-exponential functions

$$e_q(x)e_{\frac{1}{q}}(y) = e_q(x+y)_q \tag{5.44}$$

and $e_q(0)_q = 1$ we get

$$e_q\Big(t([2]_qZ + [2]_{\frac{1}{q}}iW)\Big)_q = \sum_{l=0}^{\infty}\sum_{k=0}^{\infty}\frac{H_l(Z;q)}{[l]_q!}\frac{H_k(W;\frac{1}{q})i^k}{[k]_{\frac{1}{q}}!}t^{l+k}. \tag{5.45}$$

For the right hand side, by changing the order of double sum we choose $k+l = n$ and expanding left hand side in t, we derive

$$\sum_{n=0}^{\infty}\frac{t^n\Big([2]_qZ + [2]_{\frac{1}{q}}iW\Big)_q^n}{[n]_q!} = \sum_{n=0}^{\infty}\frac{t^n}{[n]_q!}\sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}_q q^{\frac{k(k-1)}{2}}H_{n-k}(Z;q)H_k(W;\frac{1}{q})i^k, \tag{5.46}$$

for every power t^n we have identity

$$\left(Z + i\frac{W}{q}\right)_q^n = \frac{1}{[2]_q^n} \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q q^{\frac{k(k-1)}{2}} H_{n-k}(Z; q) H_k(W; \frac{1}{q}) i^k, \tag{5.47}$$

where

$$[k]_{\frac{1}{q}} = \frac{1}{q^{k-1}}[k]_q, \qquad [k]_{\frac{1}{q}} = \frac{1}{q^{\frac{k(k-1)}{2}}}[k]_q! \tag{5.48}$$

Replacing $Z = z$ and $\frac{W}{q} = w$ the desired result is obtained

$$(z + iw)_q^n = \frac{1}{[2]_q^n} \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q q^{\frac{k(k-1)}{2}} H_{n-k}(z; q) H_k(qw; \frac{1}{q}) i^k.$$

$\square$

Another proof can be done by using the following identity and using complex q-Laplace equation.

Identity 5.4 *Following identity*

$$e_q\left(-\frac{1}{[2]_q^2}\Delta_q\right)_q (z + iw)_q^n = (z + iw)_q^n, \tag{5.49}$$

holds.

Proof $(z + iw)_q^n$ is q-analytic function. Therefore, it satisfies q-Laplace equation

$$\Delta_q (z + iw)_q^n = 0.$$

Furthermore, we have

$$\begin{aligned} e_q\left(-\frac{1}{[2]_q^2}\Delta_q\right)_q (z + iw)_q^n &= e_q\left(-\frac{1}{[2]_q^2}\left((D_q^z)^2 + (D_{\frac{1}{q}}^w)^2\right)\right)_q (z + iw)_q^n \\ &= \sum_{n=0}^{\infty} \frac{1}{[n]_q!}\left(-\frac{1}{[2]_q^2}\right)^n \left((D_q^z)^2 + (D_{\frac{1}{q}}^w)^2\right)_q^n (z + iw)_q^n \end{aligned}$$

$$= \sum_{n=0}^{\infty} \frac{1}{[n]_q!}\left(-\frac{1}{[2]_q^2}\right)^n \left(\Delta_q\right)_q^n (z+iw)_q^n, \tag{5.50}$$

where $\left(\Delta_q\right)_q^n = \Delta_q \cdot \Delta_q^{(1)} \cdot \Delta_q^{(2)} \cdot ... \cdot \Delta_q^{(n-1)}$ and

$$\Delta_q = (D_q^z)^2 + (D_{\frac{1}{q}}^w)^2, \qquad \Delta_q^{(1)} = (D_q^z)^2 + q(D_{\frac{1}{q}}^w)^2, \;\; ..., \;\; \Delta_q^{(n-1)} = (D_q^z)^2 + q^{n-1}(D_{\frac{1}{q}}^w)^2.$$

Using the fact that $\left(\Delta_q\right)_q^m (z+iw)_q^n = 0, \forall m = 1, 2, ...,$ only the first term in expansion survives then we get desired result.

Using the factorization of q-exponential function

$$\begin{aligned} & e_q\left(-\frac{1}{[2]_q^2}\Delta_q\right)_q (z+iw)_q^n \\ = \;& e_q\left(-\frac{1}{[2]_q^2}(D_q^z)^2\right) e_{\frac{1}{q}}\left(-\frac{1}{[2]_q^2}(D_{\frac{1}{q}}^w)^2\right)(z+iw)_q^n \\ = \;& \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q q^{\frac{k(k-1)}{2}} i^k e_q\left(-\frac{1}{[2]_q^2}(D_q^z)^2\right) z^{n-k} e_{\frac{1}{q}}\left(-\frac{1}{[2]_q^2}(D_{\frac{1}{q}}^w)^2\right) w^k \end{aligned} \tag{5.51}$$

By using the generating function of q-Hermite Polynomials (5.42) we have the following identity:

$$H_n(x;q) = [2]_q^n e_q\left(-\frac{1}{[2]_q^2}(D_q^x)^2\right) x^n, \tag{5.52}$$

which gives

$$e_q\left(-\frac{1}{[2]_q^2}(D_q^z)^2\right) z^{n-k} = \frac{1}{[2]_q^{n-k}} H_{n-k}(z;q) \tag{5.53}$$

and

$$e_{\frac{1}{q}}\left(-\frac{1}{[2]_q^2}(D_{\frac{1}{q}}^w)^2\right) w^k = \frac{1}{[2]_{\frac{1}{q}}^k q^k} H_k(qw; \frac{1}{q}) \tag{5.54}$$

where $D^{qw}_{\frac{1}{q}} = \frac{1}{q}D^{w}_{\frac{1}{q}}$. Substituting these results into equation 5.51, we get

$$e_q\left(-\frac{1}{[2]_q^2}\Delta_q\right)_q (z+iw)_q^n = \sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}_q q^{\frac{k(k-1)}{2}} i^k \frac{1}{[2]_q^{n-k}} H_{n-k}(z;q) \frac{1}{[2]_{\frac{1}{q}}^k q^k} H_k(qw;\frac{1}{q})$$

According to identity 5.4 we get the desired result as

$$(z+iw)_q^n = \frac{1}{[2]_q^n}\sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}_q q^{\frac{k(k-1)}{2}} H_{n-k}(z;q) H_k(qw;\frac{1}{q}) i^k. \tag{5.55}$$

□

5.4. Double q-Analytic Function

Here we consider a class of complex valued functions of two complex variables, z and w, (or four real variables), analytic in these variables $\frac{\partial}{\partial \bar{z}} f = \frac{\partial}{\partial \bar{w}} f = 0$.

Definition 5.7 *A complex-valued function $f(z, w)$ of four real variables is called the double analytic in a region if the following identity holds in the region:*

$$\bar{\partial}_{z,w} f \equiv \frac{1}{2}(\partial_z + i\partial_w) f = 0, \tag{5.56}$$

where

$$\partial_z f = \frac{1}{2}(\partial_x - i\partial_y), \quad \partial_w f = \frac{1}{2}(\partial_u - i\partial_v)$$

and $z = x + iy, \quad w = u + iv$.

Definition 5.8 *A complex-valued function $f(z, w)$ of four real variables is called the double q-analytic in a region if the following identity holds in the region:*

$$\bar{D}_{z,w} f = \frac{1}{2}(D_q^z + iD^w_{\frac{1}{q}}) f = 0, \tag{5.57}$$

where

$$D_q^z f(z,w) = \frac{f(qz,w)-f(z,w)}{(q-1)z}, \qquad D_{\frac{1}{q}}^w f(z,w) = \frac{f(z,\frac{w}{q})-f(z,w)}{(\frac{1}{q}-1)w}$$

and $z = x+iy, \quad w = u+iv$.

Here we should notice that

$$D_q^z \neq \frac{1}{2}(D_q^x - iD_q^y), \quad D_q^w \neq \frac{1}{2}(D_q^u - iD_q^v).$$

The simplest set of double q-analytic functions is given by complex q-binomials

$$(z+iw)_q^n \equiv (z+iw)(z+iqw)(z+iq^2w)...(z+iq^{n-1}w) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q q^{k(k-1)/2} i^k z^{n-k} w^k$$

satisfying

$$\frac{1}{2}(D_q^z + iD_{\frac{1}{q}}^w)(z+iw)_q^n = 0.$$

Proposition 5.1 *Complex q-binomials satisfy following relation*

$$\frac{1}{2}(D_q^z - iD_{\frac{1}{q}}^w)(z+iw)_q^n = [n]_q (z+iw)_q^{n-1}.$$

Proof We have

$$\begin{aligned}
&\frac{1}{2}(D_q^z - iD_{\frac{1}{q}}^w)(z+iw)_q^n \\
&= \frac{1}{2}\left(\sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q q^{k(k-1)/2} i^k \left(D_q^z\, z^{n-k}\right) w^k - i\sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q q^{k(k-1)/2} i^k z^{n-k} \left(D_{\frac{1}{q}}^w\, w^k\right)\right) \\
&= \frac{1}{2}\left(\sum_{k=0}^{n-1} \begin{bmatrix} n \\ k \end{bmatrix}_q q^{k(k-1)/2} [n-k]_q z^{n-k-1} i^k w^k - i\sum_{k'=1}^{n} \begin{bmatrix} n \\ k' \end{bmatrix}_q q^{k'(k'-1)/2} i^{k'} z^{n-k'} \frac{w^{k'-1}}{q^{k'-1}} [k']_q\right) \\
&= \frac{1}{2}\left(\sum_{k=0}^{n-1} \begin{bmatrix} n \\ k \end{bmatrix}_q q^{k(k-1)/2} [n-k]_q z^{n-k-1} i^k w^k - i\sum_{k=0}^{n-1} \begin{bmatrix} n \\ k+1 \end{bmatrix}_q q^{k(k+1)/2} i^{k+1} z^{n-k-1} \frac{w^k}{q^k} [k+1]_q\right) \\
&= \frac{1}{2}\sum_{k=0}^{n-1}\left(\begin{bmatrix} n \\ k \end{bmatrix}_q q^{k(k-1)/2} [n-k]_q - i\begin{bmatrix} n \\ k+1 \end{bmatrix}_q q^{k(k+1)/2} i \frac{1}{q^k} [k+1]_q\right) z^{n-k-1} i^k w^k
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{2}\sum_{k=0}^{n-1} 2\left(\frac{[n]!}{[n-k-1]![k]!} q^{k(k-1)/2}\right) z^{n-k-1} i^k w^k \\
&= [n]\sum_{k=0}^{n-1} \begin{bmatrix} n-1 \\ k \end{bmatrix}_q q^{k(k-1)/2} z^{n-k-1} i^k w^k \\
&= [n](z+iw)_q^{n-1}.
\end{aligned}
$$

□

From the above result follows that any convergent power series

$$
f(z+iw)_q = \sum_{n=0}^{\infty} a_n (z+iw)_q^n
$$

determines a double q-analytic function. Since our relation (5.41) shows expansion of double q-analytic q-binomials in terms of q-Hermite polynomials, it also gives expansion of any double q-analytic function in terms of the analytic polynomials.

Examples: For $n = 1$:

$$
\begin{aligned}
(z+iw)_q^1 = z + iw &= \frac{1}{[2]_q}\sum_{k=0}^{1} \begin{bmatrix} 1 \\ k \end{bmatrix}_q q^{\frac{k(k-1)}{2}} H_{1-k}(z;q) H_k(qw;\frac{1}{q}) i^k \\
&= \frac{1}{[2]_q}\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}_q H_1(z;q)H_0(qw;\frac{1}{q}) + \begin{bmatrix} 1 \\ 1 \end{bmatrix}_q H_0(z;q)H_1(qw;\frac{1}{q})i\right).
\end{aligned} \tag{5.58}
$$

For $n = 2$:

$$
\begin{aligned}
(z+iw)_q^2 &= (z+iw)(z+iqw) = z^2 + i[2]_q zw - qw^2 \\
&= \frac{1}{[2]_q^2}\sum_{k=0}^{2} \begin{bmatrix} 2 \\ k \end{bmatrix}_q q^{\frac{k(k-1)}{2}} H_{2-k}(z;q) H_k(qw;\frac{1}{q}) i^k \\
&= \frac{1}{[2]_q^2}\left(\begin{bmatrix} 2 \\ 0 \end{bmatrix}_q H_2(z;q)H_0(qw;\frac{1}{q}) + \begin{bmatrix} 2 \\ 1 \end{bmatrix}_q H_1(z;q)H_1(qw;\frac{1}{q})i + \begin{bmatrix} 2 \\ 2 \end{bmatrix}_q qH_0(z;q)H_2(qw;\frac{1}{q})i^2\right)
\end{aligned}
$$

***q*-Holomorphic Laplacian:** Another proof of identity (5.41) can be done by noticing that q-binomial $(z+iw)_q^n$ is double q-analytic function. Then we can use the following identity and complex q-Laplace equation.

Theorem 5.1 *The following identity holds*

$$e_q(-\frac{1}{[2]_q^2}\Delta_q)_q(z+iw)_q^n = (z+iw)_q^n \tag{5.59}$$

Proof

Since $(z+iw)_q^n$ is q-analytic function, it satisfies the q-Laplace equation

$$\Delta_q(z+iw)_q^n = 0$$

and

$$\begin{aligned}
e_q(-\frac{1}{[2]_q^2}\Delta_q)_q(z+iw)_q^n &= e_q(-\frac{1}{[2]_q^2}\Big((D_q^z)^2+(D_{\frac{1}{q}}^w)^2\Big))_q(z+iw)_q^n \\
&= \sum_{n=0}^{\infty}\frac{1}{[n]_q!}\left(-\frac{1}{[2]_q^2}\right)^n\Big((D_q^z)^2+(D_{\frac{1}{q}}^w)^2\Big)_q^n(z+iw)_q^n \\
&= \sum_{n=0}^{\infty}\frac{1}{[n]_q!}\left(-\frac{1}{[2]_q^2}\right)^n\Big(\Delta_q\Big)_q^n(z+iw)_q^n, \qquad (5.60)
\end{aligned}$$

where $\Big(\Delta_q\Big)_q^n = \Delta_q\cdot\Delta_q^{(1)}\cdot\Delta_q^{(2)}\cdot\ldots\cdot\Delta_q^{(n-1)}$ and

$$\Delta_q = (D_q^z)^2+(D_{\frac{1}{q}}^w)^2, \qquad \Delta_q^{(1)} = (D_q^z)^2+q(D_{\frac{1}{q}}^w)^2, \;\; \ldots,\;\; \Delta_q^{(n-1)} = (D_q^z)^2+q^{n-1}(D_{\frac{1}{q}}^w)^2.$$

Using the fact that $\Big(\Delta_q\Big)_q^m(z+iw)_q^n = 0, \forall m = 1,2,\ldots$, only the first term in expansion survives, then we get desired result. □

Due to (5.44) we can factorize q-exponential operator function as

$$\begin{aligned}
& e_q\left(-\frac{1}{[2]_q^2}\Delta_q\right)_q(z+iw)_q^n \\
= \;& e_q\left(-\frac{1}{[2]_q^2}(D_q^z)^2\right)e_{\frac{1}{q}}\left(-\frac{1}{[2]_q^2}(D_{\frac{1}{q}}^w)^2\right)(z+iw)_q^n \\
= \;& \sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}_q q^{\frac{k(k-1)}{2}} i^k e_q\left(-\frac{1}{[2]_q^2}(D_q^z)^2\right)z^{n-k}e_{\frac{1}{q}}\left(-\frac{1}{[2]_q^2}(D_{\frac{1}{q}}^w)^2\right)w^k. \qquad (5.61)
\end{aligned}$$

By using the generating function of q-Hermite Polynomials (5.42) we have the following identity:

$$H_n(x;q) = [2]_q^n e_q\left(-\frac{1}{[2]_q^2}(D_q^x)^2\right)x^n, \tag{5.62}$$

which gives

$$e_q\left(-\frac{1}{[2]_q^2}(D_q^z)^2\right)z^{n-k} = \frac{1}{[2]_q^{n-k}}H_{n-k}(z;q), \tag{5.63}$$

and

$$e_{\frac{1}{q}}\left(-\frac{1}{[2]_q^2}(D_{\frac{1}{q}}^w)^2\right)w^k = \frac{1}{[2]_{\frac{1}{q}}^k q^k}H_k(qw;\frac{1}{q}), \tag{5.64}$$

where $D_{\frac{1}{q}}^{qw} = \frac{1}{q}D_{\frac{1}{q}}^w$. Substituting into (5.51), we get

$$e_q(-\frac{1}{[2]_q^2}\Delta_q)_q(z+iw)_q^n = \sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}_q q^{\frac{k(k-1)}{2}} i^k \frac{1}{[2]_q^{n-k}}H_{n-k}(z;q)\frac{1}{[2]_{\frac{1}{q}}^k q^k}H_k(qw;\frac{1}{q}).$$

Then, according to identity (5.59) we obtain desired result

$$(z+iw)_q^n = \frac{1}{[2]_q^n}\sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}_q q^{\frac{k(k-1)}{2}} H_{n-k}(z;q)H_k(qw;\frac{1}{q})i^k. \tag{5.65}$$

Theorem 5.2 *Double q-analytic Kampe-de Feriet binomial expansion is given by*

$$(z+iw)_q^n = \frac{1}{\left([2]_q\sqrt{-\nu t}\right)^n}\sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}_q q^{\frac{k(k-1)}{2}} K_{n-k}([2]_q\sqrt{-\nu t}z, \nu t; q)K_k([2]_{\frac{1}{q}}\sqrt{-\nu t}qw, \nu t; \frac{1}{q})i^k. \tag{5.66}$$

Proof Using the definition of q-Kampe-de Feriet polynomials (Nalci and Pashaev, 2010)

$$K_n(x,\nu t;q) = (-\nu t)^{\frac{n}{2}}H_n\left(\frac{x}{[2]_q\sqrt{-\nu t}};q\right) \tag{5.67}$$

and changing arguments as $\frac{x}{[2]_q\sqrt{-\nu t}} \equiv z$ we get

$$K_{n-k}([2]_q\sqrt{-\nu t}z, \nu t; q)(-\nu t)^{-\frac{n-k}{2}} = H_{n-k}(z\,; q)$$

and replacing by $n-k \to k,\ z \to qw,\ q \to \frac{1}{q}$

$$K_k([2]_{\frac{1}{q}}\sqrt{-\nu t}qw, \nu t; \frac{1}{q})(-\nu t)^{-\frac{k}{2}} = H_k(qw\,; \frac{1}{q}).$$

Using the above relations into the q-binomial expansion in terms of q-Hermite binomial formula (5.65) the desired result is obtained. □

As a particular case of our binomial formula, we can find q-Hermite binomial expansion for the q-analytic binomial $(x+iy)^n$ as well. If in (5.41) we replace $z \to x$ and $w \to y$, then we get

$$(x+iy)_q^n = \frac{1}{[2]_q^n}\sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}_q q^{\frac{k(k-1)}{2}} H_{n-k}(x; q) H_k(qy; \frac{1}{q}) i^k. \tag{5.68}$$

Since a q-analytic function is determined by power series in q-binomials (Pashaev and Nalci , 2014), this formula allows us to get expansion of an arbitrary q-analytic function in terms of real q-Hermite polynomials.

5.4.1. q-Binomial and q-Translation Operator

Our proof is based on representation of q-binomial as a q-translation:

$$e_{\frac{1}{q}}^{aD_q^x} x^n = (x+a)_q^n. \tag{5.69}$$

This formula can be proved by expanding q-exponential function as follows

$$
\begin{aligned}
e_{\frac{1}{q}}^{aD_q^x} x^n &= \sum_{k=0}^{\infty} \frac{a^k (D_q^x)^k}{[k]_{\frac{1}{q}}!} x^n = \sum_{k=0}^{n} \frac{a^k (D_q^x)^k}{[k]_{\frac{1}{q}}!} x^n = \sum_{k=0}^{n} a^k \frac{[n]_q[n-1]_q...[n-k+1]_q}{[k]_q!} q^{\frac{k(k-1)}{2}} x^{n-k} \\
&= \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q q^{\frac{k(k-1)}{2}} x^{n-k} a^k = (x+a)_q^n, \qquad (5.70)
\end{aligned}
$$

where $(D_q^x)^k = [n]_q[n-1]_q...[n-k+1]x^{n-k}$, $\qquad [k]_{\frac{1}{q}}! = \frac{1}{q^{\frac{k(k-1)}{2}}}[k]_q!$.

q-Binomial Expansion in terms of q-Hermite Polynomials

Complex q-binomial expansion in terms of q-Hermite polynomials we find in the next form:

$$
e_{\frac{1}{q}}^{iyD_q^x} x^n = (x+iy)_q^n = \frac{1}{[2]_q^n} \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q i^k q^{\frac{k(k-1)}{2}} H_{n-k}(x;q) H_k(qy; \frac{1}{q}). \qquad (5.71)
$$

In order to prove this formula, from the generating function for q-Hermite polynomial (5.34) we get

$$
H_n(x;q) = e_q(-\frac{1}{[2]_q^2}(D_q^x)^2)[2]_q^n x^n. \qquad (5.72)
$$

By replacing $n \to k, \quad x \to qy, \quad q \to \frac{1}{q}$ we obtain

$$
H_k(qy; \frac{1}{q}) = e_{\frac{1}{q}}(-\frac{1}{[2]_{\frac{1}{q}}^2}(D_{\frac{1}{q}}^{qy})^2)[2]_{\frac{1}{q}}^k (qy)^k. \qquad (5.73)
$$

As the next step, to both sides of Gauss' binomial formula (5.71) we apply the operator

$$
\frac{1}{[2]_q^n} e_q(-\frac{1}{[2]_q^2}(D_q^x)^2)[2]_q^{n-k} e_{\frac{1}{q}}(-\frac{1}{[2]_{\frac{1}{q}}^2}(D_{\frac{1}{q}}^{qy})^2)[2]_{\frac{1}{q}}^k q^k. \qquad (5.74)
$$

Then, by using (5.72) and (5.73), from the RHS of (5.71) we obtain

$$
\frac{1}{[2]_q^n} \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q i^k q^{\frac{k(k-1)}{2}} H_{n-k}(x;q) H_k(qy; \frac{1}{q}). \qquad (5.75)
$$

From another side the LHS of (5.71) becomes

$$e_q(-\frac{1}{[2]_q^2}(D_q^x)^2)e_{\frac{1}{q}}(-\frac{1}{[2]_{\frac{1}{q}}^2}(D_{\frac{1}{q}}^{qy})^2)e_{\frac{1}{q}}^{iyD_q^x}x^n,$$

and due to $(D_{\frac{1}{q}}^{qy})^2 e_{\frac{1}{q}}^{iyD_q^x}x^n = (\frac{i}{q})^2(D_q^x)^2 e_{\frac{1}{q}}^{iyD_q^x}x^n$ and as follows

$$e_{\frac{1}{q}}(-\frac{1}{[2]_{\frac{1}{q}}^2}(D_{\frac{1}{q}}^{qy})^2)e_{\frac{1}{q}}^{iyD_q^x}x^n = e_{\frac{1}{q}}(\frac{1}{[2]_q^2}(D_q^x)^2)e_{\frac{1}{q}}^{iyD_q^x}x^n,$$

finally we find

$$e_{\frac{1}{q}}^{iyD_q^x}e_q(-\frac{1}{[2]_q^2}(D_q^x)^2 + \frac{1}{[2]_q^2}(D_q^x)^2)_q x^n = e_{\frac{1}{q}}^{iyD_q^x}x^n = (x+iy)_q^n. \tag{5.76}$$

By changing real variables to the complex ones $x \to z$ and $y \to w$, we obtain the expansion of double q-analytic q-binomial in terms of q-Hermite polynomials (5.9).

