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RESONANCE SOLITONS AS BLACK HOLES IN MADELUNG FLUID

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Envelope solitons of the Nonlinear Schrödinger equation (NLS) under quantum potential's influence are studied. Corresponding problem is found to be integrable for an arbitrary strength, $s \neq 1$, of the quantum potential. For $s < 1$, the model is equivalent to the usual NLS with rescaled coupling constant, while for $s > 1$, to the reaction–diffusion system. The last one is related to the anti-de Sitter (AdS) space valued Heisenberg model, realizing a particular gauge fixing condition of the $(1 + 1)$-dimensional Jackiw–Teitelboim gravity. For this gravity model, by the Madelung fluid representation we derive the acoustic form of the space–time metric. The space–time points, where dispersion changes the sign, correspond to the event horizon, while the soliton solution to the AdS black hole. Moving with the above bounded velocity, it describes evolution on the one sheet hyperboloid with nontrivial winding number, and creates under collision, the resonance states which we study by the Hirota bilinear method.

Keywords: Nonlinear Schrödinger equation; soliton; black hole; quantum potential; Madelung fluid; acoustic metric; resonance; Jackiw–Teitelboim gravity.

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1. Introduction

An intimate similarity between black hole physics and hydrodynamics of supersonic acoustic flows, was first noticed by Unruh.[1] In an attempt to better understand quantum gravity, it has then been applied to investigating Hawking radiation and other phenomena.[2] Recently, by a similar approach, quantum of effects related to event horizon and ergoregion have been simulated, but in a superfluid, which contrasts with the usual liquids that allow non-dissipative motion of the flow.[3] In this case, a "superluminally" moving soliton-like inhomogeneity of the order parameter plays the role of black-holes-like quasi-equilibrium state, exhibiting an event horizon. From another site, Madelung showed in 1926 that the linear Schrödinger equation of quantum mechanics acquires a fluid mechanical form, with an addition to pressure of the so-called "quantum potential".[4] It turns out that the Madelung fluid representation has a meaning beyond the quantum mechanical probability flow, applying to nonlinear modifications of the Schrödinger equation, and generically appearing in a wide range of physical problems. Thus, from a phenomenological approach to the superfluidity of an almost ideal Bose gas, based on the Ginzburg–Gross–Pitaevskiy equation with cubic nonlinearity,[5] $\rho = |\psi|^2$ means particle number density in the condensate state, while the phase gradient is proportional to the velocity of superfluid motion $\mathbf{v_s} = \nabla \arg \psi$. In one space dimension, this model is known as the Nonlinear Schrödinger equation (NLS), possesses integrable structure and attracted much attention particularly from nonlinear optics, describing solitons propagation in optical fibers.[6] It is instructive to consider how the black-hole-like phenomena might be generalized to the Madelung fluid, thereby to the appropriate nonlinear Schrödinger type equation. Very recently an implication appeared that indeed, it can be realized at least in $1 + 1$ dimensions, where black holes of constant curvature space–time have been related to the soliton-like solutions, for dissipative version of NLS in the Reaction–Diffusion (RD) form.[7,8] These solutions called *dissipatons*, characterize completely black hole's horizon, the Hawking temperature and the causal structure. Furthermore, described in terms of elastic scattering of dissipatons, the collision of identical black holes exemplify a novel character creating a metastable state of specific lifetimes.[8] However, the unbounded dissipaton solution characteristics entail vague analytical meaning, which is why it is desirable to find a similar representation of black holes, but in terms of regular envelope solitons.

Here we show a black-hole-like phenomena, arisen from a problem of NLS soliton, subjected with the influence of quantum potential with strength bounded from below. This problem naturally generalizes a previous older problem considered by Bohm,[9] where a quantum mechanical particle is represented as the classical one, moving under the action of a classical potential force which also includes a contribution from the quantum one. If, instead of classical particle, we consider the NLS soliton, subjected with quantum potential's influence of intensity s, it could represent the stochastically quantized soliton. We find that such a problem is exactly solvable, and depending on s, reducible to the usual NLS ($s < 1$), or to the Reaction–Diffusion system ($s > 1$). When $s > 1$ by the Madelung fluid representation, we obtain a so-called acoustic metric of Jackiw–Teitelboim gravity, with simple interpretation of black hole's event horizon. Moreover, to any envelope soliton of our modified NLS, we relate some dissipaton solution of RD. Moving with a velocity of bounded above strength, this soliton describes a black hole with rich resonance scattering phenomenology, examined by the Hirota bilinear method. To analyze the causal picture, we construct an anti-de Sitter (AdS) representation of the model, which is similar to Kruskal–Szekeres coordinates, and relate the black hole to an AdS topological soliton.

2. Solitons in Quantum Potential

We begin with a problem of the NLS soliton subjected with the influence of the so-called "quantum potential" and described by the equation

$$i\partial_t\psi + \partial_x^2\psi + \frac{\Lambda}{4}|\psi|^2\psi = s\frac{\partial_x^2|\psi|}{|\psi|}\psi\,, \tag{2.1}$$

where the term

$$U_Q(x) \equiv \frac{\partial_x^2|\psi|}{|\psi|}\,, \tag{2.2}$$

on the R.H.S. represents contribution from the quantum potential. Even though Eq. (2.2) includes the space derivatives and instead of nonlinear quantum mechanics we are dealing rather with nonlinear dynamics, we preserve the historical name "quantum potential". This potential was introduced by L. de Broglie[10] and has been explored by D. Bohm[9] to make a hidden-variable theory in quantum mechanics. It is responsible for producing the quantum behavior, so that all quantum features are related to its special properties. Recently, it appears in the stochastic mechanics as the source of non-classical diffusion.[11] Relations of such a non-classical motion with the *internal* spin motion and the *zitterbewegung* are considered in a series of papers.[12]

Potential U_Q is invariant under rescaling transformations, $\psi(x,t) \to \lambda\psi(x,t)$, with complex constant $\lambda \in \mathcal{C}$, and hence does not depend on the strength of the wave, associated with a soliton, but depends only on soliton's form. Therefore, its effect could be large even for the well-separated and far enough solitons. This type of homogeneity property is the reason why quantum potential appears in attempts to nonlinear extensions of quantum mechanics[25]: a) in a stochastic quantization, allowing for the diffusion coefficient to differ $\hbar/2m$, as a result of the difference in the Planck constant[13] or the inertial mass,[14] b) in corrections from quantum gravity.[15] Auberson and Sabatier[16] showed that, depending of quantum potential's intensity, the linear Schrödinger equation linearizable in form of the Schrödinger equation or as the pair of time reversed diffusion equations. Both cases do not admit soliton solutions. However, below we find that for self-consistent potential $U = -\frac{\Lambda}{4}|\psi|^2$ in Eq. (2.1), the model has soliton solutions with a rich resonance dynamics.

