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A Fully Discrete ε-Uniform Method for Convection-Diffusion Problem on Equidistant Meshes

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Abstract

For a singularly-perturbed two-point boundary value problem, we propose an ε-uniform finite difference method on an equidistant mesh which requires no exact solution of a differential equation. We start with a full-fitted operator method reflecting the singular perturbation nature of the problem through a local boundary value problem. However, to solve the local boundary value problem, we employ an upwind method on a Shishkin mesh in local domain, instead of solving it exactly. We further study the convergence properties of the numerical method proposed and prove it nodally converges to the true solution for any ε.

Mathematics Subject Classification: 65N06, 65L10, 65L11, 65L12

Keywords: ε-uniform, singular perturbation, fitted operator method, Shishkin mesh

1 Introduction

It is well-known that the classical finite difference methods for the approximation of singularly perturbed boundary value problems problem does not work in the critical range of ε where ε is considerably small compared to the mesh parameter h. Although the centered difference approximation produces good approximations for large values of ε, the result is totally unphysical as $\varepsilon \to 0$. These deficiencies disappear if we discretize the convection term by an appropriate one-sided finite difference operator, in which case the resulting numerical method is known as the upwind method. However, the approximate solution may not converge to the true solution in the layer region where the useful information is confined. Therefore it is important to devise uniformly convergent methods that yields the numerical approximations consistent with the physical configuration of the problem in all regimes.

A considerable amount of research work has been devoted to the development of the uniformly convergent methods. In the construction of ε-uniform finite difference methods, two major approaches have generally been taken to date. The first of these involves replacing the standard finite difference operator by a difference operator which reflects the singularly perturbed nature of the differential operator. Such numerical methods are referred to, in general, as fitted operator finite difference methods, [9, 10]. Typical derivation of such methods based on the discretization of the domain into a set of equidistant subintervals and the exact solution of a local boundary value problem with an irregular data on a pair of adjacent subintervals. It is appreciated that the method use an equidistant mesh but the method overall suffers from the fact that it depends on the exact solution which is not easier to solve than the original problem.

The second major approach in the construction of ε-uniform finite difference method involves the use of a fitted mesh, a mesh that is adapted according to the singular perturbation [9, 10]. Let us concentrate on a subclass of the full-fitted meshes known as Shishkin mesh [11]. A Shishkin mesh, also called piecewise uniform full-fitted meshes, consist of a union of finite number of uniform meshes having different mesh parameters on both sides of a transition point . It turns out that a Shishkin mesh together with the simple upwind method is sufficient for the construction of an ε-uniform method [12]. These meshes can also be applied to singular perturbation problems with interior layers caused by point sources, succesfully [6]. The simplicity of Shiskin mesh is due to the use of equidistant subintervals on both side of a transition point and this property is considered to be one of its major attractions. However, it requires the precise location of the layer structure.

The algorithm investigated in this work combines these two major classes of ε-uniform finite difference methods. We start with a full-fitted operator method reflecting the singular perturbation nature of the problem through a local boundary value problem posed on an adjacent pair of subintervals. However, the local BVP (boundary value problem) has an interior layer caused by a concentrated source and instead of solving it exactly, we approximate it with the upwind method on a Shishkin-like mesh on the patch of these subintervals. The distribution of the mesh points in the subdomain is determined depending on the local flow regime. Further we prove that the resulting numerical method nodally converges to the true solution for any ε. Thus we display that it is possible to develop an ε-uniform method on a equidistant mesh without solving the local differential equation exactly.

The layout of the paper is as follows. In Section 2 and 3, we briefly recall the basic ideas of the full-fitted operator method and the full-fitted mesh method, respectively, applied to a singularly perturbed BVP. In Section 3, the application of the standard upwind method on Shishkin mesh and its convergence properties are presented for two types of source functions. Merging the ideas in Section 2 and 3, we propose a numerical method, in Section 4, on a uniform mesh which do not require the exact solution of the local BVP. Instead we display how to approximate to the solution of the local BVP conveniently, so that the resulting numerical method recovers the same convergence properties as the one using the exact solution of the local BVP. Further details related to convergence are given in Section 5, where we prove that the new algorithm nodally converges to the true solution.

2 A Fitted Operator Method On an Equidistant Mesh

Let us recall how to construct an ε-uniform method of full-fitted operator type and what its convergence properties are. Consider the following singularly perturbed boundary value problem on the unit interval $\Omega = (0, 1)$

$$\left\{ \begin{array}{l} \text{Find } u(x) \text{ such that } u(0) = u_0, \quad u(1) = u_1 \quad \text{and} \\ L_1 u = -\varepsilon u^{''} + b(x)\, u^{'} + c(x)\, u(x) = f(x), \quad \forall x \in \Omega, \end{array} \right. \tag{1}$$

under the assumptions that $b(x) \geq b_0 > 0$ and $c(x) \geq 0$, where u_0, u_1 are given constants. The formal adjoint operator L^* of L is given by $L^* = -\varepsilon u^{''} - b\, u^{'} + cu$. Define a uniform mesh $\{x_i\}_0^N$ where $x_i = i\, h$, $i = 0, 1, .., N$ and $h = 1/N$, denoted by Ω^N; the space of all mesh functions defined on Ω^N by $V(\Omega^N)$ and the discrete maximum norm for any mesh function V by $\parallel V \parallel_{\Omega^N} = \max_{0 \leq i \leq N} |V_i|$. Further define the subinterval $\Omega_i = (x_{i-1}, x_i)$. Let g_i be the local Green's function of L^* with respect to the point x_i, which is posed on a pair of subintervals containing x_i. The boundary value problem associated with g_i on the local domain $\overline{\Omega}_i \cup \overline{\Omega}_{i+1}$ reads:

$$\left\{\begin{array}{c} \text{Find } g_i \in C(\overline{\Omega}_i \cup \overline{\Omega}_{i+1}) \cap C^2(\Omega_i \cup \Omega_{i+1}) \text{ such that} \\ g_i(x_{i-1}) = 0, \quad g_i(x_{i+1}) = 0 \qquad \text{and} \\ L^* g_i = -\varepsilon g_i^{''}(x) - b g_i^{'}(x) + c\ g_i(x) = 0,\ \ \forall x \in \Omega_i \cup \Omega_{i+1}, \end{array}\right. \tag{2}$$

with the additional condition

$$\varepsilon(g_i^{'}(x_i^-) - g_i^{'}(x_i^+)) = 1. \tag{3}$$

Thus, multiplying the equation $L_1 u = f$ in (1) with g_i and integrating the resulting expression from x_{i-1} to x_{i+1}, we obtain

$$\int_{x_{i-1}}^{x_{i+1}} (Lu)\ g_i dx = \int_{x_{i-1}}^{x_{i+1}} f\, g_i dx. \tag{4}$$

