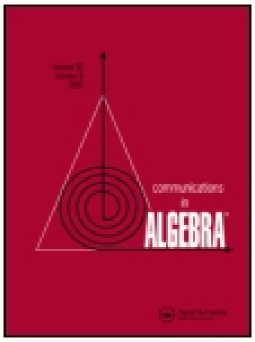

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Cofinitely Weak Supplemented Modules

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ABSTRACT

We prove that a module M is *cofinitely weak supplemented* or briefly *cws* (i.e., every submodule N of M with M/N finitely generated, has a weak supplement) if and only if every maximal submodule has a weak supplement. If M is a *cws*-module then every M-generated module is a *cws*-module. Every module is *cws* if and only if the ring is semilocal. We study also modules, whose finitely generated submodules have weak supplements.

Key Words: Cofinite submodule; Cofinitely weak supplemented module; Finitely weak supplemented module.

AMS Classification Numbers: 16D10; 16L30; 16D99.

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1. INTRODUCTION

Throughout the paper R will be an *associative ring with identity* and we will consider only *left unital R-modules.* All definitions not given here can be found in Anderson and Fuller (1992) or in Wisbauer (1991). A module M is *supplemented* (see Wisbauer, 1991), if every submodule N of M has a *supplement*, i.e., a submodule K minimal with respect to $N+K=M$. K is a supplement of N in M if and only if $N+K=M$ and $N\cap K\ll K$ (see Wisbauer, 1991). If $N+K=M$ and $N\cap K\ll M$, then K is called a *weak supplement* of N (see Lomp, 1999; Zöschinger, 1978), and clearly in this situation N is a weak supplement of K, too. M is a *weakly supplemented* module if every submodule of M has a weak supplement. A submodule N of a module M is said to be *cofinite* if M/N is finitely generated. M is called a *cofinitely supplemented* module if every cofinite submodule of M has a supplement (see Alizade et al., 2001). We call M a *cofinitely weak supplemented* module (or briefly a *cws*-module) if every cofinite submodule has (is) a weak supplement. Clearly cofinitely supplemented modules and weakly supplemented modules are cofinitely weak supplemented and a finitely generated module is weakly supplemented if and only if it is a *cws*-module.

In Sec. 2, we show that M is a *cws*-module if and only if every maximal submodule of M has a weak supplement. For the modules M with $\operatorname{Rad} M\ll M$, we give some conditions equivalent to being a *cws*-module in Theorem 2.21. It is proved by Lomp (1999) that the ring R is semilocal if and only if $_RR$ is a weakly supplemented R-module. We show that R is semilocal if and only if every R-module is *cws*.

In Sec. 3, we study *finitely weak supplemented* modules (briefly, *fws*-modules), that is the modules whose finitely generated submodules have weak supplements. Under proper conditions the sum of two *fws*-modules is a *fws*-module. We show that M is a *fws*-module if and only if every cyclic submodule of M is a weak supplement.

2. COFINITELY WEAK SUPPLEMENTED MODULES

The following lemma shows that without loss of generality, weak supplements of cofinite submodules can be regarded as finitely generated.

Lemma 2.1. *Let M be a module and U be a cofinite* (*maximal*) *submodule of M. If V is a weak supplement of U in M, then U has a finitely generated* (*cyclic*) *weak supplement in M contained in V.*

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Proof. If U is cofinite, then $V/V\cap U$ is finitely generated since $V/(V\cap U)\cong M/U$. Let $V/(V\cap U)$ be generated by elements

$$x_1 + V\cap U,\ x_2 + V\cap U, \ldots, x_n + V\cap U$$

Then for the finitely generated submodule $W = Rx_1 + Rx_2 + \cdots + Rx_n$ of V we have $W + U = W + V\cap U + U = V + U = M$ and $W\cap U \leq V\cap U \ll M$. Therefore, W is a finitely generated weak supplement of U contained in V.

