Theoretical and Mathematical Physics, Vol. 127, No. 3, pp. 779–788, 2001

SELF-DUAL VORTICES IN CHERN–SIMONS HYDRODYNAMICS

J. H. Lee[1] and O. K. Pashaev[2]

The classical theory of a nonrelativistic charged particle interacting with a $U(1)$ gauge field is reformulated as the Schrödinger wave equation modified by the de Broglie–Bohm nonlinear quantum potential. The model is gauge equivalent to the standard Schrödinger equation with the Planck constant $\hbar$ for the deformed strength $1-\hbar^2$ of the quantum potential and to the pair of diffusion–antidiffusion equations for the strength $1+\hbar^2$. Specifying the gauge field as the Abelian Chern–Simons (CS) one in 2+1 dimensions interacting with the nonlinear Schrödinger (NLS) field (the Jackiw–Pi model), we represent the theory as a planar Madelung fluid, where the CS Gauss law has the simple physical meaning of creation of the local vorticity for the fluid flow. For the static flow when the velocity of the center-of-mass motion (the classical velocity) is equal to the quantum velocity (generated by the quantum potential velocity of the internal motion), the fluid admits an N-vortex solution. Applying a gauge transformation of the Auberson–Sabatier type to the phase of the vortex wave function, we show that deformation parameter $\hbar$, the CS coupling constant, and the quantum potential strength are quantized. We discuss reductions of the model to 1+1 dimensions leading to modified NLS and DNLS equations with resonance soliton interactions.

1. Introduction

The generalization of the Schrödinger equation by the "quantum potential" nonlinear term was considered in connection with a stochastic quantization problem [1] and corrections to quantum mechanics from quantum gravity effects [2]. It also appears in the wave theory formulation of classical mechanics [3] and the dispersionless limit of nonlinear wave dynamics [4]. Sabatier [5] showed that this extension preserves the Lagrangian structure. Moreover, Auberson and Sabatier [6] found a transformation of the phase of the wave function that linearizes the model and reduces it, depending on the strength of the coupling coefficient for the quantum potential, to the Schrödinger equation with a rescaled potential or to a pair of time-reversed diffusion equations. Because of this linearization, soliton solutions of this equation do not exist [5, 6].

We recently considered the nonlinear version of the Bohm formulation of quantum mechanics [7], namely, the problem of the nonlinear Schrödinger (NLS) soliton under the influence of the quantum potential [4, 8]. Applying a phase transformation of the Auberson–Sabatier type to this problem with an overcritical strength of the quantum potential allowed reducing the problem to a pair of time-reversed reaction-diffusion equations, representing an imaginary-time version of the real q-r NLS-type system [9] ($SL(2,R)$ reduction of the Zakharov–Shabat problem). Constructing a two-soliton solution, we found a resonance character of their mutual interaction [4, 8].

In the present paper, we consider the influence of the quantum potential on the planar vortex in the (2+1)-dimensional problem for the NLS equation interacting with the Chern–Simons (CS) gauge field. Applying a transformation of the Auberson–Sabatier type to the phase of the wave function dramatically changes the parameters of the vortex configurations. In the Madelung representation, we reformulate the model as a rotational planar hydrodynamics. The self-dual limit, admitting N-vortex solutions, then has

[1]Institute of Mathematics, Academia Sinica, Taipei, Taiwan.
[2]Department of Mathematics, Izmir Institute of Technology, Cankaya-Izmir, Turkey; Joint Institute for Nuclear Research, Dubna, Moscow Oblast, Russia.

Translated from Teoreticheskaya i Matematicheskaya Fizika, Vol. 127, No. 3, pp. 432–443, June, 2001.

a simple physical interpretation as the condition for equality of the "classical velocity" (velocity of the center-of-mass) and the "quantum" velocity (velocity of the motion in the center-of-mass frame associated with the internal "spin motion" or *zitterbewegung*).

