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Exact longitudinal vibration characteristics of rods with variable cross-sections

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Abstract. Longitudinal natural vibration frequencies of rods (or bars) with variable cross-sections are obtained from the exact solutions of differential equation of motion based on transformation method. For the rods having cross-section variations as power of the sinusoidal functions of $ax + b$, the differential equation is reduced to associated Legendre equation by using the appropriate transformations. Frequency equations of rods with certain cross-section area variations are found from the general solution of this equation for different boundary conditions. The present solutions are benchmarked by the solutions available in the literature for the special case of present cross-sectional variations. Moreover, the effects of cross-sectional area variations of rods on natural characteristics are studied with numerical examples.

Keywords: Exact solution, longitudinal vibration, variable cross-section

1. Introduction

Rods with variable cross-sections are structural members designed for many different systems used in aeronautical, civil, mechanical, and other engineering applications. Since the governing differential equation of motion of such types of rods has variable coefficients due to the variable cross-sectional area, exact analytical solution of this equation of motion is only possible for particular cross-sectional area functions. Therefore, exact longitudinal vibration characteristics of non-uniform rods are covered in quite a few vibration books [1,2]. Graff [1] noticed that the equation of motion for rods with conical cross-sections is the form of spherical wave equation. Elishakoff [2] treated closed-form solutions for vibration of inhomogeneous rods considering polynomial variations in the longitudinal rigidity and the inertial coefficient, independently.

Exact analytical longitudinal vibration characteristics of non-uniform rods are studied rarely. Raman [3] reported the general solutions available for particular cross-sectional area variations of rods such as cos(x), sin(x), exp(-x^2). Eisenberger [4] presented exact element method for the title problem with polynomial variation in the cross-sectional area. Bapat [5] proposed an exact approach as the combination of closed form solution and transfer matrix method for the longitudinal vibration of conical, exponential, and catenoidal rods. Abrate [6] showed that equation of motion for the non-uniform rods with area variation of the form $A(x) = A_0[1 + \alpha(x/L)]^2$ can be transformed into classical wave equation. Kumar and Sujith [7] obtained exact analytical solutions for the longitudinal vibration of rods with cross-sectional area variations given by $A(x) = (a + bx)^n$ and $A(x) = A_0 \sin^2(ax + b)$. Horgan and Chan [8] provided exact solutions for the vibration of rods whose cross-section varies as $A(x) = A_0[1 + \alpha(x/l)]^n$ for the case $n = -1$, 1, 2 and $A(x) = A_0 \exp(-\alpha\, x/l)$. Li et al. [9] presented exact analytical solutions for longitudinal vibration of non-uniform rods with concentrated masses coupled by translational springs. In their study, cross-sectional area variations are selected as follows: $A(x) = a \exp(-bx/l)$ and $A(x) = a\,(1 + bx)^c$. Li [10] dealt

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with the longitudinal vibrations of non-uniform rods under the assumption that the mass distribution of a rod is arbitrary, and the distribution of longitudinal stiffness is expressed as a functional relation with the mass distribution and vice versa. Raj and Sujith [11] determined cross-sectional area variations of rods as $A(x) = kx^n \exp(bx^2)$, $A(x) = kx^n \exp(bx)$, and $A(x) = k \exp(bx) \exp(n \exp(mx))$ that give Kummer's hypergeometric function as solution.

In this paper, exact analytical solutions for longitudinal vibrations of non-uniform rods with area variations of the form $A(x) = A_0 \sin^n(ax + b)$ and $A(x) = A_0 \cos^n(ax + b)$ are found in terms of associated Legendre functions by using the appropriate successive transformations applied to governing differential equation. Then, frequency equations are obtained for clamped-clamped, clamped-free, and free-free end conditions. The present solutions are benchmarked by the solutions given by Kumar and Sujith [7] for $A(x) = A_0 \sin^2(ax + b)$. Furthermore, the effects of the parameters regarding the cross-sectional area variations of the rods on natural frequencies are studied with numerical examples. The results are presented in tabular form.