If U is maximal, then $V/(V\cap U)$ is a cyclic module generated by some element $x + (V\cap U)$ and $W = Rx$ is a weak supplement of U. □

A slight modification of 41.1(2) by Wisbauer (1991) shows that supplements of cofinite submodules are finitely generated. The following example shows that a weak supplement of a cofinite submodule need not be finitely generated. Firstly we need the following lemma.

Lemma 2.2. *The* $\mathbf{Z}$*-submodule* $M = \sum_{q\text{prime}} \mathbf{Z}\cdot\frac{1}{q}$*, consisting of all rational numbers with square-free denominators, is a small submodule of the* $\mathbf{Z}$*-module* $\mathbf{Q}$ *of all rational numbers:* ${}_{\mathbf{Z}}M \ll {}_{\mathbf{Z}}\mathbf{Q}$.

Proof. Suppose $M + A = \mathbf{Q}$ for some $A \leq \mathbf{Q}$. Clearly $B = \mathbf{Z}\cap M\cap A = \mathbf{Z}\cap A \neq 0$, hence $B = n\mathbf{Z}$ for some $0 \neq n \in \mathbf{Z}$. Then

$$(M/B)/[(M\cap A)/B] \cong M/(M\cap A) \cong (M + A)/A = \mathbf{Q}/A$$

is divisible.

It is not hard to see that M/B is torsion and each p-component is bounded (see Fuchs, 1970). Then the same is true for the divisible group $\mathbf{Q}/A$. Since every nonzero torsion divisible group is a direct sum of groups $\mathbf{Z}_{p^\infty}$, which are not bounded we conclude that $A = \mathbf{Q}$. □

Example 2.3. Consider $\mathbf{Q}\oplus\mathbf{Z}_p$ as a $\mathbf{Z}$-module where p is a prime. Then $\mathbf{Q}\oplus 0$ is a maximal submodule of $\mathbf{Q}\oplus\mathbf{Z}_p$, therefore it is cofinite. Let M be as in Lemma 2.2. $(M\oplus\mathbf{Z}_p)\cap(\mathbf{Q}\oplus 0) = M\oplus 0 \ll \mathbf{Q}\oplus 0 \leq \mathbf{Q}\oplus\mathbf{Z}_p$ by Lemma 2.2, therefore $M\oplus\mathbf{Z}_p$ is a weak supplement of $\mathbf{Q}\oplus 0$. Note that $M/\mathbf{Z}$ is a direct sum of the cyclic groups $\langle\frac{1}{q} + \mathbf{Z}\rangle$, q prime; therefore $M/\mathbf{Z}$ is not finitely generated and hence M is not finitely generated. So weak supplements of cofinite submodules need not be finitely generated.

Lemma 2.4. *If* $f: M \to N$ *is a homomorphism and a submodule* L *containing* Ker$f$ *is a weak supplement in M, then* $f(L)$ *is a weak supplement in* $f(M)$.

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Proof. If L is a weak supplement of K in M then $f(M)=f(L+K)=f(L)+f(K)$ and since $L\cap K\ll M$, we have $f(L\cap K)\ll f(M)$ by 5.18 Anderson and Fuller (1992). As $L\supseteq \mathrm{Ker} f$, $f(L)\cap f(K)=f(L\cap K)$. So $f(L)$ is a weak supplement of $f(K)$ in $f(M)$. □

Proposition 2.5. *A homomorphic image of a cws-module is a cws-module.*

Proof. Let $f: M\to N$ be a homomorphism and M be a *cws*-module. Suppose that X is a cofinite submodule of $f(M)$, then

$$M/f^{-1}(X)\cong (M/\mathrm{Ker} f)/(f^{-1}(X)/\mathrm{Ker} f)\cong f(M)/X.$$

Therefore $M/f^{-1}(X)$ is finitely generated. Since M is a *cws*-module, $f^{-1}(X)$ is a weak supplement in M and by Lemma 2.4, $X=f(f^{-1}(X))$ is a weak supplement in $f(M)$. □

Corollary 2.6. *Any factor module of a cws-module is a cws-module.*

We will see below that the inverse image of a *cws*-module under a small epimorphism is a *cws*-module.