5.5. q-Traveling Waves

Here as another, hyperbolic extension of q-analytic functions, we consider the q-analogue of traveling waves as a solution of the q-wave equation. Direct extension of traveling waves to q-traveling waves is not possible. This happens due to the lack of the chain rule in q-calculus and as follows, impossibility to use moving frame as an argument of the wave function. Moreover, if in the Fourier harmonics $f(x,t) = e^{i(kx-\omega t)}$, we try naively to replace the exponential function by Jackson's q-exponential, $f_q(x,t) = e_q(i(kx-\omega t))$, then we find that it doesn't work due to the absence of factorization for q-exponential function $e_q(i(kx-\omega t)) \neq e_q(ikx)\, e_q(i\omega t)$.

5.5.1. Traveling Waves

Real functions of two real variables $F(x,t) = F(x \pm ct)$ called the traveling waves, satisfy the following first order equations

$$\left(\frac{\partial}{\partial t} \mp c\frac{\partial}{\partial x}\right)F(x \pm ct) = 0. \tag{5.77}$$

It describes waves with fixed shape, prorogating with constant speed c in the left and in the right direction correspondingly. The general solution of the wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial x^2}, \tag{5.78}$$

can be written as an arbitrary superposition of these traveling waves

$$u(x,t) = F(x+ct) + G(x-ct). \tag{5.79}$$

5.5.2. q-Traveling Waves

Direct extension of traveling waves to q-traveling waves is not possible. This happens due to the absence of chain rule in q-calculus and as follows, impossibility to use moving frame as an argument of the wave function. Moreover, if we try in the Fourier harmonics $f(x,t) = e^{i(kx-\omega t)}$, replace exponential function by Jackson's q-exponential function $f(x,t) = e_q(i(kx-\omega t))$, then we find that it doesn't work due to the absence of factorization for q-exponential function $e_q(i(kx-\omega t)) \neq e_q(ikx)\, e_q(i\omega t)$.

That is why, here we propose another way. First we observe that q-binomials

$$(x \pm ct)_q^n = (x \pm ct)(x \pm qct)...(x \pm q^{n-1}ct) \tag{5.80}$$

for $n = 0, \pm 1, \pm 2, ...$, satisfy the first order one-directional q-wave equations

$$\left(D^t_{\frac{1}{q}} \mp cD^x_q\right)(x \pm ct)_q^n = 0, \tag{5.81}$$

which are hyperbolic analogs of q-analyticity (and anti-analyticity). Then, the Laurent series expansion in terms of these q-binomials determines the q-analog of traveling waves

$$f(x \pm ct)_q = \sum_{n=-\infty}^{\infty} a_n (x \pm ct)_q^n.$$

Due to (5.81) the q-binomials (5.80) satisfy the q-wave equation

$$\left((D^t_{\frac{1}{q}})^2 - c^2 (D^x_q)^2\right) u(x,t) = 0 \tag{5.82}$$

and the general solution of this equation is expressed in the form of q-traveling waves

$$u(x,t) = F(x+ct)_q + G(x-ct)_q \tag{5.83}$$

where

$$F(x+ct)_q = \sum_{n=-\infty}^{\infty} a_n (x+ct)_q^n$$

and

$$G(x-ct)_q = \sum_{n=-\infty}^{\infty} b_n (x-ct)_q^n.$$

This allows us to solve IVP for the q-wave equation

$$\left[\left(D^t_{\frac{1}{q}}\right)^2 - c^2 (D^x_q)^2\right] u(x,t) = 0, \tag{5.84}$$

$$u(x,0) = f(x), \tag{5.85}$$

$$D^t_{\frac{1}{q}} u(x,0) = g(x), \tag{5.86}$$

where $-\infty < x < \infty$, in the D'Alembert form:

$$u(x,t) = \frac{f(x+ct)_q + f(x-ct)_q}{2} + \frac{1}{2c} \int_{(x-ct)_q}^{(x+ct)_q} g(x') d_q x', \tag{5.87}$$

where the Jackson integral is

$$\int_{(x-ct)_q}^{(x+ct)_q} g(x')d_qx' = (1-q)(x+ct)\sum_{j=0}^{\infty} q^j g(q^j(x+ct))_q$$
$$- (1-q)(x-ct)\sum_{j=0}^{\infty} q^j g(q^j(x-ct))_q. \quad (5.88)$$

If the initial velocity is zero, $g(x) = 0$, the formula reduces to

$$u(x,t) = \frac{1}{2}\left(f(x+ct)_q + f(x-ct)_q\right). \quad (5.89)$$

It should be noted here that q-traveling wave is not traveling wave in the standard sense and it is not preserving shape during evolution. It can be seen from simple observation. The traveling wave polynomial $(x-ct)_q^n = (x-ct)(x-qct)(x-q^2ct)...(x-q^{n-1}ct)$ includes the set of moving frames (as zeros of this polynomial) with re-scaled set of speeds $(c, qc, q^2c, ..., q^{n-1}c)$. It means that zeros of this polynomial are moving with different speeds and therefore the shape of polynomial wave does not preserve. Only in the linear case and in the case $q = 1$, when speeds of all frames coincide, we get standard traveling wave.

5.5.3. Examples

In this section we are going to illustrate our results by several explicit solutions.

Example 1: We consider I.V.P. for the q-wave equation (5.84) with initial functions

$$u(x,0) = x^2,$$
$$D^t_{\frac{1}{q}} u(x,0) = 0. \quad (5.90)$$

Then the solution of the given I.V.P. for q-wave equation in D'Alembert form is found as

$$u(x,t) = x^2 + qc^2t^2. \quad (5.91)$$

When $q = 1$, it reduces to well-known one as superposition of two traveling wave parabolas $(x \pm ct)^2$ moving to the right and to the left with speed c. Geometrically, the meaning of q is the acceleration of our parabolas in vertical direction.

Example 2: The q-traveling wave

$$\begin{aligned} u(x,t) = (x-ct)_q^2 &= (x-ct)(x-qct) \\ &= \left(x - \frac{[2]}{2}ct\right)^2 - \frac{(q-1)^2}{4}c^2t^2 \end{aligned} \tag{5.92}$$

gives solution of I.V.P. for the q-wave equation (5.84) with initial functions

$$\begin{aligned} &u(x,0) = x^2 \\ &D^t_{\frac{1}{q}} u(x,0) = -[2]_q cx. \end{aligned} \tag{5.93}$$

If $q = 1$ in this solution, we have two degenerate zeros moving with the same speed c. In the case $q \neq 1$, two zeros are moving with different speeds c and qc. It means that, the distance between zeros is growing linearly with time as $(q-1)ct$. The solution is the parabola, moving in vertical direction with acceleration $\frac{(q-1)^2}{4}c^2$, and in horizontal direction with constant speed $\frac{[2]_q}{2}c$. The area under the curve between moving zeros $x = ct$ and $x = qct$

$$\int_{ct}^{qct} (x-ct)_q^2 dx = -\frac{(q-1)^3c^3}{6}t^3$$

is changing according to time as t^3.

For more general initial function $f(x) = x^n,\ \ n = 2, 3, ...$ we get q-traveling wave

$$u(x,t) = (x-ct)_q^n = (x-ct)(x-qct)...(x-q^{n-1}ct)$$

with n-zeros moving with speeds $c, qc, ..., q^{n-1}c$. The distance between two zeros is growing as $(q^m - q^n)ct$, and the shape of wave is changing. In parabolic case with $n = 2$, the shape of curve is not changing, but moving in horizontal direction with constant speed, and in vertical direction with constant acceleration. In contrast to this, for $n > 2$, the motion of zeros with different speeds changes the shape of the wave, and it can not be reduced to simple translation and acceleration.

Example 3: Given IVP for the q-wave equation (5.84) with initial functions as q-

trigonometric functions (Kac and Cheung, 2002)

$$
\begin{aligned}
u(x, 0) &= cos_q x, \\
D^t_{\frac{1}{q}} u(x, 0) &= sin_q x.
\end{aligned} \tag{5.94}
$$

By using the D'Alembert form (5.87), after q-integration , we get

$$
\begin{aligned}
u(x,t) &= \frac{1}{2}\left[cos_q(x+ct)_q + cos_q(x-ct)_q\right] + \frac{1}{2c}\int_{(x-ct)_q}^{(x+ct)_q} sin_q(x')d_q x' \\
&= \frac{1}{2}\left[\left(1+\frac{1}{c}\right)cos_q(x-ct)_q + \left(1-\frac{1}{c}\right)cos_q(x+ct)_q\right].
\end{aligned} \tag{5.95}
$$

Example 4: q-Gaussian Traveling Wave For the initial function in the Gaussian form: $u(x, 0) = e^{-x^2}$ in the standard case $q = 1$ we have the Gaussian traveling wave with the same shape $u(x,t) = e^{-(x-ct)^2}$ (see Figure 5.1). For the q-traveling wave with the same initial condition $u(x, 0) = e^{-x^2}$, we have solution

$$
u(x,t) = \left(e^{-(x-ct)^2}\right)_q \equiv \sum_{n=0}^{\infty} \frac{(-1)^n}{n!}(x-ct)_q^{2n}
$$

(see Figure 5.2). As we can see, during evolution the shape of the wave is changing as an amplitude is growing.

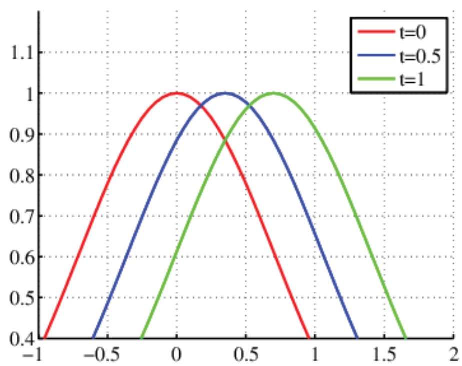

Figure 5.1. Gaussian traveling wave at time $t = 0$, $t = 0.5$ and $t = 1$

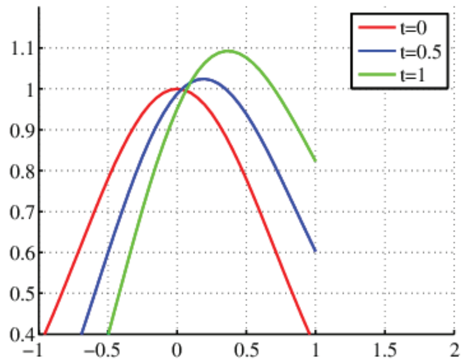

Figure 5.2. q-Gaussian traveling wave at time $t = 0$, $t = 0.5$ and $t = 1$

5.5.4. q-Traveling Waves in terms of q-Hermite Polynomials

Identity(5.26) allows us to rewrite the traveling wave binomial in terms of Hermite polynomials as

$$(x + ct)^n = \frac{1}{2^n} \sum_{k=0}^{n} \binom{n}{k} i^k H_{n-k}(x) H_k(-ict).$$

Its q-analogue for q-traveling wave binomial follows from (5.41)

$$(x + ct)_q^n = \frac{1}{[2]_q^n} \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q i^k q^{\frac{k(k-1)}{2}} H_{n-k}(x; q) H_k(-iqct, \frac{1}{q}).$$

Then, the general solution of q-wave equation (5.82) can be expressed in the form of q-Hermite polynomials

$$u(x, t) = F(x + ct)_q + G(x - ct)_q, \tag{5.96}$$

where

$$
\begin{aligned}
F(x+ct)_q &= \sum_{n=-\infty}^{\infty} a_n (x+ct)_q^n = \sum_{n=-\infty}^{\infty} a_n \frac{1}{[2]_q^n} \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q i^k q^{\frac{k(k-1)}{2}} H_{n-k}(x;q) H_k(-iqct, \frac{1}{q}), \\
G(x-ct)_q &= \sum_{n=-\infty}^{\infty} a_n (x-ct)_q^n = \sum_{n=-\infty}^{\infty} a_n \frac{1}{[2]_q^n} \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q i^k q^{\frac{k(k-1)}{2}} H_{n-k}(x;q) H_k(iqct, \frac{1}{q}).
\end{aligned}
$$

It is instructive to prove the q-traveling wave solution

$$
\left(D^t_{\frac{1}{q}} - cD^x_q\right)(x+ct)_q^n = 0
$$

by using q- Hermite binomial. We have

$$
\begin{aligned}
&(D^t_{\frac{1}{q}} - cD^x_q)(x+ct)_q^n = \left(D^t_{\frac{1}{q}} - cD^x_q\right) \frac{1}{[2]_q^n} \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q i^k q^{\frac{k(k-1)}{2}} H_{n-k}(x;q) H_k(-iqct, \frac{1}{q}) \\
&= \frac{1}{[2]_q^n} \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q i^k q^{\frac{k(k-1)}{2}} H_{n-k}(x;q) D^t_{\frac{1}{q}} H_k(-iqct, \frac{1}{q}) \\
&-c \frac{1}{[2]_q^n} \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q i^k q^{\frac{k(k-1)}{2}} D^x_q H_{n-k}(x;q) H_k(-iqct, \frac{1}{q})
\end{aligned}
$$

By recursion formula for q-Hermite polynomials

$$
D^x_q H_n(x;q) = [2]_q [n]_q H_{n-1}(x;q)
$$

we get

$$
\begin{aligned}
&(D^t_{\frac{1}{q}} - cD^x_q)(x+ct)_q^n = \frac{1}{[2]_q^n} \sum_{k'=1}^{n} \begin{bmatrix} n \\ k' \end{bmatrix}_q i^{k'} q^{\frac{k'(k'-1)}{2}} H_{n-k'}(x;q) [2]_{\frac{1}{q}} [k']_{\frac{1}{q}} H_{k'-1}(-iqct, \frac{1}{q})(-iqc) \\
&-c \frac{1}{[2]_q^n} \sum_{k=0}^{n-1} \begin{bmatrix} n \\ k \end{bmatrix}_q i^k q^{\frac{k(k-1)}{2}} [2]_q [n-k]_q H_{n-k-1}(x;q) H_k(-iqct, \frac{1}{q}) \\
&= \frac{1}{[2]_q^n} \sum_{k=0}^{n-1} \begin{bmatrix} n \\ k+1 \end{bmatrix}_q i^{k+1} q^{\frac{k(k+1)}{2}} H_{n-k-1}(x;q) [2]_{\frac{1}{q}} [k+1]_{\frac{1}{q}} H_k(-iqct, \frac{1}{q})(-iqc) \\
&-c \frac{1}{[2]_q^n} \sum_{k=0}^{n-1} \begin{bmatrix} n \\ k \end{bmatrix}_q i^k q^{\frac{k(k-1)}{2}} [2]_q [n-k]_q H_{n-k-1}(x;q) H_k(-iqct, \frac{1}{q})
\end{aligned}
$$

$$= ci^k \frac{1}{[2]_q^n} \sum_{k=0}^{n-1} \left(\begin{bmatrix} n \\ k+1 \end{bmatrix}_q i q^{\frac{k(k+1)}{2}} [2]_{\frac{1}{q}} [k+1]_{\frac{1}{q}} q - \begin{bmatrix} n \\ k \end{bmatrix}_q q^{\frac{k(k-1)}{2}} [2]_q [n-k]_q \right) H_{n-k-1}(x;q) H_k(-iqct, \frac{1}{q})$$
$$= 0.$$

The expression in parenthesis is zero due to q-combinatorial formula and $[n]_{\frac{1}{q}} = \frac{[n]_q}{q^{n-1}}$.

As a potential application of our results we should mention that an analytic function of two complex variables can be related to the tensor product of Glauber coherent states. Then, the double analytic functions, as well as the q-double analytic functions correspond to some symmetry restrictions on these states. Expansion of these states in binomial and Hermite binomials form would reflect some entanglement properties of these states. These questions are under the study now. Interesting problem also is to find the symmetry group of the q-wave equation as a q-deformation of the Lorentz group.

CHAPTER 6

Q-BINOMIAL AND Q-TRANSLATION OPERATOR

Here we introduce q-translation operator, which produces q-binomials, q-analytic and q-anti analytic functions, and q-travelling waves. A second type of q-translation operator as q-commutative (non-commutative) translation operator is also given, which produces non-commutative binomials, functions for non-commutative coordinates. We generalize these q-translations to q, p-translations for two bases. By specific choice of bases as Golden ratio we obtain Golden binomials as translation of monomials. Finally we show that all these translations can be described by first order q-difference equations.

6.1. Binomial and Translation Operator

As is well known, the translation operator $e^{a\frac{d}{dx}}$, with real number coefficient a, acting on monomial x^n, denoted by $e^{a\frac{d}{dx}}x^n$ produces binomial expansion

$$e^{a\frac{d}{dx}}x^n = (x+a)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k}a^k. \tag{6.1}$$

According to this, application of translation operator to any analytic function $f(x) = \sum_{n=0}^{\infty} a_n x^n$ gives translation of the argument

$$e^{a\frac{d}{dx}}f(x) = f(x+a). \tag{6.2}$$

If the translation coefficient is related to time variable t as $a = \pm ct$, we get the travelling waves

$$e^{\pm ct\frac{d}{dx}}f(x) = f(x \pm ct), \tag{6.3}$$

as a solution of wave equations $(\frac{\partial^2}{\partial t^2} \mp c^2 \frac{\partial^2}{\partial x^2})f(x \pm ct) = 0$, which are moving to the left and right direction correspondingly with speed c.

The translation operator with complex translation coefficient $a = \pm iy$ produces complex analytic and complex anti-analytic binomials respectively

$$e^{\pm iy\frac{d}{dx}} x^n = (x \pm iy)^n. \tag{6.4}$$

As a result, any analytic or anti-analytic complex functions can be written as the translation of real analytic function $f(x) = \sum_{n=0}^{\infty} a_n x^n$

$$e^{\pm iy\frac{d}{dx}} f(x) = f(x \pm iy). \tag{6.5}$$

By differentiating this relation in y we get the analyticity and anti-analyticity conditions correspondingly

$$\left(\frac{\partial}{\partial x} \pm i\frac{\partial}{\partial y}\right) f(x \pm iy) = 0.$$

6.2. q-Binomial and q-Translation Operator

Definition 6.1 *The q-translation operator of the first kind is defined as*

$$e_q^{aD_q^x}, \tag{6.6}$$

where $e_q(x)$ is first Jackson's q-exponential function (2.25).

Definition 6.2 *The q-translation operator of the second kind is defined as*

$$e_{\frac{1}{q}}^{aD_q^x} = E_q^{aD_q^x}, \tag{6.7}$$

where $E_q(x)$ is second Jackson's q-exponential function (2.26)

Proposition 6.1 *The first kind q-translation operator (6.6) acting on monomial x^n produces* *the binomial*

$$e_q^{aD_q^x} x^n = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q x^{n-k} a^k \tag{6.8}$$

Proof Using the definition of the first Jackson's q-exponential function we get

$$\begin{aligned} e_q^{aD_q^x} x^n &= \sum_{k=0}^{\infty} \frac{(aD_q^x)^k}{[k]_q!} x^n = \sum_{k=0}^{n} \frac{a^k}{[k]_q!} [n]_q ... [n-k+1]_q x^{n-k} \\ &= \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q x^{n-k} a^k. \end{aligned} \tag{6.9}$$

□

Proposition 6.2 *The second kind q-translation operator (6.7) acting on monomial x^n produces q-binomial*

$$e_{\frac{1}{q}}^{aD_q^x} x^n = (x+a)_q^n, \tag{6.10}$$

where

$$(x+a)_q^n = (x+a)(x+qa)...(x+q^{n-1}a) = \sum_{n=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q q^{\frac{k(k-1)}{2}} x^{n-k} a^k$$

and $yx = xy$.

