ENTROPIC TUNNELING TIME AND ITS APPLICATIONS

A Thesis Submitted to
the Graduate School of Engineering and Sciences of
İzmir Institute of Technology
in Partial Fulfillment of the Requirements for the Degree of

MASTER OF SCIENCE

in Physics

by
Tuğrul GÜNER

December 2014
İZMİR

We approve the thesis of **Tuğrul GÜNER**

Examining Committee Members:

Prof. Dr. Durmuş Ali DEMİR
Department of Physics, İzmir Institute of Technology

Assoc. Prof. Dr. Haldun SEVİNÇLİ
Department of Materials Science and Engineering, İzmir Institute of Technology

Assoc. Prof. Dr. Özgür ÇAKIR
Department of Physics, İzmir Institute of Technology

15 December 2014

Prof. Dr. Durmuş Ali DEMİR
Supervisor, Department of Physics
İzmir Institute of Technology

Prof. Dr. Nejat BULUT
Head of the Department of
Physics

Prof. Dr. Bilge KARAÇALI
Dean of the Graduate School of
Engineering and Sciences

ACKNOWLEDGMENTS

I would like to thank to my supervisor Prof.Dr. Durmuş Ali DEMİR for his endless efforts and support in this thesis. I am very grateful to benefit from his experience, and deep knowledge. It is a great honour for me to work with him. Besides, I want to thank Ozan SARGIN for his help, corrections, support, and my family, my friends, and Özge ÇAYLIOĞLU for their support, and motivation.

ABSTRACT

ENTROPIC TUNNELING TIME AND ITS APPLICATIONS

Quantum tunneling is one of the most interesting consequences of quantum behaviour. However, even though it is known and understood well, due to non-existence of a time operator in quantum mechanics, estimating what time it takes for a particle to cross a barrier remains an open question. There are some attempts like phase time, dwell time, Larmor clock, Buttiker-Landauer, and Feynman Path Integral approaches. These definitions do not agree with each other and with experiment.

In this thesis work, tunneling time problem is studied in a rather new context. Knowing time necessitates momentum, we deal with momentum state vectors and define entropy accordingly. This entropy, which gives a temperature, defines a thermal energy in the tunneling region. With this thermal energy and uncertainty principle, resulting time deviation of the particle from the classically instantaneous is stated as our tunneling time, entropic tunneling time. Moreover, we compare this tunneling time with recent experiments in detail, and find that, it is in very good agreement with the data. Then, we apply this entropic tunneling time to α - decay, STM and creation of universe from nothing to predict natural time scale of these processes.

ÖZET

ENTROPİK TÜNELLEME ZAMANI VE UYGULAMALARI

Kuantum tünelleme, kuantum mekaniğinin en ilginç sonuçlarından biridir. Ama iyi bilinmesine ve anlaşılmasına rağmen, kuantum mekaniğinde zaman operatörü yazılamayışından ötürü, bir parçacığın bariyeri ne kadar zamanda katedeceği açık bir soru olarak durmaktadır. Bu sorunu çözmek icin faz zamanı, dwell zamanı, Larmor saati, Buttiker-Landauer, ve Feynman Yol İntegrasyonu yaklaşımı gibi bazı girişimler mevcut olsada, henüz daha bir fikir birliğine varılamamıştır.

Bu tezde, tünelleme zamanı problemi oldukça yeni bir bağlamda çalışıldı. Zamanı anlamak momentumu gerekli kılar, ve dolayısıyla, entropiyi tanımlamak için momentum durum vektörleri kullanıldı. Bu entropi, ki ayrıca bir sıcaklık ifade eder, tünelleme bölgesi için termal enerji tanımlar. Bu termal enerji ve belirsizlik ilkesi neticesinde, parçacık için sonuçlanan ve klasik olarak anlık ifade edilen zamandan sapmayı temsil eden zaman tanımı bizim tünelleme zamanımız, entropik tünelleme zamanı, olarak belirlendi. Dahası, bu modeli deney ile detaylıca karşılaştırdık ve deney sonuçlarıyla oldukça iyi bir şekilde uyuştuğunu bulduk. Bunun üzerine, entropik tünelleme zamanını α-bozunumu'na, STM'e ve evrenin tünellemesi'ne, bu süreçlerin doğal zaman skalasını belirleyebilmek için uyguladık.

TABLE OF CONTENTS

LIST OF FIGURES

LIST OF TABLES

CHAPTER 1

INTRODUCTION

Quantum tunneling (for general information you can check (Griffiths, 2005), or for more detailed information (Razavy, 2003)) is a pure quantum mechanical phenomenon enabled by the uncertainty principle which can be explained simply as a penetration of subatomic particles into classically forbidden region. Ordinarily, if a subatomic particle interacts with a barrier that exceeds its energy, we would expect it to be reflected by the barrier. However, as we mentioned above, contrary to our senses, quantum mechanics implies non-vanishing probability for any particle to cross this type of barriers. This interesting result of quantum mechanics' physical relevance was first established by Gamow during his work on the α-decay (Gamow, 1928a,b, 1931). Max Born is the one who generalizes and formulates this effect. Then technological applications such as tunnel diode of Esaki (Esaki, 1976) and STM (Binning and Rohrer, 1993) followed. Today, quantum tunneling is known to enable various physical (Razavy, 2003), chemical (McMahon, 2003), biological (Lambert et al., 2013), and complex systems (Ankerhold, 2007) phenomena.

With the increasing attention to quantum tunneling and its technological applications, having still no consensus (why there is no consensus on this problem will be explained in Section.1.2 in detail) on the answer of the question, what time it takes for the particle to cross the barrier, is an important ongoing problem. In this work, we develop a novel formulation for tunneling time in order to solve this problem (Demir, 2014), and in addition to this, we apply this tunneling time formula to some well-known cases as numerical applications (Demir and Guner, 2014). In general, we divide this thesis into three main chapters, and one can follow these chapters according to given information below.

In the first chapter, we give brief information about quantum tunneling. We summarise two main approaches that are used for analytical calculations of the tunneling. These two approaches, which are Rectangular Potential Barrier as a constant potential case and WKB Approximation as a slowly varying potential case respectively, are the subsections of the first chapter. Next, we give a quick review and critics of some existing well known approaches such as Phase Time (Wigner, 1955; Hauge and Støvneng, 1989), Dwell Time, Larmor Clock (Baz', 1967a,b; Büttiker, 1983), Buttiker-Landauer Approach (Büttiker and Landauer, 1982, 1985) in one subsection and Feynman Path Integral Approach (Sokolovski and Baskin, 1987) in another subsection alone. We separate the last one from the others because it fits the experimental data better than any other tunneling time approaches as we will see in Chapter 2

explicitly.
Second chapter consists of the formulation of our approach in detail. However, in addition to development of the tunneling time formula theoretically in one section by following (Demir, 2014), we analyse the latest experiment about tunneling time in detail in the remaining section. This experiment compares several important and common approaches which have already been investigated briefly in section.1.2, with the tunneling time data that are taken by the velocity map imaging spectrometer (VMIS) and the colt-target recoil-ion momentum spectrometer (COLTRIMS) (these techniques will be explained in Section.2.2, but for more detail see the experiment section of (Landsman et al., 2013)). Thanks to this experiment, we are able to state that our formulation of tunneling time is consistent with, and is in good agreement with the experiment.
In third chapter, we apply this approach to some well known cases that are using tunneling such as *α-decay*, *STM*, and *creation of universe from nothing* by following (Demir and Guner, 2014).
In last chapter, we will conclude our work, and you can find further details in Appendix section.

1.1. Review of the Quantum Tunneling

1.1.1. Rectangular Potential Barrier

Rectangular potential barrier is the simplest case, and therefore, commonly studied while teaching quantum tunneling. According to the time-independent Schroedinger equation, exact analytical solution exists when we deal with this type of constant potential case. This is important because for any potential function $V(x)$, solving this Schroedinger equation exactly is not easy and for most of the cases, impossible. However, approximation like WKB which we will see it the following subsection in detail, will be used but only for the specific cases. For the rectangular potential barrier as one can see in Fig.1.1, we have

$$V(x) = \begin{cases} V_0, & 0 \leq x \leq L \\ 0, & \text{elsewhere} \end{cases}$$

Separating all three regions and solving as a stationary case, we have:

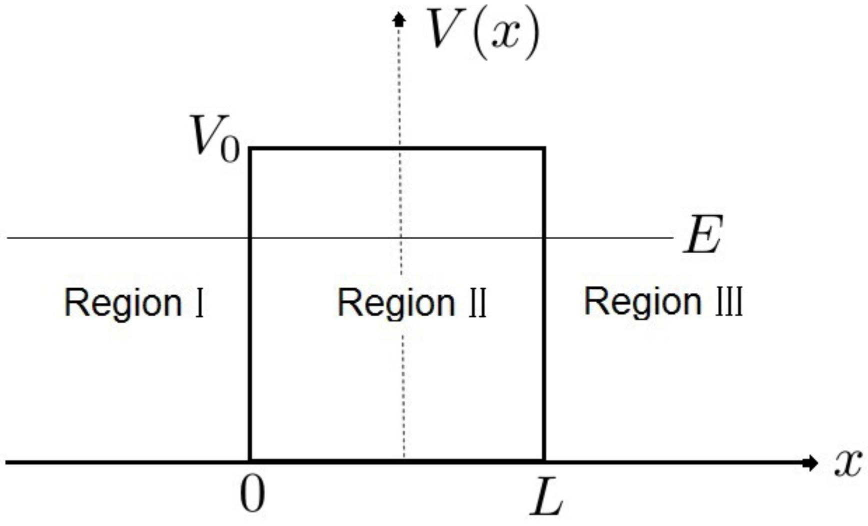

Figure 1.1. Simple Illustration of the Rectangular Potential Barrier

Region I:

$$-\frac{\hbar^2}{2m}\frac{d^2\psi_I(x)}{dx^2} = E\psi_I(x) \tag{1.1}$$

where E is energy of the particle. Then resulting differential equation is:

$$\frac{d^2\psi_I(x)}{dx^2} + k^2\psi_I(x) = 0 \tag{1.2}$$

where $k = \frac{2mE}{\hbar^2}$ with m is mass of the particle. Solution is then:

$$\psi_I = Ae^{ikx} + Re^{-ikx} \tag{1.3}$$

Region II:

$$-\frac{\hbar^2}{2m}\frac{d^2\psi_{II}(x)}{dx^2} + V_0\psi_{II}(x) = E\psi_{II}(x) \tag{1.4}$$

Differential form becomes:

$$\frac{d^2\psi_{II}(x)}{dx^2} - \kappa^2\psi_{II}(x) = 0 \tag{1.5}$$

where $\kappa = \frac{\sqrt{2m(V_0-E)}}{\hbar}$. Solution is then:

$$\psi_{II}(x) = Ce^{\kappa x} + De^{-\kappa x} \tag{1.6}$$

Region III:

This region has exactly same solution with Eq.(1.1) because $V(x) = 0$ here. Therefore, we expect a solution $\psi_{III}(x) = Fe^{ikx} + Ge^{-ikx}$ but we have to be aware that, there is no reflected particle in region III because if particle crosses the barrier, moves forward, cannot be reflected by any other potential barrier according to Fig.1.1. This simplifies the result as:

$$\psi_{III}(x) = Te^{ikx} \tag{1.7}$$

Transmission and Reflection Coefficients

We've constructed our wavefunction and investigated its analytical behaviour.

$$\psi(x) = \begin{cases} \psi_I = Ae^{ikx} + Re^{-ikx}, & x \leq 0 \\ \psi_{II} = Ce^{\kappa x} + De^{-\kappa x}, & 0 \leq x \leq L \\ \psi_{III} = Te^{ikx}, & x \geq L \end{cases}$$

Now, it is important to find transmission ($t = \frac{|T|^2}{|A|^2}$) and reflection ($r = \frac{|R|^2}{|A|^2}$) coefficients because, first, these lead to information about the probabilities that represent particle's possible status, and second, we will see in Section.1.2 that some of the known tunneling time approaches need especially this transmission coefficient to calculate tunneling time. To do so, we should use continuity conditions at the boundaries. First condition is:

$$\psi_I(0) = \psi_{II}(0) \tag{1.8}$$

Second condition is:

$$\dot{\psi_I}(0) = \dot{\psi_{II}}(0) \tag{1.9}$$

Third condition is:

$$\psi_{II}(L) = \psi_{III}(L) \tag{1.10}$$

And last condition is:

$$\dot{\psi_{II}}(L) = \dot{\psi_{III}}(L) \tag{1.11}$$

All of these Eqs.(1.8 - 1.11) lead to following equalities:

$$A + R = C + D \tag{1.12}$$

$$A - R = \frac{\kappa}{ik}(C - D) \tag{1.13}$$

$$Ce^{\kappa L} + De^{-\kappa L} = Te^{ikL} \tag{1.14}$$

$$Ce^{\kappa L} - De^{-\kappa L} = \frac{ik}{\kappa}Te^{ikL} \tag{1.15}$$

Using Eqs.(1.14) and (1.15), we express two coefficients C and D in terms of T:

$$C = \frac{T}{2}e^{(ik-\kappa)L}\left(1 + \frac{ik}{\kappa}\right) \tag{1.16}$$

$$D = \frac{T}{2}e^{(ik+\kappa)L}\left(1 - \frac{ik}{\kappa}\right) \tag{1.17}$$

If we put these C and D in Eqs.(1.12) and (1.13), we get:

$$\frac{T}{A} = \frac{4e^{-ikL}}{4\cosh\kappa L + 2i(\frac{\kappa^2-k^2}{\kappa k})\sinh\kappa L} \tag{1.18}$$

Then, transmission coefficient is,

$$t = \frac{|T|^2}{|A|^2} = \frac{4}{4\cosh^2\kappa L + (\frac{\kappa^2-k^2}{\kappa k})^2\sinh^2\kappa L} \tag{1.19}$$

Particle must be either transmitted or reflected, therefore, $t+r=1$ must be satisfied and this simplifies the calculation of reflection coefficient, which is:

$$r = \frac{|R|^2}{|A|^2} = 1 - t = \frac{4(\frac{\kappa^2+k^2}{\kappa k})^2\sinh^2\kappa L}{4\cosh^2\kappa L + (\frac{\kappa^2-k^2}{\kappa k})^2\sinh^2\kappa L} \tag{1.20}$$