Proposition 2.7 (c.f. 41.4(4) of Wisbauer, 1991)**.** *If K is a weak supplement of N in a module M and $T\ll M$, then K is weak supplement of $N+T$ in M as well.*

Proof. Let $f: M\to (M/N)\oplus(M/K)$ be defined by $f(m)=(m+N, m+K)$ and $g:(M/N)\oplus(M/K)\to(M/(N+T))\oplus(M/K)$ be defined by $g(m+N, m'+K)=(m+N+T, m'+K)$. Then f is an epimorphism as $M=N+K$ and $\mathrm{Ker} f=N\cap K\ll M$ as K is a weak supplement of N in M. So f is a small epimorphism. Now $\mathrm{Ker}\, g=(N+T)/N\oplus 0$ and $(N+T)/N=\sigma(T)\ll M/N$ since $T\ll M$, where $\sigma: M\to M/N$ is the canonical epimorphism. Therefore g is a small epimorphism. By 19.2 in Wisbauer (1991), fg is a small epimorphism, i.e., $(N+T)\cap K=\mathrm{Ker}(fg)\ll M$. Clearly $(N+T)+K=M$, so K is a weak supplement of $N+T$ in M. □

Lemma 2.8. *If $f: M\to N$ is a small epimorphism, then a submodule L of M is a weak supplement in M if and only if $f(L)$ is a weak supplement in N.*

Proof. If L is a weak supplement of K in M then by Proposition 2.7, $L+\mathrm{Ker} f$ is also a weak supplement of K and by Lemma 2.4, $f(L)=f(L+\mathrm{Ker} f)$ is a weak supplement in N.

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Now let $f(L)$ be a weak supplement of a submodule T of N, i.e., $N = f(L) + T$ and $f(L) \cap T \ll N$. Then $M = L + f^{-1}(T)$. It follows from the proof of Corollary 9.1.5 in Kasch (1982) that the inverse image of a small submodule of N is small in M. So $L \cap f^{-1}(T) \leq f^{-1}(f(L) \cap T) \ll N$. Thus $f^{-1}(T)$ is a weak supplement of L. □

A module N is called a *small cover* of a module M if there exist a small epimorphism $f: N \to M$, i.e., $\operatorname{Ker} f \ll N$ (see Lomp, 1999).

Corollary 2.9. *A small cover of a cws-module is a cws-module.*

Proof. Let N be a *cws*-module, $f: M \to N$ be a small epimorphism and L be a cofinite submodule of M. Then $N/f(L)$ is an epimorphic image of M/L under the epimorphism $\bar{f}: M/L \to N/f(L)$ defined by $\bar{f}(m + K) = f(m) + f(L)$, therefore $f(L)$ is a cofinite submodule of N. Since N is a *cws*-module, $f(L)$ is a weak supplement. By Lemma 2.8, L is also a weak supplement in M. □

Corollary 2.10. *Suppose that M is an R-module with* $\operatorname{Rad} M \ll M$ *and* $M/\operatorname{Rad} M$ *is a cws-module. Then M is a cws-module.*

To prove that an arbitrary sum of *cws*-modules is a *cws*-module, we use the following standard lemma (see 41.2 of Wisbauer, 1991).

Lemma 2.11. *Let N and U be submodules of M with cofinitely weak supplemented N and cofinite U. If $N + U$ has a weak supplement in M, then U also has a weak supplement in M.*

Proof. Let X be a weak supplement of $N + U$ in M. Then we have

$$N/[N \cap (X + U)] \cong (N + X + U)/(X + U) = M/(X + U)$$
$$\cong (M/U)/[(X + U)/U].$$

The last module is a finitely generated module, hence $N \cap (X + U)$ has a weak supplement Y in N i.e.,

$$Y + [N \cap (X + U)] = N, \quad Y \cap N \cap (X + U) = Y \cap (X + U) \ll N \leq M.$$

Now

$$M = U + X + N = U + X + Y + [N \cap (X + U)] = U + X + Y,$$

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and

$$U \cap (X+Y) \leq [X \cap (Y+U)] + [Y \cap (X+U)]$$
$$\leq [X \cap (N+U)] + [Y \cap (X+U)] \ll M$$

