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# NEAT-FLAT MODULES

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***Let R be a ring. A right R-module M is said to be neat-flat if the kernel of any epimorphism $Y \to M$ is neat in Y, i.e., the induced map $\mathrm{Hom}(S, Y) \to \mathrm{Hom}(S, M)$ is surjective for any simple right R-module S. Neat-flat right R-modules are projective if and only if R is a right $\sum$-CS ring. Every cyclic neat-flat right R-module is projective if and only if R is right CS and right C-ring. It is shown that, over a commutative Noetherian ring R, (1) every neat-flat module is flat if and only if every absolutely coneat module is injective if and only if $R \cong A \times B$, wherein A is a QF-ring and B is hereditary, and (2) every neat-flat module is absolutely coneat if and only if every absolutely coneat module is neat-flat if and only if $R \cong A \times B$, wherein A is a QF-ring and B is Artinian with $J^2(B) = 0$.***

***Key Words:*** Closed submodule; (Co)neat submodule; Extending module; Neat-flat module; *QF*-ring.

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## 1. INTRODUCTION

Throughout, $R$ is an associative ring with identity and all modules are unitary right $R$-modules. For an R-module $M$, $E(M)$, $\mathrm{Soc}(M)$ will denote the injective hull, the socle of $M$, respectively. The character module $\mathrm{Hom}_Z(M, \mathbb{Q}/\mathbb{Z})$ of $M$ is denoted by $M^+$. The Jacobson radical of the ring $R$ is denoted by $J(R)$.

A submodule $K$ of an $R$-module $M$ is called closed (in $M$) provided $K$ has no proper essential extension in $M$. When $R$ is a Dedekind domain (more generally a Prüfer domain), a submodule $K$ of an R-module $M$ is said to be pure if and only if $K \cap aM = aK$ for all $a \in R$. Inspired by this characterization of pure submodules over Dedekind domains, Honda [14] introduced neat subgroups in order to characterize closed subgroups in abelian groups. Namely, a subgroup $A$ of an abelian group $B$ is called neat in $B$ if $Ap = A \cap Bp$ for every prime $p$. A subgroup $A$ of an abelian group $B$ is closed if and only if it is neat if and only if $\mathrm{Hom}(S, B) \to \mathrm{Hom}(S, B/A) \to 0$ is surjective for each simple $R$-module $S$. Neatness over arbitrary associative rings considered by Renault [20], namely, a submodule $A$ of an $R$-module $B$ is called neat if $\mathrm{Hom}(S, B) \to \mathrm{Hom}(S, B/A) \to 0$ is surjective

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for each simple $R$-module $S$. Closed submodules are neat, but the converse is true exactly for $C$-rings (i.e., $\mathrm{Soc}(R/I) \neq 0$ for every proper essential right ideal $I$ of $R$).

A submodule $K$ of $M$ is called small in $M$ if $M \neq K + T$ for every proper submodule $T$ of $M$. As the dual of closed submodule, the submodule $K$ is called coclosed in $M$ if for every submodule $A$ of $M$ with $A \leq K$, $K/A \ll M/A$ implies $K = A$. Recently, Zöschinger showed in [28] that, over a commutative Noetherian ring $R$, closed submodules are coclosed if and only if coclosed submodules are closed if and only if $R$ is distributive. In his recent article, as a dual of neat submodule, Fuchs [10] defined a submodule $N$ of $M$ to be coneat if $\mathrm{Hom}(M, S) \to \mathrm{Hom}(N, S) \to 0$ is surjective for each simple $R$-module $S$. In that article, he proved that for an integral domain $R$ neat submodules and coneat submodules coincide if and only if every maximal ideal of $R$ is finitely generated. Crivei is also concerned with the same problem in [7], and he showed that if $R$ is a commutative ring whose maximal ideals are principal then neat and coneat submodules of every module coincide.

Recently, there is a significant interest to some classes of modules that are defined via (co) closed submodules and (co) neat submodules, (see, [7, 17, 25–28]). An $R$-module $M$ is said to be $m$-injective (weakly-injective, absolutely coneat, respectively) if it is neat (coclosed, coneat, respectively) in every extension. Note that closed submodule of an injective module is injective. $m$-injective modules are injective if and only if every neat submodule is closed (i.e., $R$ is a right $C$-ring), (see [24]). Weakly-injective modules are introduced and discussed by Zöschinger in [27, 28]). Absolutely coneat modules are introduced and studied by Crivei in [7].

Motivating by the relation between weakly-flat modules and closed submodules, we investigate the modules $M$, for which any short exact sequence ending with $M$ is neat-exact. Namely, we say $M$ is *neat-flat* if the kernel of any epimorphism $Y \to M$ is neat in $Y$, i.e., the induced map $\mathrm{Hom}(S, Y) \to \mathrm{Hom}(S, M)$ is surjective for any simple $R$-module $S$. Projective modules, weakly-flat modules, and nonsingular modules are neat-flat. In [17], the author introduced *simple-projective* modules to characterize the rings whose simple modules have projective (pre)envelope. An $R$-module $M$ is called simple-projective if for any simple right $R$-module $N$, every homomorphism $f : N \to M$ factors through a finitely generated free right $R$-module $F$.

