Benefits of averaging lateration estimates obtained using overlapped subgroups of sensor data ☆

Mustafa A. Altınkaya

İzmir Institute of Technology, Faculty of Engineering, Department of Electrical & Electronics Engineering, İzmir, Turkey

A B S T R A C T

In this paper, we suggest averaging lateration estimates obtained using overlapped subgroups of distance measurements as opposed to obtaining a single lateration estimate from all of the measurements directly if a redundant number of measurements are available. Least squares based closed form equations are used in the lateration. In the case of Gaussian measurement noise the performances are similar in general and for some subgroup sizes marginal gains are attained. Averaging laterations method becomes especially beneficial if the lateration estimates are classified as useful or not in the presence of outlier measurements whose distributions are modeled by a mixture of Gaussians (MOG) pdf. A new modified trimmed mean robust averager helps to regain the performance loss caused by the outliers. If the measurement noise is Gaussian, large subgroup sizes are preferable. On the contrary, in robust averaging small subgroup sizes are more effective for eliminating measurements highly contaminated with MOG noise. The effect of high-variance noise was almost totally eliminated when robust averaging of estimates is applied to QR decomposition based location estimator. The performance of this estimator is just 1 cm worse in root mean square error compared to the Cramér–Rao lower bound (CRLB) on the variance both for Gaussian and MOG noise cases. Theoretical CRLBs in the case of MOG noise are derived both for time of arrival and time difference of arrival measurement data.

© 2013 Elsevier Inc. All rights reserved.

1. Introduction

Localization is an important problem encountered in a diverse area of applications such as finding the employees or utilities in a working area, location aware services supplied to mobile phone users, bioinstrumentation and communicating toys [1,2]. A popular method of localization is lateration where measured distances from sensors to the point to be localized are used [3]. Some of the lateration based methods require solving a set of nonlinear equations what is usually performed by some iterative procedures as in the case of nonlinear least squares type algorithms in [4–6]. On the other hand, there are some other lateration based methods which reduce the original nonlinear problem into a linear one making a closed form solution easily obtainable such as the least squares-time difference of arrival (LS-TDOA) [7], least squares-time of arrival (LS-TOA) [8,9] and the so-called ordinary least squares-time of arrival (OLS-TOA) [5] location estimators. Another closed form location estimator is based on Cayley–Menger determinants and uses the geometric properties of tetrahedrons whose corners are defined by the unknown location and three known locations [10]. Some of the closed form location estimators reduce the non-

linear problem partially to a linear system of equations whose solution is obtained in terms of another unknown which is found solving a quadratic equation. One of the resulting two solutions of this quadratic equation is easily eliminated and substituting the solution into the linear system of equations the location is estimated. The time of arrival (TOA) based location estimator in [11], its QR decomposition based version in [4] and another one able to utilize more than necessary distance measurements [12,13] which is in contrast with the former two estimators, belong to this class of estimators. Although these methods have some performance inferiority with respect to the iterative nonlinear methods, their computational simplicity make them preferable in some applications where limited computational power is available.

One type of location estimation scenarios may supply more than necessary distance measurements. In that case either all of the measurements can be used at once to obtain a single location estimate or subgroups of measurements can be selected to obtain many estimates. In the latter, the final location estimate is found as the average of the estimates. We will call this approach as *averaging laterations* while the former one will be called as *single lateration*. The Divide-And-Conquer (DAC) approach of [14] uses averaging laterations with subgroups of distance measurements which might overlap or not. Non-overlapping subgroups necessitate a large number of measurements which property is not shared by the choice of overlapping subgroups [8,15]. In [8], TOA-based

---

averaging laterations method with overlapping subgroups of size three was given as an alternative to LS algorithm in finding an unknown location in two dimensions. Recently, in [15], the QR decomposition based TOA method of [4] was applied in regular or robust versions of averaging with overlapping subgroups of measurements. In fact, for fairness in overlapping subgroups, every possible combination of total measurements with the chosen group size is considered in [15] which is also the adopted method in this work. On the other hand, every lateration-based location estimator utilizing LS algorithm can operate in *single lateration* mode with any redundant number of measurements. However, for estimators which cannot handle excessive number of measurements which is more than their nominal number such as the estimators of [11,4], the adopted *averaging laterations* mode of operation seems to be the best candidate in utilizing all of the available measurements while assuring fairness among them.

In order to compare *averaging* and *single* lateration modes of operation, a location estimator which can utilize both nominal or more than nominal number of measurements is required. We name this property as the *scalability* of the estimator. A good example for non-scalable location estimators is the QR decomposition based lateration technique in [4] which always uses three distance measurements. Many of the location estimators having closed form solutions are scalable such as LS solutions of TOA and TDOA type problem formulations. In general, iterative location estimators mainly solving nonlinear equations are scalable too. However, closed form solution producing methods which transform the nonlinear equation systems partially to a linear system of equations, are not scalable with the exception of the methods in [12,13]. Even though iterative methods are scalable, they are not considered for averaging laterations since they are both computationally demanding already and their optimality will be disturbed unless whole data is considered. So, in the first part of our study we chose two TOA based and one time difference of arrival (TDOA) based least squares (LS) estimators for investigating averaging estimates.

Simulation experiments in our study show that generally averaging the estimates obtained with partial sensor data achieves similar performance compared to the case of using whole data in obtaining a single estimate when the measurements contain Gaussian disturbances. However, when some of the measurements are very noisy, the performance of a single lateration based estimator deteriorates significantly. Such measurements which are substantially different from the other measurements are many times called as *outliers*. The disturbing effect of outlier measurements can be eliminated by a robust location estimator [16,17]. Robust location estimators can be classified into two groups: outlier detection based or robust estimation based [17]. Outlier detection based methods eliminate detected outliers completely whereas robust estimation methods lessen their weights in the estimation. For a detailed comparison of these methods, one can refer to [17]. As a robust estimation example, recently in [18], the least median of squared errors obtained in a TDOA based LS solution is minimized over the set of every possible subgroup combination of measurements. An outlier elimination procedure is applied in [19] in order to eliminate outlier estimates obtained with minimal subgroups of measurements and with an iterative nonlinear LS solution based on first order Taylor series expansion of nonlinear localization equation. Then the final estimate is obtained as the median of the qualified estimates. Still some other robust estimation solutions exist such as the genetic algorithm based TDOA solution in [19] and the location estimate in [20] obtained by utilizing the expectation maximization algorithm for removing outlier distance measurements iteratively.

The main idea of this paper is to promote *averaging laterations* as opposed to *single lateration*. Additionally, it is demonstrated that

the averager can be easily transformed into a robust version which can handle outlier measurements. The averaging laterations approach in this paper can be considered as an extension of the work in [8] of subgroups with three measurements for TOA based lateration in two dimensions to any possible subgroup size of measurements for lateration in three dimensions. For the localization scenario with outlier measurements, the proposed robust method can be classified as an outlier detection based location estimator like one of the methods in [19] and the new robust averager used for detecting outliers resembles to the modified trimmed mean (MTM) averager defined in [21] which will be described in Section 4. However in this work varying subgroup sizes for measurements are investigated which was not considered before. The outlier statistics was a general mixture model in [19]. Here the statistics of measurement noise with outliers is modeled by a mixture of Gaussians (MOG) distribution. Other than the formerly described TOA and TDOA based LS location estimators, the non-scalable QR decomposition based lateration technique in [4] is also used in the investigation with MOG sensor noise. Furthermore, the performances of TOA and TDOA based location estimators were also compared to the theoretical performance bounds. Theoretical Cramér–Rao lower bounds (CRLB) for TOA and TDOA based location estimation with MOG sensor noise are derived.

The remaining part of the paper is organized as follows. In Section 2, the linearized TOA and TDOA based LS location estimators and the QR decomposition and TOA based location estimator [4] are described. In Section 3, the averaging laterations method of location estimation is described and its performance is investigated by simulation studies. Section 4 considers location estimation when the distance measurements have MOG noise contamination. In this section, first a robust version of averaging laterations is proposed then the performance of this method is compared to simple averaging laterations and single lateration. Section 5 includes a discussion on the proposed averaging technique, draws conclusions from the work and suggests directions of further study. Lastly, in Appendices A–C the derivations of CRLB both for the cases of Gaussian and MOG sensor noise and both for TOA and TDOA measurement data, are given.

