Some Remarks on Exp-Function Method and Its Applications-A Supplement

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Abstract　*Recently, the authors of [Commun. Theor. Phys. **56** (2011) 397] made a number of useful observations on Exp-function method. In this study, we focus on another vital issue, namely, the misleading results of double Exp-function method.*

1 Introduction

　　Nonlinear evolution equations (NEEs) that give rise to solitary waves can be seen in many areas of nonlinear physical sciences. Traveling waves of NEEs may be coupled with different frequencies and different velocities. Multi-wave solutions are crucial because they may sometimes be converted into a single soliton of very high energy that propagates over large domains of space without dispersion. Thus, a destructive wave may be occured. A tsunami (a very long ocean wave caused by an underwater earthquake) can be a good example for this kind of phenomena. Searching for exact solutions of NEEs with multi-velocities and multi-frequencies is an attractive research area. Over decades, NEEs have been analyzed for such solutions by using some elegant methods. To make mention of a few, inverse scattering method,[1] Hirota bilinear method,[2] Backlund transformation,[3] and Painlevé analysis.[4] Among those, Hirota bilinear method seems to be a very powerful technique working well for completely integrable equations. But, depending on problems, some methods work perfectly while others not.

　　Lately, Exp-function method[5] drew the attention of many researchers. Quite a few important NEEs have been tackled by this popular approach. As expected, it has been adapted, generalized and extended for different kinds of problems as well as for multi-waves (n-soliton solutions,[6−7] multi-exp function method,[8] double-exp function method[9]). In the mean time, a number of pitfalls of Exp-function method came to our attention, see Refs. [10−20]. Exp-function method assumes an ansatz (a rational combination of exponential functions) involving many unknown parameters to be specified at the stage of solving the problem. Because of a resulting highly nonlinear system of algebraic equations, computations become very lengthy and intractable by hand without a computer algebra system (such as *MATHEMATICA, MAPLE*, or

MATLAB). There should never be a blind trust in using Exp-function method. Being in eagle-eyed solving mode, one should always be able to justify why the output is a believable answer. The authors of Ref. [18] made the following useful remarks (giving examples from the literature) on the encountered shortcomings of Exp-function method:

　　(i) A careless application of Exp-function method leads to incorrect solutions;

　　(ii) A single case of Exp-function method may lead to equivalent solutions;

　　(iii) Different cases of Exp-function method may lead to equivalent solutions;

　　(iv) Some cases of Exp-function method are equivalent;

　　(v) The balancing procedure of Exp-function method seems redundant;

　　(vi) Not every generalization of Exp-function method is convenient.

　　Recently, Bekir[21] investigated multisoliton solutions to the Cahn–Allen equation

$$u_t = u_{xx} - u^3 + u\,, \tag{1}$$

by means of the ansatz (due to double-exp function method[9])

$$u = \frac{a_1\,\mathrm{e}^\xi + a_2\,\mathrm{e}^{-\xi} + a_5 + a_3\,\mathrm{e}^\eta + a_4\,\mathrm{e}^{-\eta}}{b_1\,\mathrm{e}^\xi + b_2\,\mathrm{e}^{-\xi} + b_5 + b_3\,\mathrm{e}^\eta + b_4\,\mathrm{e}^{-\eta}}\,,$$

$$\xi = c_1 x + c_2 t\,, \quad \eta = c_3 x + c_4 t\,. \tag{2}$$

The Cahn–Allen equation (1) is a reaction-diffusion type semilinear parabolic equation which has been the subject of extensive research work from numerical and analytical points of view.

　　In a previous study,[18] the misleading results of double-exp function method were already brought to discussion (item (vi) above). Taking the remarks of the authors[10−20] into account, we made a careful analysis for the findings of Ref. [21]. The author presented one-soliton

solutions for eight different cases, however, all cases (except an incorrect one) can be further reduced to two previously known ones. We observed that double–exp function method was incapable of providing any two-soliton solutions once again. Contrary to the author's[21] belief, all two-soliton solutions (except an incorrect one) for six different cases are indeed equivalent to one-soliton solutions. What is more it was mistakenly indicated that supposedly "new" two-soliton solutions to Eq. (1) were published for the first time. In the forthcoming sections, we shall refute the author's[21] argument by discussing his results in detail.