1.1.2. WKB Approximation

The WKB (Wentzel (Wentzel, 1926), Kramers (Kramers, 1926) and Brillouin (Brillouin, 1926)) method is an approximation to time-independent schrodinger equation while dealing with a potential function $V(x)$ depending on x, as you see in Fig.1.2. In this case, analytical solution of this

$$-\frac{\hbar^2}{2m}\frac{d^2\psi(x)}{dx^2} + V(x)\psi(x) = E\psi(x) \tag{1.21}$$

time-independent Schroedinger equation as we mentioned early, is not easy. Here, assumption of this WKB approximation is simple; potential function $V(x)$ slowly varies with respect to the position x. According to this, we propose a solution similar to the constant potential case as you see in Eqs.(1.3) and (1.7). Therefore, expected solution is of the form

$$\psi(x) = Ae^{\frac{i}{\hbar}\phi(x)} \tag{1.22}$$

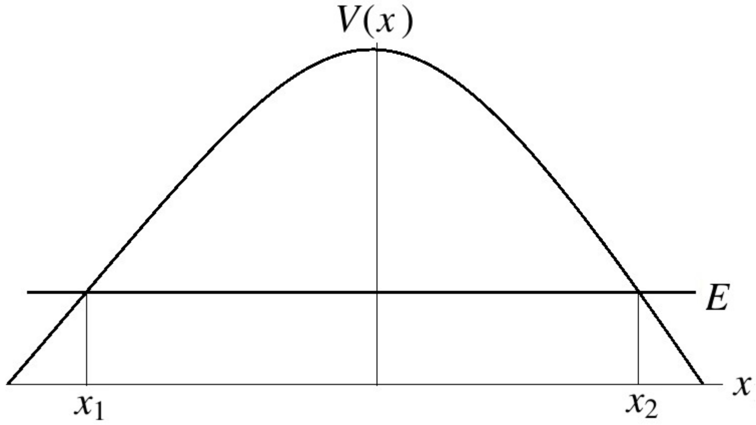

Figure 1.2. Simple Illustration of the Potential Barrier for the WKB Method

Now, putting Eq.(1.22) into (1.21) leads to the following equation for $\phi(x)$,

$$-i\hbar\frac{d^2\phi(x)}{dx^2}+\left(\frac{d\phi(x)}{dx}\right)^2+2m(V(x)-E)=0 \tag{1.23}$$

Notice that, if we set $\phi(x) = \pm\hbar kx$, or $\phi(x) = \pm i\hbar\kappa x$, then we would immediately get the solutions of the rectangular potential barrier for related regions, but with these $\phi(x)$, first term of the Eq.(1.23) vanishes. On the other hand, since Eq.(1.23) is a non-linear equation, we have to start making approximations to solve this equation. Therefore, based on this negligible property of this first term and its proportionality to $\hbar$ (where $\hbar \to 0$ in the classical limit), we treat this $\hbar$ as an indication of smallness[1] and we are able to expand the function $\phi(x)$ in the power series in terms of it,

$$\phi(x)=\phi_0(x)+\hbar\phi_I(x)+\frac{\hbar^2}{2}\phi_{II}(x)+... \tag{1.24}$$

[1]Details can be found in http://www.pha.jhu.edu/courses/171_303/WKB_notes_bransden.pdf

From now on, all we have to do is placing the above expansion into the Eq.(1.23) and by equating every coefficient of power of $\hbar$ term by term to zero, we obtain

$$\left(\frac{d\phi_0(x)}{dx}\right)^2 + 2m(V(x) - E) = 0 \tag{1.25}$$

$$-\frac{i}{2}\frac{d^2\phi_0(x)}{dx^2} + \frac{d\phi_0(x)}{dx}\frac{d\phi_I(x)}{dx} = 0 \tag{1.26}$$

$$-i\frac{d^2\phi_I(x)}{dx^2} + \frac{d\phi_0(x)}{dx}\frac{d\phi_{II}(x)}{dx} + \left(\frac{d\phi_I(x)}{dx}\right)^2 = 0 \tag{1.27}$$

We see explicitly that, $\phi_0(x)$, $\phi_I(x)$ and $\phi_{II}(x)$ will be solved sequentially because each higher order term depends on the previous terms. However, we have to notice that, we keep the $\phi_{II}(x)$ as the highest term because WKB method only considers $\phi_0(x)$ and $\phi_I(x)$ terms and this highest term enables us to show that why this and remaining higher order terms vanish. Our assumption of slowly varying potential plays its crucial role at this point. For the $E < V(x)$ case, Eq.(1.25) solved as

$$\phi_0(x) = \pm i \int^x q(x)dx \tag{1.28}$$

where $q(x) = \sqrt{2m(V(x) - E)}$. Now, we are able to solve the Eq.(1.26), and the solution is:

$$\phi_I(x) = \frac{i}{2}\log q(x) \tag{1.29}$$

As a final step, solution of the Eq.(1.27) is:

$$\phi_{II}(x) = i\frac{m}{2q(x)^3}\frac{dV(x)}{dx} - i\frac{m^2}{4}\int^x \frac{1}{q(x)^5}\left(\frac{dV(x)}{dx}\right)^2 dx \tag{1.30}$$

Here comes the main assumption, and approximation of the WKB method. We have already told that potential must be slowly varying with respect to x, and since $\phi_{II}(x) \propto \frac{dV(x)}{dx}$, this means that third term in the Eq.(1.24) is negligible

$$\left|\frac{\hbar^2}{2}\phi_{II}(x)\right| \ll 1 \tag{1.31}$$

and this leads to a constraint that must be satisfied to ensure that WKB is applicable:

$$\left|\frac{\hbar^2 m \frac{dV(x)}{dx}}{4q(x)^3}\right| \ll 1 \tag{1.32}$$

What we have left is, since ϕ_{II} and the following terms negligible, by taking the non-negligible first two terms in Eq.(1.24) and putting it into the Eq.(1.22), we obtain the wave-function solution of the WKB method,

$$\psi(x) = \frac{A}{\sqrt{q(x)}} e^{-\frac{1}{\hbar}\int^x q(x)dx} \tag{1.33}$$

of course, general solution is a linear combination, but keeping exponentially decaying term only is enough for now.

So far, we have determined the wavefunction of the particle inside the tunneling region according to the WKB method. Now, we need to find what is the transmission coefficient related to this method. However, we know that, at the turning points x_1 and x_2, $q(x_1) = q(x_2) = 0$ and this diverges the wavefunction in Eq.(1.33). Therefore, connection formula is necessary to move forward. This connection formula involves Airy functions, and contains lengthy calculations while finding the transmission coefficient. We leave this content to Appendix A, and writing down the exact result of the transmission coefficient here,

$$T_{rans} = \frac{1}{\cosh^2 \Phi} \tag{1.34}$$

where

$$\Phi = \frac{1}{\hbar}\int_{x_1}^{x_2} q(x)dx \tag{1.35}$$

is Hamilton's characteristic function (also known as Gamow factor) in units of $\hbar$ within the tunneling region.

1.2. Review of the Tunneling Time Problem

Tunneling is a genuine quantum phenomenon as we already mentioned. Solving Schroedinger equation, in WKB approximation if not exactly, gives wavefunction inside and outside the tunneling region. This, however, gives no information about tunneling time. The question of *tunneling time*, and therefore defining an average *speed* for a particle that represents the barrier crossing was first pointed out by Condon (Condon and Morse, 1931). Then, MacColl attempted to solve this problem just a year later (MacColl, 1932). Afterwards, this problem almost did not take attention up to fifties. Problem of defining a tunneling time is challenging and this is not surprising because in quantum theory time is not an operator whose expectation value can be evaluated by using the wave-function. From fifties, to date, it is also for this particular reason that various tunneling time definitions which we have already mentioned above have been introduced: Büttiker-Landauer, Larmor approach, phase time, and dwell time. However, contrary to their raison d'etre, all these time definitions utilize a sort of time operator because they all involve derivatives with respect to energy E or potential $V(x)$ (Yamada, 2004). Compared to them, the Feynman path integral (FPI) approach is particularly interesting because it involves times taken by individual classical trajectories yet tunneling time is still controversial (details can be found in Section.1.2.2). Now, we will investigate some of the main tunneling time approaches (there are many reviews about these time approaches that contain both theoretical and experimental point of views, and hence, one can find more technical details in these reviews (Olkhovsky and Recami, 1992; Landauer and Martin, 1994; Privitera et al., 2004; Winful, 2006; Maji et al., 2007), and in thesis (Eser, 2011)).

1.2.1. Phase Time, Dwell Time, Larmor Clock and Buttiker-Landauer Approaches

Phase time approach is about tracking the position of a peak of very narrow wavepacket around a determined wave number k_0. Mostly, stationary phase is considered, and therefore, notation of simple rectangular potential barrier in Section.1.1.1 is valid here. Under right circumstances and neglecting the dispersion effects (Olkhovsky and Recami, 1992; Olkhovsky et al., 2001), one can take this peak of the wave-packet as a reference point.

We have to note that, this peak of the wavepacket is consisting of the Fourier components of the form

$$\psi(x,t;k) = |T|e^{i(kx-\frac{Et}{\hbar}+\alpha(k))} \tag{1.36}$$

which dominates the surrounding phase variation around k_0, and hence, prohibits the destructive interference. Here E is energy of the particle, t is time and $\alpha(k)$ comes from the $T \rightarrow |T|e^{i\alpha(k)}$ as phase shifting term to represent contribution of transmission.
We are able to find position of the peak, where the phase is maximum, and hence, stationary as we told earlier. This will be achieved simply;

$$\frac{d}{dk}\psi(x,t;k) = \frac{d}{dk}\left(|T|e^{i(kx-\frac{Et}{\hbar}+\alpha(k))}\right) = 0 \tag{1.37}$$

Result of this derivation is,

$$x - \frac{dE}{dk}\frac{t}{\hbar} + \frac{d\alpha(k)}{dk} = 0 \tag{1.38}$$

and position x that satisfies above equation is the position of the peak wave-packet which we call it as $x = x_D(t)$. Then,

$$x_D(t) = \frac{dE}{dk}\frac{t}{\hbar} - \frac{d\alpha(k)}{dk} \tag{1.39}$$

but to understand what the equation above means, we also have to evaluate the case without the barrier. In this case, there aren't any transmission or reflection coefficients, and we must consider just a free particle that is moving as a wave-packet of the form

$$\psi(x,t;k) = Ce^{i(kx-\frac{Et}{\hbar})} \tag{1.40}$$

where C is the amplitude. Applying same procedure which is about finding the location of the peak of the wave-packet, we have

$$\frac{d}{dk}\psi(x,t;k) = \frac{d}{dk}\left(Ce^{i(kx-\frac{Et}{\hbar})}\right) = 0 \tag{1.41}$$

then resulting peak position $x = x_F$ for this free particle case is

$$x_F = \frac{dE}{dk}\frac{t}{\hbar} \tag{1.42}$$

This x_F is nothing but the first term in the Eq.(1.39), and rearranging x_D gives

$$x_D = x_F - \frac{d\alpha(k)}{dk} \tag{1.43}$$

By looking at the equation above, we immediately conclude that, the term

$$\frac{d\alpha(k)}{dk} = \delta x \tag{1.44}$$

represents the spatial delay caused by the barrier, phase shift of the particle due to the transmission. In addition to this spatial delay, one will also find a temporal delay by using equation of motion,

$$\delta\tau = \frac{\delta x}{v_G} \tag{1.45}$$

where v_G is the group velocity, and if we rewrite the Eq.(1.42) as

$$x_F = v_G t \tag{1.46}$$

then the group velocity is nothing but the

$$v_G = \frac{1}{\hbar}\frac{dE}{dk} \tag{1.47}$$

using both (1.44) and (1.47), temporal delay in Eq.(1.45) becomes

$$\delta\tau = \frac{\frac{d\alpha(k)}{dk}}{\frac{1}{\hbar}\frac{dE}{dk}} = \hbar\frac{d\alpha(k)}{dE} \tag{1.48}$$

We have found temporal delay explicitly. Now, we are able to write down the tunneling time as *phase time* which is nothing but the sum of the time that it takes a particle to move a distance without any barrier and the contribution of the barrier as $\delta\tau$,

$$\tau_t = \frac{x_2 - x_1}{v_G} + \delta\tau \tag{1.49}$$

where x_2 and x_1 here represents two distant points external to the barrier region, but barrier region be somewhere in between these two points. One important note is that, if we call x_1 a point in region I of Fig.(1.1), then $x_1 \ll 0$ and x_2 a point in region III, then $x_2 \gg L$, because this time is meaningful only when this narrow wave-packet moves completely out of the tunneling region. On the other hand, explicitly, *phase time* (1.49) is

$$\tau_t = \frac{x_2 - x_1}{v_G} + \hbar\frac{d\alpha(k)}{dE} \tag{1.50}$$

but in general, for simplicity, they use (1.45) to take into v_G parenthesis, and therefore

$$\tau_t = \frac{1}{v_G}\big(x_2 - x_1 + \frac{d\alpha(k)}{dk}\big) \tag{1.51}$$

this is transmission delay of the phase time approach. In the same way, one finds the reflection time by again adding phase term to reflection amplitude which is $R \to |R|e^{i\beta(k)}$ where $\beta(k)$ represents the phase term here. This time wavefunction is of the form

$$\psi(x,t;k)_r = |R|e^{i(-kx-\frac{Et}{\hbar}+\beta(k))} \tag{1.52}$$

and location of the peak of the wave-packet is

$$\frac{d}{dk}\psi(x,t;k)_r = \frac{d}{dk}\left(|R|e^{i(-kx-\frac{Et}{\hbar}+\beta(k))}\right) \tag{1.53}$$

from here, we get

$$-x - \frac{dE}{dk}\frac{t}{\hbar} + \frac{d\beta(k)}{dk} = 0 \tag{1.54}$$

therefore, solution for the position of the peak $x = x_{RD}$ of the reflected wave-packet is

$$x_{RD} = -\frac{dE}{dk}\frac{t}{\hbar} + \frac{d\beta(k)}{dk} \tag{1.55}$$

this time, the term representing the group velocity v_G of the without barrier case $-\frac{dE}{dk}\frac{1}{\hbar}$ is with a minus sign because reflected wave-packet moves in opposite direction. Here, $\frac{d\beta(k)}{dk} = \delta x$ term is now representing the spatial delay and leads to a

$$\delta\tau = \frac{\delta x}{v_G} = \frac{1}{v_G}\frac{d\beta(k)}{dk} \tag{1.56}$$

temporal delay. Then, one finds the reflection delay as

$$\tau_r = \frac{1}{v_G}(-x_1 - x_1 + \delta\tau) \tag{1.57}$$

if we compare this Eq.(1.57) with the transmission delay in Eq.(1.51), we immediately see that, here instead of x_2, there exists a $-x_1$, and this is simply because of starting from the point x_1, wave-packet travels and comes back to x_1 again with $-v_G$, therefore $\frac{x_2}{v_G} \to \frac{x_1}{-v_G}$. Writing the reflection delay explicitly,

$$\tau_r = \frac{1}{v_G}(-2x_1 + \frac{d\beta(k)}{dk}) \tag{1.58}$$

So far, we've found both the transmission and reflection delay for this phase time approach. However, there is some serious criticism about this approach. Mainly, we know that incoming peak of the wave-packet consists of several different energetic components. Therefore, some of these components of the Fourier may be reflected while peak of this wave-packet get transmitted. One can argue about physical validity of this approach due to whether the resulting wave-packet represents incoming one or not (Landauer and Martin, 1994). Also, again similar reasoning, some of these Fourier components may be higher than the potential V_0 of the barrier, and hence, delay becomes unconnected to tunneling or even if there exists no components exceeding the potential V_0, there will be some components very close to it, and this will cause these components to be transmitted more effectively. As a result, emerging packet will have a higher velocity which implies that, tunneling is an electron accelerator (Hauge and Støvneng, 1989; Landauer and Martin, 1994).