Decomposing the wave function

$$\psi = e^{R-iS}\,, \qquad \bar{\psi} = e^{R+iS}\,, \tag{2.3}$$

in terms of two real functions R and S, we have the system of equations

$$\partial_t S + (1-s)[\partial_x^2 R + (\partial_x R)^2] - (\partial_x S)^2 + \frac{\Lambda}{4}e^{2R} = 0\,, \tag{2.4a}$$

$$\partial_t R - \partial_x^2 S - 2\partial_x R\partial_x S = 0\,, \tag{2.4b}$$

representable as the so-called Madelung fluid. In terms of density function $\rho = |\psi|^2 = e^{2R}$, Eq. (2.4b) becomes the continuity equation form

$$\partial_t \rho + \partial_x(\rho V) = 0\,, \tag{2.5}$$

where we introduced a velocity field $V(x,t) = -2\partial_x S$. On the other hand, Eq. (2.4a) has a form of Hamilton–Jacobi equation

$$-\partial_t S + (\partial_x S)^2 + U + (s-1)U_Q = 0\,, \tag{2.6}$$

with nonlinear potential $U = -\frac{\Lambda}{4}e^{2R} = -\frac{\Lambda}{4}\rho$, and the quantum potential (2.2). Taking the gradient of Eq. (2.6) we obtain the hydrodynamic equation

$$\partial_t V + V\partial_x V + 2\partial_x[U + (s-1)U_Q] = 0\,. \tag{2.7}$$

Then, a particular particle acceleration given by the total derivative of V would obey the Newton's equation of motion

$$\frac{1}{2}\frac{dV}{dt} = -\frac{\partial}{\partial x}[U + (s-1)U_Q]\,. \tag{2.8}$$

This relation explains why U_Q is called quantum potential. In quantum mechanics $U_Q \sim \hbar^2$ and its contribution vanishes in the classical limit, when $\hbar \to 0$. Similar interpretation can be given in nonlinear dynamics, for the stationary wave's self-focusing in the cubic medium, where two forces on the R.H.S. of Eq. (2.8) determine behavior of the eikonal (wave front): a force connected with nonlinear refraction and the diffraction force from quantum potential.[6]

3. Madelung Fluid and the Reaction–Diffusion System

Since quantum potential U_Q is positive definite, its contribution to equation of motion (2.8) changes the sign at critical value $s = 1$. Therefore, we treat values of $s > 1$ and $s < 1$ separately. First we consider the case $s < 1$. Rescaling time and phase of the wave function (2.3)

$$t = (1-s)^{-\frac{1}{2}}\tilde{t}\,, \qquad S(x,t) = (1-s)^{\frac{1}{2}}\tilde{S}(x,\tilde{t})\,, \tag{3.1}$$

instead of system (2.4) we get

$$\partial_{\tilde{t}}\tilde{S} + [\partial_x^2 R + (\partial_x R)^2] - (\partial_x \tilde{S})^2 + \frac{\Lambda}{4(1-s)}e^{2R} = 0\,, \tag{3.2a}$$

$$\partial_{\tilde{t}} R - \partial_x^2 \tilde{S} - 2\partial_x R \partial_x \tilde{S} = 0\,. \tag{3.2b}$$

In terms of a new complex function $\tilde{\psi} = e^{R - i\tilde{S}}$, the system reverts to the usual NLS equation

$$i\partial_{\tilde{t}}\tilde{\psi} + \partial_x^2\tilde{\psi} + \frac{\Lambda}{4(1-s)}|\tilde{\psi}|^2\tilde{\psi} = 0\,, \tag{3.3}$$

but with rescaled coupling constant $\tilde{\Lambda} = \Lambda/(1-s)$. For $\Lambda < 0$, NLS Eq. (3.3), descriptive of the repulsive near-ideal Bose gas and the defocusing optical media, admits soliton solutions (dark soliton) only when $\tilde{\psi}$ is a nonvanishing function at the space and time infinities. This asymptotic form of the wave function corresponds to the Bose gas condensate state. It is an exact, homogeneous in space solution of equations of motion (3.3), acquiring the form

$$\psi = \left(\frac{4\rho_0(1-s)}{-\Lambda}\right)^{1/2} e^{-i\rho_0(1-s)t},$$

for original problem (2.1). The last solution implies that with growth of quantum potential's intensity s, particle's number density in the condensate state decreases as $|\psi|^2 = 4\rho_0(1-s)/(-\Lambda)$, that satisfies our intuitive idea of the decoherence's increasing. When $s = 1$, the effective density of the condensate state vanishes and dark solitons disappear from the spectrum. Consequently, quantum potential's contribution completely disappears from the system (2.5), (2.6), corresponding now to the semiclassical limit for NLS.[27]

But situation changes drastically if $s > 1$. Then, rescaling time and phase of the wave function (2.3)

$$t = (s-1)^{-\frac{1}{2}}\tilde{t}, \qquad S(x,t) = (s-1)^{\frac{1}{2}}\tilde{S}(x,\tilde{t}), \tag{3.4}$$

we get the system

$$\partial_{\tilde{t}}\tilde{S} - [\partial_x^2 R + (\partial_x R)^2] - (\partial_x \tilde{S})^2 + \frac{\Lambda}{4(s-1)} e^{2R} = 0, \tag{3.5a}$$

$$\partial_{\tilde{t}} R - \partial_x^2 \tilde{S} - 2\partial_x R \partial_x \tilde{S} = 0. \tag{3.5b}$$

However, in contrast to the previous case (3.2), this system cannot be simplified in terms of one complex function $\tilde{\psi}$. On the other hand, if we introduce two real functions (in what follows we skip the tilde sign), $e^+ = \exp(R+S)$, $-e^- = \exp(R-S)$, such that

$$-e^+e^- = e^{2R} = |\psi|^2, \qquad S = \frac{1}{2}\ln\frac{e^+}{-e^-} = \frac{1}{2i}\ln\frac{\bar{\psi}}{\psi}, \tag{3.6}$$

then we obtain the Reaction–Diffusion (RD) system

$$-\partial_t e^+ + \partial_x^2 e^+ + \frac{\Lambda}{4} e^+ e^- e^+ = 0, \tag{3.7a}$$

$$+\partial_t e^- + \partial_x^2 e^- + \frac{\Lambda}{4} e^+ e^- e^- = 0. \tag{3.7b}$$

It is worthwhile to note that unusual negative value for diffusion coefficient in the second equation (3.7b) is crucial for the existence of Hamiltonian structure and integrability of the model. The system (3.7) is time reversible $t \to -t$, $e^\pm \to e^\mp$, and invariant under the global SO(1,1) transformations, $e^\pm \to e^{\pm\alpha}e^\pm$. For $\Lambda < 0$, it admits solution

$$e^\pm = \pm\left(\frac{8}{-\Lambda}\right)^{1/2} k e^{\pm[(\frac{1}{4}v^2+k^2)t - \frac{1}{2}vx]} \cosh^{-1}[k(x - vt - x_0)], \tag{3.8}$$

with exponentially growing and decaying in time components, but with perfect solitonic shape for $O(1,1)$ scalar product

$$-e^{+}e^{-} = |\psi|^2 = \frac{8}{-\Lambda}k^2 \cosh^{-2}[k(x - vt - x_0)]\,. \tag{3.9}$$