Proof Using the definition of the second Jackson's q-exponential function we get

$$\begin{aligned} e_{\frac{1}{q}}^{aD_q^x} x^n &= \sum_{k=0}^{\infty} \frac{(aD_q^x)^k}{[k]_{\frac{1}{q}}!} x^n = \sum_{k=0}^{n} \frac{a^k}{[k]_{\frac{1}{q}}!} [n]_q ... [n-k+1]_q x^{n-k} \\ &= \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q q^{\frac{k(k-1)}{2}} x^{n-k} a^k = (x+a)_q^n, \end{aligned} \tag{6.11}$$

where we used

$$q^{\frac{k(k-1)}{2}} [k]_{\frac{1}{q}!} = [k]_q!.$$

□

According to proposition (6.1), the application of first kind of translation operator (6.6) to any function $f(x) = \sum_{n=0}^{\infty} a_n x^n$ we obtain q-function as

$$e_q^{aD_q^x} f(x) = \sum_{n=0}^{\infty} \sum_{k=0}^{n} a_n \begin{bmatrix} n \\ k \end{bmatrix}_q x^{n-k} a^k, \tag{6.12}$$

which can not be written in simplest form of zeros of polynomial.

For $a = \pm iy$ in first kind of q-translation operator and application to monomial x^n, gives q-analytic and q-anti-analytic complex q-binomial

$$e_q^{\pm iyD_q^x} x^n = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q x^{n-k} (\pm iy)^k \equiv_1 (x \pm iy)_q^n. \tag{6.13}$$

Therefore, the application of first kind of q-translation operator to any function $f(x) = \sum_{n=0}^{\infty} a_n x^n$, produces the q-analytic and q-anti-analytic functions

$$e_q^{\pm iyD_q^x} f(x) = \sum_{n=0}^{\infty} a_n \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q x^{n-k} (\pm iy)^k = \sum_{n=0}^{\infty} a_n \,_1(x \pm iy)_q^n \equiv_1 f_q(x \pm iy). \tag{6.14}$$

These functions satisfy the q-analyticity and q-anti-analyticity conditions

$$\frac{1}{2} \left(D_q^x \pm i D_q^y \right) f_q(x, iy) = 0, \tag{6.15}$$

and corresponding q-Cauchy Riemann equations

$$D_q^x u = D_q^y v, \quad D_q^x v = -D_q^y u, \tag{6.16}$$

where $f_q(x, y) = u_q(x, y) + i v_q(x, y)$.

A more interesting case is the application of second kind q-translation operator (6.7) to any function $f(x) = \sum_{n=0}^{\infty} a_n x^n$. This yields a q-function f as an expansion in q-binomials

$$e_{\frac{1}{q}}^{aD_q^x} f(x) = f(x + a)_q = \sum_{n=0}^{\infty} a_n (x + a)_q^n. \tag{6.17}$$

By choosing $a = \pm ct$ in q-translation operator and applying this to any function $f(x) =$

$\sum_{n=0}^{\infty} a_n x^n$, we get the q-travelling waves,

$$e_{\frac{1}{q}}^{\pm ctD_q^x} f(x) = f(x \pm ct)_q. \tag{6.18}$$

These q-travelling waves give the general solution of the q-wave equation studied in (Nalci Tumer and Pashaev , 2016)

$$\left(\left(D_{\frac{1}{q}}^t\right)^2 - c^2 \left(D_q^x\right)^2\right) u(x, t; q) = 0,$$

in the following form $u(x, t; q) = f(x + ct)_q + g(x - ct)_q$.

For $a = \pm iy$ in q-translation operator and application to monomial x^n, gives q-analytic and q-anti-analytic binomials (Pashaev and Nalci , 2014)

$$e_{\frac{1}{q}}^{\pm iyD_q^x} x^n = (x \pm iy)_q^n. \tag{6.19}$$

As a result, the application of q-translation operator to any function $f(x) = \sum_{n=0}^{\infty} a_n x^n$, produces the q-analytic and q-anti-analytic functions

$$e_{\frac{1}{q}}^{\pm iyD_q^x} f(x) = f(x \pm iy)_q. \tag{6.20}$$

These functions satisfy the q-analyticity and q-anti-analyticity conditions

$$\frac{1}{2}\left(D_q^x \pm iD_{\frac{1}{q}}^y\right) f(x \pm iy)_q = 0, \tag{6.21}$$

and corresponding q-Cauchy Riemann equations

$$D_q^x u = D_{\frac{1}{q}}^y v, \quad D_q^x v = -D_{\frac{1}{q}}^y u, \tag{6.22}$$

where $f(x + iy; q) = u(x, y; q) + iv(x, y; q)$.

We can generalize the above results in the following definition:

Definition 6.3 *The q,p-translation operator is defined as*

$$E_{q,p}^{yD^x_{q,p}} = e_{\frac{1}{q},\frac{1}{p}}^{yD^x_{q,p}}, \tag{6.23}$$

where (q,p)-Jackson's exponential is

$$E_{q,p}(x) = \sum_{n=0}^{\infty} \frac{1}{[n]_{q,p}!}(qp)^{\frac{n(n-1)}{2}} x^n \tag{6.24}$$

and (q,p)-number is

$$[n]_{q,p} = \frac{q^n - p^n}{q - p} = [n]_{p,q}.$$

Proposition 6.3 *Action of q,p-translation operator (6.23) to monomial x^n produces (q,p)-polynomial in the form*

$$e_{\frac{1}{q},\frac{1}{p}}^{yD^x_{q,p}} x^n = (x+y)^n_{q,p}, \tag{6.25}$$

where (q,p) polynomial is defined in the form

$$\begin{aligned}(x+y)^n_{q,p} &= \sum_{k=0}^{n} (q\,p)^{\frac{k(k-1)}{2}} \begin{bmatrix} n \\ k \end{bmatrix}_{q,p} x^{n-k}y^k \\ &= (x+q^{n-1}y)(x+q^{n-2}py)...(x+qp^{n-2}y)(x+p^{n-1}y).\end{aligned} \tag{6.26}$$

As a special case when $q=\varphi,\ p=\varphi' = -\frac{1}{\varphi}$, where φ is the Golden ratio, we obtain Fibonomial (Pashaev and Nalci, 2012) as translation of monomial

$$e_{\frac{1}{\varphi},-\varphi}^{yD_{\varphi,-\frac{1}{\varphi}}} x^n = (x+y)^n_{\varphi,-\frac{1}{\varphi}}, \tag{6.27}$$

where

$$e_{\frac{1}{\varphi},-\varphi}^{x} = \sum_{n=0}^{\infty} \frac{x^n}{[n]_{\frac{1}{\varphi},-\varphi}!}$$

$$(x+y)_F \equiv (x+y)^n_{\varphi,-\frac{1}{\varphi}} = \sum_{k=0}^{n} (-1)^{\frac{k(k-1)}{2}} \begin{bmatrix} n \\ k \end{bmatrix}_{\varphi,-\frac{1}{\varphi}} x^{n-k} y^k,$$

and q-binomial coefficients are Fibonomial

$$\begin{bmatrix} n \\ k \end{bmatrix}_{\varphi,-\frac{1}{\varphi}} = \frac{F_n!}{F_{n-k}!F_k!},$$

where $F_n \equiv [n]_F = \frac{\varphi^n - (\varphi')^n}{\varphi - \varphi'}$.

6.3. Non-Commutative Translations

Definition 6.4 *q-commutative translation operator is defined as*

$$e_q^{aD^x_{\frac{1}{q}}}, \tag{6.28}$$

where $ax = qxa$.

Proposition 6.4 *Application of q-commutative translation operator to monomial x^n gives non-commutative binomial*

$$e_q^{aD^x_{\frac{1}{q}}} x^n = (x+a)^n = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q x^{n-k} a^k, \tag{6.29}$$

where $ax = qxa$.

Proof By using the definition of q-exponential function we have

$$e_q^{aD^x_{\frac{1}{q}}} x^n = \sum_{k=0}^{\infty} \frac{(aD^x_{\frac{1}{q}})^k}{[k]_q!} x^n \tag{6.30}$$

and using the non-commutativity of coordinates $ax = qxa$, we obtain the k-th derivative as

$$\begin{aligned}
(aD^x_{\frac{1}{q}})^k x^n &= \underbrace{(aD^x_{\frac{1}{q}})(aD^x_{\frac{1}{q}})...(aD^x_{\frac{1}{q}})}_{\text{k-times}}\, x^n \\
&= \underbrace{(aD^x_{\frac{1}{q}})(aD^x_{\frac{1}{q}})...(aD^x_{\frac{1}{q}})}_{\text{k-1 times}}[n]_q x^{n-1} a \\
&= [n]_q[n-1]_q...[n-k+1]_q x^{n-k} a^k. \qquad (6.31)
\end{aligned}$$

Putting this into the definition of q-exponential function we get the desired result. □

The q-commutative translation operator with complex translation coefficient $a = \pm iy$ produces q-analytic and q-anti-analytic binomials respectively for non-commutative coordinates $yx = qxy$

$$e_q^{\pm iyD^x_{\frac{1}{q}}} x^n = (x \pm iy)^n = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q x^{n-k} (\pm iy)^k. \qquad (6.32)$$

Thus any q-analytic or q-anti-analytic complex functions of q-commutative coordinates $yx = qxy$ can be written as the translation of real analytic function $f(x) = \sum_{n=0}^{\infty} a_n x^n$

$$e_q^{\pm iyD^x_{\frac{1}{q}}} f(x) = f(x \pm iy) = \sum_{n=0}^{\infty} a_n (x \pm iy)^n. \qquad (6.33)$$

By taking D^y_q derivative of this relation, we get the q-analyticity and q-anti-analyticity conditions for q-commutative coordinates correspondingly

$$\left(D^x_{\frac{1}{q}} \pm iD^y_q\right) f(x \pm iy) = 0, \qquad yx = qxy, \qquad (6.34)$$

where the q-derivatives are acting on non commutative complex q-binomial as follows

$$\begin{aligned}
D^x_{\frac{1}{q}}(x + iy)^n &= [n]_q \left(\frac{x}{q} + iy\right)^{n-1} \\
D^y_q (x + iy)^n &= i[n]_{\frac{1}{q}} (x + iqy)^{n-1}. \qquad (6.35)
\end{aligned}$$

The commutator relation of corresponding q-derivatives for function $f(x+iy) = \sum_{n=0}^{\infty} a_n(x+iy)^n$ is found as

$$\left[D_q^y, D_{\frac{1}{q}}^x\right] = D_q^y D_{\frac{1}{q}}^x - q D_{\frac{1}{q}}^x D_q^y = 0 \tag{6.36}$$

By choosing $a = \pm ct$ in q-commutative translation operator and applying this to the monomial x^n we get q-travelling binomial of non-commutative time t and space $x : tx = qxt$

$$e_q^{\pm ct D_{\frac{1}{q}}^x} x^n = (x \pm ct)^n. \tag{6.37}$$

As a result, the application of q-commutative translation operator to any function $f(x) = \sum_{n=0}^{\infty} a_n x^n$, produces the general solution of q-wave equation of q-commutative variables x, t :

$$\left(\left(D_q^t\right)^2 - q\left(c D_{\frac{1}{q}}^x\right)^2\right) u(x,t) = 0 \tag{6.38}$$

as superposition of travelling waves with $tx = qxt$

$$u(x,t) = F(x+ct) + G(x-ct) \tag{6.39}$$

where

$$F(x+ct) = \sum_{n=0}^{\infty} a_n (x+ct)^n, \qquad G(x-ct) = \sum_{n=0}^{\infty} a_n (x-ct)^n.$$

We can generalize the above results in the following definition:

Definition 6.5 *The qp-commutative translation operator is defined as*

$$e_{q,p}^{iy D_{\frac{1}{q},\frac{1}{p}}} \tag{6.40}$$

where (q,p)*-Jackson's exponential is*

$$e_{q,p}(x) = \sum_{n=0}^{\infty} \frac{1}{[n]_{q,p}!} x^n \tag{6.41}$$

and (q,p)*-number is*

$$[n]_{q,p} = \frac{q^n - p^n}{q - p}.$$

Proposition 6.5 *Action of qp-noncommutative translation operator (6.41) to monomial* x^n *produces complex* (q,p)*-polynomial with* $yx = qpxy$ *in the form*

$$e_{q,p}^{iyD_{\frac{1}{q},\frac{1}{p}}} x^n = (x+iy)^n, \tag{6.42}$$

where

$$(x+iy)^n = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_{q,p} x^{n-k}(iy)^k, \qquad yx = q\,p\,x\,y. \tag{6.43}$$

As a special case when $q = \varphi$ and $p = \varphi' = -\frac{1}{\varphi}$, where φ is the Golden ratio, we obtain Fibonomial as translation of monomial with $yx = -xy$

$$e_{\varphi,-\varphi'}^{iyD_{\frac{1}{\varphi},-\varphi}} x^n = (x+iy)^n_{\varphi,-\frac{1}{\varphi}} = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_{\varphi,-\frac{1}{\varphi}} x^{n-k}(iy)^k, \tag{6.44}$$

and q-binomial coefficients are

$$\begin{bmatrix} n \\ k \end{bmatrix}_{\varphi,-\frac{1}{\varphi}} = \frac{F_n!}{F_{n-k}!F_k!},$$

where $F_n \equiv [n]_{\varphi,\varphi'} = [n]_F = \frac{\varphi^n - (\varphi')^n}{\varphi - \varphi'}$.

CHAPTER 7

COHERENT STATES AND GENERALIZED MEHLER FORMULA

Here by applying evolution operator at time $t = 1$ to Glauber coherent states we introduce a new type of quantum states as Hermite coherent states, characterized by Hermite polynomials, which can be normalized by using the known Mehler formula. Then the evolution operator at arbitrary time t generates more general Kampe-de Feriet coherent states, characterized by Kampe-de Feriet polynomials. In order to normalize Kampe-de Feriet coherent states we introduce the generalization of the known Mehler formula. Then we introduce corresponding Fock-Bargmann representation for these new coherent states. By using the generating function of Bernoulli polynomials we construct Bernoulli coherent states and related Fock-Bargmann representation. By using q-evolution operator for q-holomorphic heat equation we obtain q-analogues of Hermite and Kampe-de Feriet coherent states and corresponding Fock- Bargmann representations. The q-translation operator allows us to get double q-analytic q-Coherent states from analytic q-Coherent states.

7.1. Coherent States

Definition 7.1 *The Glauber coherent state in the basis of Fock states $|n\rangle$ is defined as (Perelomov, 1986)*

$$|\alpha\rangle = e^{-\frac{|\alpha|^2}{2}} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle \,, \tag{7.1}$$

where α is an arbitrary complex number and $|n\rangle = \frac{(a^+)^n}{\sqrt{n!}} |0\rangle$.

The Fock-Bargmann representation of coherent state $|\alpha\rangle$ is defined by scalar product of this state with coherent state $|z\rangle$:

$$\langle z|\alpha\rangle = e^{-\frac{(|z|^2+|\alpha|^2)}{2}} \sum_{m=0}^{\infty} \frac{\bar{z}^m}{\sqrt{m!}} \langle m| \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle = e^{-\frac{(|z|^2+|\alpha|^2)}{2}} \sum_{n=0}^{\infty} \frac{(\bar{z}\alpha)}{n!} = e^{-\frac{(|z|^2+|\alpha|^2)}{2}} e^{\bar{z}\alpha} = e^{-\frac{(|z|^2+|\alpha|^2)}{2}} G(\alpha, \bar{z}),$$

where $G(\alpha, z) = e^{z\alpha}$ analytic in z and α.

7.2. Holomorphic Heat Equation

Definition 7.2 *For complex α, the holomorphic heat equation in space of analytic functions is defined as*

$$\left(\frac{\partial}{\partial t} + \frac{1}{4}\frac{\partial^2}{\partial \alpha^2}\right)\Phi(\alpha, t) = 0 \tag{7.2}$$

with evolution operator

$$U(t) = e^{-\frac{1}{4}t\frac{d^2}{d\alpha^2}} \tag{7.3}$$

so that

$$\Phi(\alpha, t) = U(t)\Phi(\alpha, 0).$$

7.3. Hermite Coherent States

Proposition 7.1 *Applying evolution operator $U(1)$ to monomial $(2\alpha)^n$ we get the Hermite polynomials $H_n(\alpha)$:*

$$H_n(\alpha) = e^{-\frac{1}{4}\frac{d^2}{d\alpha^2}}(2\alpha)^n. \tag{7.4}$$

In this section, we consider the coherent states which are not normalized

$$|\alpha\rangle = \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle\,. \tag{7.5}$$

Definition 7.3 *Applying evolution operator $U(1)$ to not normalized coherent state $|\alpha\rangle$ we de-*

fine Hermite coherent state $|H(\alpha)\rangle$ *as*

$$e^{-\frac{1}{4}\frac{d^2}{d\alpha^2}}|\alpha\rangle = \sum_{n=0}^{\infty} \frac{H_n(\alpha)}{2^n \sqrt{n!}} |n\rangle \equiv |H(\alpha)\rangle \,, \tag{7.6}$$

characterized by analytic Hermite polynomials in α,

Proposition 7.2 *The action of bosonic operators* a, a^+, *which satisfy the commutation relation* $[a, a^+] = 1$, *to Hermite coherent state is expressed in the following form:*

$$a\,|H(\alpha)\rangle = \left(\alpha - \frac{1}{2}\frac{d}{d\alpha}\right)|H(\alpha)\rangle \,, \tag{7.7}$$

$$a^+\,|H(\alpha)\rangle = \frac{d}{d\alpha}\,|H(\alpha)\rangle \,. \tag{7.8}$$

Proof Action of annihilation operator to n-particle states $a\,|n\rangle = \sqrt{n}\,|n-1\rangle$ gives

$$a\,|H(\alpha)\rangle = a \sum_{n=0}^{\infty} \frac{H_n(\alpha)}{2^n \sqrt{n!}} |n\rangle = \sum_{n=0}^{\infty} \frac{H_n(\alpha)}{2^n \sqrt{n!}} \sqrt{n}\,|n-1\rangle = \sum_{n=0}^{\infty} \frac{H_{n+1}(\alpha)}{2^{n+1} \sqrt{n!}} |n\rangle \,, \tag{7.9}$$

by using two and three term recurrence relation for Hermite polynomials

$$\frac{d}{d\alpha} H_n(\alpha) = 2n H_{n-1}(\alpha) \tag{7.10}$$

$$H_{n+1}(\alpha) = 2\alpha H_n(\alpha) - 2n H_{n-1}(\alpha) \tag{7.11}$$

we get desired result

$$\begin{aligned} a\,|H(\alpha)\rangle &= \sum_{n=0}^{\infty} \frac{\alpha H_n(\alpha)}{2^n \sqrt{n!}} |n\rangle - \sum_{n=0}^{\infty} \frac{\frac{d}{d\alpha} H_n(\alpha)}{2^{n+1} \sqrt{n!}} |n\rangle \\ &= \alpha\,|H(\alpha)\rangle - \frac{1}{2}\frac{d}{d\alpha}\,|H(\alpha)\rangle \,. \end{aligned} \tag{7.12}$$

Another way to prove this proposition is starting with eigenvalue problem for annihilation operator

$$a\,|\alpha\rangle = \alpha\,|\alpha\rangle \tag{7.13}$$

and multiplying both sides by evolution operator $U(1) = e^{-\frac{1}{4}\frac{d^2}{d\alpha^2}}$ so that we have

$$
\begin{aligned}
e^{-\frac{1}{4}\frac{d^2}{d\alpha^2}} a\,|\alpha\rangle &= e^{-\frac{1}{4}\frac{d^2}{d\alpha^2}} \alpha\,|\alpha\rangle \\
a\,|H(\alpha)\rangle &= \underbrace{\left[e^{-\frac{1}{4}\frac{d^2}{d\alpha^2}}, \alpha\right]}_{*}|\alpha\rangle + \alpha\,\underbrace{e^{-\frac{1}{4}\frac{d^2}{d\alpha^2}}\,|\alpha\rangle}_{|H(\alpha)\rangle}
\end{aligned}
\tag{7.14}
$$

We need to calculate commutator $*$

$$
* = \left[\sum_{n=0}^{\infty}\left(-\frac{1}{4}\right)^n \frac{1}{n!}\left(\frac{d^2}{d\alpha^2}\right)^n, \alpha\right] = \sum_{n=0}^{\infty}\left(-\frac{1}{4}\right)^n \frac{1}{n!}\left[\left(\frac{d^2}{d\alpha^2}\right)^n, \alpha\right] \tag{7.15}
$$

by using the known commutation relation $[\frac{d}{dx}, x] = 1$,

$$
\left[\frac{d}{dx}, x\right] = 1 \quad \left[\frac{d^2}{dx^2}, x\right] = 2\frac{d}{dx}, \quad ..., \quad \left[\frac{d^n}{dx^n}, x\right] = n\frac{d^{n-1}}{dx^{n-1}}, \tag{7.16}
$$

so the commutator $*$ becomes

$$
\begin{aligned}
\left[e^{-\frac{1}{4}\frac{d^2}{d\alpha^2}}, \alpha\right] &= \sum_{n=1}^{\infty}\left(-\frac{1}{4}\right)^n \frac{1}{n!} 2n \frac{d^{2n-1}}{d\alpha^{2n-1}} \\
&= 2\sum_{n=0}^{\infty}\left(-\frac{1}{4}\right)^{n+1} \frac{1}{n!}\frac{d^{2n+1}}{d\alpha^{2n+1}} = -\frac{1}{2}\frac{d}{d\alpha}e^{-\frac{1}{4}\frac{d^2}{d\alpha^2}}.
\end{aligned}
\tag{7.17}
$$

As a final result in (7.14) we obtain

$$
a\,|H(\alpha)\rangle = \left(\alpha - \frac{1}{2}\frac{d}{d\alpha}\right)|H(\alpha)\rangle\,. \tag{7.18}
$$

Similarly, using the action of creation operator to n-particle state $a^+\,|n\rangle = \sqrt{n+1}\,|n+1\rangle$, and the recurrence relations we have

$$
\begin{aligned}
a^+ |H(\alpha)\rangle &= \sum_{n=0}^{\infty} \frac{H_n(\alpha)}{2^n \sqrt{n!}} a^+ |n\rangle = \sum_{n=0}^{\infty} \frac{H_n(\alpha)}{2^n \sqrt{n!}} \sqrt{n+1}\, |n+1\rangle \\
& \sum_{n=1}^{\infty} \frac{H_{n-1}(\alpha)}{2^{n-1} \sqrt{(n-1)!}} \sqrt{n}\, |n\rangle = \sum_{n=1}^{\infty} \frac{\frac{d}{d\alpha} H_n(\alpha)}{2^n \sqrt{n!}} |n\rangle \\
&= \frac{d}{d\alpha} |H(\alpha)\rangle \, . \qquad (7.19)
\end{aligned}
$$

□

As a result, we obtain the eigenvalue problem for Hermite coherent state with eigenvalue α, which are eigenvectors of a superposition of bosonic operators

$$
B\,|H(\alpha)\rangle = \frac{2}{\sqrt{3}} \alpha\, |H(\alpha)\rangle \, , \qquad (7.20)
$$

where

$$
B \equiv \frac{2}{\sqrt{3}} \left(a + \frac{1}{2} a^+ \right)
$$

and $[B, B^+] = 1$.