Another well-known approach, dwell time, is first introduced by Smith (Smith, 1960), and it is an average time spent by a particle during its motion in the specific bounded region. It is defined as

$$\tau^D = \frac{1}{v_{in}} \int_{x_1}^{x_2} |\psi(x)|^2 dx \tag{1.59}$$

where v_{in} is the incoming flux. This Eq.(1.59) tells us that, dwell time is the average of probability of finding the particle in the region of interest over the flux that represents the same region but without any interaction, which is then simply the free wave-packet group velocity $v_{in} = v_G$ for this particular region.

Calculating dwell time is straightforward. For the case of simple rectangular potential barrier in Section.1.1.1, we know that in tunneling region $0 < x < L$, wavefunction is

$$\psi(x) = Ce^{\kappa x} + De^{-\kappa x} \tag{1.60}$$

where C is (1.16), D is (1.17) and $\kappa = \frac{\sqrt{2m(V_0 - E)}}{\hbar}$. We continue by calculating the probability density of the tunneling region using (1.60) with the turning points $x_1 = 0$, and $x_2 = L$

$$|\psi(x)|^2 = |C|^2 e^{2\kappa x} + |D|^2 e^{-2\kappa x} + C^* D + C D^* \tag{1.61}$$

one will simply calculate these coefficients, and result is

$$\left.\begin{aligned} |C|^2 &= \frac{|T|^2}{4} e^{-2\kappa L} \frac{(k^2 + \kappa^2)}{\kappa^2} \\ |D|^2 &= \frac{|T|^2}{4} e^{2\kappa L} \frac{(k^2 + \kappa^2)}{\kappa^2} \\ C^* D &= \frac{|T|^2}{4} \left(1 - 2\frac{ik}{\kappa} - \frac{k^2}{\kappa^2}\right) \\ C D^* &= \frac{|T|^2}{4} \left(1 + 2\frac{ik}{\kappa} - \frac{k^2}{\kappa^2}\right) \end{aligned}\right\} \tag{1.62}$$

now, integrating the probability density in Eq.(1.61)

$$\int_0^L |\psi(x)|^2 dx = \frac{|C|^2}{2\kappa}(e^{2\kappa L} - 1) - \frac{|D|^2}{2\kappa}(e^{-2\kappa L} - 1) + (C^* D + C D^*) L \tag{1.63}$$

and putting the coefficients in (1.62), we get

$$\int_0^L |\psi(x)|^2 dx =$$
$$\frac{|T|^2}{8\kappa}(e^{2\kappa L}-1)e^{-2\kappa L}\frac{(k^2+\kappa^2)}{\kappa^2} - \frac{|T|^2}{8\kappa}(e^{-2\kappa L}-1)e^{2\kappa L}\frac{(k^2+\kappa^2)}{\kappa^2} + \frac{|T|^2}{4}\left(2-2\frac{k^2}{\kappa^2}\right)L \tag{1.64}$$

if we simplify this result,

$$\int_0^L |\psi(x)|^2 dx = \frac{|T|^2}{4\kappa^3}\left((k^2+\kappa^2)\sinh(2\kappa L) + 2L\frac{(\kappa^2-k^2)}{\kappa^2}\right) \tag{1.65}$$

where $|T|^2$ is the transmission coefficient in Eq.(1.19). Solution is then

$$\int_0^L |\psi(x)|^2 dx =$$
$$\frac{4\kappa^2 k^2}{4(4\kappa^2 k^2\cosh^2\kappa L + (\kappa^2-k^2)^2\sinh^2\kappa L)}\left(\frac{(k^2+\kappa^2)}{\kappa^3}\sinh(2\kappa L) + 2L\frac{(\kappa^2-k^2)}{\kappa^2}\right) \tag{1.66}$$

So, dwell time $\tau^D = \frac{1}{v_{in}}\int_0^L |\psi(x)|^2 dx$ is found by dividing (1.66) to $v_{in} = v_G = \frac{\hbar k}{m}$, also dividing and multiplying by κ

$$\tau^D = \frac{mk}{\hbar\kappa}\left(\frac{(\kappa^2+k^2)\sinh 2\kappa L + 2\kappa L(\kappa^2-k^2)}{4\kappa^2 k^2\cosh^2 2\kappa L + (\kappa^2-k^2)^2\sinh^2 2\kappa L}\right) \tag{1.67}$$

this is the *dwell time* for the tunneling region.

Main criticism about this approach is that as we have already mentioned above, this time in Eq.(1.67) measures how much time the particle spends inside the barrier region. Therefore, one cannot make a specific conclusion about which this time corresponds to: *reflection time* or *transmission time* (Büttiker and Landauer, 1982; Leavens and Aers, 1989).

Now, we will investigate the Büttiker - Landauer approach. This approach is known as one type of clock approaches in literature. Simply, idea is summarised as: there is a time modulated barrier, and particle interacts with this barrier by absorbing or emitting quanta. Hence, by analysing this external effect of modulating barrier to tunneling particle, one will get the information of tunneling time. This is possible because of the existing dependency of tunneling time to modulation frequency. Here, there are two cases, in low frequency

modulation, particle sees a static barrier because its crossing time is lower than the period, and hence, it sees only a part of the modulation cycle. The other case, higher frequency modulation, this time particle sees several modulation cycles because, unlike the previous case, crossing time is now higher than the period, and as a result, particle interacts with the time-averaged barrier.
We begin with defining the potential function

$$V(t) = V_0 + V_L \cos wt \tag{1.68}$$

here, V_0 represents the static potential, and V_L represents the amplitude of this modulation of the barrier. Then, Hamiltonian becomes

$$H = \frac{p^2}{2m} + V_0 + V_L \cos wt \tag{1.69}$$

we know the solution of energy eigenfunctions $\phi_E(x) = \psi_{II} = Ce^{\kappa x} + De^{-\kappa x}$ which have Hamiltonian $H = \frac{p^2}{2m} + V_0$ already from the Section.1.1.1. Therefore, we need to add the time evolution part by using

$$i\hbar \frac{\partial}{\partial t} \phi_t(t) = \hat{H} \phi_t(t) \tag{1.70}$$

which is

$$i\hbar \frac{\partial}{\partial t} \phi_t(t) = (E + V_L \cos wt) \phi_t(t) \tag{1.71}$$

therefore, solution is nothing but the

$$\phi_t(t) = e^{-\frac{iE}{\hbar} t} e^{-\frac{iV_L}{\hbar w} \sin wt} \tag{1.72}$$

as a result, wave-function is found as

$$\psi(x,t) = \phi_E(x) e^{-\frac{iE}{\hbar} t} e^{-\frac{iV_L}{\hbar w} \sin wt} \tag{1.73}$$

we have to express $\psi(x,t)$ so that, it contains the related absorbed and emitted quanta $E \pm n\hbar w$ caused by the time modulated barrier. Therefore, we use the identity

$$e^{\frac{x}{2}(\frac{c^2-1}{c})} = \sum_{n=-\infty}^{n=\infty} J_n(x)c^n \tag{1.74}$$

for the last term $e^{-\frac{iV_L}{\hbar w}\sin wt}$ in Eq.(1.73) where $J_n(x)$ is the Bessel function of first kind. After expanding $\sin wt = \frac{e^{iwt}-e^{-iwt}}{2i}$, then we immediately get this term by $x \to \frac{V_L}{\hbar w}$, and $c \to e^{-iwt}$. What we get is

$$e^{-\frac{iV_L}{\hbar w}\sin wt} = e^{-\frac{iV_L}{\hbar w}\left(\frac{e^{iwt}-e^{-iwt}}{2i}\right)} = \sum_{n=-\infty}^{n=\infty} J_n\left(\frac{V_L}{\hbar w}\right)e^{-inwt} \tag{1.75}$$

and resulting wavefunction is

$$\psi(x,t) = \phi_E(x)e^{-\frac{iE}{\hbar}t}\sum_{n=-\infty}^{n=\infty} J_n\left(\frac{V_L}{\hbar w}\right)e^{-inwt} \tag{1.76}$$

now, we will understand by looking at (1.76) that, particle can absorb or emit quanta as $\pm n\hbar w$ due to this interaction with the time modulated barrier. However, to simplify this expansion, we take care of only the neighbourhood absorbed or emitted quanta energies, which are $E \pm \hbar w$. This assumption leads us to conclude that, there are three wave-functions with the energies E, $E+\hbar w$, and $E-\hbar w$. Here, wave-function with energy E represents the case that particle is under static potential effect only. On the other hand, since the amplitude of the modulation V_L is small, we will expand the $\left(\frac{V_L}{\hbar w}\right)$ term while getting these three wave-functions. According to the fact that $\frac{V_L}{\hbar w} \ll 1$, then

$$J_n\left(\frac{V_L}{\hbar w}\right) \sim \left(\frac{V_L}{\hbar w}\right)^n \frac{1}{\Gamma(n+1)} \tag{1.77}$$

one gets

$$J_n\left(\frac{V_L}{\hbar w}\right) \propto \left(\frac{V_L}{\hbar w}\right)^n \tag{1.78}$$

Last step is about determining the $\phi_E(x)$ term in the wave-function (1.76). This term is simply calculated in same way that we did in the Section.1.1.1 while dealing with the simple rectangular potential barrier case. However, here, there are three wave-functions and we label related parameters using the same notation in Section.1.1.1 (k, κ, T) with the energy E as $k_\pm$, $\kappa_\pm$, and $T_\pm$ that (k_+, κ_+, T_+) corresponds to $E + \hbar w$, (k_-, κ_-, T_-) corresponds to $E - \hbar w$ energies.
One further assumption is, we consider $\hbar w \ll E$ and $\hbar w \ll V_0 - E$ since $V_L \cos wt$ is a small perturbation. Therefore,

$$k_\pm = \frac{\sqrt{2m(E \pm \hbar w)}}{\hbar} = \frac{\sqrt{2mE(1 \pm \frac{\hbar w}{E})}}{\hbar} \tag{1.79}$$

which will be written as

$$k_\pm = \frac{\sqrt{2mE}}{\hbar}\sqrt{1 \pm \frac{\hbar w}{E}} \approx \frac{\sqrt{2mE}}{\hbar}(1 \pm \frac{\hbar w}{2E}) \tag{1.80}$$

result is then

$$k_\pm \approx k(1 \pm \frac{\hbar w}{2E}) \tag{1.81}$$

where $E = \frac{\hbar^2 k^2}{2m}$. Making the same calculations for the $\kappa_\pm$

$$\kappa_\pm = \frac{\sqrt{2m(V_0 - (E \pm \hbar w))}}{\hbar} = \frac{\sqrt{2m(V_0 - E)(1 \mp \frac{\hbar w}{V_0 - E})}}{\hbar} \tag{1.82}$$

with using same approach again, we get

$$\kappa_\pm \approx \kappa(1 \mp \frac{\hbar w}{2(V_0 - E)}) \tag{1.83}$$

where $V_0 - E = \frac{\hbar^2 \kappa^2}{2m}$.
Solving for the related transmission amplitudes which is multiplied with the related trans-

mitted wave-functions, then we immediately get

$$T_{\pm} = T\frac{V_L}{2\hbar w}(e^{\pm\tau w} - 1) \tag{1.84}$$

from here, we calculate the relative intensity by

$$|T_{\pm}|^2 = |T|^2\left(\frac{V_L}{2\hbar w}\right)^2(e^{\pm w(\frac{mL}{\hbar\kappa})} - 1)^2 \tag{1.85}$$

where $|T|^2$ is the transmission coefficient in Eq.(1.19). Büttiker-Landauer defined this crossing time for the simple rectangular potential barrier as you see above, $\tau_T^{BL} = \frac{mL}{\hbar\kappa}$ where L is the barrier width $x_2 - x_1$. This time they found measures the duration of interaction between particle and time modulated barrier.

For the opaque barriers (when κL is large), T_+ grows exponentially, but T_- decays in same speed. These authors then define the tunneling time for this case as the difference between $|T_+|^2$ and $|T_-|^2$,

$$\tanh w\tau = \frac{|T_+|^2 - |T_-|^2}{|T_+|^2 + |T_-|^2} \tag{1.86}$$

and one will determine tunneling time τ by measuring the $|T_+|^2$ and $|T_-|^2$.

Last approach that we will investigate in this subsection is the Larmor clock. Again this is the same type of approach belongs to the clock type approaches like Büttiker-Landauer approach. However, in this case, applying the method of Baz‘ which was proposed in the paper (Baz’, 1967a,b) to calculate collision times based on the Larmor precession, Rybachenko (Rybachenko, 1967) computed the tunneling time. It considers a charged particle which is tunneling through a barrier under the effect of a weak magnetic field $\vec{B} = B_0\hat{z}$ in $\hat{z}$ direction. Here, spin-$1/2$ particles are polarized in $\hat{x}$ direction, which you see in Fig.1.3[2]. As a result, while particle is crossing the barrier, due to the existence of this infinitesimal magnetic field, its spin axis will rotate, and hence, this rotation will provide the necessary information for

[2]This image is taken from the (Olkhovsky and Recami, 1992)

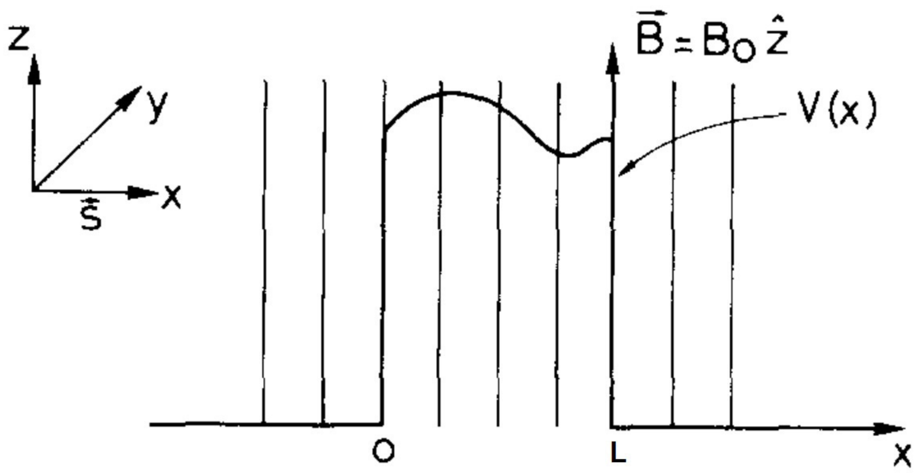

Figure 1.3. Set-up of the Larmor clock approach

the tunneling time. we get

$$\tau_{Tra} = \frac{\theta_{Tra}}{w} \tag{1.87}$$

for average time corresponding to transmission

$$\tau_{Ref} = \frac{\theta_{Ref}}{w} \tag{1.88}$$

for average time corresponding to reflection. Here $w = -\frac{eB_0}{2mc}$ is the Larmor frequency, θ_{Tra} corresponds to angular rotation by transmitted particles, and θ_{Ref} corresponds to angular rotation by reflected particles. With this additional action of the magnetic field, Hamiltonian becomes in the region $0 \leq x \leq L$ with turning points $x_1 = 0$, and $x_2 = L$

$$\hat{H}_L = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V(x) - \mu\vec{B}\hat{\sigma} \tag{1.89}$$

then, the solution of the time-independent Schroedinger equation

$$\hat{H}_L\psi(x,s) = E\psi(x,s) \tag{1.90}$$

where $\psi(x, s)$ contains the spin wave-function. Wave-function solution for this region inside the barrier is changed due to the energy shift $-\mu\vec{B}\hat{\sigma}$, but here, our main concern is about understanding what happens to transmission and reflection coefficients. By switching on the magnetic field, one changes both coefficients of the wave-function inside the barrier region, and also change both transmission T and reflection R coefficients. According to Rybachenko (Rybachenko, 1967), this change will be written as:

$$\hat{T} = \hat{I}T + \mu\vec{B}\hat{\sigma}\frac{\delta T}{\delta E} \tag{1.91}$$

$$\hat{R} = \hat{I}R + \mu\vec{B}\hat{\sigma}\frac{\delta R}{\delta E} \tag{1.92}$$

where $\hat{I}$ is identity matrix. Here, $\frac{\delta}{\delta E} = \frac{d}{dE} - \left(\frac{\partial}{\partial E}\right)$.