By analogy with dissipative structures in the pattern formation theory we called this dissipative soliton solution as *dissipaton*.[7] Using (3.6) and (3.8) we find one-soliton solution of Eq. (2.1)

$$\psi = \left(\frac{8}{-\Lambda}\right)^{1/2} k \frac{e^{-i[(\frac{1}{4}v^2 + k^2)t - \frac{1}{2}vx]}}{\cosh k(x - vt)}\,, \tag{3.10}$$

where the value $s = 2$ is fixed. For this particular value the rescaling factors in Eq. (3.4) become unity, and Eq. (2.1) acquires the canonical form

$$i\partial_t\psi + \partial_x^2\psi + \frac{\Lambda}{4}|\psi|^2\psi = 2\frac{\partial_x^2|\psi|}{|\psi|}\psi\,, \tag{3.11}$$

which we call the *resonance Nonlinear Schrödinger equation* (RNLS). As we show in Sec. 5, dispersive part of the energy density for this equation is sign indefinite and allows creation of soliton resonances. We recall that under fixed sign of the nonlinearity (in optical fibers the nonlinearity coefficient never changes its sign), say $\Lambda < 0$, the focusing and defocusing NLS has strictly negative and positive definite dispersion correspondingly. Moreover, the relative sign between dispersion and nonlinearity is crucial for the existence of bright and dark solitons respectively. Now, along with focusing and defocusing NLS, Eq. (3.11) can be considered as the third integrable version of NLS, mixing both cases.

4. Geometrical and Gravitational Interpretation

Dissipatons, relating to black hole solutions of the Jackiw–Teitelboim gravity[8] provide interesting tool to study nonperturbative sector of the general relativity. Defining two-dimensional metric tensor in terms of Einstein–Cartan zweibein fields

$$g_{\mu\nu} = e^a_\mu e^b_\nu \eta_{ab} = \frac{1}{2}(e^+_\mu e^-_\nu + e^+_\nu e^-_\mu)\,, \tag{4.1}$$

where $e^{\pm}_\mu = e^0_\mu \pm e^1_\mu = (e^{\pm}_0, e^{\pm}_1)$, $\eta_{ab} = \mathrm{diag}(-1,1)$, one can formulate the gravity model as the noncompact BF gauge field theory with $\mathrm{SO}(2,1)$ Poincaré gauge group.[17] We fix the gauge freedom by conditions

$$e^{\pm}_0 = \pm\frac{\partial}{\partial x}e^{\pm}\,, \qquad e^{\pm}_1 \equiv e^{\pm}\,, \tag{4.2}$$

such that

$$g_{00} = -\frac{\partial e^+}{\partial x}\frac{\partial e^-}{\partial x}\,, \qquad g_{11} = e^+e^-\,, \qquad g_{01} = \frac{1}{2}\left(\frac{\partial e^+}{\partial x}e^- - e^+\frac{\partial e^-}{\partial x}\right), \tag{4.3}$$

implying identification $x_0 \equiv t$, $x_1 \equiv x$. It follows that when $e^{\pm}$ satisfy Eqs. (3.7), this metric describes two-dimensional pseudo-Riemannian space–time with a constant curvature Λ ("cosmological term"):

$$R = g^{\mu\nu} R_{\mu\nu} = \Lambda \,. \tag{4.4}$$

This, low-dimensional ("lineal") gravity model is known as the Jackiw–Teitelboim model[18] though it was proposed before in Ref. 19. It turns out that the model describes also S-wave sector of extremal $D = 4$ supersymmetric black hole in dilaton's coupled gravity model.[20] When curvature vanishes, $\Lambda = 0$, the nonlinear term in Eq. (2.1) and the system (3.7) disappears. Then, the system (3.7) decouples into a pair of linear heat equation and the time-reversal one, while (2.1) reduces to the unusual modification of the linear Schrödinger equation:

$$i\partial_t \psi + \partial_x^2 \psi - 2\frac{\partial_x^2 |\psi|}{|\psi|}\psi = 0 \,, \tag{4.5}$$

with sign indefinite dispersion. It is worth to note that the classical theories corresponding to the usual quantum mechanical Schrödinger equation and to the modified model (4.5) are equivalent, since the "quantum potential" is proportional to $\hbar^2$, and in the $\hbar \to 0$ limit both models lead to the same Hamilton–Jacobi equations. Furthermore, the model (4.5) is also relevant to the black hole solutions of the CGHS string-inspired gravitational theory.[21]

Representation (4.3) allows us to establish a correspondence between geometrical and physical characteristics of the model. In terms of ψ variable the metric tensor (4.1) is given by

$$g_{00} = 2\left(\frac{\partial|\psi|}{\partial x}\right)^2 - \frac{\partial\bar{\psi}}{\partial x}\frac{\partial\psi}{\partial x} \,, \qquad g_{11} = -|\psi|^2 \,, \qquad g_{01} = \frac{i}{2}\left(\frac{\partial\bar{\psi}}{\partial x}\psi - \bar{\psi}\frac{\partial\psi}{\partial x}\right) , \tag{4.6}$$

so that g_{00} component has meaning of the dispersive part of energy density, while g_{11} and g_{01}, of the mass and momentum densities correspondingly. For one-dissipaton solution (3.8) or one-soliton solution (3.10) the mass, momentum and energy conserved quantities

$$M = -\int_{-\infty}^{\infty} e^+ e^- \, dx = \int_{-\infty}^{\infty} |\psi|^2 \, dx \,, \tag{4.7a}$$

$$P = -\int_{-\infty}^{\infty} (e^+ \partial_x e^- - \partial_x e^+ e^-) dx = i\int_{-\infty}^{\infty} (\partial_x \bar{\psi}\psi - \bar{\psi}\partial_x \psi) dx \,, \tag{4.7b}$$

$$\begin{aligned} E &= 2\int_{-\infty}^{\infty} \left[\partial_x e^+ \partial_x e^- - \frac{\Lambda}{8}(e^+ e^-)^2\right] dx \\ &= 2\int_{-\infty}^{\infty} \left[\partial_x \bar{\psi}\partial_x \psi - 2\partial_x|\psi|\partial_x|\psi| - \frac{\Lambda}{8}|\psi|^4\right] dx \,, \end{aligned} \tag{4.7c}$$

become

$$M = \frac{16}{-\Lambda}|k|\,, \qquad P = Mv\,, \qquad E = \frac{Mv^2}{2} + \frac{\Lambda^2}{384}M^3\,. \tag{4.8}$$

This means that one dissipaton/soliton (3.8)/(3.10) can be interpreted as non-relativistic quasi-particle of non-negative mass M and momentum P, with positive rest energy $E_0 = E(v = 0) = \frac{\Lambda^2}{384}M^3$. As we show in the next section, this energy allows decay of the soliton at rest.