Therefore $X+Y$ is a weak supplement of U in M. □

Proposition 2.12. *An arbitrary sum of cws-modules is a cws-module.*

Proof. Let $M=\sum_{i\in I} M_i$ where each submodule M_i is cofinitely weak supplemented and N be a cofinite submodule of M. Then M/N is generated by some finite set $\{x_1+N,\ x_2+N, \ldots, x_k+N\}$ and therefore $M=Rx_1+Rx_2+\cdots+Rx_k+N$. Since each x_i is contained in the sum $\sum_{j\in F_i} M_j$ for some finite subset F_i of I, $Rx_1+Rx_2+\cdots+Rx_k \leq \sum_{j\in F} M_j$ j for some finite subset $F=\{i_1,\ i_2, \ldots,\ i_r\}$ of I. Then $M=N+\sum_{t=1}^{r} M_{i_t}$. Since $M=M_{i_r}+(N+\sum_{t=1}^{r-1} M_{i_t})$ has a trivial weak supplement 0 and M_{i_r} is a *cws*-module, $N+\sum_{t=1}^{r-1} M_{i_t}$ has a weak supplement in M by Lemma 2.11. Similarly $N+\sum_{t=1}^{r-2} M_{i_t}$ has a weak supplement in M and so on. Continuing in this way we will obtain (after we have used Lemma 2.11 r times) at last that N has a weak supplement in M. □

Let M and N be R-modules. If there is an epimorphism $f\colon M^{(\Lambda)} \to N$ for some set Λ, then N is said to be an *M-generated* module.

The following corollary follows from Proposition 2.12 and Proposition 2.5.

Corollary 2.13. *If M is a cws-module, then any M-generated module is a cws-module.*

The class of *cws*-modules is strictly wider than the class of the weakly supplemented modules as the following example shows.

Example 2.14. Let p be a prime integer and consider the **Z**-module $M=\bigoplus_{i=1}^{\infty}\langle a_i\rangle$ which is the direct sum of cyclic subgroups $\langle a_i\rangle$ of order p^i. Since each $\langle a_i\rangle$ is local and therefore is a *cws*-module, M is a *cws*-module by Proposition 2.12. We will show that M is not weakly supplemented.

Let $T=pM$ and suppose that T has a weak supplement L, i.e., $M=T+L$ and $N=T\cap L \ll M$. Then $N \ll E(M)$ as well, where $E(M)$ is an injective hull of M. Since the injective hull $E(N)$ of N is a direct summand of $E(M)$, $N \ll E(N)$. It follows from Theorem 4 by Leonard (1966) that if a torsion abelian group is small in its injective hull then it is

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bounded. Therefore N must be bounded, i.e., $p^n N = 0$ for some positive integer n. Then, as $pL \leq L \cap pM = L \cap T = N$,

$$p^{n+1}M = p^{n+1}T + p^n(pL) \leq p^{n+1}T + p^n N = p^{n+1}T$$

Therefore $p^{n+1}a_{n+2} = p^{n+1}b$ for some $b \in T = pM$. Since $b = pc$ for some $c = (m_i a_i)_{i=1}^{\infty} \in M$, we have

$$0 \neq p^{n+1}a_{n+2} = p^{n+1}(pm_{n+2}a_{n+2}) = m_{n+2}p^{n+2}a_{n+2} = 0$$

This contradiction implies that M is not a weakly supplemented module.

Now we are going to prove that a module is cofinitely weak supplemented if and only if every maximal submodule has a weak supplement. Firstly we need the following lemma.