2. The equation of motion

A non-uniform cross-sectioned rod is considered. The governing partial differential equation for the longitudinal vibration of a rod with varying cross-section can be written as [12, p.447]

$$\frac{\partial}{\partial x}\left[E\,A\,(x)\,\frac{\partial u(x,t)}{\partial x}\right] = \rho A(x)\frac{\partial^2 u(x,t)}{\partial t^2} \text{ on } L_1 < x < L_2 \tag{1}$$

where $u(x,t)$ is the axial displacement of the rod at distance x and time t. E, $A(x)$, and ρ are Young's modulus, cross-sectional area and density of the rod, respectively. If the rod is clamped at the end $x = L_1$ and free at the end $x = L_2$, the boundary conditions are

$$u(L_1, t) = 0 \tag{2}$$

$$EA(x)\left.\frac{\partial u(x,t)}{\partial x}\right|_{x=L_2} = 0 \tag{3}$$

The boundary conditions for a clamped-clamped or free-free rod can be written considering Eqs (2) or (3). If the boundary conditions of the rod are homogeneous as in Eqs (2) and (3), the solution of Eq. (1) is assumed as

$$u(x,t) = U(x)T(t) \tag{4}$$

where $U(x)$ is displacement as function of x and $T(t) = \exp(i\omega\, t)$ in which ω is the circular natural frequency of the harmonic vibrations. Eq. (1) is reduced to following ordinary differential equation for $U(x)$ by using Eq. (4)

$$\frac{d}{dx}\left[E\,A\,(x)\,\frac{dU(x)}{dx}\right] + \omega^2 \rho\, A(x)\, U(x) = 0 \tag{5}$$

For the sake of clarity, Eq. (5) is written in open-form as

$$\frac{d^2U(x)}{dx^2} + \frac{1}{A(x)}\frac{dA(x)}{dx}\frac{dU(x)}{dx} + \beta^2\, U(x) = 0 \tag{6}$$

where

$$\beta^2 = \omega^2 \frac{\rho}{E} \tag{7}$$

Since Eq. (6) has variable coefficient, its exact analytical solution can be found only for particular area functions.

3. General solutions for the selected cross-sectional area functions

Cross-sectional area variation of the rod is selected as

$$A(x) = A_0 \sin^n(ax + b) \tag{8}$$

Having an inspiration from the functional transformation $\xi = \sin(x)$ given by Polyanin and Zaitsev [13, Eq. (2,1,6,29)] for a differential equation which corresponds to Eq. (6) with $A(x) = A_0 \cos^n(x)$, the functional transformation for the present case is considered as

$$\xi = \cos(ax + b) \tag{9}$$

By using the transformation written in Eq. (9), Eq. (6) along with Eq. (8) is reduced to

$$a^2(1 - \xi^2)\frac{d^2U(\xi)}{d\xi^2} - a^2(1 + n)\xi\frac{dU(\xi)}{d\xi} + \beta^2 U(\xi) = 0 \tag{10}$$

The following form is assumed for the solution of Eq. (10)

$$U(\xi) = f(\xi)g(\xi) \tag{11}$$

Substituting Eq. (11) into Eq. (10), the following equation is obtained:

$$\begin{aligned} &(1 - \xi^2)\frac{d^2f(\xi)}{d\xi^2} - \left[(1 + n)\,\xi + 2(\xi^2 - 1)\frac{1}{g(\xi)}\frac{dg(\xi)}{d\xi}\right]\frac{df(\xi)}{d\xi} \\ &+ \left[\frac{\beta^2}{a^2} - (1 + n)\,\xi\frac{1}{g(\xi)}\frac{dg(\xi)}{d\xi} - (\xi^2 - 1)\frac{1}{g(\xi)}\frac{dg^2(\xi)}{d\xi^2}\right] f(\xi) = 0 \end{aligned} \tag{12}$$

Now, the problem is reduced to find $g(\xi)$, in order to write Eq. (12) as the associated Legendre differential equation [14–16] expressed for the function $f(\xi)$ in the following form,

$$(1 - \xi^2)\frac{d^2f(\xi)}{d\xi^2} - 2\xi\frac{df(\xi)}{d\xi} + \left[l(l + 1) - \frac{m^2}{1 - \xi^2}\right] f(\xi) = 0 \tag{13}$$