The Hamiltonian for Hermite coherent states is obtained as

$$
H = \frac{\hbar\omega}{2} (B^+ B + B B^+) = \frac{1}{6m} \left((3m\omega x)^2 + p^2 \right), \qquad (7.21)
$$

where

$$
\begin{aligned}
B &= \frac{3m\omega x + ip}{\sqrt{6\hbar m\omega}} \\
B^+ &= \frac{3m\omega x - ip}{\sqrt{6\hbar m\omega}} \qquad (7.22)
\end{aligned}
$$

and $p = -i\frac{d}{dx}$.

We find the fluctuation deviation in x and p in the following form:

$$\Delta x = \sqrt{\langle x^2 \rangle - \langle x \rangle^2} = \sqrt{\frac{\hbar}{6m\omega}}, \tag{7.23}$$

$$\Delta p = \sqrt{\langle p^2 \rangle - \langle p \rangle^2} = \sqrt{\frac{3\hbar m\omega}{2}}, \tag{7.24}$$

where $\langle x \rangle = \langle H(\alpha)|x|H(\alpha)\rangle$.

As an important result, Hermite coherent state has minimal Heisenberg uncertainty relation

$$\Delta x \Delta p = \frac{\hbar}{2} \tag{7.25}$$

If we compare Hermite coherent states with Glauber coherent states, then we note that both states minimize the Heisenberg uncertainty relation (7.25). But for the coherent states we have

$$\Delta x = \sqrt{\frac{\hbar}{2m\omega}}$$

$$\Delta p = \sqrt{\frac{\hbar m\omega}{2}}$$

Comparing with (7.23), (7.24) we find that position uncertainty in Hermite state is squeezed by factor $\sqrt{3}$. This indicate that Hermite coherent states are the squeezed states.

In order to find the normalized Hermite coherent state we use the Mehler Identity (Mehler, 1866).

Identity 7.1 *The Mehler formula is defined as*

$$E(x, y) = \frac{1}{\sqrt{1 - u^2}} e^{\frac{-u^2(x^2+y^2)+2uxy}{1-u^2}} = \sum_{n=0}^{\infty} \frac{H_n(x)H_n(y)}{n!} \left(\frac{u}{2}\right)^n, \tag{7.26}$$

where $|u| < 1$.

Proof Using the Gaussian integral

$$\int_{-\infty}^{\infty} e^{-\xi^2} d\xi = \sqrt{\pi} \tag{7.27}$$

we can write

$$e^{-x^2} = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-\xi^2+2i\xi x} d\xi. \tag{7.28}$$

By substituting this relation into the Rodrigues' formula for the Hermite polynomials we obtain integral representation for Hermite polynomials

$$\begin{aligned} H_n(x) &= e^{x^2}\left(-\frac{d}{dx}\right)^n e^{-x^2} = e^{x^2}\left(-\frac{d}{dx}\right)^n \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} e^{-\xi^2+2i\xi x}d\xi \\ &= \frac{1}{\sqrt{\pi}}e^{x^2}\int_{-\infty}^{\infty}(-1)^n(2i\xi)^n e^{-\xi^2+2i\xi x}d\xi \\ &= \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty}(-1)^n(2i\xi)^n e^{-(\xi-ix)^2}d\xi. \end{aligned} \tag{7.29}$$

So, by using the integral representation of Hermite polynomials we have

$$\begin{aligned} \sum_{n=0}^{\infty}\frac{H_n(x)H_n(y)}{n!}\left(\frac{u}{2}\right)^n &= \frac{1}{\pi}\int_{-\infty}^{\infty}d\xi\int_{-\infty}^{\infty}d\eta\; e^{-(\xi-ix)^2-(\eta-iy)^2}\sum_{n=0}^{\infty}\frac{(-2u\xi\eta)^n}{n!} \\ &= \frac{1}{\pi}\int_{-\infty}^{\infty}d\xi\int_{-\infty}^{\infty}d\eta\; e^{-(\xi-ix)^2-(\eta-iy)^2}e^{-2u\xi\eta} \\ &= \frac{1}{\pi}\int_{-\infty}^{\infty}d\xi\; e^{-(\xi-ix)^2-2iu\xi y+u^2\xi^2}\underbrace{\int_{-\infty}^{\infty}d\eta\; e^{-((\eta-iy)+u\xi)^2}}_{\sqrt{\pi}} \\ &= \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty}e^{-(1-u^2)\xi^2+2i\xi(x-uy)+x^2}d\xi \\ &= \frac{1}{\sqrt{\pi}}\sqrt{\frac{\pi}{1-u^2}}e^{\frac{-(x-uy)^2+x^2(1-u^2)}{1-u^2}} = \frac{1}{\sqrt{1-u^2}}e^{\frac{-u^2(x^2+y^2)+2uxy}{1-u^2}} \end{aligned} \tag{7.30}$$

where we have used the Gaussian integral

$$\int_{-\infty}^{\infty}e^{-ax^2+bx+c}dx = \sqrt{\frac{\pi}{a}}e^{\frac{b^2}{4a}+c}.$$

$\square$

Using the Mehler formula for $u = \frac{1}{2},\; x = \bar{\beta}$ and $y = \alpha$, we find inner product of two Hermite coherent states in the following form

$$\langle H(\beta)|H(\alpha)\rangle = \sum_{n=0}^{\infty}\frac{H_n(\bar{\beta})H_n(\alpha)}{n!}\left(\frac{1}{2^2}\right)^n = \frac{2}{\sqrt{3}}e^{-\frac{1}{3}(\alpha^2+\bar{\beta}^2-4\alpha\bar{\beta})} \neq 0. \tag{7.31}$$

This form shows that Hermite coherent states are not orthogonal.

By putting $\alpha = \beta$, as a result we find normalized Hermite Coherent state as

$$|H(\alpha)\rangle = \frac{\sqrt[4]{3}}{\sqrt{2}}\, e^{\frac{1}{6}(\alpha^2+\bar{\alpha}^2-4|\alpha|^2)} \sum_{n=0}^{\infty} \frac{H_n(\alpha)}{2^n\sqrt{n!}}\,|n\rangle\,, \tag{7.32}$$

where $\langle H(\alpha)|H(\alpha)\rangle = 1$.

The probability of Hermite coherent state $|H(\alpha)\rangle$ being in the state $|n\rangle$ is

$$P(n) = |\langle n|H(\alpha)\rangle|^2 = \frac{\sqrt{3}}{2}\left|e^{\frac{1}{3}(\alpha^2+\bar{\alpha}^2)-4|\alpha|^2}\right| \frac{H_n(\alpha)H_n(\bar{\alpha})}{2^{2n}n!}. \tag{7.33}$$

The Fidelity for Hermite Coherent states, which measure the closeness of two quantum states is found as

$$|\langle H(\beta)|H(\alpha)\rangle|^2 = \frac{4}{3}e^{-\frac{1}{3}((\alpha-\bar{\beta})^2+(\bar{\alpha}-\beta)^2-2(\alpha\bar{\beta}+\bar{\alpha}\beta)}, \tag{7.34}$$

By applying bra vector $\langle x|$ to the normalized Hermite coherent state (7.32) we obtain

$$\langle x|H(\alpha)\rangle = \frac{\sqrt[4]{3}}{\sqrt{2}}\, e^{\frac{1}{6}(\alpha^2+\bar{\alpha}^2-4|\alpha|^2)} \sum_{n=0}^{\infty} \frac{H_n(\alpha)}{2^n\sqrt{n!}}\,\langle x|n\rangle\,, \tag{7.35}$$

and writing the coordinate representation of n particle state

$$\langle x|n\rangle = \psi_n(x) = \left(\frac{\omega}{\pi\hbar}\right)^{\frac{1}{4}} \frac{1}{\sqrt{2^n\,n!}} H_n\left(\sqrt{\frac{\omega}{\hbar}}x\right) e^{-\frac{\omega}{2\hbar}x^2} \tag{7.36}$$

we get the wave function

$$\langle x|H(\alpha)\rangle = \frac{\sqrt[4]{3}}{\sqrt{2}}\, e^{\frac{1}{6}(\alpha^2+\bar{\alpha}^2-4|\alpha|^2)} \left(\frac{\omega}{\pi\hbar}\right)^{\frac{1}{4}} e^{-\frac{\omega}{2\hbar}x^2} \sum_{n=0}^{\infty} \frac{1}{2^n\sqrt{2^n n!}} H_n\left(\sqrt{\frac{\omega}{\hbar}}x\right) H_n(\alpha). \tag{7.37}$$

By using Mehler formula (7.26) with $u = \frac{1}{\sqrt{2}}$ the normalized Hermite coherent state in coordinate representation is obtained as Gaussian function with origin determined by complex

α

$$\langle x|H(\alpha)\rangle = \left(\frac{3\omega}{\pi\hbar}\right)^{\frac{1}{4}} e^{\frac{1}{6}(3\alpha^2+\bar{\alpha}^2-4|\alpha|^2)} e^{-\frac{3\omega}{2\hbar}\left(x-\frac{2}{3}\sqrt{\frac{2\hbar}{\omega}}\alpha\right)^2}. \tag{7.38}$$

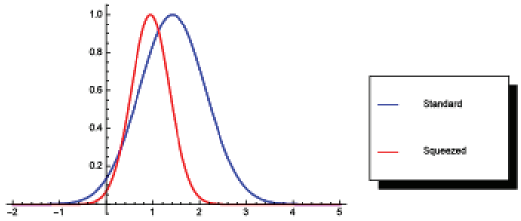

Figure 7.1. Squeezed Hermite state in coordinate representation

We have indicated above and now we can see that the Hermite coherent state is the squeezed coherent state

$$\langle x|H(\alpha)\rangle = \underbrace{(\sqrt{3})^{x\frac{d}{dx}}}_{M^x_{\sqrt{3}}}\langle x|\alpha\rangle,$$

where squeezing factor is written in terms of dilatation operator M^x_q with $q = \sqrt{3}$.

Definition 7.4 *The generating function for Hermite polynomials is defined as (Arfken and Weber, 2005)*

$$e^{-t^2+2xt} = \sum_{n=0}^{\infty} H_n(x)\frac{t^n}{n!} \tag{7.39}$$

Proposition 7.3 *We have the following identity*

$$e^{-\left(\frac{a^+}{2}\right)^2}\left|\frac{\alpha}{2}\right\rangle = |H(\alpha)\rangle \tag{7.40}$$

Proof Using the definition of Hermite coherent state (7.6) and the generating function for Hermite polynomials (7.39), we get desired results as follows

$$
\begin{aligned}
|H(\alpha)\rangle &= \sum_{n=0}^{\infty} \frac{H_n(\alpha)}{2^n \sqrt{n!}} |n\rangle = \sum_{n=0}^{\infty} \frac{H_n(\alpha)}{2^n \sqrt{n!}} \frac{(a^+)^n}{\sqrt{n!}} |0\rangle = \sum_{n=0}^{\infty} \frac{H_n(\alpha)}{n!} \left(\frac{a^+}{2}\right)^n |0\rangle \\
&= e^{-\left(\frac{a^+}{2}\right)^2 + \alpha \frac{a^+}{2}} |0\rangle = e^{-\left(\frac{a^+}{2}\right)^2} e^{\alpha \frac{a^+}{2}} |0\rangle = e^{-\left(\frac{a^+}{2}\right)^2} \sum_{n=0}^{\infty} \frac{\alpha^n}{2^n} \frac{(a^+)^n}{n!} |0\rangle \\
&= e^{-\left(\frac{a^+}{2}\right)^2} \sum_{n=0}^{\infty} \frac{\alpha^n}{2^n \sqrt{n!}} \frac{(a^+)^n}{\sqrt{n!}} |0\rangle = e^{-\left(\frac{a^+}{2}\right)^2} \sum_{n=0}^{\infty} \frac{\alpha^n}{2^n \sqrt{n!}} |n\rangle = e^{-\left(\frac{a^+}{2}\right)^2} \left|\frac{\alpha}{2}\right\rangle . \qquad (7.41)
\end{aligned}
$$

□

Proposition 7.4 *The eigenvalue problem for Hermite coherent state is found as*

$$
2e^{-\left(\frac{a^+}{2}\right)^2} a\, e^{\left(\frac{a^+}{2}\right)^2} |H(\alpha)\rangle = \alpha\, |H(\alpha)\rangle . \qquad (7.42)
$$

Proof Applying operator $e^{\left(\frac{a^+}{2}\right)^2}$ to both sides of equation (7.40),

$$
\left|\frac{\alpha}{2}\right\rangle = e^{\left(\frac{a^+}{2}\right)^2} |H(\alpha)\rangle \qquad (7.43)
$$

and then annihilation operator a

$$
\begin{aligned}
a \left|\frac{\alpha}{2}\right\rangle &= a\, e^{\left(\frac{a^+}{2}\right)^2} |H(\alpha)\rangle \\
\frac{\alpha}{2} \left|\frac{\alpha}{2}\right\rangle &= a\, e^{\left(\frac{a^+}{2}\right)^2} |H(\alpha)\rangle \\
e^{-\left(\frac{a^+}{2}\right)^2} \frac{\alpha}{2} \left|\frac{\alpha}{2}\right\rangle &= e^{-\left(\frac{a^+}{2}\right)^2} a\, e^{\left(\frac{a^+}{2}\right)^2} |H(\alpha)\rangle \\
\alpha\, |H(\alpha)\rangle &= 2e^{-\left(\frac{a^+}{2}\right)^2} a\, e^{\left(\frac{a^+}{2}\right)^2} |H(\alpha)\rangle . \qquad (7.44)
\end{aligned}
$$

□

Definition 7.5 *Another definition for the Glauber coherent state is application of unitary displacement operator* $D(\alpha) = e^{\alpha a^+ - \bar{\alpha} a}$ *to the vacuum state* $|0\rangle$

$$
|\alpha\rangle = D(\alpha)\, |0\rangle . \qquad (7.45)
$$

Definition 7.6 *The displacement operator for the Hermite coherent state is defined as*

$$|H(\alpha)\rangle = \mathcal{D}(H(\alpha))\,|0\rangle \equiv e^{-\left(\frac{a^+}{2}\right)^2} e^{\frac{\alpha}{2}a^+ - \frac{\bar{\alpha}}{2}a}\,|0\rangle\,. \tag{7.46}$$

Proof Using the definition of coherent state $D(\alpha)\,|0\rangle = |\alpha\rangle$ we write

$$\begin{aligned} D\left(\frac{\alpha}{2}\right)|0\rangle &= \left|\frac{\alpha}{2}\right\rangle \\ e^{-\left(\frac{a^+}{2}\right)^2} e^{\frac{\alpha}{2}a^+ - \frac{\bar{\alpha}}{2}a}\,|0\rangle &= e^{-\left(\frac{a^+}{2}\right)^2}\left|\frac{\alpha}{2}\right\rangle \\ \underbrace{e^{-\left(\frac{a^+}{2}\right)^2} e^{\frac{\alpha}{2}a^+ - \frac{\bar{\alpha}}{2}a}}_{\mathcal{D}(H(\alpha))}\,|0\rangle &= |H(\alpha)\rangle \end{aligned} \tag{7.47}$$

□

By using the definition of Hermite coherent states and the unitary displacement operator $D(\alpha)$

$$\begin{aligned} |H(\alpha)\rangle &= e^{-\frac{1}{4}\frac{d^2}{d\alpha^2}}\,|\alpha\rangle = e^{-\frac{1}{4}\frac{d^2}{d\alpha^2}} D(\alpha)\,|0\rangle \\ &= e^{-\frac{1}{4}\frac{d^2}{d\alpha^2}} e^{\alpha(-\frac{\bar{\alpha}}{2}+a^+)} e^{-\bar{\alpha}a}\,|0\rangle \end{aligned} \tag{7.48}$$

and the property

$$e^{-\frac{1}{4}\frac{d^2}{d\alpha^2}} e^{\alpha k} = e^{-\frac{1}{4}k^2} e^{\alpha k} \tag{7.49}$$

we get our Hermite coherent states as squeezed states

$$|H(\alpha)\rangle = \underbrace{e^{-\frac{1}{4}(a^+ - \frac{\bar{\alpha}}{2})^2}}_{S(\xi)}\,|\alpha\rangle\,, \tag{7.50}$$

where $S(\xi) \equiv e^{-\frac{1}{4}(a^+ - \frac{\bar{\alpha}}{2})^2}$. (This squezing operator is not unitary, but can be extended to the unitary one.) If we denote b, b^+ in terms of boson creation and annihilation operators as follows

$$b^+ \equiv a^+ - \frac{\bar{\alpha}}{2}, \quad b \equiv a - \frac{\alpha}{,}2 \tag{7.51}$$

the it satisfies the same commutation relation $[a, a^+] = [b, b^+] = 1$. As a result we get the $SU(1,1)$ algebra

$$\begin{aligned} & K_- = \frac{b^2}{2}, \quad K_+ = \frac{(b^+)^2}{2}, \quad K_0 = \frac{1}{4}(b^+b + bb^+) \\ & [K_0, K_\pm] = \pm K_\pm, \quad [K_+, K_-] = -2K_0 \end{aligned} \tag{7.52}$$

wiht Casimir operator

$$C = K_0^2 - K_1^2 - K_2^2 = K_0(K_0 + I) + K_-K_+ \tag{7.53}$$

where

$$K_\pm = K_1 \pm iK_2, \quad K_1 = \frac{K_+ + K_-}{2}, \quad K_2 = \frac{K_+ - K_-}{2i}, \tag{7.54}$$

$$[K_1, K_2] = -iK_0, \quad [K_2, K_0] = iK_1, \quad [K_0, K_1] = iK_2. \tag{7.55}$$

This shows that our Hermite coherent states are $SU(1,1)$ generalized coherent states, representing squeezed coherent states.

Fock-Bargmann Representation of Hermite Coherent State By using scalar product of coherent state $|z\rangle$ with Hermite coherent state $|H(\alpha)\rangle$, and the generating function for Hermite polynomials (7.39) we get

$$\begin{aligned} \langle z|H(\alpha)\rangle &= \frac{\sqrt[4]{3}}{\sqrt{2}} e^{-\frac{|z|^2}{2}} e^{\frac{1}{6}(\alpha^2+\bar{\alpha}^2-4|\alpha|^2)} \sum_{m=0}^{\infty} \frac{\bar{z}^m}{\sqrt{m!}} \langle m| \sum_{n=0}^{\infty} \frac{H_n(\alpha)}{2^n\sqrt{n!}} |n\rangle \sum_{n=0}^{\infty} \frac{\bar{z}^n H_n(\alpha)}{2^n n!} \\ &= \frac{\sqrt[4]{3}}{\sqrt{2}} e^{-\frac{|z|^2}{2}} e^{\frac{1}{6}(\alpha^2+\bar{\alpha}^2-4|\alpha|^2)} e^{-\frac{\bar{z}^2}{4}+\alpha\bar{z}}, \end{aligned} \tag{7.56}$$

which gives Fock-Bargmann representation of Hermite coherent states, where $G(\alpha, z) = e^{-\frac{z^2}{4}+\alpha z}$ is analytic in α, z. This representation can be considered as generating function of analytic Hermite polynomials and is a function of two complex variables.

7.4. Kampe-de Feriet Coherent State

Proposition 7.5 *The Kampe-de Feriet polynomials are found acting by the operator* $U(t) = e^{-\frac{1}{4}t\frac{d^2}{d\alpha^2}}$ *to monomial* $(2\alpha)^n$ *as follows:*

$$K_n(\alpha, t) = e^{-\frac{1}{4}t\frac{d^2}{d\alpha^2}}(2\alpha)^n. \tag{7.57}$$

Definition 7.7 *Applying operator* $U(t)$ *to coherent state* $|\alpha\rangle$, *we introduce the Kampe-de Feriet Coherent states*

$$e^{-\frac{1}{4}t\frac{d^2}{d\alpha^2}}|\alpha\rangle = \sum_{n=0}^{\infty}\frac{K_n(\alpha, t)}{2^n\sqrt{n!}}|n\rangle \equiv |K(\alpha, t)\rangle\,. \tag{7.58}$$

We obtain the eigenvalue problem for Kampe-de Feriet coherent states in the following form

$$b\,|K(\alpha, t)\rangle = \frac{1}{\sqrt{1-\frac{t^2}{4}}}\alpha\,|K(\alpha, t)\rangle \tag{7.59}$$

where

$$b = \frac{1}{\sqrt{1-\frac{t^2}{4}}}\left(a + \frac{t}{2}a^+\right)$$

$[b, b^+] = 1$

The Hamiltonian for harmonic oscillator determined by b operators is written as harmonic oscillator with variable mass $\mu(t) \equiv m\frac{2+t}{2-t}$

$$H = \hbar\omega(b^+b + \frac{1}{2}) = \frac{p^2}{2\mu(t)} + \frac{\mu(t)\omega^2x^2}{2},$$

or with parametric frequency $\omega(t) \equiv \omega\frac{2+t}{2-t}$

$$H = \hbar\omega(t)(b^+b + \frac{1}{2}) = \frac{p^2}{2m} + \frac{m\omega^2(t)x^2}{2}.$$

Fluctuation deviations in x and p are

$$\begin{aligned}\Delta x &= \sqrt{\langle x^2\rangle - \langle x\rangle^2} = \frac{1}{2A\omega(1+\frac{t}{2})},\\ \Delta p &= \sqrt{\langle p^2\rangle - \langle p\rangle^2} = \frac{m}{2A(1-\frac{t}{2})}\end{aligned}$$

where

$$A \equiv \sqrt{\frac{m}{2\hbar\omega(1-\frac{t^2}{4})}}.$$

As an important result, Kampe-de Feriet coherent states minimize Heisenberg uncertainty relation

$$\Delta x \Delta p = \frac{\hbar}{2}.$$

Comparison with Glauber coherent states shows that in both cases we have the minimal uncertainty relation, but for the Kampe-de Feriet states the coordinate uncertainty is squeezed by factor $\sqrt{\frac{2-t}{2+t}}$, depending on parameter t and it vanishes for $t \to 2$.