Lemma 2.15. *Let U and K be submodules of N such that K is a weak supplement of a maximal submodule M of N. If $K + U$ has a weak supplement in N, then U has a weak supplement in N.*

Proof. Let X be a weak supplement of $K + U$ in N. If $K \cap (X + U) \subseteq K \cap M \ll N$ then $X + K$ is a weak supplement of U since

$$U \cap (X + K) \leq X \cap (K + U) + K \cap (X + U) \ll N$$

Now suppose $K \cap (X + U) \nsubseteq K \cap M$. Since $K/(K \cap M) \cong (K + M)/M = N/M$, $K \cap M$ is a maximal submodule of K. Therefore $(K \cap M) + [K \cap (X + U)] = K$. Then X is a weak supplement of U in N since $U \cap X \leq (K + U) \cap X \ll N$ and

$$N = X + U + K = X + U + (K \cap M) + [K \cap (X + U)] = X + U$$

as $K \cap (X + U) \leq X + U$ and $K \cap M \ll N$. So in both cases there is weak supplement of U in N. □

For a module N, let Γ be the set of all submodules K such that K is a weak supplement for some maximal submodule of N and let $cws(N)$ denote the sum of all submodules from Γ. As usual $cws(N) = 0$ if $\Gamma = \emptyset$.

Theorem 2.16. *For a module N, the following statements are equivalent.*

1. *N is a cws-module.*
2. *Every maximal submodule of N has a weak supplement.*
3. *$N/cws(N)$ has no maximal submodules.*

Proof. (1) ⇒ (2) is obvious since every maximal submodule is cofinite.

(2) ⇒ (3): Suppose that there is a maximal submodule $M/cws(N)$ of $N/cws(N)$. Then M is a maximal submodule of N. By (2), there is a weak supplement K of M in N. Then $K \in \Gamma$, therefore $K \leq cws(N) \leq M$. Hence $N = M + K = M$. This contradiction shows that $N/cws(N)$ has no maximal submodules.

(3) ⇒ (1): Let U be a cofinite submodule of N. Then $U + cws(N)$ is also cofinite. If $N/[U + cws(N)] \neq 0$, then by Theorem 2.8 of Anderson and Fuller (1992), there is a maximal submodule $M/[U + cws(N)]$ of the finitely generated module $N/[U + cws(N)]$. It follows that M is a maximal submodule of N and $M/cws(N)$ is a maximal submodule of $N/cws(N)$. This contradicts (3). So $N = U + cws(N)$. Now N/U is finitely generated, say by elements $x_1 + U, x_2 + U, \ldots, x_m + U$, therefore $N = U + Rx_1 + Rx_2 + \cdots + Rx_m$. Each element x_i $(i = 1, 2, \ldots, m)$ can be written as $x_i = u_i + c_i$, where $u_i \in U$, $c_i \in cws(N)$. Since each c_i is contained in the sum of finite number of submodules from Γ, $N = U + K_1 + K_2 + \cdots + K_n$ for some submodules $K_1, K_2, \ldots, K_n$ of N from Γ. Now $N = (U + K_1 + K_2 + \cdots + K_{n-1}) + K_n$ has a weak supplement, namely 0. By Lemma 2.15, $U + K_1 + K_2 + \cdots + K_{n-1}$ has a weak supplement. Continuing in this way (applying Lemma 2.15 n times) we obtain that U has a weak supplement in N. □

Recall that a module M is *cofinitely supplemented* if every cofinite submodule of M has a supplement in M.

The following example shows that *cws*-modules need not be cofinitely supplemented.

Example 2.17. Consider the ring,

$$R = \mathbf{Z}_{p,q} = \left\{ \frac{a}{b} \mid a, b \in Z,\ b \neq 0,\ (p, b) = 1,\ (q, b) = 1 \right\}$$

The left module ${}_RR$ is (cofinitely) weak supplemented, but is not (cofinitely) supplemented (See Remark 3.3 of Lomp, 1999).

It is known (see 41.1(5) of Wisbauer, 1991 and Lemma 1.1 of Keskin, 2000) that for every supplement submodule K of a module M, $\operatorname{Rad} K = K \cap \operatorname{Rad} M$ and that for a weak supplemented module M the last equality implies that K is a supplement.