If we apply these transmission and reflection operators in (1.91) and (1.92) respectively, we will get the shifted spin wave-function according to the spin wave-function of the incident particle. Then, by using this shift, one finds the rotation angles as

$$\theta_{Tra} = 2\mu B_0 \text{Im}\left(\frac{\delta \ln |T|}{\delta E}\right) \tag{1.93}$$

$$\theta_{Ref} = 2\mu B_0 \text{Im}\left(\frac{\delta \ln |R|}{\delta E}\right) \tag{1.94}$$

Using these angular rotations in the times we give in Eqs.(1.87) and (1.88), we immediately get

$$\tau_{Tra} = \hbar \text{Im}\left(\frac{\delta \ln |T|}{\delta E}\right) \tag{1.95}$$

$$\tau_{Ref} = \hbar \text{Im}\left(\frac{\delta \ln |R|}{\delta E}\right) \tag{1.96}$$

These are the corresponding Larmor clock for both transmission τ_{Tra} and reflection τ_{Ref} times.

Before ending this subsection, we should give brief criticism about these clock type of approaches. As you notice that, one will introduce many degrees of freedom as a clock. We learned time modulated barrier, and spin degree of freedom achieve this, but there is no limit

here. Which one of the clock approaches leads to a better tunneling time, or which one is true, is an unknown question among these approaches.
As we have pointed earlier in the introduction, obviously there is no consensus on the tunneling time. This is because, there are several approaches existing in the literature, and all of them have some serious disadvantages as we already mentioned while criticising the related approaches above.

1.2.2. Feynman Path Integral Approach

Feynman Path Integral approach is proposed by Sokolovski and Baskin in 1987, and it is conceptually based on the Feynman Path Integration which makes this approach distinctive according to the other approaches we have investigated above. Here, classical trajectories are considered over all paths in the region Ω, and due to this classicality, this approach is also known as *semi-classical approach*. This certain region Ω includes all possible paths between x_1 and x_2 with time $t_2 - t_1$ under the influence of the potential $V(x)$. Note that, t_1 is the time when particle is in x_1, and t_2 is the time when this moving particle is detected at the x_2. Therefore, any particle that is following a classical path **x**(t) in this region Ω will be represented by a time expression that gives the time spent by a particle as you see in Fig.1.4[3], which is

$$\tau_{\Omega}^{cl} = \int_{t_1}^{t_2} \Theta(x(t))dt \tag{1.97}$$

where $\Theta_{\Omega}(x(t))$ is a functional depending on the related path of the particle, and it obeys

$$\Theta_{\Omega}(x(t)) = \begin{cases} 1, & x(t) \in \Omega \\ 0, & \text{otherwise} \end{cases}$$

[3]This image is taken from the (Sokolovski and Baskin, 1987)

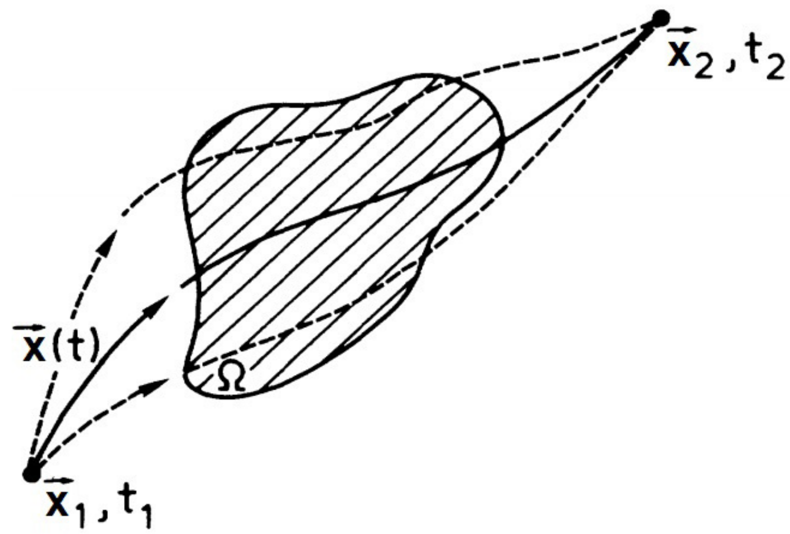

Figure 1.4. Simple illustration of the domain Ω that contains the classical trajectories of Feynman Path Integral approach

this condition above leads to a more general form for the definition of time. In one-dimensional case, Eq.(1.97) will be written as

$$\tau_{\Omega}^{cl} = \int_{t_1}^{t_2} dt \int_{\Omega} \delta(x - x(t))dx \tag{1.98}$$

and thanks to this delta function $\delta(x - x(t))$ property, we are able to ensure that only paths that are belonging to this Ω region are chosen. From here, by taking the mean value for the functional $\tau_{\Omega}^{cl}[x(\cdot)]$, which also means taking the time average of those paths, we get

$$\tau_{\Omega} = \langle \tau_{\Omega}^{cl}[x(\cdot)] \rangle \tag{1.99}$$

Here, for any functional $G[x(\cdot)]$, mean value is taken as

$$\langle G[x(\cdot)] \rangle = \frac{\int Dx(\cdot) G[x(\cdot)] e^{\frac{i}{\hbar} S[x(\cdot)]}}{\int Dx(\cdot) e^{\frac{i}{\hbar} S[x(\cdot)]}} \tag{1.100}$$

where $S[x(\cdot)]$ is the action, and explicitly

$$S[x(\cdot)] = \int_{t_1}^{t_2} L dt = \int_{t_1}^{t_2} \left(\frac{1}{2} m\dot{x}(\cdot)^2 - V(x(\cdot), t \right) dt \tag{1.101}$$

to calculate Eq.(1.99), Sokolovski and Baskin have used the time functional form in Eq.(1.98), and then putting it into the mean value definition in Eq.(1.100), they have got

$$\tau_\Omega = \langle \tau_\Omega^{cl}[x(\cdot)] \rangle = i\hbar \int_{t_1}^{t_2} dt \int_\Omega \frac{\delta \ln \langle x_2, t_2 | x_1, t_1 \rangle}{\delta V(x)} \tag{1.102}$$

where $\langle x_2, t_2 | x_1, t_1 \rangle$ is the propagator which takes particle from x_1 to x_2 with the time interval $t_2 - t_1$. Also $\frac{\delta}{\delta V}$ is a functional derivative, and it is defined for any function $G[f(x)]$ as

$$\frac{\delta G[V(x)]}{\delta V(y)} = \lim_{\epsilon \to 0} \frac{G[V(x) + \epsilon \delta(x - y)] - G[V(x)]}{\epsilon} \tag{1.103}$$

However, this equation above (1.102) is a generalized way of calculating the time using Feynman Path Integral formulation. From here, using the same calculations, now, we have to find an expression for the tunneling time. It is necessary to distinguish reflection and transmission amplitudes for this case, and this will be done by considering particle that is starting its motion far from the left of the barrier, and it is detected after a long time in the right of the barrier. We follow Sokolovski and Baskin, and then set

$$\left.\begin{aligned} x_1 &\to -\infty \\ \tau &\to \infty \\ \frac{|x_1|}{\tau} &= v \\ x_2 &> a \end{aligned}\right\} \tag{1.104}$$

which satisfies the conditions we have mentioned above. With the help of these conditions, authors found

$${}^{T}\tau_\Omega = i\hbar \int_{x_1}^{x_2} \frac{\delta \ln (T e^{i\alpha})}{\delta V(x)} \tag{1.105}$$

is the transmission time and

$$^{R}\tau_{\Omega} = i\hbar \int_{x_1}^{x_2} \frac{\delta \ln{(Re^{i\beta})}}{\delta V(x)} \tag{1.106}$$

is the reflection time where T is the transmission amplitude, R is the reflection amplitude, α and β are the phase shifts for the transmission and reflection respectively.
As you notice by looking at these transmission and reflection times that, corresponding times are complex. Not only these but also the generalized time τ_{Ω} in Eq.(1.102) is complex in general. This complexity of resulting times make sense and they are expected because, we know that for the tunneling process, momentum is imaginary, and therefore, particle has both imaginary velocity and time while following the classical path. Since this approach is a time average of these classical paths inside the region Ω, having complex time is not a surprising result. Here, main criticism about this approach is based on these facts; complex time and also, the other important criticisms are, this resulting time is a average time of the all paths inside the region Ω (Fertig, 1990; Martin, 1996), and there will be important consequences of the interference of these paths (Fertig, 1993; Sokolovski and Connor, 1991, 1993; Mugnai and Ranfagni, 1992) which cannot be controlled.

CHAPTER 2

ENTROPIC TUNNELING TIME

2.1. Derivation of the Entropic Tunneling Time

First, we have to answer the question of how time reveals itself. In other words, what is the key ingredient of expressing time ? In classical realm, the time elapsed while a particle moves from one point to another cannot be determined without knowing its momentum at each point in between. Therefore, holding the information of momentum is the answer (in this section, the content that we are covering is available in (Demir, 2014)). This is because momentum is the generator of translations and, with strict energy conservation, it becomes $\sqrt{2m\,(E - V(x))}$ for a material point moving along x axis with mass m, potential energy $V(x)$ and total energy E. The tunneling region, bounded by the turning points x_1 and $x_2 > x_1$ at which $V(x_1) = E = V(x_2)$ and between which $E < V(x)$, has the particle diffusing with imaginary momentum. This necessitates time to turn into pure imaginary direction $dt \to i d\tau$ where $d\tau = (m/q(x))\,dx$ (see Zeh (2007) for time arrow) with

$$q(x) = \sqrt{2m\,(V(x) - E)} \tag{2.1}$$

being the momentum modulus in tunneling region. The crucial point is that tunneling, being instantaneous in real time, violates causality. Tunneling time, however, must be real and finite (Landauer, 1989) as was discussed in (Low and Mende, 1991; Demir, 1998; Grossmann, 2000) by kinematic approaches (spatial discretum causes Zeno effect (Demir and Sargın, 2014)).

In search for a phenomenological approach that can evade problems with tunneling process, we develop a novel formulation building upon the three observations below:

1. For a proper formulation of tunneling time, we consider only those states having definite momentum, and among them we specialize to that state which diffuses the particle from x_1 to x_2 straight. (Constant potentials and smooth potentials admitting WKB approximation accommodate such states.)

2. Imaginary time takes propagators in quantum mechanics into partition functions in equilibrium statistical mechanics (Feynman and Hibbs, 1965), and thus, we resort to

statistical methods in defining the tunneling time. (Probabilistic interpretation enables statistical methods for individual quantum states.)

3. While the energy E of the particle is fixed everywhere, the thermal energy is not, and it facilitates thus formulation of a real tunneling time representative of the entropy production in tunneling region.

These elements give rise to a consistent framework in which tunneling time can be formulated in verity and its application to most recent attoclock measurements exhibits remarkable agreement with experimental data.

The wave-function describing diffusion from x_1 to x where $x_1 < x < x_2$

$$\psi(x) \propto \frac{1}{\sqrt{q(x)}} e^{-\frac{1}{\hbar}\int_{x_1}^{x} q(x')dx'} \tag{2.2}$$

is an approximate solution of the Schroedinger equation in WKB approximation as we already analysed in Section.1.1.2. This wave-function is an approximate eigenfunction of the momentum operator with approximate eigenvalue $iq(x)$. Its deviation from exact momentum eigenfunction, proportional to $dV(x)/dx$, is small in the WKB regime (see Section.1.1.2). In effect, particles tunnel with a nearly definite momentum under smooth potentials.

The imaginary time, when interpreted as inverse temperature, is known to convert quantum mechanical propagators into statistical mechanical partition functions (Feynman and Hibbs, 1965). This fact can be interpreted to indicate that, statistical methods can provide correct framework for addressing tunneling time problem. In statistical approach, entropy is the core observable and it is proportional to $-p \log p$ for an event with probability p. This Boltzmann definition takes on different forms depending on the system under consideration. It is taken, for instance, $(p - p^r)/(r - 1)$ for non-extensive statistical systems (Tsallis, 1988), $-\log p$ for applications in medical data analysis (Pincus et al., 1991), and $(1/(r-1)) \log p^r$ for entanglement in quantum systems (Renyi, 1961). For quantum tunneling, following (Demir, 2014), we take it in the form

$$S(p) = k_B p \log(1 - \log p) \tag{2.3}$$

where k_B is the Boltzmann constant. Just as the Boltzmann entropy, this novel entropy vanishes at the points of impossibility ($p = 0$) and certainty ($p = 1$). Its loglog form, which exhibits double exponential Gompertz distribution (Gompertz, 1825) even for uniform p, can have requisite distributional structure to account for the diffusive tunneling dynamics stemming from the evanescent wave in (2.2).

In (2.3), p is the probability that the tunneling particle diffuses till x_2 after entering the barrier region at x_1. It is necessarily proportional to $\psi^\dagger(x_2)\psi(x_2)$. Moreover, irrespective of whether the particle tunnels to reach $x = x_2$ or is reflected to reappear at $x = x_1$, at the beginning of scattering, it must be situated at $x = x_1$ with certainty. This condition is fulfilled by taking

$$p = \frac{\psi^\dagger(x_2)\psi(x_2)}{\psi^\dagger(x_1)\psi(x_1)} = e^{-2\Phi} \tag{2.4}$$

where

$$\Phi = \frac{1}{\hbar}\int_{x_1}^{x_2} q(x)dx \tag{2.5}$$

is Hamilton's characteristic function in units of $\hbar$ within the tunneling region. It follows from (2.2). The use of (2.4) in (2.3) returns the entropy produced during diffusion form x_1 to x_2. Thermodynamically, energy rate of change of entropy gives reciprocal temperature, and for the tunneling entropy in (2.3) it reads

$$\frac{1}{T} \quad = \quad \frac{\partial S}{\partial E} = -2\frac{k_B}{\hbar}\tau \exp\left(-2\Phi\right)\left(\frac{1}{1+2\Phi} + \log\frac{1}{1+2\Phi}\right) \tag{2.6}$$

where

$$\tau = \int_{x_1}^{x_2} \frac{mdx}{q(x)} \tag{2.7}$$

is the modulus of the classical tunneling time (which is pure imaginary as mentioned above equation (2.1)). It is not surprising that the temperature T is proportional to the reciprocal τ (Feynman and Hibbs, 1965).

