

Nondegeneracy of the ground state for nonrelativistic Lee model

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In the present work, we first briefly sketch the construction of the nonrelativistic Lee model on Riemannian manifolds, introduced in our previous works. In this approach, the renormalized resolvent of the system is expressed in terms of a well-defined operator, called the principal operator, so as to obtain a finite formulation. Then, we show that the ground state of the nonrelativistic Lee model on compact Riemannian manifolds is nondegenerate using the explicit expression of the principal operator that we obtained. This is achieved by combining heat kernel methods with positivity improving semi-group approach and then applying these tools directly to the principal operator, rather than the Hamiltonian, without using cut-offs. © 2014 AIP Publishing LLC. [<http://dx.doi.org/10.1063/1.4892763>]

I. INTRODUCTION

The Lee model¹ is a well-known nontrivial toy model for describing the interaction of two relativistic chargeless spinless fermions, called V and N (e.g., “nucleons”), with a scalar chargeless boson, called θ (e.g., “pion”), through the only allowable process:

$$V \rightleftharpoons N + \theta, \quad (1)$$

where the energies of the nucleons are assumed to be independent of their momenta (i.e., the recoil of the nucleon is neglected). There are two conserved quantities Q_1 and Q_2 , which correspond to the total number of fermions and the difference between the number of N and θ particles, respectively. Due to absence of antiparticles in the model (crossing symmetry is violated), and the above conserved quantities, the Fock space splits into a direct sum of invariant Hilbert spaces spanned by the restricted basis vectors labeled by the eigenvalues of Q_1 and Q_2 . This makes the model solvable since once Q_1 and Q_2 are fixed only a finite number of particles couples to each other. In addition to that, it is one of the first models where the coupling constant, mass, and wave function renormalization can be carried out in a nonperturbative way. An extensive discussion of this model can be found in the textbook of Schweber.²

Following the idea of the original Lee model, its several variations have been studied. One simplified version is obtained by regarding the fermions so heavy that their momenta are completely neglected and the bosons are assumed to be nonrelativistic (i.e., their energy momentum relation satisfies $E = \frac{p^2}{2m} + m$ in the nonrelativistic approximation). In this model, only an additive renormalization of the mass difference of the localized (or static) fermion states is sufficient and performed in a closed form. This static version of the Lee model has been discussed for the first two boson sectors in Henley and Thirring’s book³ in detail. An extension of this model onto the Riemannian

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manifolds has been discussed in Refs. 4 and 5 and the present work is a continuation of those works for understanding the ground state structure of the system in depth.

In the literature, there is a great deal of work devoted to some pathologies of Lee model, which appears when the renormalized coupling constant is greater than some critical value, and their possible resolutions are worked out. Essentially, this happens because of a new state of the V -particle having an energy that is below the mass of the “normal” V -particle.^{6–8} However, our interest here is to focus on the nonperturbative nature of the model as such and the issue of introducing a physical V -particle will not be addressed.

Our approach in the present work largely follows the unpublished work by Rajeev.⁹ There, he introduced a new nonperturbative formulation of renormalization for some simple nonrelativistic quantum mechanical and quantum field theoretical models. In his approach, the basic idea is to work out the resolvent of the Hamiltonian in the Fock space formalism and thereby identifying the divergent part explicitly and nonperturbatively. After removing the divergent part of the problem via this new renormalization procedure, a finite formulation of the model is accomplished. The resolvent contains the inverse of a new operator, called principal operator $\Phi(E)$. In this way, the whole renormalization procedure is carried out in the resolvent formalism without worrying about the self-adjoint extension of the Hamiltonian and its domain issues directly, which is essential for describing point interactions rigorously.¹⁰ Since the resolvent includes all the information about the spectrum of the problem, the bound states can then be found from its poles. Point interactions in quantum mechanics, nonrelativistic Lee model, and a model where the bosons are interacting through the two-body point delta potentials have been studied from this point of view in Ref. 9.

Following the ideas developed in Ref. 9, we have extended the nonrelativistic Lee model defined on flat spaces onto Riemannian manifolds with the help of heat kernel techniques.^{4,5} At the end, the resolvent is expressed in terms of the principal operator without an explicit expression of the renormalized Hamiltonian. In other words, we have obtained the analog of Krein’s formula for the resolvent. Furthermore, we have proved in Ref. 11 that there exists a densely defined self-adjoint Hamiltonian operator corresponding to the resolvent we found and that the ground state energy is bounded from below. The principal operator approach for Lee model on Riemannian manifolds can also be extended to interacting bosons on two dimensional manifolds.¹²

In the study of the bound states in quantum mechanics and quantum field theory, one of the main problems is to prove the uniqueness of the ground state. In general, the proof is not so trivial (non-uniqueness could also lead to interesting physics). Nevertheless, for sufficiently regular potentials, the proof is given by Courant and Hilbert¹³ by *implicitly* assuming all the regularity conditions on the potential in the context of Sturm-Liouville systems. A modern and a more rigorous treatment of the nondegeneracy for the ground state has actually been first given by Glimm and Jaffe¹⁴ in the context of quantum field theory for self-interacting bosons ($\lambda(\phi^4)_2$ interaction) by using the infinite dimensional extension of Perron-Frobenius theorem¹⁵ and positivity arguments from semi-group theory.¹⁶ Then, the applicability of these techniques in proving the nondegeneracy of the ground states to the Schrödinger operators has been developed by Simon and Höegh-Krohn¹⁷ for some regular class of potentials and this new modern version of the proof is also given in the textbooks.^{16,18}

The problem is mathematically formulated in the following way: For a given self-adjoint Hamiltonian H describing the dynamics of the quantum mechanical or quantum field theoretical system, one must prove that Hamiltonian is bounded from below and the infimum of the spectrum is an eigenvalue. The eigenvector corresponding to that eigenvalue is then called the ground state. This completes the proof for the existence of the ground state. Then, one must prove that the eigenspace corresponding to that eigenvalue is one dimensional, i.e., the ground state is nondegenerate (unique up to complex multiples). The proof is essentially based on the Hilbert space generalization of the Perron-Frobenius theorem developed for nonnegative matrices. If we have a quantum mechanical model, then the Hilbert space is L^2 over some measure space. If we have a purely bosonic quantum field theoretical model, then the positivity arguments are most naturally introduced in the tensor product space obtained from the coordinate representation, so called Q space.^{16,19} In this representation, the concept of positivity becomes clear, in the present work, we will also be using the natural

L^2 -space on the manifold and the symmetrized tensor products thereof. For positivity, in general it is easier to work with a bounded operator e^{-tH} rather than H itself. The largest eigenvalue of e^{-tH} will become e^{-tE_0} if E_0 is the ground state energy for H . Then, assuming the Hamiltonian operator is self-adjoint and bounded from below and ground state corresponds to an eigenvector, the positivity improving property of the operator e^{-tH} for all $t > 0$ (which will be defined in Sec. IV) is equivalent to the statement that the ground state energy E_0 of a self-adjoint Hamiltonian H , is non-degenerate (which is also equivalent to the associated eigenvector being strictly positive).

After the inspiring work of Glimm and Jaffe,¹⁴ Gross²⁰ and Faris²¹ extended it to the models involving fermions and bosons in a more abstract framework where the Hilbert space is not a standard L^2 space. Since then, the nondegeneracy of the ground state for several quantum field theory models, such as polaron models, spin-boson models, the van Hove model, the Wigner-Weisskopf model, and non-relativistic quantum electrodynamics have been proved and discussed extensively from several point of views.²²⁻²⁹

In the framework of our formulation for finitely many N point delta interactions in two and three dimensional Riemannian manifolds,³⁰ we have proved that the resolvent after the renormalization procedure is given by a kind of Krein's formula and expressed in terms of an $N \times N$ matrix Φ , called principal matrix. The matrix $\Phi(E)$ includes all the information about the bound state spectrum, the values of which are found by solving the equation $\Phi(E)A = 0$. In other words, the zero modes of the principal matrix correspond to the bound state spectrum of the problem. Since we do not have an expression for the renormalized Hamiltonian, the proof for the nondegeneracy of the ground state cannot be given in the same way as developed for the regular potentials.¹⁶ Nevertheless, the principal matrix allows us to prove the uniqueness of the ground state, via the Perron-Frobenius theorem applied directly to the principal matrix. Since the ground state eigenvector is expressed in terms of the eigenvector of the principal operator associated to its minimum eigenvalue and their degeneracies are equal, this proves the claim.

In this work, we will generalize the arguments developed for the point interactions to prove the nondegeneracy of the ground state of the Lee model defined on Riemannian manifolds, the construction of which were already established in our previous studies^{4,5} by extending the ideas introduced in Ref. 30. Although the basic idea in proving the nondegeneracy of the ground state in this model is similar to the one which we developed for point interactions, the proof requires the use of the positivity arguments. The main difference between our method and the one given for other field theory models¹⁶ is that we have no formal expression of the Hamiltonian after our nonperturbative renormalization procedure. Hence, we cannot apply the positivity arguments for the semi-group generated by the Hamiltonian operator, instead we will directly use them for the principal operator using the results given in Ref. 11 and then we will be able to prove the nondegeneracy of the ground state without using any cut-off.