2. The localization problem and some linearized estimators

We define a localization setup which will emphasize the main motivation in the paper. So, let us assume that we have *N* sensors located uniformly on a circle placed at the ceiling and the location to be determined is placed on the floor. Note that this hypothetical placement of the objects does not disturb the applicability of the method in a problem of determining the position of an aircraft using the distance measurements from several base stations like in [11] or determining the location of employees in an office environment like in [22].

The true distance from the ith sensor to the unknown location, $\mathbf{p} = (p_x, p_y, p_z)^T$, can be given as

$$d_{t_i} = \|\mathbf{p} - \mathbf{a}_i\| = \sqrt{(p_x - x_i)^2 + (p_y - y_i)^2 + (p_z - z_i)^2} \qquad (1)$$

where $\mathbf{a}_i = (x_i, y_i, z_i)^T$ is the location of the ith sensor and $(\cdot)^T$ denotes the transposition operation. The measurement is modeled as

$$d_i = d_{t_i} + \sigma_i \epsilon_i \qquad (2)$$

where ϵ_i is a zero mean Gaussian random variable with unity variance and σ_i is a constant. From a geometrical point of view three and four measurements are required for 2-dimensional (2-D) and 3-D localization, respectively. The distance measurements are generally obtained indirectly, computing the distance traveled by an

electromagnetic or acoustic wave between the sensor and the unknown location in a measured time. In some cases, these time measurements contain a common unknown time shift since the common TOA or the common time of departure of the traveling wave is not known. Then, the TDOA is known which in turn corresponds to difference of distance measurements defined as

$$d_{i,j} = d_i - d_j. \tag{3}$$

2.1. TOA-based LS location estimation

Most of the linear location estimators are essentially based on squaring (1) for two different values of i and subtracting one of them from the other. So, when absolute distances that is distances from the sensors to the unknown location are available, as an example one can obtain

$$d_{t_i}^2 - d_{t_1}^2 = 2\big((x_i - x_1), (y_i - y_1), (z_i - z_1)\big)\mathbf{p} + \mathbf{a}_i^T\mathbf{a}_i - \mathbf{a}_1^T\mathbf{a}_1 \tag{4}$$

and defining the squared length of the sensor vectors as

$$k_i = \mathbf{a}_i^T\mathbf{a}_i \quad \text{for } i = 1, 2, \ldots, N,$$

the linear TOA estimate of location can be obtained solving the LS equation as

$$\hat{\mathbf{p}}_{\text{LS-TOA}} = \big(\mathbf{A}^T\mathbf{A}\big)^{-1}\mathbf{A}^T\mathbf{b} \tag{5}$$

where

$$\mathbf{A} = \begin{bmatrix} (x_1 - x_2) & (y_1 - y_2) & (z_1 - z_2) \\ \vdots & \vdots & \vdots \\ (x_1 - x_N) & (y_1 - y_N) & (z_1 - z_N) \end{bmatrix} \tag{6}$$

and

$$\mathbf{b} = \frac{1}{2}\begin{bmatrix} d_2^2 - d_1^2 - k_2 + k_1 \\ \vdots \\ d_N^2 - d_1^2 - k_N + k_1 \end{bmatrix}. \tag{7}$$

2.2. TDOA-based LS location estimation

When relative distances of the sensors to the unknown location that is the differences of their distances relative to the distance of a reference point, e.g. the first sensor, are known, one can construct a set of $N - 1$ equations discarding the noise components in the measurements as

$$(d_{i,1} + d_{t_1})^2 = d_{t_i}^2. \tag{8}$$

Then one way of obtaining a linear estimate of location is first to solve an LS equation to find an estimate in terms of the reference point and later using the constraint (1) in order to obtain the unconditional location estimate [7,23].

Another way of solution is to obtain a joint LS estimate for the unknown location and its distance to the reference point. Following an easy manipulation a linear set of equations can be obtained as follows [24]:

$$\underbrace{\begin{bmatrix} (x_1 - x_2) & (y_1 - y_2) & (z_1 - z_2) & d_{2,1} \\ \vdots & \vdots & \vdots & \vdots \\ (x_1 - x_N) & (y_1 - y_N) & (z_1 - z_N) & d_{N,1} \end{bmatrix}}_{\mathbf{A}'}\underbrace{\begin{bmatrix} \mathbf{p} \\ d_{t_1} \end{bmatrix}}_{\mathbf{p}'}$$
$$= \underbrace{\frac{1}{2}\begin{bmatrix} k_2 - k_1 - d_{2,1}^2 \\ \vdots \\ k_N - k_1 - d_{N,1}^2 \end{bmatrix}}_{\mathbf{b}'}. \tag{9}$$

Then

$$\hat{\mathbf{p}}'_{\text{LS-TDOA}} = \big(\mathbf{A}'^T\mathbf{A}'\big)^{-1}\mathbf{A}'^T\mathbf{b}'. \tag{10}$$

2.3. An alternative TOA-based LS location estimator

A modified version of TOA-based location estimate can be obtained by using the mid-point of the sensor locations found by averaging the coordinates of the sensors defined by $\mathbf{a}_{\text{av}} = (x_{\text{av}}, y_{\text{av}}, z_{\text{av}})^T$ as a reference point [5]. We form an $N \times 1$ column vector $\mathbf{s}$ whose ith component is given by

$$s_i = \frac{1}{2}\big(d_{i,\text{av}}^2 - d_i^2\big)$$

where d_i and $d_{i,\text{av}}$ denote the measured distance from the ith sensor to $\mathbf{p}$ and to $\mathbf{a}_{\text{av}}$, respectively. Afterwards an $N \times 3$ data matrix $\mathbf{D}$ is formed as

$$\mathbf{D} = \begin{bmatrix} (x_1 - x_{\text{av}}) & (y_1 - y_{\text{av}}) & (z_1 - z_{\text{av}}) \\ (x_2 - x_{\text{av}}) & (y_2 - y_{\text{av}}) & (z_2 - z_{\text{av}}) \\ \vdots & \vdots & \vdots \\ (x_N - x_{\text{av}}) & (y_N - y_{\text{av}}) & (z_N - z_{\text{av}}) \end{bmatrix}. \tag{11}$$

The ordinary LS (OLS) estimate of $\mathbf{p}$ [5] is obtained as

$$\hat{\mathbf{p}}_{\text{OLS}} = \big(\mathbf{D}^T\mathbf{D}\big)^{-1}\mathbf{D}^T\mathbf{s}_* \tag{12}$$

where $\mathbf{s}_* = \mathbf{s} + \mathbf{D}\mathbf{a}_{\text{av}}$.

2.4. QR decomposition and TOA-based location estimator

Now, we consider the partially linearized TOA-based location estimator in [4] which utilizes QR decomposition (QRD). Actually this method is aimed for finding the intersection of n hyperspheres in $\mathcal{R}^n$. Since our observations are in $\mathcal{R}^3$ only the locations of 3 sensors can be used at each estimation. So, we start with

$$-\mathbf{A}^T = \mathbf{Q}\begin{bmatrix} \mathbf{R} \\ \mathbf{0}^T \end{bmatrix} \tag{13}$$

where $\mathbf{A}$ is as defined in (6) with only three sensor positions. In (13), $\mathbf{Q}$, $\mathbf{R}$ and $\mathbf{0}^T$ are a 3×3 orthogonal matrix, the 2×2 upper-triangular coefficient matrix and a 1×2 zero vector, respectively. Consequently, the location will be found as

$$\mathbf{p} = \mathbf{Q}\begin{bmatrix} \mathbf{y} \\ z \end{bmatrix} + \mathbf{a}_1 \tag{14}$$

where $\mathbf{y}$ is a 2-vector and z is a scalar. Since this translation and rotation/reflection operations preserve the Euclidean length, the distances of sensors to the unknown location become

$$\|\mathbf{y} - \mathbf{r}_j\|_2^2 + z^2 = d_j^2, \quad j = 2, 3, \tag{15}$$

and

$$\|\mathbf{y}\|_2^2 + z^2 = d_1^2 \tag{16}$$

where $\mathbf{r}_j$ is the jth column of the matrix, $\mathbf{R}$. Substituting the distance of the first sensor to the unknown location given by (16) into (15) the linear system of equations

$$\begin{bmatrix} r_{11} & 0 \\ r_{12} & r_{22} \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \tag{17}$$

is obtained where r_{ij} is the (i, j)th element of $\mathbf{R}$ and, y_1 and y_2 are the corresponding elements of the vector, $\mathbf{y}$. Here c_1 and c_2 are given as

$$c_j = \frac{1}{2}\big(d_1^2 - d_j^2 + \|\mathbf{r}_j\|_2^2\big), \quad j = 1, 2. \tag{18}$$

After solving the linear system in (17), the last coordinate of the unknown location in the transformed system will be found as the solution to the quadratic equation in (16) given as:

$$z = \pm\sqrt{d_1^2 - \|\mathbf{y}\|_2^2}. \tag{19}$$

Only one of the solutions above is valid. This solution can be easily chosen using the constraints on (14) defined by the localization problem.