In Sec. 2, we reformulate the classical dynamics of a charged particle interacting with an Abelian gauge field as an NLS-type wave equation. Properly deforming the strength of the quantum potential, we recover the standard Schrödinger equation, where the deformation parameter plays the role of the Planck constant. In Sec. 3, we specify the gauge field as an Abelian CS one interacting with the NLS equation and derive the corresponding rotational Madelung-type hydrodynamics and its dispersionless limit and deformations. In Sec. 4, we study the quantum velocity and its properties. For the static flow moving with a velocity equal to the quantum velocity, we reduce the problem to the Liouville equation and describe the corresponding vortex configurations. From the conditions for nonsingularity and single-valuedness, we find the quantization condition for the coupling constants. In Sec. 5, we consider dimensional reduction to the one-dimensional NLS equation and its modification by the quantum potential. In the conclusion, we briefly discuss our results.

2. Nonlinear wave equation of classical dynamics

The classical dynamics of a charged nonrelativistic particle in the U(1) gauge field $A_\mu = (A_0, \mathbf{A})$ with the Hamilton function

$$H = \frac{\mathbf{p}^2}{2m} + \frac{e}{c}A_0 + U$$

is described by the Hamilton–Jacobi equation

$$\frac{\partial S}{\partial t} + H(\nabla S, A_0, \mathbf{A}, U) = 0 \tag{2.1}$$

with the momentum

$$\mathbf{p} = \nabla S + \frac{e}{c}\mathbf{A}.$$

Combining (2.1) with the Liouville equation $\partial\rho/\partial t + \nabla(\rho\mathbf{V}) = 0$ for the density ρ of the integral invariant in the gradient dynamic system

$$\dot{\mathbf{x}} = \mathbf{V} = \frac{1}{m}\mathbf{p} = \frac{1}{m}\left(\nabla S + \frac{e}{c}\mathbf{A}\right), \tag{2.2}$$

we obtain the system of equations

$$\frac{\partial S}{\partial t} + \frac{1}{2m}\left(\nabla S + \frac{e}{c}\mathbf{A}\right)^2 + \frac{e}{c}A_0 + U = 0, \tag{2.3a}$$

$$\frac{\partial \rho}{\partial t} + \nabla(\rho\mathbf{V}) = 0. \tag{2.3b}$$

This classical system is representable in the wave form. Introducing the complex wave function ("order parameter")

$$\psi = \sqrt{\rho}\, e^{iS}, \tag{2.4}$$

we rewrite system (2.3) as the single nonlinear wave equation

$$iD_0\psi + \frac{1}{2m}\mathbf{D}^2\psi - U\psi = \frac{1}{2m}\frac{\Delta|\psi|}{|\psi|}\psi, \tag{2.5}$$

where $D_0 = \partial_t + eA_0/c$ and $\mathbf{D} = \nabla + e\mathbf{A}/c$. The last equation has the form of the Schrödinger equation (without a Planck constant) modified by the so-called quantum potential term in the right-hand side. It admits all the usual solutions of classical mechanics but does not allow superpositions of these solutions. Because system (2.3) describes the formal semiclassical limit of the quantum mechanical Schrödinger equation, Eq. (2.5) can be considered its dispersionless limit. In fact, wave equation (2.5) is covariant under the gauge transformations

$$\psi \to \psi e^{i\alpha}, \qquad \mathbf{A} \to \mathbf{A} - \frac{c}{e}\nabla\alpha,$$

generating the shift of the classical action $S \to S + \alpha$. Being U(1) gauge invariant, the additional term in the right-hand side of (2.5) then completely compensates the corresponding gauge-invariant contribution from the dispersion in the left-hand side.