The energy E of the particle is strictly constant throughout its motion inside and outside the tunneling region. It is known with certainty. Then, in view of energy-time uncertainty, only an infinite time lapse can be associated with E. In contrast to E, however, the thermal energy k_BT varies as in (2.6) and physically a finite time interval can well be associated with it. The finite time interval must be nothing but the time elapsed between the disappearance of the particle at x_1 and its reappearance at x_2. In other words, the time interval has every reason to qualify as the tunneling time Δt provided that tunneling process is properly completed. Again, following (Demir, 2014), this requirement is assured by the uncertainty product

$$\Big(T_{rans} \times 2\pi k_B T\Big) \times \Big(\Delta t\Big) = \frac{\hbar}{2} \tag{2.8}$$

where the factor of 2π converts $k_B T$ into Matsubara frequency (Matsubara, 1955) of the particle. In this equation, T_{rans} (1.34) is transmission coefficient from x_1 to x_2. It must be present in (2.8) because in case it vanishes ($T_{rans} \to 0$) the tunneling time must tend to infinity ($\Delta t \to \infty$) in complete consistency with the nonoccurrence of tunneling. In the WKB regime (details of this transmission coefficient corresponding to WKB approach can be found in Appendix (A))

$$T_{rans} = \frac{1}{\cosh^2 \Phi} \tag{2.9}$$

so that the tunneling time in (2.8) takes the form

$$\Delta t = -\frac{\tau}{2\pi} \cosh^2 \Phi \exp\left(-2\Phi\right) \left(\frac{1}{1+2\Phi} + \log \frac{1}{1+2\Phi}\right) \tag{2.10}$$

which is proportional to the classical classical time τ. In the classical limit, where $\Phi \to \infty$, one gets $\Delta t \to \infty$. This infinite limit affirms that the entropic tunneling time is a genuine real time (Low and Mende, 1991), not the norm of an imaginary time like τ.

2.2. Test of the Entropic Tunneling Time by Experiment

We now study if the entropic tunneling time (2.10) agrees with experimental data. The electric fields of high-intensity lasers reshape Coulomb potential in atoms to generate a potential barrier through (see Fig.2.1) which electrons can tunnel to continuum (Keldysh, 1965). At the peak value $\mathcal{E}$ of the electric field, an atomic electron possesses the potential energy

$$V(\xi) = -\frac{Z}{\xi} - \mathcal{E}\xi \tag{2.11}$$

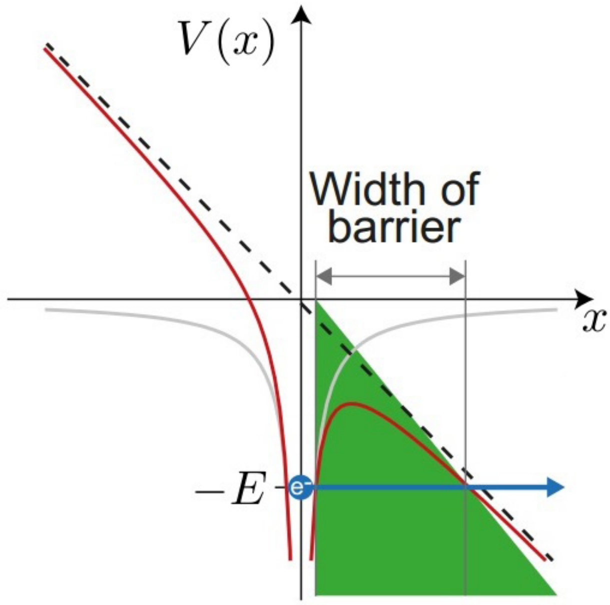

Figure 2.1. Illustration of the potential barrier that is reshaped by the high-intensity laser.

in atomic units (see Appendix B.1 for the detailed calculations of atomic units representation). The Coulomb potential, proportional to the atomic number Z, turns to linear Stark form at large ξ ($\xi = \frac{x}{a_0}$ where a_0 is the Bohr radius). The electron energy E_I remains unchanged up to second order in $\mathcal{E}\xi$ by symmetry. Advancements in ultrafast science, where strong laser fields are used to ionize atoms, are able to observe the tunneling phenomenon and measure the tunneling time (Steinberg et al., 1993; Uiberacker et al., 2007; Eckle et al., 2008; Haessler et al., 2010; Smirnova et al., 2009). In spite of various factors affecting the experiments (Lein, 2012; Torlina et al., 2014; Shafir et al., 2012; Sabbar et al., 2014), improving on previous single-particle tunneling time measurements (Eckle et al., 2008), Landsman and her collaborators have recently performed a refined measurement of the tunneling time of electrons in He atom (Landsman et al., 2013). We base our phenomenological analysis below on the data from this recent experiment.

In the experimental set-up, laser intensity $3.5152 \times 10^{16}\ \mathrm{W/cm^2}\mathcal{E}^2$ is varied from $0.730 \times 10^{14}\ \mathrm{W/cm^2}$ to $7.50 \times 10^{14}\ \mathrm{W/cm^2}$ by varying the peak electric field $\mathcal{E}$ from 0.046 to 0.146 in atomic units (curiously, experiment reports results at $\mathcal{E} = 0.04$, too)(for the details see Appendix B.3). The electron energy $E_I = -0.8941\ \mathrm{a.u.}$ is the first ionization energy of He. Momentum distribution of continuum electrons are obtained by cold-target recoil-ion momentum spectrometer (COLTRIMS) and velocity map imaging spectrometer (VMIS) (see the experiment section of (Landsman et al., 2013) for details). The VMIS is used particularly at low laser intensities. Quantum tunneling is ensured to be the dominant ionization mechanism by keeping Keldysh parameter small. The experimental results are given in Fig. 3 of (Landsman et al., 2013). Depicted in Fig. 3 (a) are different tunneling times that we have already investigated in Section.1.2 contrasted with experiment's own results. Given in Fig. 3 (b) and (d) are tunneling times as functions of the peak electric field $\mathcal{E}$ and barrier width (approximated as $E_I/\mathcal{E}$ a.u. in the experiment). The experiment concludes that among all widely-used tunneling time definitions only the FPI time comes closest to its data points (see Fig. 3 (b) and (d)).

For testing our model, we pick up the potential energy (2.11) and use parameters of the experiment in the formulae leading to Δt in (2.10). By increasing $\mathcal{E}$ from 0.05 a.u. to 0.09 a.u., one finds turning points as: ξ_L ($\xi_L = \frac{\hat{E}_I+\sqrt{\hat{E}_I^2-4\mathcal{E}Z}}{2\mathcal{E}}$) increases from 2.616 a.u. to 3.387 a.u., ξ_R ($\xi_R = \frac{\hat{E}_I-\sqrt{\hat{E}_I^2-4\mathcal{E}Z}}{2\mathcal{E}}$) decreases from 15.290 a.u. to 6.561 a.u. (see Appendix B.2 for detailed calculations of these turning points), and correspondingly, modulus of the classical imaginary time τ decreases from 693.045 as to 396.845 as. At these extremes, the entropic tunneling time Δt decreases from 73.066 as to 12.365 as. Numerically, thus, it takes attosecond times for particle to wend the laser-induced nano-width He barrier. This theoretical prediction affirms the attoclock experiments studying tunneling time (Steinberg et al., 1993). Basically, attosecond tunneling times must be the characteristic time scale in nano-width eV-barriers.

We plot in Figs.(2.2) and (2.3) the entropic tunneling time (2.10) as a function of the peak electric field $\mathcal{E}$ and experiment's barrier width $E_I/\mathcal{E}$ (which is not the true barrier width $\xi_R - \xi_L$). The minimum entropic tunneling time $(\Delta t)_{min}$, shown by filled triangles in Figs. 2.2 and 2.3, is superimposed on Fig. 3 (b) and (d) of (Landsman et al., 2013) [1]. The figure ensures that the minimum of the entropic tunneling time is indeed the shortest time among all theoretical predictions and experimental data (see Fig. 3 (a) of (Landsman et al., 2013)). The FPI time (Sokolovski and Baskin, 1987; Fertig, 1990; Martin, 1996; Sokolovski et al., 1994), the theoretical prediction coming closest to the experimental data, overlies data in the left panel and crosses over data in the right panel. The entropic time, although what is plotted in Figs. 2.2 and 2.3 is only its minimum, performs even better than the FPI time as indicated especially by the right panel. Overall, what is interesting is that, in both panels of Figs. 2.2 and 2.3, the minimum entropic time Δt stays congruent to the experimental data by closely following it from a distance. Barring various factors affecting the experiment (Lein, 2012; Torlina et al., 2014; Shafir et al., 2012; Sabbar et al., 2014; Keldysh, 1965; Landsman

[1] digitized by using `http://arohatgi.info/WebPlotDigitizer/`

et al., 2013)), actual value of the entropic time has every reason to agree with the experiment.
The entropic tunneling time Δt is depicted by filled triangles. It is seen to stay congruent

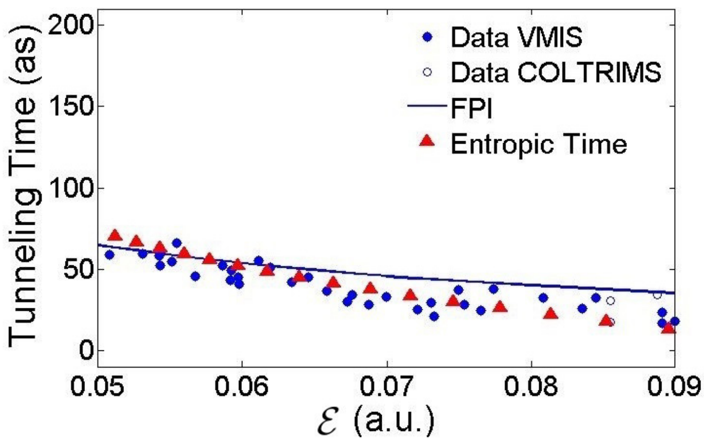

Figure 2.2. Tunneling times as functions of peak electric field $\mathcal{E}$.

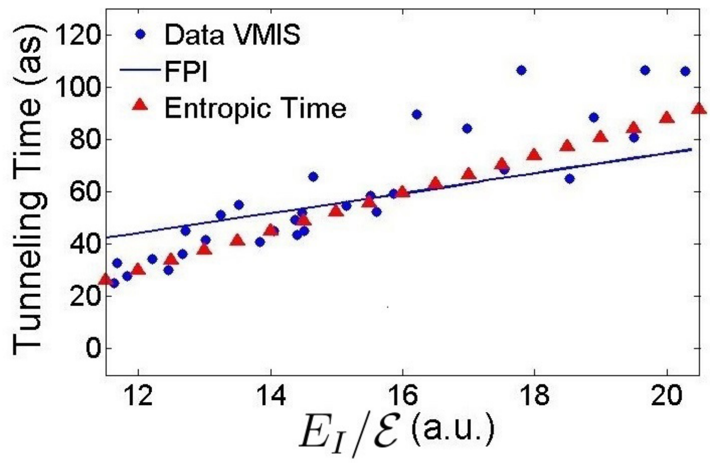

Figure 2.3. Tunneling times as functions of experimental barrier width $E_I/\mathcal{E}$.

to the experimental data throughout. The right panel specifically shows how the minimum entropic time adheres to the experimental data while the FPI time diverges away from data at the asymptotics [2].

In summary, in the present study, we have founded a new tunneling time formula and verified it with most recent experimental data. The formula works. It has far-reaching consequences in basic sciences and technological applications because knowing the lower bound means knowing the highest speed with which the reaction under concern proceeds.

[2] Data of the experiment

CHAPTER 3

APPLICATIONS OF THE ENTROPIC TUNNELING TIME

3.1. α - decay

α - decay is a type of radioactive decay and a well known process. In addition, stability of nucleus has a strict line, and this provides an important information about when α-decay becomes a dominant process for the unstable nuclei. As you see in the Fig.3.1[1], if

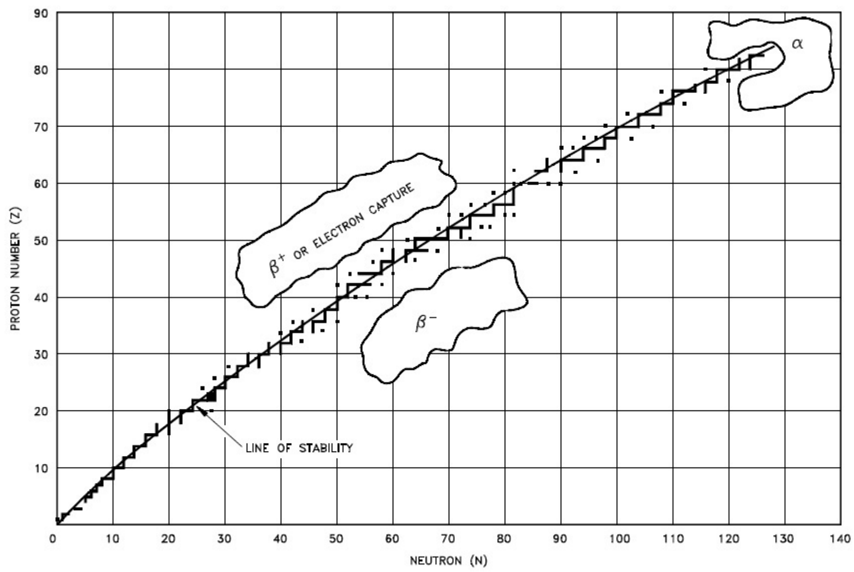

Figure 3.1. Stability curve of the radioactivity

the mass exceeds 83 or neutron number is higher than 126, nucleus starts to emit α particles to become more stable.

How this α escapes from the nucleus was a challenging problem. But it was understood that,

[1] this image is taken from the `http://knowledgepublications.com/doe/doe_nuclear_physics_detail.htm`

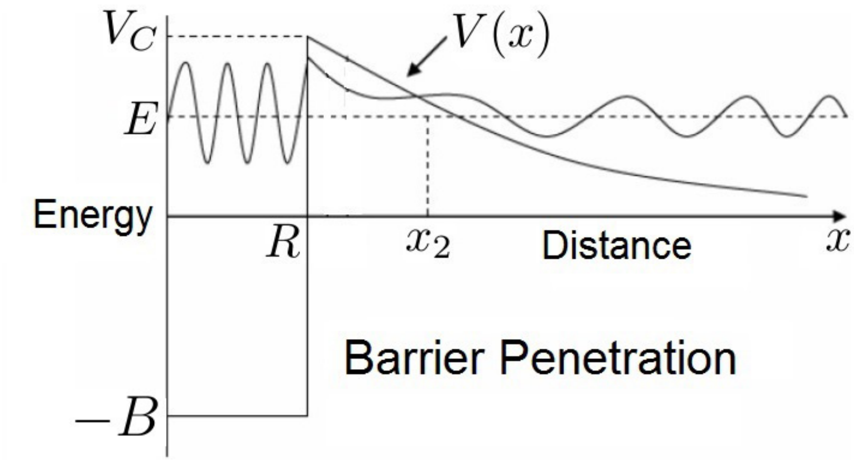

Figure 3.2. Visualization of the α-decay process.

in the nucleus, there exists strong nuclear force which is very strong but effective in very short range, and electromagnetic force that is trying to push the protons out of the nucleus. Around some region close to the radius of the nucleus where this strong force and electromagnetic force becomes comparable as you see in the Fig.(3.2)[2], hence, Coulomb potential governing outside radius of the nucleus provides $V(x)$ has the shape of the potential barrier which quantum tunneling can take place. Of course, we have to note that this α particle has positive energy and is not in a bound state which forbids this decay. One will model the α-decay as:

$$ {}^{A}_{Z}P \rightarrow {}^{A-4}_{Z-2} D + \alpha \tag{3.1} $$

where P represents parent nucleus, D represents daughter nucleus, and we know that α particle is an isotope of He which having a stable He nucleus, has atomic mass number $A = 4$ and atomic number $Z = 2$.