The article is organized as follows. In Section 2, it is shown that neat-flat modules coincide with simple-projective modules over arbitrary rings. Next, we give the main properties of the class of neat-flat $R$-modules. The right socle of $R$ is zero if and only if neat-flat modules coincide with the modules that have zero socle. A ring $R$ is a right $C$-ring if and only if neat-flat modules are weakly-flat. We also investigate the rings over which neat-flat modules are projective. Namely, we prove that, (1) every neat-flat module is projective if and only if $R$ is a right $\sum$-$CS$ ring; (2) every finitely generated neat-flat module is projective if and only if $R$ is a right $C$-ring and every finitely generated free right $R$-module is extending; and (3) every cyclic right $R$-module is projective if and only if $R$ is right $CS$ and right $C$-ring.

In Section 3, it is shown that, over a commutative Noetherian ring $R$, (1) every neat-flat module is flat if and only if every absolutely coneat module is injective if and only if $R \cong A \times B$, wherein $A$ is $QF$-ring and $B$ is hereditary; and (2) every neat-flat module is absolutely coneat if and only if every absolutely coneat module is neat-flat if and only if every neat-flat module is weakly-injective if and only if every absolutely coneat module is weakly-flat if and only if $R \cong A \times B$, wherein $A$ is $QF$-ring and $B$ is Artinian with $J^2(B) = 0$.

In Section 4, localization of neat exact sequences and neat-flat modules are investigated. It is shown that, over a commutative $N$-ring $R$, (1) a short exact sequence $0 \to A \to B \to C \to 0$ is neat exact, i.e., $A$ is neat in $B$ if and only if $0 \to A_P \to B_P \to C_P \to 0$ is neat exact for each maximal ideal $P$ of $R$; and (2) a module $M$ is neat-flat if and only if, for all maximal ideals $P$ of $R$, $M_P$ is neat-flat $R_P$-module.

For the unexplained concepts and results, we refer the reader to [1, 4] and [16].

## 2. NEAT-FLAT MODULES

Let $\mathbb{E} : 0 \to K \xrightarrow{f} L \xrightarrow{g} M \to 0$ be a short exact sequence. $\mathbb{E}$ is called *neat exact* if $f(K)$ is a neat submodule of $L$. In this case, $f$ and $g$ are called neat monomorphism and neat epimorphism, respectively. By definition, the class of neat exact sequences is projectively generated by the class of simple $R$-modules. Hence neat-exact sequences form a proper class in the sense of Bushbaum, (see[4, 10.8]). For the following lemma we refer to [18, Proposition 1.12-1.13]. The proof is included for completeness.

**Lemma 2.1.** *The following statements are equivalent for a right $R$-module $M$:*

(1) *$M$ is neat-flat*;
(2) *Every exact sequence $0 \to A \to B \to M \to 0$ is neat exact*;
(3) *There exists a neat exact sequence $0 \to K \to F \to M \to 0$ with $F$ projective*;
(4) *There exists a neat exact sequence $0 \to K \to F \to M \to 0$ with $F$ neat-flat.*

*Proof.* (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) are clear.

(4) $\Rightarrow$ (1) Let $0 \to A \to B \xrightarrow{g} M \to 0$ be any short exact sequence. We claim that $g$ is a neat epimorphism, i.e., $\mathrm{Ker}(g)$ is a neat submodule of $B$. By (4), there exists a neat exact sequence $0 \to K \xrightarrow{f} F \xrightarrow{s} M \to 0$ with $F$ neat-flat. Considering the pullback of $g$ and $s$, we obtain a commutative diagram with exact rows

$$\begin{array}{ccccccccc}
0 & \longrightarrow & A & \xrightarrow{\alpha} & B' & \xrightarrow{t} & F & \longrightarrow & 0 \\
 & & \Vert & & \downarrow{\scriptstyle u} & & \downarrow{\scriptstyle s} & & \\
0 & \longrightarrow & A & \xrightarrow{\beta} & B & \xrightarrow{g} & M & \longrightarrow & 0 .
\end{array}$$

Since $F$ is neat-flat, $\alpha(A)$ is neat in $B'$. As $\alpha(A)$ is neat in $B'$ and $f(K)$ is neat in $F$, we have $\beta(A)$ is neat in $B$ by [4, 10.1]. This completes the proof. □

**Remark 2.2.**

(1) Clearly, if $\mathrm{Soc}(M)$ is projective, then $M$ is neat-flat. In particular, if $M$ has no simple submodules, then $M$ is neat-flat.
(2) Obviously, projective modules are neat-flat. On the other hand, the infinite direct product of the ring of integers $\mathbb{Z}$ is neat-flat, but not projective.
(3) Note that a simple right $R$-module is neat-flat if and only if it is projective. Thus $R$ is a semisimple Artinian ring if and only if every right R-module is neat-flat.