We integrate (4) by parts, and then use the continuity of $u^{'}$ and the condition (3), respectively, to get

$$-\varepsilon g_i'(x_{i-1}) u(x_{i-1}) + u(x_i) + \varepsilon g_i'(x_{i+1}) u(x_{i+1}) = \int_{x_{i-1}}^{x_{i+1}} f\, g_i dx. \tag{5}$$

In general, it is difficult to evaluate each $g_i^{'}$ exactly, so we need further approximation to convert (5) to a working scheme. In the simplest case where b, c and f are constant in (x_{i-1}, x_{i+1}), it is possible to compute g_i explicitly. Denoting them by b_i, c_i and f_i, respectively, we solve (2) exactly and then, substitute the exact solution of g_i into the equation (5). Thus, we get the following the difference equation;

$$\left\{\begin{array}{l} \text{Find } U \in V(\Omega^N) \text{ such that } U_0 = u_0,\ U_N = u_1, \text{ and } 1 \le i \le N-1, \\ -\dfrac{e^{\gamma_i + \rho_i}}{1 + e^{2\gamma_i}} U_{i-1} + U_i - \dfrac{e^{\gamma_i - \rho_i}}{1 + e^{2\gamma_i}} U_{i+1} = \dfrac{f_i}{c_i} \left(\dfrac{e^{2\gamma_i} - e^{\gamma_i - \rho_i} - e^{\gamma_i + \rho_i} + 1}{1 + e^{2\gamma_i}} \right), \end{array}\right. \tag{6}$$

where $U_i \approx u(x_i)$, $\gamma_i = \frac{\left(\sqrt{b_i^2 + 4 c_i \varepsilon}\right) h}{2\varepsilon}$ and $\rho_i = \frac{b_i h}{2\varepsilon}$. This is a variant of the El-Mistikawy-Werle scheme [10] for which, we have the following error estimate in [8].

Theorem 1 *The fitted operator finite difference method* (6) *with the uniform mesh* Ω^N*, is* $\varepsilon-$*uniform for the problem (1). Moreover, the solution* u *of (1) and the solution* U_O *of (6) satisfy the following* $\varepsilon-$*uniform error estimate*

$$\sup_{0<\varepsilon\le 1} \parallel u - U_O \parallel_{\Omega^N} \le C N^{-2},$$

where C *is a constant independent of* ε*.*

Proof: See [5]. ■

Although the fitted operator method (6) converges ε-uniformly in the discrete maximum norm, it is based on the exact solution of the local boundary value problem (2), which is not much easier to solve than the problem (1). This can be seen as a major drawback of this method.

3 Upwind difference method on a Shiskin mesh

It is well-known that a piecewise uniform fitted mesh only turns out to be sufficient for the construction of an $\varepsilon-uniform$ method. A simple example of a piecewise uniform mesh is constructed on the interval $\Omega = (0, 1)$ as follows: Choose a point $1-\tau$ satisfying $0 < \tau \leq 1/2$ and assume that $N = 2^r$, for some $r \geq 2$. The point $1-\tau$ divides Ω into the two subintervals $(0, 1-\tau)$ and $(1-\tau, 1)$. The corresponding piecewise uniform mesh is constructed by dividing both $(0, 1-\tau)$ and $(1-\tau, 1)$ into $N/2$ equal subintervals denoted by Ω_τ^N. Thus the fitted piecewise uniform mesh $\Omega_\tau^N = \{x_i\}_0^N$ is defined such that its points satisfy the following relations:

$$x_0 = 0, \quad \text{and} \quad x_i - x_{i-1} = \begin{cases} h_1 = \dfrac{2(1-\tau)}{N} & \text{for } 0 < i \leq N/2, \\ h_2 = \dfrac{2\tau}{N} & \text{for } N/2 < i \leq N, \end{cases}$$

where $\tau = \min\left\{\frac{1}{2}, \frac{\varepsilon}{b_0}\ln N\right\}$. Note that whenever N is sufficiently large, τ

Figure 1: The piecewise uniform mesh Ω_τ^8

takes the value 1/2, in which case the mesh Ω_τ^N becomes uniform with N equal-sized subintervals. For all other permissible values of τ, $0 < \tau < 1/2$, the subinterval $(1-\tau, 1)$ is smaller than the subinterval $(0, 1-\tau)$. In such cases the mesh is piecewise uniform rather than uniform.

3.1 Irregular source function

Next we consider a singular perturbation problem with a concentrated source which is crucial to the development of the numerical method in the next section:

$$\begin{cases} \text{Find } u(x) \text{ such that } u(0) = u_0, \quad u(1) = u_1 \qquad \text{and} \\ L_2 u = -\varepsilon u''(x) - b u'(x) + c\, u(x) = f(x) + \delta_d(x), \quad \forall x \in \Omega, \end{cases} \tag{7}$$

where f is the smooth component of the source function and δ_d is the shifted $Dirac-$delta function; $\delta_d(x) = \delta(x-d)$ with $d \in (0, 1)$ and $0 < \varepsilon \leq 1$. The problem (7) has to be understood in a distributional context. The solution u typically has an exponential boundary layer at the outflow boundary $x = 0$ and an internal layer at $x = d$ caused by the concentrated source. To approximate the problem (7), we employ a Shishkin mesh and design it in a special way to resolve both the boundary and the internal layers. To construct such mesh, take three points τ, d and $d+\tau$, which divide the domain $\overline{\Omega}$ into the four subdomains $I_1 = [0, \tau]$, $I_2 = [\tau, d]$, $I_3 = [d, d+\tau]$ and $I_4 = [d+\tau, 1]$ where τ satisfies the condition $\tau = \min\left\{\frac{1}{4}, \frac{\varepsilon}{b_0} \ln N\right\}$. The corresponding piecewise

