01-02b-II

Доклади на Българската академия на науките
Comptes rendus de l'Académie bulgare des Sciences

Tome 65, No 1, 2012

MATHEMATIQUES
Algèbre

COMPLETELY COTORSION MODULES

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(*Submitted by Academician S. Troyanski on August 2, 2011*)

Abstract

We show that any finitely generated projective cotorsion left module over a ring of left pure global dimension at most 1 is a direct sum of indecomposable direct summands. We deduce that such a ring is left cotorsion and semiperfect if and only if its left cotorsion envelope is finitely presented. Some extensions of this result are also discussed.

Key words: hereditary rings, cotorsion envelopes, flat convers, semiperfect rings, semilocal rings, pure-injective modules

2010 Mathematics Subject Classification: 16D50, 16D90, 16L30

1. Introduction. Let R be a left hereditary ring. It has been shown in [8] that R is semilocal (i.e., R/J is semisimple, where J denotes the Jacobson radical of R) whenever its left cotorsion envelope $C = C({}_R R)$ is countably generated. And it has been deduced that R is a semiperfect left cotorsion ring if and only if C is finitely generated. The key idea of this result is to prove, by means of Set Theoretic counting arguments, that the left cotorsion envelope of a left hereditary ring is a finite direct sum of indecomposable modules when it is countably generated. As it is pointed out in [8], it seems that this result might be true under a more general hypothesis. Namely, the authors ask in ([8], Question 10) whether the left cotorsion envelope of a ring R with left pure global dimension at most 1 is a direct sum of indecomposable direct summands provided it is countably (or finitely) generated.

The purpose of this note is to adapt the techniques in [4,13] to show that, if a finitely generated projective cotorsion module C fails to be a direct sum of

The author has been supported by TUBITAK Program 2219.

indecomposable direct summands, then there exists a (countably presented) pure homomorphic image of it which is not cotorsion. In particular, we deduce that any finitely generated projective cotorsion left module over a ring R with left pure global dimension at most 1 is a direct sum of indecomposables. Following the notation in [4], we will say that a module is completely cotorsion provided that any pure quotient (in particular, the module itself) is cotorsion. We next show that any cotorsion left module over a ring with left pure global dimension at most 1 is completely cotorsion. This allows us to obtain a partial positive answer to ([8], Question 10) in the particular case in which the left cotorsion envelope of the ring is finitely presented. Indeed, we characterize these rings. We show that a ring with left pure-global dimension at most 1 is left cotorsion and semiperfect if and only if its left cotorsion envelope is finitely presented.

In the last part of this note, we extend the above results for rings of smaller cardinality. We show that any finitely generated flat completely cotorsion module over a ring R of cardinality strictly smaller than $2^{2^{\aleph_0}}$ is a direct sum of indecomposable direct summands. As a consequence, we deduce that ([8], Question 10) is true for those rings. Moreover, this extension suggests that the following question may have a positive answer:

Question 1. *Is any finitely generated completely cotorsion flat left R-module a direct sum of indecomposable direct summands?*

We want to remark that the above extension does not need the assumption of the (Generalized) Continuum Hypothesis.

Throughout this paper, all rings will be unitary and associative. By a module M we will always mean a unitary left module over a ring R, unless otherwise stated. We will denote by R-Mod the category of left modules over a ring R. We refer to [1,9,15] for any undefined concept used along this paper.

2. Main results. We begin this section by recalling some well known facts about cotorsion modules. Let R be a unitary ring. A left R-module C is called *cotorsion* if $\mathrm{Ext}^1_R(F, C) = 0$ for any flat left R-module F. A homomorphism $u : M \to C$ from a module M to a cotorsion module C is called a cotorsion preenvelope of M if any other morphism from M to a cotorsion module factors through u. A cotorsion preenvelope $u : M \to C$ is called a cotorsion envelope if, moreover, it satisfies that any endomorphism f of C which satisfies $f \circ u = u$ is an isomorphism. The existence of cotorsion envelopes of modules has been proved in [2] (see also [9]). We will denote the cotorsion envelope of a module M by $C(M)$. As it is noted in [15], any cotorsion envelope $u : M \to C(M)$ is a monomorphism with flat cokernel. In particular, any module is a pure submodule of its cotorsion envelope, and the cotorsion envelope of a flat module is always flat. A module M is called *indecomposable* if its only direct summands are 0 and M. A module M is said to be a *direct sum of indecomposable submodules* if there exists a family $\{M_i \mid i \in I\}$ of indecomposable submodules of M such that $M = \bigoplus_{i \in I} M_i$. An independent family of submodules $\{M_i \mid i \in I\}$ of a module M

