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Application of the Exp-function method to the (2 + 1)-dimensional Boiti–Leon–Pempinelli equation using symbolic computation

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This paper deals with the so-called Exp-function method for studying a particular nonlinear partial differential equation (PDE): the (2 + 1)-dimensional Boiti–Leon–Pempinelli equation. The method is constructive and can be carried out in a computer with the aid of a computer algebra system. The obtained generalized solitary wave solutions contain more arbitrary parameters compared with the earlier works, and thus, they are wider. This means that our method is effective and powerful for constructing exact and explicit analytic solutions to nonlinear PDEs.

Keywords: Exp-function method; (2 + 1)-dimensional Boiti–Leon–Pempinelli equation; solitary wave solutions

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1. Introduction

Many fundamental problems in nonlinear sciences are described by nonlinear partial differential equations (NPDEs). It is also believed that many problems of future technologies will be described in the same way. Physical problems and the increasing availability of technology (including computer algebra systems such as MATHEMATICA, MAPLE, or MATLAB) have motivated the development of many analytic methods for finding exact and explicit solutions of such equations. Thus, in the last four decades or so, many different methods have been presented in the open literature to look for exact solutions of these equations. To mention some, Darboux transformation [33], Hirota bilinear method [27], inverse scattering transformation [2], symmetry method [9], Weierstrass function method [43], Jacobi elliptic function method [29], Sine–Cosine function [40], Tanh–Coth function [32], F-expansion method [1], homotopy perturbation method [15,16,23], variational iteration method [14,22,38], Adomian decomposition method [3,17,18], (G'/G)-expansion method [4,6,7,35,36,39], and so forth. However, most of the methods have some restrictions for application purposes.

Since the nonlinear phenomena is so complex, it is still an open problem to find more powerful techniques to deduce more new exact solutions to the nonlinear equations of modelling problems. Recently, He and Wu [25] introduced the so-called Exp-function method, a relatively new method. The Exp-function 'method' consists of trying rational combinations of exponential functions as an 'ansatz' to find exact solutions of the ordinary differential equation (ODE) for travelling waves of the original equation. It can be used as an alternative method for obtaining approximate and analytic solutions of many NPDEs. The Exp-function method was found to be one of the most effective methods to search for exact solutions of NPDEs [5,24,26,34,42]. It has been successfully extended by the authors [19,45] to study some nonlinear equations with variable coefficients. Moreover, being less restrictive and concise, it is modified by the researchers [12,13,46,47] to analyse some special nonlinear differential-difference equations.

In this paper, we consider the following $(2 + 1)$-dimensional Boiti–Leon–Pempinelli (BLP) equation [8]

$$\begin{cases} u_{ty} - (u^2 - u_x)_{xy} - 2v_{xxx} = 0, \\ v_t - v_{xx} - 2uv_x = 0, \end{cases} \tag{1}$$

and apply the Exp-function method to the BLP equation to construct new and interesting exact solutions. The integrability of Equation (1) was shown in [8]. In [21], it was also observed that Equation (1) was Hamiltonian. Both Boiti *et al.* and Garagash [8,21] derived Bäcklund transformation for Equation (1). The fact that the sine-Gordon equation or the sinh-Gordon equation can be derived from the BLP equation by a certain transformation was pointed out in [21]. A great deal of research work has been invested in the literature for the study of BLP equation. To mention some, Weiss [41] has shown that the BLP system to be solvable in terms of a commuting set of nonlinear Hamiltonian ordinary differential equations, which are related to the fixed points of the Bäcklund transformation for this system. Yurov [44] studied the Darboux and Laplace transformations for the BLP equation, as well as demonstrated that it can be reduced to the Burgers (and anti-Burgers) equation in a one-dimensional limit. By using a further extended tanh method, Lü and Zhang [31] constructed explicit exact solutions for the BLP equation. Huang and Zhang [28] obtained several types of exact travelling wave solutions to the BLP equation by implementing an improved projective Riccati equation method. Based on the variable separation solution and by selecting appropriate multi-valued functions, Dai and Ni [10] investigated novel interactions between a special bell-like semi-foldon and a special peakon-like semi-foldon for the BLP equation. Inspired by the extended tanh function method and the direct method of symmetry reduction, Lü [30] presented a Burgers equation-based constructive method for solving NPDEs and had taken the BLP equation as an example to illustrate the algorithm. By giving some types of general solutions of a first-order nonlinear ODE with six degree and presenting a new generalized algebra method to find more exact solutions of NPDEs, Ren *et al.* [37] obtained many types of solutions for the BLP equation. The extended tanh method has been successfully applied to the BLP equation by Feng *et al.* [20] to search for exact and explicit solutions. Recently, a new mapping method is used by Dai and Wang [11] to obtain the variable separation solutions (with two arbitrary functions) of the BLP equation, and so forth.

2. The Exp-function method

To begin with, suppose that we have a NPDE for $u(x, y, t)$ in the form

$$P(u, u_t, u_x, u_y, u_{tt}, u_{tx}, u_{ty}, u_{xx}, u_{xy}, u_{yy}, \ldots) = 0, \tag{2}$$

where P is a polynomial in its arguments. We summarize the algorithm as follows:

Step 1 By taking $u(x, y, t) = U(\zeta)$, $\zeta = kx + my + wt$, where k, m, and w are arbitrary non-zero constants, look for travelling wave solutions of Equation (2) and transform it to the ODE

$$Q(U, U', U'', \ldots) = 0, \tag{3}$$

where the prime denotes derivative with respect to ζ.