In order to find normalized Kampe-de Feriet coherent states and their coordinate representation we introduce the generalized Mehler Formula.

Theorem 7.1 *Generalized Mehler formula is introduced as bilinear generating function for two Kampe-de Feriet polynomials*

$$\sum_{n=0}^{\infty}\left(\frac{v}{2}\right)^n \frac{K_n(x,t)K_n(y,\tau)}{n!} = \frac{1}{\sqrt{1-t\tau v^2}} e^{\frac{-v^2(\tau x^2+ty^2)+2vxy}{1-t\tau v^2}}, \tag{7.60}$$

where $|v| < \frac{1}{\sqrt{t\tau}}$.

Proof By using the relation between Hermite and Kampe-de Feriet polynomials

$$\begin{aligned}K_n(x,t) &= \left(\sqrt{t}\right)^n H_n\left(\frac{x}{\sqrt{t}}\right)\\ K_n(y,\tau) &= \left(\sqrt{\tau}\right)^n H_n\left(\frac{y}{\sqrt{\tau}}\right),\end{aligned} \tag{7.61}$$

and replacing by $\frac{x}{\sqrt{t}} \equiv \xi$ $\frac{y}{\sqrt{\tau}} \equiv \eta$, the Mehler formula is written as

$$\sum_{n=0}^{\infty}\left(\frac{u}{2}\right)^n \frac{H_n(\xi)H_n(\eta)}{n!} = \sum_{n=0}^{\infty}\left(\frac{u}{2}\right)^n (t\tau)^{-\frac{n}{2}} \frac{K_n(\sqrt{t}\xi,t)K_n(\sqrt{\tau}\eta,\tau)}{n!} = \frac{1}{\sqrt{1-u^2}} e^{\frac{-u^2(\xi^2+\eta^2)+2u\xi\eta}{1-u^2}}. \tag{7.62}$$

If we rewrite the above formula in terms of x and y variables, we obtain

$$\sum_{n=0}^{\infty}\left(\frac{u}{2\sqrt{t\tau}}\right)^n \frac{K_n(x,t)K_n(y,\tau)}{n!} = \frac{1}{\sqrt{1-u^2}} e^{\frac{-u^2\left(\frac{x^2}{t}+\frac{y^2}{\tau}\right)+2u\frac{xy}{\sqrt{t\tau}}}{1-u^2}}. \tag{7.63}$$

and by denoting $u = \sqrt{t\tau}v$ we get the desired result. □

Special Cases:

i. If we choose $t = \tau = 1$ in the generalized Mehler formula and use the relation

$$K_n(x,t) = \left(\sqrt{t}\right)^n H_n\left(\frac{x}{\sqrt{t}}\right) \tag{7.64}$$

we obtain the Mehler formula (7.26)

$$\sum_{n=0}^{\infty}\left(\frac{v}{2}\right)^n \frac{H_n(x)H_n(y)}{n!} = \frac{1}{\sqrt{1-v^2}} e^{\frac{-v^2(x^2+y^2)+2vxy}{1-v^2}}. \tag{7.65}$$

ii. If we choose $t = \tau = 0$, the application of evolution operator to monomials produces monomials and the generalized Mehler formula transforms into exponential function

$$\sum_{n=0}^{\infty}\left(\frac{v}{2}\right)^n \frac{x^n y^n}{n!} = e^{2vxy}. \tag{7.66}$$

iii. For $t = 0$ and $\tau = 1$ the generalized Mehler formula produces the generating function of Hermite polynomials

$$\sum_{n=0}^{\infty}\left(\frac{v}{2}\right)^n \frac{x^n H_n(y,\tau)}{2^n n!} = \sum_{n=0}^{\infty} \frac{z^n H_n(y)}{n!} = e^{-4z^2+4zy}, \tag{7.67}$$

where $U(1)(2y)^n = H_n(y)$ and $\frac{vx}{2} \equiv z$.

iv. For $\tau = 0$ and t-arbitrary we get generating function of Kampe-de Feriet polynomials

$$\sum_{n=0}^{\infty}\left(\frac{v}{2}\right)^n \frac{y^n K_n(x,t)}{n!} = e^{-(vy)^2 t+2vxy} = \sum_{n=0}^{\infty} \frac{z^n}{n!} K_n(x,t) = e^{-4z^2 t+4xz}, \tag{7.68}$$

where $\frac{vy}{2} \equiv z$.

v. For $\tau = 1$ and t-arbitrary we have

$$\sum_{n=0}^{\infty} \left(\frac{v}{2}\right)^n \frac{K_n(x,t)H_n(y)}{n!} = \frac{1}{1 - tv^2} e^{\frac{-v^2(x^2+y^2)+2vxy}{1-tv^2}}. \tag{7.69}$$

vi. For $x = y$ we have

$$\sum_{n=0}^{\infty} \left(\frac{v}{2}\right)^n \frac{K_n(x,t)K_n(x,\tau)}{n!} = \frac{1}{\sqrt{1 - t\tau v^2}} e^{\frac{-(vx)^2(\tau+t)+2vx^2}{1-t\tau v^2}}. \tag{7.70}$$

vii. For $t = \tau$ we get

$$\sum_{n=0}^{\infty} \left(\frac{v}{2}\right)^n \frac{K_n(x,t)K_n(y,t)}{n!} = \frac{1}{\sqrt{1 - (tv)^2}} e^{\frac{-v^2t(x^2+y^2)+2vxy}{1-(tv)^2}}. \tag{7.71}$$

viii. For $x = y$ and $t = \tau$ we have

$$\sum_{n=0}^{\infty} \left(\frac{v}{2}\right)^n \frac{(K_n(x,t))^2}{n!} = \frac{1}{\sqrt{1 - (tv)^2}} e^{\frac{-2t(vx)^2+2vx^2}{1-(tv)^2}}. \tag{7.72}$$

Applying bra vector $\langle x|$ to Kampe-de Feriet coherent state (7.58) and using (7.36) we obtain

$$\langle x|K(\alpha,t)\rangle = C\left(\frac{\omega}{\pi\hbar}\right)^{\frac{1}{4}} \sum_{n=0}^{\infty} \left(\frac{1}{2\sqrt{2}}\right)^n \frac{H_n\left(\sqrt{\frac{\omega}{\hbar}}x\right)K_n(\alpha,t)}{n!}. \tag{7.73}$$

Using the special case of generalized Mehler formula (7.69) for $v = \frac{1}{\sqrt{2}}$ we get coordinate representation of Kampe-de Feriet coherent state

$$\langle x|K(\alpha,t)\rangle = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{\frac{\alpha^2}{2+t}} e^{-\frac{2+t}{2-t}\frac{m\omega}{2\hbar}\left(x - \sqrt{\frac{2\hbar}{m\omega}}\frac{2\alpha}{2+t}\right)^2}$$

where $|t| < 2$.

For $\alpha = \alpha_1 + i\alpha_2$, the Gaussian probability distribution function is

$$|\langle x|K(\alpha,t)\rangle|^2 = \left(\frac{m\omega}{\pi\hbar}\right)^{1/2} e^{2\left(\frac{\alpha_1^2-\alpha_2^2}{2+t}+\frac{4\alpha_2^2}{4-t^2}\right)} e^{-\frac{2+t}{2-t}\frac{m\omega}{\hbar}(x-\sqrt{\frac{2\hbar}{m\omega}}\frac{2}{2+t}\alpha_1)^2}.$$

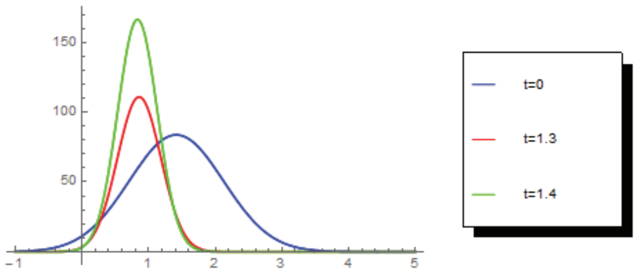

Figure 7.2. Gaussian probability distribution of Kampe-de Feriet coherent state

Normalized Kampe-De Feriet Coherent States

For $\nu = \frac{1}{2}$, $x = \bar{\alpha}$ and $y = \beta$, the generalized Mehler formula allows us to get the inner product of two different Kampe-de Feriet coherent states

$$\langle K(\alpha,t)|K(\beta,\tau)\rangle = \sum_{n=0}^{\infty} \frac{K_n(\bar{\alpha},t)K_n(\beta,\tau)}{2^{2n}n!} = \frac{1}{\sqrt{1-t\tau\frac{1}{4}}} e^{\frac{-\frac{1}{4}(\tau\bar{\alpha}^2+t\beta^2)+\bar{\alpha}\beta}{1-t\tau\frac{1}{4}}}. \tag{7.74}$$

If $\tau = t$ and $\beta = \alpha$, the inner product is written in the form

$$\langle K(\alpha,t)|K(\alpha,t)\rangle = \sum_{n=0}^{\infty} \frac{K_n(\bar{\alpha},t)K_n(\alpha,t)}{2^{2n}n!} = \frac{1}{\sqrt{1-\left(\frac{t}{2}\right)^2}} e^{\frac{-\frac{t}{4}(\bar{\alpha}^2+\alpha^2)+|\alpha|^2}{1-(\frac{1}{2})^2}}. \tag{7.75}$$

As a result, the normalized Kampe-de Feriet coherent state is defined in the following form

$$|K(\alpha,t)\rangle = C\sum_{n=0}^{\infty} \frac{K_n(\alpha,t)}{2^n\sqrt{n!}}|n\rangle, \tag{7.76}$$

where

$$C = \left(\frac{e^{\frac{-\frac{t}{4}(\alpha^2+\bar{\alpha}^2)+|\alpha|^2}{1-\frac{t^2}{4}}}}{\sqrt{1-\frac{t^2}{4}}} \right)^{-1/2}. \tag{7.77}$$

Fock-Bargmann Representation of Kampe-De Feriet Coherent State

The Fock-Bargmann representation of Kampe-de Feriet coherent state $|K(\alpha, t)\rangle$ is

$$\langle z|K(\alpha, t)\rangle = e^{-\frac{|z|^2}{2}} C \sum_{n=0}^{\infty} \frac{\bar{z}^n K_n(\alpha, t)}{n!} = Ce^{\bar{z}^2 t + \bar{z}\alpha} = CG(\bar{z}, \alpha; t) \tag{7.78}$$

where the corresponding analytic Fock Bargmann representation of Kampe de Feriet Coherent state

$$G(z, \alpha; t) = e^{z^2 t + z\alpha}$$

depends on two complex variables z, α.

This function satisfies the holomorphic heat equation is found in the following form

$$\frac{\partial}{\partial t} \langle z|K(\alpha, t)\rangle = \frac{\partial^2}{\partial \alpha^2} \langle z|K(\alpha, t)\rangle \tag{7.79}$$

as the plane wave solution in the form

$$G(z, \alpha; t) = e^{z^2 t + z\alpha}.$$

7.5. Bernoulli Coherent State

The generating function of Bernoulli polynomials is

$$\frac{te^{tx}}{e^t - 1} = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!}. \tag{7.80}$$

By using the relation for Bernoulli polynomials

$$B_{n+1}(x+1) - B_n(x) = nx^{n-1}, \tag{7.81}$$

we find the following representation

$$B_n(x) = \left(e^{\frac{d}{dx}} - 1\right)^{-1} \frac{d}{dx} x^n. \tag{7.82}$$

Then, by application of Bernoulli operator $\left(e^{\frac{d}{d\alpha}} - 1\right)^{-1} \frac{d}{d\alpha}$ to coherent state $|\alpha\rangle$ we introduce Bernoulli coherent states

$$\left(e^{\frac{d}{d\alpha}} - 1\right)^{-1} \frac{d}{d\alpha} |\alpha\rangle = \sum_{n=0}^{\infty} \frac{1}{\sqrt{n!}} \underbrace{\left(e^{\frac{d}{d\alpha}} - 1\right)^{-1} \frac{d}{d\alpha} \alpha^n}_{B_n(\alpha)} |n\rangle = \sum_{n=0}^{\infty} \frac{B_n(\alpha)}{\sqrt{n!}} |n\rangle \equiv |B(\alpha)\rangle . \tag{7.83}$$

In order to normalize the state, we use the normalization condition $\langle B(\alpha)|B(\alpha)\rangle = 1$ which gives $C = \frac{1}{\sqrt{\sum_{n=0}^{\infty} \frac{B_n(\bar{\alpha})B_n(\alpha)}{n!}}}$. Finally the normalized Bernoulli coherent state is found as

$$|B(\alpha)\rangle = \frac{1}{\sqrt{\sum_{n=0}^{\infty} \frac{B_n(\bar{\alpha})B_n(\alpha)}{n!}}} \sum_{n=0}^{\infty} \frac{B_n(\alpha)}{\sqrt{n!}} |n\rangle \tag{7.84}$$

Fock-Bargmann Representation of Bernoulli Coherent State The Fock-Bargmann representation of Bernoulli coherent state is

$$\begin{aligned} \langle z|B(\alpha)\rangle &= \frac{e^{-\frac{|z|^2}{2}}}{\sqrt{\sum_{n=0}^{\infty} \frac{B_n(\bar{\alpha})B_n(\alpha)}{n!}}} \sum_{m=0}^{\infty} \frac{\bar{z}^m}{\sqrt{m!}} \langle m| \sum_{n=0}^{\infty} \frac{B_n(\alpha)}{\sqrt{n!}} |n\rangle = \frac{e^{-\frac{|z|^2}{2}}}{\sqrt{\sum_{n=0}^{\infty} \frac{B_n(\bar{\alpha})B_n(\alpha)}{n!}}} \sum_{n=0}^{\infty} \frac{\bar{z}^n B_n(\alpha)}{n!} \\ &= \frac{e^{-\frac{|z|^2}{2}}}{\sqrt{\sum_{n=0}^{\infty} \frac{B_n(\bar{\alpha})B_n(\alpha)}{n!}}} \frac{\bar{z}e^{\alpha\bar{z}}}{e^{\bar{z}} - 1} = \frac{e^{-\frac{|z|^2}{2}}}{\sqrt{\sum_{n=0}^{\infty} \frac{B_n(\bar{\alpha})B_n(\alpha)}{n!}}} G(\bar{z}, \alpha), \end{aligned} \tag{7.85}$$

where the corresponding analytic function in α, z is found as

$$G(\alpha, z) = \frac{ze^{\alpha z}}{e^z - 1},$$

which is generating function of analytic Bernoulli polynomials as function of two complex variables α, z.

7.6. q-Coherent States

Definition 7.8 *q-Holomorphic heat equation is defined as*

$$\left(D_q^t + \frac{1}{[2]_q^2}(D_q^\alpha)^2\right)\phi(\alpha, t) = 0, \tag{7.86}$$

with q-evolution operator (Nalci and Pashaev, 2010)

$$U(t; q) = e_q\left(-\frac{1}{[2]_q^2}t(D_q^\alpha)^2\right). \tag{7.87}$$

Proposition 7.6 *Applying evolution operator $U(1; q)$ to monomial with q-numbers coefficients $([2]_q\alpha)^n$ we get the q-Hermite polynomials $H_n(\alpha; q)$: (Nalci and Pashaev, 2010)*

$$H_n(\alpha; q) = e_q\left(-\frac{1}{[2]_q^2}(D_q^\alpha)^2\right)([2]_q\alpha)^n. \tag{7.88}$$

Proposition 7.7 *Applying evolution operator $U(t; q)$ to monomial with q-numbers coefficients $([2]_q\alpha)^n$ we get the q-Kampe-de Feriet polynomials*

$$K_n(\alpha, t; q) = e_q\left(-\frac{1}{[2]_q^2}t(D_q^\alpha)^2\right)([2]_q\alpha)^n. \tag{7.89}$$

Definition 7.9 *Analytic q-coherent state (Arik and Coon, 1976), (Vitiello, 2012), (Vitiello, 2009) and (Vitiello, 2008) is defined as*

$$|\alpha; q\rangle = e^{-\frac{|\alpha|^2}{2}} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{[n]_q!}} |n\rangle. \tag{7.90}$$

Definition 7.10 *Applying q-evolution operator to analytic q-coherent state we introduce q-Hermite coherent state, which is analytic in α, in the following form*

$$|H(\alpha;q)\rangle \equiv e_q\left(-\frac{1}{[2]_q^2}(D_q^\alpha)^2\right)|\alpha;q\rangle = C\sum_{n=0}^{\infty} e_q\left(-\frac{1}{[2]_q^2}(D_q^\alpha)^2\right)\frac{\alpha^n}{\sqrt{[n]_q!}}\,|n\rangle = C\sum_{n=0}^{\infty}\frac{H_n(\alpha;q)}{[2]_q^2\sqrt{[n]_q!}}\,|n\rangle\,,$$

where the normalization constant is

$$C = \frac{1}{\sqrt{\sum_{n=0}^{\infty}\frac{H_n(\bar{\alpha};q)H_n(\alpha;q)}{[2]_q^{2n}[n]_q!}}}.$$

The Fock-Bargmann representation of this state is $|H(\alpha;q)\rangle$

$$\langle z;q|H(\alpha;q)\rangle = \frac{e^{-\frac{|z|^2}{2}}}{\sqrt{\sum_{n=0}^{\infty}\frac{H_n(\bar{\alpha};q)H_n(\alpha;q)}{[2]_q^{2n}[n]_q!}}}e_q\left(-\frac{\bar{z}^2}{[2]_q^2}\right)e_q(\bar{z}\alpha),$$

where the corresponding analytic Fock-Bargmann representation is found as generating function of q-Hermite polynomials with two complex variables z,α

$$G(z,\alpha;q) = e_q\left(-\frac{z^2}{[2]_q^2}\right)e_q(z\alpha).$$

Definition 7.11 *Action of q-evolution operator $U(t;q)$ to analytic q-coherent state $|\alpha;q\rangle$ produces q-Kampe-de Feriet coherent state*

$$e_q\left(-\frac{1}{[2]_q^2}t(D_q^\alpha)^2\right)|\alpha;q\rangle = \sum_{n=0}^{\infty} e_q(t(D_q^\alpha)^2)\frac{\alpha^n}{\sqrt{[n]_q!}}\,|n\rangle = \sum_{n=0}^{\infty}\frac{K_n(\alpha,t;q)}{\sqrt{[n]_q!}}\,|n\rangle \equiv |K(\alpha,t;q)\rangle\,. \quad (7.91)$$

The Fock-Bargmann representation of this state is

$$\langle z\,;q|K(\alpha,t\,;q)\rangle = C\sum_{m=0}^{\infty}\frac{\bar{z}^m}{\sqrt{[m]_q!}}\,\langle m|\sum_{n=0}^{\infty}\frac{K_n(\alpha,t\,;q)}{\sqrt{[n]_q!}}\,|n\rangle = C\sum_{n=0}^{\infty}\frac{\bar{z}^n K_n(\alpha,t;q)}{[n]_q!}\,|n\rangle = Ce_q(\bar{z}^2 t)e_q(\bar{z}\alpha).$$

The corresponding Fock-Bargmann representation is written

$$G(z,\alpha,t) = e_q(z^2 t)e_q(z\alpha) \tag{7.92}$$

as generating function of q-Kampe de Feriet polynomials with two complex variables z,α.

And the normalization constant is calculated

$$C = \frac{e^{-\frac{|z|^2}{2}}}{\sqrt{\sum_{n=0}^{\infty} \frac{K_n(\bar{\alpha},t;q)K_n(\alpha,t\,;q)}{[n]!}}}. \tag{7.93}$$

As an application we write q-Heat equation with q-Kampe de Feriet q-coherent states solution

$$D_q^t \left\langle z\,; q|K(\alpha,t\,;q)\right\rangle = (D_q^{\alpha})^2 \left\langle z\,; q|K(\alpha,t\,;q)\right\rangle. \tag{7.94}$$

7.6.1. q-Translation Operators and q-Coherent States

Definition 7.12 *Action of the translation operator $e^{i\beta\frac{d}{d\alpha}}$ of complex α and β on complex monomial α^n produces the double analytic binomial*

$$e^{i\beta\frac{d}{d\alpha}}\alpha^n = (\alpha + i\beta)^n, \tag{7.95}$$

where $(\alpha + i\beta)^n$ is analytic in both α,β and it is a double analytic function

$$\frac{1}{2}\left(\frac{\partial}{\partial\alpha} + i\frac{\partial}{\partial\beta}\right)(\alpha + i\beta)^n = 0.$$

Definition 7.13 *Applying the q-commutative translation operator $e_q^{i\beta D_{\frac{1}{q}}^{\alpha}}$ to complex monomial α^n gives q-commutative binomial*

$$e_q^{i\beta D_{\frac{1}{q}}^{\alpha}}\alpha^n = (\alpha + i\beta)^n, \tag{7.96}$$

where q-commutative binomial is defined as

$$(\alpha + i\beta)^n = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q \alpha^{n-k} (i\beta)^k, \tag{7.97}$$

which is q-commutative ($\beta\alpha = q\alpha\beta$) double analytic binomial.