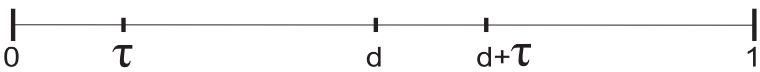

Figure 2: Subdomains for the discretization of the problem (7)

uniform mesh is established by dividing each subdomain into $N/4$ equidistant subintervals (Figure 2). The resulting mesh $\Omega^N_{\tau-d}$ is described by $x_0 = 0$ and

$$x_i - x_{i-1} = \begin{cases} h_1 & \text{for} \quad 0 < i \le N/4 \quad \text{or} \quad N/2 < i \le 3N/4, \\ h_2 & \text{for} \quad N/4 < i \le N/2 \quad \text{or} \quad 3N/4 < i \le N, \end{cases} \tag{8}$$

where $h_1 = \frac{4\tau}{N}$ and $h_2 = \frac{4(d-\tau)}{N}$. We approximate to (7) by using the upwind method on the piecewise uniform mesh described in (8):

$$\begin{cases} \text{Find } U \in V(\Omega^N_{\tau-d}) \text{ such that } U_0 = 0,\, U_N = 0 \text{ and} \\ \\ -\varepsilon D^+ D^- U_i - b_i\, D^+ U_i + c_i\, U_i = f_i + \Delta_{d,i}, \quad i = 1, 2, .., N-1, \end{cases} \tag{9}$$

where

$$\Delta_{d,i} = \begin{cases} \dfrac{1}{h_{i+1}} & \text{if } d \in [x_i, x_{i+1}) \\ 0 & \text{otherwise,} \end{cases}$$

is an approximation of the shifted $Dirac-$delta function and $b_i = \lim_{x \to x_i^-} b(x)$. The solution of the numerical method (9) converges nodally to the solution of (7):

Theorem 2 *The fitted mesh finite difference method* (9) *with the piecewise uniform fitted mesh* $\Omega^N_{\tau-d}$ *is* $\varepsilon-$*uniform for the problem* (7) *provided that* τ *is chosen to satisfy the condition* $\tau = \min\{\frac{1}{4}, \frac{\varepsilon}{b_0} \ln N\}$ *above. Moreover, the solution* u *of* (7) *and the solution* U_D *of* (9) *satisfy the following* $\varepsilon-$*uniform error estimate*

$$\sup_{0<\varepsilon\le 1} \| u - U_D \|_{\Omega^N_{\tau-d}} \le C N^{-1} \ln N,$$

where C *is a constant independent of* ε*.*

Proof: See [6].■

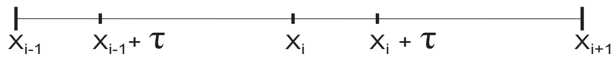

Figure 3: The subintervals of the local domain $\overline{\Omega}_i \cup \overline{\Omega}_{i+1}$

4 An ε-Uniform Numerical Method on an Equisditant Mesh without using the Exact Solution

Let us try to solve the problem (1) by an ε-uniform difference method on the uniform mesh Ω^N, as it is described in Section 2. The problem (2) is equivalent to the following one: Find the local function g_i, defined with respect to the mesh point x_i, on $\overline{\Omega}_i \cup \overline{\Omega}_{i+1}$ such that

$$\begin{cases} L^* g_i = -\varepsilon g_i''(x) - b g_i'(x) + c g_i(x) = \delta_{x_i}(x), \quad \forall x \in \Omega_i \cup \Omega_{i+1}, \\ g_i(x_{i-1}) = 0, \quad g_i(x_{i+1}) = 0, \end{cases} \tag{10}$$

where $\Omega_i = (x_{i-1}, x_i)$. The equation (10) should be read in the sense of distributions. Multiplying the equation $L_1 u = f$ with g_i, integrating the resulting expression from x_{i-1} to x_{i+1} and using the integration by parts and the continuity of u, respectively, we get the following identity

$$-\varepsilon g_i'(x_{i-1}) U_{i-1} + U_i + \varepsilon g_i'(x_{i+1}) U_{i+1} = f_i \int_{x_{i-1}}^{x_{i+1}} g_i \, dx. \tag{11}$$

However the evaluation of $g_i'(x_{i-1})$, $g_i'(x_{i+1})$ and $\int_{x_{i-1}}^{x_{i+1}} g_i dx$ requires the exact solution of (10) which may be difficult as much as the original problem (1). Therefore, we approximate the local Green's function g_i by a fitted mesh method as it is described in Section 3.1 and then use the resulting approximations in place of g_i's in (11).

In that context, we reformulate the method in section (3.1) on the union of fixed subintervals Ω_i and Ω_{i+1}: Divide the local domain $\overline{\Omega}_i \cup \overline{\Omega}_{i+1}$ into the four subintervals $[x_{i-1}, x_{i-1} + \tau]$, $[x_{i-1} + \tau, x_i]$, $[x_i, x_i + \tau]$ and $[x_i + \tau, x_{i+1}]$ (Figure 3) each has $M/4$ mesh elements, where $\tau = \min\left\{\frac{h}{2}, \frac{\varepsilon}{b_i} \ln M\right\}$. The corresponding mesh parameters becomes $h_1^* = \frac{4\tau}{M}$ and $h_2^* = \frac{4}{M}(h - \tau)$. Thus, the Shiskin's the fitted mesh $\Omega_{i,\tau}^{M/2} \cup \Omega_{i+1,\tau}^{M/2} = \{x_j^*\}_0^M$ is defined by $x_0^* = x_{i-1}$ and

$$x_j^* - x_{j-1}^* = \begin{cases} h_1^* & 0 < j \le M/4 \quad \text{or} \quad M/2 < j \le 3M/4 \\ h_2^* & M/4 < j \le M/2 \text{ or } \ 3M/4 < j \le M. \end{cases} \tag{12}$$

The discrete problem for (10), using the upwind difference operator on the specified mesh (12), is given by

$$\begin{cases} \text{Find } G \in V(\Omega_{i,\tau}^{M/2} \cup \Omega_{i+1,\tau}^{M/2}) \text{ such that } \; G_0 = 0, \;\; G_M = 0 \text{ and} \\ \\ -\varepsilon D_*^+ D_*^- G_j - b_i\, D_*^+ G_j + c_i G_j = \Delta_{x_i,j}, \qquad j = 1, 2, .., M-1, \end{cases} \tag{13}$$

where we mean G_j^i by G_j with $G_j \approx g_i(x_j^*)$, and

$$D_*^+ v_j = \frac{v_{j+1} - v_j}{h_{j+1}^*}, \quad D_*^- v_j = \frac{v_j - v_{j-1}}{h_j^*}, \quad \text{and} \quad \Delta_{x_i,j} = \begin{cases} \dfrac{1}{h_{j+1}^*}, & x_i \in [x_j^*, x_{j+1}^*) \\ 0, & \text{otherwise.} \end{cases}$$