The paper is organized as follows. In Sec. II, we will first present a very brief overview of the basic results for the construction of the nonrelativistic Lee model on $D = 2, 3$ dimensional Riemannian manifolds and then give a more detailed analysis on the ground state of the problem. Finally, we prove that the ground state wave function is strictly positive, so that ground state is nondegenerate for compact Riemannian manifolds with Ricci curvature bounded below. The proof for noncompact manifolds is technically much more challenging and exceeds the present skills of the authors. We comment that compactness can be thought of as a kind of infrared regularization, from this point of view, such a restriction should not be an essential handicap in understanding *only the ultraviolet complications* of these problems.

In this paper, we will use the notations $\langle \cdot | \cdot \rangle$ or $\langle \cdot, \cdot \rangle$ to denote the inner product and $\| | \cdot \rangle \|$ or $\| \cdot \|$ to denote the associated norms.

II. SUMMARY OF THE CONSTRUCTION OF NONRELATIVISTIC LEE MODEL ON MANIFOLDS

In order to make our discussion reasonably self-contained, first we shortly give the important results of our approach for the renormalization of the model presented in Refs. 4 and 5 and add some new comments about the ground state of the problem.

The regularized Hamiltonian in a D (2 or 3) dimensional Riemannian manifold (\mathcal{M}, g) is formally given by

$$H^\epsilon = H_0 + H_{I,\epsilon} = 1 \otimes \int_{\mathcal{M}} \phi_g^\dagger(x) \left(-\frac{1}{2m} \nabla_g^2 + m \right) \phi_g(x) d_g^D x + \mu(\epsilon) \left(\frac{1 - \sigma_3}{2} \right) \otimes 1 + \lambda \int_{\mathcal{M}} K_\epsilon(x, a) [\sigma_- \otimes \phi_g(x) + \sigma_+ \otimes \phi_g^\dagger(x)] d_g^D x, \quad (2)$$

where ϵ is a cut-off parameter (we use the units such that $\hbar = c = 1$), λ is the coupling constant, and ∇_g^2 is the Laplace-Beltrami operator. Also, $d_g^D x$ is the Riemannian volume element and x, y refer to points on the manifold \mathcal{M} . The Hilbert space is $\mathbb{C} \otimes \mathcal{F}_b$, where \mathcal{F}_b is the bosonic Fock space. The function $K_\epsilon(x, a)$ is the heat kernel on a Riemannian manifold with metric structure g and it converges to the Dirac delta function $\delta_g(x, a)$ around the point a on the manifold as we take the limit $\epsilon \rightarrow 0^+$.³¹ The creation and annihilation operators $\phi_g^\dagger(x), \phi_g(x)$ defined on (\mathcal{M}, g) obey the following canonical commutation relations:

$$[\phi_g(x), \phi_g^\dagger(y)] = \delta_g(x, y). \quad (3)$$

Also, the matrices σ_\pm given in the Hamiltonian are the standard Pauli spin-flip matrices. Similar to the flat space case, the coefficient $\mu(\epsilon)$ denotes the bare mass difference between the V particle (neutron) and the N particle (proton). The conserved charge Q_2 is

$$-\left(\frac{1 + \sigma_3}{2} \right) \otimes 1 + 1 \otimes \int_{\mathcal{M}} \phi_g^\dagger(x) \phi_g(x) d_g^D x, \quad (4)$$

which makes the model solvable. Since $\mathbb{C} \otimes \mathcal{F}_b = \mathcal{F}_b \oplus \mathcal{F}_b$, we can represent the regularized Hamiltonian as a 2×2 block matrix:

$$H^\epsilon - E = \begin{pmatrix} H_0 - E & \lambda \int_{\mathcal{M}} K_\epsilon(x, a) \phi_g^\dagger(x) d_g^D x \\ \lambda \int_{\mathcal{M}} K_\epsilon(x, a) \phi_g(x) d_g^D x & H_0 - E + \mu(\epsilon) \end{pmatrix}. \quad (5)$$

For given $Q_1 = 1$ and $Q_2 = n$, the above Hamiltonian acts on the sector $\mathcal{F}_{n+1} \oplus \mathcal{F}_n$, where \mathcal{F}_n stands for the symmetrized n tensor product of the one particle Hilbert spaces. If we suppose that the regularized resolvent is of the following form,

$$R^\epsilon(E) = (H^\epsilon - E)^{-1} = \begin{pmatrix} a_\epsilon & b_\epsilon^\dagger \\ b_\epsilon & d_\epsilon \end{pmatrix}^{-1} = \begin{pmatrix} \alpha_\epsilon & \beta_\epsilon^\dagger \\ \beta_\epsilon & \delta_\epsilon \end{pmatrix}, \quad (6)$$

one can find $\alpha_\epsilon, \beta_\epsilon$, and δ_ϵ in terms of a_ϵ, b_ϵ , and d_ϵ given in Eq. (5) in two apparently different but equivalent ways (see the Appendix in Ref. 9 for the explicit computation),

$$\begin{aligned} \alpha_\epsilon &= (H_0 - E)^{-1} + (H_0 - E)^{-1} b_\epsilon^\dagger \Phi_\epsilon^{-1}(E) b_\epsilon (H_0 - E)^{-1}, \\ \beta_\epsilon &= -\Phi_\epsilon^{-1}(E) b_\epsilon (H_0 - E)^{-1}, \\ \delta_\epsilon &= \Phi_\epsilon^{-1}(E), \\ b_\epsilon &= \lambda \int_{\mathcal{M}} K_\epsilon(x, a) \phi_g(x) d_g^D x, \end{aligned} \quad (7)$$

where

$$\Phi_\epsilon(E) = H_0 - E + \mu(\epsilon) - \lambda^2 \int_{\mathcal{M}^2} K_\epsilon(x, a) K_\epsilon(y, a) \phi_g(x) (H_0 - E)^{-1} \phi_g^\dagger(y) d_g^D x d_g^D y, \quad (8)$$

called regularized *principal operator*. By using the eigenfunction expansion of the creation and the annihilation operators with the commutation relations (3), we find⁴

$$(H_0 - E)^{-1} \phi_g^\dagger(y) = \int_{\mathcal{M}} \int_0^\infty \phi_g^\dagger(y) e^{-t(H_0 - E + m)} K_t(y, y') dt d_g^D y', \quad (9)$$

and

$$\phi_g(x)(H_0 - E)^{-1} = \int_{\mathcal{M}} \int_0^\infty e^{-t(H_0 - E + m)} K_t(x, x') \phi_g(x') dt d_g^D x'. \quad (10)$$

If we normal order the principal operator (8) using the above relations (9) and (10), its singular structure becomes transparent,

$$\begin{aligned} \Phi_\epsilon(E) = H_0 - E - \lambda^2 \int_{\epsilon/2}^\infty \int_{\mathcal{M}^2} K_t(x, a) K_t(y, a) \phi_g^\dagger(x) e^{-(t-\epsilon/2)(H_0+2m-E)} \phi_g(y) d_g^D x d_g^D y dt \\ + \mu(\epsilon) - \lambda^2 \int_\epsilon^\infty K_t(a, a) e^{-(t-\epsilon)(H_0+m-E)} dt \end{aligned} \quad (11)$$

(we warn the reader that there is a typo in the corresponding equation of this result in Refs. 5 and 11, which does not change the final expression of the principal operator). It is now easy to see that the last term in Eq. (11) is divergent due to the short time asymptotic expansion of the diagonal heat kernel given by

$$K_t(x, x) \sim \frac{1}{(4\pi t/2m)^{D/2}} \sum_{k=0}^\infty u_k(x, x) (t/2m)^k, \quad (12)$$

for every x in any D dimensional Riemannian manifold without boundary³² and the functions $u_k(x, x)$ are scalar polynomials in the curvature tensor of the manifold and its covariant derivatives at x . This suggests that by choosing $\mu(\epsilon)$,

$$\mu(\epsilon) = \mu + \lambda^2 \int_\epsilon^\infty K_t(a, a) e^{-t(m-\mu)} dt, \quad (13)$$

where μ being the experimentally measured bound state energy of the composite state which consists of a boson and the attractive heavy neutron at the center a , and substituting it in (11) and then taking the limit $\epsilon \rightarrow 0^+$, we get the following finite expression:

$$\begin{aligned} \Phi(E) = H_0 - E + \mu + \lambda^2 \int_0^\infty K_t(a, a) [e^{-t(m-\mu)} - e^{-t(H_0+m-E)}] dt \\ - \lambda^2 \int_0^\infty \int_{\mathcal{M}^2} K_t(x, a) K_t(y, a) \phi_g^\dagger(x) e^{-t(H_0+2m-E)} \phi_g(y) d_g^D x d_g^D y dt. \end{aligned} \quad (14)$$

Therefore, we have a well-defined explicit formula for the full resolvent of the Hamiltonian in terms of the inverse of the principal operator $\Phi(E)$ and the free resolvent, namely,

$$R(E) = \begin{pmatrix} \alpha(E) & \beta^\dagger(E) \\ \beta(E) & \delta(E) \end{pmatrix}, \quad (15)$$

where

$$\begin{aligned} \alpha(E) &= (H_0 - E)^{-1} + (H_0 - E)^{-1} \phi^\dagger(a) \Phi^{-1}(E) \phi(a) (H_0 - E)^{-1}, \\ \beta(E) &= -\Phi^{-1}(E) \phi(a) (H_0 - E)^{-1}, \\ \delta(E) &= \Phi^{-1}(E). \end{aligned} \quad (16)$$

III. GROUND STATE

For compact and connected Riemannian manifolds, the spectral theorem for Laplace-Beltrami operator³³ states that:

There exists a complete orthonormal system of C^∞ eigenfunctions $f_\sigma(x)$ in L^2 ,

$$-\nabla_g^2 f_\sigma(x) = \sigma f_\sigma(x), \quad (17)$$

with purely discrete spectrum $\{\sigma_l\} = \{0 = \sigma_0 \leq \sigma_1 \leq \sigma_2 \leq \dots \leq \sigma_N \leq \dots\}$, where σ_l tending to infinity as $l \rightarrow \infty$ and each eigenvalue has finite multiplicity.