5. Resonance Dispersion and Black Holes

We are now in a position to show that an interaction of the above introduced quasi-particles leads to creation and annihilation processes, forming resonance states. Existence of these states relates to the sign-indefinite form of dispersive part of energy density (4.7c), written in terms of variables (3.6)

$$\varepsilon_0 \equiv 2(\partial_x\bar\psi\partial_x\psi - 2\partial_x|\psi|\partial_x|\psi|) = 2[(\partial_x S)^2 - (\partial_x R)^2]e^{2R}\,. \tag{5.1}$$

For comparison we recall that in the usual NLS case, when $s = 0$, one has positive-definite dispersion energy

$$\varepsilon_0 \equiv 2\partial_x\bar\psi\partial_x\psi = 2[(\partial_x S)^2 + (\partial_x R)^2]e^{2R}\,. \tag{5.2}$$

The space–time points where function ε_0 in Eq. (5.1) changes the sign are solutions of equations

$$(\partial_x S)^2 - (\partial_x R)^2 = (\partial_x S - \partial_x R)(\partial_x S + \partial_x R) = 0\,, \tag{5.3}$$

or $\partial_x S = \pm\partial_x R$. Comparing Eq. (5.1) with Eq. (4.6), we find that dispersion energy density has geometrical meaning of the metric tensor component $\varepsilon_0 = -2g_{00}$. Therefore, conditions (5.3) are equivalent to the existence of the event horizon at the space–time points (x_H, t_H), where g_{00} change the sign. This relates resonance dispersion of the RNLS (3.11) with the existence of the event horizon in two-dimensional space–time, indicating on nontrivial causal structure and the corresponding black hole type phenomena. In fact, if we calculate the metric (4.6) for one-soliton solution (3.10),

$$ds^2 = \frac{8}{-\Lambda}\left[(k^2\tanh^2 k(x - vt) - \frac{1}{4}v^2)(dt)^2 - (dx)^2 - v\,dx\,dt\right]|\psi|^2\,, \tag{5.4}$$

then for $|v| \le 2|k| \equiv |v_{\max}|$, it shows a horizon singularity at

$$\tanh k(x - vt) = \pm\frac{v}{2k}\,. \tag{5.5}$$

Thus, a black hole soliton cannot move faster than the maximal value of the velocity $|v_{\max}| = 2|k|$ and event horizons are located at the distances $\pm\tanh^{-1}|v/2k|$ from the soliton's center. The corresponding metric can be transformed to the Schwarzschild type form and shows the causal structure in terms of Kruskal–Szekeres coordinates.[8]

Comparing (5.1) with (4.7c) and (4.8) one can see that quantum potential's contribution to the total energy is negative. For one-soliton solution (3.10) we have $-2\partial_x S = v$ and only the first term in (5.1) contributes to the kinetic energy, while quantum potential contributes only to the rest energy E_0. Due to the positive sign of nonlinearity ($\Lambda < 0$), the resulting rest energy remains positive. This is the reason why the resonance behavior could occur for our model. Indeed, decay of a soliton at rest on the pair of solitons with positive energies is allowed only if the rest energy is positive. Since the rest energy satisfies inequality

$$E_0 = \frac{\Lambda^2}{384}M^3 = \frac{\Lambda^2}{384}(M_1 + M_2)^3 > \frac{\Lambda^2}{384}(M_1^3 + M_2^3) = E_{0(1)} + E_{0(2)}\,, \tag{5.6}$$

such that $\Delta E_0 = E_0 - (E_{0(1)} + E_{0(2)}) > 0$, it allows creation of two solitons. Then, the first three conservation laws for the decaying process are

$$M = M_1 + M_2\,, \qquad P = P_1 + P_2\,, \qquad E = E_1 + E_2\,, \tag{5.7}$$

where we have to use asymptotic form (4.8) for separated solitons. From the mass conservation law it follows that defect of the mass always vanishes, $\Delta M = M - (M_1 + M_2) = 0$. Further, from conservation of momentum we find that velocity of decaying soliton coincides with the one for the center-of-mass $v = (M_1 v_1 + M_2 v_2)/(M_1 + M_2)$. Substituting to the energy conservation law we obtain the following constraint on velocities of the decay's product

$$|v_1 - v_2| = -\frac{\Lambda}{8}(M_1 + M_2)\,. \tag{5.8}$$

In Sec. 7 we apply this constraint to the soliton dynamics and find that for two-soliton solution it corresponds to the resonance creation condition.

6. Hydrodynamical Interpretation

The Madelung fluid representation gives simple hydrodynamical explanation for the existence of resonance states and the event horizon. To proceed, we introduce the fluid density $\rho \equiv e^{2R} = |\psi|^2$, according to Eq. (3.6). Since an envelope soliton is characterized by two motions, the center-of-mass motion and internal oscillations in the envelope, we define the corresponding local velocities. Velocity $V \equiv -2\partial_x S$ (see Eq. (2.5)), of the center-of-mass motion and velocity $V_Q \equiv \partial_x \rho/\rho$, of an internal motion, referred to the center-of-mass frame. The last one is similar to the "quantum velocity" describing stochastic diffusion[11] or *the zitterbewegung* motion,[12] generated by quantum potential. If for the standard NLS ($s = 0$) the dispersive energy density (5.2) is just the sum of kinetic energies of these two motions

$$\varepsilon_0 = \left(\frac{\rho V^2}{2} + \frac{\rho V_Q^2}{2}\right), \tag{6.1}$$

then, in contrast, for RNLS (3.8) the density (5.1) is given by their difference

$$\varepsilon_0 = \left(\frac{\rho V^2}{2} - \frac{\rho V_Q^2}{2}\right), \tag{6.2}$$

that is the origin of black hole type behavior. In our hydrodynamical representation, metric tensor (4.6) acquires the form

$$g_{00} = \frac{1}{4}\rho(V_Q^2 - V^2)\,, \qquad g_{11} = -\rho\,, \qquad g_{01} = \frac{1}{2}\rho V\,. \tag{6.3}$$

It is similar to the ADM splits of a $(1+1)$-dimensional Lorentzian space–time corresponding to the so-called acoustic metric, derived by Unruh[1] for the sound waves in a fluid.[2] Hence, the event horizon defined by $g_{00} = 0$ appears at a point where velocities of the center-of-mass and the internal motion are equal $V = \pm V_Q$. For one-soliton solution (3.10), velocity of the center-of-mass is a constant, coinciding with the soliton propagation velocity $V = v$, while the "quantum" velocity is the bounded function $V_Q = -2k\tanh k(x - vt)$, such that $|V_Q| \leq 2|k|$. As a consequence, compensation of velocities and vanishing of g_{00} is possible only in this region. Thus, velocity compensation condition $V = \pm V_Q$ becomes equivalent to soliton's event horizon condition (5.5).

To derive the black hole we first rewrite metric (6.3) in the moving frame $(\xi, t) = (x - vt, t)$ with a constant velocity v. Then, in terms of new "shifted" local velocity $W(\xi, t) = 2v - V(\xi, t)$ it acquires convenient form

$$\tilde{g}_{00} = \frac{1}{4}\rho(V_Q^2 - W^2)\,, \qquad \tilde{g}_{11} = -\rho\,, \qquad \tilde{g}_{01} = -\frac{1}{2}\rho W\,. \tag{6.4}$$

For one-soliton solution (3.10), when $W = v$, this metric becomes of the form (6.3) (with v instead of V), but with stationary space–time geometry $\rho = \rho(\xi)$, $V_Q = V_Q(\xi)$.