Step 2 Integrate Equation (3), if possible, term by term one or more times. This yields constant(s) of integration. The integration constant(s) can be set to zero for simplicity.

Step 3 Suppose the solution $U(\zeta)$ of Equation (3) can be expressed in the form

$$U(\zeta) = \frac{a_c \exp(c\zeta) + \cdots + a_{-d} \exp(-d\zeta)}{b_p \exp(p\zeta) + \cdots + b_{-q} \exp(-q\zeta)}, \tag{4}$$

where c, d, p, and q are unknown positive integers to be determined, a_i and b_j are unknown constants.

Step 4 Determine the highest order nonlinear term and the linear term of highest order in Equation (3) and express them in terms of Equation (4). Then, in the resulting terms, balance the highest order Exp-function to determine c and p, and the lowest order Exp-function to determine d and q.

Step 5 Substitute Equation (4) into Equation (3) and equate the coefficients of $\exp(n\zeta)$ to zero, and obtain a *system* of algebraic equations for a_i, b_j, k, m, and w. Then, to determine these constants, solve the system with the aid of a computer algebra system.

Step 6 Substitute the values solved in Step 5 into expression (4) and find the travelling wave solutions of Equation (2). Then, it is necessary to substitute them into the original Equation (2) to assure the correctness of the solutions.

3. Solutions to the BLP equation by the Exp-function method

To seek for the travelling wave solutions to the BLP equation (1), we first make the transformation $u(x, y, t) = U(\zeta), v(x, y, t) = V(\zeta), \zeta = kx + my + wt$, where k, m, and w are constants to be determined later. Then, we get the system of ODEs

$$\begin{aligned} wmU'' &= km(U^2 - kU')'' + 2k^3 V''', \\ wV' &= k^2 V'' + 2kUV', \end{aligned} \tag{5}$$

where the prime denotes derivative with respect to ζ. By integrating the first equation in Equation (5) twice and setting the constants of integration to zero, we get

$$V = \frac{m}{2k} U + \frac{m}{2k^3} \int (wU - kU^2) \, d\zeta. \tag{6}$$

Moreover, substituting Equation (6) into the second equation of Equation (5), we obtain

$$k^4 U'' - 2k^2 U^3 + 3kwU^2 - w^2 U = 0. \tag{7}$$

Now, we consider the ansatz

$$U(\zeta) = \frac{a_c \exp(c\zeta) + \cdots + a_{-d} \exp(-d\zeta)}{b_p \exp(p\zeta) + \cdots + b_{-q} \exp(-q\zeta)} \tag{8}$$

for the solution of Equation (7) and balance the terms U'' and U^3. By a simple calculation, we have

$$U'' = \frac{k_1 \exp[(c + 3p)\zeta] + \cdots}{k_2 \exp[4p\zeta] + \cdots} \tag{9}$$

and

$$U^3 = \frac{k_3 \exp[3c\zeta] + \cdots}{k_4 \exp[3p\zeta] + \cdots} = \frac{k_3 \exp[(3c + p)\zeta] + \cdots}{k_4 \exp[4p\zeta] + \cdots}, \tag{10}$$

where k_is are determined coefficients for simplicity. Balancing highest order of Exp-function in Equations (9) and (10), we have

$$c + 3p = 3c + p, \tag{11}$$

which leads to the result

$$p = c. \tag{12}$$

Similarly, from the ansatz (8), we have

$$U'' = \frac{\cdots + l_1 \exp[-(d + 3q)\zeta]}{\cdots + l_2 \exp[-4q\zeta]} \tag{13}$$

and

$$U^3 = \frac{\cdots + l_3 \exp[-3d\zeta]}{\cdots + l_4 \exp[-3q\zeta]} = \frac{\cdots + l_3 \exp[-(3d + q)\zeta]}{\cdots + l_4 \exp[-4q\zeta]}, \tag{14}$$

where l_is are determined coefficients for simplicity. Balancing lowest order of Exp-function in Equations (13) and (14), we have

$$-(d + 3q) = -(3d + q), \tag{15}$$

which leads to the result

$$q = d. \tag{16}$$

We can freely choose the values of c and d in general. However, the final solution does not strongly depend on the values of c and d [18,38].

Case 1. $p = c = 1, d = q = 1$.
   In this case, the solution of Equation (7) can be expressed as

$$U(\zeta) = \frac{a_1 \exp(\zeta) + a_0 + a_{-1} \exp(-\zeta)}{b_1 \exp(\zeta) + b_0 + b_{-1} \exp(-\zeta)}. \tag{17}$$