This implies that the spectrum of the second quantized free Hamiltonian H_0 in the given sector must also be discrete. Since our main interest here is in the bound state spectrum, we will require from now on that $m > \mu$, which guarantees that heavy particles do not decay. Moreover, although we have no explicit expression for the renormalized Hamiltonian after our renormalization procedure, we have proved in Ref. 11 that there exists a densely defined self-adjoint Hamiltonian operator associated with the renormalized resolvent. Hence, the spectrum of the Hamiltonian must be real.

The poles of the resolvent $R(E)$ in (15) give the bound state spectrum of the Hamiltonian operator. For compact manifolds, it turns out that the bound state of the system in the given sector must only come from the roots of the equation,

$$\Phi(E)|\Psi\rangle = 0, \quad (18)$$

since the free resolvent has no poles in the given sector (e.g., the spectrum of H_0 starts from $(n + 1)m > nm + \mu$ in the $(n + 1)$ boson sector). This is due to the fact that the bound state energy is below $nm + \mu$, which will be shown by a variational ansatz at the end of this section. In other words, the zero modes of the principal operator are responsible for the bound state of the system.

It is well known that the contour integral of the resolvent around any pole of it in the complex E plane is the projection operator \mathbb{P}_k onto the associated eigenspace of the Hamiltonian, given by Riesz integral formula,

$$\mathbb{P}_k = -\frac{1}{2\pi i} \oint_{\Gamma_k} R(E) dE, \quad (19)$$

where Γ_k is a small contour enclosing the isolated eigenvalue E_k .¹⁶ Let us first consider the first sector of the resolvent and choose the contour Γ_0 , enclosing only the ground state energy E_{gr} . Then, the contour integral gives the projection onto the eigenspace associated with $(n + 1)$ boson sector, represented by $|\Psi_0^{(n+1)}\rangle \langle\Psi_0^{(n+1)}|$.

We have proved that the principal operator $\Phi(E)$ is a self-adjoint holomorphic family of type A in the sense of Kato,¹¹ so that we can apply the generalized spectral theorem for its inverse, namely,

$$\Phi^{-1}(E) = \sum_k \frac{1}{\omega_k(E)} |\omega_k(E)\rangle \langle\omega_k(E)|, \quad (20)$$

where $\omega_k(E)$ and $|\omega_k(E)\rangle$ are the eigenvalues and the eigenvectors of the principal operator, respectively. We have removed any possible continuous part of the resolution of our operator, since as we will prove below in Sec. IV, the principal operator $\Phi(E)$ only has a discrete spectrum. Let $\omega_0(E)$ be the minimum eigenvalue of the principal operator for all E by assuming it exists for the moment (which we will prove in Sec. IV). Then, Feynman-Hellman theorem applied to ω_0 implies that (which is valid even for degenerate states) the flow of ω_0 with respect to E is

$$\begin{aligned} \frac{\partial \omega_0}{\partial E} &= \langle\omega_0(E)| \frac{\partial \Phi(E)}{\partial E} |\omega_0(E)\rangle = -\left(1 + \lambda^2 \int_0^\infty t K_t(a, a) \left\| e^{-\frac{t}{2}(H_0 - \mu + E)} |\omega_0(E)\rangle \right\|^2 dt \right. \\ &\quad \left. + \lambda^2 \int_0^\infty t \left\| e^{-\frac{t}{2}(H_0 - \mu + E)} \int_{\mathcal{M}} K_t(x, a) \phi_g(x) |\omega_0(E)\rangle d_g^D x \right\|^2 dt \right). \end{aligned} \quad (21)$$

From this equation, the positivity property of the heat kernel $K_t(x, y) > 0$ for all $x, y \in \mathcal{M}$ and $t > 0$ leads to the following inequality:

$$\frac{\partial \omega_0}{\partial E} < 0. \quad (22)$$

As a consequence of this important fact (22), the ground state energy must correspond to the zero of the minimum eigenvalue $\omega_0(E)$. Expanding $\omega_0(E)$ near the ground state energy E_{gr} ,

$$\omega_0(E) = \omega_0(E_{gr}) + (E - E_{gr}) \frac{\partial \omega_0(E)}{\partial E} \Big|_{E_{gr}} + \dots = (E - E_{gr}) \frac{\partial \omega_0(E)}{\partial E} \Big|_{E_{gr}} + \dots, \quad (23)$$

and using the residue theorem in (19), we obtain

$$|\Psi_0^{(n+1)}\rangle \langle \Psi_0^{(n+1)}| = (H_0 - E_{gr})^{-1} \phi^\dagger(a) \left(-\frac{\partial \omega_0(E)}{\partial E} \Big|_{E_{gr}} \right)^{-1} |\omega_0(E_{gr})\rangle \langle \omega_0(E_{gr})| \phi(a) (H_0 - E_{gr})^{-1}, \quad (24)$$

where

$$|\omega_0(E_{gr})\rangle = \int_{\mathcal{M}^n} \psi_0(x_1, \dots, x_n) |x_1 \cdots x_n\rangle d_g^D x_1 \cdots d_g^D x_n \quad (25)$$

is a generic form of the ground state of the principal operator. Then, by repeated applications of the formula (9) and (10), we can shift all the creation operators $\phi_g^\dagger(x)$ to the leftmost

$$(H_0 - E)^{-1} \phi_g^\dagger(a) \phi_g^\dagger(x_1) \cdots \phi_g^\dagger(x_n) = \int_{\mathcal{M}^{n+1}} \phi_g^\dagger(y_1) \cdots \phi_g^\dagger(y_{n+1}) \int_0^\infty e^{-t(H_0 - E + (n+1)m)} \\ \times K_t(y_1, a) K_t(y_2, x_1) \cdots K_t(y_{n+1}, x_n) dt d_g^D y_1 \cdots d_g^D y_{n+1}, \quad (26)$$

and all the annihilation operators $\phi_g(x)$ to the rightmost

$$\phi_g(a) \phi_g(x_1) \cdots \phi_g(x_n) (H_0 - E)^{-1} = \int_{\mathcal{M}^{n+1}} \int_0^\infty e^{-t(H_0 - E + (n+1)m)} K_t(y_1, a) K_t(y_2, x_1) \cdots K_t(y_{n+1}, x_n) \\ \times \phi_g(y_1) \cdots \phi_g(y_{n+1}) dt d_g^D y_1 \cdots d_g^D y_{n+1} \quad (27)$$

in Eq. (24), so that we can read off the ground state vector $|\Psi_0^{(n+1)}\rangle$ in the $(n + 1)$ boson sector:

$$|\Psi_0^{(n+1)}\rangle = \int_{\mathcal{M}^{n+1}} \Psi_0(y_1, \dots, y_{n+1}) |y_1 \cdots y_{n+1}\rangle d_g^D y_1 \cdots d_g^D y_{n+1} \\ = \int_{\mathcal{M}^{n+1}} \int_{\mathcal{M}^n} \frac{1}{(n+1)} \sum_{\sigma \in (1 \cdots (n+1))} \int_0^\infty e^{-t((n+1)m - E_{gr})} K_t(y_{\sigma(1)}, a) K_t(y_{\sigma(2)}, x_1) \cdots K_t(y_{\sigma(n+1)}, x_n) \\ \times \psi_0(x_1, \dots, x_n) \left(-\frac{\partial \omega_0(E)}{\partial E} \Big|_{E_{gr}} \right)^{-1/2} |y_{\sigma(1)} \cdots y_{\sigma(n+1)}\rangle dt d_g^D y_1 \cdots d_g^D y_{n+1} d_g^D x_1 \cdots d_g^D x_n. \quad (28)$$

Here the sum runs over all cyclic permutations σ of $(123 \dots (n + 1))$ since the wave function Ψ_0 must be symmetric. Similarly, we can compute the ground state in the n boson sector from the residue integral, and obtain

$$|\Psi_0^{(n)}\rangle = \left(-\frac{\partial \omega_0(E)}{\partial E} \Big|_{E_{gr}} \right)^{-1/2} \int_{\mathcal{M}^n} \psi_0(y_1, \dots, y_n) |y_1 \cdots y_n\rangle d_g^D y_1 \cdots d_g^D y_n. \quad (29)$$

Hence, the ground state of the system is given in the following form,

$$|\Psi_0\rangle = \begin{pmatrix} |\Psi_0^{(n+1)}\rangle \\ |\Psi_0^{(n)}\rangle \end{pmatrix}, \quad (30)$$

where $|\Psi_0^{(n+1)}\rangle$ and $|\Psi_0^{(n)}\rangle$ are given in (28) and (29), respectively. Notice that if the right hand side of the contour integral of the resolvent only includes one dimensional projection operators, so is the left hand side, which will be of fundamental importance in our proof for the nondegeneracy of the ground state.