2 One-Soliton Solutions to Eq. (1) Obtained by the Double Exp-Function Method

The author[21] found one-soliton solutions for eight different cases. These solutions (numbered as (I.1)–(I.8) in Ref. [21]) were given as follows:

$$u_1(x,t) = \frac{\pm b_5}{b_5 + b_4\,\mathrm{e}^{-(c_3 x + c_4 t)}}, \quad c_3 = \frac{1}{\sqrt{2}}, \quad c_4 = \frac{3}{2}, \quad (3)$$

$$u_2(x,t) = \frac{\pm b_5}{b_5 + b_2\,\mathrm{e}^{-(c_1 x + c_2 t)}}, \quad c_1 = \frac{1}{\sqrt{2}}, \quad c_2 = \frac{3}{2}, \quad (4)$$

$$u_3(x,t) = \frac{\pm b_5}{b_5 + b_1\,\mathrm{e}^{(c_1 x + c_2 t)}}, \quad c_1 = \frac{1}{\sqrt{2}}, \quad c_2 = -\frac{3}{2}, \quad (5)$$

$$u_4(x,t) = \frac{\pm b_4\,\mathrm{e}^{-(c_3 x + c_4 t)}}{b_5 + b_4\,\mathrm{e}^{-(c_3 x + c_4 t)}}, \quad c_3 = \frac{1}{\sqrt{2}}, \quad c_4 = -\frac{3}{2}, \quad (6)$$

$$u_5(x,t) = \frac{\pm b_4\,\mathrm{e}^{-(c_3 x + c_4 t)}}{b_2 + b_5 + b_4\,\mathrm{e}^{-(c_3 x + c_4 t)}},$$
$$c_3 = \frac{1}{\sqrt{2}}, \quad c_4 = -\frac{3}{2}, \quad (7)$$

$$u_6(x,t) = \frac{\pm b_4 \pm b_5}{b_2\,\mathrm{e}^{-(c_1 x + c_2 t)} + b_5 + b_4},$$
$$c_1 = \frac{1}{\sqrt{2}}, \quad c_2 = \frac{3}{2}, \quad (8)$$

$$u_7(x,t) = \frac{\pm b_4\,\mathrm{e}^{-(c_3 x + c_4 t)}}{b_3\,\mathrm{e}^{(c_3 x + c_4 t)} + b_4\,\mathrm{e}^{-(c_3 x + c_4 t)}},$$
$$c_3 = \frac{1}{2\sqrt{2}}, \quad c_4 = -\frac{3}{4}, \quad (9)$$

$$u_8(x,t) = \frac{-a_5(a_5 - b_5)\,\mathrm{e}^{-(c_3 x + c_4 t)} + b_3 a_5}{\pm a_5(a_5 - b_5)\,\mathrm{e}^{-(c_3 x + c_4 t)} + b_3 b_5},$$
$$c_3 = \frac{1}{\sqrt{2}}, \quad c_4 = -\frac{3}{2}. \quad (10)$$

Unfortunately, the author left the reader with the impression that Eqs. (3)–(10) were distinct one-soliton solutions. However, by direct substitution, we figure out that $u_8(x,t)$ does not satisfy Eq. (1). Moreover, the remaining functions (3)–(9) can be readily simplified in the form