As mentioned above, Eq. (2.5) does not contain a Planck constant. But if we consider the contribution from the quantum potential in the right-hand side of Eq. (2.5) deformed by a constant $\hbar^2$,

$$iD_0\psi + \frac{1}{2m}\mathbf{D}^2\psi - U\psi = \frac{1}{2m}(1-\hbar^2)\frac{\Delta|\psi|}{|\psi|}\psi, \tag{2.6}$$

then in terms of the new wave function

$$\chi = \sqrt{\rho}\, e^{iS/\hbar}, \tag{2.7}$$

we recover the standard linear Schrödinger equation

$$i\hbar D_0\chi + \frac{\hbar^2}{2m}\mathbf{D}^2\chi - U\chi = 0,$$

where $\hbar$ plays the role of the Planck constant. For $\hbar \neq 0$, Eq. (2.6) is therefore gauge equivalent to the Schrödinger equation; for $\hbar = 0$, it reduces to nonlinear wave equation (2.5) of classical mechanics. Moreover, for $\hbar = \pm 1$, it reduces directly to the linear Schrödinger equation or its complex conjugate.

On the other hand, if the deformation of Eq. (2.5) appears with the opposite sign as

$$iD_0\psi + \frac{1}{2m}\mathbf{D}^2\psi - U\psi = \frac{1}{2m}(1+\hbar^2)\frac{\Delta|\psi|}{|\psi|}\psi, \tag{2.8}$$

then it cannot be linearized to the form of the Schrödinger equation by transformation (2.7) in the same classical limit $\hbar = 0$ as for Eq. (2.6). However, Eq. (2.8) can be reduced to Eq. (2.6) by the formal analytic substitution of the purely imaginary value $\hbar \to i\hbar$ for the Planck constant (in quantum mechanics, a similar continuation to the classically inaccessible region leads to an exponentially decaying (growing) wave function). Then, written in terms of the two real functions

$$Q^{\pm} = \sqrt{\rho}\, e^{\pm S/\hbar},$$

Eq. (2.8) and its complex conjugate become the pair of decoupled diffusion–antidiffusion equations

$$\pm\hbar D_0 Q^{\pm} + \frac{\hbar^2}{2m}\mathbf{D}^2 Q^{\pm} - UQ^{\pm} = 0,$$

similar to the one Schrödinger considered in 1931 [10]. From the above consideration, we see that the Schrödinger equation perturbed by a quantum potential includes the classical mechanics ($\hbar = 0$), the quantum mechanics ($\hbar = \pm|\hbar|$), and the pair of diffusion–antidiffusion equations ($\hbar = i|\hbar|$) as particular cases.

3. Chern–Simons hydrodynamics

The semiclassical limit was recently applied to the defocusing NLS equation

$$i\hbar\partial_t\chi + \frac{\hbar^2}{2m}\Delta\chi + 2g|\chi|^2\chi = 0, \quad g < 0, \tag{3.1}$$

in one [11] and two space dimensions [12]; it provides an analytic tool for describing shock waves in nonlinear optics and vortices in a superfluid. Decomposing the wave function as in (2.7), we derive the quantum deformation of the Hamilton–Jacobi equation by the quantum potential or, after differentiation with respect to the space coordinates, the Madelung fluid. In the formal semiclassical limit $\hbar \to 0$ (before shocks appear), we neglect the contribution from the quantum potential, and the fluid becomes the Euler system. In terms of wave function (2.4), we then have the dispersionless NLS equation

$$i\partial_t\psi + \frac{1}{2m}\Delta\psi + 2g|\psi|^2\psi = \frac{1}{2m}\frac{\Delta|\psi|}{|\psi|}\psi. \tag{3.2}$$

The quantum deformation of Eq. (3.2) in the form

$$i\partial_t\psi + \frac{1}{2m}\Delta\psi + 2g|\psi|^2\psi = (1-\hbar^2)\frac{1}{2m}\frac{\Delta|\psi|}{|\psi|}\psi,$$

reformulated for wave function (2.7), again leads to original equation (3.1).