Generically, the metric (6.4) contains off-diagonal terms. The time synchronization in this space–time is possible if function $2W/(V_Q^2 - W^2)$ is integrable. Then we define new time coordinate $d\tau = dt - 2W/(V_Q^2 - W^2)d\xi$ and obtain Schwarzschild type black hole metric

$$ds^2 = \rho\left[\frac{1}{4}(V_Q^2 - W^2)(d\tau)^2 - \frac{V_Q^2}{V_Q^2 - W^2}(d\xi)^2\right]. \tag{6.5}$$

From this metric, following the same arguments as for black holes, the Hawking temperature can be derived. In particular, for one-soliton solution all calculations can be done in explicit form. Synchronization of the stationary metric is given by the above transformation of time, integrated as

$$\tau = f(\xi, t) = t + \frac{v}{2k^3(1-\gamma^2)}\left[-k\xi + \frac{1}{2|\gamma|}\ln\left|\frac{|\gamma| + \tanh k\xi}{|\gamma| - \tanh k\xi}\right|\right],$$

where $|\gamma| \equiv |v/2k| < 1$. Then, the Hawking temperature is $T_H = \frac{1}{2\pi}k^2(1-\gamma^2)$. For $|v| = 2|k|$ it vanishes similarly to the extremal black hole.[8]

7. Resonance Interaction of Solitons

Our task is now to apply the Hirota bilinear method and construct two soliton/dissipaton solution of (3.8)/(3.7). It is straightforward to show that RD system

(3.7) admits the following bilinear representation:

$$(\pm D_t - D_x^2)(G^\pm \circ F) = 0\,, \qquad D_x^2(F \circ F) = -2G^+G^-\,, \tag{7.1}$$

where three new real functions $G^\pm$ and F are defined by $e^\pm = (-8/\Lambda)^{1/2}G^\pm/F$, with the corresponding form for the product $e^+e^- = (8/\Lambda)\partial_x^2(\log F)$.

The following solution of system (7.1)

$$G^\pm = \pm e^{\eta_1^\pm}\,, \qquad F = 1 + e^{\eta_1^+ + \eta_1^- + \phi_{1,1}}\,, \qquad e^{\phi_{1,1}} = (k_1^+ + k_1^-)^{-2}\,, \tag{7.2}$$

where $\eta_1^\pm \equiv k_1^\pm x \pm (k_1^\pm)^2 t + \eta_1^{\pm(0)}$, and $k_1^\pm$, $\eta_1^{\pm(0)}$, are arbitrary constants, gives the one-dissipaton of Eqs. (3.7). In terms of redefined parameters, $k \equiv (k_1^+ + k_1^-)/2$, $v \equiv -(k_1^+ - k_1^-)$ it acquires the form (3.8). Depending on the boundary conditions, three types of dissipaton solutions exist. We consider one-dissipaton boundary conditions and compare them with the horizon condition (5.5). In the space of parameters (v, k) there exist critical value $v_c = 2k$, such that when $v < v_c$, for solution (7.2) one has $e^\pm \to 0$ at infinities. So, the vanishing boundary conditions for a dissipaton are equivalent to the black hole (BH) existence. We call the corresponding heavy quasi-particle as the BH dissipaton. At the critical value $v = 2k$ the solution is a kink steady state in the moving frame $e^\pm = \pm k e^{\pm k\xi_0}(1 \mp \tanh k\xi)$, with constant asymptotics $e^\pm \to \pm 2k e^{\pm k\xi_0}$ for $x \to \mp\infty$ and $e^\pm \to \pm 0$ for $x \to \pm\infty$. In the last case we have the *extremal* black hole or the EBH dissipaton. In the over-critical case $v > v_c$, $e^\pm \to \pm\infty$ for $x \to \mp\infty$ and $e^\pm \to \pm 0$ for $x \to \pm\infty$, no black hole exists and we have very fast and light quasi-particles called the LD.

For two-dissipaton solution we have

$$G^\pm = \pm\left[e^{\eta_1^\pm} + e^{\eta_2^\pm} + \frac{(k_1^\pm - k_2^\pm)^2}{(k_{21}^{\pm\mp}k_{11}^{+-})^2}e^{\eta_1^+ + \eta_1^- + \eta_2^\pm} + \frac{(k_1^\pm - k_2^\pm)^2}{(k_{12}^{\pm\mp}k_{22}^{+-})^2}e^{\eta_2^+ + \eta_2^- + \eta_1^\pm}\right], \tag{7.3a}$$

$$\begin{aligned} F = 1 &+ \frac{e^{\eta_1^+ + \eta_1^-}}{(k_{11}^{+-})^2} + \frac{e^{\eta_1^+ + \eta_2^-}}{(k_{12}^{+-})^2} + \frac{e^{\eta_2^+ + \eta_1^-}}{(k_{21}^{+-})^2} + \frac{e^{\eta_2^+ + \eta_2^-}}{(k_{22}^{+-})^2} \\ &+ \frac{(k_1^+ - k_2^+)^2(k_1^- - k_2^-)^2}{(k_{12}^{+-}k_{21}^{+-}k_{11}^{+-}k_{22}^{+-})^2}e^{\eta_1^+ + \eta_1^- + \eta_2^+ + \eta_2^-}\,, \end{aligned} \tag{7.3b}$$

where $k_{ij}^{ab} \equiv k_i^a + k_j^b$, $\eta_i^\pm \equiv k_i^\pm x \pm (k_i^\pm)^2 t + \eta^{\pm(0)}$. The degenerate case of this solution, when $k_1^+ = k_1^- \equiv p_1$, $k_2^+ = k_2^- \equiv p_2$, can be simplified in the form

$$e^\pm = \pm\left(\frac{8}{-\Lambda}\right)^{1/2} p_+p_- \frac{p_1\cosh\theta_2 e^{\pm p_1^2 t} + p_2\cosh\theta_1 e^{\pm p_2^2 t}}{p_-^2\cosh\theta_+ + p_+^2\cosh\theta_- + 4p_1p_2\cosh(p_+p_-t)}\,, \tag{7.4}$$

where $p_\pm \equiv p_1 \pm p_2$, $\theta_\pm \equiv \theta_1 \pm \theta_2$, $\theta_i \equiv p_i(x - x_{0i})$, $(i = 1, 2)$. It describes a collision of two dissipatons with identical amplitudes $p_+/2$, moving in opposite directions with equal velocities $|v| = |p_-|$, and creating the resonance bound state. The lifetime of this state,[8] $\Delta T \approx 2p_2 d/p_+p_-$, linearly depends on the relative distance d, where $x_{01} = 0$, $x_{02} = d$.

In a more general case, treatable analytically, when $k_i^{\pm} > 0$, $(i = 1, 2)$, and $k_1^+ - k_1^- > 0$, $k_2^+ - k_2^- > 0$, $k_1^+ - k_2^- > 0$, $k_2^+ - k_1^- < 0$, solution (7.3) describes collision of two dissipatons with amplitudes $k_{12}^{+-}/2$ and $k_{21}^{+-}/2$ and velocities $v_{12} = -(k_1^+ - k_2^-)$ and $v_{21} = -(k_2^+ - k_1^-)$, correspondingly. Depending on the relative position's shift, also in this general case the resonance states can be created.