As we have already mentioned Gamow is the one who attempted to solve this problem first.

[2]This image is a modified version of the figure taken from the `http://ictwiki.iitk.ernet.in/wiki/index.php/Spontaneous_Decay_Processes_-_Alpha_Decay`

Now, we are starting to analyse the solution [3] with first considering the potential energy function which is of the form

$$V(x) = \begin{cases} \frac{Z_1 Z_2 e^2}{4\pi\epsilon_0 x}, & x \geq R \\ -B, & x < R \end{cases}$$

where B is the nuclear binding energy, and R is radius of the Parent nucleus. We set $l = 0$ for the centrifugal potential $\frac{l(l+1)\hbar^2}{2mx^2}$ for simplicity. By looking at this potential above, we immediately find the turning point which $E = V(x_2)$, and hence

$$x_2 = \frac{Z_1 Z_2 e^2}{4\pi\epsilon_0 E} \tag{3.2}$$

To calculate tunneling time (2.10)(Demir, 2014; Demir and Guner, 2014) for the α-decay of any radioactive element like Uranium, we have to find Gamow factor Φ of Eq.(1.35). Gamow factor is written for this case with $x_1 = R$ as

$$\Phi = \frac{1}{\hbar}\int_R^{x_2} \sqrt{2m_\alpha(\frac{Z_1 Z_2 e^2}{4\pi\epsilon_0 x} - E)}dx \tag{3.3}$$

where m_α is the mass of the α particle, and if we set

$$\frac{Z_1 Z_2 e^2}{4\pi\epsilon_0 x} = \frac{E x_2}{x} \tag{3.4}$$

we can simplify the term inside of the square root. Therefore, Eq.(3.3) becomes

$$\Phi = \frac{1}{\hbar}\int_R^{x_2} \sqrt{2m_\alpha E(\frac{x_2}{x} - 1)}dx \tag{3.5}$$

---

[3] further details can be found in `https://fenix.tecnico.ulisboa.pt/downloadFile/3779576931836/Ex8Serie1/`

simplified Φ becomes

$$\Phi = \frac{\sqrt{2m_\alpha E}}{\hbar} \int_R^{x_2} \sqrt{\frac{x_2}{x} - 1}dx \tag{3.6}$$

this integral is solved easily when we set

$$x \equiv x_2 \sin^2 u \tag{3.7}$$

which leads to $dx = 2x_2 \cos u \sin u du$, and then

$$\Phi = \frac{\sqrt{2m_\alpha E}}{\hbar} \int \left(\sqrt{\frac{x_2}{x_2 \sin^2 u} - 1} \right) 2x_2 \cos u \sin u du \tag{3.8}$$

using the $\cos^2 u = 1 - \sin^2 u$ for the term in square root, then integration reduces to

$$\Phi = \frac{\sqrt{2m_\alpha E}}{\hbar} 2x_2 \int \cos^2 u du \tag{3.9}$$

resulting integral is simply $\int \cos^2 u du = \frac{1}{2}(u + \sin u \cos u)$, and using (3.7), we get for u as

$$u \equiv \arcsin \sqrt{\frac{x}{x_2}} \tag{3.10}$$

then, analytical solution for the Gamow factor Φ becomes

$$\Phi = \frac{\sqrt{2m_\alpha E}}{\hbar} 2x_2 \frac{1}{2} \left[\arcsin \sqrt{\frac{x}{x_2}} \Bigg|_R^{x_2} + \sin \left(\arcsin \sqrt{\frac{x}{x_2}} \right) \cos \left(\arcsin \sqrt{\frac{x}{x_2}} \right) \Bigg|_R^{x_2} \right] \tag{3.11}$$

and applying integral limits, we get the total result

$$\phi = \frac{\sqrt{2m_\alpha E}}{\hbar} x_2 \left[\frac{\pi}{2} - \arcsin \sqrt{\frac{R}{x_2}} - \sqrt{\frac{R}{x_2}} \cos \left(\arcsin \sqrt{\frac{R}{x_2}} \right) \right] \tag{3.12}$$

In addition to this calculation of Φ, we have to also calculate τ in (2.7) to be able to get the numerical value of our tunneling time (2.10). We write τ for the α-decay as

$$\tau = \int_R^{x_2} \frac{m_\alpha}{q(x)} dx = m_\alpha \int_R^{x_2} \frac{1}{\sqrt{2m_\alpha(\frac{Z_1 Z_2 e^2}{4\pi\epsilon_0 x} - E)}} dx \tag{3.13}$$

following the same calculation of Φ above, what we get is

$$\tau = \frac{m_\alpha}{\sqrt{2m_\alpha E}} \int_R^{x_2} \frac{1}{\sqrt{\frac{x_2}{x} - 1}} dx \tag{3.14}$$

with the help of Mathematica (Wolfram Research, 2010), we get the result for this integral as

$$\tau = \frac{m_\alpha}{\sqrt{2m_\alpha E}} \left[-x\sqrt{\frac{x_2}{x} - 1}\Bigg|_R^{x_2} - \frac{x_2}{2} \arctan\left(\frac{(2x - x_2)\sqrt{\frac{x_2}{x} - 1}}{2(x - x_2)} \right)\Bigg|_R^{x_2} \right] \tag{3.15}$$

applying integral limits

$$\tau = \frac{m_\alpha}{\sqrt{2m_\alpha E}} \left[\frac{x_2}{4\pi} + R\sqrt{\frac{x_2}{R} - 1} + \frac{x_2}{2} \arctan\left(\frac{(2R - x_2)\sqrt{\frac{x_2}{R} - 1}}{2(R - x_2)} \right) \right] \tag{3.16}$$

gives us the analytical expression of τ. Before applying our tunneling time approach to this α-decay via Gamow factor Φ, and τ, we have to mention that, E must be determined by using the binding energy formula. This formula is necessary because this type of nuclear processes involve change in the binding energy, and one will calculate the energy of the α particle by considering it as absorbing the released energy that determines its kinetic energy. Therefore, released energy, and hence, kinetic energy of the α particle is

$$E = (m_{parent} - m_{daughter} - m_\alpha)c^2 \tag{3.17}$$

where m_{parent} is the atomic mass of the parent nucleus, and $m_{daughter}$ is the atomic mass of resulting daughter nucleus. Knowing the energy of α particle is not enough, and also, we have to determine radius of the related nucleus by using an empirical formula [4]

$$\text{Approximated radius of the nucleus:} R \approx 1.07 \times 10^{-15}\text{m} \times A^{\frac{1}{3}} \tag{3.18}$$

where A is the atomic mass number. Putting numerical values of these parameters x_2, R, and E into the expressions of τ, and Φ, we obtain the tunneling times (2.10) for α-decays of different isotopes of nucleus Uranium (U), Thorium (Th), Polonium (Po), Protactinium (Pa), and Bismuth(Bi) as you see in Table 3.1 and in Fig.(3.3) [5].

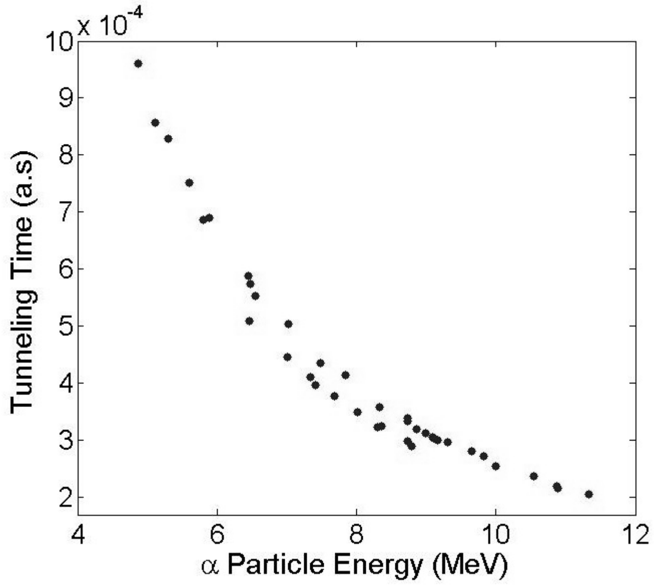

Figure 3.3. Plot of the α-decay tunneling times

[4]details can be found in `http://physics.bu.edu/py106/notes/RadioactiveDecay.html`

[5]Mass of the α-particle is taken as $m_\alpha = 4.001506\text{u}$.

Table 3.1. Calculated tunneling times for α-decay processes of different elements. Here extra term $t_c = \frac{x_2 - R}{c}$ represents the time it takes for a photon to travel this barrier region. *fm* is femtometer, *as* is attosecond, and *u* is atomic mass unit.

Parent Nucleus	m_{parent}(u)	R(fm)	$m_{daughter}$(u)	E(MeV)	Δt(as)	t_c(as)
$^{218}_{92}$U	218.023540	6.439800	214.011500	9.826900	0.000273	0.000066
$^{220}_{92}$U	220.024720	6.459400	216.011062	11.336000	0.000206	0.000055
$^{222}_{92}$U	222.026090	6.478900	218.013284	10.541000	0.000238	0.000060
$^{224}_{92}$U	224.027605	6.498300	220.015748	9.656100	0.000281	0.000068
$^{226}_{92}$U	226.029339	6.517600	222.018468	8.736300	0.000339	0.000077
$^{228}_{92}$U	228.031374	6.536800	224.021467	7.837000	0.000414	0.000088
$^{230}_{92}$U	230.033940	6.555800	226.024903	7.025400	0.000504	0.000101
$^{232}_{92}$U	232.037156	6.574800	228.028741	6.445200	0.000588	0.000112
$^{234}_{92}$U	234.040952	6.593600	230.033139	5.883600	0.000690	0.000125
$^{236}_{92}$U	236.045568	6.612300	232.038055	5.603700	0.000752	0.000132
$^{238}_{92}$U	238.050788	6.631000	234.043601	5.299600	0.000828	0.000141
$^{240}_{92}$U	240.056592	6.649500	236.049870	4.865800	0.000960	0.000155
$^{210}_{90}$Th	210.015075	6.360000	206.003827	9.088000	0.000306	0.000072
$^{212}_{90}$Th	212.012980	6.380100	208.001840	8.987300	0.000313	0.000073
$^{214}_{90}$Th	214.011500	6.400100	210.000495	8.861300	0.000321	0.000074
$^{216}_{90}$Th	216.011062	6.420000	211.999794	9.106700	0.000304	0.000071
$^{218}_{90}$Th	218.013284	6.439800	214.000108	10.887000	0.000216	0.000056
$^{220}_{90}$Th	220.015748	6.459400	216.003533	9.990100	0.000255	0.000063
$^{222}_{90}$Th	222.018468	6.478900	218.007140	9.162700	0.000300	0.000071
$^{224}_{90}$Th	224.021467	6.498300	220.011028	8.333300	0.000358	0.000080
$^{226}_{90}$Th	226.024903	6.517600	222.015375	7.483500	0.000436	0.000091
$^{228}_{90}$Th	228.028741	6.536800	224.020212	6.551500	0.000554	0.000107
$^{230}_{90}$Th	230.033134	6.555800	226.025410	5.800500	0.000687	0.000124
$^{232}_{90}$Th	232.038055	6.574800	228.0310703	5.110800	0.000857	0.000143
$^{190}_{84}$Po	189.995101	6.151300	185.984239	8.727900	0.000300	0.000070
$^{192}_{84}$Po	191.991335	6.172800	187.980874	8.353800	0.000325	0.000074
$^{194}_{84}$Po	193.988186	6.194200	189.978082	8.020800	0.000350	0.000077
$^{196}_{84}$Po	195.985535	6.215400	191.975785	7.690600	0.000378	0.000082
$^{198}_{84}$Po	197.983389	6.236500	193.974012	7.342600	0.000411	0.000086
$^{200}_{84}$Po	199.981799	6.257400	195.972774	7.014200	0.000447	0.000091
$^{214}_{91}$Pa	214.020920	6.400100	210.00944	9.304400	0.000298	0.000070
$^{220}_{91}$Pa	220.02188	6.459400	216.008720	10.872000	0.000220	0.000057
$^{224}_{91}$Pa	224.025626	6.498300	220.014763	8.728900	0.000334	0.000076
$^{230}_{91}$Pa	230.034541	6.555800	226.026098	6.471300	0.000575	0.000110
$^{186}_{83}$Bi	185.996600	6.107800	181.985670	8.791400	0.000291	0.000068
$^{188}_{83}$Bi	187.99227	6.129700	183.981870	8.296900	0.000324	0.000073
$^{192}_{83}$Bi	191.98546	6.172800	187.976010	7.410700	0.000398	0.000084
$^{196}_{83}$Bi	195.980667	6.215400	191.972230	6.465700	0.000509	0.000100

3.2. STM

STM (Scanning Tunneling Microscope) is an important instrument that was developed by Binning and Rohrer in 1981 and they earned the Nobel Prize in Physics in 1986 for this invention. This instrument images surfaces at the atomic level by using quantum tunneling. One can use STM with two different ways: *constant current* and *constant height*. First one provides better resolution since tip arranges itself to make current fixed, and latter one is faster but only appropriate for the atomically flat surfaces since the vertical position of the tip remain unchanged. One can visualize this set up as in Fig.3.4[6]. As you see in

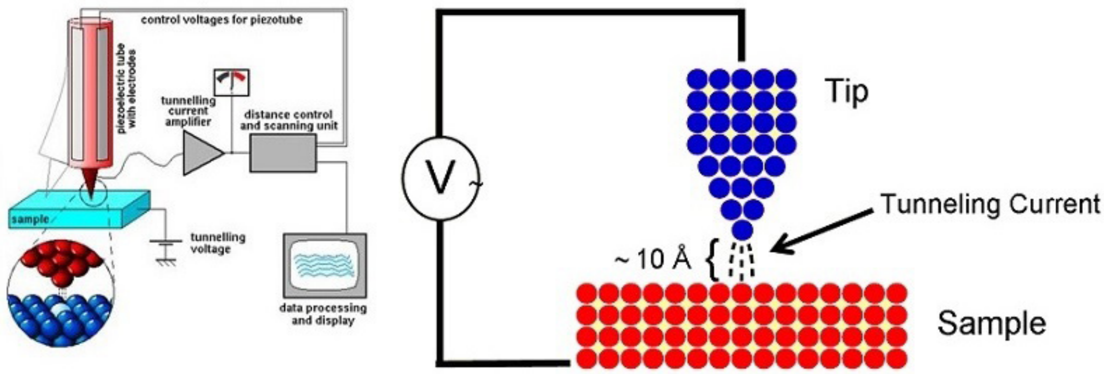

Figure 3.4. Illustration of the STM set-up and tip-sample interaction

this figure, a conducting tip brought very close to the conducting sample. Conductivity is necessary because these free electrons form the tunneling current when bias voltage is applied because this bias voltage make electrons to start tunneling between this gap of tip and sample. This gap which also means a barrier width $x_2 - x_1$ is typically ≤ 10Å for the best resolution because as you see in Eq.(1.19) that, transmission coefficient ($t \approx e^{-2\kappa L}$) decays exponentially as barrier width L ($x_2 - x_1$) increases, and therefore, is very sensitive to change in any distance of tip from the sample.