By equating the coefficient of $df(\xi)/d\xi$ in Eqs (12) and (13), $g(\xi)$ is found as

$$g(\xi) = (\xi^2 - 1)^{\frac{1-n}{4}} \tag{14}$$

Then, equating the coefficient of $f(\xi)$ in Eqs (12) and (13) along with Eq. (14), l and m in Eq. (13) are obtained as

$$l = \frac{-a + \sqrt{a^2n^2 + 4\beta^2}}{2a} \tag{15}$$

$$m = \frac{n - 1}{2} \tag{16}$$

Solution of Eq. (13) is available in the literature [14–16] as

$$f(\xi) = C_1P_l^m(\xi) + C_2Q_l^m(\xi) \tag{17}$$

where $P_l^m(\xi)$ and $Q_l^m(\xi)$ are associated Legendre functions of the first and second kind, respectively. Therefore, substituting Eqs (14) and (17) into Eq. (11), the general solution of Eq. (10) is obtained as

$$U(\xi) = (\xi^2 - 1)^{\frac{1-n}{4}}[C_1P_l^m(\xi) + C_2Q_l^m(\xi)] \tag{18}$$

Using the transformation expressed in Eq. (9), the general solution of Eq. (6) for the cross-sectional area variation function defined by Eq. (8) is found as follows:

$$U(x) = (\cos^2(ax + b) - 1)^{\frac{1-n}{4}}[C_1P_l^m(\cos(ax + b)) + C_2Q_l^m(\cos(ax + b))] \tag{19}$$

Similarly, if the cross-sectional area variation of the rod is selected as

$$A(x) = A_0 \cos^n(ax + b) \tag{20}$$

Equation (10) is found by using the following functional transformation

$$\xi = \sin(ax + b) \tag{21}$$

In this case, general solution of Eq. (6) along with Eq. (21) is written as

$$U(x) = (\sin^2(ax + b) - 1)^{\frac{1-n}{4}}[C_1P_l^m(\sin(ax + b)) + C_2Q_l^m(\sin(ax + b))] \tag{22}$$

4. Frequency equations for the selected cross-sectional area functions

Frequency equations of the rods with the cross-sectional area function represented by Eq. (8) is found for three possible boundary conditions by using the general solution given by Eq. (19) as follows:

For a clamped-clamped rod, the boundary conditions are

$$U(L_1) = 0 \tag{23a}$$

$$U(L_2) = 0 \tag{23b}$$

Using the boundary conditions given in Eqs (23a) and (23b) in Eq. (19), the frequency equation is found as follows:

$$P_l^m(\cos(aL_1 + b))\, Q_l^m(\cos(aL_2 + b)) - P_l^m(\cos(aL_2 + b))\, Q_l^m(\cos(aL_1 + b)) = 0 \tag{24}$$

For a clamped-free rod, the boundary conditions are

$$U(L_1) = 0 \tag{25a}$$

$$U'(L_2) = 0 \tag{25b}$$

The prime used throughout this paper represents differentiation with respect to x. Using the boundary conditions given in Eqs (25a) and (25b) in Eq. (19), the frequency equation is obtained as follows:

$$\begin{aligned}&P_l^m(\cos(aL_1 + b))[(n - 2l - 1)\cos(aL_2 + b)Q_l^m(\cos(aL_2 + b)) + 2(l + m)Q_{l-1}^m(\cos(aL_2 + b))] -\\ &Q_l^m(\cos(aL_1 + b))[(n - 2l - 1)\cos(aL_2 + b)P_l^m(\cos(aL_2 + b)) + 2(l + m)P_{l-1}^m(\cos(aL_2 + b))] = 0\end{aligned} \tag{26}$$