Example 1: Action of this operator to analytic coherent state $|\alpha\rangle$ produces double analytic coherent state with q-commutativity of α and β :

$$e_q^{i\beta D^{\alpha}_{\frac{1}{q}}} |\alpha\rangle = \sum_{n=0}^{\infty} \frac{(\alpha + i\beta)^n}{\sqrt{n!}} |n\rangle = |\alpha + i\beta\rangle\,. \tag{7.98}$$

Example 2: Action of this operator to analytic q-coherent state $|\alpha; q\rangle$ produces double analytic q-coherent state with q-commutative α and β :

$$e_q^{i\beta D^{\alpha}_{\frac{1}{q}}} |\alpha; q\rangle = \sum_{n=0}^{\infty} \frac{(\alpha + i\beta)^n}{\sqrt{[n]_q!}} |n\rangle = |\alpha + i\beta; q\rangle\,. \tag{7.99}$$

Definition 7.14 *The second kind q-translation operator (6.7) is defined as*

$$e_{\frac{1}{q}}^{i\beta D^{q}_{\alpha}}. \tag{7.100}$$

Action of this operator on monomial α^n gives

$$e_{\frac{1}{q}}^{i\beta D^{\alpha}_{q}} \alpha^n = (\alpha + i\beta)^n_q, \tag{7.101}$$

where $(\alpha + i\beta)^n_q$ is double q-analytic binomial

$$\frac{1}{2}\left(D^{\alpha}_q + iD^{\beta}_{\frac{1}{q}}\right)(\alpha + i\beta)^n_q = 0.$$

Proposition 7.8 *Application of q-translation operator to analytic coherent state $|\alpha\rangle$ produces* *double q-analytic coherent state*

$$e_{\frac{1}{q}}^{i\beta D_\alpha^q}|\alpha\rangle = \sum_{n=0}^{\infty} e_{\frac{1}{q}}^{i\beta D_\alpha^q}\frac{\alpha^n}{\sqrt{n!}}|n\rangle = \sum_{n=0}^{\infty}\frac{(\alpha+i\beta)_q^n}{\sqrt{n!}}|n\rangle\,. \tag{7.102}$$

This double q-analytic coherent state in Fock-Bargmann representation is written as follows

$$\left\langle z|(\alpha+i\beta)_q\right\rangle = \sum_{n=0}^{\infty}\frac{\bar{z}^n(\alpha+i\beta)_q^n}{n!}, \tag{7.103}$$

and corresponding q-analytic Fock-Bargmann representation is constructed as

$$\left\langle \bar{z}|(\alpha+i\beta)_q\right\rangle = \sum_{n=0}^{\infty}\frac{(z\alpha+iz\beta)_q^n}{n!} = e(z\alpha+iz\beta)_q. \tag{7.104}$$

Proposition 7.9 *Application of q-translation operator to analytic q-coherent state $|\alpha;q\rangle$ produces double q-analytic coherent state*

$$e_{\frac{1}{q}}^{i\beta D_\alpha^q}|\alpha;q\rangle = \sum_{n=0}^{\infty} e_{\frac{1}{q}}^{i\beta D_\alpha^q}\frac{\alpha^n}{\sqrt{[n]_q!}}|n\rangle = \sum_{n=0}^{\infty}\frac{(\alpha+i\beta)_q^n}{\sqrt{[n]_q!}}|n\rangle \equiv \left|(\alpha+i\beta)_q;q\right\rangle. \tag{7.105}$$

This double q-analytic coherent state $|\alpha;q\rangle$ in Fock-Bargmann representation is written as follows

$$\left\langle z\,;q|(\alpha+i\beta)_q\,;q\right\rangle = \sum_{n=0}^{\infty}\frac{\bar{z}^n(\alpha+i\beta)_q^n}{[n]_q!} = e_q(\bar{z}\alpha+i\bar{z}\beta)_q, \tag{7.106}$$

and corresponding double q-analytic q-coherent state in q-Fock Bargmann representation is constructed as

$$\left\langle \bar{z}\,;q|(\alpha+i\beta)_q\,;q\right\rangle = \sum_{n=0}^{\infty}\frac{(z\alpha+iz\beta)_q^n}{[n]!} = e_q(z\alpha+iz\beta)_q = e_q(z\alpha)e_{\frac{1}{q}}(iz\beta). \tag{7.107}$$

As a result, we obtain double q-analytic function from analytic q-coherent state.

CHAPTER 8

GOLDEN QUANTUM CALCULUS

The Binet-Fibonacci formula for Fibonacci numbers is treated as a q-number (and q-operator) with Golden ratio bases $q = \varphi$ and $Q = -1/\varphi$, and the corresponding Fibonacci or Golden quantum calculus is introduced. Quantum harmonic oscillator for this Golden calculus is derived so that its spectrum is given just by Fibonacci numbers. The ratio of successive energy levels is found as the Golden sequence and for asymptotic states in the limit $n \to \infty$ it appears as the Golden ratio. That is why we call this oscillator as the Golden oscillator. By double Golden bosons, the Golden angular momentum and its representation in terms of Fibonacci numbers and the Golden ratio are derived.

8.1. Golden q-Calculus

In (Q, q) calculus we have the number

$$[n]_{Q,q} = \frac{Q^n - q^n}{Q - q}. \tag{8.1}$$

If we choose $Q = \varphi = \frac{1+\sqrt{5}}{2}$ and $q = \varphi' = \frac{1-\sqrt{5}}{2} = -\frac{1}{\varphi}$. Then (8.1) becomes Binet's formula for Fibonacci numbers as (φ, φ')-numbers :

$$F_n = \frac{\varphi^n - \varphi'^n}{\varphi - \varphi'} = [n]_{\varphi,\varphi'} \equiv [n]_F. \tag{8.2}$$

This definition can be extended to arbitrary real number x,

$$F_x \equiv [x]_{\varphi,\varphi'} \equiv [x]_F = \frac{\varphi^x - \varphi'^x}{\varphi - \varphi'} = \frac{\varphi^x - (-\frac{1}{\varphi})^x}{\varphi + \frac{1}{\varphi}}, \tag{8.3}$$

though due to negative sign for the second base, it is not a real number for general x

$$F_x = \frac{1}{\varphi + \frac{1}{\varphi}} \left(\varphi^x - e^{i\pi x} \frac{1}{\varphi^x} \right) = \frac{1}{\sqrt{5}} \left(\varphi^x - e^{i\pi x} \frac{1}{\varphi^x} \right). \tag{8.4}$$

Instead of real number x we can also consider complex numbers $z = x + iy$,

Example : It is easy to see that

$$\lim_{n\to\infty} \frac{[n+1]_F}{[n]_F} = \lim_{n\to\infty} \frac{F_{n+1}}{F_n} = \varphi. \tag{8.5}$$

The addition formula for Golden numbers is given in the form

$$[n+m]_F = F_{n+m} = \varphi^n F_m + \left(-\frac{1}{\varphi}\right)^m F_n. \tag{8.6}$$

By using (8.2) we can get

$$\varphi^N = \varphi F_N + F_{N-1}, \qquad \varphi'^N = \varphi' F_N + F_{N-1}, \tag{8.7}$$

and the above formula (8.6) can be rewritten as

$$\begin{aligned} F_{n+m} &= F_n F_{m-1} + F_{n+1} F_m \\ &= F_{n-1} F_m + F_n F_{m+1}. \end{aligned} \tag{8.8}$$

The substraction formula can be obtained from it by changing m by $-m$ as

$$F_{n-m} = [n-m]_F = \varphi^n [-m]_F + \left(-\frac{1}{\varphi}\right)^{-m} [n]_F \tag{8.9}$$

or by using the equality

$$[-n]_F = -(-1)^{-n} [n]_F \tag{8.10}$$

it can be also written

$$\begin{aligned}[n-m]_F &= \left(-\frac{1}{\varphi}\right)^{-m} ([n]_F - \varphi^{n-m}[m]_F) \\ &= \left(-\frac{1}{\varphi}\right)^{-m} F_n - \frac{\varphi^n}{(-1)^m} F_m. \end{aligned} \tag{8.11}$$

or

$$F_{n-m} = \left(-\frac{1}{\varphi}\right)^{-m} F_n - \frac{\varphi^n}{(-1)^m} F_m. \tag{8.12}$$

Definition 8.1 *Higher Fibonacci Numbers are*

$$F_n^{(m)} \equiv \frac{(\varphi^m)^n - (\varphi'^m)^n}{\varphi^m - \varphi'^m} = [n]_{\varphi^m,\varphi'^m} \tag{8.13}$$

and $F_n^{(1)} \equiv F_n$.

By definition, the multiplication rule for Golden numbers is given by next formula

$$[nm]_{\varphi,-\frac{1}{\varphi}} = F_{nm} = [n]_{\varphi,-\frac{1}{\varphi}} [m]_{\varphi^n,(-\frac{1}{\varphi})^n} = F_n F_m^{(n)}, \tag{8.14}$$

and the division rule is

$$\begin{aligned}\left[\frac{m}{n}\right]_{\varphi,\varphi'} &= \frac{[m]_{\varphi,\varphi'}}{[n]_{\varphi^{m/n},\varphi'^{m/n}}} = \frac{[m]_{\varphi^{1/n},\varphi'^{1/n}}}{[n]_{\varphi^{1/n},\varphi'^{1/n}}} \\ F_{\frac{m}{n}} &= \frac{F_m}{F_n^{(\frac{m}{n})}} = \frac{F_m^{(\frac{1}{n})}}{F_n^{(\frac{1}{n})}}. \end{aligned} \tag{8.15}$$

Higher Fibonacci numbers can be written in terms of ratio of Fibonacci numbers as follows

$$F_n^{(m)} = \frac{F_{mn}}{F_m}. \tag{8.16}$$

From definition (8.2) we have the following relation

$$F_{-n} = (-1)^{n+1} F_n. \tag{8.17}$$

For any real x, y

$$\begin{aligned} [x+y]_F &= \varphi^x [y]_F + \left(-\frac{1}{\varphi}\right)^y [x]_F \\ &= \varphi^y [x]_F + \left(-\frac{1}{\varphi}\right)^x [y]_F \end{aligned} \tag{8.18}$$

which are written in terms of Fibonacci numbers as follows

$$\begin{aligned} F_{x+y} &= \varphi^x F_y + (-\frac{1}{\varphi})^y F_x \\ &= \varphi^y F_x + \left(-\frac{1}{\varphi}\right)^x F_y. \end{aligned} \tag{8.19}$$

For real x, we have the Fibonacci recurrence relation

$$[x]_F = [x-1]_F + [x-2]_F \Rightarrow F_x = F_{x-1} + F_{x-2}. \tag{8.20}$$

Example : Golden π is

$$F_\pi = [\pi]_F \simeq= 4,73068 + 0,0939706i.$$

8.1.1. Fibonacci and Golden Derivative

We define the Fibonacci derivative operator

$$F_{x\frac{d}{dx}} = \frac{q^{x\frac{d}{dx}} - (-\frac{1}{q})^{x\frac{d}{dx}}}{q + q^{-1}} = [x\frac{d}{dx}]_F. \tag{8.21}$$

and the Golden derivative operator as

$$F_{x\frac{d}{dx}} = \frac{\varphi^{x\frac{d}{dx}} - \varphi'^{x\frac{d}{dx}}}{\varphi - \varphi'} = [x\frac{d}{dx}]_F. \tag{8.22}$$

Then the Golden derivative of arbitrary function $f(x)$ is given by

$$F_{x\frac{d}{dx}} f(x) = D_F f(x) = \frac{f(\varphi x) - f(-\frac{x}{\varphi})}{(\varphi + \frac{1}{\varphi})x} = \frac{\left(M_\varphi - M_{-\frac{1}{\varphi}}\right) f(x)}{(\varphi + \frac{1}{\varphi}x)}. \tag{8.23}$$

Here, arguments are scaled by the Golden ratio: $x \to \varphi x$ and $x \to -\frac{x}{\varphi}$. This scaling can be written in terms of Golden dilatation operator

$$M_\varphi f(x) = f(\varphi x), \tag{8.24}$$

where $f(x)$- smooth function. Its operator form can also be written as

$$M_\varphi = \varphi^{x\frac{d}{dx}} = \left(\frac{1+\sqrt{5}}{2}\right)^{x\frac{d}{dx}}. \tag{8.25}$$

A function $A(x)$ is called Golden periodic function if

$$D_F A(x) = 0. \tag{8.26}$$

This implies

$$A(\varphi x) = A(-\frac{1}{\varphi}x). \tag{8.27}$$

As an example we have:

$$A(x) = \sin\left(\frac{\pi}{\ln\varphi} \ln|x|\right) \tag{8.28}$$

Example 1: Application of Golden derivative operator D_F on x^n gives

$$D_F x^n = F_n x^{n-1}$$

or

$$F_n = \frac{D_F x^n}{x^{n-1}},$$

so it generates Fibonacci numbers.

Example 2:

$$D_F e^x = \sum_{n=0}^{\infty} \frac{F_n}{n!} x^n$$

or

$$D_F e^x = \frac{e^{\varphi x} - e^{-\frac{x}{\varphi}}}{\varphi + \frac{1}{\varphi}} = \frac{2e^{\frac{x}{2}} \sinh \frac{\sqrt{5}}{2} x}{\sqrt{5} x} = \sum_{n=0}^{\infty} \frac{F_n}{n!} x^n.$$

For $x = 1$ it gives next summation formula

$$\sum_{n=0}^{\infty} \frac{F_n}{n!} = e^{\frac{1}{2}} \frac{\sinh \frac{\sqrt{5}}{2}}{\frac{\sqrt{5}}{2}}. \tag{8.29}$$

8.1.2. Golden Leibnitz Rule

We derive the Golden Leibnitz rule

$$D_F(f(x)g(x)) = D_F f(x) g(\varphi x) + f(-\frac{x}{\varphi}) D_F g(x). \tag{8.30}$$

By symmetry, the second form of the Leibnitz rule can be derived as

$$D_F(f(x)g(x)) = D_F f(x) g(-\frac{x}{\varphi}) + f(\varphi x) D_F g(x). \tag{8.31}$$

These formulas can be rewritten in explicitly symmetrical form :

$$D_F(f(x)g(x)) = D_F f(x)\left(\frac{g(\varphi x) + g(-\frac{x}{\varphi})}{2}\right) + D_F g(x)\left(\frac{f(\varphi x) + f(-\frac{x}{\varphi})}{2}\right). \tag{8.32}$$

More general form of Golden Leibnitz formula is given with arbitrary α ,

$$D_F(f(x)g(x)) = \left(\alpha f(-\frac{x}{\varphi})) + (1-\alpha)f(\varphi x)\right)D_F g(x) + \left(\alpha g(\varphi x) + (1-\alpha)g(-\frac{x}{\varphi}))\right)D_F f(x).$$

Now we may compute the golden derivative of the quotient of $f(x)$ and $g(x)$. From (8.30) we have

$$D_F\left(\frac{f(x)}{g(x)}\right) = \frac{D_F f(x)g(\varphi x) - D_F g(x)f(\varphi x)}{g(\varphi x)g(-\frac{x}{\varphi})}. \tag{8.33}$$

However, if we use (8.31), we get

$$D_F\left(\frac{f(x)}{g(x)}\right) = \frac{D_F f(x)g(-\frac{x}{\varphi}) - D_F g(x)f(-\frac{x}{\varphi})}{g(\varphi x)g(-\frac{x}{\varphi})}. \tag{8.34}$$

In addition to the formulas (8.33) and (8.34) one may determine one more representation in symmetrical form

$$D_F\left(\frac{f(x)}{g(x)}\right) = \frac{1}{2}\frac{D_F f(x)(g(-\frac{x}{\varphi}) + g(\varphi x)) - D_F g(x)(f(-\frac{x}{\varphi}) + f(\varphi x))}{g(\varphi x)g(-\frac{x}{\varphi})}. \tag{8.35}$$

In particular applications one of these forms could be more useful than others.

8.1.3. Golden Taylor Expansion

Theorem 8.1 *Let the Golden derivative operator D_F is a linear operator on the space of polynomials, and*

$$P_n(x) \equiv \frac{x^n}{F_n!} \equiv \frac{x^n}{F_1F_2...F_n}$$

satisfy the following conditions :

(i) $P_0(0) = 1$ *and* $P_n(0) = 0$ *for any* $n \geq 1$ *;*

(ii) deg $P_n = n$ *;*

(iii) $D_F P_n(x) = P_{n-1}(x)$ *for any* $n \geq 1$ *, and* $D_F(1) = 0$. *Then, for any polynomial* $f(x)$ *of degree N,one has the following Taylor formula :*

$$f(x) = \sum_{n=0}^{N}(D_F^n f)(0)P_n(x) = \sum_{n=0}^{N}(D_F^n f)(0)\frac{x^n}{F_n!}.$$

In the limit $N \to \infty$ (when it exists) this formula can determine some new function

$$f_F(x) = \sum_{n=0}^{\infty}(D_F^n f)(0)\frac{x^n}{F_n!} \tag{8.36}$$

which we can call the Golden (or Fibonacci) function.

Example : (Golden Exponential) The Golden exponential functions are

$$e_F^x \equiv \sum_{n=0}^{\infty}\frac{x^n}{F_n!} \;; \quad E_F^x \equiv \sum_{n=0}^{\infty}(-1)^{\frac{n(n-1)}{2}}\frac{x^n}{F_n!}, \tag{8.37}$$

and for $x = 1$, we get the Fibonacci natural base as follows

$$e_F^x \equiv \sum_{n=0}^{\infty}\frac{1}{F_n!} \equiv e_F.$$

These functions are entire analytic functions. For the second function explicitly we have

$$E_F^x = 1 + \frac{x}{F_1!} - \frac{x^2}{F_2!} - \frac{x^3}{F_3!} + \frac{x^4}{F_4!} + \frac{x^5}{F_5!} - \frac{x^6}{F_6!} - \frac{x^7}{F_7!} + \frac{x^8}{F_8!} + \frac{x^9}{F_9!} - ... \tag{8.38}$$

The Golden derivative of these exponential functions are found

$$D_F e_F^{kx} = k e_F^{kx},$$

$$D_F E_F^{kx} = kE_F^{-kx}$$

for arbitrary constant k (or F-periodic function). Then these two functions give the general solution of the hyperbolic F-oscillator

$$(D_F^2 - k^2)\phi(x) = 0, \tag{8.39}$$

as

$$\phi(x) = Ae_F^{kx} + Be_F^{-kx}, \tag{8.40}$$

and elliptic F-oscillator

$$(D_F^2 + k^2)\phi(x) = 0, \tag{8.41}$$

$$\phi(x) = AE_F^{kx} + BE_F^{-kx}. \tag{8.42}$$

We have next Euler formulas

$$e_F^{ix} = \cos_F x + i \sin_F x, \tag{8.43}$$

$$E_F^{ix} = \cosh_F x + i \sinh_F x, \tag{8.44}$$

and relations

$$Cosh_F x = \cos_F x, \tag{8.45}$$

$$Sinh_F x = \sin_F x, \tag{8.46}$$

where

$$Cosh_F x \equiv \frac{E_F^x + E_F^{-x}}{2}, \quad Sinh_F x \equiv \frac{E_F^x - E_F^{-x}}{2}. \tag{8.47}$$

We notice here that these relations are valid due to alternating character of second exponential function.

Example : (F-Oscillator)

For F-oscillator

$$D_F^2 x + \omega^2 x = 0 \tag{8.48}$$

the general solution is

$$x(t) = aE_F^{\omega t} + bE_F^{-\omega t} = a' Cosh_F \omega t + b' Sinh_F \omega t = a' \cos_F \omega t + b' \sin_F \omega t \tag{8.49}$$

8.1.4. Golden Binomial

Golden Binomial we define as

$$(x+y)_F^n = (x + \varphi^{n-1} y)(x - \varphi^{n-3} y)...(x + (-1)^{n-1} \varphi^{-n+1} y) \tag{8.50}$$

and it has n-zeros at the Golden ratio powers

$$\frac{x}{y} = -\varphi^{n-1}, \quad \frac{x}{y} = -\varphi^{n-3}, \quad ..., \frac{x}{y} = -\varphi^{-n+1}.$$

For Golden binomial next expansion is valid

$$
\begin{aligned}
(x+y)_F^n \equiv (x+y)_{\varphi,-\frac{1}{\varphi}}^n &= \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_F (-1)^{\frac{k(k-1)}{2}} x^{n-k} y^k \\
&= \sum_{k=0}^{n} \frac{F_n!}{F_{n-k}!F_k!} (-1)^{\frac{k(k-1)}{2}} x^{n-k} y^k
\end{aligned}
\tag{8.51}
$$

The proof is easy by induction.