To apply our tunneling time approach in Eq.(2.10) (Demir, 2014; Demir and Guner, 2014) to

[6]These images are taken from the (left)`http://www.iap.tuwien.ac.at/www/surface/STM_Gallery/stm_schematic.html`, and (right)`http://www.personal.psu.edu/ewh10/ResearchBackground.htm`

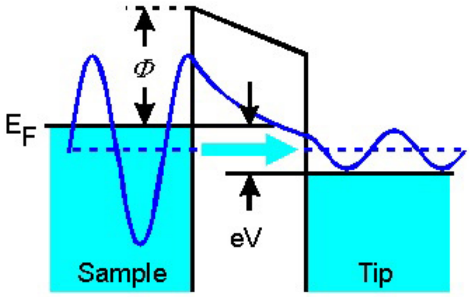

Figure 3.5. Simple illustration of the tunneling process in STM

STM, we have to understand how we deal with this system first. Most of the studies consider potential function as constant, and hence, we treat this system as an approximately a rectangular potential barrier (details and other approaches can be found in (Pitarke et al., 1989) and (Olesen et al., 1996)). This simple approach is illustrated in Fig.3.5[7]. According to this approach, we use κ in Section.1.1.1,

$$\kappa = \frac{\sqrt{2m(\phi - E)}}{\hbar} \tag{3.19}$$

where ϕ is the work-function here, E is electron's energy, and m is the mass of electron. ϕ is work-function because it represents the height that is corresponding to the interval between E_F (fermi-energy level) and top of the barrier. This work-function is approximately taken as average

$$\phi = \frac{1}{2}(\phi_S + \phi_T) \tag{3.20}$$

[7]This picture is taken from the `http://www2.fkf.mpg.de/ga/research/stmtutor/stmtheo.html`

where ϕ_S is the work-function of the sample, and ϕ_T is the work function of the tip. Therefore, Φ in Eq.(1.35) found as

$$\Phi = \frac{\sqrt{2m(\phi - E)}}{\hbar} \int_{x_1}^{x_2} dx \tag{3.21}$$

and if we set turning points as $x_1 = 0$, and $x_2 = L$, result becomes

$$\Phi = \frac{\sqrt{2m(\phi - E)}}{\hbar} L \tag{3.22}$$

from here, we approximately treat the system as [8]

$$\Phi \approx \frac{\sqrt{2m\phi}}{\hbar} L \tag{3.23}$$

since applied bias voltage is small compared to the work-function $\mathrm{eV} \ll \phi$. In other words, if we look at the κ term in (3.19), we see that there is energy of the particle E in terms of eV due to applied voltage V, and if we keep bias voltage very small, then $\phi - E$ term becomes dominated by the work-function ϕ which leads to a Φ solution as we seen in (3.23) or we simply consider that electrons are having energies oscillating around the fermi-energy level E_F that allows us to take $E = E_F = 0$ approximately.
In addition to Φ, we also need to calculate τ (2.7) in this case appropriate for STM where $x_1 = 0$, and $x_2 = L$, which is

$$\tau = \int_0^L \frac{m dx}{q(x)} \tag{3.24}$$

and where $q(x) = \sqrt{2m(\phi - E)} \approx \sqrt{2m\phi}$, then τ becomes

$$\tau \approx \frac{m}{\sqrt{2m\phi}} L \tag{3.25}$$

[8] http://www2.fkf.mpg.de/ga/research/stmtutor/stmtheo.html

We apply our tunneling time formula in (2.10) to STM using these Φ (3.23), and τ (3.25). To obtain numerical data, we consider different samples, and three different tip-sample distances L(1Å, 5Å, and 8Å) as you see in Table 3.2 and Fig.(3.6-3.8), and we take Tungsten (W) as tip material which has a work-function as $\phi_T = 4.55\mathrm{eV}$. We should note that the time it takes for photon to travel these widths are $t_c = 0.3333\mathrm{as}$ for 1Å, $t_c = 1.667\mathrm{as}$ for 5Å, and $t_c = 2.667\mathrm{as}$ for 8Å respectively.

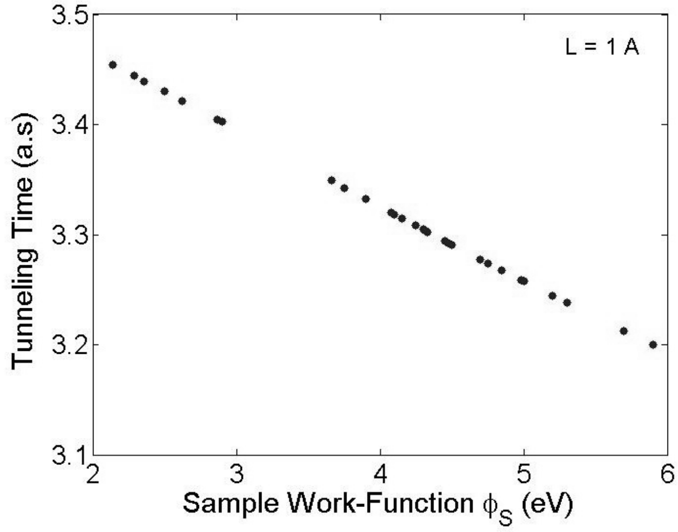

Figure 3.6. Plot of the STM tunneling times for 1Å

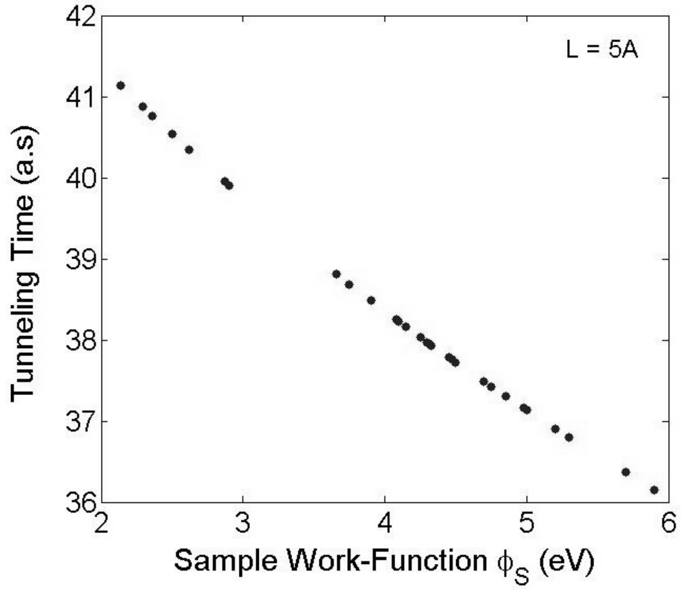

Figure 3.7. Plot of the STM tunneling times for 5Å

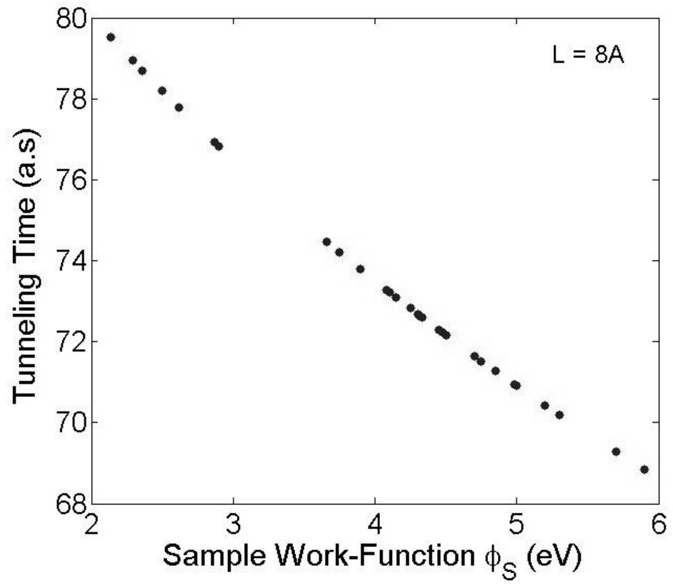

Figure 3.8. Plot of the STM tunneling times for 8Å

Table 3.2. Calculated tunneling times for different samples in STM.

Sample	ϕ_S(eV)	Width (Å)	Δt(as)	Sample	ϕ_S(eV)	Width (Å)	Δt(as)
Ag	4.30	1	3.3046	Al	4.15	1	3.3149
Ag	4.30	5	37.9723	Al	4.15	5	38.1620
Ag	4.30	8	72.6869	Al	4.15	8	73.0915
As	3.75	1	3.3426	Au	5.3	1	3.2379
As	3.75	5	38.6866	Au	5.3	5	36.7961
As	3.75	8	74.2140	Au	5.3	8	70.1892
B	4.45	1	3.2944	Ba	2.62	1	3.4217
B	4.45	5	37.7864	Ba	2.62	5	40.3404
B	4.45	8	72.2906	Ba	2.62	8	77.7812
Be	4.98	1	3.2588	Bi	4.31	1	3.3039
Be	4.98	5	37.1569	Bi	4.31	5	37.9598
Be	4.98	8	70.9530	Bi	4.31	8	72.6602
C	5	1	3.2575	Ca	2.87	1	3.4043
C	5	5	37.1339	Ca	2.87	5	39.9501
C	5	8	70.9044	Ca	2.87	8	76.9354
Cd	4.08	1	3.3197	Li	2.9	1	3.4022
Cd	4.08	5	38.2518	Li	2.9	5	39.9043
Cd	4.08	8	73.2833	Li	2.9	8	76.8362
Ti	4.33	1	3.3026	Cr	4.5	1	3.2910
Ti	4.33	5	37.9349	Cr	4.5	5	37.7252
Ti	4.33	8	72.6069	Cr	4.5	8	72.1603
Cs	2.14	1	3.4543	Cu	4.7	1	3.2775
Cs	2.14	5	41.1347	Cu	4.7	5	37.4842
Cs	2.14	8	79.5104	Cu	4.7	8	71.6479
Eu	2.5	1	3.4300	Fe	4.75	1	3.2741
Eu	2.5	5	40.5332	Fe	4.75	5	37.4249
Eu	2.5	8	78.2000	Fe	4.75	8	71.5219
Ga	4.32	1	3.3032	Hg	4.475	1	3.2927
Ga	4.32	5	37.9473	Hg	4.475	5	37.7557
Ga	4.32	8	72.6335	Hg	4.475	8	72.2253
K	2.29	1	3.4443	Se	5.9	1	3.1997
K	2.29	5	40.8798	Se	5.9	5	36.1550
K	2.29	8	78.9542	Se	5.9	8	68.8368
Mg	3.66	1	3.3489	Mn	4.1	1	3.3184
Mg	3.66	5	38.8086	Mn	4.1	5	38.2260
Mg	3.66	8	74.4757	Mn	4.1	8	73.2283
Na	2.36	1	3.4395	Ni	5.20	1	3.2444
Na	2.36	5	40.7629	Ni	5.20	5	36.9073
Na	2.36	8	78.6997	Ni	5.20	8	70.4246
Pb	4.25	1	3.3080	Pt	5.70	1	3.2122
Pb	4.25	5	38.0351	Pt	5.70	5	36.3638
Pb	4.25	8	72.8208	Pt	5.70	8	69.2766
Si	4.85	1	3.2674	Zn	3.90	1	3.3322
Si	4.85	5	37.3075	Zn	3.90	5	38.4865
Si	4.85	8	71.2724	Zn	3.90	8	73.7854

3.3. Creation of Universe from Nothing

Understanding origin of the universe, and also, trying to provide a satisfactory explanation for the "Universe from nothing" are still ongoing studies in physics. In this section, we want to apply our tunneling time formula (2.10) (Demir, 2014; Demir and Guner, 2014) to the universe that emerges by tunneling at the beginning. This idea of tunneling universe is a point of view towards understanding origin of the universe, and explaining the "Universe from nothing". To do so, first (details can be found in (Stenger, 2006)), we have to understand how this tunneling occurs, and we need to get the Hamiltonian representing this initial stage of the universe to find corresponding potential. Following Atkatz (Atkatz, 1994), one can write action for FRW universe as

$$S \equiv \int dt L = \frac{3\pi c^4}{4G_N} \int dt \left[-\frac{1}{c^2} \left(\frac{\partial a}{\partial t} \right)^2 a + a \left(1 - \frac{a^2}{a_0^2} \right) \right] \tag{3.26}$$

where a is scale factor that represents the radius, $a_0 = \sqrt{\frac{3c^2}{\Lambda}}$, c is speed of light, Λ is cosmological constant, and G_N is Newton's gravitational constant. Therefore, Lagrangian is

$$L = \frac{3\pi c^4}{4G_N} \left[-\frac{1}{c^2} \left(\frac{\partial a}{\partial t} \right)^2 a + a \left(1 - \frac{a^2}{a_0^2} \right) \right] \tag{3.27}$$

and note that, $k = 1$ which means we have considered FRW universe as closed, and depending on the variable a. From here, using our classical mechanics knowledge, we get canonical momentum via Lagrangian above.