is called a *local direct summand* of M if $\bigoplus_{i\in F} M_i$ is a direct summand of M for any finite subset $F \subseteq I$. If the whole direct sum $\bigoplus_{i\in I} M_i$ is a direct summand of M, we will say that the local direct summand is a direct summand. Recall that a left R-module P is called a *progenerator* in R-Mod if it is a finitely generated projective generator in the category (see e.g. ([1], Chapter 6)). A module M is called *countably presented* if there exists an epimorphism $p : R^{(\mathbb{N})} \to M$ with countably generated kernel.

We can now prove our main result.

Theorem 2. *Let C be a finitely generated projective cotorsion module. If C is not a direct sum of indecomposable direct summands, then there exists a (countably presented) pure quotient of C which fails to be cotorsion.*

Proof. Let $p : R^n \to C$ be an epimorphism from a free R-module of finite rank onto C. As R^n is a progenerator, it induces a category equivalence between R-Mod and $\mathrm{End}_R(R^n)$-Mod (see ([1], Theorem 22.2, Corollary 22.4)). Under this category equivalence, C becomes a cyclic $\mathrm{End}_R(R^n)$-projective module. So we may assume without loss of generality that ${}_RC$ is already a cyclic module.

Let us assume that ${}_RC$ is not a direct sum of indecomposable direct summands and we will construct a pure submodule L such that C/L is not cotorsion. By ([10], Theorem 2.17), there exists a local direct summand $\bigoplus_I C_i$ of C which is not a direct summand. Note that, as C is finitely generated, we may choose I to be countable (since no countable direct subsum of $\bigoplus_I C_i$ can be a direct summand of C). Moreover, as $\bigoplus_I C_i$ is a pure submodule of C, the factor $C/\bigoplus_I C_i$ is also flat. This means that C is a cotorsion preenvelope of $\bigoplus_I C_i$ (since C is cotorsion and $C/\bigoplus_I C_i$ is flat). Therefore, C contains a cotorsion envelope of $\bigoplus_I C_i$, say $C(\bigoplus_I C_i)$, which is a direct summand of C. Let us write $C = C(\bigoplus_I C_i) \oplus C'$ for some direct summand C' of C. Adding C' to the local direct summand $\bigoplus_I C_i$ if necessary, we may also assume that C is, indeed, the cotorsion envelope of $\bigoplus_I C_i$.

Let us write $I = \bigcup_{n\in\mathbb{N}} I_n$ as a disjoint union of countably many infinite subsets of I. By Zorn's Lemma, there exists a maximal family $\mathcal{K}$ of infinite subsets of I in respect to the following properties:

(i) $\{I_n\}_{n\in\mathbb{N}} \subseteq \mathcal{K}$,

(ii) $|K \cap K'| < \infty$ if $K, K' \in \mathcal{K}$ with $K \neq K'$.

Let us call $S = \mathrm{End_R(C)}$. And let us fix, for any $i \in I$, idempotents $e_i \in S$ such that

$$C_i = Ce_i \text{ and } \bigoplus_{j\neq i} C_i \subseteq C(1-e_i).$$

Let us note that the idempotents $\{e_i\}_{i\in I}$ are pairwise orthogonal by construction. We can also fix, for any $K \in \mathcal{K}$, cotorsion envelopes

$$C_K = C(\bigoplus_{i\in K} C_i) \text{ and } C'_K = C(\bigoplus_{i\in I\setminus K} C_i)$$

such that $C = C_K \oplus C'_K$. And again there will exist idempotents $e_K \in S$ such that $C_K = Ce_K$ and $C'_K = C(1-e_K)$. Let us also note that, as $C_i \subseteq C_K$ for any $i \in K$ and $C_i \subseteq C'_K$ when $i \notin K$, we get that $e_i e_K = e_i$ if $i \in K$ whereas $e_i e_K = 0$ if $i \notin K$.