Interacting with the CS gauge field in 2+1 dimensions, NLS model (3.1) is called the Jackiw–Pi (JP) model and describes anyonic phenomena [13]. The semiclassical limit of anyons requires studying this model in the limit as $\hbar \to 0$ or, similarly to the case of Eq. (3.2), its perturbations by the quantum potential.

To describe the deformed theory, we consider the Lagrangian

$$L = \frac{\kappa}{2}\epsilon^{\mu\nu\lambda}A_\mu\partial_\nu A_\lambda + \frac{i}{2}(\bar\psi D_0\psi - \psi\overline{D}_0\bar\psi) - \frac{1}{2m}|\mathbf{D}\psi|^2 + (1-\hbar^2)\frac{1}{2m}(\nabla|\psi|)^2 + g|\psi|^4, \tag{3.3}$$

where $D_\mu = \partial_\mu + ieA_\mu/c$, leading to the system of equations of motion

$$iD_0\psi + \frac{1}{2m}\mathbf{D}^2\psi + 2g|\psi|^2\psi = (1-\hbar^2)\frac{1}{2m}\frac{\Delta|\psi|}{|\psi|}\psi, \tag{3.4a}$$

$$\partial_1 A_2 - \partial_2 A_1 = \frac{e}{\kappa c}\bar\psi\psi, \tag{3.4b}$$

$$\partial_0 A_j - \partial_j A_0 = -\frac{ie}{2mc\kappa}\epsilon_{jk}(\bar\psi D_k\psi - \psi\overline{D}_k\bar\psi). \tag{3.4c}$$

Factoring the wave function ψ given by Eq. (2.4) and introducing the new function χ given by Eq. (2.7), we obtain the JP model:

$$i\hbar\left(\partial_0 + \frac{ie}{\hbar c}A_0\right)\chi + \frac{\hbar^2}{2m}\left(\nabla + \frac{ie}{\hbar c}\mathbf{A}\right)^2\chi + 2g|\chi|^2\chi = 0, \tag{3.5a}$$

$$\partial_1 A_2 - \partial_2 A_1 = \frac{e}{\kappa c}\bar\chi\chi, \tag{3.5b}$$

$$\partial_0 A_j - \partial_j A_0 = -\frac{ie\hbar}{2mc\kappa}\epsilon_{jk}\left[\bar\chi\left(\partial_k + \frac{ie}{\hbar c}A_k\right)\chi - \chi\left(\partial_k - \frac{ie}{\hbar c}A_k\right)\bar\chi\right]. \tag{3.5c}$$

The corresponding Lagrangian

$$L = \frac{\kappa}{2}\epsilon^{\mu\nu\lambda}A_\mu\partial_\nu A_\lambda + \frac{i\hbar}{2}\left[\bar{\chi}\left(\partial_0 + \frac{ie}{\hbar c}A_0\right)\chi - \chi\left(\partial_0 - \frac{ie}{\hbar c}A_0\right)\bar{\chi}\right] -$$

$$- \frac{\hbar^2}{2m}\left(\nabla - \frac{ie}{\hbar c}\mathbf{A}\right)\bar{\chi}\left(\nabla + \frac{ie}{\hbar c}\mathbf{A}\right)\chi + g|\chi|^4 \tag{3.6}$$

follows from (3.3). In system of equations (3.5), the deformation parameter $\hbar$ plays a role similar to the Planck constant. Both systems (3.4) and (3.5) admit the same hydrodynamic (Madelung-type) representation. From Eq. (3.4a), we obtain the quantum Hamilton–Jacobi equation

$$\frac{\partial S}{\partial t} + \left[\frac{mV^2}{2} + \frac{e}{c}A_0 - 2g\rho - \frac{\hbar^2}{2m}\frac{\Delta\sqrt{\rho}}{\sqrt{\rho}}\right] = 0 \tag{3.7}$$

and continuity equation (2.3b), where we introduce local velocity field (2.2) Then Eqs. (3.5b) and (3.5c) become

$$\partial_1 A_2 - \partial_2 A_1 = \frac{e}{\kappa c}\rho, \tag{3.8}$$

$$\partial_0 A_j - \partial_j A_0 = \frac{e}{\kappa c}\epsilon_{jk}\rho V_k. \tag{3.9}$$