Application of Golden derivative to the Golden binomial gives

$$D_F^x (x+y)_F^n = F_n (x+y)_F^{n-1},$$

$$D_F^y (x+y)_F^n = F_n (x-y)_F^{n-1}.$$

It means

$$D_F^x \frac{(x+y)_F^n}{F_n!} = \frac{(x+y)_F^{n-1}}{F_{n-1}!},$$

$$D_F^y \frac{(x+y)_F^n}{F_n!} = \frac{(x-y)_F^{n-1}}{F_{n-1}!},$$

$$\left(D_F^y\right)^n (x+y)_F^n.$$

For $n = 2k$ we have

$$\left(D_F^y\right)^{2k} (x+y)_F^{2k} = (-1)^k F_{2k}!,$$

and for $n = 2k + 1$ we get

$$\left(D_F^y\right)^{2k+1} (x + y)_F^{2k+1} = (-1)^k F_{2k+1}!.$$

In terms of Golden binomial we introduce the Golden polynomials

$$P_n(x) = \frac{(x - a)_F^n}{F_n!} \tag{8.52}$$

where $n = 1, 2, ...,$ and $P_0(x) = 1$ with property

$$D_F^x P_n(x) = P_{n-1}(x). \tag{8.53}$$

For even and odd polynomials we have products

$$P_{2n}(x) = \frac{1}{F_{2n}!} \prod_{k=1}^{n} (x - (-1)^{n+k} \varphi^{2k-1} a)(x + (-1)^{n+k} \varphi^{-2k+1} a), \tag{8.54}$$

$$P_{2n+1}(x) = \frac{(x - (-1)^n a)}{F_{2n+1}!} \prod_{k=1}^{n} (x - (-1)^{n+k} \varphi^{2k} a)(x - (-1)^{n+k} \varphi^{-2k} a). \tag{8.55}$$

By using (8.7) it is easy to find

$$\varphi^{2k} + \frac{1}{\varphi^{2k}} = F_{2k} + 2F_{2k-1}, \tag{8.56}$$

$$\varphi^{2k+1} - \frac{1}{\varphi^{2k+1}} = F_{2k+1} + 2F_{2k}. \tag{8.57}$$

Then we can rewrite our polynomials in terms of just Fibonacci numbers

$$P_{2n}(x) = \frac{1}{F_{2n}!}\prod_{k=1}^{n}(x^2 - (-1)^{n+k}(F_{2k-1} + 2F_{2k-2})xa - a^2), \tag{8.58}$$

$$P_{2n+1}(x) = \frac{(x - (-1)^n a)}{F_{2n+1}!}\prod_{k=1}^{n}(x^2 - (-1)^{n+k}(F_{2k} + 2F_{2k-1})xa + a^2). \tag{8.59}$$

First few polynomials are

$$P_1(x) = (x - a) \tag{8.60}$$

$$P_3(x) = \frac{1}{2}(x + a)(x^2 - 3xa + a^2) \tag{8.61}$$

$$P_5(x) = \frac{1}{2 \cdot 3 \cdot 5}(x - a)(x^2 + 3xa + a^2)(x^2 - 7xa + a^2) \tag{8.62}$$

$$P_7(x) = \frac{1}{2 \cdot 3 \cdot 5 \cdot 8 \cdot 13}(x + a)(x^2 - 3xa + a^2)(x^2 + 7xa + a^2)(x^2 - 18xa + a^2) \tag{8.63}$$

$$... \tag{8.64}$$

$$P_2(x) = (x^2 - xa - a^2) \tag{8.65}$$

$$P_4(x) = \frac{1}{2\cdot 3}(x^2 + xa - a^2)(x^2 - 4xa - a^2) \tag{8.66}$$

$$P_6(x) = \frac{1}{2\cdot 3\cdot 5\cdot 8}(x^2 - xa - a^2)(x^2 + 4xa - a^2)(x^2 - 11xa - a^2) \tag{8.67}$$

$$... \tag{8.68}$$

8.1.5. Noncommutative Golden Ratio and Golden Binomials

By choosing $q = -\frac{1}{\varphi}$ and $Q = \varphi$, in general Q-commutative q-binomial (Nalci Tumer and Pashaev , in preparation), where φ is the Golden section, we obtain the Binet-Fibonacci Binomial formula with Golden non-commutative plane ($yx = \varphi xy$). (It should be compared with Golden ratio $b = \varphi a$).

$$\begin{aligned}(x+y)^n_{-\frac{1}{\varphi}} &= (x+y)(x+(-\frac{1}{\varphi})y)(x+(-\frac{1}{\varphi})^2 y)...(x+(-\frac{1}{\varphi})^{n-1}y)\\ &= \sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_{\varphi,-\frac{1}{\varphi}}(-\frac{1}{\varphi})^{\frac{k(k-1)}{2}}x^{n-k}y^k\\ &= \sum_{k=0}^{n}\frac{F_n!}{F_k!F_{n-k}!}\left(-\frac{1}{\varphi}\right)^{\frac{k(k-1)}{2}}x^{n-k}y^k,\end{aligned} \tag{8.69}$$

where F_n are Fibonacci numbers.

8.1.6. Golden Pascal Triangle

The Golden binomial coefficients are defined by

$$\begin{bmatrix}n\\k\end{bmatrix}_F = \frac{[n]_F!}{[n-k]_F![k]_F!} = \frac{F_n!}{F_{n-k}!F_k!} \tag{8.70}$$

with n and k being nonnegative integers, $n \geq k$ and are called the Fibonomials. Using the addition formula for Golden numbers (8.6), we write following expression

$$F_n = F_{n-k+k} = (-\frac{1}{\varphi})^k F_{n-k} + \varphi^{n-k} F_k,$$

and from (8.7) it can be written as follows

$$\begin{aligned} F_n &= F_{n-k-1}F_k + F_{n-k}F_{k+1} \\ &= F_{n-k}F_{k-1} + F_{n-k+1}F_k. \end{aligned} \tag{8.71}$$

With the above definition (8.70)we have next recursion formulas

$$\begin{aligned} \begin{bmatrix} n \\ k \end{bmatrix}_F &= \frac{(-\frac{1}{\varphi})^k [n-1]_F!}{[k]_F![n-k-1]_F!} + \frac{\varphi^{n-k}[n-1]_F!}{[n-k]_F![k-1]_F!} \\ &= (-\frac{1}{\varphi})^k \begin{bmatrix} n-1 \\ k \end{bmatrix}_F + \varphi^{n-k} \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_F \qquad (8.72) \\ &= \varphi^k \begin{bmatrix} n-1 \\ k \end{bmatrix}_F + (-\frac{1}{\varphi})^{n-k} \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_F. \qquad (8.73) \end{aligned}$$

These two rules determine the multiple Golden Pascal triangle, where $1 \leq k \leq n-1$. Then, we can construct Golden Pascal triangle as follows

$$\begin{array}{ccccccccccccc} & & & & & & 1 & & & & & & \\ & & & & & \swarrow & & \searrow & & & & & \\ & & & & 1 & & & & 1 & & & & \\ & & & \swarrow & & \searrow -\frac{1}{\varphi} & & \varphi \swarrow & & \searrow & & & \\ & & 1 & & & & [2]_F & & & & 1 & & \\ & \swarrow & & \searrow (-\frac{1}{\varphi})^2 & & \varphi \swarrow & & \searrow -\frac{1}{\varphi} & & \varphi^2 \swarrow & & \searrow & \\ & & \ldots & & & & \ldots & & & & \ldots & & \end{array}$$

Figure 8.1. Golden Pascal triangle

8.1.7. Remarkable Limit

From Golden binomial expansion (8.51) we have

$$\begin{aligned}(1+y)_F^n &= \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_F (-1)^{\frac{k(k-1)}{2}} y^k \\ &= \sum_{k=0}^{n} \frac{F_n!}{F_{n-k}!F_k!} (-1)^{\frac{k(k-1)}{2}} y^k. \end{aligned} \tag{8.74}$$

Then

$$\left(1+\frac{y}{\varphi^n}\right)_F^n = \sum_{k=0}^{n} \frac{F_n!}{F_{n-k}!F_k!} (-1)^{\frac{k(k-1)}{2}} \frac{y^k}{\varphi^{nk}} \tag{8.75}$$

or by opening Fibonomials and taking limit

$$\lim_{n\to\infty}\left(1+\frac{y}{\varphi^n}\right)_F^n = \sum_{k=0}^{\infty} \frac{1}{F_k!} (-1)^{\frac{k(k-1)}{2}} \frac{y^k}{\varphi^{\frac{k(k-1)}{2}}(\varphi+\frac{1}{\varphi})^k} \tag{8.76}$$

$$\lim_{n\to\infty}\left(1+\frac{y}{\varphi^n}\right)_F^n = \sum_{k=0}^{\infty} \frac{1}{[k]_{-\varphi^2}!} \left(\frac{y\varphi}{\varphi^2+1}\right)^k \tag{8.77}$$

where we introduced q-number, $[k]_q = 1 + q + ... + q^{k-1}$, with base $q = -\varphi^2$, so that

$$[k]_{-\varphi^2} = 1 + (-\varphi^2) + ... + (-\varphi^2)^{k-1} = \frac{(-\varphi^2)^k - 1}{(-\varphi^2) - 1}. \tag{8.78}$$

The last expression allow us to rewrite the limit in terms of Jackson q-exponential function $e_q(x)$ with $q = -\varphi^2$,

$$\lim_{n\to\infty}\left(1+\frac{y}{\varphi^n}\right)_F^n = e_{-\varphi^2}\left(\frac{y\varphi}{\varphi^2+1}\right) \tag{8.79}$$

or finally we have remarkable limit

$$\lim_{n\to\infty}\left(1+\frac{y}{\varphi^n}\right)_F^n = e_{-\varphi^2}\left(\frac{y}{\sqrt{5}}\right). \tag{8.80}$$

In particular case it gives

$$\lim_{n\to\infty}\left(1+\frac{\sqrt{5}}{\varphi^n}\right)_F^n = e_{-\varphi^2}(1). \tag{8.81}$$

8.1.8. Golden Integral

8.1.8.1. Golden Antiderivative

Definition 8.2 *The function $G(x)$ is Golden antiderivative of $g(x)$ if $D_F G(x) = g(x)$.*
It is denoted by

$$G(x) = \int g(x) d_F x. \tag{8.82}$$

$$D_F G(x) = 0 \Rightarrow G(x) = C - constant$$

or

$$D_F G(x) = 0 \Rightarrow G(\varphi x) = G(-\frac{x}{\varphi})$$

is called the Golden 'periodic' function.

8.1.8.2. Golden-Jackson Integral

By inverting equation (8.23) and expanding inverse operator we find Jackson type representation for anti-derivative.

Definition 8.3 *We introduce Jackson type anti-derivative as*

$$G(x) = \int g\left(\frac{x}{\varphi}\right) d_Q x = (1 - Q)x \sum_{k=0}^{\infty} Q^k f\left(\frac{x}{\varphi} Q^k\right) \tag{8.83}$$

where $Q \equiv -\frac{1}{\varphi^2}$.

8.2. Golden Quantum Oscillator

Now we construct quantum oscillator with spectrum in the form of Fibonacci numbers. Since in this oscillator the base in commutation relations is φ-Golden ratio, we called it as Golden oscillator. The algebraic relations for Golden Oscillator are

$$bb^+ - \varphi b^+ b = (-\frac{1}{\varphi})^N \tag{8.84}$$

or

$$bb^+ + \frac{1}{\varphi} b^+ b = \varphi^N, \tag{8.85}$$

where N is the hermitian number operator and φ is the deformation parameter. The bosonic Golden-oscillator is defined by three operators b^+, b and N which satisfy the commutation relations:

$$[N, b^+] = b^+, \qquad [N, b] = -b. \tag{8.86}$$

By using the definition of number operator with basis φ we find following equalities

$$[N+1]_F - \varphi[N]_F = (-\frac{1}{\varphi})^N \tag{8.87}$$

$$[N+1]_F + \frac{1}{\varphi}[N]_F = \varphi^N, \tag{8.88}$$

where

$$[N]_F = \frac{\varphi^N - (-\frac{1}{\varphi})^N}{\varphi + \frac{1}{\varphi}}$$

is the Fibonacci number operator. Here operator $(-1)^N = e^{i\pi N}$.

By comparison the above operator relations with algebraic relations (8.84) and (8.85) we have

$$b^+b = [N]_F, \qquad bb^+ = [N+1]_F.$$

Here we should note that the number operator N is not equal to b^+b as in ordinary case. By using the property of Fibonacci numbers (8.7) the algebraic relations (8.84) and (8.85) are equivalent to Fibonacci rule for operators

$$F_{N+1} = F_N + F_{N-1}.$$

Proposition 8.1 *We have following commutator relation*

$$\begin{aligned}[][N]_F, b^+] &= \{[N]_F - [N-1]_F\}b^+ \\ &= b^+\{[N+1]_F - [N]_F\} \end{aligned} \tag{8.89}$$

Proposition 8.2 *We have following equality for* $n = 0, 1, 2, ..$

$$[[N]_F^n, b^+] = \{[N]_F^n - [N-1]_F^n\}b^+ \tag{8.90}$$

Proof By using mathematical induction to show the above equality is not difficult.

□

Corollary 8.1 *For any function expandable to power series (analytic)* $F(x) = \sum_{n=0}^{\infty} c_n x^n$ *we have the following relation*

$$\begin{aligned}[F([N]_F), b^+] &= \{F([N]_F) - F([N-1]_F)\}b^+ \\ &= b^+\{F([N+1]_F) - F([N]_F)\}\end{aligned} \tag{8.91}$$

and

$$b^+ F([N+1]_F) = F([N]_F)b^+ \tag{8.92}$$

or

$$F(N)b^+ = b^+ F(N+1). \tag{8.93}$$

By using the eigenvalues of the Number operator

$$N|n\rangle_F = n|n\rangle_F ,$$

$$[N]_F |n\rangle_F = F_N |n\rangle_F = [n]_F |n\rangle_F = F_n |n\rangle_F$$

we get Fibonacci numbers as eigenvalues of $[N]$-operator, where we call F_N as Fibonacci operator and we denote $|n\rangle_{\varphi,-\frac{1}{\varphi}} \equiv |n\rangle_F$.

The basis of the Fock space is defined by repeated action of the creation operator b^+ on the vacuum state, which is annihilated by $b\,|0\rangle_F = 0$

$$|n\rangle_F = \frac{(b^+)^n}{\sqrt{F_1 \cdot F_2 \cdot ...F_n}}\,|0\rangle_F\,, \tag{8.94}$$

where $[n]_F! = F_1 \cdot F_2 \cdot ...F_n$.

In the limit

$$\lim_{n\to\infty} \frac{F(n+1)}{F(n)} = \lim_{n\to\infty} \frac{[n+1]_F}{[n]_F} = \frac{1+\sqrt{5}}{2} \equiv \varphi =\approx 1,6180339887,$$

which is the Golden ratio.

The number operator N for Fibonacci case is written in two different forms according to even or odd eigenstates $N\,|n\rangle_F = n\,|n\rangle_F$. For $n = 2k$, we get

$$N = \log_\varphi\left(\frac{\sqrt{5}}{2}F_N + \sqrt{\frac{5}{4}F_N^2 + 1}\right), \tag{8.95}$$

and for $n = 2k + 1$,

$$N = \log_\varphi\left(\frac{\sqrt{5}}{2}F_N - \sqrt{\frac{5}{4}F_N^2 - 1}\right), \tag{8.96}$$

where $[N]_F$ is Fibonacci number operator defined as

$$[N]_F = \frac{\varphi^N - (-\frac{1}{\varphi})}{\varphi - (-\frac{1}{\varphi})} = F_N.$$

As a result, the Fibonacci numbers are the example of (q, Q) numbers with two basis and one of the base is Golden Ratio.This is why we called the corresponding q- oscillator as a Golden oscillator or Binet-Fibonacci Oscillator. The Hamiltonian for q-Binet-Fibonacci oscillator is written as a Fibonacci number operator

$$H = \frac{\hbar\omega}{2}(b^+b + bb^+) = \frac{\hbar\omega}{2}\left([N+1]_F + [N]_F\right) = \frac{\hbar\omega}{2}F_{N+2},$$

where $bb^{+} = [N+1]_F = F_{N+1}, \quad b^{+}b = [N]_F = F_N$. According to the Hamiltonian, the energy spectrum of this oscillator is written in terms of Fibonacci numbers sequence,

$$E_n = \frac{\hbar\omega}{2}\left([n]_{\varphi,-\frac{1}{\varphi}} + [n+1]_{\varphi,-\frac{1}{\varphi}}\right) = \frac{\hbar\omega}{2}\left(F_n + F_{n+1}\right) = \frac{\hbar\omega}{2}F_{n+2}, \tag{8.97}$$

$$E_n = \frac{\hbar\omega}{2}F_{n+2}.$$

A first energy eigenvalues

$$E_0 = \frac{\hbar\omega}{2}F_2 = \frac{\hbar\omega}{2},$$

which is exactly the same ground state as in the ordinary case. Higher energy excited states are given by Fibonacci sequence

$$E_1 = \frac{\hbar\omega}{2}F_3 = \hbar\omega, \quad E_2 = \frac{3\hbar\omega}{2}, \quad E_3 = \frac{5\hbar\omega}{2}, \ldots$$

In Figure 8.2 we show the quantum Fibonacci tree for this oscillator.

The difference between two consecutive energy levels of our oscillator is found as

$$\triangle E_n = E_{n+1} - E_n = \frac{\hbar\omega}{2}F_{n+1}.$$

Then the ratio of two successive energy levels $\frac{E_{n+1}}{E_n}$ gives the Golden sequence, and for the limiting case of higher excited states $n \to \infty$ it is the Golden ratio

$$\lim_{n\to\infty}\frac{E_{n+1}}{E_n} = \lim_{n\to\infty}\frac{F_{n+3}}{F_{n+2}} = \lim_{n\to\infty}\frac{[n+3]_F}{[n+2]_F} = \frac{1+\sqrt{5}}{2} = \varphi \approx 1,6180339887.$$

This property of asymptotic states to relate each other by a Golden ratio, leads us to call this oscillator as a Golden oscillator.

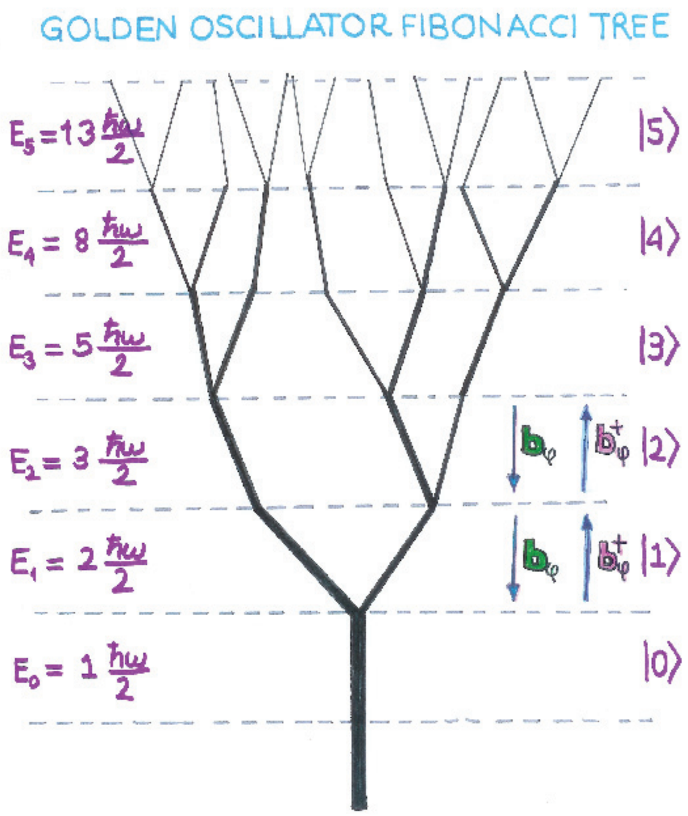

Figure 8.2. Quantum Fibonacci tree for Golden oscillator

We have the following relations between q- creation and annihilation operators and standard creation and annihilation operators

$$b^+ = a^+\sqrt{\frac{F_{N+1}}{N+1}} = \sqrt{\frac{F_N}{N}}a^+ \tag{8.98}$$

$$b = \sqrt{\frac{F_{N+1}}{N+1}}a = a\sqrt{\frac{F_N}{N}}, \tag{8.99}$$

which we call nonlinear unitary transformation, where $[a, a^{+}] = 1$.