We take $b_1 = 1$ in Equation (17) for simplicity. Then, substituting Equation (17) into Equation (7), we have

$$\frac{1}{A}[C_0 + C_1 \exp(\zeta) + C_2 \exp(2\zeta) + C_3 \exp(3\zeta) + C_4 \exp(4\zeta) + C_5 \exp(5\zeta) + C_6 \exp(6\zeta)] = 0, \tag{18}$$

where

$$A = (\exp(2\varsigma) + b_{-1} + b_0 \exp(\varsigma))^3,$$

$$C_0 = -2k^2 a_{-1}^3 + 3kwa_{-1}^2 b_{-1} - w^2 a_{-1} b_{-1}^2,$$

$$C_1 = -6k^2 a_{-1}^2 a_0 + 6kwa_{-1}a_0 b_{-1} + k^4 a_0 b_{-1}^2 - w^2 a_0 b_{-1}^2 + 3kwa_{-1}^2 b_0$$
$$\quad - k^4 a_{-1} b_{-1} b_0 - 2w^2 a_{-1} b_{-1} b_0,$$

$$C_2 = 3kwa_{-1}^2 - 6k^2 a_{-1}a_0^2 - 6k^2 a_{-1}^2 a_1 - 4k^4 a_{-1}b_{-1} - 2w^2 a_{-1}b_{-1} + 3kwa_0^2 b_{-1}$$
$$\quad + 6kwa_{-1}a_1 b_{-1} + 4k^4 a_1 b_{-1}^2 - w^2 a_1 b_{-1}^2 + 6kwa_{-1}a_0 b_0 - k^4 a_0 b_{-1} b_0 - 2w^2 a_0 b_{-1} b_0$$
$$\quad + k^4 a_{-1} b_0^2 - w^2 a_{-1} b_0^2,$$

$$C_3 = 6kwa_{-1}a_0 - 2k^2 a_0^3 - 12k^2 a_{-1}a_0 a_1 - 6k^4 a_0 b_{-1} - 2w^2 a_0 b_{-1} + 6kwa_0 a_1 b_{-1}$$
$$\quad + 3k^4 a_{-1}b_0 - 2w^2 a_{-1}b_0 + 3kwa_0^2 b_0 + 6kwa_{-1}a_1 b_0 + 3k^4 a_1 b_{-1} b_0$$
$$\quad - 2w^2 a_1 b_{-1} b_0 - w^2 a_0 b_0^2,$$

$$C_4 = 4k^4 a_{-1} - w^2 a_{-1} + 3kwa_0^2 + 6kwa_{-1}a_1 - 6k^2 a_0^2 a_1 - 6k^2 a_{-1}a_1^2 - 4k^4 a_1 b_{-1}$$
$$\quad - 2w^2 a_1 b_{-1} + 3kwa_1^2 b_{-1} - k^4 a_0 b_0 - 2w^2 a_0 b_0 + 6kwa_0 a_1 b_0$$
$$\quad + k^4 a_1 b_0^2 - w^2 a_1 b_0^2,$$

$$C_5 = k^4 a_0 - w^2 a_0 + 6kwa_0 a_1 - 6k^2 a_0 a_1^2 - k^4 a_1 b_0 - 2w^2 a_1 b_0 + 3kwa_1^2 b_0,$$

$$C_6 = -w^2 a_1 + 3kwa_1^2 - 2k^2 a_1^3.$$

Equating the coefficients of $\exp(j\varsigma)$ to zero in Equation (18) and solving the resulting algebraic system for $a_1, a_0, a_{-1}, b_0, b_{-1}, k, m$, and w, we get the following solution sets (denoted in curly brackets from now on) and the corresponding generalized solitary wave solutions to Equation (1):

Case 1.1

$$\left\{ a_{-1} = 0, b_{-1} = \frac{-a_0^2 \mp ka_0 b_0}{k^2}, a_1 = \mp k, w = \mp k^2, a_0 = a_0, b_0 = b_0, k = k, m = m \right\}, \quad (19)$$

$$u_1(x, y, t) = \frac{-k^2 \exp(kx + my - k^2 t)}{a_0 + k(b_0 + \exp(kx + my - k^2 t))},$$

$$v_1(x, y, t) = -\frac{m}{2}, \quad (20)$$

$$u_2(x, y, t) = \frac{-k^2 \exp(kx + my + k^2 t)}{a_0 - k\left(b_0 + \exp(kx + my + k^2 t)\right)},$$

$$v_2(x, y, t) = \frac{m}{2} + \frac{m(kb_0 - a_0)}{a_0 - k(b_0 + \exp(kx + my + k^2 t))}, \quad (21)$$

where a_0, b_0, k, and m are arbitrary constants.

Case 1.2

$$\left\{ a_{-1} = 0, a_1 = 0, b_{-1} = 0, b_0 = \mp \frac{a_0}{k}, w = \mp k^2, a_0 = a_0, k = k, m = m \right\}, \quad (22)$$

$$u_3(x, y, t) = \frac{ka_0}{k\exp(kx + my - k^2t) - a_0},$$

$$v_3(x, y, t) = \frac{ma_0}{k\exp(kx + my - k^2t) - a_0}, \tag{23}$$

$$u_4(x, y, t) = \frac{ka_0}{k\exp(kx + my + k^2t) + a_0},$$

$$v_4(x, y, t) = 0, \tag{24}$$

where a_0, k, and m are arbitrary constants.