Assuming b_i and c_i are piecewise constants in $\Omega_i \cup \Omega_{i+1}$, the equation (13) becomes a constant coefficient difference equation whose exact solution is possible. To solve (13), we combine terms with the same indices together and obtain a three-point difference scheme:

$$(-\lambda_j^*) G_{j+1} + \left(\frac{h_{j+1}^*}{h_j^*} + \lambda_j^* + \frac{c_i\; h_{j+1}^*\; h_j^*}{\varepsilon} \right) G_j + \left(-\frac{h_{j+1}^*}{h_j^*} \right) G_{j-1} = \Delta_{x_i,j}, \tag{14}$$

where $j = 1, 2, ..., M-1$ and λ_j^* is defined by

$$\lambda_j^* = \begin{cases} \lambda_1 & 1 \le j \le M/4 \quad\;\; \text{or } \; M/2 < j \le 3M/4, \\ \lambda_2 & M/4 < j \le M/2 \; \text{ or } \; 3M/4 < j \le M-1, \end{cases} \tag{15}$$

where $\lambda_1 = 1 + \frac{b_i h_1^*}{\varepsilon}$, $\lambda_2 = 1 + \frac{b_i h_2^*}{\varepsilon}$. At the transition points $x_{i-1} + \tau$, x_i, $x_i + \tau$ and at the interior points of the subregions, the difference equation (14) can explicitly be written, as follows:

$$\begin{aligned} (-\lambda_1) G_{M/4+1} + \left(\frac{h_2^*}{h_1^*} + \lambda_1 + \frac{c_i\; h_1^*\; h_2^*}{\varepsilon} \right) G_{M/4} + \left(-\frac{h_2^*}{h_1^*} \right) G_{M/4-1} &= 0 \quad if \;\; j = M/4 \\ (-\lambda_2) G_{M/2+1} + \left(\frac{h_1^*}{h_2^*} + \lambda_2 + \frac{c_i\; h_1^*\; h_2^*}{\varepsilon} \right) G_{M/2} + \left(-\frac{h_1^*}{h_2^*} \right) G_{M/2-1} &= \frac{h_2^*}{\varepsilon} \quad if \;\; j = M/2 \\ (-\lambda_1) G_{3M/4+1} + \left(\frac{h_2^*}{h_1^*} + \lambda_1 + \frac{c_i\; h_1^*\; h_2^*}{\varepsilon} \right) G_{3M/4} + \left(-\frac{h_2^*}{h_1^*} \right) G_{3M/4-1} &= 0 \quad if \;\; j = 3M/4 \\ (-\lambda_j^*) G_{j+1} + \left(1 + \lambda_j^* + Z_j^* \right) G_j + (-1) G_{j-1} = 0 \text{ otherwise,} & \end{aligned} \tag{16}$$

where Z_j^* is defined by

$$Z_j^* = \begin{cases} Z_1 & 1 \le j \le M/4 \quad\;\; \text{or } \; M/2 < j \le 3M/4, \\ Z_2 & M/4 < j \le M/2 \; \text{ or } \; 3M/4 < j \le M-1, \end{cases} \tag{17}$$

where $Z_1 = \frac{c\ (h_1^*)^2}{\varepsilon}$, $Z_2 = \frac{c\ (h_2^*)^2}{\varepsilon}$. Let the roots of the characteristic polynomial of the last difference equation be r_1 and r_2 outside the layer region and, r_3

and r_4 inside the layer region. The roots of difference equations are explicitly given as follows:

$$\begin{aligned} r_{1,2} &= \frac{1+Z_1+\lambda_1 \pm \sqrt{(1+Z_1+\lambda_1)^2-4\lambda_1}}{2\lambda_1}, \\ r_{3,4} &= \frac{1+Z_2+\lambda_2 \pm \sqrt{(1+Z_2+\lambda_2)^2-4\lambda_2}}{2\lambda_2}. \end{aligned}$$

Let us state the form of the solution of the difference equation (13) in terms of the roots of the characteristic polynomial;

$$G_j^i = \begin{cases} a_1 r_1^j + a_2 r_2^j & \text{if } 0 \le j \le M/4 \\ a_3 r_3^j + a_4 r_4^j & \text{if } M/4 \le j \le M/2 \\ a_5 r_1^j + a_6 r_2^j & \text{if } M/2 \le j \le 3M/4 \\ a_7 r_3^j + a_8 r_4^j & \text{if } 3M/4 \le j \le M. \end{cases} \tag{18}$$

The coefficients a_i, $i = 1, .., 8$ are to be determined and we need eight equations to solve the resulting system. The boundary conditions $G_0 = G_M = 0$ gives us two equations and three equations comes from the difference equations (16) written at the transition points $x^*_{M/4}$, $x^*_{M/2}$ and $x^*_{3M/4}$, respectively. Finally, the other three equations are obtained by imposing the continuity of the difference solution at transition points;

$$\begin{aligned} a_1 r_1^{M/4} + a_2 r_2^{M/4} &= a_3 r_3^{M/4} + a_4 r_4^{M/4} \\ a_3 r_3^{M/2} + a_4 r_4^{M/2} &= a_5 r_1^{M/2} + a_6 r_2^{M/2} \\ a_5 r_1^{3M/4} + a_6 r_2^{3M/4} &= a_7 r_3^{3M/4} + a_8 r_4^{3M/4}. \end{aligned}$$