Lemma 2.18. *Let M be an R-module and U be a cofinite submodule of M. If U has a weak supplement V in M and for every finitely generated* *submodule K of V,* $\operatorname{Rad} K = K \cap \operatorname{Rad} M$*, then U has a finitely generated supplement in M.*

Proof. V is a weak supplement of U in M, i.e., $U + V = M$ and $U \cap V \ll M$. Since M/U is finitely generated, by Lemma 2.1, U has a finitely generated weak supplement $K \leq V$ in M, i.e., $M = U + K$ and $U \cap K \ll M$. Then $U \cap K \leq \operatorname{Rad} M$. Therefore $U \cap K \leq K \cap \operatorname{Rad} M = \operatorname{Rad} K$. But $\operatorname{Rad} K \ll K$ by 10.4 of Anderson and Fuller (1992), so $U \cap K \ll K$, i.e., K is a supplement of U in M. □

Theorem 2.19. *Let M be an R-module such that for every finitely generated submodule K of M,* $\operatorname{Rad} K = K \cap \operatorname{Rad} M$*. Then M is cofinitely weak supplemented if and only if M is cofinitely supplemented.*

Proof. Let U be a cofinite submodule of M. Since M is a *cws*-module, U has a weak supplement N in M and by Lemma 2.18, U has a supplement. Hence M is cofinitely supplemented.

The converse statement is obvious. □

Corollary 2.20. *Let M be a finitely generated module such that for every finitely generated submodule N of M,* $\operatorname{Rad} N = N \cap \operatorname{Rad} M$*. Then M is weakly supplemented if and only if M is supplemented. Furthermore in this case every finitely generated submodule of M is a supplement.*

Proof. The first statement follows from Theorem 2.19 as in a finitely generated module, every submodule is cofinite. If N is a finitely generated submodule then N has a weak supplement K, therefore $N + K = M$ and $N \cap K \leq N \cap \operatorname{Rad} M = \operatorname{Rad} N \ll N$, i.e., N is a supplement of K. □

Theorem 2.21. *Let M be an R-module with* $\operatorname{Rad} M \ll M$*. Then the following statements are equivalent.*

1. *M is a cws-module.*
2. *$M/\operatorname{Rad} M$ is a cws-module.*
3. *Every cofinite submodule of $M/\operatorname{Rad} M$ is a direct summand.*
4. *Every maximal submodule of $M/\operatorname{Rad} M$ is a direct summand.*
5. *Every maximal submodule of $M/\operatorname{Rad} M$ is a weak supplement.*
6. *Every maximal submodule of M is a weak supplement.*

Proof. $(1) \Rightarrow (2)$: By Corollary 2.6.

$(2) \Rightarrow (3)$ is obvious since $\operatorname{Rad}(M/\operatorname{Rad} M) = 0$.

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(3) ⇒ (4): Maximal submodules are cofinite.
(4) ⇒ (5) is obvious.
(5) ⇒ (6): By Lemma 2.8.
(6) ⇒ (1) holds for every module M by Theorem 2.16. □

Corollary 3.2 by Lomp (1999) gives that a ring R is semilocal if and only if ${}_RR$ is a weak supplemented R-module. Since every left R-module is ${}_RR$-generated, by Corollary 2.13 we have the following corollary.

Corollary 2.22. *Let R be a ring. Then R is semilocal if and only if every R-module is a cws-module.*

3. FINITELY WEAK SUPPLEMENTED MODULES

Let M be a module. If every finitely generated submodule of M has a weak supplement in M, then M is called a *finitely weak supplemented* or briefly, an *fws*-module.

To prove that under proper conditions the sum of two *fws*-modules is a *fws*-module, we give the following modification of 41.2 from Wisbauer (1991).