Case 1.3

$$\left\{a_{-1} = \mp 2kb_{-1}, a_1 = 0, b_0 = 0, a_0 = 0, w = \mp 2k^2, b_{-1} = b_{-1}, k = k, m = m\right\}, \tag{25}$$

$$u_5(x, y, t) = -\frac{2kb_{-1}}{b_{-1} + \exp(2(kx + my - 2k^2t))},$$

$$v_5(x, y, t) = -\frac{2mb_{-1}}{b_{-1} + \exp(2(kx + my - 2k^2t))}, \tag{26}$$

$$u_6(x, y, t) = \frac{2kb_{-1}}{b_{-1} + \exp(2(kx + my + 2k^2t))},$$

$$v_6(x, y, t) = 0, \tag{27}$$

where b_{-1}, k, and m are arbitrary constants.

Case 2 $p = c = 2, d = q = 1$.
  Then, the trial function (8) becomes

$$U(\zeta) = \frac{a_2\exp(2\zeta) + a_1\exp(\zeta) + a_0 + a_{-1}\exp(-\zeta)}{b_2\exp(2\zeta) + b_1\exp(\zeta) + b_0 + b_{-1}\exp(-\zeta)}. \tag{28}$$

We set $b_2 = 1$ for simplicity, and thus, Equation (28) takes the form

$$U(\zeta) = \frac{a_2\exp(2\zeta) + a_1\exp(\zeta) + a_0 + a_{-1}\exp(-\zeta)}{\exp(2\zeta) + b_1\exp(\zeta) + b_0 + b_{-1}\exp(-\zeta)}. \tag{29}$$

Substituting Equation (29) into Equation (7), we get the equation

$$(\exp(3\zeta) + b_{-1} + \exp(\zeta)b_0 + \exp(2\zeta)b_1)^{-3} \sum_{j=0}^{9} C_j\exp(j\zeta) = 0. \tag{30}$$

We omit to display the coefficients C_j explicitly. Then, equating the coefficients of $\exp(j\zeta)$ to zero and solving the resulting algebraic system for a_2, a_1, a_0, a_{-1}, b_1, b_0, b_{-1}, k, m, and w, we get the following solution sets and the corresponding generalized solitary wave solutions to Equation (1):

Case 2.1

$$\left\{ a_{-1} = 0, a_2 = 0, b_{-1} = 0, a_1 = \frac{1}{2}\left(\mp kb_1 - \sqrt{k^2(b_1^2 - 4b_0)} \right), a_0 = \mp kb_0, \right.$$
$$\left. w = \mp k^2, b_1 = b_1, b_0 = b_0, k = k, m = m \right\},$$
(31)

$$u_7(x, y, t) = \frac{-kb_0 - \frac{1}{2}\left(kb_1 + \sqrt{k^2\left(b_1^2 - 4b_0\right)} \right) \exp(kx + my - k^2t)}{\exp(2(kx + my - k^2t)) + b_0 + b_1 \exp(kx + my - k^2t)},$$

$$v_7(x, y, t) = -\frac{m\left(2kb_0 + \left(kb_1 + \sqrt{k^2\left(b_1^2 - 4b_0\right)} \right) \exp(kx + my - k^2t) \right)}{2k(b_0 + (b_1 + \exp(kx + my - k^2t)) \exp(kx + my - k^2t))},$$
(32)

$$u_8(x, y, t) = \frac{kb_0 + \frac{1}{2}\left(kb_1 - \sqrt{k^2\left(b_1^2 - 4b_0\right)} \right) \exp(kx + my + k^2t)}{\exp(2(kx + my + k^2t)) + b_0 + b_1 \exp(kx + my + k^2t)},$$

$$v_8(x, y, t) = 0,$$
(33)

where b_0, b_1, k, and m are arbitrary constants.

Case 2.2

$$\left\{ a_{-1} = 0, a_2 = \mp k, b_{-1} = 0, a_1 = \frac{1}{2}\left(\mp kb_1 - \sqrt{k^2\left(b_1^2 - 4b_0\right)} \right), a_0 = 0, \right.$$
$$\left. w = \mp k^2, b_1 = b_1, b_0 = b_0, k = k, m = m \right\},$$
(34)

$$u_9(x, y, t) = \frac{-k \exp(2(kx + my - k^2t)) - \frac{1}{2}\left(kb_1 + \sqrt{k^2\left(b_1^2 - 4b_0\right)} \right) \exp(kx + my - k^2t)}{\exp(2(kx + my - k^2t)) + b_0 + b_1 \exp(kx + my - k^2t)},$$

$$v_9(x, y, t) = \frac{-m}{2},$$
(35)

$$u_{10}(x, y, t) = \frac{k \exp(2(kx + my + k^2t)) + \frac{1}{2}\left(kb_1 - \sqrt{k^2\left(b_1^2 - 4b_0\right)} \right) \exp(kx + my + k^2t)}{\exp(2(kx + my + k^2t)) + b_0 + b_1 \exp(kx + my + k^2t)},$$

$$v_{10}(x, y, t) = -\frac{m\left(kb_0 + \left(\sqrt{k^2\left(b_1^2 - 4b_0\right)} - k \exp(kx + my + k^2t) \right) \exp(kx + my + k^2t) \right)}{2k(b_0 + (b_1 + \exp(kx + my + k^2t)) \exp(kx + my + k^2t))},$$
(36)

where b_0, b_1, k, and m are arbitrary constants.