3. The averaging method

When more than necessary distance measurements are available either all of the measurements can be used at once to obtain a single location estimate as described in the previous section or overlapping subgroups of measurements can be used to obtain many estimates. In the latter, the final location estimate is found as the average of those intermediate estimates

$$\hat{\mathbf{p}}_{\mathrm{av}} = \frac{1}{M} \sum_{m=1}^{M} \hat{\mathbf{p}}_m^{(n)} \tag{20}$$

where the available N distance measurements are used to obtain overlapping subgroups with n members. Then, the total number of estimates to be used in averaging, M, will be found as the number of n combinations out of N given by [25]:

$$M = C_n^N = \binom{N}{n} = \frac{N!}{(N-n)!n!}. \tag{21}$$

This was also mentioned in [8] for location estimation in two dimensions where $n = 3$. Whether the subgroups of measurements do overlap or not is of great importance and overlapping assures the symmetry in selecting the measurements. Additionally, the total number of measurements is usually not large enough to make selection without replacement possible.

3.1. Simulation study

Scalable closed form location estimators described in Sections 2.1, 2.2 and 2.3 are used to compare the performances of the *single lateration* and *averaging laterations* methods of location estimation defined in Section 1. Since obtaining general results is desirable, one should consider random placement of sensors with respect to the location to be estimated. This randomness can be assured either moving the sensors randomly or changing the location to be estimated. In this study, we preferred moving the location to be estimated on a plane and keeping the location of the sensors which are placed on some higher altitude, e.g. at the ceiling, constant. Secondly, the structure of the sensor locations is to be chosen if a structure in sensor placement is wanted. Realizing that an arbitrary placement of sensors would produce only particular results, a structured sensor placement is preferred. Furthermore since measurements from more sensors are wanted, for any number of sensors, a fair placement which will assure an even contribution from each measurement can be attained if the sensors are placed on a circle.

The OLS and LS-TOA location estimators will successfully estimate the location if the rank of the matrices $\mathbf{A}$ and $\mathbf{D}$ are three, respectively. Correspondingly, the LS-TDOA location estimator needs a rank of four for the matrix $\mathbf{A}'$ since it needs to find one more unknown parameter. For that reason none of the subgroups of the sensor locations must be on a plane. Otherwise an uncertainty in the estimation of the height coordinate would be present for any method of location estimation; one estimate being on the floor and

the other one above the ceiling. This requirement of the model will be ensured by introducing a random perturbation to the height of the sensor locations.

In our simulations, the sensors are placed on a circle at the ceiling which is 3 m high. The radius of the circle is 2 m. The coordinate of the ith sensor is

$$\mathbf{a}_i = \left(x_i, \sqrt{4 - x_i^2}, 3 + \eta_i\right)^T \tag{22}$$

where η_i is a zero mean Gaussian random variable with standard deviation of 2 m. The standard deviation of the Gaussian noise in the distance measurements is 5 cm. Actually, it is not true that the sensors are placed on a circle because of the large variance in their z-coordinates. But this fact does not affect the generality of the selected sensor points. On the contrary, the estimation problem would be easier if the sensors were planary located. With a slight modification in the algorithms for linearized LS estimators, their 2-D versions can be obtained. In that case, firstly the x and y coordinates of the unknown location will be estimated. After that a quadratic equation is solved in order to find two z coordinates where the one being below the plane of the sensors is chosen as the true solution.

In order to decide what kind of simulations to conduct consider the preliminary simulation results obtained with LS-TOA, OLS-TOA and LS-TDOA location estimators given in Tables 1, 2 and 3, respectively. In these tables results both with Gaussian and MOG measurement noise are given. The tabulated results correspond to the most advantageous unknown location which is $\mathbf{p}_1 = (0, 0, 0)^T$ m and $\mathbf{p}_2 = (2, 0, 0)^T$ m for LS-TOA and OLS-TOA and for LS-TDOA location estimators, respectively. The tables also depict the CRLBs which can be obtained using the procedures described in appendices. As an example consider Table 1 and the results in it obtained with Gaussian measurement noises. Although theoretically four measurements are enough for obtaining an LS-TOA location estimate, only with five or more measurements valid estimates are obtained. This means that for comparing single and averaging laterations methods at least a total of six distance measurements are needed in order to have the minimum number of five measurements in a subgroup. Furthermore, the subgroup size, n, also affects the performance. As N gets large, averaging becomes more advantageous compared to single lateration. In order to investigate the dependence on the subgroup size more measurements than the minimum applicable number are needed. In the sequel we adopted using ten distance measurements for assessing that dependence.

The results in Table 2 with OLS-TOA estimator are very similar to the ones in Table 1, obtained with LS-TOA method. For the LS-TDOA method, the results in Table 3, depict both some similar and some dissimilar behaviors when compared to LS-TOA and OLS-TOA estimators. With a subgroup size of $n = N - 1$ a similar behavior is observed which means that as N gets large, the averaging method becomes more advantageous, starting being superior to single lateration at $N = 11$. On the other hand, its performance with a nominal subgroup size of six measurements is still considerably worse than the performance of single lateration even for the very large number of measurements, $N = 15$, for which value the total number of estimates in averaging, M, is very large already. This number in turn is proportional to the increase in computational requirement. This means that with LS-TDOA estimator the benefits to be obtained with averaging laterations method is not worth of the required additional computational burden. So, we preferred not using that estimator in the subsequent simulation studies.

3.1.1. Performance of single lateration OLS estimator with 6 sensors

Firstly, OLS based laterations are performed with six sensors. For this simulation only, the standard deviation of the Gaussian

Table 1
RMSE of LS-TOA estimator (cm).

N	Gaussian noise				MOG noise					
	Single	Averaging		CRLB	Single	Robust averaging				CRLB
		$n = N-1$	$n = 5$			$n = 4$	$n = 5$	$n = 6$	$n = 7$	
4	4102	–	–	8.87	4150	–	–	–	–	8.98
5	41.25	1267	–	7.99	64.05	74.11	–	–	–	8.08
6	16.54	18.38	18.38	7.25	33.51	47.46	34.89	–	–	7.36
7	12.72	12.52	15.09	6.68	26.99	36.53	25.95	27.52	–	6.79
8	11.19	10.85	11.82	6.30	25.36	31.04	20.33	25.76	25.71	6.41
9	10.42	10.02	10.74	5.89	24.45	30.99	16.43	23.60	25.71	6.00
10	9.84	9.44	9.43	5.60	22.66	27.96	11.45	17.24	23.25	5.71
11	9.36	9.00	8.76	5.35	21.32	23.94	9.51	12.80	19.39	5.47
12	9.05	8.70	8.14	5.12	20.75	18.72	8.41	10.38	15.13	5.24

Table 2
RMSE of OLS-TOA estimator (cm).

N	Gaussian noise				MOG noise					
	Single	Averaging		CRLB	Single	Robust averaging				CRLB
		$n = N-1$	$n = 6$			$n = 4$	$n = 5$	$n = 6$	$n = 7$	
4	4102	–	–	8.87	4150	–	–	–	–	8.98
5	42.03	1267	–	7.99	62.68	74.11	–	–	–	8.08
6	15.30	18.08	18.08	7.25	32.15	47.46	33.83	–	–	7.36
7	11.74	12.00	14.96	6.68	24.88	36.53	25.69	25.74	–	6.79
8	10.11	10.16	11.66	6.30	22.91	31.04	20.85	25.08	23.51	6.41
9	9.19	9.17	10.54	5.89	22.16	30.99	16.67	23.29	24.33	6.00
10	8.48	8.45	9.31	5.60	18.99	27.96	11.62	16.89	21.88	5.71
11	7.88	7.86	8.68	5.35	17.75	23.94	9.56	13.08	18.91	5.47
12	7.48	7.46	8.09	5.12	16.79	18.72	8.46	10.64	15.14	5.24

Table 3
RMSE of LS-TDOA estimator (cm).