(4) By [22, Lemma 2.3(a)], every nonsingular module is weakly-flat. Since weakly-flat modules are neat-flat, nonsingular modules are neat-flat.

The following observation is useful for the further characterization of neat-flat modules.

**Lemma 2.3.** *Let $R$ be a ring. An $R$-module $M$ is simple-projective if and only if $M$ is neat-flat.*

*Proof.* Suppose $M$ is simple-projective and $s : R^{(I)} \to M$ be an epimorphism. Let $S$ be simple right $R$-module and $f : S \to M$ be a homomorphism. As $M$ is simple-projective $f$ factors through a finitely generated free module, i.e., there are homomorphisms $h : S \to R^n$ and $g : R^n \to M$ such that $f = gh$. Since $R^n$ is projective, there is a homomorphism $t : R^n \to R^{(I)}$ such that $g = st$. We get the following diagram:

$$\begin{array}{ccc} R^n & \xleftarrow{\;h\;} & S \\ \downarrow{\scriptstyle t} & \searrow{\scriptstyle g} & \downarrow{\scriptstyle f} \\ R^{(I)} & \xrightarrow{\;s\;} & M\,. \end{array}$$

Then $f = gh = sth$, and so the induced map $\mathrm{Hom}(S, R^{(I)}) \to \mathrm{Hom}(S, M) \to 0$ is surjective. Therefore, the sequence $0 \to \mathrm{Ker}s \to R^{(I)} \stackrel{s}{\to} M \to 0$ is neat exact. Hence $M$ is neat-flat by Lemma 2.1(3).

Conversely, let $M$ be a neat-flat module. Then there is a neat exact sequence $0 \to K \to F \stackrel{g}{\to} M \to 0$ with $F$ free by Lemma 2.1. Let $S$ be a simple module and $f : S \to M$ be any homomorphism. Then there is a homomorphism $h : S \to F$ such that $f = gh$. As $S$ is finitely generated, $h(S) \leq H$ for some finitely generated free submodule of $F$. Then we get $f = gh = (gi)h'$, where $i : H \to F$ is the inclusion and $h' : S \to H$ is the homomorphism defined as $h'(x) = h(x)$ for each $x \in S$. Therefore, $f$ factors through $H$, and so $M$ is simple projective. □

From the proof of the lemma above, we have the following corollary.

**Corollary 2.4.** *If $M$ is a neat-flat right $R$-module, then any simple submodule of $M$ is isomorphic to a minimal right ideal of $R$.*

Let $M$ be a module with $\mathrm{Soc}(M) = 0$. Then $\mathrm{Hom}(S, M) = 0$ for any simple right $R$-module $S$, and so $M$ is neat-flat. Corollary 2.4 yields the following corollary.

**Corollary 2.5.** *Let $R$ be a ring. The following statements are equivalent*:

(1) $\mathrm{Soc}(R_R) = 0$;
(2) *An $R$-module $M$ is neat-flat if and only if* $\mathrm{Soc}(M_R) = 0$.

**Proposition 2.6.** *The class of neat-flat $R$-modules is closed under extensions, direct sums, pure submodules, and direct summands.*

*Proof.* By Lemma 2.3 and [17, Proposition 2.4]. □

Recall that a ring $R$ is called a right $C$-ring if $\mathrm{Soc}(M) \neq 0$ for every (cyclic) singular $R$-module $M$. Left perfect rings, right semiartinian rings and almost perfect domains are right $C$-rings. $R$ is a right $C$-ring if and only if neat submodules are closed if and only if $m$-injective modules are injective, (see [24, Lemma 4], [11, Theorem 5]).

Following, Zöschinger [28], a right $R$-module $M$ is called *weakly-flat* if the kernel of any epimorphism $Y \to M \to 0$ is a closed submodule of $Y$. Every nonsingular module is weakly-flat, and the converse is true exactly when the underlying ring is nonsingular (see, [22, Lemma 2.3]).

**Proposition 2.7.** *A ring $R$ is a right $C$-ring if and only if neat-flat are $R$-modules are weakly-flat.*

*Proof.* Necessity is clear. For the sufficiency suppose an $R$-module $M$ is $m$-injective. We claim that $M$ is injective. Consider the exact sequence $0 \to M \hookrightarrow E(M) \to E(M)/M \to 0$. By [6, Theorem 3], $\mathrm{Soc}(E(M)/M) = 0$, and so $E(M)/M$ is neat-flat. Now, $M$ is closed in $E(M)$ by the hypothesis. Therefore, $M$ is injective, so $R$ is a right $C$-ring by [24, Lemma 4]. □

**Corollary 2.8.** *A ring $R$ is right $C$-ring and right nonsingular if and only if neat-flat modules are nonsingular.*

The following result is a generalization of [28, Satz 1.1].

**Proposition 2.9.** *Let $R$ be a right $C$-ring and $M$ be a right $R$-module. The following statements are equivalent*:

(1) *$M$ is weakly-flat*;
(2) *$M$ is neat-flat*;
(3) $\mathrm{Soc}(M) = M.\mathrm{Soc}(R_R)$.