We bring together these eight equations by rewriting them in the matrix form

$$\mathbf{A}\ \mathbf{x} = \mathbf{b} \tag{19}$$

where

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ \omega_1 & \omega_2 & -\omega_3 & -\omega_4 & 0 & 0 & 0 & 0 \\ k_1 & k_2 & -\omega_3 r_3 \lambda_1 & -\omega_4 r_4 \lambda_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \omega_3 & \omega_4 & -\omega_1 & -\omega_2 & 0 & 0 \\ 0 & 0 & k_3 & k_4 & -\omega_1 r_1 \lambda_2 & -\omega_2 r_2 \lambda_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & \omega_1^3 & \omega_2^3 & -\omega_3^3 & -\omega_4^3 \\ 0 & 0 & 0 & 0 & k_5 & k_6 & -\omega_3^3 r_3 \lambda_1 & -\omega_4^3 r_4 \lambda_1 \\ 0 & 0 & 0 & 0 & 0 & 0 & \omega_3^4 & \omega_4^4 \end{bmatrix}$$

$$\mathbf{x} = \begin{bmatrix} a_1\ a_2\ a_3\ a_4\ a_5\ a_6\ a_7\ a_8 \end{bmatrix}^T, \qquad \mathbf{b} = \begin{bmatrix} 0\ 0\ 0\ 0\ \dfrac{h_2^*}{\varepsilon}\ 0\ 0\ 0 \end{bmatrix}^T,$$

with
$\omega_1 = r_1^{M/4},\ \omega_2 = r_2^{M/4},\ \omega_3 = r_3^{M/4},\ \omega_4 = r_4^{M/4},$

$$k_1 = \omega_1 \left(Y_1 - r_1^{-1}\ \frac{h_2^*}{h_1^*} \right), \quad k_2 = \omega_2 \left(Y_1 - r_2^{-1}\ \frac{h_2^*}{h_1^*} \right), \quad k_3 = \omega_3^2 \left(Y_2 - r_3^{-1}\ \frac{h_1^*}{h_2^*} \right),$$
$$k_4 = \omega_4^2 \left(Y_2 - r_4^{-1}\ \frac{h_1^*}{h_2^*} \right), \quad k_5 = \omega_1^3 \left(Y_1 - r_1^{-1}\ \frac{h_1^*}{h_2^*} \right), \quad k_6 = \omega_2^3 \left(Y_1 - r_2^{-1}\ \frac{h_1^*}{h_2^*} \right),$$
$$Y_1 = \frac{h_2^*}{h_1^*} + \lambda_1 + \sqrt{Z_1\ Z_2}, \quad Y_2 = \frac{h_1^*}{h_2^*} + \lambda_2 + \sqrt{Z_1\ Z_2}.$$

Solving the linear system (19) and substituting the coefficients $a_1, \ldots, a_8$ into (18), we get the solution of the difference equation (13) in an explicit manner:

$$G_j^i = \begin{cases} -\dfrac{A_1 h_1^*(r_1^j - r_2^j)}{A_{14}} & \text{if} \quad 0 \le j \le M/4 \\ \dfrac{(h_2^*)^2\ (A_9 r_3^j + A_{10} r_4^j)}{A_{14}} & \text{if} \quad M/4 \le j \le M/2 \\ \dfrac{(A_{12} r_1^j + A_{13} r_2^j)}{A_{14}} & \text{if} \quad M/2 \le j \le 3M/4 \\ \dfrac{(\varepsilon\ \omega_1\ \omega_2\,(r_1 - r_2)\,r_3 r_4\ \omega_3\ \omega_4^{-3}(h_2^*)^3) r_3^j}{A_{14}} & \\ -\dfrac{(\varepsilon\ \omega_1\ \omega_2\,(r_1 - r_2)\,r_3^{-1}\ r_4\ \omega_3 \omega_4^3 (h_2^*)^3) r_4^j}{A_{14}} & \text{if} \quad 3M/4 \le j \le M, \end{cases} \tag{20}$$

where
$A_1 = (\varepsilon + b\ h_1^*)\ (h_2^*)^2\ r_1\ r_2\ r_3(r_3 - r_4),$
$A_2 = -(r_2(r_4-2)+1) r_3 \omega_3 r_4 + (1-r_2)\omega_3 r_4 + r_2\ r_3^2\ \omega_4 r_4 - \omega_4 r_3 (1+r_2(2r_4-1)-r_4),$
$A_3 = (r_1\omega_2 - (r_2 - 1)\ \omega_1 - \omega_2)\ \ r_3\ r_4\,(r_3\omega_4 - (r_4 - 1)\ \omega_3 - \omega_4)\,,$
$A_4 = \omega_1 r_1 r_2((r_2 - 1) r_3^2 r_4\ \omega_4 - r_3\omega_4((r_2 - 2) r_4 + 1) + \omega_3\ r_4 - r_3\omega_3\,(r_2\,(r_4 - 1)$
$\quad - r_4 + 2)\ \ r_4) - \omega_1\ r_2\,(r_3(r_4 - 1)\omega_4 - r_3 r_4 \omega_3 + \omega_3 r_4) + r_1\omega_2\ A_2$
$\quad + r_1^2\ r_2\ r_3\ r_4\ \omega_2\,((r_4 - 1)\ \omega_3 - (r_3 - 1)\omega_4))\,,$
$A_5 = h_1^*\ h_2^*\ r_1\ r_2\ r_3\ r_4\,(-r_1\ \omega_2(\omega_3 - \omega_4) + \omega_2\,(r_3\ \omega_4 - (r_4 - 2)\ \omega_3 - 2\ \omega_4)$
$\quad + \omega_1\,((r_2 + r_4 - 2)\ \omega_3 - r_3\ \omega_4 + (2 - r_2)\omega_4)),$
$A_6 = b\,((\omega_2 - \omega_1)(r_3 - 1)(r_4 - 1)\,(r_3\ \omega_4 - \omega_3\ r_4)\,r_1 r_2 (h_1^*)^3 - A_3 r_1 r_2 (h_1^*)^2 h_2^*$
$\quad - A_3 r_1 r_2 h_1^* (h_2^*)^2 - (\omega_3 - \omega_4)(r_1\ \omega_2 - \omega_1\ r_2)(h_2^*)^3 (r_1 - 1)(r_2 - 1) r_3 r_4)\,,$
$A_7 = \varepsilon(A_6 + c\ h_1^*\ h_2\,(-(r_3(r_4 - 1)\omega_4\ \ - r_3 r_4\ \omega_3 + \omega_3\ r_4)\ (\omega_2 - \omega_1) r_1 r_2 (h_1^*)^2$
$\quad + (\omega_3 - \omega_4)(r_1\ (r_2 - 1)\ \omega_2 - r_1\ r_2\ \omega_1 + \omega_1\ r_2)(h_2^*)^2 r_3 r_4 + A_5)),$
$A_8 = \varepsilon^2\,((\omega_2 - \omega_1)(r_3 - 1)(r_4 - 1)\,(r_3\ \omega_4 - \omega_3\ r_4)\,(h_1^*)^2\ r_1 r_2 + A_4\ h_1^*\ h_2^*$
$\quad - ((\omega_3 - \omega_4)\ (r_1\ \omega_2 - \omega_1\ r_2)\ (h_2^*)^2 (r_1 - 1)(r_2 - 1)\ r_3\ r_4))\,,$
$A_9 = (\varepsilon(h_2^*(\ (r_1 - 1)\ r_2\ \omega_1 - (r_2 - 1)\ r_1\ \omega_2) - (-h_1^*\ r_1\ r_2\ (\omega_2 - \omega_1))\ (r_4 - 1))$
$\quad + h_1^*\ (-h_1^*\ r_1\ r_2\ (\omega_2 - \omega_1))\ (b + c\ h_2^* - b\ r_4))\,r_3\ r_4 \omega_3^{-1},$
$A_{10} = (\varepsilon(-h_2^*(\ (r_1 - 1)\ r_2\ \omega_1 - (r_2 - 1)\ r_1\ \omega_2) + (-h_1^*\ r_1\ r_2\ (\omega_2 - \omega_1))\ (r_3 - 1)),$
$\quad - h_1^*\ (-h_1^*\ r_1\ r_2\ (\omega_2 - \omega_1))\ (b + c\ h_2^* - b\ r_3)) r_3\ r_4 \omega_4^{-1})$
$A_{11} = (b\ r_1^2\ r_2\ r_3\ r_4\ \omega_2\,((r_4 - 1)\ \omega_3 - (r_3 - 1)\omega_4))/(\omega_2 r_1^2\ r_2\ r_3\ r_4),$