Case 2.3

$$\{ a_{-1} = 0, a_2 = \mp 2k, b_{-1} = 0, a_1 = 0, a_0 = 0, b_1 = 0, w = \mp 2k^2, b_0 = b_0, k = k, m = m \},$$
(37)

$$u_{11}(x, y, t) = -\frac{2k \exp(2(kx + my - 2k^2t))}{b_0 + \exp(2(kx + my - 2k^2t))},$$

$$v_{11}(x, y, t) = -m, \tag{38}$$

$$u_{12}(x, y, t) = \frac{2k \exp(2(kx + my + 2k^2 t))}{b_0 + \exp(2(kx + my + 2k^2 t))},$$

$$v_{12}(x, y, t) = m - \frac{2mb_0}{b_0 + \exp(2(kx + my + 2k^2 t))}, \tag{39}$$

where b_0, k, and m are arbitrary constants.

Case 2.4

$$\{a_{-1} = \mp 3kb_{-1}, a_2 = 0, a_1 = 0, a_0 = 0, b_1 = 0, b_0 = 0, w = \mp 3k^2, b_{-1} = b_{-1},$$
$$k = k, m = m\}, \tag{40}$$

$$u_{13}(x, y, t) = -\frac{3kb_{-1}}{b_{-1} + \exp(3(kx + my - 3k^2 t))},$$

$$v_{13}(x, y, t) = -\frac{3mb_{-1}}{b_{-1} + \exp(3(kx + my - 3k^2 t))}, \tag{41}$$

$$u_{14}(x, y, t) = \frac{3kb_{-1}}{b_{-1} + \exp(3(kx + my + 3k^2 t))},$$

$$v_{14}(x, y, t) = 0, \tag{42}$$

where b_{-1}, k, and m are arbitrary constants.

Case 2.5

$$\{a_{-1} = 0, a_2 = \mp 3k, b_{-1} = 0, a_1 = \mp 3kb_1, a_0 = \mp 3kb_0, w = \mp 3k^2, b_1 = b_1,$$
$$b_0 = b_0, k = k, m = m\}, \tag{43}$$

$$u_{15,16}(x, y, t) = \frac{\mp 3k \exp(2(kx + my \mp 3k^2 t)) \mp 3kb_0 \mp 3kb_1 \exp(kx + my \mp 3k^2 t)}{\exp(2(kx + my \mp 3k^2 t)) + b_0 + b_1 \exp(kx + my \mp 3k^2 t)},$$

$$v_{15,16}(x, y, t) = \mp \frac{3m}{2}, \tag{44}$$

where b_0, b_1, k, and m are arbitrary constants.

Case 2.6

$$\{a_{-1} = 0, a_2 = \mp 3k, a_1 = 0, a_0 = 0, b_1 = 0, b_0 = 0, w = \mp 3k^2, b_{-1}$$
$$= b_{-1}, k = k, m = m\}, \tag{45}$$

$$u_{17}(x, y, t) = \frac{-3k \exp(3(kx + my - 3k^2 t))}{b_{-1} + \exp(3(kx + my - 3k^2 t))},$$

$$v_{17}(x, y, t) = \frac{-3m}{2}, \tag{46}$$

$$u_{18}(x, y, t) = \frac{3k \exp(3(kx + my + 3k^2 t))}{b_{-1} + \exp(3(kx + my + 3k^2 t))},$$

$$v_{18}(x, y, t) = \frac{3m}{2} - \frac{3mb_{-1}}{b_{-1} + \exp(3(kx + my + 3k^2 t))}, \tag{47}$$

where b_{-1}, k, and m are arbitrary constants.

Case 3 $p = c = 2, d = q = 2.$

Then, the trial function (8) becomes

$$U(\zeta) = \frac{a_2 \exp(2\zeta) + a_1 \exp(\zeta) + a_0 + a_{-1} \exp(-\zeta) + a_{-2} \exp(-2\zeta)}{b_2 \exp(2\zeta) + b_1 \exp(\zeta) + b_0 + b_{-1} \exp(-\zeta) + b_{-2} \exp(-2\zeta)}. \tag{48}$$

There are some free parameters in Equation (48), so we set $b_2 = 1$, $b_1 = 0$, and $b_{-1} = 0$ for simplicity, and thus, Equation (48) takes the form

$$U(\zeta) = \frac{a_2 \exp(2\zeta) + a_1 \exp(\zeta) + a_0 + a_{-1} \exp(-\zeta) + a_{-2} \exp(-2\zeta)}{\exp(2\zeta) + b_0 + b_{-2} \exp(-2\zeta)}. \tag{49}$$

By the same manipulations as in the previous cases, we obtain the following solution sets and the corresponding generalized solitary wave solutions to Equation (1).