*Proof.* (1) $\Leftrightarrow$ (2) By Proposition 2.7.

(2) $\Rightarrow$ (3) Let $S$ be simple submodule of $M$. Then the inclusion map $i : S \to M$ factors through $R$ by Lemma 2.3. That is, there are homomorphisms $f : S \to R$ and $g : R \to M$ such that $gf = i$. As $S$ is simple, $f(S) = A_R$ is a simple right ideal of $R$. Therefore $S = i(S) = gf(S) = g(A) = g(R)A \leq M.Soc(R_R)$. Hence $\mathrm{Soc}(M) \leq M.\mathrm{Soc}(R_R)$. The reverse containment is clear.

(3) $\Rightarrow$ (2) Suppose $M \cong F/K$ for some free module $F$ and a submodule $K$ of $F$. Assume $K$ is not closed in $F$. Then there is a submodule $T$ of $F$ containing $K$ essentially. Now $\mathrm{Soc}(T/K) \neq 0$, because $T/K$ is singular and $R$ is right $C$-ring. Let $A$ be a complement of $K$ in $F$. Then $A \oplus K$ is essential in $F$, and so $\mathrm{Soc}(F) = \mathrm{Soc}(A) \oplus \mathrm{Soc}(K)$. We get $\mathrm{Soc}(\frac{F}{K}) = (\frac{F}{K})\mathrm{Soc}(R_R) = \frac{(\mathrm{Soc}(F)+K)}{K} = \frac{(\mathrm{Soc}(A)+K)}{K}$. Therefore $\frac{T}{K} \cap [\frac{(\mathrm{Soc}(A)+K)}{K}] \neq 0$, and this implies $A \cap K \neq 0$, a contradiction. Hence $K$ is a closed submodule of $F$, and so $M$ is weakly-flat. □

A module $M$ is said to be extending or a $CS$-module if every closed submodule of $M$ is a direct summand of $M$. $R$ is a right $CS$ ring if $R_R$ is $CS$. $M$ is called $\sum$-$CS$ module if every direct sum of copies of $M$ is $CS$, (see [8]). The $\sum$-$CS$ rings were first introduced and termed as co-$H$-rings in [19].

**Theorem 2.10.** *Let R be a ring. The following statements are equivalent*:

(1) *Every neat-flat R-module is projective*;
(2) *R is a right $\sum$-CS ring.*

*Proof.* (1) ⇒ (2) Let $P$ be a projective $R$-module and $N$ be a closed submodule of $P$. Then $P/N$ is neat-flat by Lemma 2.1, and so $P/N$ is projective by (1). Therefore, the sequence $0 \to N \to P \to P/N \to 0$ splits, and so $N$ is a direct summand of $P$. Hence $R$ is a $\sum$-$CS$ ring.

(2) ⇒ (1) Every right $\sum$-$CS$ ring is both right and left perfect by [19, Theorem 3.18]. Hence, $R$ is a right $C$-ring by [1, Theorem 28.4]. Let $M$ be a neat-flat $R$-module. Then there is a neat exact sequence $0 \to K \hookrightarrow P \to M \to 0$ with $P$ projective by Lemma 2.1. Since $R$ is a right $C$-ring, $K$ is closed in $P$ by [11, Theorem 5]. By the assumption, $K$ is direct summand in $P$, and so $M$ is projective. □

**Theorem 2.11.** *Let R be a ring. The following statements are equivalent*:

(1) *Every finitely generated neat-flat R-module is projective*;
(2) *R is a right C-ring and every finitely generated free R-module is extending.*

*Proof.* (1) ⇒ (2) Let $I$ be an essential right ideal of $R$ with $\mathrm{Soc}(R/I) = 0$. Then $\mathrm{Hom}(S, R/I) = 0$ for each simple $R$-module $S$, and hence $I$ is neat ideal of $R$. So $R/I$ is neat-flat by Lemma 2.1. But it is projective by (1), and so $I$ is direct summand of $R$. This contradicts with essentiality of $I$ in $R$. So that $R$ is a right $C$-ring.

Let $F$ be a finitely generated free $R$-module and $K$ a closed submodule of $F$. Since every closed submodule is neat, $F/K$ is neat-flat by Lemma 2.1. Then $F/K$ is projective by (1), and so $K$ is a direct summand of $F$.

(2) ⇒ (1) Let $M$ be a finitely generated neat-flat $R$-module. Then there is an exact sequence $0 \to \mathrm{Ker}(f) \hookrightarrow F \to M \to 0$ with $F$ finitely generated free $R$-module. By Lemma 2.1 $\mathrm{Ker}(f)$ is a neat submodule of $F$. Since $R$ is a $C$-ring, $\mathrm{Ker}(f)$ is a closed submodule of $F$ by [11, Theorem 5]. Then $0 \to \mathrm{Ker}(f) \hookrightarrow F \to M \to 0$ is a split exact sequence. Hence $M$ is projective. □

Following the proof of Theorem 2.11, we obtain the following corollary.