$$A_{12} = \big(h_2^*)^2(\omega_1^{-2}\omega_2 r_1 r_3 r_4)(h_1^*)^2 r_2((-c(\omega_3-\omega_4)h_2^*)+A_{11}$$
$$+\,\varepsilon(h_2^*\,(\omega_3-\omega_4))\,(r_2-1)+h_1^*\,((r_4-1)\omega_3-r_3\omega_4+\omega_4)),$$
$$A_{13} = \big(h_2^*)^2(\omega_1^{2}\omega_2^{-1} r_2 r_3 r_4)(-h_1^*)^2 r_1((c(\omega_3-\omega_4)h_2^*)+A_{11}$$
$$+\,\varepsilon(h_2^*\,(\omega_3-\omega_4))(r_1-1)-h_1^*\,((r_4-1)\omega_3-r_3\omega_4+\omega_4))\,,$$
$$A_{14} = -(\omega_2-\omega_1)\,c\,h_1^*+b\,((r_2-1)\;\omega_1-r_1\;\omega_2+\omega_2)\;((b\,((r_4-1)\;\omega_3-r_3\;\omega_4+\omega_4)$$
$$-\;(\omega_2-\omega_1)\;c\;h_2^*)\;\;r_1\;r_2\;r_3\;r_4)\;(h_1^*\;h_2^*)^2+A_7+A_8.$$

In the simple case where the mesh is uniform, the finite difference solution (20) reduces to the following appropriate form:

$$G_j^i \;=\; \begin{cases} \dfrac{M\xi_3^{M/2}\left(r_1^j-r_2^j\right)}{\xi_4} & \text{if} \quad 0\le j\le M/4 \\ \dfrac{M\xi_3^{M/2}\left(r_3^j-r_4^j\right)}{\xi_4} & \text{if} \quad M/4\le j\le M/2 \\ \dfrac{-\left(\xi_3\;\xi_2\right)^{M/2}r_1^j+M\left(\xi_3^{-1}\;\xi_2\right)^{-M/2}r_2^j}{\xi_4} & \text{if} \quad M/2\le j\le 3M/4 \\ \dfrac{-\left(\xi_3\;\xi_2\right)^{M/2}r_3^j+M\left(\xi_3^{-1}\;\xi_2\right)^{-M/2}r_4^j}{\xi_4} & \text{if} \quad 3M/4\le j\le M, \end{cases} \tag{21}$$

where

$\xi_1 = h\sqrt{4c_i^2h^2+4b_ic_iMh+M^2\left(b_i^2+4c_i\varepsilon\right)}$, $\quad \xi_2=(1-h)/(1+h)$,
$\xi_3 = M(2b_ih+\varepsilon M)$, $\quad \xi_4=\Big(\left(\xi_1(1/h-1)\right)^{M/2}+\left(\xi_1(1/h+1)\right)^{M/2}\Big)\,(\xi_1/h)$.

Now we replace $g_i'(x_{i-1})$ and $g_i'(x_{i+1})$ in (11) by using G_j in their one-sided approximations; that is

$$g_i'(x_{i-1})\approx D^+G_0=\frac{G_1-G_0}{h_1^*}, \qquad g_i'(x_{i+1})\approx D^-G_M=\frac{G_M-G_{M-1}}{h_2^*},$$

which yields the ultimate numerical method that:

$$-\varepsilon D^+G_0\,\tilde{U}_{i-1}+\tilde{U}_i+\varepsilon D^-G_M\,\tilde{U}_{i+1}=f_i\;\int_{x_{i-1}}^{x_{i+1}}G^i\,dx. \tag{22}$$

The method (22) is remarkable in the sense that it requires no exact solution at all. In the implementation stage, we may use the approximations G_j^is, directly from solution of the difference equation (13). Thus, we do not even need to find the explicit expressions for G in (20) from the implementation point of view, as we just need them to prove that the method (22) is ε-uniform convergent, in the next section.