For a free-free rod, the boundary conditions are

$$U'(L_1) = 0 \tag{27a}$$

$$U'(L_2) = 0 \tag{27b}$$

Using the boundary conditions given in Eqs (27a) and (27b) in Eq. (19), the frequency equation is written as follows:

$$\begin{aligned}&[(n - 2l - 1)\cos(aL_1 + b)P_l^m(\cos(aL_1 + b)) + 2(l + m)P_{l-1}^m(\cos(aL_1 + b))]\\ &[(n - 2l - 1)\cos(aL_2 + b)Q_l^m(\cos(aL_2 + b)) + 2(l + m)Q_{l-1}^m(\cos(aL_2 + b))] -\\ &[(n - 2l - 1)\cos(aL_1 + b)Q_l^m(\cos(aL_1 + b)) + 2(l + m)Q_{l-1}^m(\cos(aL_1 + b))]\\ &[(n - 2l - 1)\cos(aL_2 + b)P_l^m(\cos(aL_2 + b)) + 2(l + m)P_{l-1}^m(\cos(aL_2 + b))] = 0\end{aligned} \tag{28}$$

Frequency equations of the rods with the cross-sectional area function represented by Eq. (20) is obtained by changing the all cos(*) in Eqs (24), (26), and (28) with sin(*), where * represents any terms.

5. Comparisons and discussions of numerical examples

In order to demonstrate the correctness of the present solutions, numerical values of non-dimensional natural frequencies (βL) given by Kumar and Sujith [7] for clamped-free rods with cross-sectional area variation of the form $A(x) = A_0 \sin^2(ax + b)$ are used in comparisons presented in Table 1. In this study, length of the rod is described by $L = L_2 - L_1$. It is seen from Table 1 that very small differences exist between the present results and the numerical results provided by Kumar and Sujith [7]. On the other hand, the non-dimensional natural frequencies obtained by using the frequency equation given by Kumar and Sujith [7] as

$$[a/\tan(aL + b)]\, \tan(L\sqrt{a^2 + \beta^2}) = \sqrt{a^2 + \beta^2} \tag{29}$$

are exactly the same with the present results given in Table 1. Therefore, the numerical non-dimensional natural frequencies presented in reference [7] have some numerical errors.

Table 1
Comparison of non-dimensional natural frequencies of clamped-free rods with $A(x) = A_0 \sin^2(ax + b)$ for various values of a ($L_1 = 0$, $L_2 = 1$, $b = 1$)

Mode	$a = 1$ [7]	$a = 1$ Present	$a = 2$ [7]	$a = 2$ Present
1	1.517638	1.517637	2.148560	2.148560
2	4.702145	4.702145	5.535762	5.535762
3	7.848311	7.848311	8.632812	8.632811
4	10.991620	10.991621	11.694640	11.694641
5	14.134120	14.134123	14.757860	14.757858
6	17.276280	17.276282	17.830600	17.830596

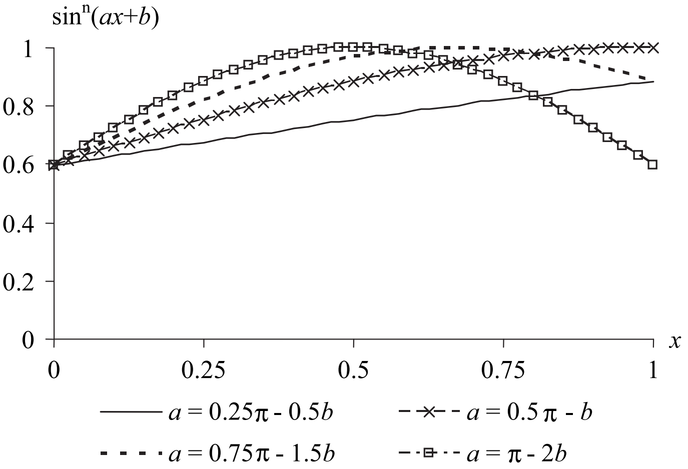

Fig. 1. Graphical illustrations of $\sin^n(ax + b)$ for $b = 1$, $n = 3$, and different values of the parameter a.