$$u_1(x,t) = \frac{\pm 1}{1 + A_1\,\mathrm{e}^{-((1/\sqrt{2})x + (3/2)t)}},$$

$$A_1 = \frac{b_4}{b_5}, \quad b_5 \neq 0, \quad (11)$$

$$u_2(x,t) = \frac{\pm 1}{1 + A_2\,\mathrm{e}^{-((1/\sqrt{2})x + (3/2)t)}},$$

$$A_2 = \frac{b_2}{b_5}, \quad b_5 \neq 0, \quad (12)$$

$$u_3(x,t) = \frac{\pm 1}{1 + A_3\,\mathrm{e}^{((1/\sqrt{2})x - (3/2)t)}},$$

$$A_3 = \frac{b_1}{b_5}, \quad b_5 \neq 0, \quad (13)$$

$$u_4(x,t) = \frac{\pm 1}{1 + A_4\,\mathrm{e}^{((1/\sqrt{2})x - (3/2)t)}},$$

$$A_4 = \frac{b_5}{b_4}, \quad b_4 \neq 0, \quad (14)$$

$$u_5(x,t) = \frac{\pm 1}{1 + A_5\,\mathrm{e}^{((1/\sqrt{2})x - (3/2)t)}},$$

$$A_5 = \frac{b_2 + b_5}{b_4}, \quad b_4 \neq 0, \quad (15)$$

$$u_6(x,t) = \frac{\pm 1}{1 + A_6\,\mathrm{e}^{-((1/\sqrt{2})x + (3/2)t)}},$$

$$A_6 = \frac{b_2}{b_4 + b_5}, \quad b_4 + b_5 \neq 0, \quad (16)$$

$$u_7(x,t) = \frac{\pm 1}{1 + A_7\,\mathrm{e}^{((1/\sqrt{2})x - (3/2)t)}},$$

$$A_7 = \frac{b_3}{b_4}, \quad b_4 \neq 0. \quad (17)$$

Now, by a close inspection, it can be seen that Eqs. (11), (12), and (16) are identical as well as Eqs. (13), (14), (15), and (17). Actually, only two exact solutions were derived:

$$u_{(1)}^{\pm}(x,t) = \frac{\pm 1}{1 + C_1\,\mathrm{e}^{-\xi}},$$
$$\xi = (1/\sqrt{2})x + (3/2)t, \quad (18)$$

$$u_{(2)}^{\pm}(x,t) = \frac{\pm 1}{1 + C_2\,\mathrm{e}^{\xi}},$$
$$\xi = (1/\sqrt{2})x - (3/2)t. \quad (19)$$

Taking, $C_1 = 0$ and $C_2 = 0$ give trivial solutions $u = \pm 1$. The functions (18) and (19) admit singularity for $C_1 < 0$ and $C_2 < 0$. Assuming $C_1 > 0$ and $C_2 > 0$, Eqs. (18) and (19) can be rewritten as

$$u_{(1)}^{\pm}(x,t) = \pm \frac{1}{2}\Big[1 + \tanh\Big(\frac{\xi + \xi_0}{2}\Big)\Big],$$
$$\xi = (1/\sqrt{2})x + (3/2)t, \quad \xi_0 = -\ln C_1,$$

$$u_{(2)}^{\pm}(x,t) = \pm \frac{1}{2}\Big[1 - \tanh\Big(\frac{\xi + \xi_0}{2}\Big)\Big],$$
$$\xi = (1/\sqrt{2})x - (3/2)t, \quad \xi_0 = \ln C_2,$$

which obviously lead to one-wave fronts (kink/antikink); for example (see Fig. 1), $u_{(1)}^{+}(\xi \to -\infty) = 0$ and
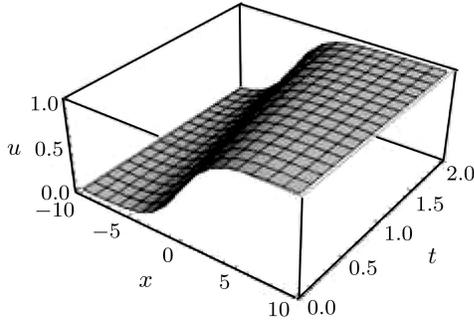
$$u^+_{(1)}(\xi \to \infty) = 1.$$

Fig. 1 A profile of one-wave front solution $u^+_{(1)}$ for $C_1 = 1$.