N	Gaussian noise				MOG noise					
	Single	Averaging		CRLB	Single	Robust averaging				CRLB
		$n = N-1$	$n = 5$			$n = 5$	$n = 6$	$n = 7$	$n = 8$	
5	2765	–	–	11.05	3019	–	–	–	–	11.22
6	170.1	17750	–	10.00	227	289.3	–	–	–	10.16
7	74.06	97.52	97.52	9.20	97.40	167.8	98.98	–	–	9.35
8	49.42	51.44	66.50	8.62	78.35	139.7	80.03	78.06	–	8.79
9	41.92	42.45	55.69	8.03	75.94	119.4	75.55	76.01	75.78	8.18
10	36.92	37.02	49.66	7.60	70.82	117.3	67.35	71.34	70.87	7.77
11	32.53	32.51	43.67	7.32	65.79	110.7	58.51	66.35	67.60	7.49
12	28.87	28.88	40.23	6.97	66.20	108.1	52.84	61.09	68.39	7.15
13	27.03	26.96	37.53	6.62	61.70	104.8	47.68	53.60	62.58	6.80
14	24.89	24.85	36.39	6.41	62.64	102.8	44.04	48.48	58.14	6.60
15	23.38	23.27	34.89	6.19	59.33	103.9	41.69	44.22	51.30	6.37

perturbation in (22) is 1 m. The locations to be estimated is a grid of square shape with an edge-length of 4 m. The localization experiment is repeated for 100 000 different realizations of sensor heights and distance measurements.

In the whole investigated area the estimates are generally unbiased and the magnitude of the bias is smaller than 0.7 mm. So, variance is more effective in the mean square error (MSE). That is why the root MSE (RMSE) is chosen as the performance measure for the estimates. In Fig. 1, the RMSE of location estimates obtained with the single lateration method is plotted. The minimum RMSE is obtained for a point which is approximately under the center of the circle on which the sensors are located uniformly at the circumference. This minimum RMSE is 15.04 cm and the maximum RMSE is 18.66 cm which is obtained at a point which is approximately 2.83 m away from the projection of the mid-point of the sensors on the floor.

Fig. 1 depicts that the RMSE performance of the single lateration location estimator is a function of the distance from the projection of the mid-point of the sensors on the floor. This behavior is shared by the averaging laterations location estimator

which is not shown. So, in the sequel 2-D graphics is utilized for comparing the performances of estimators. In the simulations, the y-coordinate of the unknown location is set to zero and the RMSE values of the estimators are plotted as a function of the x-coordinate of the unknown location. The CRLB for the estimation which can be found following the procedures in the appendices are also shown in the subsequent figures.

3.1.2. Comparing performances of single lateration and averaging laterations

The case of 6 distance measurements In Fig. 2 the performances of single and averaging laterations are plotted when there are six sensors. The RMSE of averaging laterations is approximately 2.6 cm larger than the one of single lateration. The handicap of the averaging laterations method in this simulation was that the subspace gain in moving from $N = 5$ to $N = 6$ is very large compared to the variance reduction of using the average of six different estimates. Since this effect is expected to be large below a critical subspace size, increasing the number of measurements beyond a
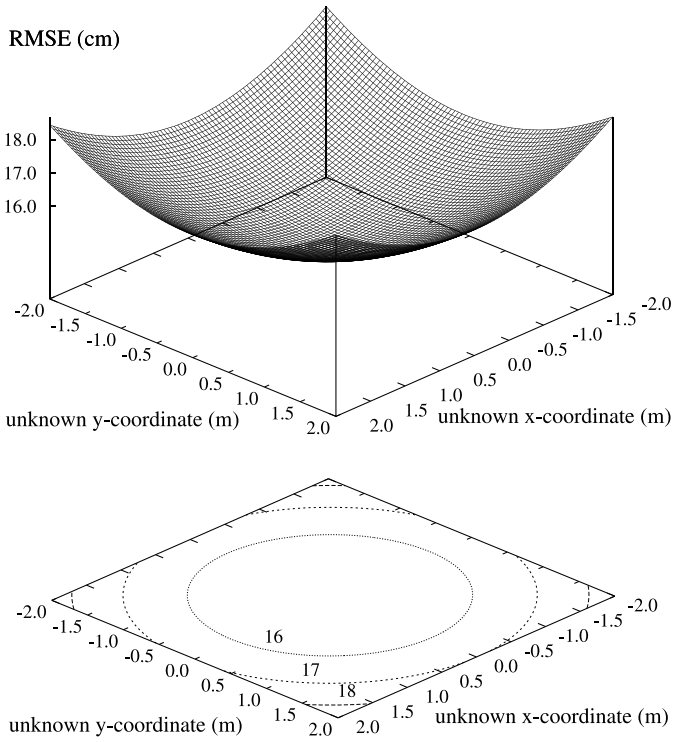
Fig. 1. RMSE of OLS location estimate (6 distance measurements, 100 000 noise realizations).
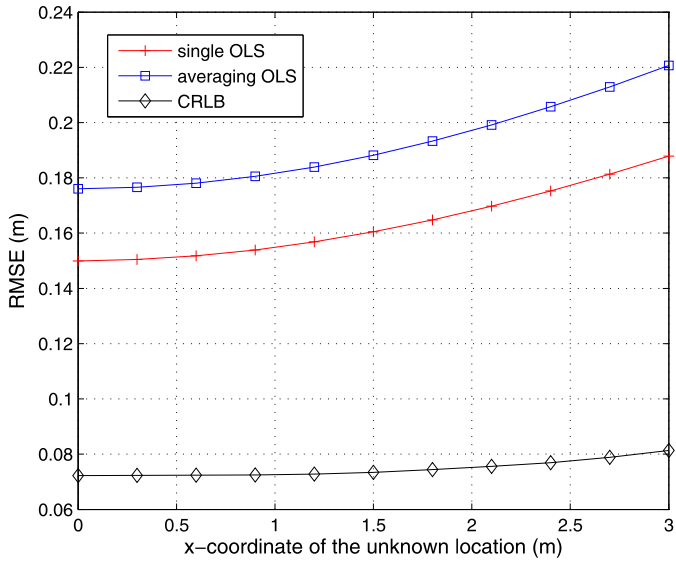

Fig. 2. RMSE of OLS location estimate versus the x-coordinate of the unknown location ($p_y = 0$ m, 6 distance measurements, 10 000 noise realizations).

critical value is important in order to make a fair comparison of the single and averaging laterations methods.

The case of 10 distance measurements In order to investigate the situation when there is a sufficiently large number of measurements in the sense of available subspace dimension, the number of sensors is increased to ten. In this case, there are more than one possible ways of averaging. The averaging can be performed among the estimates obtained with five-tuples, six-tuples, seven-tuples, eight-tuples or nine-tuples of distance measurements. We can give the number of estimates used in averaging by n combinations out of 10, C_n^{10}, defined in (21), where $n = 5, \ldots, 9$. In Fig. 3, the RMSE performances of the single and averaging laterations are plotted
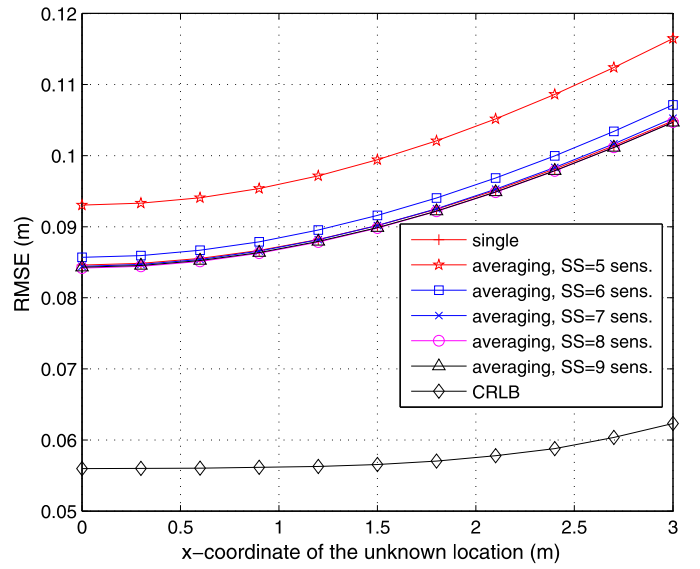

Fig. 3. RMSE of OLS location estimate versus the x-coordinate of the unknown location ($p_y = 0$ m, 10 distance measurements, 10 000 noise realizations).

as a function of the x-coordinate of the unknown location. The figure depicts that averaging OLS lateration estimates of five sensors and six sensors result in larger RMSE values compared to the case of single lateration with ten sensors. However, starting with the average of OLS estimates with seven distance measurements, the averaging slightly improves the estimation performance. The best performance is obtained for a subgroup size of eight measurements. Increasing the size of the subgroups to nine decreases the performance slightly which is still better than the estimate with single lateration.