Lemma 3.1. *Let M be a finitely generated module, M_1 and U be finitely generated submodules and M_1 be a fws-module . If M_1+U has a weak supplement X in M such that $M_1\cap(X+U)$ is finitely generated, then U has a weak supplement in M.*

Proof. X is a weak supplement of M_1+U in M, so $M_1+U+X=M$ and $(M_1+U)\cap X\ll M$. $M_1\cap(U+X)\leq M_1$ and by the assumption $M_1\cap(U+X)$ is finitely generated and M_1 is a *fws*-module. So $M_1\cap(X+U)$ has a weak supplement Y in M_1, that is, $[M_1\cap(X+U)]+Y=M_1$ and $M_1\cap(X+U)\cap Y=(X+U)\cap Y\ll M_1$. We get

$$M_1 + U + X = [M_1 \cap (X + U)] + Y + U + X = U + X + Y$$

and

$$\begin{aligned} U \cap (X + Y) &\leq [(U + Y) \cap X] + [(U + X) \cap Y] \\ &\leq [(U + M_1) \cap X] + [(U + X) \cap Y] \ll M \end{aligned}$$

This means that $X+Y$ is a weak supplement of U in M. □

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We do not know whether the sum of two *fws*-modules is always a *fws*-module, however the following statement holds (c.f. 41.3 of Wisbauer, 1991). Recall that a module is *coherent* if it is finitely generated and every finitely generated submodule is finitely presented.

Proposition 3.2. *Let M be an R-module and $M = M_1 + M_2$, with M_1, M_2 finitely generated and finitely weak supplemented. Suppose that either*

(i) *M is coherent, or*
(ii) *M is self-projective and $M_1 \cap M_2 = 0$.*

Then M is a fws-module.

Proof. Let U be a finitely generated submodule of M.

(i) Clearly $M_1 + M_2 + U$ has the trivial weak supplement. Let us consider the submodule $M_2 \cap (M_1 + U + 0)$.

If M is coherent, then by 26.1 of Wisbauer (1991), $M_2 \cap (M_1 + U + 0)$, as an intersection of finitely generated submodules, is finitely generated and by Lemma 3.1, $M_1 + U$ has a weak supplement X in M. By Lemma 2.1, we may consider X as a finitely generated submodule of M.

Now $M_1 \cap (X + U)$ is also finitely generated by 26.1 from Wisbauer (1991) and applying again Lemma 3.1, we conclude that U has a weak supplement in M.

(ii) If M is self-projective and $M_1 \cap M_2 = 0$, then

$$(M_1 + U)/[M_2 \cap (M_1 + U)] \cong M/M_2 \cong M_1$$

is M-projective by 18.1 of Wisbauer (1991) and by 16.12 of Anderson and Fuller (1992), $(M_1 + U)$-projective. Now we can say $M_2 \cap (M_1 + U)$ is a direct summand of $M_1 + U$ and so finitely generated. Hence by Lemma 3.1, $M_1 + U$ has a weak supplement X in M. By Lemma 2.1, we may consider X as a finitely generated submodule of M.

Now

$$(X + U)/[M_1 \cap (X + U)] \cong (M_1 + X + U)/M_1 = M/M_1 \cong M_2$$

is M-projective by 18.1 of Wisbauer (1991) and by 16.12 of Anderson and Fuller (1992), it is $(X + U)$-projective. So $M_1 \cap (X + U)$ is a direct summand of $X + U$ and so is finitely generated. Therefore by Lemma 3.1, U has a weak supplement. This completes the proof. □

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Proposition 3.3. *If M is a fws-module and $f: M \to N$ is an epimorphism with finitely generated* Ker$f$ *then N is also a fws-module.*

Proof. If K is a finitely generated submodule of N then clearly $f^{-1}(K)$ is finitely generated as Ker$f$ is finitely generated. Therefore $f^{-1}(K)$ is a weak supplement in M. By Lemma 2.4, $K = f(f^{-1}(K))$ is a weak supplement in N. □

Corollary 3.4. *Let M be a finitely generated fws-module which is self projective or coherent. Then for any finitely generated submodule $K \leq M^n$, with positive integer n, the factor module M^n/K is a fws-module.*

Proof. By Proposition 3.2, M^n is a *fws*-module and by Proposition 3.3, M^n/K is a *fws*-module. □

Proposition 3.5. *Let $f: M \to N$ be a small epimorphism. Then M is a fws-module if and only if N is a fws-module.*

Proof. If M is a *fws*-module M then, by Lemma 2.8, N is a *fws*-module since for every finitely generated submodule K of N there is a finitely generated submodule L of M with $f(L) = K$.