3 Two-Soliton Solutions to Eq. (1) Obtained by the Double Exp-Function Method

The author[21] derived two-soliton solutions for six different cases. These solutions (numbered as (II.1)–(II.6) in Ref. [21]) were presented as follows:

$$u_9(x,t) = \frac{\pm b_5}{b_2 \, \mathrm{e}^{-(c_1 x + c_2 t)} + b_5 + b_4 \, \mathrm{e}^{-(c_3 x + c_4 t)}} \,,$$

$$c_1 = c_3 = \frac{1}{\sqrt{2}} \,, \quad c_2 = c_4 = \frac{3}{2} \,, \tag{20}$$

$$u_{10}(x,t) = \frac{\pm b_5}{b_2 \, \mathrm{e}^{-(c_1 x + c_2 t)} + b_5 + b_3 \, \mathrm{e}^{(c_3 x + c_4 t)}} \,,$$

$$c_1 = -\frac{1}{\sqrt{2}} \,, \quad c_2 = \frac{3}{2} \,, \quad c_3 = \frac{1}{\sqrt{2}} \,, \quad c_4 = -\frac{3}{2} \,, \tag{21}$$

$$u_{11}(x,t) = \frac{\pm b_5}{b_1 \, \mathrm{e}^{(c_1 x + c_2 t)} + b_5 + b_4 \, \mathrm{e}^{-(c_3 x + c_4 t)}} \,,$$

$$c_1 = -\frac{1}{\sqrt{2}} \,, \quad c_2 = -\frac{3}{2} \,, \quad c_3 = \frac{1}{\sqrt{2}} \,, \quad c_4 = \frac{3}{2} \,, \tag{22}$$

$$u_{12}(x,t) = \frac{\pm b_5}{b_1 \, \mathrm{e}^{(c_1 x + c_2 t)} + b_5 + b_3 \, \mathrm{e}^{(c_3 x + c_4 t)}} \,,$$

$$c_1 = c_3 = \frac{1}{\sqrt{2}} \,, \quad c_2 = c_4 = -\frac{3}{2} \,, \tag{23}$$

$$u_{13}(x,t) = \frac{\pm b_4 \, \mathrm{e}^{-(c_3 x + c_4 t)}}{b_2 \, \mathrm{e}^{-(c_1 x + c_2 t)} + b_3 \, \mathrm{e}^{(c_3 x + c_4 t)} + b_4 \, \mathrm{e}^{-(c_3 x + c_4 t)}} \,,$$

$$c_1 = \frac{1}{\sqrt{2}} + c_3 \,, \quad c_2 = \frac{3}{2} + c_4 \,, \tag{24}$$

$$u_{14}(x,t) = \frac{\pm b_4 \, \mathrm{e}^{-(c_3 x + c_4 t)}}{b_2 \, \mathrm{e}^{-(c_1 x + c_2 t)} + b_4 \, \mathrm{e}^{-(c_3 x + c_4 t)}} \,,$$

$$c_1 = -\frac{1}{2\sqrt{2}} \,, \quad c_2 = \frac{3}{4} \,, \quad c_3 = \frac{1}{2\sqrt{2}} \,, \quad c_4 = -\frac{3}{4} \,. \tag{25}$$

Again, the author left the reader with the impression that Eqs. (20)–(25) were "two-soliton" solutions. But this is not the case. First, by direct substitution, we figure out that $u_{13}(x,t)$ does not satisfy Eq. (1). Second, the remaining functions (20), (21), (22), (23), and (25) can be

easily simplified in the form

$$u_9(x,t) = \frac{\pm b_5}{b_2 \, \mathrm{e}^{-((1/\sqrt{2})x + (3/2)t)} + b_5 + b_4 \, \mathrm{e}^{-((1/\sqrt{2})x + (3/2)t)}}$$

$$= \frac{\pm b_5}{b_5 + (b_2 + b_4) \, \mathrm{e}^{-((1/\sqrt{2})x + (3/2)t)}}$$

$$= \frac{\pm 1}{1 + A_9 \, \mathrm{e}^{-((1/\sqrt{2})x + (3/2)t)}} \,,$$

$$A_9 = \frac{b_2 + b_4}{b_5} \,, \quad b_5 \neq 0 \,, \tag{26}$$

$$u_{10}(x,t) = \frac{\pm b_5}{b_2 \, \mathrm{e}^{-(-(1/\sqrt{2})x + (3/2)t)} + b_5 + b_3 \, \mathrm{e}^{((1/\sqrt{2})x - (3/2)t)}}$$