Now, we can completely exclude the vector potentials $\mathbf{A}$ from consideration in favor of velocity field (2.2). We note that this field is an explicitly gauge-invariant variable. From Eqs. (3.9) and (3.7), we then derive the Euler equation for the velocity $\mathbf{V}$,

$$\frac{\partial \mathbf{V}}{\partial t} + (\mathbf{V}\nabla)\mathbf{V} = -\frac{1}{m}\nabla P \tag{3.10}$$

with the pressure

$$P = -2g\rho - \frac{\hbar^2}{2m}\frac{\Delta\sqrt{\rho}}{\sqrt{\rho}}. \tag{3.11}$$

In terms of our hydrodynamic variables, CS Gauss law (3.8) becomes

$$\nabla \times \mathbf{V} = \frac{e^2}{m\kappa c^2}\,\rho, \tag{3.12}$$

which has the simple meaning of a rotational fluid such that the local vorticity is nonzero at any point of the fluid with a nonvanishing density ρ. System of equations (2.3b) and (3.10)–(3.12) determines the Madelung fluid for our model. Like the velocity field in (2.2), it is explicitly $U(1)$ gauge invariant. Moreover, continuity equation (2.3b) in this system is not independent. It appears as a consistency condition for CS Gauss law (3.12) during the evolution. To verify this, it is sufficient to simply differentiate (3.12) with respect to time and substitute the result in (3.10). We therefore have a hydrodynamic model defined by two equations:

$$\begin{aligned} &\frac{\partial \mathbf{V}}{\partial t} + (\mathbf{V}\nabla)\mathbf{V} = -\frac{1}{m}\nabla\left(-2g\rho - \frac{\hbar^2}{2m}\frac{\Delta\sqrt{\rho}}{\sqrt{\rho}}\right), \\ &\nabla \times \mathbf{V} = \frac{e^2}{m\kappa c^2}\rho. \end{aligned} \tag{3.13}$$

The semiclassical or dispersionless limit of this model as $\hbar \to 0$ is given by

$$\frac{\partial \mathbf{V}}{\partial t} + (\mathbf{V}\nabla)\mathbf{V} = -\frac{1}{m}\nabla(-2g\rho),$$

$$\nabla \times \mathbf{V} = \frac{e^2}{m\kappa c^2}\rho.$$

The nonlinear wave form of these equations follows directly from system (3.4), and the Lagrangian follows from Eq. (3.3) with $\hbar = 0$.

4. Quantum velocity and stationary flow

The quantum potential was recently interpreted in terms of the velocity of internal motion [14]. In that approach, a decomposition of the nonrelativistic local velocity into two parts, one parallel and the other orthogonal to the momentum, was obtained based on the Pauli current. The first part, determined by ∇S and interpreted as the "classical" part, corresponds to the velocity of the center-of-mass. The second part, called the "quantum" part, is the velocity of motion in the center-of-mass frame (the internal "spin motion" or Schrödinger's *zitterbewegung*). The contribution of the quantum potential to Lagrangian (3.6) then has the simple physical meaning of the kinetic energy for the spin motion,

$$\frac{\hbar^2}{2m}(\nabla|\chi|)^2 = \frac{\hbar^2}{8m}\left(\frac{\nabla\rho}{\rho}\right)^2 = \frac{m\mathbf{V}_\mathrm{q}^2}{2},$$

where the "quantum" velocity is defined by

$$\mathbf{V}_\mathrm{q} = \frac{\nabla\rho\times\mathbf{s}}{m\rho}.$$

For planar motion in the xy plane, $\nabla_z = 0$ and $s_x = s_y = 0$. Therefore, $s_z = \hbar/2$, and we have