We will now demonstrate that the zero of the minimum eigenvalue of the principal operator $\omega_0(E)$ occurs for a value below $nm + \mu$, hence it is enough to study this operator family for the sector $E \leq nm + \mu$. Let us now make the following variational ansatz,

$$|\omega_{var}\rangle = \frac{1}{V(\mathcal{M})^{n/2}} \frac{1}{\sqrt{n!}} \int_{\mathcal{M}^n} \phi_g^\dagger(y_1) \cdots \phi_g^\dagger(y_n) |0\rangle d_g^D y_1 \cdots d_g^D y_n, \quad (31)$$

and choose $E = E_{var} = nm + \mu$. Here $V(\mathcal{M})$ denotes the volume of the compact Riemannian manifold \mathcal{M} . By the variational principle, the lowest eigenvalue of the principal operator $\Phi(E_{var})$

satisfies the inequality:

$$\omega_0(E_{var}) \leq \langle \omega_{var} | \Phi(E_{var}) | \omega_{var} \rangle. \quad (32)$$

We note that $\langle \omega_{var} | H_0 | \omega_{var} \rangle = nm$ and $e^{-t(H_0+m-E_{var})} | \omega_{var} \rangle = e^{-t(m-\mu)} | \omega_{var} \rangle$. This implies that

$$\omega_0(E_{var}) \leq \langle \omega_{var} | U(E_{var}) | \omega_{var} \rangle. \quad (33)$$

Then, we get

$$\omega_0(E_{var}) \leq -\lambda^2 \int_0^\infty \langle \phi_g(K_t(\cdot, a)) \omega_{var} | e^{-t(H_0+2m-E_{var})} | \phi_g(K_t(\cdot, a)) \omega_{var} \rangle dt, \quad (34)$$

where we have defined the following compact notation, which will be useful in Sec. IV:

$$\phi_g(K_t(\cdot, a)) = \int_{\mathcal{M}} K_t(y, a) \phi_g(y) d^D y. \quad (35)$$

Since

$$\begin{aligned} |\phi_g(K_t(\cdot, a)) \omega_{var}\rangle &= \frac{\sqrt{n}}{V(\mathcal{M})^{n/2}} \frac{1}{\sqrt{(n-1)!}} \int_{\mathcal{M}} K_t(x, a) \int_{\mathcal{M}^{n-1}} \phi_g^\dagger(y_1) \dots \phi_g^\dagger(y_{n-1}) |0\rangle d_g^D y_1 \dots d_g^D y_{n-1} d_g^D x \\ &= \frac{\sqrt{n}}{V(\mathcal{M})^{n/2}} \frac{1}{\sqrt{(n-1)!}} \int_{\mathcal{M}^{n-1}} \phi_g^\dagger(y_1) \dots \phi_g^\dagger(y_{n-1}) |0\rangle d_g^D y_1 \dots d_g^D y_{n-1}, \end{aligned} \quad (36)$$

and the operator $e^{-t(H_0+2m-(nm+\mu))}$ acting on this wave function brings a multiplicative factor $e^{-t(m-\mu)}$, we have

$$\omega_0(E_{var}) \leq \langle \Psi_{var} | U(E_{var}) | \Psi_{var} \rangle = -\frac{n\lambda^2}{(m-\mu)V(\mathcal{M})}, \quad (37)$$

after taking the inner product and integrating over t . This expression is strictly negative, so that in order to find the solution $\omega_0(E_{gr}) = 0$ we must reduce E below E_{var} due to (22). As a result, the ground state energy indeed is below $nm + \mu$ and corresponds to the zero of ω_0 , as claimed.

We have also proved in Refs. 4, 5, and 11 that the Hamiltonian is bounded from below and this lower bound is given by

$$E_* = nm + \mu - (n\lambda^2 C)^{\frac{2}{4-D}}, \quad (38)$$

where C is a positive constant.

IV. POSITIVITY AND NONDEGENERACY OF THE GROUND STATE

In this section we will show that the ground state wave function of the nonrelativistic Lee model defined in two and three dimensional compact Riemannian manifold can be chosen as positive, and as a consequence of this it is nondegenerate. The key idea behind this is to study some positivity properties of the semi-group $e^{-t\Phi}$ generated by the principal operator Φ rather than the Hamiltonian, which is usually the standard method used to prove the nondegeneracy of the ground states for some field theory and quantum models.¹⁶ We know from Sec. III that all the information about the ground state of the system is hidden in the principal operator, that is, the solutions of the zeros of its minimum eigenvalue give the ground state energy. Therefore, it is natural to study the positivity properties of the semi-group $e^{-t\Phi}$ instead of e^{-tH} .

Let us first remind some terminology of the positivity¹⁶ in Hilbert space $\mathcal{H} = L^2(\mathcal{M}, d\mu)$. A function $\psi \in L^2(\mathcal{M}, d\mu)$ is called *positive* if it is nonnegative almost everywhere and is not the zero function ($\psi \geq 0$). It is called *strictly positive* if $\psi > 0$ almost everywhere. A bounded operator A on $L^2(\mathcal{M}, d\mu)$ is called *positivity preserving* if $A\psi$ is positive whenever ψ is positive. In order to show the nondegeneracy of the ground state, we need a slightly stronger positivity property: A is called *positivity improving* if $A\psi$ is strictly positive whenever ψ is positive. A bounded operator A on $L^2(\mathcal{M}, d\mu)$ is positivity improving if and only if $(f, Ag) > 0$ for all positive functions $f, g \in L^2(\mathcal{M}, d\mu)$. We may use this to show that the Laplace-Beltrami operator on a manifold generates a positivity improving semi-group.

The notion of positivity can be extended onto symmetric (bosonic) Fock spaces $\mathcal{F}_b(\mathcal{H}) = \bigoplus_{n=0}^{\infty} \mathcal{S}_n \otimes^n L^2(\mathcal{M}, d\mu)$, that is, if the real-valued functions on one-particle Hilbert spaces $L^2(\mathcal{M}, d\mu)$ are positive, then it implies that the function on the symmetric Fock space, constructed from the one-particle Hilbert spaces is also positive. For a fermionic system, this would not be true due to the minus signs under permutations.

A. Existence of the ground state

In general, if we are given a self-adjoint Hamiltonian H , which is bounded from below, and the infimum of the spectrum is an eigenvalue, then we say that the ground state exists for the given model Hamiltonian H . However, we do not have an explicit expression for the Hamiltonian yet, we have shown that there exists a self-adjoint Hamiltonian associated to the renormalized resolvent and the bound state spectrum is bounded from below.¹¹ It suffices to prove that the infimum of the spectrum must be an eigenvalue for the existence of the ground state. We will prove this as follows: We first prove that the infimum of the spectrum of the principal operator Φ is an eigenvalue, that is,

$$\inf \sigma(\Phi(E)) = \omega_0(E), \quad (39)$$

for all $E \leq nm + \mu$. Then it follows that the infimum of the spectrum of the Hamiltonian is an eigenvalue due to the fact that the first eigenvalue ω_0 of the principal operator is a monotonically decreasing function of E (see Eq. (22)) and the unique solution for the zero of this eigenvalue corresponds to the ground state of the model.

For simplicity, let us decompose the principal operator $\Phi(E)$ into three parts:

$$\Phi(E) = K_0(E) + K_1(E) + U(E), \quad (40)$$

where

$$\begin{aligned} K_0(E) &= H_0 - E + \mu, \\ K_1(E) &= \lambda^2 \int_0^\infty K_t(a, a) (e^{-(m-\mu)t} - e^{-(H_0+m-E)}) dt, \\ U(E) &= -\lambda^2 \int_0^\infty \int_{\mathcal{M}^2} K_t(x, a) K_t(y, a) \phi_g^\dagger(x) e^{-t(H_0+2m-E)} \phi_g(y) d_g^D x d_g^D y dt. \end{aligned} \quad (41)$$

As a consequence of the compactness of the manifold, the essential spectrum of H_0 (or K_0) is empty, that is,

$$\sigma_{ess}(K_0) = \sigma_{ess}(H_0) = \emptyset. \quad (42)$$

Since the kinetic part K_1 is a function of H_0 , we expect that the essential spectrum of $K_0 + K_1$ is also empty. In order to show this explicitly, we will first prove that K_1 is a relatively compact perturbation of the kinetic part H_0 by showing that $(K_0 - z)^{-1} K_1$ for some z in the resolvent set of K_0 is a trace-class operator when raised to a certain power, say $4n$ (every trace class operator is compact³⁴). We first remark that

$$\text{tr}_{\mathcal{F}_n}(e^{-tH_0}) = \left[\text{tr}_{\mathcal{H}} \left(e^{-t(-\frac{1}{2m}\nabla_g^2 + m)} \right) \right]^n = e^{-tnm} \left(\int_{\mathcal{M}} K_t(x, x) d_g^D x \right)^n, \quad (43)$$

where $\text{tr}_{\mathcal{F}_n}$ stands for the trace over the symmetrized n tensor product of one particle Hilbert spaces. For compact manifolds, Ricci curvature is bounded from below, i.e., $\text{Ric}(\cdot, \cdot) \geq -\kappa g(\cdot, \cdot)$, assuming $\kappa \geq 0$ to cover the most general case, the upper bound of the diagonal heat kernel for any $t > 0$ and $x \in \mathcal{M}$ is given by

$$K_t(a, a) \leq \frac{1}{V(\mathcal{M})} + \frac{C}{(t/2m)^{D/2}}, \quad (44)$$

where C depends on κ and the diameter, and the volume of the manifold.³⁵ Then, the operator e^{-tH_0} is trace class, hence a compact operator for all $t \in (0, \infty)$. In our proof these facts will be essential.