$$= \frac{\pm b_5}{b_5 + (b_2 + b_3) \, \mathrm{e}^{((1/\sqrt{2})x - (3/2)t)}}$$

$$= \frac{\pm 1}{1 + A_{10} \, \mathrm{e}^{(1/\sqrt{2})x - (3/2)t}} \,,$$

$$A_{10} = \frac{b_2 + b_3}{b_5} \,, \quad b_5 \neq 0 \,, \tag{27}$$

$$u_{11}(x,t) = \frac{\pm b_5}{b_1 \, \mathrm{e}^{(-(1/\sqrt{2})x - (3/2)t)} + b_5 + b_4 \, \mathrm{e}^{-((1/\sqrt{2})x + (3/2)t)}}$$

$$= \frac{\pm b_5}{b_5 + (b_1 + b_4) \, \mathrm{e}^{-((1/\sqrt{2})x + (3/2)t)}}$$

$$= \frac{\pm 1}{1 + A_{11} \, \mathrm{e}^{-((1/\sqrt{2})x + (3/2)t)}} \,,$$

$$A_{11} = \frac{b_1 + b_4}{b_5} \,, \quad b_5 \neq 0 \,, \tag{28}$$

$$u_{12}(x,t) = \frac{\pm b_5}{b_1 \, \mathrm{e}^{((1/\sqrt{2})x - (3/2)t)} + b_5 + b_3 \, \mathrm{e}^{((1/\sqrt{2})x - (3/2t))}}$$

$$= \frac{\pm b_5}{b_5 + (b_1 + b_3) \, \mathrm{e}^{((1/\sqrt{2})x - (3/2)t)}}$$

$$= \frac{\pm 1}{1 + A_{12} \, \mathrm{e}^{((1/\sqrt{2})x - (3/2)t)}} \,,$$

$$A_{12} = \frac{b_1 + b_3}{b_5} \,, \quad b_5 \neq 0 \,, \tag{29}$$

$$u_{14}(x,t) = \frac{\pm b_4 \, \mathrm{e}^{-((1/2\sqrt{2})x - (3/4)t)}}{b_2 \, \mathrm{e}^{-(-(1/2\sqrt{2})x + (3/4)t)} + b_4 \, \mathrm{e}^{-((1/2\sqrt{2})x - (3/4)t)}}$$

$$= \frac{\pm b_4}{b_2 \, \mathrm{e}^{((1/\sqrt{2})x - (3/2)t)} + b_4}$$

$$= \frac{\pm 1}{1 + A_{14} \, \mathrm{e}^{((1/\sqrt{2})x - (3/2)t)}} \,,$$

$$A_{14} = \frac{b_2}{b_4} \,, \quad b_4 \neq 0 \,. \tag{30}$$

At a first glance, we see that Eqs. (26) and (28) are the same and equivalent to Eq. (18) while Eqs. (27), (29), and (30) are the same and equivalent (19). In addition, the functions (26)–(30) cannot represent two-soliton solutions because each depends on only one wave variable; either $\xi = (1/\sqrt{2})x + (3/2)t$ or $\eta = (1/\sqrt{2})x - (3/2)t$ but not

both.

4 Analysis of Eq. (1) by the Truncated (Painlevé) Expansion Method

The solutions (18) and (19) are not general since they contain only one arbitrary constant while Eq. (1) has the second order. Now, let us study Eq. (1) by means of truncated (Painlevé) expansion method[4] for wider solutions (involving more arbitrary constants). To this end, we seek for solutions in the form of infinite series

$$u(x,t) = \phi^{-p} \sum_{j=0}^{\infty} u_j \phi^j = \frac{u_0}{\phi^p} + \frac{u_1}{\phi^{p-1}} + \cdots + u_p + \cdots, \quad (31)$$

where $\phi = \phi(x,t)$ and $u_j = u_j(x,t)$ are analytic functions of (x,t) and p is a positive integer. Substituting Eq. (31) into Eq. (1) and balancing the exponents of the leading order terms, we observe that $p = 1$ and thus we can truncate the summation after the first two terms. Truncated expansion is