4. Non-Gaussian distance noise and robust averaging

Sometimes the noise component in the distance measurements can diverge from Gaussianity. This event can be a consequence of a contemporary failure of a certain sensor or a disturbance in measurements or an erroneous transfer of the measurements. One way of modeling such an outlier is to adopt a MOG noise model for the distance measurement:

$$d_i = d_{t_i} + (1 - q)\sigma_i\epsilon_i + q(L\sigma_i)\delta_i \qquad (23)$$

where ϵ_i and δ_i are zero mean Gaussian random variables with unity standard deviation, q is a positive real number which is much smaller than one and L is a large number, e.g. 100. Then the distance measurement will have a Gaussian contamination with a nominal standard deviation σ_i. Occasionally, the additive noise will come from a Gaussian distribution of much higher standard deviation compared to the nominal one. In such a scenario, any estimator which is a function of a measurement vector containing a large error in one or more components will produce a more noisy location estimate compared to the estimators relying on purely nominal Gaussian noise contaminated vectors. When multiple location estimates obtained with subgroups of total measurements are available, the estimates that are based on subgroups containing an outlier will be highly erroneous. On the other hand, the estimates that are based on purely nominal Gaussian noise contamination will exhibit nominal errors. Instead of simple arithmetic averaging, a robust averaging operation which discards the outliers of the estimates can attain approximately the performance of the final location estimates obtained when no outliers are present. Robust methods are known for offering successful results in the case of data with outliers [16].

In this work, we adopt an outlier detection based robust method and for that we suggest a robust averager that follows the steps below in order to decide whether to use a particular estimate in averaging or not:

1. Obtain simple arithmetic average of estimates;
2. Order the estimates according to their distance from the average;
3. Discard the estimates whose distances are farther away from the average compared to a certain multiple of the median distance.

This averager resembles to the MTM averager defined in [21] and later also formulated in [26]. Adopting the formulation in [26] we can express our robust averager as:

$$\hat{\mathbf{p}}_{\text{rob-av}}^{(n)} = \frac{\sum_{m=1}^{M} \xi_m \hat{\mathbf{p}}_m^{(n)}}{\sum_{m=1}^{M} \xi_m} \tag{24}$$

where

$$\xi_m = \begin{cases} 1 & \text{if } \gamma_m < \theta \text{ median}(\gamma_i) \\ 0 & \text{else} \end{cases} \quad \text{for } i = 1, \ldots, M \tag{25}$$

and

$$\gamma_m = \left\| \mathbf{p}_m^{(n)} - \mathbf{p}_{\text{av}}^{(n)} \right\|.$$

Here θ denotes a user defined constant factor which determines whether or not a particular estimate will be counted in the robust average when multiplied by the median distance from the centroid of estimates. Instead of the centroid the original MTM in [21,26] was using the median of 1-D location parameter values.

4.1. Simulations with non-Gaussian data

In the simulations with MOG distance noise the nominal standard deviation of the Gaussian noise in the distance measurements is 5 cm as before but with 0.01 probability this standard deviation becomes 1 m.

Again some preliminary results with LS-TOA, OLS-TOA and LS-TDOA location estimators in the case of MOG measurement noise are given in Tables 1, 2 and 3, respectively. These tables show that again a minimum number of five measurements are needed for reasonable estimates for TOA based estimators whereas this number is six for LS-TDOA based location estimator. These minimum subgroup sizes are also important for the tabulated results of robust averaging. Since a smaller subgroup size is more advantageous as will be explained in Section 4.1.2, the smallest RMSE values in the case of LS-TOA, OLS-TOA and LS-TDOA estimators are obtained for these minimum subgroup sizes. Considering the last line of Table 1, LS-TOA based estimator with the proposed robust averaging attains an RMSE of 8.41 cm which is only 3.17 cm larger than the CRLB when $N = 12$ and $n = 5$. OLS-TOA exhibits similar performance and LS-TDOA based estimator achieves its best performance in the case of $N = 15$ and $n = 6$. Note that CRLB values in the MOG noise cases are only slightly larger than the case of Gaussian noise. However, the performance of the estimators significantly deteriorate with single lateration and averaging laterations methods utilizing ordinary arithmetic averaging. The subsequent simulation studies investigate appropriate choices to regain this performance loss with robust averaging applied in averaging laterations.

4.1.1. Comparison of simple and robust averaging

In Fig. 4, RMSE results of LS-TOA location estimator are plotted. Without averaging the minimum RMSE is smaller than 10 cm in the Gaussian noise case and it increases to 22 cm in the MOG case. The simple arithmetic averaging of (20) brings no good. However
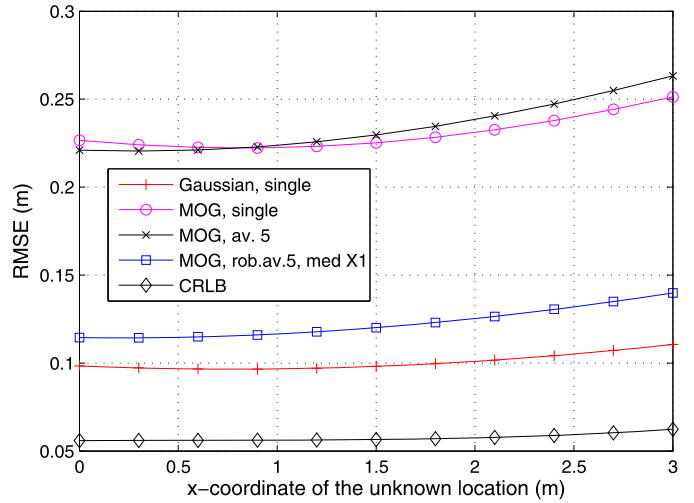

Fig. 4. RMSE of LS-TOA location estimate versus the *x*-coordinate of the unknown location (Gaussian, MOG, MOG averaging with 5 sensors and robust MOG averaging with 5 sensors median times trimming factor one, $p_y = 0$ m, 10 distance measurements, 10 000 noise realizations).

the robust averager makes the estimator attain a minimum RMSE value which is smaller than 12 cm, a performance only 2 cm worse than the one in pure Gaussian noise contamination. The CLRB for Gaussian measurement noise which is 5.60 cm for the coordinate $\mathbf{p}_1 = (0, 0, 0)^T$ m is also plotted in this figure. The difference of the CRLBs for Gaussian and MOG noises is indistinguishable as shown in Tables 1, 2 and 3. So, only one of the CRLB curves is plotted in the figures for the sake of clarity of the illustration.

4.1.2. Dependence on the size of subgroups

The effects of subgroup size are investigated next. Firstly, let us define the probability $P(H_i)$ as the probability that i of the measurements come from the Gaussian pdf of high variance. Then

$$P(H_0) = C_0^{10}(1-q)^{10}q^0 = 0.99^{10} \approx 0.9, \tag{26}$$

and

$$P(H_1) = C_1^{10}(1-q)^9 q = 10 \times 0.99^{10} \times 0.1 \approx 0.1. \tag{27}$$

Given the condition that one of the ten measurements comes from the Gaussian pdf of high variance, the number of five-tuples which does not contain that sample is C_5^9. Since the total number of five-tuples is C_5^{10}, the probability that a subgroup contains the sample having high variance, given that one of the measurements comes from the pdf with high variance, can be given as

$$P(R_5 \mid H_1) = \left(1 - \frac{C_5^9}{C_5^{10}}\right) = 0.5 \tag{28}$$

where the event $\{R_j\}$ corresponds to the case that subgroups of size j contain the sample having the high variance. Similarly, one can find the corresponding conditional probabilities for subgroups of size 6 and 7, as 0.6 and 0.7, respectively.