Without loss of generality, let $z = 0$ for simplicity. Then using the integral representation of the operator $K_0^{-1} K_1$

$$\int_0^\infty \int_0^1 t K_t(a, a) e^{-tu(H_0+\mu-E)} e^{-tm} du dt, \tag{45}$$

the trace of $4n$ th power of this operator becomes

$$\int_{\mathbb{R}_+^{4n}} \int_0^1 \int_0^1 \dots \int_0^1 t_1 \dots t_{4n} \text{tr}_{\mathcal{H}_s^{(n)}} \left(e^{-(t_1 u_1 + \dots + t_{4n} u_{4n})(H_0+\mu-E)} \right) K_{t_1}(a, a) \dots K_{t_{4n}}(a, a) \times e^{-(t_1 + \dots + t_{4n})m} du_1 du_2 \dots du_{4n} dt_1 \dots dt_{4n}. \tag{46}$$

After scaling the variables $t_i u_i$ to t_i , we get

$$\int_{\mathbb{R}_+^{4n}} \int_0^1 \int_0^1 \dots \int_0^1 \frac{t_1 \dots t_{4n}}{u_1^2 \dots u_{4n}^2} \text{tr}_{\mathcal{H}_s^{(n)}} \left(e^{-(t_1 + \dots + t_{4n})(H_0+\mu-E)} \right) K_{t_1/u_1}(a, a) \dots K_{t_{4n}/u_{4n}}(a, a) \times e^{-(t_1/u_1 + \dots + t_{4n}/u_{4n})m} du_1 du_2 \dots du_{4n} dt_1 \dots dt_{4n}. \tag{47}$$

Using Eq. (43), the above result becomes

$$\int_{\mathbb{R}_+^{4n}} \int_0^1 \int_0^1 \dots \int_0^1 \frac{t_1 \dots t_{4n}}{u_1^2 \dots u_{4n}^2} e^{-(t_1 + \dots + t_{4n})(nm+\mu-E)} \left(\int_{\mathcal{M}} K_{t_1 + \dots + t_{4n}}(x, x) d_g^D x \right)^n \times K_{t_1/u_1}(a, a) \dots K_{t_{4n}/u_{4n}}(a, a) e^{-(t_1/u_1 + \dots + t_{4n}/u_{4n})m} du_1 du_2 \dots du_{4n} dt_1 \dots dt_{4n}. \tag{48}$$

From the upper bound of the diagonal heat kernel given in (44), an upper bound of Eq. (48) is obtained

$$(V^n(\mathcal{M})C^{5n}(2m)^{5nD/2}) \int_{\mathbb{R}_+^{4n}} \int_0^1 \int_0^1 \dots \int_0^1 \frac{t_1 \dots t_{4n}}{u_1^2 \dots u_{4n}^2} e^{-(t_1 + \dots + t_{4n})(nm+\mu-E)} \times \frac{(u_1 \dots u_{4n})^{D/2}}{(t_1 \dots t_{4n})^{D/2}} \frac{1}{(t_1 + \dots + t_{4n})^{nD/2}} e^{-(t_1/u_1 + \dots + t_{4n}/u_{4n})m} du_1 du_2 \dots du_{4n} dt_1 \dots dt_{4n}, \tag{49}$$

by taking account of only the most singular terms, that is, we disregard contributions coming from the volume term. As one can check, those terms that we dropped behave much better. Using the arithmetic-mean inequality,

$$\frac{1}{t_1 + \dots + t_{4n}} \leq \frac{1}{(4n)^{nD/2} (t_1 \dots t_{4n})^{D/8}}, \tag{50}$$

Eq. (49) is less than

$$(V^n(\mathcal{M})C^{5n}(2m)^{5nD/2}) \left(\int_0^1 \int_0^\infty u^{\frac{D}{2}-2} \frac{e^{-t(nm+\mu-E+\frac{m}{u})}}{t^{7/8}} dt du \right)^{4n}. \tag{51}$$

Evaluating the t -integral and using the fact that $E \leq nm + \mu$, the upper bound to Eq. (49) becomes

$$(V^n(\mathcal{M})C^{5n}(2m)^{5nD/2}) \left(\int_0^1 u^{\frac{D}{2}-2} \left(\frac{u}{m}\right)^{1/8} \Gamma(1/8) du \right)^{4n}, \tag{52}$$

which is finite, i.e.,

$$\text{tr}_{\mathcal{F}_n}([K_0^{-1} K_1]^{4n}) < \infty. \tag{53}$$

Hence the essential spectra of the K_0 and $K_0 + K_1$ must coincide due to classical Weyl's theorem:¹⁶

$$\sigma_{ess}(K_0 + K_1) = \sigma_{ess}(K_0) = \emptyset. \tag{54}$$

We should now prove that the same is true for the potential part, that is $U(E)$ is a relatively compact perturbation of K_0 , hence by Weyl's theorem again their essential spectra must coincide,¹⁶ that is,

$$\sigma_{ess}(K_0 + U) = \sigma_{ess}(K_0) = \emptyset, \quad (55)$$

and this means that

$$\sigma_{ess}(\Phi) = \emptyset. \quad (56)$$

In other words, everything in the spectrum of the principal operator is an eigenvalue.

We will now show explicitly that the operator $K_0^{-1}U$ can be approximated by a sequence of finite rank operators in the norm topology (which is sufficient to establish compactness). For that purpose, we choose the following basis,

$$|\varphi_{\sigma_1} \dots \varphi_{\sigma_n}\rangle = \frac{1}{\sqrt{n!} \sqrt{\prod_{i=1}^n n_i!}} \sum_{\mathcal{P}} \mathcal{P} |\varphi_{\sigma_1} \dots \varphi_{\sigma_n}\rangle, \quad (57)$$

where the sum runs over all possible permutations for the n -tuple $(1, 2, \dots, n)$ and $|\varphi_{\sigma_1} \dots \varphi_{\sigma_n}\rangle$ is given in terms of the one-particle eigenstates, i.e., $|\varphi_{\sigma_1}^{(1)}\rangle |\varphi_{\sigma_2}^{(2)}\rangle \dots |\varphi_{\sigma_n}^{(n)}\rangle$ (the upper indices refer to the particle label). Here, the permutation operator \mathcal{P} is assumed to act on the particle indices. Then, any n -particle state can be expanded in terms of this orthonormal basis;

$$|\psi^{(n)}\rangle = \sum_{\sigma_1, \sigma_2, \dots, \sigma_n} \psi(\sigma_1, \sigma_2, \dots, \sigma_n) |\varphi_{\sigma_1}, \dots, \varphi_{\sigma_n}\rangle. \quad (58)$$

Hence, when we write $\psi(\sigma_1, \dots, \sigma_n)$, whenever there are coincidences of these labels, the appropriate combinatoric factor is taken into account. Thus,

$$\sum_{\sigma_1, \dots, \sigma_n} |\psi(\sigma_1, \dots, \sigma_n)|^2 = 1. \quad (59)$$

Let us now write down $\hat{U}(E) = [H_0 - E + m]^{-1}U(E)$ in an eigenfunction expansion using the basis we refer above. An easy computation shows that

$$\begin{aligned} \hat{U}(E)|\psi^{(n)}\rangle &= \lambda^2 \sum_{\sigma': \sigma_1 \sigma_2, \dots, \sigma_n} \frac{1}{n} \sum_{(12\dots n)} \int_0^\infty \int_0^1 f_{\sigma'}(a) t e^{-tu(\frac{\sigma'}{2m} + m - \mu)} e^{-t(\sum_i \frac{\sigma_i}{2m} + (n+1)m - E)} \\ &\quad \times f_{\sigma_1}(a) \psi(\sigma', \sigma_2, \dots, \sigma_n) |\varphi_{\sigma_1}, \varphi_{\sigma_2}, \dots, \varphi_{\sigma_n}\rangle du dt, \end{aligned} \quad (60)$$

here the symbol $\sum_{(12\dots n)}$ refers to sum over cyclic permutations for the symmetrization and the matrix elements in this basis can easily be read. Now we introduce the finite rank truncations of this operator, simply by cutting-off at the N th eigenvalue for each block, so that we have its finite dimensional approximation,

$$\begin{aligned} \hat{U}_N(E)|\psi^{(n)}\rangle &= \lambda^2 \sum_{[\sigma': \sigma_1 \sigma_2, \dots, \sigma_n] \leq \sigma_N} \frac{1}{n} \sum_{(12\dots n)} \int_0^\infty \int_0^1 f_{\sigma'}(a) s e^{-ts(\frac{\sigma'}{2m} + m - \mu)} e^{-t(\sum_i \frac{\sigma_i}{2m} + (n+1)m - E)} \\ &\quad \times f_{\sigma_1}(a) \psi(\sigma', \sigma_2, \dots, \sigma_n) |\varphi_{\sigma_1}, \varphi_{\sigma_2}, \dots, \varphi_{\sigma_n}\rangle du dt, \end{aligned} \quad (61)$$

where the symbol underneath reflects the fact that all the sums over the eigenvalues are upto the N th eigenvalue σ_N . We will show that U_N strongly converges to U , that is,

$$\|U(E) - U_N(E)\| \rightarrow 0, \quad (62)$$

as $N \rightarrow \infty$. The difference will have various blocks, so that we may represent these operators as block sums as follows:

$$\sum_{\sigma' > \sigma_N} \sum_{\sigma_1, \dots, \sigma_n} \oplus_{i=1}^n \sum_{\sigma' < \sigma_N} \sum_{\sigma_i > \sigma_N} \sum_{[\sigma_1, \dots, \sigma_{i-1}] < \sigma_N} \sum_{\sigma_{i+1}, \dots, \sigma_n}. \quad (63)$$