$$u(x,t) = \frac{u_0}{\phi} + u_1, \quad (32)$$

where we assume $u_0 \neq 0$. By substituting Eq. (32) into Eq. (1) and equating coefficients of the powers of ϕ to zero, we end up with the following set of equations:

$$u_0^3 - 2u_0\phi_x^2 = 0, \quad (33)$$

$$3u_0^2 u_1 - u_0\phi_t + 2\phi_x(u_0)_x + u_0\phi_{xx} = 0, \quad (34)$$

$$-u_0 + 3u_0 u_1^2 + (u_0)_t - (u_0)_{xx} = 0, \quad (35)$$

$$(u_1)_t - (u_1)_{xx} - u_1 + u_1^3 = 0. \quad (36)$$

Now, we solve for u_0 in Eq. (33), substitute it into Eq. (34), solve for u_1, and substitute it into Eq. (35) to get

$$u_0 = \pm\sqrt{2}\phi_x, \quad u_1 = \pm\frac{1}{3\sqrt{2}}\frac{\phi_t}{\phi_x} \mp \frac{1}{\sqrt{2}}\frac{\phi_{xx}}{\phi_x},$$

$$\phi_t^2 - 6\phi_x^2 + 6\phi_x\phi_{xt} - 6\phi_t\phi_{xx} + 9\phi_{xx}^2 - 6\phi_x\phi_{xxx} = 0. \quad (37)$$

Hence, Eq. (1) admits Painlevé–Backlund transformations in the form

$$u(x,t) = \pm\frac{\sqrt{2}\phi_x}{\phi} \pm \frac{1}{3\sqrt{2}}\frac{\phi_t}{\phi_x} \mp \frac{1}{\sqrt{2}}\frac{\phi_{xx}}{\phi_x}, \quad (38)$$

where $\phi = \phi(x,t)$ is a solution of the third equation in Eq. (37). On the other hand, it is evident from Eq. (36) that u_1 is a solution of Eq. (1). Hence, we can take the constant solutions $u_1 = 0$ and $u_1 = \pm1$ into consideration. If we set $u_1 = 0$ in Eq. (37), solve the resulting constant coefficient linear partial differential equation for $\phi = \phi(x,t)$, plug the result into Eq. (32) or Eq. (38), we obtain a solution of Eq. (1) as

$$u_\pm^{(1)}(x,t)$$
$$= \pm\frac{C_1 e^{(1/\sqrt{2})x+(3/2)t} - C_2 e^{-(1/\sqrt{2})x+(3/2)t}}{C_1 e^{(1/\sqrt{2})x+(3/2)t} + C_2 e^{-(1/\sqrt{2})x+(3/2)t} + C_3}. \quad (39)$$

For $u_1 = \pm1$, by a very similar work, we get another solution of Eq. (1) as

$$u_\pm^{(2)}(x,t) = \pm\frac{C_1 e^{\sqrt{2}x} - C_3}{C_1 e^{\sqrt{2}x} + C_2 e^{(1/\sqrt{2})x-(3/2)t} + C_3}. \quad (40)$$

Above, C_1, C_2, and C_3 denote arbitrary real constants. Indeed, Eqs. (39) and (40) can be further simplified as follows:

$$u_\pm^{(1)}(x,t) = \pm\frac{A_1 e^{(1/\sqrt{2})x+(3/2)t} - A_2 e^{-(1/\sqrt{2})x+(3/2)t}}{A_1 e^{(1/\sqrt{2})x+(3/2)t} + A_2 e^{-(1/\sqrt{2})x+(3/2)t} + 1},$$

$$A_1 = \frac{C_1}{C_3}, \quad A_2 = \frac{C_2}{C_3}, \quad C_3 \neq 0, \quad (41)$$

$$u_\pm^{(2)}(x,t) = \pm\frac{B_1 e^{\sqrt{2}x} - 1}{B_1 e^{\sqrt{2}x} + B_2 e^{(1/\sqrt{2})x-(3/2)t} + 1},$$

$$B_1 = \frac{C_1}{C_3}, \quad B_2 = \frac{C_2}{C_3}, \quad C_3 \neq 0. \quad (42)$$