Now if N is a *fws*-module and L is a finitely generated submodule of M, then $f(L)$ is finitely generated and therefore is a weak supplement in N. By Lemma 2.8, L is a weak supplement in N. □

Corollary 3.6. *If $L \ll M$ and M/L is a fws-module, then M is a fws-module.*

Lemma 3.7. *If every cyclic submodule of a module M is a direct summand of M then every finitely generated submodule of M is a direct summand of M.*

Proof. Let a submodule L of M be generated by elements $x_1, x_2, \ldots, x_n$. By induction on n we will prove that L is a direct summand of M. If $n = 1$ then L is cyclic and therefore is a direct summand. Assume that every submodule of M generated by less than n elements is a direct summand. The cyclic submodule Rx_1 is direct summand of M, i.e., $M = (Rx_1) \oplus K$ for some $K \leq M$. By modular law $L = (Rx_1) \oplus (K \cap L)$. Then $K \cap L$ is generated by elements $p(x_2), p(x_3), \ldots, p(x_n)$, where $p: L \to K \cap L$ is the standard projection. Therefore $K \cap L$ is a direct summand of M. Then we have $K = (K \cap L) \oplus T$ for some $T \leq K$ and we have $M = (Rx_1) \oplus (K \cap L) \oplus T = L \oplus T$. So L is a direct summand of M. □

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Theorem 3.8. *The following are equivalent for a module M with $\operatorname{Rad} M \ll M$.*

1. *M is a fws-module.*
2. *Every cyclic submodule of M has (is) a weak supplement.*
3. *Every cyclic submodule of $M/\operatorname{Rad} M$ is a direct summand.*
4. *Every finitely generated submodule of $M/\operatorname{Rad} M$ is a direct summand.*
5. *$M/\operatorname{Rad} M$ is a fws-module.*

Proof. (1) $\Rightarrow$ (2) and (4) $\Rightarrow$ (5) are obvious.

(2) $\Rightarrow$ (3): Every cyclic submodule $R(m + \operatorname{Rad} M)$ equals to $\sigma(Rm)$, where $\sigma : M \to M/\operatorname{Rad} M$ is the canonical epimorphism, therefore, is a weak supplement in $M/\operatorname{Rad} M$ by Lemma 2.8. Since $\operatorname{Rad}(M/\operatorname{Rad} M) = 0$, every cyclic submodule of $M/\operatorname{Rad} M$ is a direct summand.

(3) $\Rightarrow$ (4): By Lemma 3.7.

(5) $\Rightarrow$ (1): By Corollary 3.6. □

To give an example of a *fws*-module which is not weakly supplemented we will consider Von Neumann regular rings.

R is called a *Von Neumann regular* ring if for all $a \in R$ there exists $x \in R$ such that $axa = a$. For brevity we will refer to these rings as *regular* rings. Every semisimple ring is regular, but a regular ring need not be semisimple, for example the direct product $R = F^I$ where F is a field and I is an infinite set (see Kasch, 1982, p. 264). Note that every weakly supplemented module with zero radical is semisimple (see Corollary 2.3 of Lomp, 1999).

Proposition 3.9. *A regular ring which is not semisimple is a fws-module over itself, but is not weakly supplemented.*

Proof. Let N be a finitely generated left ideal of R. By Theorem 3.3.16 of Ribenboim (1967), N is a direct summand, i.e., $R = N \oplus K$ and $K \cap N = 0 \ll R$ for some left ideal K of R. This shows that K is a weak supplement of N. Therefore ${}_RR$ is finitely weak supplemented.

Now suppose that ${}_RR$ is weakly supplemented and let L be any left ideal of R. Then $L + K = M$ and $L \cap K \ll R$ for some $K \leq R$. By Theorem 3.3.18 of Ribenboim (1967), $\operatorname{Rad} {}_RR = 0$ and so $L \cap K = 0$. It follows that L is a direct summand. So every left module of ${}_RR$ is a direct summand, i.e., R is semisimple. This is a contradiction. □

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