As a simplest example we consider conditions for decay of BH dissipaton at rest $(v = 0)$ on two dissipatons (3.8) with parameters (k_1, v_1) and (k_2, v_2). From the mass, momentum and energy conservation laws (5.6) and condition (5.7) we obtain the following relations $v_1^2 = 4k_2^2$, $v_2^2 = 4k_1^2$. It is obvious that two possibilities exist:

(a) $|v_1| = |v_2|$. In this case $|k_1| = |k_2|$, and both dissipatons have equal masses $M_1 = M_2 = M/2$ and velocities, satisfying the critical values $v_i^2 = 4k_i^2$, $(i = 1, 2)$, clearly corresponding to two EBH.

(b) $|v_1| > |v_2|$ (without lose of generality). In this case $v_1^2 > 4k_1^2$ and $v_2^2 < 4k_2^2$, so that the initial BH decays on one BH and one LD dissipaton.

The process of creation of resonant BH is illustrated in Fig. 1. Figure 2 shows interaction of two BH dissipatons by exchange of LD. A more complicated interaction, creating two resonant BHs (the Feynman diagram with four vertices) is shown in Fig. 3. Detailed description of various interactions simulated by MATHEMATICA would be published elsewhere. We would like to just emphasize here that the resonance of dissipatons/solitons occurs when coefficient in the last term of Eq. (7.3b) vanishes or become infinite, similarly to the resonance equation considered in Ref. 22. As easy to check by direct substitution, this condition is satisfied under the conservation laws (5.6) constrained with Eq. (5.8).

8. Black Holes as Topological Solitons

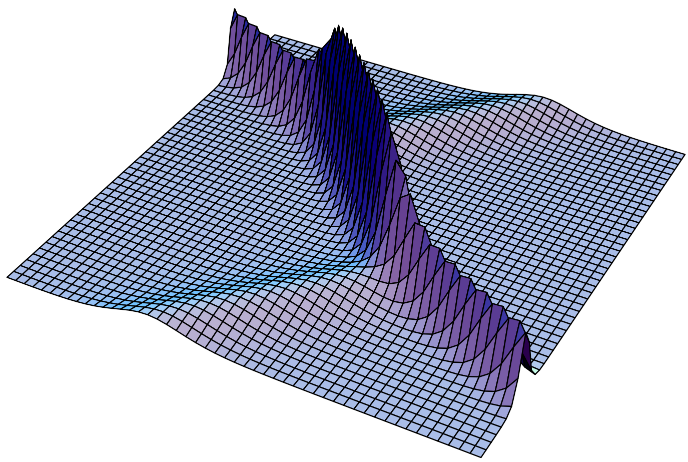
The gauge fixing condition (4.2) defines classical SO(2, 1)/SO(1, 1) Heisenberg model on the anti-de Sitter space $(\Lambda < 0)$,

$$\partial_0 \mathbf{s} = \mathbf{s} \wedge \partial_x^2 \mathbf{s}\,, \tag{8.1}$$

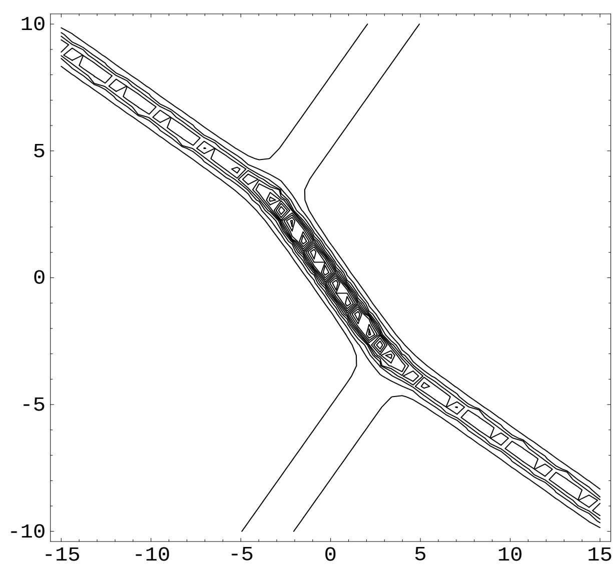
where $e_\mu^{\pm}$ play the role of local coordinates in the tangent plane $\partial_\mu \mathbf{s} = (\frac{-\Lambda}{8})^{1/2}(e_\mu^+ \mathbf{n}_- - e_\mu^- \mathbf{n}_+)$, so that the metric tensor is

$$g_{\mu\nu} = \left(\frac{2}{-\Lambda}\right)(\partial_\mu \mathbf{s} \partial_\nu \mathbf{s}) \tag{8.2}$$

and $\mathbf{s}^2 = -(S^1)^2 + (S^2)^2 - (S^3)^3 = -1$. Below we establish gauge equivalence of RD system (3.7) to Eq. (8.1), allowing us to construct an exact solution for the last one. The solution provides simple geometrical visualization of the event horizon position and allow us to interpret the black hole as a topological soliton. Furthermore, under



(a)



(b)

Fig. 1. (a) 3D plot of two dissipatons resonance-type collision for $k_1^+ = 2$, $k_1^- = 1$, $k_2^+ = 0.001$, $k_2^- = 0.001$ in the (x,t)-plane. (b) Contour plot of two dissipatons collision with BH resonance in the (x,t)-plane.

collision they show the similar resonance properties as dissipatons of (3.7). In the matrix representation for $S \in \mathrm{SO}(2,1)$ we have

$$S = i \begin{pmatrix} S^3 & S^- \\ S^+ & -S^3 \end{pmatrix} = (\mathbf{s}, \tau) = g\tau_3 g^{-1}\,, \tag{8.3}$$

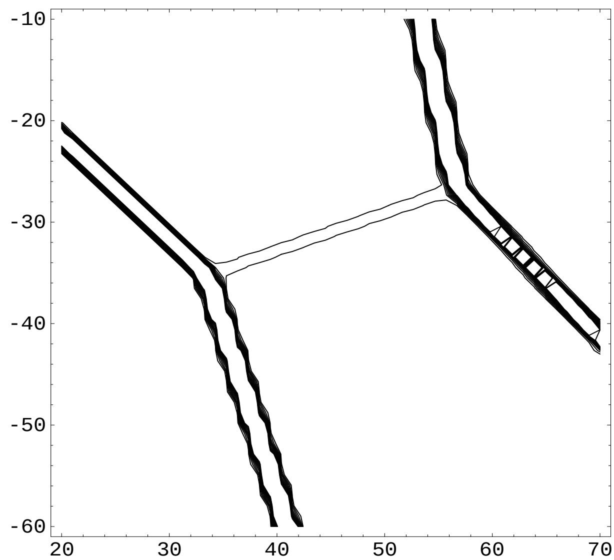


Fig. 2. Contour plot of two BH dissipatons exchange-type collision for $k_1^+ = 2$, $k_1^- = 1$, $k_2^+ = -1.7$, $k_2^- = -1.9$ and $d = 50$ in the (x, t)-plane.