**Corollary 2.12.** *Every cyclic neat-flat R-module is projective if and only if R is right CS and right C-ring.*

A module $N$ is called *semiartinian* if every nonzero homomorphic image of $N$ contains a simple module.

**Remark 2.13.** Let $M$ be an $R$-module. Then the socle series $\{S_\alpha\}$ of $M$ is defined as $S_1 = \mathrm{Soc}(M)$, $S_\alpha/S_{\alpha-1} = \mathrm{Soc}(M/S_{\alpha-1})$, and for a limit ordinal $\alpha$, $S_\alpha = \cup_{\beta<\alpha} S_\beta$. Put $S = \cup\{S_\alpha\}$. Then, by construction $M/S$ has zero socle. $M$ is semiartinian if and only if $S = M$ (see, for example, [8]).

From the proof of Theorem 2.10, we see that the condition that every free $R$-module is extending implies $R$ is a right $C$-ring. In the following example, we show that, if every finitely generated free $R$-module is extending, then $R$ need not be a right $C$-ring. Hence the right $C$-ring condition in 2.11 is not superfluous.

**Example 2.14.** Let $R$ be the ring of all linear transformations (written on the left) of an infinite dimensional vector space over a division ring. Then $R$ is prime, regular, right self-injective and $\mathrm{Soc}(R_R) \neq 0$ by [13, Theorem 9.12]. As $R$ is a prime ring, $\mathrm{Soc}(R_R)$ is an essential ideal of $R_R$. Let $S$ be as in Remark 2.13, for $M = R$. Then $S \neq R$, by [5, Lemma 1(2)]. Since $R/S$ has zero socle, $S$ is a neat submodule of $R_R$. On the other hand, $S$ is not a closed submodule of $R$, otherwise $S$ would be a direct summand of $R$ because $R$ is right self injective (i.e., extending). Therefore, $R$ is not a right $C$-ring. Also, as $R$ is right self injective $R^n$ is injective, and so extending for every $n \geq 1$.

## 3. N-RINGS

A commutative domain $R$ is called an $N$-domain if every maximal ideal of $R$ is finitely generated. These domains are characterized as those domains $R$, over which coneat submodules and neat submodules coincide (see, [10]). A ring $R$ is called a right $N$-ring if every maximal right ideal of $R$ is finitely generated.

**Remark 3.1.** An $R$-module $M$ is said to be FP-injective or absolutely pure if it is pure in every extension, i.e., $\mathrm{Ext}^1(N, M) = 0$ for each finitely presented $R$-module $N$. If $R$ is a right $N$-ring, then it is easy to see that every pure submodule is neat. So that, in this case, any flat (resp. FP-injective) module is neat-flat (resp. m-injective). An $R$-module $M$ is said to be pure-injective if $M$ is injective relative to all pure exact sequences. The character module $M^+$ of an $R$-module $M$ is pure injective left $R$-module, and every $R$-module $M$ is a pure submodule of the pure injective $R$-module $M^{++}$ (see [9, Proposition 5.3.7]).

The following result will be used in the sequel.

**Theorem 3.2** ([3, Theorem 1]). *The following statements are equivalent*:

(1) *$R$ is a right coherent ring*;
(2) *$M_R$ is FP-injective if and only if $M^+$ is a flat module*;
(3) *$M_R$ is FP-injective if and only if $M^{++}$ is an injective right $R$-module*;
(4) *${}_RM$ is flat if and only if $M^{++}$ is a flat left $R$-module.*

**Definition 3.3.** An $R$-module $M$ is called *max-flat* if $\mathrm{Tor}^1_R(M, R/I) = 0$ for every maximal left ideal $I$ of $R$ (see [26]).

Note that an $R$-module $M$ is max-flat if and only if $M^+$ is $m$-injective by the standard isomorphism $\mathrm{Ext}^1(S, M^+) \cong \mathrm{Tor}_1(M, S)^+$, for all simple left $R$-module $S$.

Using the similar arguments of [26, Theorem 4.5], one can prove the following lemma. The proof is omitted.

**Lemma 3.4.** *Let $R$ be a right $N$-ring. The following statements hold*:

(1) *An $R$-module $M$ is m-injective if and only if $M^+$ is max-flat*;

(2) *An R-module M is m-injective if and only if* $M^{++}$ *is m-injective*;
(3) *An R-module M is a max-flat left R-module if and only if* $M^{++}$ *is a max-flat left R-module.*

**Proposition 3.5.** *Assume that every neat-flat R-module is flat. Then the following statements hold*:

(1) *Every m-injective R-module is FP-injective*;
(2) *For every left R-module M, M is max-flat if and only if M is flat.*

*Proof.* (1) Let $M$ be an $m$-injective $R$-module. By [6, Theorem 3], $\mathrm{Soc}(E(M)/M) = 0$, and so $E(M)/M$ is a neat-flat $R$-module. Then $E(M)/M$ is flat by our hypothesis. Hence $M$ is a pure submodule of $E(M)$, and so $M$ is an FP-injective module.