In Fig. 5 the RMSE curves of LS-TOA location estimator with different subgroup sizes in robust averaging are shown. The best performance is achieved with a subgroup size of 5 distance measurements which attains an RMSE value of less than 12 cm, only less than 2 cm worse compared to the case of Gaussian sensor noise and single lateration. With a subgroup size of 6 and 7 measurements, the lowest RMSE values are 17 cm and 23 cm, respectively. Note that, with subgroup size of 7 measurements, the performance is even worse than the one of the single lateration estimator. These performances are in accordance with the order of
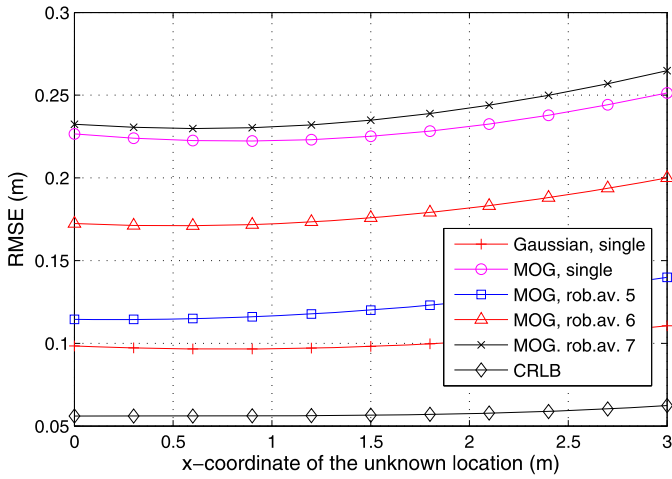
Fig. 5. RMSE of LS-TOA location estimate versus the *x*-coordinate of the unknown location (Gaussian, MOG, MOG robust median times trimming factor of one with 5, 6, 7 sensor averaging, $p_y = 0$ m, 10 distance measurements, 10 000 noise realizations).
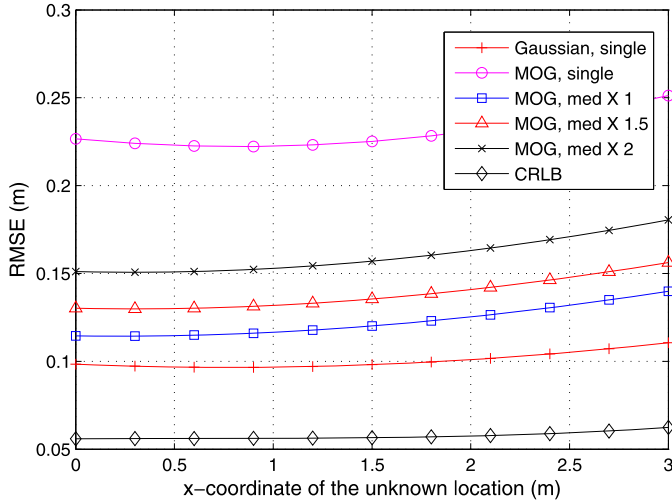

Fig. 6. RMSE of LS-TOA location estimate versus the *x*-coordinate of the unknown location (Gaussian, MOG, MOG robust median with 5 sensor averaging and median trimming factor of 1, 1.5 and 2, 10 distance measurements, 10 000 noise realizations).

the calculated probabilities of the subgroups containing one distance measurement of high variance.

4.1.3. Dependence on the trimming factor of robust averager

Fig. 6 shows the effect of changing the threshold on the distances from the average of the estimates in robust averaging. Again a subgroup size of 5 is chosen for the distance measurements. The thresholds applied to the estimators are 1, 1.5 and 2 times the distance of the median of distances. Note that a threshold factor of 1 corresponds to trimming, i.e. discarding, the estimates that are farther away from the arithmetic average compared to the median value of the sorted distances from the average. The higher that factor, the farther away estimates from the arithmetic average of estimates are taken into account. The best performance is achieved when the trimming factor is one.

TOA-based LS estimators considered in this study were suitable for investigating several parameters in averaging laterations method. But they share the property that their performance is significantly worse compared to theoretical lower bound on the variance of estimates. So, the performance of QRD based closed form but non-scalable estimator is investigated next.
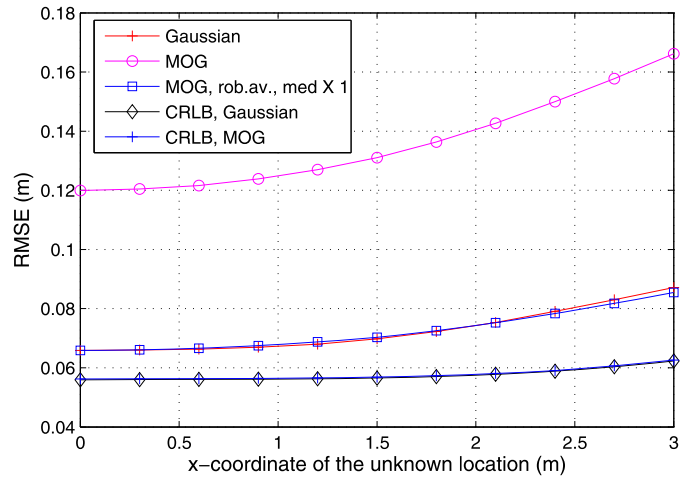

Fig. 7. RMSE of QRD location estimate versus the *x*-coordinate of the unknown location (Gaussian, MOG, MOG robust median with trimming factor of one, 10 distance measurements, 10 000 noise realizations).

4.1.4. Robust averaging for QRD based location estimator

The QRD based location estimator cannot use more than three measurements at once. However, it can use subgroups of size three formed from those measurements. So, it is inherently, an averaging laterations type location estimator when $N > 3$.

The performance of the QRD location estimator is shown in Fig. 7 for the cases of Gaussian and MOG data. In the Gaussian noise case the estimator achieved an RMSE of smaller than 7 cm, which is only 1 cm worse than the CRLB [28] when *x*-coordinate of the unknown location is zero. The final QRD location estimate is obtained as the average of C_3^{10} intermediate estimates. In the case of MOG distance measurement noise its performance gets significantly worse, as much as 12 cm RMSE for $x = 0$. When robust averaging is applied, it reaches nearly the performance in the case of Gaussian noise which is approximately 6.6 cm RMSE at $x = 0$ m and 8.6 cm RMSE at $x = 3$ m. In this figure also the CRLB curves in the cases of Gaussian and MOG noises are shown. The CRLB when the noise in distance measurements comes from a MOG distribution is only slightly larger than the CRLB in the Gaussian case. However, Figs. 4–7 show that the effect of MOG noise on the performance of considered estimators is significant unless it is mitigated with a suitable technique such as the considered robust averaging.

In [15], for the same problem the QRD based location estimator with standard α-trimmed-mean (α-TM) averaging was used. That robust averager was discarding $\alpha\%$ of estimates farthest away from the arithmetic average or centroid of them. Note that for 50% of trimming, which means keeping the estimates with distances that are smaller than or equal to the median distance from the centroid, that averager is equivalent to the MTM robust averager in (24) with $\theta = 1$.

Calculation of the CRLB for the Gaussian and MOG contamination of distance measurements are given in Appendices A and B.

5. Discussion and conclusion

In order to consider averaging in lateration based location estimation, the first requirement is that the size of the subgroups formed from the measurements should be large enough that a further increase in that size brings little subspace gain compared to the variance reduction which will be offered by averaging estimates. In the simulations with TOA-based LS estimators, averaging estimates formed by five measurements performed significantly worse than the single lateration estimate of six measurements, but

starting with a subgroup size of seven measurements, the contribution of subspace gain was smaller which made averaging advantageous.

When the noise in measurements was Gaussian with a reasonable variance, the gain offered by averaging was limited. However obtaining a group of estimates offer the possibility of classifying those estimates easily as being useful or not. This type of classification is more difficult for sensor measurements. In particular, when some of the measurements can contain a noise component of high variance, single lateration methods and simple arithmetic averaging of lateration estimates are severely affected, increasing the RMSE from 10 cm in Gaussian noise case to 22 cm in the MOG noise case used for modeling occasional high variance of the measurement noise. On the contrary, the proposed robust averaging of lateration estimates method was very effective in eliminating that occasional noise, making the RMSE as low as 12 cm.

In the MOG noise case a smaller subgroup size of estimates was more successful, which was required in order to have a large proportion of subgroups not contaminated by a noise component of high variance. The non-scalable QRD-based location estimator which is more successful compared to the considered TOA-based LS location estimator was utilized for comparing simple arithmetic and robust averaging in the Gaussian and MOG noise cases. The performance of that estimator in MOG noise with robust averaging was indistinguishable from its performance in Gaussian noise with simple averaging, both of which were just 1 cm worse than the CRLB. This behavior is expectable since the CRLBs in the Gaussian noise case and in the MOG noise case which are derived in Appendices A and B, respectively, are not distinguishable for all practical purposes.