Case 3.1

$$\left\{ a_{-2} = \mp 2kb_{-2}, a_2 = 0, a_{-1} = 0, a_1 = 0, a_0 = \mp kb_0 - \sqrt{k^2 \left(b_0^2 - 4b_{-2}\right)}, \right.$$
$$\left. w = \mp 2k^2, b_{-2} = b_{-2}, b_0 = b_0, k = k, m = m \right\}, \tag{50}$$

$$u_{19}(x, y, t) = \frac{-2kb_{-2} \exp(-2(kx + my - 2k^2t)) - kb_0 - \sqrt{k^2 \left(b_0^2 - 4b_{-2}\right)}}{b_0 + \exp(2(kx + my - 2k^2t)) + b_{-2} \exp(-2(kx + my - 2k^2t))},$$

$$v_{19}(x, y, t) = -\frac{m \left(2kb_{-2} + \left(kb_0 + \sqrt{k^2 \left(b_0^2 - 4b_{-2}\right)}\right) \exp(2(kx + my - 2k^2t))\right)}{k(\exp(4(kx + my - 2k^2t)) + b_{-2} + b_0 \exp(2(kx + my - 2k^2t)))}, \tag{51}$$

$$u_{20}(x, y, t) = \frac{2kb_{-2} \exp(-2(kx + my + 2k^2t)) + kb_0 - \sqrt{k^2 \left(b_0^2 - 4b_{-2}\right)}}{\exp(2(kx + my + 2k^2t)) + b_{-2} \exp(-2(kx + my + 2k^2t)) + b_0},$$

$$v_{20}(x, y, t) = 0, \tag{52}$$

where b_0, b_{-2}, k, and m are arbitrary constants.

Case 3.2

$$\left\{ a_{-2} = 0, a_2 = \mp 2k, a_{-1} = 0, a_1 = 0, a_0 = \mp kb_0 - \sqrt{-4k^2 b_{-2} + k^2 b_0^2}, w = \mp 2k^2, \right.$$
$$\left. b_{-2} = b_{-2}, b_0 = b_0, k = k, m = m \right\}, \tag{53}$$

$$u_{21}(x, y, t) = \frac{-2k \exp(2(kx + my - 2k^2t)) - kb_0 - \sqrt{k^2 \left(b_0^2 - 4b_{-2}\right)}}{b_0 + \exp(2(kx + my - 2k^2t)) + b_{-2} \exp(-2(kx + my - 2k^2t))},$$

$$v_{21}(x, y, t) = -m, \tag{54}$$

$$u_{22}(x, y, t) = \frac{2k \exp(2(kx + my + 2k^2t)) + kb_0 - \sqrt{k^2 \left(b_0^2 - 4b_{-2}\right)}}{b_0 + \exp(2(kx + my + 2k^2t)) + b_{-2} \exp(-2(kx + my + 2k^2t))},$$

$$v_{22}(x, y, t) = -\frac{m\left(\dfrac{-k\exp(4(kx+my+2k^2t))+kb_{-2}+\sqrt{k^2\left(b_0^2-4b_{-2}\right)}}{\exp(2(kx+my+2k^2t))}\right)}{k(\exp(4(kx+my+2k^2t))+b_{-2}+b_0\exp(2(kx+my+2k^2t)))}, \quad (55)$$

where b_0, b_{-2}, k, and m are arbitrary constants.

Case 3.3

$$\{a_{-2}=\mp 4kb_{-2}, a_2=0, a_{-1}=0, a_1=0, a_0=0, b_0=0, w=\mp 4k^2,$$
$$b_{-2}=b_{-2}, k=k, m=m\}, \quad (56)$$

$$u_{23}(x,y,t)=-\frac{4kb_{-2}}{b_{-2}+\exp(4(kx+my-4k^2t))},$$

$$v_{23}(x,y,t)=-\frac{4mb_{-2}}{b_{-2}+\exp(4(kx+my-4k^2t))}, \quad (57)$$

$$u_{24}(x,y,t)=\frac{4kb_{-2}}{b_{-2}+\exp(4(kx+my+4k^2t))},$$

$$v_{24}(x,y,t)=0, \quad (58)$$

where b_{-2}, k, and m are arbitrary constants.

Case 3.4

$$\{a_{-2}=0, a_2=\mp 4k, a_{-1}=0, a_1=0, a_0=0, b_0=0, w=\mp 4k^2,$$
$$b_{-2}=b_{-2}, k=k, m=m\}, \quad (59)$$

$$u_{25}(x,y,t)=-\frac{4k}{1+b_{-2}\exp(-4(kx+my-4k^2t))},$$

$$v_{25}(x,y,t)=-2m, \quad (60)$$

$$u_{26}(x,y,t)=\frac{4k}{1+b_{-2}\exp(-4(kx+my+4k^2t))},$$

$$v_{26}(x,y,t)=2m-\frac{4mb_{-2}}{b_{-2}+\exp(4(kx+my+4k^2t))}, \quad (61)$$

where b_{-2}, k, and m are arbitrary constants.

Case 4 $p=c=3, d=q=2.$
   Then, the trial function (8) becomes

$$V(\zeta)=\frac{a_3\exp(3\zeta)+a_2\exp(2\zeta)+a_1\exp(\zeta)+a_0+a_{-1}\exp(-\zeta)+a_{-2}\exp(-2\zeta)}{b_3\exp(3\zeta)+b_2\exp(2\zeta)+b_1\exp(\zeta)+b_0+b_{-1}\exp(-\zeta)+b_{-2}\exp(-2\zeta)}. \quad (62)$$

There are some free parameters in Equation (62), so we set $b_3=1$, $b_2=0$, $b_1=0$, and $b_{-1}=0$ for simplicity, and thus Equation (62) takes the form

$$V(\zeta)=\frac{a_3\exp(3\zeta)+a_2\exp(2\zeta)+a_1\exp(\zeta)+a_0+a_{-1}\exp(-\zeta)+a_{-2}\exp(-2\zeta)}{\exp(3\zeta)+b_0+b_{-2}\exp(-2\zeta)}. \quad (63)$$

By the same procedure, we obtain the following solution sets and the corresponding generalized solitary wave solutions to Equation (1).