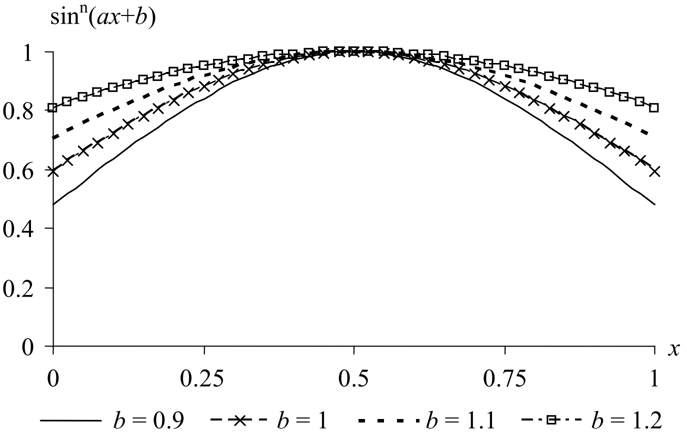

Fig. 2. Graphical illustrations of $\sin^n(ax + b)$ for $a = \pi–2b$, $n = 3$, and different values of the parameter b.

As numerical applications of the present frequency equations obtained for clamped-clamped, clamped-free, and free-free end conditions, the truncated sinus waves shown in Figs 1–3 are determined for the functional part of the cross-sectional area variations. Non-dimensional natural frequencies for aforementioned boundary conditions of the rods illustrated in Figs 1–3 are found and given in Tables 2–10.

It is evident in Table 2 that when the value of a increases within the given range, non-dimensional frequencies decrease. In Table 3, non-dimensional frequencies are found to increase with increasing b. Table 4 has the similar

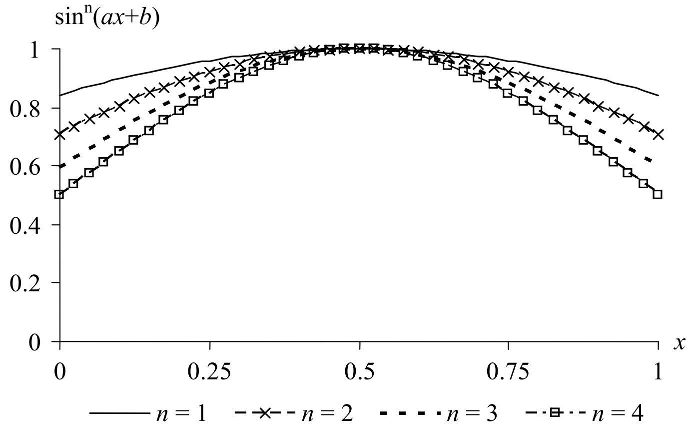

Fig. 3. Graphical illustrations of $\sin^n(ax+b)$ for $a = \pi–2b$, $b = 1$, and different values of the parameter n.

Table 2
Non-dimensional natural frequencies of clamped-clamped rods with $A(x) = A_0 \sin^3(ax+b)$ for various values of a ($L_1 = 0$, $L_2 = 1$, $b = 1$)

Mode	$a = 0.25\pi - 0.5b$	$a = 0.5\pi - b$	$a = 0.75\pi - 1.5b$	$a = \pi - 2b$
1	3.124175	3.066848	2.965843	2.821246
2	6.274524	6.246492	6.198613	6.133696
3	9.419010	9.400401	9.368810	9.326456
4	12.562046	12.548111	12.524504	12.492985
5	15.704504	15.693364	15.674510	15.649387
6	18.846674	18.837393	18.821697	18.800803

Table 3
Non-dimensional natural frequencies of clamped-clamped rods with $A(x) = A_0 \sin^3(ax+b)$ for various values of b ($L_1 = 0$, $L_2 = 1$, $a = \pi - 2b$)

Mode	$b = 0.9$	$b = 1.0$	$b = 1.1$	$b = 1.2$
1	2.693969	2.821246	2.925733	3.008704
2	6.080621	6.133696	6.180034	6.218531
3	9.292421	9.326456	9.356605	9.381897
4	12.467826	12.492985	12.515398	12.534268
5	15.629399	15.649387	15.667244	15.682303
6	18.784210	18.800803	18.815649	18.828182