We claim that

$$C_K \cap (C_{K_1} + \cdots + C_{K_n}) = \bigoplus_{i \in K \cap (K_1 \cup \cdots \cup K_n)} C_i$$

if $K, K_1, \ldots, K_n \in \mathcal{K}$ with $K \neq K_1, \ldots, K_n$. Reasoning by induction, it is enough to prove it for $n = 1$. So let $K, K' \in \mathcal{K}$ with $K \neq K'$. Clearly, $\bigoplus_{i \in K \cap K'} C_i \subseteq C \cap C'$. As $K \cap K'$ is a finite set, we may let $e = \sum_{i \in K \cap K'} e_i$. Then $e = e^2$ is an idempotent of S, because the idempotents $\{e_i \,|i \in I\}$ are pairwise orthogonal. Moreover, as $e_i e_K = e_i$ if $i \in K$ and $e_i e_K = 0$ if $i \notin K$, we get that

$$e'_K = e_K - e \text{ and } e'_{K'} = e_{K'} - e$$

are also idempotents and

$$Ce_K = Ce'_K \oplus Ce \text{ and } Ce_{K'} = Ce'_{K'} \oplus Ce.$$

We claim that $Ce'_K \cap Ce'_{K'} = 0$. Assume on the contrary that $Ce'_K \cap Ce'_{K'} \neq 0$. Then the homomorphism $f : Ce'_K \to C$ defined by the rule $f(x) = x(1 - e'_{K'})$ is not injective since $Ce'_K \cap Ce'_{K'} \subseteq \mathrm{Ker}(p)$. However, it is clear that $f|_{\bigoplus_{i \in K \setminus K'} C_i}$ is a monomorphism with flat cokernel. And, as Ce'_K is the cotorsion envelope of $\bigoplus_{i \in K \setminus K'} C_i$, this means that f must be a monomorphism, because Ce'_K is the cotorsion envelope of $\bigoplus_{i \in K \setminus K'} C_i$, and therefore the embedding of $\bigoplus_{i \in K \setminus K'} C_i$ into Ce'_K is strongly pure-essential in the sense of [7]. A contradiction which shows that $Ce'_K \cap Ce'_{K'} = 0$ and therefore,

$$C_K \cap C_{K'} = \bigoplus_{i \in K \cap K'} C_i.$$

Arguing now by induction, we get that

$$C_K \cap (C_{K_1} + \cdots + C_{K_n}) = \bigoplus_{i \in K \cap (K_1 \cup \cdots \cup K_n)} C_i$$

whenever $K, K_1, \ldots, K_n \in \mathcal{K}$ with $K \neq K_1, \ldots, K_n$ and hence, $\sum_{K_1, \ldots, K_n} C_K$ is a direct summand of C for any finite family of elements $K_1, \ldots, K_n \in \mathcal{K}$ (since $K \cap (K_1 \cup \cdots \cup K_n)$ is finite). Therefore, $\sum_{K \in \mathcal{K}} C_K$ is a directed union of direct summands of C and thus, $C / \sum_{K \in \mathcal{K}} C_K$ is flat. Let us call $L = \bigoplus_I C_i$. We have shown that

$$\frac{\sum_{K \in \mathcal{K}} C_K}{L} \cong \bigoplus_{K \in \mathcal{K}} \frac{C_K + L}{L}.$$

Note that each $(C_K + L)/L \cong C_K/(C_K \cap L) \neq 0$ because $C_K = C(\bigoplus_{i\in K} C_i)$ is finitely generated but $C_K \cap L = \bigoplus_K C_i$ cannot be finitely generated (since K is infinite).