5 Convergence Properties

In order to investigate the convergence properties of the numerical method (22), we shall recall some well-known results that is needed to prove the method under consideration converges uniformly in ε. Let us first rewrite the exact scheme (6), whose derivation uses the exact solution of local Green's problem (2), for the problem (1) in the upwind form:

$$\begin{cases} \text{Find } U \in V(\Omega^N) \text{ such that } \quad U_0 = u_0,\ U_N = u_1, \text{ and } 1 \le i \le N-1, \\ -\varepsilon\, B_D(\rho_i,\gamma_i)\, D^+D^-U_i + b_i\; B_C(\rho_i,\gamma_i)\, D^-U_i + c_i\; B_R(\rho_i,\gamma_i)\, U_i = f_i, \end{cases} \tag{23}$$

where

$$\begin{aligned} B_D(\rho_i,\gamma_i) &= \frac{h^2\ c_i}{\varepsilon}\ \frac{e^{\gamma_i}}{e^{2\gamma_i+\rho_i} + e^{\rho_i} - e^{\gamma_i} - e^{\gamma_i+2\rho_i}}, \\ B_C(\rho_i,\gamma_i) &= \frac{h\ c_i}{b_i}\ \frac{e^{\gamma_i}\ (e^{2\ \rho_i} - 1)}{e^{2\gamma_i+\rho_i} + e^{\rho_i} - e^{\gamma_i} - e^{\gamma_i+2\rho_i}}, \\ B_R(\rho_i,\gamma_i) &= 1. \end{aligned}$$

On the other hand, consider a difference scheme of the form

$$\begin{cases} \text{Find } \hat{U} \in V(\Omega^N) \text{ such that } \quad \hat{U}_0 = u_0, \quad \hat{U}_N = u_1, \quad \text{and} \\ -\varepsilon\, \hat{\sigma}_i\ D^+D^-\hat{U}_i + \hat{\eta}_i\, b_i\, D^-\hat{U}_i + \hat{\theta}_i\, c_i\, \hat{U}_i = f_i, \qquad 1 \le i \le N-1, \end{cases} \tag{24}$$

where $\hat{\sigma}_i > 0$, $\hat{\eta}_i \gg 0$ and $\hat{\theta}_i > 0$. Farrell derived sufficient conditions for uniform convergence of schemes written of the form (24) in [2] and showed that the schemes of type (24) whose coefficients are close to the coefficients of the method (23) are also uniformly convergent. In that context, let us rewrite the numerical method (22), doing some algebric manipulations, in the form of (24):

$$-\varepsilon\, \sigma_i D^+D^-\tilde{U}_i + \eta_i\, b_i\, D^-\tilde{U}_i + \theta_i\, c_i\, \tilde{U}_i = f_i, \qquad 1 \le i \le N-1, \tag{25}$$

where

$$\sigma_i = \frac{h^2}{\varepsilon}\,\frac{T_2}{T_3}, \qquad \eta_i = \frac{h^2}{2\ \varepsilon\ \rho_i}\,\frac{T_1 - T_2}{T_3}, \qquad \theta_i = \frac{1}{c_i}\left(\frac{1 - T_1 - T_2}{T_3}\right), \tag{26}$$

and

$$\begin{aligned} T_1(\varepsilon, b_i, c_i, h, M) &= \varepsilon\, D^+G_0, \\ T_2(\varepsilon, b_i, c_i, h, M) &= -\varepsilon\, D^-G_M, \\ T_3(\varepsilon, b_i, c_i, h, M) &= \int_{x_{i-1}}^{x_{i+1}} G^i\, dx. \end{aligned} \tag{27}$$

Then, to prove the method (22) is uniformly convergent, it is enough to prove that the coefficients σ_i, η_i and θ_i in (24) can be made arbitrarily close to the coefficients of the numerical method (23). That is, for uniform convergence, we need to prove that

$$\lim_{M\to\infty} \sigma_i(\varepsilon, b_i, c_i, h, M) = B_D(\rho_i,\gamma_i), \tag{28}$$

$$\lim_{M\to\infty} \eta_i(\varepsilon, b_i, c_i, h, M) = B_C(\rho_i, \gamma_i), \tag{29}$$

$$\lim_{M\to\infty} \theta_i(\varepsilon, b_i, c_i, h, M) = B_R(\rho_i, \gamma_i). \tag{30}$$

Since G^i is a strictly positive function whose integral from x_{i-1} to x_{i+1} is also strictly positive, we can first evaluate $\lim_{M\to\infty} T_i$ for $i = 1, 2, 3$ in (27), respectively, and then combine them to find the limits (28)-(30). We present the following proofs for uniform cases. The non-uniform cases are similar but longer. So we omitted them.

Lemma 1 *Let $T_1(\varepsilon, b_i, c_i, h, M)$ be given as in (27), that is,*

$$T_1(\varepsilon, b_i, c_i, h, M) = \varepsilon\, D^+G_0. \tag{31}$$

If ρ_i and γ_i are fixed, then we have

$$\lim_{M\to\infty} T_1(\varepsilon, b_i, c_i, h, M) = -\frac{e^{\gamma_i+\rho_i}}{1+e^{2\gamma_i}}. \tag{32}$$

Proof: Consider the uniform case where $\tau = h/2$. The mesh parameters become $h_1^* = h_2^* = 2h/M$ and $\lambda_1 = \lambda_2 = 1 + \frac{2b_ih}{M\varepsilon}$. Using these, we rewrite T_1 by rearranging the terms and using the explicit solution of G^i in (20):

$$T_1 = \frac{G_1 - G_0}{h_1^*} = \frac{M\xi_3^{M/2}\,(r_1 - r_2)}{h_1^*\xi_4} = -\frac{\xi_1\; M^2(M(2b_ih + \varepsilon M))^{\frac{M-2}{2}}}{\xi_4\; h}.$$

Using the fact that $\lim_{M\to\infty}(1+\frac{x}{M})^M = e^x$ for any $x \in \Re$, a calculation leads to

$$\lim_{M\to\infty} \frac{G_1 - G_0}{h_1^*} = -\frac{e^{\frac{\left(b^2+4c\varepsilon+b\;\sqrt{b^2+4c\varepsilon}\right)h}{2\varepsilon\sqrt{b^2+4c\varepsilon}}}}{1+e^{\frac{h\sqrt{b^2+4c\varepsilon}}{\varepsilon}}} = -\frac{e^{\gamma_i+\rho_i}}{1+e^{2\gamma_i}}. \qquad \blacksquare$$

Lemma 2 *Let $T_2(\varepsilon, b_i, c_i, h, M)$ be given as in (27), that is,*

$$T_2(\varepsilon, b_i, c_i, h, M) = -\varepsilon\, D^-G_M. \tag{33}$$

If ρ_i and γ_i are fixed, then we have

$$\lim_{M\to\infty} T_2(\varepsilon, b_i, c_i, h, M) = -\frac{e^{\gamma_i-\rho_i}}{1+e^{2\gamma_i}}. \tag{34}$$