$$(V_\mathrm{q})_x = \frac{\hbar}{2m}\frac{\partial_y\rho}{\rho}, \qquad (V_\mathrm{q})_y = -\frac{\hbar}{2m}\frac{\partial_x\rho}{\rho}$$

or

$$(\mathbf{V}_\mathrm{q})_i = \frac{\hbar}{2m}\epsilon_{ij}\frac{\partial_j\rho}{\rho} \tag{4.1}$$

for the components of the quantum velocity. Differentiating Eq. (4.1) with respect to time and using continuity equation (2.3b), we obtain

$$\partial_0\mathbf{V}_\mathrm{q} + (\mathbf{V}\nabla)\mathbf{V}_\mathrm{q} = 0,$$

which means that $\mathbf{V}_\mathrm{q}$ propagates with the main flow velocity $\mathbf{V}$, i.e., it is the velocity of the inner motion. Moreover, by direct computation from Eq. (4.1), we obtain the no-divergence condition for the quantum velocity flow, $\nabla(\rho\mathbf{V}_\mathrm{q}) = 0$. This condition applied to continuity equation (2.3b) for a flow propagating with the quantum velocity

$$\mathbf{V} = \pm\mathbf{V}_\mathrm{q}, \tag{4.2}$$

which has the meaning of a special planar motion with equal velocities of the classical (center-of-mass) and the quantum (internal) motions, leads to stationary flow, $\partial_0\rho = 0$.

On the other hand, for the stationary flow where $\partial_0\mathbf{V} = 0$ and

$$\frac{\kappa g}{e^2} = \pm\frac{\hbar}{2mc^2}, \tag{4.3}$$

Madelung fluid equation (3.13) can be rewritten as

$$\frac{m}{2}\nabla_j(\mathbf{V}-\mathbf{V}_\mathrm{q})(\mathbf{V}+\mathbf{V}_\mathrm{q}) - \frac{e^2}{\kappa c^2}\rho\epsilon_{jk}(\mathbf{V}\mp\mathbf{V}_\mathrm{q})_k \mp \frac{\hbar}{2}\nabla_j[\nabla\times(\mathbf{V}\mp\mathbf{V}_\mathrm{q})] = 0,$$

which is identically satisfied by Eq. (4.2). Deriving this equation, we use the identity

$$\frac{\hbar^2}{2m}\frac{\nabla\sqrt{\rho}}{\sqrt{\rho}} = \frac{m\mathbf{V}_\mathrm{q}^2}{2} - \frac{\hbar}{2}[\partial_1(\mathbf{V}_\mathrm{q})_2 - \partial_2(\mathbf{V}_\mathrm{q})_1]$$

and CS Gauss law (3.12). Therefore, under condition (4.2), we need only satisfy the vorticity condition in (3.13) for the quantum velocity,

$$\nabla \times \mathbf{V}_{\mathrm{q}} = \pm \frac{e^2}{\kappa m c^2}\, \rho,$$

in order to obtain the Liouville equation in the form

$$\Delta \log \rho = \mp \frac{2e^2}{\kappa \hbar c^2}\, \rho$$

from definition (4.1).

We stress that the Liouville equation in our model has the meaning of the vorticity condition for the quantum flow. Solutions of the model are well known [13, 15]. We only mention that in the polar symmetric case (for the minus sign), we obtain the solution

$$\rho = 4\frac{\kappa \hbar c^2 N^2}{e^2 r^2} \left[\left(\frac{r}{r_0}\right)^N + \left(\frac{r_0}{r}\right)^N \right]^{-2},$$

which is regular for $N \geq 1$ and can be ontained from the general solution

$$\rho = \alpha \frac{|\zeta'(z)|^2}{(1 + |\zeta(z)|^2)^2},$$

where

$$\zeta(z) = \frac{c_N}{(z - z_0)^N}, \qquad z = x + iy.$$

There are two physical conditions in the original $(\psi, \mathbf{A})$ formulation that restrict our solution. It follows from the regularity of the gauge potential $\mathbf{A}$ that the phase of χ given by (2.7) (see Eqs. (3.5)) must be