where $S^\pm = S^1 \pm S^2$, $S^2 = -I$, $\det S = 1$, and Eq. (8.1) acquires the standard form

$$\partial_t S = \frac{1}{2i}[S, \partial_x^2 S]\,, \tag{8.4}$$

with the Lax pair

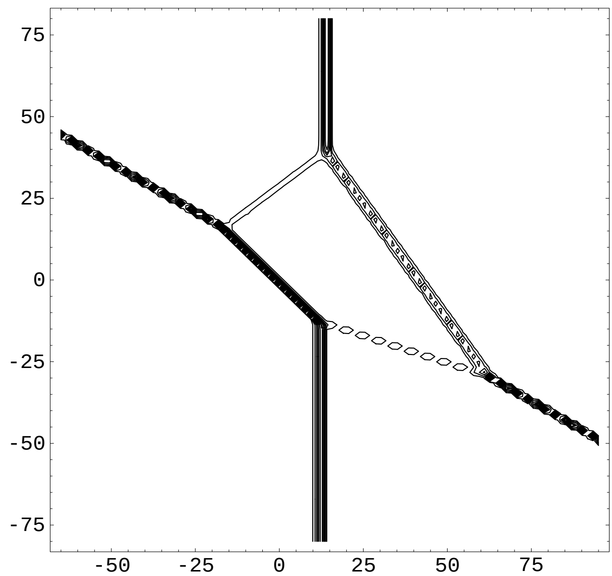
$$J_1^{HM} = \frac{i}{4}\lambda S\,, \qquad J_0^{HM} = \frac{i}{8}\lambda^2 S + \frac{\lambda}{4} S\partial_x S\,. \tag{8.5}$$

This model is gauge equivalent (in the sense of integrable models) to the resonance NLS (3.10) and RD (3.7). Although the Lax pair for the first one has been derived,[23] it has quite complicated structure, while for the second one it is just of a real Zakharov–Shabat form

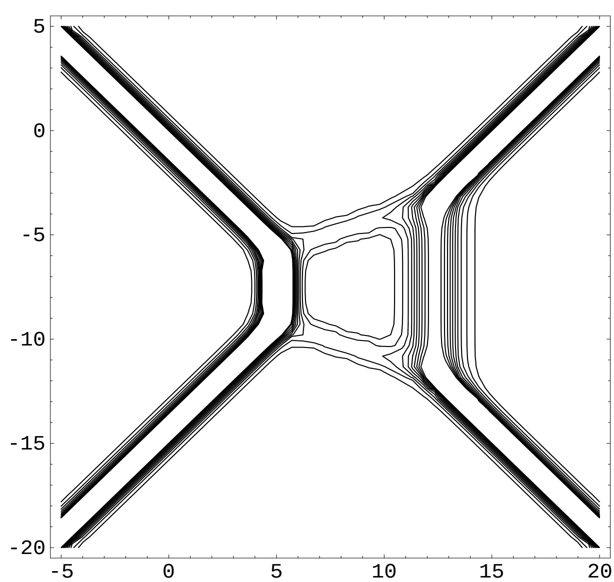
$$J_1^{RD} = \begin{pmatrix} \frac{1}{4}\lambda & q^- \\ q^+ & -\frac{1}{4}\lambda \end{pmatrix}, \qquad J_0^{RD} = \begin{pmatrix} \frac{1}{8}\lambda^2 - q^+q^- & -(\partial_x - \frac{1}{2}\lambda)q^- \\ (\partial_x + \frac{1}{2}\lambda)q^+ & -\frac{1}{8}\lambda^2 + q^+q^- \end{pmatrix}, \tag{8.6}$$

for two independent functions $q^\pm \equiv (\frac{-\Lambda}{8})^{1/2} e^\pm$. The gauge equivalence is implemented by non-Abelian transformation $J_\mu^{HM} = gJ_\mu^{RD}g^{-1} - \partial_\mu g g^{-1}$, $(\mu = 0, 1)$, where matrix $g(x,t)$ is a solution of the linear problem $\partial_\mu g = gJ_\mu^{RD}(\lambda = 0)$. To construct the "magnetic" analog of dissipaton (3.9) we solve this system first. The result is

$$g(x,t) = \begin{pmatrix} \tanh z - \gamma & \frac{1}{\cosh z} e^{\gamma z - k^2(1-\gamma^2)t} \\ -\frac{1}{\cosh z} e^{-\gamma z + k^2(1-\gamma^2)t} & \tanh z + \gamma \end{pmatrix}, \tag{8.7}$$



(a)



(b)

Fig. 3. (a) Contour plot of two BH dissipatons four vertex-type collision for $k_1^+ = 2$, $k_1^- = 1$, $k_2^+ = 1$, $k_2^- = 0.3$ and $d = 40$ in the (x, t)-plane. (b) Contour plot of creation of two resonant BH dissipatons for $k_1^+ = 2$, $k_1^- = 1$, $k_2^+ = 1$, $k_2^- = 2$ and $d = 15$.

where $z \equiv k(x - vt)$, $\gamma \equiv v/2k$, $\det g = 1 - \gamma^2$. Then, from Eq. (8.3) we find solution of Eq. (8.4),

$$S^3 = -1 + \frac{2}{1-\gamma^2}\cosh^{-2} z\,, \tag{8.8a}$$

$$S^- = \frac{2}{1-\gamma^2}\cosh^{-1} z(\tanh z - \gamma)e^{\gamma z - k^2(1-\gamma^2)t}\,, \tag{8.8b}$$

$$S^+ = \frac{2}{1-\gamma^2}\cosh^{-1} z(\tanh z + \gamma)e^{-\gamma z + k^2(1-\gamma^2)t}\,, \tag{8.8c}$$

describing magnetic (curved) analog of dissipaton (3.8). Indeed, in moving frame with velocity v, $z = \text{const.}$, and the S^3 component is time-independent, while S^- and S^+ are decaying and growing with time. As well as for dissipaton (3.8), properties of solution (8.8) essentially depend on the velocity v. We have the following cases.

(a) $\gamma^2 < 1$, or $|v| < 2|k|$, then $-1 \le S^3 \le \frac{1+\gamma^2}{1-\gamma^2}$. At any fixed time t, components S^+ and S^- vanish when $z \to \pm\infty$, while $S^3 \to -1$. Due to the boundary value $\mathbf{s} = (0, 0, -1)$, when $z \to \pm\infty$, the real line $R = \{z\}$ is compactified. Since the hyperboloid $\mathbf{s}^2 = -(S^1)^2 + (S^2)^2 - (S^3)^2 = -1$ has topology of cylinder $R \times S^1$, solution (8.8) describes $S^1 \to S^1$ mapping of degree one. Therefore, solution (8.8) is a topological soliton, traveling with a constant velocity v. When $z = z_H$, so that $\tanh z = \pm\gamma$, one of the components S^+ or S^- vanishes and in the metric (8.2) component $g_{00} = 0$. Hence, we have the event horizon at the same position as for dissipaton in Eq. (5.5) (see Fig. 4). Since any topological soliton configuration crosses one of the lines $S^+ = 0$ and $S^- = 0$ at least once, intersection points correspond to the event horizon. The last result relates the black hole solution with topological soliton on the hyperboloid.