(2) Assume $M$ is a max-flat left $R$-module. Then $M^+$ is $m$-injective, and so it is FP-injective by (1). But $M^+$ pure injective by [9, Proposition 5.3.7], so $M^+$ is injective. Then $M$ is flat by [21, Theorem 3.52]. The converse statement is clear. □

**Proposition 3.6.** *Let R be a ring. Consider the following statements*:

(1) *R is a right N-ring and every neat-flat R-module is flat*;
(2) *An R-module M is m-injective if and only if* $M^+$ *is flat*;
(3) *An R-module M is m-injective if and only if M is FP-injective, and R is right coherent.*

*Then* (1) $\Rightarrow$ (2) $\Leftrightarrow$ (3).

*Proof.* (1) $\Rightarrow$ (3) By Proposition 3.5(1), every $m$-injective $R$-module is FP-injective. On the other hand, every FP-injective $R$-module is $m$-injective since every simple $R$-module is finitely presented by (1). Then, for every $R$-module $M$, $M$ is FP-injective if and only if $M$ is $m$-injective, if and only if $M^+$ is max-flat by Theorem 3.4(2), if and only if $M^+$ is a flat module by Proposition 3.5(2). Hence $R$ is a right coherent ring by [3, Theorem 1]. This proves (3).

(2) $\Rightarrow$ (3) Let $M$ be a left $R$-module. We claim that, $M$ is a flat $R$-module if and only if $M^{++}$ is a flat module. If $M$ is flat, then $M^+$ is injective by [21, Theorem 3.52], and so $M^{++}$ is flat left $R$-module by (2). Conversely, if $M^{++}$ is a flat module, then $M$ is flat since $M$ is a pure submodule of $M^{++}$ by [9, Proof of Proposition 5.3.9.], and flat modules are closed under pure submodules (see, [16, Corollary 4.86]). So $R$ is a right coherent ring by Theorem 3.2. The last part of (3) follows by (2) and Theorem 3.2 again.

(3) $\Rightarrow$ (2) By Theorem 3.2. □

**Proposition 3.7.** *A finite direct product of left C-rings is also a left C-ring.*

*Proof.* Assume $R$ is a finite direct product of the left $C$-rings $R_1, R_2 \ldots R_n$. We will show that $\mathrm{Soc}(R/I) \neq 0$ for each essential left ideal $I$ of $R$. By assumption, $I = I_1 \times I_2 \times \cdots \times I_n$, where $I_i \leq R_i$ for $i = 1, 2, \ldots, n$. Since $I$ is essential in $R$, $I_i$ is essential in $R_i$ for $i = 1, 2, \ldots, n$. Then $\mathrm{Soc}(R_i/I_i) \neq 0$ for $i = 1, 2, \ldots, n$. $\mathrm{Soc}(R/I) \cong \prod_i^n \mathrm{Soc}(R_i/I_i) \neq 0$, as desired. □

Set $Sa(M) := \Sigma_{M_i \in \Lambda} M_i$, where $\Lambda$ is the class of all semiartinian submodules $M_i$ of $M$. Then $M/Sa(M)$ is neat-flat for each $R$-module $M$, because $\mathrm{Soc}(M/Sa(M)) = 0$ by [15, pp. 238].

Note that (1) two-sided hereditary Noetherian rings are $C$-ring by [4, 10.15(3)], and (2) noetherian semiartinian rings are artinian by [23, Proposition 3.1].

**Remark 3.8.** Let $R$ be a ring and $e$ be a central idempotent in $R$. Then for a right $R$-module $M$ one has, $M = Me \oplus M(1-e)$. It can be easily verified that, $M$ is a neat-flat (flat) $R$-module if and only if $Me$ is a neat-flat (flat) $eR$-module and $M(1-e)$ is a neat-flat (flat) $(1-e)R$-module.

**Theorem 3.9.** *Let $R$ be a commutative Noetherian ring. The following statements are equivalent*:

(1) *Every neat-flat module is flat*;
(2) *Every absolutely coneat module is FP-injective*;
(3) *$R \cong A \times B$, wherein $A$ is $QF$-ring and $B$ is hereditary.*

*Proof.* (1) $\Leftrightarrow$ (2) By [2, Lemma 4.4].

(1) $\Rightarrow$ (3) By the assumption, $R/Sa(R)$ is projective and $Sa(R)$ is direct summand of $R$, i.e. $R \cong A \times B$, where $A = Sa(R)$ is artinian, and $\mathrm{Soc}(B) = 0$ as $\mathrm{Soc}(R) \leq Sa(R)$. By Remark 3.8, we can assume $R$ is artinian or $\mathrm{Soc}(R) = 0$. In the former case, every neat-flat module is projective by the assumption, and hence $R$ is a $QF$-ring by Theorem 2.10 and [19, Theorem 4.4]. In the later case, let $I$ be an ideal of $R$. Since $\mathrm{Soc}(R) = 0$, we have $\mathrm{Soc}(I) = 0$. Then, $I$ is flat by (1) and Corollary 2.5. But $R$ is Noetherian, and so $I$ is finitely generated. Therefore, $I$ is projective, and so $R$ is hereditary.