$$\frac{S}{\hbar} = (N - 1)\theta, \qquad \theta = \tan^{-1} \frac{x_2}{x_1}$$

and N must be an integer for single-valued χ. But single-valuedness of the original function ψ given by (2.4) in system (3.4) requires that the product $(N - 1)\hbar$ be integer-valued. This, in turn, requires that for any integer N, the deformation parameter $\hbar$ be integer-valued, $\hbar = n$. As a consequence of (4.3), we obtain the quantization condition

$$\frac{\kappa g}{e^2} = \pm \frac{n}{2mc^2}, \quad n = 1, 2, 3, \dots .$$

This relation means that the CS coupling constant and the quantum potential strength must be quantized,

$$\kappa = n \frac{e^2}{2gmc^2}, \qquad 1 - \hbar^2 = 1 - n^2 = (1 - n)(1 + n).$$

We now present the Lagrangian formulation of our fluid model given by system (3.13). Excluding the vector potentials A_μ from Lagrangian (3.6), we obtain

$$L = \frac{\kappa m^2 c^2}{2e^2} \epsilon_{\mu\nu\lambda} V_\mu \partial_\nu V_\lambda - \rho V_0 - \rho \frac{m\mathbf{V}^2}{2} - \rho \frac{m{\mathbf{V}_{\mathrm{q}}}^2}{2} + g\rho^2,$$

where V_0 plays the role of the Lagrange multiplier. The Hamilton function (constrained by the CS Gauss law)

$$H = \int \rho \frac{m\mathbf{V}^2}{2} + \rho \frac{m{\mathbf{V}_{\mathrm{q}}}^2}{2} - g\rho^2$$

is simply interpreted as the sum of the kinetic energies of the classical and quantum motions plus the self-interaction energy. It is easy to verify that it vanishes for self-dual flow (4.2) with fixed constants (4.3).

5. Reduction to 1+1 dimensions

For a one-dimensional flow, for example, in the x direction with $\partial_2 = 0$, system (3.13) reduces to

$$\partial_0 V_1 + V_1 \partial_1 V_1 = \frac{1}{m}\partial_1 \left(2g\rho + \frac{\hbar^2}{2m}\frac{\partial_1^2\sqrt{\rho}}{\sqrt{\rho}}\right), \tag{5.1a}$$

$$\partial_0 V_2 + V_1 \partial_1 V_2 = 0, \tag{5.1b}$$

$$\partial_1 V_2 = \frac{e^2}{\kappa m c^2}\rho. \tag{5.1c}$$

Substituting $\partial_1 V_2$ from Eq. (5.1c) in (5.1b), we find that the velocity field component V_2 is decoupled completely from Eq. (5.1a) and is determined by the first-order system of equations

$$\partial_1 V_2 = \frac{e^2}{\kappa m c^2}\rho, \qquad \partial_0 V_2 = -\frac{e^2}{\kappa m c^2}\rho V_1.$$

Then the compatibility condition for this system is just the continuity equation for the one-dimensional flow,

$$\partial_0 \rho + \partial_1(\rho V_1) = 0. \tag{5.2}$$

Equations (5.1a) and (5.2) determine the Madelung fluid in one space dimension. Rewriting them for the wave function

$$\chi = \sqrt{\rho}\,\exp\left(\frac{i}{\hbar}\int_{-\infty}^{x} V_1\right),$$

we obtain the NLS model

$$i\hbar\partial_0\chi + \frac{\hbar^2}{2m}\partial_1^2\chi + 2g|\chi|^2\chi = 0.$$