(b) $\gamma^2 > 1$, or $|v| > 2|k|$. In this case $-\frac{1+\gamma^2}{1-\gamma^2} \le S^3 \le -1$. At $z \to \pm\infty$ one of the components S^+ or S^- grows exponentially. Moreover, asymptotics at $+\infty$ and $-\infty$ are orthogonal. Therefore, solution (8.8) has trivial winding. It can be considered as the turning traveling wave, which never cross the asymptotic lines at finite distance. So, in this case event horizon and black holes do not exist.

From the above analysis one can see that the existence of the event horizon closely relates to topologically nontrivial solutions of model (8.1). If we calculate metric (8.2) for one-soliton solution (8.8)

$$ds^2 = \frac{8}{-\Lambda}(k^2\cosh^{-2} z)[(k^2(\tanh^2 z - \gamma^2))(dt)^2 - k^{-2}(dz)^2 - 2\gamma\, dz\, dt]\,, \tag{8.9}$$

in the synchronized frame (z, T), defined by (see Eq. (6.6))

$$T = f(z,t) = t + \frac{\gamma}{k^2(1-\gamma^2)}\left[-z + \frac{1}{2|\gamma|}\ln\left|\frac{|\gamma| + \tanh z}{|\gamma| - \tanh z}\right|\right],$$

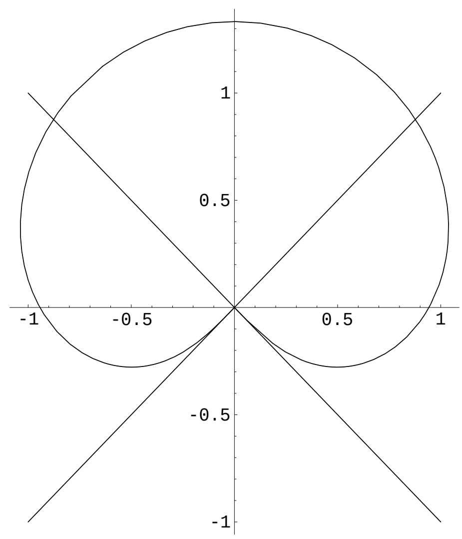


Fig. 4. Parametric plot of topological soliton projection on (S^1, S^2) plane for $k = 1$, $\gamma = 0.5$. Positions of the BH event horizon correspond to intersection points with $S^+ = 0$ and $S^- = 0$ lines, while asymptotics on $\pm\infty$ to the beginning of coordinates $(0, 0)$.

it becomes of the black hole form

$$ds^2 = \frac{8}{-\Lambda}(k^2 \cosh^{-2} z)\left[(k^2(\tanh^2 z - \gamma^2))(dT)^2 - \frac{\tanh^2 z}{k^2(\tanh^2 - \gamma^2)}(dz)^2\right] \quad (8.10)$$

and shows a horizon singularity at $\tanh z = \pm\gamma$, only if $|v| \le 2|k| \equiv |v_{\max}|$. Consequently, the topological soliton cannot move faster than the maximal value of velocity $|v_{\max}| = 2|k|$. The event horizon singularity is removable in the Kruskal–Szekeres (KS) coordinates, defined in our case as

$$v = e^{r_H^2(R+T)}, \qquad u = -e^{r_H^2(R-T)}, \qquad (r > r_H);$$

$$v = e^{r_H^2(R+T)}, \qquad u = +e^{r_H^2(R-T)}, \qquad (r > r_H),$$

where $R = 2^{-1} r_H^{-2} \ln|1 - r_H^2/r^2|$, $r = |k| \cosh^{-1} z$, $r_H^2 = k^2(1-\gamma^2)$. Then, the metric has regular form of the anti-de Sitter space[8,24]

$$ds^2 = \frac{8}{-\Lambda}\frac{du\, dv}{(1 - uv)^2},$$

where the black hole event horizon corresponds to the diagonal lines $u = 0$ and $v = 0$. These diagonal lines are equivalent to the above conditions for the spin vector, $S_+ = 0$ and $S_- = 0$. Thus, soliton (8.8) provides a simple geometrical visualization of event horizon's position, the same as in the KS coordinates, and allow treating of black holes as the topological solitons. It also exhibits a global meaning of black holes in the JT gravity. Since dissipatons of (3.7) create the resonance states under collisions, we expect that similar states would also appear for solitons collision in Eq. (8.1). Due to the equivalence of our coordinates with KS ones for asymptotically isolated soliton, they can be useful tool in construction of causal picture for black holes collision. This question is under further investigation.

9. Conclusions

The black hole picture with its resonance interaction as described above can be applied to physical models of slowly varying quasi-monochromatic wave, in nonlinear media with the sign of indefinite dispersion. The $(1+1)$-dimensional case can be realized particularly from nonlinear optics. The great variety of optical solitons is due to the various properties of media involved, including nonlinearity, material and geometric dispersion, passive or active properties, etc. The crucial role in soliton properties plays the group-velocity dispersion of the optical fibers, depending not only on the glass material involved, but also on the fiber waveguide property. According to the sign of dispersion, there are two types of known NLS — defocusing and focusing cases, that correspondingly admit "dark" and "bright" solitons. Since the wave function is a complex quantity, the quadratic dispersion, in general, consists of two parts: phase dispersion and absolute value dispersion. The former corresponds to geometrical optics, while the latter is responsible for diffraction. But for both focusing and defocusing cases, contributions from the phase and the absolute value dispersions have the same sign (positive and negative correspondingly). Here we have considered a nonlinear media with the sign of indefinite quadratic dispersion, resulting in competition between opposite sign contributions from the phase and the absolute value dispersion. However, in principle, another mechanism could exist to change the sign of dispersion, for example by multiplication of dispersion with some function of space–time. In a very recent paper Clarke *et al.*[28] consider a *dispersion-decreasing* fiber, being an efficient tool for compression of optical pulses, and described it by NLS type equation with variable dispersion coefficient. Simulation of a wave pulse passing a point, where the dispersion coefficient changes its sign from focusing to defocusing, demonstrates that in the focusing region the pulse keeps a soliton-like shape until it is close to the zero-dispersion point. However, after the passage of this point, depending on the pulse's energy, it decays into radiation or into a double-humped structure. Although this model is different from the one that we consider here, the qualitative behavior of soliton solution is quite similar. In both models dispersion changes the sign and the single soliton, passing the point where it is happening, decays on the pair of pulses (solitons, in our case). It indicates on interpretation of points where dispersion changes the sign with event horizon and the relevance of black hole's language for the corresponding *dispersion-managed* type phenomena in nonlinear optics.

Finally, we note that application of quantum potential to planar system with Chern–Simons vortex/soliton configuration, leads to a new phenomena such as quantization of the potential strength s and the Chern–Simons coupling constant.[26] In $1+1$ dimensions,[29] this model reduces to the problem of NLS soliton in quantum potential, studied here.

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