We will now estimate the norm of the each term in the block sum after applying the norm inequality for each term represented by this block splitting. Since removing the restrictions in the indices, after

we take the absolute values in the norms, increases the value of the sum, we will instead estimate the norm of the following sum,

$$\sum_{\sigma' > \sigma_N} \sum_{\sigma_1, \dots, \sigma_n} \oplus_i \sum_{\sigma_i > \sigma_N} \sum_{\sigma_1, \dots, \hat{\sigma}_i, \dots, \sigma_n} \sum_{\sigma'} \tag{64}$$

which provides an upper bound. The index $\hat{\sigma}_i$ means that the sum over σ_i is omitted. The norm square of the first sum turns out to be

$$\begin{aligned} & \lambda^4 \int_0^\infty \int_0^\infty \int_0^1 \int_0^1 \sum_{\sigma_1, \dots, \sigma_n} \left(\sum_{\sigma' > \sigma_N} f_{\sigma'}(a) \psi(\sigma', \sigma_{(2)}, \dots, \sigma_{(n)}) e^{-\frac{t_1 u_1 \sigma'}{2m}} \right) \\ & \times \left(\sum_{\sigma'' > \sigma_N} f_{\sigma''}(a) \psi(\sigma'', \sigma_{(2)}, \dots, \sigma_{(n)}) e^{-\frac{t_2 u_2 \sigma''}{2m}} \right) |f_{\sigma_{(1)}}(a)|^2 t_1 t_2 e^{-(t_1 u_1 + t_2 u_2)(m-\mu)} \\ & \times e^{-(t_1+t_2)(\sum_i \frac{\sigma_i}{2m} + (n+1)m-E)} du_2 du_1 dt_2 dt_1, \end{aligned} \tag{65}$$

where the parenthesis in the indices $\sigma_{(i)}$ refers to the cyclic permutations. For the term coming from the action of the Hamiltonian, which gives $\sum_i \sigma_i$, if we only keep the index which gives $\sigma_{(1)}$, that becomes an upper bound. Moreover, the upper bound for the above sums inside the bracket can be easily found as

$$\begin{aligned} & \left| \sum_{\sigma > \sigma_N} f_\sigma(a) \psi(\sigma, \sigma_{(2)}, \dots, \sigma_{(n)}) e^{-\frac{t\sigma}{2m}} \right| \leq e^{-t\sigma_N/4m} \left| \sum_{\sigma > \sigma_N} f_\sigma(a) \psi(\sigma, \sigma_{(2)}, \dots, \sigma_{(n)}) e^{-t\sigma/4m} \right| \\ & \leq e^{-t\sigma_N/4m} \sum_{\sigma} \left| f_\sigma(a) \psi(\sigma, \sigma_{(2)}, \dots, \sigma_{(n)}) e^{-t\sigma/4m} \right| \\ & \leq e^{-t\sigma_N/4m} \left[\sum_{\sigma} |f_\sigma(a)|^2 e^{-t\sigma/2m} \right]^{1/2} \left[\sum_{\sigma} |\psi(\sigma, \sigma_{(2)}, \dots, \sigma_{(n)})|^2 \right]^{1/2}, \end{aligned} \tag{66}$$

where we have applied the Cauchy-Schwarz inequality in the last line. The sum $\sum_{\sigma} |f_\sigma(a)|^2 e^{-t\sigma/2m}$ is the eigenfunction expansion of the heat kernels $K_{tu}(a, a)$. We also note that the two sums over the wave function combine to give the norm of the wave function, the left-over index $\sigma_{(1)}$ again combines with $|f_{\sigma_{(1)}}(a)|^2$ and $e^{-(t_1+t_2)\sigma_{(1)}}$ to give another heat kernel. As a result we obtain the following upper bound for Eq. (65),

$$\begin{aligned} & \lambda^4 \int_0^\infty \int_0^\infty \int_0^1 \int_0^1 K_{t_1+t_2}(a, a) K_{t_1 u_1}^{1/2}(a, a) K_{t_2 u_2}^{1/2}(a, a) t_1 t_2 \\ & \times e^{-(t_1 u_1 + t_2 u_2)\sigma_N/4m} e^{-(t_1 u_1 + t_2 u_2)(m-\mu)} e^{-(t_1+t_2)(m-E)} \|\psi\|^2 du_2 du_1 dt_2 dt_1. \end{aligned} \tag{67}$$

Due to the upper bound of the diagonal heat kernel (44), the above expression is bounded above by

$$\begin{aligned} & (2m)^D \lambda^4 C^2 \int_0^\infty \int_0^\infty \int_0^1 \int_0^1 \frac{1}{(t_1+t_2)^{D/2}} \frac{1}{(t_1 u_1)^{D/4}} \frac{1}{(t_2 u_2)^{D/4}} t_1 t_2 \\ & \times e^{-(t_1 u_1 + t_2 u_2)\sigma_N/4m} e^{-(t_1 u_1 + t_2 u_2)(m-\mu)} e^{-(t_1+t_2)(m-E)} \|\psi\|^2 du_2 du_1 dt_2 dt_1. \end{aligned} \tag{68}$$

Now we scale the t_1, t_2 variables by u_1, u_2 , respectively, to get, after simplifications,

$$\begin{aligned} & (2m)^D \lambda^2 C^2 \int_0^\infty \int_0^\infty \int_0^1 \int_0^1 \frac{t_1^{1-\frac{D}{4}} t_2^{1-\frac{D}{4}}}{(t_1 u_2 + t_2 u_1)^{D/2} (u_1 u_2)^{\frac{(4-D)}{2}}} e^{-(t_1+t_2)\sigma_N/4m} e^{-(t_1+t_2)(m-\mu)} \\ & \times e^{-(t_1/u_1 + t_2/u_2)(m-E)} \|\psi\|^2 du_2 du_1 dt_2 dt_1. \end{aligned} \tag{69}$$

Using the arithmetic-mean inequality,

$$\frac{1}{(t_1 u_2 + t_2 u_1)^{D/2}} \leq \frac{1}{2^{D/2}} \frac{1}{(t_1 t_2 u_1 u_2)^{D/4}}, \tag{70}$$

we have decoupled the terms, and obtain the following upper bound to the norm in the first term:

$$\frac{(2m)^{D/2}\lambda^2 C \|\psi\|}{2^{D/4}} \left[\int_0^\infty t^{1-\frac{D}{2}} e^{-t(\frac{\sigma_N}{4m}+m-E)} \left(\int_0^1 \frac{e^{-(t/u)(m-E)}}{u^{2-\frac{D}{4}}} du \right) dt \right]. \tag{71}$$

Since

$$\int_0^1 \frac{e^{-(t/u)(m-E)}}{u^{2-\frac{D}{4}}} du \leq \int_0^\infty \frac{e^{-(t/u)(m-E)}}{u^{2-\frac{D}{4}}} du = \Gamma\left(1 - \frac{D}{4}\right) (m - E)^{\frac{D}{4}-1} t^{\frac{D}{4}-1}, \tag{72}$$

the total result (71) is bounded from above by

$$\frac{(2m)^{D/2}\lambda^2 C \|\psi\|}{2^{D/4}} \Gamma^2\left(1 - \frac{D}{4}\right) (m - E)^{\frac{D}{4}-1} (\sigma_N)^{\frac{D}{4}-1}, \tag{73}$$

which goes to 0 as $N \rightarrow \infty$. Note that here, we are using the important fact about the eigenvalues, that they are of finite multiplicities, hence there is no infinite subsequence which remains bounded as we let $N \rightarrow \infty$. Let us now consider the other sum (we have n -identical such terms),

$$\begin{aligned} & \int_0^\infty \int_0^\infty \int_0^1 \int_0^1 \sum_{\sigma_i > \sigma_N} \sum_{\sigma_1, \dots, \sigma_n} \left[\sum_{\sigma'} f_{\sigma'}(a) \psi(\sigma', \sigma_{(2)}, \dots, \sigma_{(n)}) e^{-\frac{t_1 u_1 \sigma'}{2m}} \right] |f_{\sigma_{(1)}}(a)|^2 \\ & \times \left[\sum_{\sigma''} f_{\sigma''}(a) \psi(\sigma'', \sigma_{(2)}, \dots, \sigma_{(n)}) e^{-\frac{t_2 u_2 \sigma''}{2m}} \right] e^{-(t_1 u_1 + t_2 u_2)(m-\mu)} e^{-(t_1+t_2)(\sum_i \sigma_i + (n+1)m-E)} du_2 du_1 dt_2 dt_1. \end{aligned} \tag{74}$$