We note that with the minus sign for the quantum deformation in system (3.4), corresponding to the replacement $\hbar^2 \to -\hbar^2$, the one-dimensional reduction of the fluid is given by system (5.2), (5.1a), where we must change the sign of the quantum potential contribution in Eq. (5.1a). The result is that for the wave function

$$\psi = \sqrt{\rho}\,\exp\left(i\int_{-\infty}^{x} V_1\right),$$

we obtain the NLS model modified by the quantum potential,

$$i\partial_0\psi + \frac{1}{2m}\partial_1^2\psi + 2g|\psi|^2\psi = (1+\hbar^2)\frac{1}{2m}\frac{\partial_1^2|\psi|}{|\psi|}\psi. \tag{5.3}$$

In terms of the two real functions

$$Q^{\pm} = \sqrt{\rho}\,\exp\left(\pm\frac{1}{\hbar}\int_{-\infty}^{x} V_1\right),$$

we have the "dissipative" (reaction-diffusion) version of the NLS model

$$\pm\hbar\partial_0 Q^{\pm} + \frac{\hbar^2}{2m}\partial_1^2 Q^{\pm} + 2gQ^{+}Q^{-}Q^{\pm} = 0.$$

This analogy allows deriving a bilinear representation for (5.3). The solution for the wave function ψ is represented in terms of the three real functions $G^{\pm}$ and F,

$$\psi = \frac{(G^+)^{(1-i\hbar)/2}(G^-)^{(1+i\hbar)/2}}{F}, \qquad \bar{\psi} = \frac{(G^+)^{(1+i\hbar)/2}(G^-)^{(1-i\hbar)/2}}{F},$$

where the functions satisfy the bilinear system of equations

$$\left(\pm\hbar D_t - \frac{\hbar^2}{2m}D_x^2\right)(G^{\pm} * F) = 0,$$

$$\frac{\hbar^2}{2m}D_x^2(F * F) = 2gG^+G^-.$$

The hydrodynamic variables are then given by

$$V_1 = \frac{\hbar}{2m}\partial_1 \log\frac{G^-}{G^+}, \qquad \rho = \frac{\hbar^2}{2mg}\partial_1^2(\log F).$$

Constructing one- and two-soliton solutions, we find that the soliton dynamics of Eq. (5.3), in contrast to the NLS case, have a rich resonance phenomenology [4, 8]. Another reduction of (2+1)-dimensional model (3.4) leads to a DNLS-type equation and its reaction-diffusion analogue. Details of the reduction procedure and resonance soliton interactions will be published elsewhere.

6. Conclusion

We have reformulated the classical dynamics of a nonrelativistic particle interacting with an Abelian gauge field as a nonlinear wave equation with a quantum potential. Considering deformations of this equation, we have found two cases depending on the sign of deformation. For one sign, we obtained the standard Schrödinger model with the deformation parameter playing the role of the Planck constant. For the other sign, we obtained a diffusion–antidiffusion equation. Specifying the gauge field as the CS one and adding a cubic nonlinear term to the Schrödinger equation, we found the dispersionless limit of the JP model, which could be a useful description of the semiclassical limit of the anyon. The deformation of this model is equivalent to the standard JP model, which we represented as the rotational hydrodynamics of a Madelung-type fluid. A special flow in this fluid with equal velocities of the classical and quantum motions leads to the Liouville equation admitting vortex configurations. A similar equation, as previously found [4, 8], defines the event horizon for a solution of the black-hole type in the one-dimensional NLS with quantum potential (5.3). Moreover, in terms of the wave function, it is exactly the CS self-(antiself-)duality condition [13]. In fact, because the self-duality equations are first-order equations, they can be interpreted in terms of velocity fields. We therefore hope that our interpretation can be applied to other models as well.

Acknowledgments. One of the authors (O. K. P.) thanks Professor Pierre Sabatier for the useful discussion and Professor Fon-Che Liu and the Institute of Mathematics, Academia Sinica, Taipei, for the warm hospitality.

This work was supported in part by the Izmir Institute of Technology, Turkey, and the Institute of Mathematics, Academia Sinica, Taipei, Taiwan.

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