Table 4
Non-dimensional natural frequencies of clamped-clamped rods with $A(x) = A_0 \sin^n(ax+b)$ for various values of n ($L_1 = 0$, $L_2 = 1$, $a = \pi - 2b$, $b = 1$)

Mode	$n = 1$	$n = 2$	$n = 3$	$n = 4$
1	3.033658	2.926836	2.821246	2.717003
2	6.228475	6.178607	6.133696	6.093840
3	9.388171	9.355384	9.326456	9.301424
4	12.538877	12.514409	12.492985	12.474621
5	15.685954	15.666425	15.649387	15.634849
6	18.831208	18.814955	18.800803	18.788757

tendency with Table 2 for the parameter n. Table 5 shows very complicated effects of the parameter a on non-dimensional frequencies. It can be seen from Table 6 that when the value of b increases within the given range, non-dimensional frequencies except first one decrease. Table 7 has the opposite tendency with Table 6. Tables 8, 9, and 10 present opposite tendency with Table 2, 3, and 4, respectively.

Table 5
Non-dimensional natural frequencies of clamped-free rods with $A(x) = A_0 \sin^3(ax + b)$ for various values of a ($L_1 = 0$, $L_2 = 1$, $b = 1$)

Mode	$a = 0.25\pi - 0.5b$	$a = 0.5\pi - b$	$a = 0.75\pi - 1.5b$	$a = \pi - 2b$
1	1.448094	1.410623	1.444102	1.547041
2	4.673935	4.663228	4.679553	4.743064
3	7.831015	7.824694	7.834857	7.875459
4	10.979190	10.974696	10.982032	11.011552
5	14.124430	14.120941	14.126673	14.149801
6	17.268341	17.265490	17.270189	17.289183

Table 6
Non-dimensional natural frequencies of clamped-free rods with $A(x) = A_0 \sin^3(ax + b)$ for various values of b ($L_1 = 0$, $L_2 = 1$, $a = \pi - 2b$)

Mode	$b = 0.9$	$b = 1$	$b = 1.1$	$b = 1.2$
1	1.524289	1.547041	1.560065	1.566753
2	4.771651	4.743064	4.726399	4.717714
3	7.896713	7.875459	7.863580	7.857574
4	11.027629	11.011552	11.002673	10.998220
5	14.162600	14.149801	14.142767	14.139250
6	17.299778	17.289183	17.283374	17.280475

Table 7
Non-dimensional natural frequencies of clamped-free rods with $A(x) = A_0 \sin^n(ax + b)$ for various values of n ($L_1 = 0$, $L_2 = 1$, $a = \pi - 2b$, $b = 1$)

Mode	$n = 1$	$n = 2$	$n = 3$	$n = 4$
1	1.568123	1.560155	1.547041	1.529023
2	4.715830	4.726102	4.743064	4.766484
3	7.856372	7.863537	7.875459	7.892108
4	10.997350	11.002678	11.011552	11.023967
5	14.138571	14.142783	14.149801	14.159624
6	17.279918	17.283392	17.289183	17.297288

Table 8
Non-dimensional natural frequencies of free-free rods with $A(x) = A_0 \sin^3(ax + b)$ for various values of a ($L_1 = 0$, $L_2 = 1$, $b = 1$)

Mode	$a = 0.25\pi$–$0.5b$	$a = 0.5\pi$–b	$a = 0.75\pi$–$1.5b$	$a = \pi$–$2b$
1	3.171275	3.237508	3.331630	3.470891
2	6.298229	6.333403	6.387328	6.475015
3	9.434833	9.458556	9.495522	9.557056
4	12.573918	12.591784	12.619786	12.666805
5	15.714004	15.728324	15.750829	15.788776
6	18.854591	18.866537	18.885339	18.917112

Table 9
Non-dimensional natural frequencies of free-free rods with $A(x) = A_0 \sin^3(ax + b)$ for various values of b ($L_1 = 0$, $L_2 = 1$, $a = \pi - 2b$)