We have two cases, as a result of symmetrization: one is that the restricted index shows up inside the wave function ψ , or it remains outside thereby it becomes the index of the eigenfunction $f_{\sigma_{(i)}}(a)$. If it is inside the wave function, we replace the last exponential sum by $e^{-(t_1+t_2)(\sigma_N/4m+(n+1)m-E)}$ (factor of 2 is for convenience only). If it comes with the eigenfunction, we split the eigenvalue part into two equal pieces and replace the first one by $e^{-(t_1+t_2)(\sigma_N/4m)}$ and keep the remaining piece inside the sum to combine with the eigenfunction again. After this replacements, we remove the restriction on the sum in both cases. As a result, by applying a Cauchy-Schwarz inequality to the mixed expression with σ', σ'' -terms, we find

$$\begin{aligned} & \leq \int_0^\infty \int_0^\infty \int_0^1 \int_0^1 e^{-(t_1\sigma_N/4m+t_2\sigma_N/4m)} K_{t_1 u_1}^{1/2}(a, a) K_{t_2 u_2}^{1/2}(a, a) e^{-(t_1+t_2)(n+1)m-E} \\ & \quad \times K_{(t_1+t_2)/2}(a, a) du_2 du_1 dt_2 dt_1. \end{aligned} \tag{75}$$

If we now use the heat kernel estimates (44) and again consider the most singular part with the arithmetic-mean inequality, the integrals become decoupled, so that the norm itself becomes smaller than

$$\frac{C\lambda^2(2m)^{D/2}}{2^{D/4}} \int_0^\infty t^{1-\frac{D}{2}} e^{-t(\sigma_N/4m+(n+1)m-E)} dt \int_0^1 \frac{1}{u^{D/4}} du \|\psi\|. \tag{76}$$

Thus each one of these terms (there are n of them) will go to zero, since

$$\leq \frac{4C\lambda^2(2m)^{D/2}}{2^{D/4}(D-4)} \Gamma\left(2 - \frac{D}{2}\right) (\sigma_N/4m + (n+1)m - E)^{\frac{D}{2}-2} \|\psi\| \rightarrow 0 \text{ as } N \rightarrow \infty. \tag{77}$$

This implies that the bottom of the spectrum of the principal operator is indeed an eigenvalue, say $\omega_0(E)$, whether it is above the free part or not is of no concern, that is,

$$\inf \sigma(\Phi) = \omega_0(E), \tag{78}$$

for all $E \leq nm + \mu$. Hence, the infimum of the spectrum of the Hamiltonian associated with the renormalized resolvent is an eigenvalue, which completes the proof of the existence of ground state wavefunction as a normalizable state.

B. Nondegeneracy of the ground state

In order to prove the nondegeneracy of the ground state, we need the following theorem¹⁶ applied to the principal operator.

Let Φ be a self-adjoint operator that is bounded from below. Suppose that $e^{-t\Phi}$ is positivity preserving for all $t > 0$ and $\omega_0(E) = \inf \sigma(\Phi)$ is an eigenvalue. Then the following are equivalent:

- (a) ω_0 is a simple eigenvalue with a strictly positive eigenvector.
- (b) $e^{-t\Phi}$ is positivity improving for all $t > 0$.

Therefore, having disposed of the preliminary steps given in Secs. II–IV A, we now only need to prove that the semi-group $e^{-t\Phi}$ is positivity improving for all $t > 0$ since we know that Φ is a self-adjoint operator which is bounded from below. One can easily check that e^{-tK_0} is a positivity improving semi-group for all $t > 0$ due to the following theorem:¹⁶

Let A be an operator on a complex Hilbert space with a distinguished complex conjugation and which obeys $A \geq cI$ for some $c > 0$. Then, $e^{-td\Gamma(A)}$ is positivity improving for all $t > 0$, where $d\Gamma(A)$ is the second quantization of A .

As a consequence of the above theorem, the semi-group generated by the second quantization of this operator, e^{-tK_0} is positivity improving for all $t > 0$, since $-\frac{1}{2m}\nabla_g^2 + mI \geq mI$. Another way of showing this is based on the idea that the semi-group generated by K_0 can be expressed in terms of the heat kernel $K_t(x, y)$ which is strictly positive as long as $t > 0$.

Before investigating the positivity property of the semi-group generated by the remaining part of the principal operator, we make the following observation now. *If e^{-tA} is positivity improving and e^{-tB} is positivity preserving for all $t > 0$, then the product $e^{-sA}(e^{-tB})$ is positivity improving for all $t > 0$.*

The second real-valued kinetic operator K_1 is a positive self-adjoint operator on the domain $D(K_1) = \{\psi \in L^2(M, d\mu) \mid \int k_1^2 |\psi|^2 d\mu < \infty\}$. One can now show that $\psi \in D(K_1)$ if $\psi \in D(H_0) = D(K_0)$. This can be seen by using the spectral theorem and the fact that K_1 is a function of the positive self-adjoint operator H_0 . In other words, the domain of K_1 contains the domain of H_0 , i.e., $D(K_1) \supset D(H_0)$. Also, one can show that quadratic form domain of K_1 includes the quadratic form domain of H_0 . This can be seen easily now, since, as shown in Sec. IV A, $K_0 + K_1$ is a relatively compact perturbation of K_0 . Both of them are positive operators, so that they naturally define positive quadratic forms. The operators K_0 and K_1 , defined via the same spectral measure, obviously commute.

We will now show that e^{-tK_1} is positivity improving for all $t > 0$. The compactness of a Riemannian manifold implies that it is complete as a Riemannian manifold and it has a Ricci curvature tensor bounded from below, i.e., $Ric(\cdot, \cdot) \geq -\kappa g(\cdot, \cdot)$. As a result of the theorem proven by Cheeger and Yau,³⁶ the heat kernel has the following lower bound,

$$K_t(x, y) \geq K_t^\kappa(d_g(x, y)), \quad (79)$$

where K_t^κ is the heat kernel for the simply connected complete Riemannian manifold of constant sectional curvature $-\kappa$ and $d_g(x, y)$ refers to the geodesic distance on the manifold \mathcal{M} . In particular, we choose $K_t^\kappa(d_g(x, y))$ as the heat kernel for the Hyperbolic manifold \mathbb{H}^D . In three dimensions, since we have an explicit expression of the heat kernel,³¹ the lower bound to it is simply

$$K_t(a, a) \geq \frac{e^{-\kappa t/2m}}{(4\pi t/2m)^{3/2}}. \quad (80)$$

We add and subtract this lower bound to the heat kernel in K_1 and then split it to the following two parts K_r and K_p , defined as

$$\begin{aligned}
 K_r(E) &= \lambda^2 \int_0^\infty \underbrace{\left(K_t(a, a) - \frac{e^{-\kappa t/2m}}{(4\pi t/2m)^{3/2}} \right)}_{\geq 0} (e^{-t(m-\mu)} - e^{-t(H_0+m-E)}) dt, \\
 K_p(E) &= \lambda^2 \int_0^\infty \frac{e^{-\kappa t/2m}}{(4\pi t/2m)^{3/2}} (e^{-t(m-\mu)} - e^{-t(H_0+m-E)}) dt \\
 &= \frac{\lambda^2}{(4\pi)^{3/2}} (2m)^{3/2} \left[\left(H_0 - E + m + \frac{\kappa}{2m} \right)^{1/2} - \left(m + \frac{\kappa}{2m} - \mu \right)^{1/2} \right], \tag{81}
 \end{aligned}$$

where E is real and $E < nm + \mu$. Since all the projections in their associated projection – valued measures of K_r and K_p commute, we have

$$e^{-tK_1(E)} = e^{-tK_r(E)-tK_p(E)} = e^{-tK_p(E)} e^{-tK_r(E)}. \tag{82}$$

We will first prove that the semi-group generated by $K_p(E)$ is actually *positivity improving* for all $t > 0$ in three dimensions,

$$e^{-tK_p(E)} = e^{t\lambda^2 C \sqrt{m+\frac{\kappa}{2m}-\mu}} e^{-t\lambda^2 C \sqrt{H_0+\frac{\kappa}{2m}+m-E}}, \tag{83}$$

where $C = \left(\frac{2m}{4\pi}\right)^{3/2}$. If we apply the subordination identity to the last piece,

$$e^{-t\lambda^2 C \sqrt{H_0+m+\frac{\kappa}{2m}-E}} = t \frac{C\lambda^2}{2\sqrt{\pi}} \int_0^\infty \frac{e^{-t^2 C^2 \lambda^4 / 4u} e^{-uH_0} e^{-u(\frac{\kappa}{2m}+m-E)}}{u^{3/2}} du, \tag{84}$$

which is *explicitly positivity improving* for all $t > 0$ since e^{-tH_0} is so and everything else is positive in the integration. We will now remark that the remaining part, which is given by $K_r(E)$ is actually positivity preserving for all $t > 0$. This can be proven by the use of Beurling-Deny criteria:¹⁶