It turns out that $A_1 = 1$ and $A_2 = 0$ lead to Eq. (18) while $A_1 = 0$ and $A_2 = 1$ lead to Eq. (19). Moreover, Eq. (42) could not be obtained by double Exp-function method. Thus, we are convinced that Bekir obtained less results (only one-wave fronts) which are special cases of ours. All these exact solutions may be used to analyze nonlinear wave phenomena that are described by the Cahn–Allen equation. We observe that the reaction-diffusion equation (1) supports nontrivial steady-state (time independent) solutions if we take $C_3 = 0$ in Eq. (39) or $C_2 = 0$ in Eq. (40). It is evident that Eq. (39) describes interacting two wave fronts (see Fig. 2).
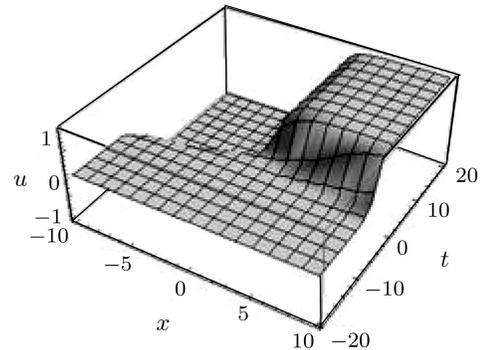

Fig. 2 A profile of interacting two wave fronts $u_+^{(1)}(x,t)$ for $A_1 = A_2 = 1$.

As mentioned in Sec. 1, there are some papers to discuss N-soliton solutions by means of Exp-function method. The author of Ref. [6] generalized Exp-function method to the famous Korteweg-de Vries (KdV) equation and obtained the previously known 2-soliton, and 3-soliton solutions in a straightforward way. The same author also discuss the advantages, as well as the drawbacks of the proposed generalized method. Later, the authors of Ref. [7] chose a KdV equation with variable coefficients

to illustrate a further generalization of the work.[6] As a result, 1-soliton, 2-soliton, and 3-soliton solutions are obtained, from which the uniform formula of N-soliton solutions is also derived. Quite recently, the authors of Ref. [8] did a very interesting and sophisticated work to generalize the basic Exp-function method for exact multiple wave solutions of nonlinear partial differential equations. The approach, named as a multiple Exp-function method, oriented towards the ease of use and capability of computer algebra systems and provides a direct and systematic solution procedure that generalizes Hirota's perturbation scheme. Via multiple Exp-function method, it is observed that the $(3 + 1)$-dimensional potential Yu-Toda-Sasa-Fukuyama equation yields exact explicit one-wave, two-wave, and three-wave solutions, which include one-soliton, two-soliton and three-soliton type solutions. In the mean time, the authors of Refs. [7–8] have verified their results by simulation as well. Via a detailed analysis, we have not witnessed any misleading results in the just-stated fine works. As a result, unlike the others,[18,21] it was successfully shown in these studies that Exp-function method may be an effective mathematical tool for generating N-soliton solutions if implemented in a carefull manner.

5 Conclusion

In this study, it is shown that double Exp-function method is inconvenient in particular when reaction and/or diffusion is involved. In a recent work,[21] we observed that the obtained one-soliton solutions disguised themselves as the obtained two-soliton solutions. We demonstrated that the author[21] derived only two one-wave solutions (no two-wave solution at all) out of fourteen. It is unfortunate that the author[21] did not check his results according to the list of common errors.[11] Alternately, we applied truncated (Painlevé) expansion method to Eq. (1) in an elegant and straightforward way to find possible more general closed-form solutions, namely, Eqs. (39) and (40). Our former solution corresponds to a coalescence of two wave fronts from which single solitary wave solutions follows as special cases. In fact, the explicit form of the Cahn–Allen equation's $N(\geqslant 3)$-soliton solution is apparently not known. It is not completely integrable. The exact solution (40) (see Ref. [22]) can also be obtained with a nonclassical method of symmetry reduction[23] or via Hirota bilinear method.[24]

Acknowledgments

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