8.2.1. Golden Angular Momentum

Double Golden Oscillator algebra $su_F(2)$, determines the Golden Quantum angular momentum operators, defined as

$$J_{+}^{F} = b_1^{+} b_2, \quad J_{-}^{F} = b_2^{+} b_1, \quad J_z^{F} = \frac{N_1 - N_2}{2},$$

and satisfying commutation relations

$$[J_{+}^{F}, J_{-}^{F}] = (-1)^{N_2} F_{2J_z} = -(-1)^{N_1} F_{-2J_z}, \tag{8.100}$$

$$[J_z^{F}, J_{\pm}^{F}] = \pm J_{\pm}^{F}, \tag{8.101}$$

where the Binet-Fibonacci operator is

$$F_N = \frac{\varphi^N - (-\frac{1}{\varphi})^N}{\varphi + \frac{1}{\varphi}} = [N]_F.$$

The Golden quantum angular momentum operators $J_{\pm}^{F}$ may be written in terms of Fibonacci sequence and standard quantum angular momentum operators $J_{\pm}$ as

$$J_{+}^{F} = J_{+} \sqrt{\frac{F_{N_1+1}}{N_1 + 1}} \sqrt{\frac{F_{N_2}}{N_2}} = \sqrt{\frac{F_{N_1}}{N_1}} \sqrt{\frac{F_{N_2+1}}{N_2 + 1}} J_{+} \tag{8.102}$$

$$J_{-}^{F} = J_{-} \sqrt{\frac{F_{N_1}}{N_1}} \sqrt{\frac{F_{N_2+1}}{N_2 + 1}} = \sqrt{\frac{F_{N_1+1}}{N_1 + 1}} \sqrt{\frac{F_{N_2}}{N_2}} J_{-}. \tag{8.103}$$

The Casimir operator for Binet-Fibonacci case is

$$
\begin{aligned}
C^F &= (-1)^{-J_z}\left(F_{J_z}F_{J_z+1} + (-1)^{-N_2}J_-^F J_+^F\right) \\
&= (-1)^{-J_z}\left(-F_{J_z}F_{J_z-1} + (-1)^{-N_2}J_+^F J_-^F\right).
\end{aligned} \tag{8.104}
$$

The angular momentum operators $J_\pm^F$ and J_z^F act on state $|j,m\rangle_F$:

$$
J_+^F\,|j,m\rangle_F = \sqrt{F_{j-m}F_{j+m+1}}\,|j,m+1\rangle_F\,, \tag{8.105}
$$

$$
J_-^F\,|j,m\rangle_F = \sqrt{F_{j+m}F_{j-m+1}}\,|j,m-1\rangle_F\,, \tag{8.106}
$$

$$
J_z^F\,|j,m\rangle_F = m\,|j,m\rangle_F\,. \tag{8.107}
$$

The eigenvalues of Casimir operator C_j^F are determined by product of two successive Fibonacci numbers

$$
C_j^F = (-1)^{-j}F_jF_{j+1},
$$

then the asymptotic ratio of two successive eigenvalues of Casimir operator gives Golden Ratio

$$
\lim_{j\to\infty}\frac{(-1)^{-j}F_jF_{j+1}}{(-1)^{-j+1}F_{j-1}F_j} = -\varphi^2.
$$

We can also construct representation of our F-deformed angular momentum algebra in terms of double Golden boson representation b_1, b_2. The actions of F-deformed angular momentum operators to the state $|n_1,n_2\rangle_F$ are given as follows :

$$J_+^F \, |n_1, n_2\rangle_F = b_1^+ \, b_2 \, |n_1, n_2\rangle_F = \sqrt{F_{n_1+1} F_{n_2}} \, |n_1 + 1, n_2 - 1\rangle_F \, , \tag{8.108}$$

$$J_-^F \, |n_1, n_2\rangle_F = b_2^+ \, b_1 \, |n_1, n_2\rangle_F = \sqrt{F_{n_1} F_{n_2+1}} \, |n_1 - 1, n_2 + 1\rangle_F \, , \tag{8.109}$$

$$J_z^F \, |n_1, n_2\rangle_F = \frac{1}{2}(N_1 - N_2) \, |n_1, n_2\rangle_F = \frac{1}{2}(n_1 - n_2) \, |n_1, n_2\rangle_F \, . \tag{8.110}$$

The above expressions reduce to the familiar ones (8.105)-(8.107) provided we define

$$j \equiv \frac{n_1 + n_2}{2}, \qquad m \equiv \frac{n_1 - n_2}{2}$$

$$|n_1, n_2\rangle_F \equiv |j, m\rangle_F \, ,$$

and substitute

$$n_1 \to j + m, \qquad n_2 \to j - m.$$

8.2.2. Symmetrical $su_{i\varphi}(2)$ Quantum Algebra

As an example of symmetrical q-deformed $su_q(2)$ algebra we choose the base as $q_i = i\varphi$ and $q_j = i\frac{1}{\varphi}$, then our complex equation for base becomes

$$(i\varphi)^2 = i(i\varphi) - 1.$$

The φ- deformed symmetrical angular momentum operators remain the same as $J_{\pm}^{(s)}, J_{z}^{(s)}$. The symmetrical quantum algebra with base $(i\varphi, \frac{i}{\varphi})$ becomes

$$[J_{+}^{\varphi}, J_{-}^{\varphi}] = [2J_{z}]_{\frac{i}{\varphi}} = [2J_{z}]_{i\varphi,\frac{i}{\varphi}}(-1)^{(\frac{1}{2}-J_{z})}, \tag{8.111}$$

where

$$[2J_{z}]_{\frac{i}{\varphi}} = \frac{\varphi^{2J_{z}} - \varphi^{-2J_{z}}}{\varphi - \varphi^{-1}}$$

and

$$[J_{z}^{(s)}, J_{\pm}^{(s)}] = \pm J_{\pm}^{(s)}. \tag{8.112}$$

8.2.3. $\tilde{su}_{F}(2)$ Algebra

One of the special cases of symmetrical $\tilde{su}_{(q,Q)}(2)$ algebra is constructed by choosing Binet-Fibonacci case $(q_{i} = \varphi, \;\; q_{j} = -\frac{1}{\varphi})$. The generators of $\tilde{su}_{F}(2)$ algebra $\tilde{J}_{\pm}^{\varphi}, \tilde{J}_{z}^{\varphi}$ are given in terms of double bosons b_{1}, b_{2} as follows :

$$\begin{aligned} \tilde{J}_{+}^{F} &= (-1)^{-\frac{N_{2}}{2}} b_{1}^{+} b_{2}, \\ \tilde{J}_{-}^{F} &= b_{2}^{+} b_{1} (-1)^{-\frac{N_{2}}{2}}, \\ \tilde{J}_{z}^{F} &= J_{z}. \end{aligned} \tag{8.113}$$

satisfying anti-commutation relation

$$\tilde{J}_{+}^{F}\tilde{J}_{-}^{F} + \tilde{J}_{-}^{F}\tilde{J}_{+}^{F} = \{\tilde{J}_{+}^{F}, \tilde{J}_{-}^{F}\} = [2J_{z}]_{F}, \tag{8.114}$$

and $[\tilde{J}^F_z, \tilde{J}^F_\pm] = \pm\tilde{J}^F_\pm$. The Casimir operator is written in the following forms

$$
\begin{aligned}
\tilde{C}^F &= (-1)^{J_z}\{F_{j_z}F_{j_z+1} - \tilde{J}^F_-\tilde{J}^F_+\} \\
&= (-1)^{J_z}\{\tilde{J}^F_+\tilde{J}^F_- - F_{j_z}F_{j_z-1}\}.
\end{aligned} \tag{8.115}
$$

The actions of the F-deformed angular momentum operators to the states $|j,m\rangle_F$ are

$$
\begin{aligned}
&\tilde{J}^F_+\,|j,m\rangle_F = (-1)^{\frac{j-m}{2}}\sqrt{F_{j-m}F_{j+m+1}}\,|j,m+1\rangle_F\,, \\
&\tilde{J}^F_-\,|j,m\rangle_F = (-1)^{\frac{j-m}{2}}\sqrt{F_{j+m}F_{j-m+1}}\,|j,m-1\rangle_F\,, \\
&\tilde{J}^F_z\,|j,m\rangle_F = m\,|j,m\rangle_F\,.
\end{aligned} \tag{8.116}
$$

And the eigenvalues of Casimir operators are given by

$$
\begin{aligned}
\tilde{C}^F\,|j,m\rangle_F &= \{(-1)^m F_m F_{m+1} - (-1)^j F_{j-m}F_{j+m+1}\}\,|j,m\rangle_F \\
&= \{(-1)^j F_{j-m+1}F_{j+m} - (-1)^m F_m F_{m-1}\}\,|j,m\rangle_F\,.
\end{aligned}
$$

CHAPTER 9

CONCLUSION

In the present thesis, we studied quantum calculus of classical Heat-Burgers' hierarchy and quantum coherent states. We constructed random walk on q-lattice as Fermat partition and obtained corresponding q-heat equation with specific q-dependence for time and space variables. In order to find exact solution of this equation we introduced a new family of q-exponential functions which produces Jackson's q-exponential functions for weighted number $N = 0$ and $N = 1$. The solution of q-heat equation is found in terms of our q-exponential functions. We obtained q-oscillator hierarchy by using this solution and it allows us to get a family of q-heat equations. Then the specific case of random walk on q-lattice produced q-heat equation with symmetrical q-derivatives in space variable and its exact solution was written as symmetrical q-exponential and symmetrical q-trigonometric functions.

We introduced a new type of q-diffusive heat equation, including standard derivatives in time and space, with nonsymmetric q-extension of the diffusion term. The polynomial solutions of this equation was written as generalized Kampe-de Feriet polynomials, corresponding dynamical symmetry and description in terms of Bell polynomials. Written in relative gradient variables this system appeared as the q-viscous Burgers' equation and its one, two and multiple shock soliton solutions are found and studied their mutual interactions for different values of q. We found that due to specific dependence of the group velocity on wave number, in addition to fusion of the solitons as in usual Burgers equation, a new process of fission of shock solitons with higher amplitude takes place. The q-semiclassical expansion of these equations in terms of Bernoulli polynomials was derived as corrections in power of $\ln q$. We get the corresponding Bäcklund transformations of q-viscous Burgers' equations.

We introduced a new class of complex valued function of complex argument which we called q-analytic functions satisfying q-Cauchy-Riemann equations and get the real and imaginary parts as q-harmonic functions. These q-analytic functions are not classical analytic functions but we proved that some class of these complex functions are considered as a generalized analytic functions. As an example we obtained that the complex q-binomial functions are generalized analytic functions by satisfying D-bar problem and their integral representation is written. In terms of these functions the complex q-analytic fractal, satisfying the self-similar q-difference equation is derived. As an application we constructed a new type of quantum states as q-analytic coherent states and corresponding q-analytic Fock-Bargmann representation. In this representation, quantum q-oscillator problem is solved and shown that the wave functions of quantum states are given by complex q-binomials.

The concept of q-analytic function was extended to expansion of q-binomial in terms of q-Hermite polynomials which are analytic in two complex arguments. In this representation, we introduced a new class of complex functions of two complex arguments, called double q-analytic functions. As hyperbolic extension, we described the q-analogue of traveling waves, which are not preserving their shape during evolution. We studied q-wave equation and solved in the q-Hermite polynomial form.

By introducing q-translation operator we obtained q-binomials, q-analytic and q-anti analytic functions, and q-travelling waves. Another type q-translation operator, called q-commutative (non-commutative) translation operator, was introduced. Then we represented non-commutative binomials, functions for non-commutative coordinates. Then we generalized these q-translations to q, p-translations for two bases. By specific choice of bases as Golden ratio, Fibonomials are constructed as translation of monomials. We described all these translations by the first order q-difference equations.

Based on acting by evolution operator, we introduced a new type of quantum states as Hermite coherent states and Kampe-de Feriet coherent states, characterized by Hermite polynomials and Kampe-de Feriet polynomials correspondingly. We generalized the known Mehler formula in order to normalize these coherent states. Their Fock-Bargmann representations were written. By using the generating function of Bernoulli polynomials, we get Bernoulli coherent states and related Fock-Bargmann representation. Then q-analogue of coherent states are introduced.

We introduced Golden quantum calculus. By Fibonacci and Golden derivatives we derived main ingredients of these calculus as Golden Leibnitz rule, Taylor expansion, Golden binomial and Golden integral. As an application of Golden quantum calculus, we studied Golden quantum oscillator and its angular momentum representations.

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APPENDIX A

D-BAR REPRESENTATION OF COMPLEX Q-BINOMIAL

In this section, we are going to prove q-complex binomial representation (4.50), (4.51)
.

A.1. Generalized Cauchy formula

For non-analytic function $\Phi(z)$, the next generalized Cauchy formula is valid (Vekua, 1962), (Ablowitz and Fokas, 1997)

$$\Phi(z) = \frac{1}{2\pi i}\oint_\Gamma \frac{\Phi(\zeta)d\zeta}{\zeta - z} - \frac{1}{\pi}\int\int_G \frac{\partial\Phi}{\partial\bar{\zeta}}\frac{d\xi d\eta}{\zeta - z}, \tag{A.1}$$

where $\zeta = \xi + i\eta$. First we are going to check this formula for non-analytic function

$$\Phi_n(z) = x + iq^n y = \frac{1+q^n}{2}z + \frac{1-q^n}{2}\bar{z}, \tag{A.2}$$

with

$$\frac{\partial\Phi_n}{\partial\bar{z}} = \frac{1-q^n}{2} = \frac{[n]_q}{2}(1-q).$$

For the disk of radius R we have:

1. The line integral part in the above generalized Cauchy formula gives

$$\frac{1}{2\pi i}\oint_\Gamma \frac{\frac{1+q^n}{2}\zeta + \frac{1-q^n}{2}\bar{\zeta}}{\zeta - z}d\zeta =$$
$$\frac{1}{2\pi i}\frac{1+q^n}{2}\oint_\Gamma d\zeta + \frac{1}{2\pi i}\frac{1+q^n}{2}z\oint_\Gamma \frac{d\zeta}{\zeta - z} + \frac{1}{2\pi i}\frac{1-q^n}{2}\oint_\Gamma \frac{\bar{\zeta}d\zeta}{\zeta - z}. \tag{A.3}$$

The first integral vanishes, while the second one gives $2\pi i$ so that we have

$$\frac{1+q^n}{2}z + \frac{1}{2\pi i}\frac{1-q^n}{2}\oint_\Gamma \frac{\bar{\zeta} d\zeta}{\zeta - z}. \tag{A.4}$$

By substitution $\zeta = Re^{i\theta}$ the last integral becomes

$$\frac{1}{2\pi i}\frac{1-q^n}{2}\int_0^{2\pi} \frac{i d\theta}{Re^{i\theta} - z}. \tag{A.5}$$

Then, rewriting it in terms of $u = e^{i\theta}$ we get contour integral along the unit circle

$$\frac{1}{2\pi i}\frac{1-q^n}{2}\oint_{|u|=1} \frac{du}{u(Ru - z)}. \tag{A.6}$$

By the residues theorem this integral vanishes

$$\frac{1-q^n}{2R}\left[\frac{R}{-z} + \frac{R}{z}\right] = 0. \tag{A.7}$$

As a result for the line integral we obtain

$$LI = \frac{1+q^n}{2}z. \tag{A.8}$$

2. The double integral part in polar coordinates $\zeta = \xi + i\eta = re^{i\theta}$ is

$$-\frac{1}{\pi}\int\int_G \frac{1-q^n}{2}\frac{d\xi d\eta}{\zeta - z} = -\frac{1-q^n}{2\pi}\int_0^{2\pi}\int_0^R \frac{r dr d\theta}{re^{i\theta} - z}. \tag{A.9}$$

By substitution $u = e^{i\theta}$ we rewrite the angle part of integral as the contour integral along the unit circle $|u| = 1$

$$-\frac{1}{\pi}\int\int_G \frac{1-q^n}{2}\frac{d\xi d\eta}{\zeta - z} = -\frac{1-q^n}{2\pi i}\int_0^R dr \oint_{|u|=1} \frac{du}{u(u - z/r)}. \tag{A.10}$$

By the residues theorem the contour integral for $|z| > r$ is

$$\oint_{|u|=1} \frac{du}{u(u - z/r)} = 2\pi i \left(-\frac{r}{z}\right) \tag{A.11}$$

and for $|z| < r$ it vanishes. Thus the double integral for $r > |z|$ also vanishes so that the range of integration in r is going from 0 to $|z|$,

$$\frac{1 - q^n}{z} \int_0^{|z|} r dr = \frac{1 - q^n}{z} \frac{|z|^2}{2}. \tag{A.12}$$

Then finally for the double integral we get

$$DI = \frac{1 - q^n}{2} \bar{z}. \tag{A.13}$$

Adding the line and the double integrals (A.8), (A.13) together we obtain desired formula (A.2):$LI + DI = \frac{1+q^n}{2} z + \frac{1-q^n}{2} \bar{z}$.

A.2. Generalized analytic function

For $\Phi_n(z)$ in (A.2) as a generalized analytic function, we have the D-bar equation

$$\frac{\partial \Phi_n}{\partial \bar{z}} = \frac{(1 - q^n)}{(1 + q^n)z + (1 - q^n)\bar{z}} \Phi_n(z) = A_n(z, \bar{z}) \Phi_n(z), \tag{A.14}$$

where

$$A_n(z, \bar{z}) = \frac{(1 - q^n)}{(1 + q^n)z + (1 - q^n)\bar{z}}.$$

Representation (4.45) for this function is

$$\Phi_n(z, \bar{z}) = \omega(z)\, e^{\frac{1}{2\pi i} \int \int_D \frac{A_n(\zeta, \bar{\zeta})}{\zeta - z} \, d\zeta \wedge d\bar{\zeta}}. \tag{A.15}$$

To check it we are going to calculate this integral explicitly and find holomorphic function $\omega(z)$ for the disk of radius R.

The double integral in exponential is

$$I = \frac{1}{2\pi i}\int\int_D \frac{A_n(\zeta,\bar{\zeta})}{\zeta - z}\, d\zeta \wedge d\bar{\zeta} = -\frac{1-q^n}{\pi}\int\int_D \frac{d\xi\, d\eta}{[(1+q^n)\zeta + (1-q^n)\bar{\zeta}][\zeta - z]}$$

where $\zeta = \xi + i\eta$ and $D = \{\zeta : |\zeta| \le R\}$ or in the polar coordinates $\zeta = re^{i\theta}$,

$$I = \frac{q^n-1}{\pi}\int_0^R\int_0^{2\pi} \frac{dr\, d\theta}{[(1+q^n)e^{i\theta} + (1-q^n)e^{-i\theta}][re^{i\theta} - z]} = \frac{q^n-1}{\pi}\int_0^R \frac{dr}{r} I_0$$

where by complex substitution $u = e^{i\theta}$ we have contour integral around unit circle

$$I_0 = \frac{1}{i}\oint_{|u|=1} \frac{du}{u}\frac{1}{[(1+q^n)u + (1-q^n)\frac{1}{u}][u - \frac{z}{r}]}, \tag{A.16}$$

or

$$I_0 = \frac{1}{i(1+q^n)}\oint_{|u|=1} \frac{du}{[u^2 + \frac{1-q^n}{1+q^n}][u - \frac{z}{r}]}. \tag{A.17}$$

For the base $0 < q < 1$ the integrand has two simple poles inside of the unit circle at $u = \pm i\sqrt{\frac{1-q^n}{1+q^n}}$ and for $|z| < r$, one more simple pole at $u = z/r$. Then by the residues theorem

$$I_0 = \frac{2\pi}{1+q^n}\begin{cases} -\frac{1}{\frac{1-q^n}{1+q^n} + \frac{z^2}{r^2}}, & |z| > r, \\ 0, & |z| < r. \end{cases} \tag{A.18}$$

Substituting to integral I we get

$$I = 2\frac{1-q^n}{1+q^n}\int_0^R \frac{dr}{r}\begin{cases} -\frac{1}{\frac{1-q^n}{1+q^n} + \frac{z^2}{r^2}}, & |z| > r, \\ 0, & |z| < r, \end{cases} \tag{A.19}$$

or

$$I = 2\frac{1-q^n}{1+q^n}\int_0^z \frac{dr}{r}\frac{1}{\frac{1-q^n}{1+q^n}+\frac{z^2}{r^2}}. \tag{A.20}$$

By elementary integration

$$I = \ln(r^2 + \frac{1-q^n}{1+q^n}z^2)\Big|_0^z = \ln\frac{(1+q^n)z + (1-q^n)\bar{z}}{(1+q^n)z}, \tag{A.21}$$

and for (A.15) then we find

$$\Phi_n(z,\bar{z}) = \omega(z)\, e^I = \frac{1+q^n}{2}z + \frac{1-q^n}{2}\bar{z}, \tag{A.22}$$

where the analytic function

$$\omega(z) = \frac{1+q^n}{2}z.$$

A.3. Complex q-binomial as generalized analytic function

The above results can be applied now for the complex q-binomial degree n,

$$(x+iy)_q^n = (x+iy)(x+iqy)...(x+iq^{n-1}y).$$

Denoting

$$\Phi(z) = \Phi_0(z)\,\Phi_1(z)\,...\,\Phi_{n-1}(z),$$

where $\Phi_n(z) = x + iq^n y = \frac{1+q^n}{2}z + \frac{1-q^n}{2}\bar{z}$, we have

$$\frac{\partial}{\partial \bar{z}}\Phi(z,\bar{z}) = \Phi(z,\bar{z})(1-q)\sum_{k=1}^{n-1}\frac{[k]_q}{(1+q^k)z+(1-q^k)\bar{z}} = A(z,\bar{z})\Phi(z,\bar{z}),$$

where

$$A(z,\bar{z}) = (1-q)\sum_{k=1}^{n-1}\frac{[k]_q}{(1+q^k)z+(1-q^k)\bar{z}} = \sum_{k=1}^{n-1}A_n(z,\bar{z}).$$

By the above calculations for the double integral in a disk of radius R, ($\zeta = \xi + i\eta$), we obtain

$$\begin{aligned}\frac{1}{2\pi i}\int\int_D \frac{A(\zeta,\bar{\zeta})}{\zeta - z}d\zeta\wedge d\bar{\zeta} &= \frac{1}{\pi}\sum_{k=1}^{n-1}(q^k-1)\int\int_D \frac{d\xi d\eta}{((1+q^k)\zeta+(1-q^k)\bar{\zeta})(\zeta-z)} \\ &= \sum_{k=1}^{n-1}\ln\frac{(1+q^k)z+(1-q^k)\bar{z}}{(1+q^k)z}. \qquad \text{(A.23)}\end{aligned}$$

Then

$$\begin{aligned}\Phi(z,\bar{z}) = (x+iy)_q^n &= \omega(z)\, e^{\frac{1}{2\pi i}\int\int_D \frac{A(\zeta,\bar{\zeta})}{\zeta-z}d\zeta\wedge d\bar{\zeta}} \\ &= \omega(z) e^{\ln\prod_{k=1}^{n-1}\left(\frac{\frac{1+q^k}{2}z+\frac{1-q^k}{2}\bar{z}}{\frac{1+q^k}{2}z}\right)} \\ &= \omega(z)\prod_{k=1}^{n-1}\frac{\frac{1+q^k}{2}z+\frac{1-q^k}{2}\bar{z}}{\frac{1+q^k}{2}z} \qquad \text{(A.24)} \\ &= \omega(z)\prod_{k=1}^{n-1}\frac{2(x+iq^k y)}{(1+q^k)z} \qquad \text{(A.25)}\end{aligned}$$

or

$$(x+iy)_q^n = \omega(z)\frac{2^n}{z^n\prod_{k=0}^{n-1}(1+q^n)}(x+iy)_q^n \qquad \text{(A.26)}$$

As a result we find the next form for the analytic function

$$\omega(z) = \left(\frac{z}{2}\right)^n \prod_{k=0}^{n-1}(1 + q^k). \tag{A.27}$$

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