Let H be a self-adjoint positive operator on L^2 . The quadratic form $(\psi, H\psi)$ is extended to all of L^2 by setting it equal to infinity when $\psi \notin Q(H)$. Then, the semi-group e^{-tH} generated by a self-adjoint, positive operator H is positivity preserving for all $t > 0$ if H satisfies the following condition for all ψ in the Hilbert space:

$$\langle |\psi|, H|\psi| \rangle \leq \langle \psi, H\psi \rangle. \tag{85}$$

In our case, this condition for K_r can be checked as follows:

$$\begin{aligned}
 \langle |\psi|, K_r(E)|\psi| \rangle &= \lambda^2 \langle |\psi|, \int_0^\infty \underbrace{\left(K_t(a, a) - \frac{e^{-\kappa t/2m}}{(4\pi t/2m)^{3/2}} \right)}_{\geq 0} (e^{-t(m-\mu)} - e^{-t(H_0+m-E)}) |\psi| dt \\
 &= \lambda^2 \int_0^\infty \left(K_t(a, a) - \frac{e^{-\kappa t/2m}}{(4\pi t/2m)^{3/2}} \right) (\langle |\psi|, |\psi| \rangle e^{-(m-\mu)t} - \langle e^{-t(H_0+m-E)/2} |\psi|, e^{-t(H_0+m-E)/2} |\psi| \rangle) dt \\
 &\leq \lambda^2 \int_0^\infty \left(K_t(a, a) - \frac{e^{-\kappa t/2m}}{(4\pi t/2m)^{3/2}} \right) (\langle \psi, \psi \rangle e^{-(m-\mu)t} - \langle |e^{-t(H_0+m-E)/2} \psi|, |e^{-t(H_0+m-E)/2} \psi| \rangle) dt, \tag{86}
 \end{aligned}$$

where we have used the fact that H_0 is self-adjoint and $e^{-t(H_0+m-E)/2}$ satisfies the Beurling-Deny criteria, since it is positivity improving for all $t > 0$. Then, by using the self-adjointness of H_0 once more, the last line in the above equation becomes

$$\begin{aligned}
 &\lambda^2 \int_0^\infty \left(K_t(a, a) - \frac{e^{-\kappa t/2m}}{(4\pi t/2m)^{3/2}} \right) (\langle \psi, \psi \rangle e^{-(m-\mu)t} - \langle e^{-t(H_0+m-E)/2} \psi, e^{-t(H_0+m-E)/2} \psi \rangle) dt \\
 &= \lambda^2 \int_0^\infty \left(K_t(a, a) - \frac{e^{-\kappa t/2m}}{(4\pi t/2m)^{3/2}} \right) (\langle \psi, \psi \rangle e^{-(m-\mu)t} - \langle \psi, e^{-t(H_0+m-E)} \psi \rangle) dt \\
 &= \langle \psi, K_r(E)\psi \rangle. \tag{87}
 \end{aligned}$$

This same condition in two dimensions requires more care, because the heat equation on \mathbb{H}^2 is given by an integral expression which is hard to estimate. Davies and Mandouvalos³⁷ have obtained the sharp upper and lower bounds of the heat kernel on hyperbolic manifolds, which give rise to remarkable consequences for us. This lower bound of the heat kernel for \mathbb{H}^2 is given by

$$K_t(x, x) = \lim_{x \rightarrow y} K_t(x, y) \geq \frac{c e^{-\kappa t/8m}}{(t/2m) \left(1 + \frac{\kappa t}{2m}\right)^{1/2}} \geq \frac{c e^{-3\kappa t/8m}}{(t/2m)}, \tag{88}$$

since $(1 + \kappa t/2m)^{-1/2} \geq e^{-\kappa t/4m}$ for all $t \geq 0$ and here c is a positive dimensionless constant.

Using the lower bound (88), the same decomposition of K_1 in a two dimensional compact Riemannian manifold leads to the following expression for $K_p(E)$:

$$\lambda^2 c(2m) \ln \left(\frac{H_0 + m + \frac{3\kappa}{8m} - E}{m - \mu + \frac{3\kappa}{8m}} \right). \tag{89}$$

The semi-group generated by this operator becomes

$$e^{-tK_p(E)} = \left(\frac{H_0 + m + \frac{3\kappa}{8m} - E}{m - \mu + \frac{3\kappa}{8m}} \right)^{-\lambda^2 ct}, \tag{90}$$

and this can be written as

$$\left(\frac{H_0 + m + \frac{3\kappa}{8m} - E}{m - \mu + \frac{3\kappa}{8m}} \right)^{-\lambda^2 ct} = \frac{1}{\Gamma(\lambda^2 ct)} \int_0^\infty \frac{e^{-s(H_0 - E + m + \frac{3\kappa}{8m})/(m - \mu + \frac{3\kappa}{8m})}}{s^{1 - \lambda^2 ct}} ds, \tag{91}$$

so that the semi-group e^{-tK_p} in two dimensions is *positivity improving* for all $t > 0$, as well. Similar to the three dimensional case, Beurling-Deny criteria for e^{-tK_r} in two dimensions can be easily checked, hence it is positivity preserving for all $t > 0$. This leads to the conclusion that e^{-tK_1} is positivity improving for all $t > 0$.

Let us recall the Trotter-Kato product formula:³⁴

Let A and B be two self-adjoint operators which are bounded from below, and assume that the sum $A + B$ is self-adjoint on a common domain. Then we have

$$e^{-t(A+B)} = \text{s-lim}_{N \rightarrow \infty} \left(e^{-tA/N} e^{-tB/N} \right)^N. \tag{92}$$

We will now apply this theorem to the principal operator $\Phi(E) = K_1(E) + K_0(E) + U(E)$ (note the change of ordering). We have proved in our previous work that the principal operator is a self adjoint operator for real E .¹¹ We also note that $K_1(E) > 0$ explicitly for $E < nm + \mu$ and thanks to the estimate of the bottom of the ground state (38), we have $K_0(E) + U(E) > 0$ as long as $E \leq E_*$. Moreover, we have also shown that the difference of the principal operator corresponding to the two different values of E is bounded. Similarly, we have

$$\|U(E_1) - U(E_2)\| < |E_1 - E_2| \lambda^2 n \int_0^\infty t K_{2t}(a, a) e^{-tnm} dt. \tag{93}$$

Hence for all values of $E_* < E < nm + \mu$, we write

$$\begin{aligned} (f^{(n)}, [K_0(E) + U(E)] f^{(n)}) &= (f^{(n)}, [H_0 - E_* + \mu + U(E_*) - E + E_* + U(E) - U(E_*)] f^{(n)}) \\ &> \underbrace{(f^{(n)}, (K_0(E_*) + U(E_*)) f^{(n)})}_{>0} - |E_* - E| \left(1 + n\lambda^2 \int_0^\infty t K_{2t}(a, a) e^{-tnm} dt \right), \end{aligned} \tag{94}$$

where $f^{(n)}$ is any n -particle state. It shows that for $E < nm + \mu$, $K_0(E) + U(E)$ is bounded from below.

Thus we may now apply the Trotter-Kato formula:

$$e^{-t\Phi(E)} = \text{s-lim}_{N \rightarrow \infty} \left(e^{-K_1(E)t/N} e^{-(K_0(E)+U(E))t/N} \right)^N. \quad (95)$$

Note that here we may rewrite $U(E)$ in the following way:

$$U(E) = -\lambda^2 \int_0^\infty \phi^\dagger(K_t(a, \cdot)) e^{-t(H_0-E+2m)} \phi(K_t(a, \cdot)) dt. \quad (96)$$

We will now assure that the semi-group $e^{-t(K_0(E)+U(E))}$ is positivity preserving for all $t > 0$. We again resort to the Beurling-Deny criteria and check this condition only for $U(E)$ since it is obviously true for $K_0(E)$ -part:

$$\langle |\psi\rangle, U(E)|\psi\rangle = -\lambda^2 \int_0^\infty \langle e^{-t(H_0-E+2m)/2} \phi_g(K_t(a, \cdot))|\psi\rangle, e^{-t(H_0-E+2m)/2} \phi_g(K_t(a, \cdot))|\psi\rangle dt. \quad (97)$$

Using the positivity of $K_t(a, x)$ and the fact that $e^{-t(H_0-E+2m)}$ generates a positivity improving semi-group for all $t > 0$, one can now check that

$$e^{-t(H_0-E+2m)/2} \phi_g(K_t(a, \cdot))|\psi\rangle \geq |e^{-t(H_0-E+2m)/2} \phi_g(K_t(a, \cdot))\psi|. \quad (98)$$

Thanks to the minus sign in front, we have now

$$\begin{aligned} \langle |\psi\rangle, U(E)|\psi\rangle &\leq -\lambda^2 \int_0^\infty \langle |e^{-t(H_0-E+2m)/2} \phi_g(K_t(a, \cdot))\psi|, |e^{-t(H_0-E+2m)/2} \phi_g(K_t(a, \cdot))\psi| \rangle dt \\ &\leq -\lambda^2 \int_0^\infty \langle e^{-t(H_0-E+2m)/2} \phi_g(K_t(a, \cdot))\psi, e^{-t(H_0-E+2m)/2} \phi_g(K_t(a, \cdot))\psi \rangle dt = \langle \psi, U(E)\psi \rangle. \end{aligned} \quad (99)$$

Thus, we conclude that the semi-group $e^{-t(K_0(E)+U(E))/N}$ in the Trotter-Kato product formula is positivity preserving for all $t > 0$. Since the first factor in the Trotter-Kato product expansion of $\Phi(E)$ is positivity improving and the second factor is positivity preserving for all $t > 0$, their product $e^{-tK_1(E)/N} e^{-t(K_0(E)+U(E))/N}$ is positivity improving for all $t > 0$. As a result, the principal operator $\Phi(E)$ generates a positivity improving semi-group for all $t > 0$ due to the fact that the strong limit of the sequence of positivity improving operators are positivity improving for all $t > 0$. Thus the eigenvalue corresponding to the bottom of the spectrum of principal operator is simple, i.e., nondegenerate and its associated eigenvector is strictly positive.

Hence, due to Eqs. (28) and (29), the ground state wave function of the model is strictly positive so that the ground state of the original model is nondegenerate.

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