Let us define the homomorphism

$$\varphi : \frac{\sum_{K\in\mathcal{K}} C_K}{L} \to C/L$$

by the rule

$$\varphi|_{C_K+L/L} = \begin{cases} 0 & \text{if } K \neq I_n \text{ for any } n \in \mathbb{N}; \\ 1_{C_K+L/L} & \text{if } K = I_n \text{ for some } n \in \mathbb{N}. \end{cases}$$

We claim that C/L cannot be cotorsion. Assume on the contrary that C/L is cotorsion. As $C/\sum_{K\in\mathcal{K}} C_K$ is flat, there would exist a homomorphism $\psi : C \to C/L$ such that $\psi|_{\sum_{K\in\mathcal{K}} C_K} = \varphi \circ \pi$, where $\pi : \sum_{K\in\mathcal{K}} C_K \to \sum_{K\in\mathcal{K}} C_K/L$ is the canonical projection. Call $q : C \to C/L$ the canonical projection. Note that L is a pure submodule of C. So, as we are assuming that C is finitely presented, there exists a $\delta : C \to C$ such that

$$\begin{tikzcd} & C \arrow[d, "\psi"] \arrow[dl, dashed, "\delta"'] \\ C \arrow[r, "q"] & C/L \end{tikzcd}$$

$q \circ \delta = \psi$. On the other hand, ${}_RC$ is a cyclic module, say $C = Rx$ for some $x \in C$. Let us call $y = \delta(x) \in C$. Then for any $K \in \mathcal{K}$ there exists an $r_K x = xe_K \in Rx$ and we obtain that

$$r_K y + L = q \circ \delta(r_K x) = \psi(r_K x) = \psi(xe_K) = \varphi\pi(xe_K) = \varphi(e_K + L).$$

So $r_K y + L = L$ if $K \neq I_n$ for any $n \in \mathbb{N}$ whereas $r_K y + L = r_K x + L$ if $K = I_n$ for some $n \in \mathbb{N}$. Let us fix a $K = I_n$ for some $n \in \mathbb{N}$. Then, there exists an $l_K \in L$ such that $r_K y = xe_K + l_K$. As $L = \bigoplus_{i\in I} Ce_i$, there exists a finite subset $\{i_1, \ldots, i_t\} \subseteq I$ such that $l_K = r_1 xe_{i_1} + \cdots + r_t xe_{i_t}$. Choosing an $i \in K\backslash\{i_1, \ldots, i_t\}$ and an element $r_i \in R$ such that $xe_i = r_i x$, we get that

$$\begin{aligned} r_i ye_i &= \delta(r_i x)e_i = \delta(xe_i)e_i = \delta(xe_i e_K) = \delta(r_i r_K x)e_i \\ &= r_i r_K ye_i = r_i xe_K e_i = xe_i e_K e_i = xe_i. \end{aligned}$$

Therefore, we obtain that $r_i y r_i = xe_i \neq 0$, for every $i \in K$ but a finite number. Reasoning analogously, we can show that, when $K \neq I_n$ for all n, $r_i y r_i = 0$ for every $i \in K$ but a finite number. Let us fix, for any $n \in \mathbb{N}$, an element $i_n \in I_n$ such that $r_{i_n} ye_{i_n} = xe_{i_n}$ and let us call $B = \{i_n \mid n \in \mathbb{N}\}$. Clearly, B is an infinite subset of I. So, by the maximality of $\mathcal{K}$, there exists a subset $K \in \mathcal{K}$ such that $K \cap B$ is infinite. Note that $K \neq I_n$ for any n, since $B \cap I_n = \{i_n\}$ is finite. But then, $r_i y e_i = 0$ for all $i \in B \cap K$ but a finite number. A contradiction since $B \cap K$ is infinite and $r_i x e_i \neq 0$ for every $i \in K$ by construction. Therefore, C/L cannot be cotorsion. $\square$

Recall that a ring R is said to have left pure-global dimension at most 1 if any pure submodule of a pure-projective left R-module is again pure-projective or, equivalently, if every pure quotient of a pure-injective left R-module is again pure-injective. It is well known that any countable ring has both left and right pure-global dimension at most 1.