Case 4.1

$$\{a_{-2} = 0, a_3 = \mp 5k, a_{-1} = 0, a_2 = 0, a_0 = 0, a_1 = 0, b_0 = 0, w = \mp 5k^2,$$
$$b_{-2} = b_{-2}, k = k, m = m\}, \tag{64}$$

$$u_{27}(x, y, t) = -\frac{5k}{1 + b_{-2} \exp(-5(kx + my - 5k^2 t))},$$

$$v_{27}(x, y, t) = -\frac{5m}{2}, \tag{65}$$

$$u_{28}(x, y, t) = \frac{5k}{1 + b_{-2} \exp(-5(kx + my + 5k^2 t))},$$

$$v_{28}(x, y, t) = \frac{5m}{2} - \frac{5mb_{-2}}{b_{-2} + \exp(5(kx + my + 5k^2 t))}, \tag{66}$$

where b_{-2}, k, and m are arbitrary constants.

Case 4.2

$$\{a_{-2} = \mp 5kb_{-2}, a_3 = 0, a_{-1} = 0, a_2 = 0, a_0 = 0, a_1 = 0, b_0 = 0, w = \mp 5k^2,$$
$$b_{-2} = b_{-2}, k = k, m = m\}, \tag{67}$$

$$u_{29}(x, y, t) = -\frac{5kb_{-2}}{b_{-2} + \exp(5(kx + my - 5k^2 t))},$$

$$v_{29}(x, y, t) = -\frac{5mb_{-2}}{b_{-2} + \exp(5(kx + my - 5k^2 t))}, \tag{68}$$

$$u_{30}(x, y, t) = \frac{5kb_{-2}}{b_{-2} + \exp(5(kx + my + 5k^2 t))},$$

$$v_{30}(x, y, t) = 0, \tag{69}$$

where b_{-2}, k, and m are arbitrary constants.

Case 5 $p = c = 3, d = q = 3.$
   Then, the trial function (8) becomes

$$U(\zeta) = \frac{\begin{array}{c} a_3 \exp(3\zeta) + a_2 \exp(2\zeta) + a_1 \exp(\zeta) + a_0 + a_{-1} \exp(-\zeta) \\ + a_{-2} \exp(-2\zeta) + a_{-3} \exp(-3\zeta) \end{array}}{\begin{array}{c} b_3 \exp(3\zeta) + b_2 \exp(2\zeta) + b_1 \exp(\zeta) + b_0 + b_{-1} \exp(-\zeta) \\ + b_{-2} \exp(-2\zeta) + b_{-3} \exp(-3\zeta) \end{array}}. \tag{70}$$

There are some free parameters in Equation (70), so we set $b_3 = 1, b_2 = 0, b_1 = 0, b_{-2} = 0$, and $b_{-1} = 0$ for simplicity, and thus, Equation (70) takes the form

$$U(\zeta) = \frac{\begin{array}{c} a_3 \exp(3\zeta) + a_2 \exp(2\zeta) + a_1 \exp(\zeta) + a_0 + a_{-1} \exp(-\zeta) \\ + a_{-2} \exp(-2\zeta) + a_{-3} \exp(-3\zeta) \end{array}}{\exp(3\zeta) + b_0 + b_{-3} \exp(-3\zeta)}. \tag{71}$$

By the same procedure, we obtain the following solution sets and the corresponding generalized solitary wave solutions to Equation (1).

Case 5.1

$$\left\{ a_{-3} = 0, a_3 = \mp 3k, a_{-2} = 0, a_2 = 0, a_0 = \frac{3}{2}\left(\mp kb_0 - \sqrt{k^2\left(b_0^2 - 4b_{-3}\right)}\right), \right.$$

$$\left. a_1 = 0, a_{-1} = 0, w = \mp 3k^2, b_{-3} = b_{-3}, b_0 = b_0, k = k, m = m \right\}, \tag{72}$$

$$u_{31}(x, y, t) = \frac{-3k\exp(3(kx + my - 3k^2t)) - \frac{3}{2}\left(kb_0 + \sqrt{k^2\left(b_0^2 - 4b_{-3}\right)}\right)}{b_0 + \exp(3(kx + my - 3k^2t)) + b_{-3}\exp(-3(kx + my - 3k^2t))},$$

$$v_{31}(x, y, t) = -\frac{3m}{2}, \tag{73}$$

$$u_{32}(x, y, t) = \frac{3k\exp(3(kx + my + 3k^2t)) + \frac{3}{2}\left(kb_0 - \sqrt{k^2\left(b_0^2 - 4b_{-3}\right)}\right)}{b_0 + \exp(3(kx + my + 3k^2t)) + b_{-3}\exp(-3(kx + my + 3k^2t))},$$

$$v_{32}(x, y, t) = -\frac{3m\left(\dfrac{-k\exp(6(kx + my + 3k^2t)) + kb_{-3} + \sqrt{k^2\left(b_0^2 - 4b_{-3}\right)}}{\exp(3(kx + my + 3k^2t))}\right)}{2k(\exp(6(kx + my + 3k^2t)) + b_{-3} + b_0\exp(3(kx + my + 3k^2t)))}, \tag{74}$$

where b_0, b_{-3}, k, and m are arbitrary constants.