Now, let us discuss the computational complexities of the estimators in this work. For an overdetermined linear system of m equations and n unknowns, the complexity of the LS algorithm is about order of $O(mn^2)$ [27] which means that for 3-D location estimation by lateration using N sensors the LS-TOA and OLS-TOA methods have a complexity of $O(9N)$ whereas the complexity of LS-TDOA based estimator is $O(16N)$, neglecting the decrease in the number of equations due to subtracting equations and assuming that N is sufficiently large. On the other hand, the complexity with QRD is approximately $O(2mn^2)$. So, the complexity of the QRD based method in (13) is $C_3^N O(54)$. For the LS-TOA and OLS-TOA methods where averaging laterations lead to similar performance as single lateration when $N = 10$ and subgroup size is seven, we have a computational complexity at the order of $C_7^{10} O(9 \times 7)$ and for the robust averaging with $N = 10$ and subgroup size $n = 5$, the computational complexity is about $C_5^{10} O(9 \times 5)$.

Finally, let us mention some possible directions for future research which can focus on outlier detection procedures together with or separately from studies on the measurement noise nature. In this work, the parameter θ defining the distance threshold from the centroid for rejecting the outliers was chosen empirically to assure good rejection performance in a worst case scenario. A future study can be optimizing the value of this parameter for a given MOG noise statistic. Another possible research direction might be to consider tracking of an object. In that case, more severely contaminated measurements can be tolerable while detecting the target and this will affect the outlier detection procedure and require a corresponding adaptation.

Appendix A. The CRLB for TOA model with Gaussian noise in distance measurements

When N measured distances obey (1) and (2), assuming independence of the measurements and identical variance, σ^2, the joint probability density function (pdf) of distance measurements conditioned on the unknown location is given as [7]:

$$f(\mathbf{d} \mid \mathbf{p}) = \left(\frac{1}{\sqrt{2\pi}\sigma}\right)^N \exp\left\{-\frac{1}{2\sigma^2}\sum_{i=1}^{N}(d_i - \|\mathbf{p} - \mathbf{a}_i\|)^2\right\}. \quad (A.1)$$

For finding the CRLB the partial derivative of the natural logarithm of this pdf with respect to the components of $\mathbf{p}$ will be found. One finds

$$\frac{\partial \ln(f(\mathbf{d} \mid \mathbf{p}))}{\partial p_x} = \frac{1}{\sigma^2}\sum_{i=1}^{N}\frac{(d_i - d_{t_i})}{d_{t_i}}(p_x - x_i). \quad (A.2)$$

The other partial derivatives are found similarly. So, the Fisher information matrix (FIM) [28] can be given as:

$$\mathbf{I_p} = E\left[\left(\frac{\partial \ln(f(\mathbf{d} \mid \mathbf{p}))}{\partial \mathbf{p}}\right)\left(\frac{\partial \ln(f(\mathbf{d} \mid \mathbf{p}))}{\partial \mathbf{p}}\right)^T\right]. \quad (A.3)$$

The CRLB for the variance of the location estimate will be found as:

$$\mathrm{var}(\hat{\mathbf{p}}) \geqslant \sum_{i=1}^{3}[\mathbf{I_p}]_{i,i}^{-1}. \quad (A.4)$$

Appendix B. The CRLB for TOA model with MOG noise in distance measurements

When N measured distances obey (1) and (23), assuming independent and identical pdf for the distance measurements with the same nominal Gaussian variance, σ^2, the pdf of distance measurements conditioned on the unknown location will be given by:

$$f(d_i \mid \mathbf{p}) = \frac{1-q}{\sqrt{2\pi}\sigma}\exp\left\{-\frac{1}{2\sigma^2}(d_i - \|\mathbf{p} - \mathbf{a}_i\|)^2\right\}$$
$$+ \frac{q}{\sqrt{2\pi}L\sigma}\exp\left\{-\frac{1}{2(L\sigma)^2}(d_i - \|\mathbf{p} - \mathbf{a}_i\|)^2\right\}. \quad (B.1)$$

Since the noise components of the distance measurements are independent, their joint pdf conditioned on the unknown location is obtained as the product of the individual conditional pdfs and one can obtain

$$\frac{\partial \ln(f(\mathbf{d} \mid \mathbf{p}))}{\partial p_x} = \frac{\partial \ln(f(d_1 \mid \mathbf{p}))}{\partial p_x} + \cdots + \frac{\partial \ln(f(d_N \mid \mathbf{p}))}{\partial p_x}. \quad (B.2)$$

Let us express (B.1) as

$$f(d_i \mid \mathbf{p}) = \frac{1}{\sqrt{2\pi}\sigma}\left[\underbrace{(1-q)\exp\left\{-\frac{1}{2\sigma^2}(d_i - \|\mathbf{p} - \mathbf{a}_i\|)^2\right\}}_{A_{i,1}}\right.$$
$$\left. + \underbrace{\frac{q}{L}\exp\left\{-\frac{1}{2(L\sigma)^2}(d_i - \|\mathbf{p} - \mathbf{a}_i\|)^2\right\}}_{A_{i,2}}\right]. \quad (B.3)$$

Then one can find

$$\frac{\partial \ln(f(d_i \mid \mathbf{p}))}{\partial p_x}$$
$$= \frac{1}{(A_{i,1} + A_{i,2})\sigma^2}\left[\left(A_{i,1} + \frac{A_{i,2}}{L^2}\right)\left(\frac{d_i - d_{t_i}}{d_{t_i}}\right)(p_x - x_i)\right]. \quad (B.4)$$

We use this result in (B.2) and then using (A.3) the FIM is obtained. The CRLB follows easily putting FIM into (A.4).

Appendix C. CRLB for TDOA under MOG measurement noise

Let us assume that there are N measurements but due to an unknown common constant in each of these measurements, the differences of distances are known obeying

$$d_i - d_1 = d_{t_i} - d_{t_1} + (1-q)\sigma_i \epsilon_i + q(L\sigma_i)\delta_i$$
$$- (1-q)\sigma_1 \epsilon_1 - q(L\sigma_1)\delta_1. \tag{C.1}$$

Define

$$\mathbf{d}' = \begin{bmatrix} (d_2 - d_1) & (d_3 - d_1) & \cdots & (d_N - d_1) \end{bmatrix}^T \tag{C.2}$$

and

$$\mathbf{d}'_t = \begin{bmatrix} (d_{t_2} - d_{t_1}) & (d_{t_3} - d_{t_1}) & \cdots & (d_{t_N} - d_{t_1}) \end{bmatrix}^T \tag{C.3}$$

where $\{d_{t_i}, \ i = 1, \ldots, N\}$ and $\{d_i, \ i = 1, \ldots, N\}$ denote noisy and noiseless distance measurements defined in (1) and (2), respectively. Since each measurement comes from a MOG distribution, each element of the vector $\mathbf{d}'$ is the sum of two variables with MOG pdf. However a more advantageous representation of the same vector is a mixture of multivariate Gaussian pdfs. So, the likelihood function can be given as

$$f(\mathbf{d}' \mid \mathbf{p}) = \sum_{l=0}^{2^N-1} \pi(l)\big((2\pi)^{N-1}|\mathbf{K}_l|\big)^{-1/2}$$
$$\times \exp\left\{-\frac{1}{2}(\mathbf{d}' - \mathbf{d}'_t)^T \mathbf{K}_l^{-1}(\mathbf{d}' - \mathbf{d}'_t)\right\}. \tag{C.4}$$

A particular mixture probability $\pi(l)$ can be obtained by counting the number of ones in the binary representation of the number l. $l = 0$ corresponds to the case that all measurement noises come from the nominal Gaussian pdf with variance σ^2, and $l = 5$ means that d_1 and d_3 have variance $L^2\sigma^2$. Denoting the number of ones in the binary representation of l as n_1, the probability $\pi(l)$ is given by:

$$\pi(l) = (1-q)^{N-n_1} q^{n_1}.$$

As an example, $\pi(0) = (1-q)^N$ and $\pi(3) = (1-q)^{N-2} q^2$. Again, for finding the CRLB the partial derivative of the natural logarithm of this pdf with respect to the components of $\mathbf{p}$ will be found. One finds

$$\frac{\partial \ln(f(\mathbf{d}' \mid \mathbf{p}))}{\partial p_x}$$
$$= \frac{\sum_{l=0}^{2^N-1} h(l)(\mathbf{d}' - \mathbf{d}'_t)^T \mathbf{K}_l^{-1} \left(\frac{p_x - x_i}{d_{t_i}} - \frac{p_x - x_1}{d_{t_1}}\right)}{\sum_{l=0}^{2^N-1} h(l)} \tag{C.5}$$

where

$$h(l) = \pi(l)|\mathbf{K}_l|^{-1/2} \exp\left\{-\frac{1}{2}(\mathbf{d}' - \mathbf{d}'_t)^T \mathbf{K}_l^{-1}(\mathbf{d}' - \mathbf{d}'_t)\right\}$$

and

$$\mathbf{B} = \frac{\sum_{l=0}^{2^N-1} h(l)\mathbf{K}_l^{-1}}{\sum_{l=0}^{2^N-1} h(l)}.$$