Proof: We use the same arguments as in the proof of Lemma 1. For the case where $\tau = h/2$, use again the difference solution G^i in (20) and $h_1^* = h_2^* = 2h/M$;

$$T_2 = \frac{G_M - G_{M-1}}{h_2^*} = -\frac{\left(-\left(\xi_3\;\xi_2\right)^{M/2} r_3^{M-1} + M\left(\xi_3^{-1}\;\xi_2\right)^{-M/2} r_4^{M-1}\right)}{h_2^*\xi_4} = \frac{\xi_1\;\varepsilon^{\frac{M}{2}-1}M^M}{\xi_4\;h},$$

which yields

$$\lim_{M\to\infty}\frac{G_M-G_{M-1}}{h_2^*}=-\frac{e^{\frac{\left(b^2+4c\varepsilon-b\ \sqrt{b^2+4c\varepsilon}\right)h}{2\varepsilon\sqrt{b^2+4c\varepsilon}}}}{1+e^{\frac{h\sqrt{b^2+4c\varepsilon}}{\varepsilon}}}=-\frac{e^{\gamma_i-\rho_i}}{1+e^{2\gamma_i}}. \qquad \blacksquare$$

Lemma 3 *Let $T_3(\varepsilon,b_i,c_i,h,M)$ be given as in (27), that is,*

$$T_3(\varepsilon,b_i,c_i,h,M)=\int_{x_{i-1}}^{x_{i+1}}G^i dx. \tag{35}$$

If ρ_i and γ_i are fixed, then we have

$$\lim_{M\to\infty}T_3(\varepsilon,b_i,c_i,h,M)=\frac{1}{c_i}\left(\frac{e^{2\gamma_i}-e^{\gamma_i+\rho_i}-e^{\gamma_i-\rho_i}+1}{1+e^{2\gamma_i}}\right). \tag{36}$$

Proof: Use the explicit solution of G^i in (20) and the composite trapezium quadrature rule to integrate (35):

$$\int_{x_{i-1}}^{x_{i+1}}G^i dx=\int_{x_{i-1}}^{x_{i-1}+\tau}G^i dx+\int_{x_{i-1}+\tau}^{x_i}G^i dx+\int_{x_i}^{x_i+\tau}G^i dx+\int_{x_i+\tau}^{x_{i+1}}G^i dx, \tag{37}$$

which considerably simplifies in the uniform case and we get

$$\begin{aligned}\lim_{M\to\infty}\int_{x_{i-1}}^{x_{i+1}}G^i dx &= \frac{e^{\frac{(\gamma_i-\rho_i)}{2}}\left(-b_i\ e^{\gamma_i}+b_i+\left(1+e^{\gamma_i}-2e^{\frac{(\gamma_i-\rho_i)}{2}}\right)\frac{2\varepsilon\gamma_i}{h}\right)}{2c_i\left(1+e^{2\gamma_i}\right)\frac{2\varepsilon\gamma_i}{h}}\\ &= \frac{1}{c_i}\left(\frac{e^{2\gamma_i}-e^{\gamma_i+\rho_i}-e^{\gamma_i-\rho_i}+1}{1+e^{2\gamma_i}}\right). \qquad \blacksquare\end{aligned}$$

Corollary 1 *If γ_i and ρ_i are fixed, then the coefficients σ_i, η_i and θ_i in (25) converges to the coefficients of the numerical method (23) . That is,*

$$\lim_{M\to\infty}\sigma_i(\varepsilon,b_i,c_i,h,M) = B_D(\rho_i,\gamma_i), \tag{38}$$

$$\lim_{M\to\infty}\eta_i(\varepsilon,b_i,c_i,h,M) = B_C(\rho_i,\gamma_i), \tag{39}$$

$$\lim_{M\to\infty}\theta_i(\varepsilon,b_i,c_i,h,M) = B_R(\rho_i,\gamma_i). \tag{40}$$

Proof: Recall the definition of σ_i from (26), and use Lemma 2 and Lemma 3, to get

$$\lim_{M\to\infty}\sigma_i=\lim_{M\to\infty}\frac{h^2}{\varepsilon}\frac{T_2}{T_3}=\frac{h^2}{\varepsilon}\frac{\lim\limits_{M\to\infty}T_2}{\lim\limits_{M\to\infty}T_3}=\frac{h^2}{\varepsilon}\frac{\dfrac{e^{\gamma_i-\rho_i}}{e^{2\gamma_i}+1}}{\dfrac{1}{c_i}\dfrac{e^{2\gamma_i}+1-e^{\gamma_i-\rho_i}-e^{\gamma_i+\rho_i}}{e^{2\gamma_i}+1}}$$

$$= \frac{h^2 c_i}{\varepsilon} \frac{e^{\gamma_i - \rho_i}}{e^{2\gamma_i} + 1 - e^{\gamma_i - \rho_i} - e^{\gamma_i + \rho_i}} = \frac{h^2\ c_i}{\varepsilon} \frac{e^{\gamma_i}}{e^{2\gamma_i + \rho_i} + e^{\rho_i} - e^{\gamma_i} - e^{\gamma_i + 2\rho_i}} = B_D(\rho_i, \gamma_i).$$

$$\lim_{M\to\infty} \eta_i = \lim_{M\to\infty} \frac{h}{b_i} \frac{(T_1 - T_2)}{T_3} = \frac{h}{b_i} \frac{\lim_{M\to\infty} (T_1 - T_2)}{\lim_{M\to\infty} T_3} = B_C(\rho_i, \gamma_i).$$

$$\lim_{M\to\infty} \theta_i = \lim_{M\to\infty} \frac{1}{c_i} \left(\frac{1 - T_1 - T_2}{T_3} \right) = \frac{1}{c_i} \frac{\lim_{M\to\infty} (1 - T_1 - T_2)}{\lim_{M\to\infty} T_3} = B_R(\rho_i, \gamma_i). \blacksquare$$

Theorem 3 *The solution of the difference equation (25) converges, in the discrete maximum norm, to the exact solution of the problem (1) uniformly in* ε.

Proof: See [2] .■

6 Conclusion

We considered an ε-uniform numerical method for a singularly-perturbed two-point boundary value problem. The method proposed is significant in the sense that, although it is uses an equidistant mesh, it requires no exact solution of the local differential equation which reflects the singular perturbation nature of the problem. We further proved the method proposed converges to the true solution uniformly in ε.

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