Case 5.2

$$\left\{ a_{-3} = \mp 3kb_{-3}, a_3 = 0, a_{-2} = 0, a_2 = 0, a_0 = \frac{3}{2}\left(\mp kb_0 - \sqrt{k^2\left(b_0^2 - 4b_{-3}\right)}\right), \right.$$

$$\left. a_1 = 0, a_{-1} = 0, w = \mp 3k^2, b_{-3} = b_{-3}, b_0 = b_0, k = k, m = m \right\}, \tag{75}$$

$$u_{33}(x, y, t) = \frac{-3kb_{-3}\exp(-3(kx + my - 3k^2t)) - \frac{3}{2}\left(kb_0 + \sqrt{k^2\left(b_0^2 - 4b_{-3}\right)}\right)}{b_0 + \exp(3(kx + my - 3k^2t)) + b_{-3}\exp(-3(kx + my - 3k^2t))},$$

$$v_{33}(x, y, t) = -\frac{3m\left(2kb_{-3} + \left(kb_0 + \sqrt{k^2\left(b_0^2 - 4b_{-3}\right)}\right)\exp(3(kx + my - 3k^2t))\right)}{2k(\exp(6(kx + my - 3k^2t)) + b_{-3} + b_0\exp(3(kx + my - 3k^2t)))}, \tag{76}$$

$$u_{33}(x, y, t) = \frac{3kb_{-3}\exp(-3(kx + my + 3k^2t)) + \frac{3}{2}\left(kb_0 - \sqrt{k^2\left(b_0^2 - 4b_{-3}\right)}\right)}{b_0 + \exp(3(kx + my + 3k^2t)) + b_{-3}\exp(-3(kx + my + 3k^2t))},$$

$$v_{33}(x, y, t) = 0, \tag{77}$$

where b_0, b_{-3}, k, and m are arbitrary constants.

Case 5.3

$$\{ a_{-3} = 0, a_3 = \mp 6k, a_{-2} = 0, a_2 = 0, a_0 = 0, a_1 = 0, a_{-1} = 0, b_0 = 0, w = \mp 6k^2,$$

$$b_{-3} = b_{-3}, k = k, m = m \}, \tag{78}$$

$$u_{34}(x, y, t) = -\frac{6k}{1 + b_{-3}\exp(-6(kx + my - 6k^2t))},$$

$$v_{34}(x, y, t) = -3m, \tag{79}$$

$$u_{35}(x, y, t) = \frac{6k}{1 + b_{-3}\exp(-6(kx + my + 6k^2t))},$$

$$v_{35}(x, y, t) = 3m - \frac{6mb_{-3}}{b_{-3} + \exp(6(kx + my + 6k^2t))}, \tag{80}$$

where b_{-3}, k, and m are arbitrary constants.

Case 5.4

$$\{a_{-3} = \mp 6kb_{-3}, a_3 = 0, a_{-2} = 0, a_2 = 0, a_0 = 0, a_1 = 0, a_{-1} = 0, b_0 = 0, w = \mp 6k^2,$$

$$b_{-3} = b_{-3}, k = k, m = m\}, \tag{81}$$

$$u_{36}(x, y, t) = -\frac{6kb_{-3}}{b_{-3} + \exp(6(kx + my - 6k^2t))},$$

$$v_{36}(x, y, t) = -\frac{6mb_{-3}}{b_{-3} + \exp(6(kx + my - 6k^2t))}, \tag{82}$$

$$u_{37}(x, y, t) = \frac{6kb_{-3}}{b_{-3} + \exp(6(kx + my + 6k^2t))},$$

$$v_{37}(x, y, t) = 0, \tag{83}$$

where b_{-3}, k, and m are arbitrary constants.

Remark 1 It is important that one should be aware of the limitations of each of the existing methods and there is no guarantee that they will succeed for a specific nonlinear problem. Any of these techniques can have some advantages and disadvantages. The Exp-function method provides exponential function solutions in a neat form from which we can construct solitary and periodic wave solutions by setting the parameters as special values. It is entirely algorithmic and involves a large amount of tedious calculations that can become virtually unmanageable if attempted manually. However, it does not require a large amount of run-time with the aid of a computer algebra system. The Exp-function method, by assuming the solution of the equation in exponential form with many parameters, entails the solution of several sets of nonlinear algebraic equations that sometimes constitute inconsistent systems.

4. Conclusion

Thanks to the improvement in analytic techniques via the rapid advance in computer technology, most of the PDEs arising from applications that were previously intractable can now be routinely solved. In this work, we used the Exp-function method to solve the (2+1)-dimensional BLP equation. Some of the obtained generalized solitary wave solutions have known physical understandings, while others require further insight into their hidden meanings. We verified the obtained solutions by putting them back into the original Equation (1) with the aid of MATHEMATICA; it provides an extra measure of confidence in the results. We predict that the Exp-function method can be found widely applicable in searching for exact and explicit solutions of nonlinear differential equations.

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