The other partial derivatives are found similarly leading to:

$$\frac{\partial \ln(f(\mathbf{d}' \mid \mathbf{p}))}{\partial \mathbf{p}} = \mathbf{GB}(\mathbf{d}' - \mathbf{d}'_t) \tag{C.6}$$

where

$$\mathbf{G} = \begin{bmatrix} \left(\frac{p_x - x_2}{d_{t_2}} - \frac{p_x - x_1}{d_{t_1}}\right) & \cdots & \left(\frac{p_x - x_N}{d_{t_N}} - \frac{p_x - x_1}{d_{t_1}}\right) \\ \left(\frac{p_y - y_2}{d_{t_2}} - \frac{p_y - y_1}{d_{t_1}}\right) & \cdots & \left(\frac{p_y - y_N}{d_{t_N}} - \frac{p_y - y_1}{d_{t_1}}\right) \\ \left(\frac{p_z - z_2}{d_{t_2}} - \frac{p_z - z_1}{d_{t_1}}\right) & \cdots & \left(\frac{p_z - z_N}{d_{t_N}} - \frac{p_z - z_1}{d_{t_1}}\right) \end{bmatrix}.$$

Substituting (C.6) into (A.3) one finds FIM as

$$\mathbf{I_p} = E\big[\mathbf{GB}(\mathbf{d}' - \mathbf{d}'_t)(\mathbf{d}' - \mathbf{d}'_t)^T \mathbf{B}^T \mathbf{G}^T\big]$$

and CRLB is obtained as in (A.4). Note that for Gaussian measurement noises $\mathbf{B}$ reduces to

$$\mathbf{K}^{-1} = \begin{bmatrix} (N-1) & -1 & \cdots & -1 \\ -1 & (N-1) & \ddots & \vdots \\ \vdots & \ddots & \ddots & -1 \\ -1 & \cdots & -1 & (N-1) \end{bmatrix} \tag{C.7}$$

and FIM becomes

$$\mathbf{I_p} = E\big[\mathbf{GK}^{-1}(\mathbf{d}' - \mathbf{d}'_t)(\mathbf{d}' - \mathbf{d}'_t)^T \mathbf{K}^{-1}\mathbf{G}^T\big].$$

If the random component in the z-coordinates of the sensors are removed FIM reduces to

$$\mathbf{I_p} = \mathbf{GK}^{-1}\mathbf{G}^T$$

which is the same as the FIM found in [7].

References

[1] H. Liu, H. Darabi, P. Banerjee, J. Liu, Survey of wireless indoor positioning techniques and systems, IEEE Trans. Syst. Man Cybern., Part C Appl. Rev. 37 (2007) 1067–1080.

[2] Y. Gu, A. Lo, I. Niemegeers, A survey of indoor positioning systems for wireless personal networks, IEEE Commun. Surv. Tutor. 11 (2009) 13–31.

[3] J. Hightower, G. Borriello, Location sensing techniques, University of Washington, Computer Science and Engineering, 2001, UW-CSE-01-07-01.

[4] I.D. Coope, Reliable computation of the points of intersection of n spheres in R^n, ANZIAM J. 42 (E) (2000) C461–C477.

[5] W. Navidi, W.S. Murphy Jr., W. Hereman, Statistical methods in surveying by trilateration, Comput. Statist. Data Anal. 27 (1998) 209–227.

[6] A. Beck, P. Stoica, Exact and approximate solutions of source localization problems, IEEE Trans. Signal Process. 56 (5) (2005) 1770–1777.

[7] Y.T. Chan, K.C. Ho, A simple and efficient estimator for hyperbolic location, IEEE Trans. Signal Process. 42 (8) (1994) 1905–1915.

[8] J.J. Caffery Jr., A new approach to the geometry of TOA location, in: Proc. IEEE Vehicular Technology Conf., 2000, pp. 24–28.

[9] K.W. Cheung, H.C. So, W.-K. Ma, Y.T. Chan, Least squares algorithms for time-of-arrival-based mobile location, IEEE Trans. Signal Process. 52 (4) (2004) 1121–1128.

[10] F. Thomas, L. Ros, Revisiting trilateration for robot localization, IEEE Trans. Robot. 21 (1) (2005) 93–101.

[11] D.E. Manolakis, Efficient solution and performance analysis of 3-D position estimation by trilateration, IEEE Trans. Aerosp. Electron. Syst. 32 (4) (1996) 1239–1248.

[12] Y. Zhou, An efficient least squares trilateration algorithm for mobile robot localization, in: 2009 IEEE/RSJ Int. Conf. on Intelligent Robots and Systems, October 11–15, St. Louis, USA, 2009.

[13] Y. Zhou, A closedform algorithm for the leastsquares trilateration problem, Robotica 29 (3) (2011) 375–389.

[14] J.S. Abel, A divide and conquer approach to least squares estimation, IEEE Trans. Aerosp. Electron. Syst. 26 (2) (1990) 423–427.

[15] M.A. Altınkaya, Alpha-trimmed means of multiple location estimates, in: 21st IEEE Signal Processing and Communications Conference (SIU13), Kyrenia, North Cyprus, April 24–26, 2013 (in Turkish).

[16] P.J. Huber, The 1972 Wald Lecture. Robust statistics: A review, Ann. Math. Statist. 43 (4) (1972) 1041–1067.

[17] N.L. Knight, J. Wang, A comparison of outlier detection procedures and robust estimation methods in GPS positioning, J. Navig. 62 (2009) 699–709.

[18] R. Pfeil, S. Schuster, P. Scherz, A. Stelzer, G. Stelzhammer, A robust position estimation algorithm for a local positioning measurement system, in: IEEE MTT-S International Microwave Workshop on Wireless Sensing, Local Positioning and RFID, IMWS 2009, Croatia, 2009.

[19] K. Fallahi, C.-T. Cheng, M. Fattouche, Robust positioning systems in the presence of outliers under weak GPS signal conditions, IEEE Syst. J. 6 (3) (2012) 699–709.
[20] J.N. Ash, R.L. Moses, Outlier compensation in sensor network self-localization via the EM algorithm, in: International Conference on Acoustics, Speech and Signal Processing, vol. 4, ICASSP, 2005, pp. 749–752.
[21] Y.H. Lee, Generalized median filtering and related nonlinear filtering techniques, IEEE Trans. Signal Process. 33 (3) (1985) 672–683.
[22] R. Want, A. Hopper, V. Falcão, J. Gibbons, The active badge location system, ACM Trans. Inf. Syst. 10 (1) (1992) 91–102.
[23] Y.T. Chan, H.Y.C. Hang, P. Ching, Exact and approximate maximum likelihood localization algorithms, IEEE Trans. Veh. Technol. 55 (1) (2006) 10–16.
[24] A. Kushki, K.N. Plataniotis, A.N. Venetsanopoulos, WLAN Positioning Systems, Cambridge Univ. Press, 2012.
[25] A.L. Garcia, Probability, Statistics and Random Processes for Electrical Engineering, 3rd ed., Pearson Education, Inc., 2009.
[26] G.R. Arce, Nonlinear Signal Processing: A Statistical Approach, John Wiley and Sons, Inc., 2005.
[27] G.H. Golub, C.F. Van Loan, Matrix Computations, 3rd ed., The Johns Hopkins University Press, 1996.
[28] Harry L. Van Trees, Detection, Estimation, and Modulation Theory, John Wiley and Sons, Inc., 1968.

Mustafa A. Altınkaya received the B.S., M.S. and Ph.D. degrees in electrical and electronics engineering from Boğaziçi University, Istanbul, Turkey, in 1987, 1990 and 1996, respectively.

From 1989 to 1990 he was a research associate in TUBITAK Marmara Research Center, Kocaeli, Turkey. From 1991 to 1996 he was a research and teaching assistant in Boğaziçi University. He worked as a part-time instructor in Air Force Academy, Naval Academy and Boğaziçi University during periods from 1995 to 1998. In 1998 he joined the Electrical and Electronics Engineering Department at Izmir Institute of Technology where he has been an assistant professor.

He is the author of more than 20 publications. His research interests are mainly on statistical signal processing and its applications in telecommunications, source separation, structural dynamic systems, and cheminformatics.