Mode	$b = 0.9$	$b = 1.0$	$b = 1.1$	$b = 1.2$
1	3.606218	3.470891	3.361601	3.276077
2	6.568999	6.475015	6.405441	6.355043
3	9.624896	9.557056	9.508098	9.473321
4	12.719210	12.666805	12.629357	12.602950
5	15.831289	15.788776	15.758539	15.737291
6	18.952812	18.917112	18.891788	18.874025

Table 10
Non-dimensional natural frequencies of free-free rods with $A(x) = A_0 \sin^n(ax + b)$ for various values of n ($L_1 = 0$, $L_2 = 1$, $a = \pi - 2b$, $b = 1$)

Mode	$n = 1$	$n = 2$	$n = 3$	$n = 4$
1	3.250523	3.360328	3.470891	3.582093
2	6.342610	6.406605	6.475015	6.547675
3	9.465160	9.509269	9.557056	9.608466
4	12.596870	12.630355	12.666805	12.706197
5	15.732444	15.759385	15.788776	15.820605
6	18.869993	18.892514	18.917112	18.943781

6. Conclusions

Exact analytical solutions for the free longitudinal vibration of rods with cross-sectional area variation of the form $A(x) = A_0 \sin^n(ax + b)$ and $A(x) = A_0 \cos^n(ax + b)$ are found by transformation method. Also, frequency equations for clamped-clamped, clamped-free, and free-free end conditions are obtained. The present results for clamped-free non-uniform rods are verified by the results available in the literature. Also, numerical non-dimensional frequency parameters for the rods of the some forms determined by pondering are presented in tabular form.

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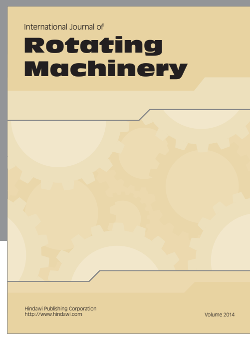
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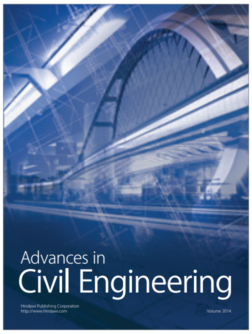
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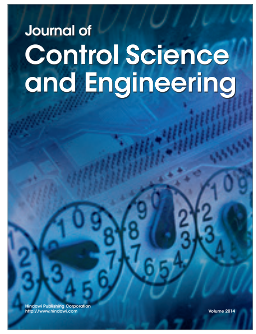
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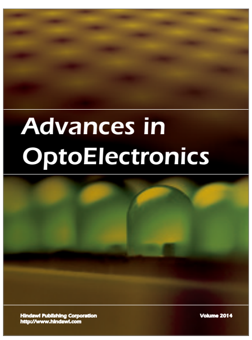
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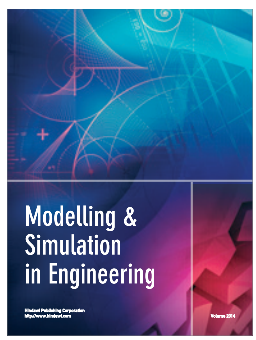
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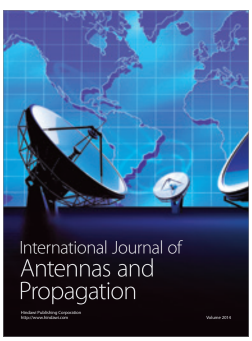
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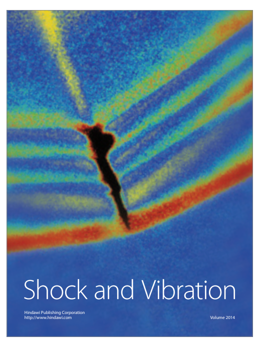
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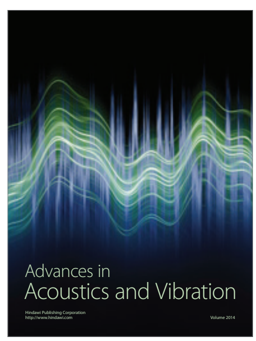
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