(3) $\Rightarrow$ (1) Assume that $R \cong A \times B$, wherein $A$ is $QF$-ring and $B$ is hereditary. Let $M$ be a neat-flat $R$-module. Since $M = MA \oplus MB$, $MA$ is a neat-flat $A$-module and $MB$ is a neat-flat $B$-module, by Remark 3.8. Then $MA$ is a projective $A$-module by Theorem 2.10, and $MB$ is a flat $B$-module by Corollary 2.8 and [12, Proposition 2.3]. Therefore, $M$ is a flat $R$-module. □

Recall that an $R$-module $M$ is said to be weakly-injective if $M$ is coclosed in every extension. $M$ is weakly-injective if and only if $M$ is coclosed in its injective hull (see [27]). Clearly, weakly-injective modules are absolutely coneat.

**Theorem 3.10.** *Let $R$ be a commutative noetherian ring. The following statements are equivalent*:

(1) *Every weakly-flat module is weakly-injective*;
(2) *Every weakly-injective module is weakly-flat*;
(3) *Every neat-flat module is absolutely coneat*;
(4) *Every absolutely coneat module is neat-flat*;
(5) *Every neat-flat module is weakly-injective*;
(6) *Every absolutely coneat module is weakly-flat*;
(7) *$R \cong A \times B$, wherein $A$ is $QF$-ring and $B$ is artinian with $J^2(B) = 0$.*

*Proof.* (1) ⇔ (2) ⇔ (7) By [28, Satz 3.8].

(5) ⇒ (3) and (6) ⇒ (4) are clear.

(3) ⇒ (4) Let $M$ be an absolutely coneat $R$-module. Then $M^+$ is neat-flat by [2, Proposition 4.3]. By (3), $M^+$ is absolutely coneat. Again by [2, Proposition 4.3], $M^{++}$ is neat-flat. Since $M$ is a pure submodule of $M^{++}$, $M$ is neat-flat by Proposition 2.6.

(4) ⇒ (3) Let $M$ be a neat-flat $R$-module. Then $M^+$ is absolutely coneat by [2, Proposition 4.3]. By (4), $M^+$ is neat-flat. Again by [2, Proposition 4.3], $M^{++}$ is absolutely coneat. Since $M$ is a pure submodule of $M^{++}$, $M$ is absolutely coneat by [2, Proposition 3.6].

(7) ⇒ (5) A finite direct product of $C$-rings is also a left $C$-ring by Proposition 3.7, and so $R$ is a $C$-ring. Then neat-flat $R$-modules are weakly-flat and, by [28, Satz 3.8], neat-flat $R$-modules are weakly-injective.

(3) ⇒ (7) First we shall prove that, every finitely generated weakly-flat $R$-module is weakly-injective. Let $N$ be a finitely generated weakly-flat $R$-module and $N \leq M$ any extension of $N$. Then $N$ is neat-flat, and absolutely coneat by (3). Then $NI = N \cap MI$ for each maximal ideal $I$ of $R$ by [10]. Since $N$ is finitely generated, it is coatomic (i.e., every submodule $U \lneqq N$ lies in a maximal submodule of $N$). Hence $N$ is coclosed in $M$ by [27, Lemma A.3(b)]. Then $N$ is weakly-injective.

The rest of the proof follows as in proof of ($i' \Rightarrow iii$) of Satz 3.8 in [28].

(4) ⇒ (6) By the equivalence of (4) ⇔ (7), $R \cong A \times B$, wherein $A$ is a $QF$-ring and $B$ is artinian with $J^2(R) = 0$. Now, $R$ is a $C$-ring by Proposition 3.7. Then neat-flat $R$-modules are weakly-flat. Therefore, the claim follows by (4). □

## 4. LOCALIZATION OF NEAT-FLAT MODULES

In this section, we shall consider localization of neat exact sequences and neat-flat modules on commutative $N$-rings.

For an $R$-module $M$ and a prime ideal $P$ of a commutative ring $R$, as usual, $M_P$ will be denote the localization of $M$ at $P$. The elements of $M_P$ are of the form $\frac{m}{s}$, where $m \in M$ and $s \in R\backslash P$. $M_P$ turns out to be an $R_P$-module with multiplication $\frac{r}{s}.\frac{m}{s'} = \frac{rm}{ss'}$, where $\frac{r}{s} \in R_P$, $\frac{m}{s'} \in M_P$.

A submodule $A$ of $B$ is neat in $B$ if and only if the following hold: if for $b \in B$ and for a maximal ideal $P$, we have $Pb \leq A$, then there is an element $a \in A$ such that $P(b - a) = 0$, (see [10, Lemma 2.1]).