Proposition 3. *Let C be a flat cotorsion module over a ring R with left pure-global dimension at most* 1. *Then any pure homomorphic image of C is also cotorsion.*

Proof. Let C' be a pure epimorphic image of C with the corresponding epimorphism $\pi : C \to C'$ and let F be a flat module. We must show that $\mathrm{Ext}^1(\mathrm{F}, \mathrm{C}') = 0$. Let $p : R^{(I)} \to F$ be an epimorphism from a free module onto F, and let $u : K \to R^{(I)}$ be the kernel of p. The result reduces to prove that any morphism $f : K \to C'$ extends to $R^{(I)}$. As R has left pure-global dimension at most 1, K is a pure projective module. So there exists a $g : K \to C$ such that $\pi \circ g = f$. And, as C is cotorsion, there is an $h : R^{(I)} \to C$ such that $h \circ u = g$. Therefore, $\pi \circ h$ is the desired extension of f. $\square$

Corollary 4. *Let R be a ring of left pure global dimension at most* 1. *If C is a finitely generated projective cotorsion module, then C is a finite direct sum of indecomposable direct summands.*

We can now prove our promised partial positive answer to ([8], Question 10).

Corollary 5. *Let R be a ring of left pure global dimension at most* 1. *If $C({}_RR)$ is finitely presented, then R is semiperfect and left cotorsion.*

Proof. If $C({}_RR)$ is finitely presented, then so is $C({}_RR)/R$. But as $C({}_RR)/R$ is flat, this means that it is projective. Therefore, R_R is a direct summand of $C({}_RR)$ and therefore, cotorsion. The result now follows from the above corollary. $\square$

Our last proposition shows that the above corollary can be extended for rings of cardinality strictly smaller than $2^{2^{\aleph_0}}$.

Proposition 6. *Let ${}_RC$ be a finitely generated flat completely cotorsion module. If $|{}_RC| \lneq 2^{2^{\aleph_0}}$, then C is a direct sum of indecomposable direct summands.*

Proof. Assume on the contrary that C is not a direct sum of indecomposable direct summands. Following the notation used in the proof of Theorem 2, we can construct a pure submodule L of C and a local direct summand $\bigoplus_{K \in \mathcal{K}} (C_K + L)/L$ of C/L consisting of $|\mathcal{K}|$ nonzero direct summands. But $|\mathcal{K}| = 2^{\aleph_0}$ by ([3], Lemma 3.2) and therefore, $\bigoplus_{K \in \mathcal{K}} (C_K + L)/L$ consists of $2^{\aleph_0}$ non-zero direct summands of C/L. As we are assuming that C/L is cotorsion, the cotorsion envelope of $\bigoplus_{K \in \mathcal{K}'} (C_K + L)/L$ is a direct summand of C/L for any subset $\mathcal{K}' \subseteq \mathcal{K}$ and this means that C/L has at least $2^{2^{\aleph_0}}$ different direct summands. Therefore, the set

C/L has cardinality at least $2^{2^{\aleph_0}}$. A contradiction which shows that C is a direct sum of indecomposable direct summands. $\square$

As a consequence, we get the following positive answer to ([8], Question 10) for rings of cardinality strictly smaller than $2^{2^{\aleph_0}}$:

Corollary 7. *Let R be a ring with left pure-global dimension at most 1 and cardinality strictly smaller than $2^{2^{\aleph_0}}$. If $C({}_RR)$ is finitely generated, then it is a finite direct sum of indecomposable direct summands.*

We would like to close this paper by noting that Proposition 6 also shows that the next question has a positive answer for rings of cardinality strictly smaller than $2^{2^{\aleph_0}}$.

Question 8. *Is any finitely generated flat completely cotorsion module a direct sum of indecomposable direct summands? In particular, is any finitely generated flat cotorsion left module over a ring of left pure-global dimension at most 1 a direct sum of indecomposables?*

Acknowledgement. This paper has been written during the visit of the author in the University of Murcia, Spain. The author would like to thank Pedro A. Guil Asensio for his assistance and Tübitak for their financial support.

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