We can also rephrase the definition of neat submodule in terms of systems of equations to make the resemblance to purity more transparent: if the maximal ideal $P$ is generated by the elements $r_i$ ($i \in I$), then we consider the system of equations

$$r_i x = a_i \in A, \ (i \in I)$$

with the single unknown $x$ and constants in $A$.

**Lemma 4.1** ([10, Lemma 2.2]). *A is neat in B if and only if such systems are solvable in A, whenever they are solvable in B.*

Let $R$ be a commutative ring and $M$ a finitely presented $R$-module. It is well known that $M$ is projective if and only if $M_P$ is a free $R_P$-module for each prime ideal $P$ of $R$, if and only if $M_P$ is a free $R_P$-module for each maximal ideal $P$ of $R$.

**Lemma 4.2.** *Let $R$ be a commutative $N$-ring. Then, a short exact sequence $0 \to A \to B \to C \to 0$ is neat exact if and only if $0 \to A_P \to B_P \to C_P \to 0$ is neat exact for each maximal ideal $P$ of $R$.*

*Proof.* ($\Rightarrow$) Assume that $0 \to A \xrightarrow{f} B \to C \to 0$ is a neat exact sequence of $R$-modules and $P$ is a maximal ideal of $R$. We show that the exact sequence

$$0 \to A_P \xrightarrow{f_P} B_P \to C_P \to 0$$

is neat exact of $R_P$-modules. Assume that $I$ is an index set, and

$$\frac{r_i}{s_i}x = \frac{f(a_i)}{s_i'} \in f_P(A_P), \qquad r_i \in R_P,\, s_i,\, s_i' \in R\backslash P,\, a_i \in A,\, i \in I$$

is a system of equations which is solvable in $B_P$, i.e., $\frac{r_i}{s_i}\frac{b}{l} = \frac{f(a_i)}{s_i'}$ for some $b \in B$, $l \in R\backslash P$. Thus for each $i \in I$, there exists an element $t_i \in R\backslash P$ such that $t_i r_i s_i' b = t_i s_i l f(a_i) \in f(A)$. Now, consider the system of equations $t_i r_i s_i' x = t_i s_i l f(a_i) \in f(A)$ which is solvable in $B$. Since $f(A)$ is a neat submodule of $B$, by Lemma 4.1, there exists an $f(a) \in f(A)$ such that $t_i r_i s_i' f(a) = t_i s_i l f(a_i)$ for each $i \in I$. Thus $\frac{r_i}{s_i}\frac{f(a)}{l} = \frac{f(a_i)}{s_i'}$, i.e., the system of equations $\frac{r_i}{s_i}x = \frac{f(a_i)}{s_i'}$ is solvable in $f_P(A_P)$. Therefore, $f_P(A_P)$ is a neat submodule of $B_P$ by Lemma 4.1.

($\Leftarrow$) Assume that $0 \to A \to B \to C \to 0$ is not a neat-exact sequence of $R$-modules but $0 \to A_P \to B_P \to C_P \to 0$ is neat exact for each maximal ideal $P$ of $R$. Then there is a simple $R$-module $S = R/P$ where $P$ is maximal ideal of $R$ such that $\mathrm{Hom}(S, B) \to \mathrm{Hom}(S, C)$ is not surjective. By the hypothesis, the natural homomorphism

$$\mathrm{Hom}_{R_P}(S_P, B_P) \to \mathrm{Hom}_{R_P}(S_P, C_P)$$

is an epimorphism. Since $S$ is finitely presented, we have the commutative diagram

$$\begin{array}{ccccc} \mathrm{Hom}_{R_P}(S_P, B_P) & \longrightarrow & \mathrm{Hom}_{R_P}(S_P, C_P) & \longrightarrow & 0 \qquad (*) \\ \downarrow \cong & & \downarrow \cong & & \\ \mathrm{Hom}_R(S, B)_P & \longrightarrow & \mathrm{Hom}_R(S, C)_P & \longrightarrow & 0 \qquad (**) \end{array}$$

by [21, Lemma 4.87]. Since the $(*)$ row is exact, the $(**)$ row is also exact.

Note that for a maximal ideal $Q \neq P$, $S_Q = R_Q \otimes_R S = 0$. Therefore, $\mathrm{Hom}_R(S, B)_Q = \mathrm{Hom}_R(S, C)_Q = 0$. Then $\mathrm{Hom}_R(S, B)_P \to \mathrm{Hom}_R(S, C)_P$ is an epimorphism for every maximal ideal $P$. Thus, by [21, Lemma 4.90], $\mathrm{Hom}_R(S, B) \to \mathrm{Hom}_R(S, C)$ is an epimorphism. This contradict with our assumption, and hence $0 \to A \to B \to C \to 0$ is a neat exact sequence of $R$-modules. □

**Corollary 4.3.** *Let $R$ be a commutative $N$-ring. A module $M$ is a neat-flat $R$-module if and only if, for all maximal ideals $P$ of $R$, $M_P$ is a neat-flat $R_P$-module.*

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