$$p_c = \frac{\partial L}{\partial \dot{a}} = \frac{3\pi c^4}{4G_N} \left(-\frac{2a}{c^2} \frac{\partial a}{\partial t} \right) \tag{3.28}$$

where $\dot{a} = \frac{\partial a}{\partial t}$. Then, one gets the Hamiltonian using the canonical momentum p_c we have found above

$$H = p_c \dot{a} - L = p_c \dot{a} - \frac{3\pi c^4}{4G_N} \left[-\frac{1}{c^2} \dot{a}^2 a + a \left(1 - \frac{a^2}{a_0^2} \right) \right] \tag{3.29}$$

and by looking at the Eq.(3.28), if we represent $\dot{a}$ in terms of canonical momentum p_c

$$\dot{a} = \frac{c^2 p_c}{-\left(\frac{3\pi c^4}{4G_N}\right)2a} \tag{3.30}$$

if we put (3.30), in Hamiltonian form (3.29), what we get is

$$H = \frac{c^2 p_c^2}{-\left(\frac{3\pi c^4 a}{2G_N}\right)} - \frac{3\pi c^4}{4G_N}\left[-\frac{ac^4 p_c^2}{\left(-\frac{3\pi c^4 a}{2G_N}\right)^2} + a\left(1 - \frac{a^2}{a_0^2}\right)\right] \tag{3.31}$$

combining all terms leads to a

$$H = -\frac{c^2 p_c^2 G_N}{3\pi c^4 a} - \frac{3\pi c^4}{4G_N} a\big(1 - \frac{a^2}{a_0^2}\big) \tag{3.32}$$

and then

$$H = -\frac{G_N}{3\pi c^4}\left[\frac{c^2 p_c^2}{a} + 4\left(\frac{3\pi c^4}{4G_N}\right)^2 a\big(1 - \frac{a^2}{a_0^2}\big)\right] \tag{3.33}$$

from here, one will express term that contains p_c in terms of $\dot{a}$ using the (3.30), and as a result, one gets

$$H = -\frac{3\pi c^4}{4G_N}\left[\frac{1}{c^2} a\left(\frac{da}{dt}\right)^2 + a\big(1 - \frac{a^2}{a_0^2}\big)\right] \tag{3.34}$$

according to this Hamiltonian, if we set $H = 0$, we immediately get

$$\frac{1}{c^2}\left(\frac{da}{dt}\right)^2 + \big(1 - \frac{a^2}{a_0^2}\big) = 0 \tag{3.35}$$

which is one of the Friedmann equations with $k = 1$. On the other hand, if we use canonical quantization by

$$p_c = -i\hbar\frac{\partial}{\partial a} \tag{3.36}$$

then Hamiltonian (3.33) becomes

$$H = -\frac{G_N}{3\pi c^4}\left[- \frac{c^2\hbar^2}{a}\frac{\partial^2}{\partial a^2} + 4\left(\frac{3\pi c^4}{4G_N}\right)^2 a\big(1 - \frac{a^2}{a_0^2}\big)\right] \tag{3.37}$$

and multiplying and dividing this Hamiltonian by $2m$, and taking into $\frac{1}{a}$ and c^2 parenthesis,

$$H = -\frac{2mG_N c^2}{3\pi c^4 a}\left[- \frac{\hbar^2}{2m}\frac{\partial^2}{\partial a^2} + \frac{4}{2mc^2}\left(\frac{3\pi c^4}{4G_N}\right)^2 a^2\big(1 - \frac{a^2}{a_0^2}\big)\right] \tag{3.38}$$

when we set $H\psi = E_n\psi = 0$,

$$-\frac{2mG_N c^2}{3\pi c^4 a}\left[- \frac{\hbar^2}{2m}\frac{\partial^2}{\partial a^2} + \frac{4}{2mc^2}\left(\frac{3\pi c^4}{4G_N}\right)^2 a^2\big(1 - \frac{a^2}{a_0^2}\big)\right]\psi = 0 \tag{3.39}$$

this Hamiltonian that is having zero eigenvalue for the energy which represents empty space without any energy at the beginning of the universe, is Wheeler-DeWitt equation, and wave-function ψ here is the wave-function of the universe (Hartle and Hawking, 1983).
We can write inside the bracket of Hamiltonian (3.39) in terms of potential and kinetic energy terms.

$$H = -\frac{2mG_N c^2}{3\pi c^4 a}\left[- \frac{\hbar^2}{2m}\frac{\partial^2}{\partial a^2} + V(a)\right]\psi = 0 \tag{3.40}$$

Therefore,

$$V(a) = \frac{4}{2mc^2}\left(\frac{3\pi c^4}{4G_N}\right)^2 a^2\big(1 - \frac{a^2}{a_0^2}\big) \tag{3.41}$$

is the potential function that we are looking for. We illustrate this potential as Fig.(3.9)[9] now, to calculate tunneling time for universe, first, we have to calculate Φ in (2.5) again, re-writing it in terms of a, taking turning points as $x_1 = 0$, $x_2 = a_0$, and setting $E = 0$ because of the

[9]This image is taken from the Atkatz (Atkatz, 1994)

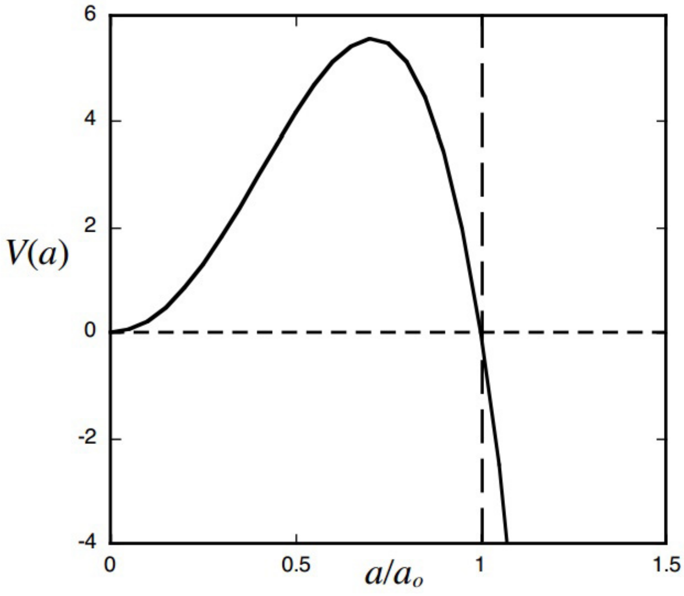

Figure 3.9. Illustration of the potential $V(a)$

Wheeler-DeWitt condition, what we get is

$$\Phi = \frac{1}{\hbar}\int_0^{a_0} \sqrt{2mV(a)}da = \frac{1}{\hbar}\int_0^{a_0} \sqrt{2m\frac{4}{2mc^2}\left(\frac{3\pi c^4}{4G_N}\right)^2 a^2\big(1-\frac{a^2}{a_0^2}\big)}da \tag{3.42}$$

this leads to

$$\Phi = \frac{1}{\hbar}\frac{2}{c}\left(\frac{3\pi c^4}{4G_N}\right)\int_0^{a_0} \sqrt{a^2\big(1-\frac{a^2}{a_0^2}\big)}da \tag{3.43}$$

this integral is taken simply by arranging it as

$$\Phi = \frac{1}{\hbar}\left(\frac{3\pi c^3}{2G_N}\right)\int_0^{a_0} \frac{a}{a_0}\sqrt{a^2-a_0^2}da \tag{3.44}$$

and then setting $a_0^2 - a^2 \equiv u$ gives,

$$\Phi = \frac{1}{\hbar}\left(\frac{3\pi c^3}{2G_N}\right)\left(-\frac{1}{3a_0}\big(a_0^2-a^2\big)^{\frac{3}{2}}\Big|_0^{a_0}\right) \tag{3.45}$$

then solution is

$$\Phi = \frac{1}{\hbar}\left(\frac{3\pi c^3}{2G_N}\right)\frac{a_0^2}{3} \tag{3.46}$$

but we already set $a_0^2 = \frac{3}{\Lambda}$ due to empty space, and

$$\Phi = \frac{1}{\hbar}\left(\frac{3\pi c^3}{2G_N}\right)\frac{1}{\Lambda} \tag{3.47}$$

Since there exists ambiguity on the numerical value of Λ, the precise value of which is unknown, we have to leave our tunneling time formula (2.10) as a function of Λ.
To apply our tunneling time formula (2.10), also, we have to evaluate τ in (2.7), which results as

$$\tau = m\frac{2G_N}{3\pi c^3}\int_0^{a_0}\frac{da}{\sqrt{a^2(1-\frac{a^2}{a_0^2})}} \tag{3.48}$$

and this ntegral is obviously a divergent one. To move on, first, we need to put this integral into non-dimensional form by making $\frac{a}{a_0} = \eta$ which

$$\int_0^{a_0}\frac{da}{\sqrt{a^2(1-\frac{a^2}{a_0^2})}} = \int_0^1\frac{d\eta}{\sqrt{\eta^2(1-\eta^2)}} \tag{3.49}$$

and second, we call this divergent integral

$$\int_0^1\frac{d\eta}{\sqrt{\eta^2(1-\eta^2)}} = C_0 \tag{3.50}$$

and this C_0 is estimated by the Mathematica (Wolfram Research, 2010) approximately as

$$C_0 = 9.59068 \quad \text{(with the error estimation } 0.40208\text{)} \tag{3.51}$$

when we express mass m in terms of planck mass

$$m = \frac{3\pi}{2}\left(\frac{\hbar c}{G_N}\right)^{\frac{1}{2}} \tag{3.52}$$

we get for the τ as

$$\tau = \frac{3\pi}{2}\left(\frac{\hbar c}{G_N}\right)^{\frac{1}{2}} \frac{2G_N}{3\pi c^3} C_0 \tag{3.53}$$

which simplifies to

$$\tau = \left(\frac{\hbar G_N}{c^5}\right)^{\frac{1}{2}} C_0 \tag{3.54}$$

if we put (3.47), and (3.54) into our tunneling time formula in (2.10), we get analytical

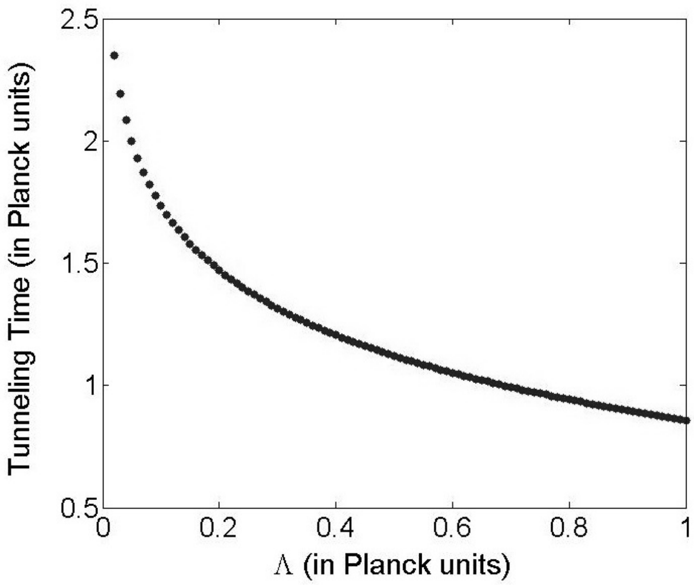

Figure 3.10. Plot of the tunneling time of the universe with respect to cosmological constant Λ in Planck units

expression as a function of Λ,

$$\Delta t = -\frac{\left(\frac{\hbar G_N}{c^5}\right)^{\frac{1}{2}}}{2\pi} C_0 \cosh^2\left(\frac{1}{\hbar}\left(\frac{3\pi c^3}{2G_N}\right)\frac{1}{\Lambda}\right)\exp\left(\frac{-2}{\hbar}\left(\frac{3\pi c^3}{2G_N}\right)\frac{1}{\Lambda}\right)\times \left(\frac{1}{1+\left(\frac{2}{\hbar}\left(\frac{3\pi c^3}{2G_N}\right)\frac{1}{\Lambda}\right)} + \log\frac{1}{1+\left(\frac{2}{\hbar}\left(\frac{3\pi c^3}{2G_N}\right)\frac{1}{\Lambda}\right)}\right) \tag{3.55}$$

however, in cosmological scale, we have to use planck units where $\hbar = G_N = c = 1$, and as a result

$$\Delta t = -\frac{C_0}{2\pi}\cosh^2\left(\frac{3\pi}{2\Lambda}\right)\exp\left(\frac{-3\pi}{\Lambda}\right)\left(\frac{1}{1+\left(\frac{3\pi}{\Lambda}\right)} + \log\frac{1}{1+\left(\frac{3\pi}{\Lambda}\right)}\right) \tag{3.56}$$

where Λ is dimensionless here. This tunneling time is the expression of how long it takes for the universe to tunnel from its origin to the critical radius a_0. If we find Λ precisely, then this tunneling time formula will give us the Planck order of this time. You find in Fig.3.10 tunneling times of the universe as a function of cosmological constant Λ in Planck units.

CHAPTER 4

CONCLUSION

In this thesis work, we proposed a new formulation for tunneling time. Unlike the well-known approaches as we already mentioned in Section.1.2, this entropic tunneling time approach contains statistical methods due to deterministic behaviour of the time, and also due to correlation between propagator and partition function that exists when time flows in imaginary direction. In addition to statistical nature, quantum mechanical contribution as uncertainty principle played a fundamental role while constructing the entropic tunneling time. Then, after deriving the tunneling time formula, we analysed the latest tunneling time experiment (Landsman et al., 2013) in detail, and compared our tunneling time expression with its data. As a result of this comparison, we found that, our theoretical predictions fit well to the data. As the next step, we apply this tunneling time formula to α-decay, STM, and creation of universe from nothing, and our results indicate femtosecond-attosecond times at characteristic sizes. Consequently, the entropic tunneling time formalism, with direct applications given here, can serve as a useful tool to define time costs of various biological, chemical, physical and technological processes.

REFERENCES

Ankerhold, J. (2007). *Quantum Tunneling in Complex Systems: The Semiclassical Approach*. Springer.

Atkatz, D. (1994). Quantum cosmology for pedestrians. *American Journal of Physics 62*(7), 619–627.

Baz', A. (1967a). *Sov. J. Nucl. Phys 4*, 182.

Baz', A. (1967b). *Sov. J. Nucl. Phys 5*, 161.

Binning, G. and H. Rohrer (1993). Scanning tunneling microscopy. In *Scanning Tunneling Microscopy*, pp. 40–54. Springer.

Brillouin, L. (1926). La mécanique ondulatoire de schrödinger; une méthode générale de résolution par approximations successives. *CR Acad. Sci 183*(11), 24–26.

Büttiker, M. (1983). Larmor precession and the traversal time for tunneling. *Physical Review B 27*(10), 6178.

Büttiker, M. and R. Landauer (1982). Traversal time for tunneling. *Physical Review Letters 49*(23), 1739.

Büttiker, M. and R. Landauer (1985). Traversal time for tunneling. *Physica Scripta 32*(4), 429.

Condon, E. and P. Morse (1931). Quantum mechanics of collision processes i. scattering of particles in a definite force field. *Reviews of Modern Physics 3*(1), 43.

Demir, D. A. (1998). Real-time tunneling. *arXiv preprint quant-ph/9809036*.

Demir, D. A. (2014, October). Loglog.entropic tunneling time. submitted.

Demir, D. A. and T. Guner (2014). Tunneling times of selected phenomena. to be submitted.

Demir, D. A. and O. Sargın (2014). Tunneling in polymer quantization and the quantum zeno effect. *Physics Letters A*.

Eckle, P., A. Pfeiffer, C. Cirelli, A. Staudte, R. Dörner, H. Muller, M. Büttiker, and U. Keller (2008). Attosecond ionization and tunneling delay time measurements in helium. *science 322*(5907), 1525–1529.

Esaki, L. (1976). Discovery of the tunnel diode. *Electron Devices, IEEE Transactions on 23*(7), 644–647.

Eser, M. C. (2011). *Tunneling Times*. Ph. D. thesis, Eastern Mediterranean University (EMU).

Fertig, H. (1990). Traversal-time distribution and the uncertainty principle in quantum tunneling. *Physical review letters 65*(19), 2321.

Fertig, H. (1993). Path decomposition and the traversal-time distribution in quantum tunneling. *Physical Review B 47*(3), 1346.

Feynman, R. P. and A. R. Hibbs (1965). *Quantum mechanics and path integrals*. McGraw-Hill.

Gamow, G. (1928a). Quantum theory of atomic nucleus.

Gamow, G. (1928b). The quantum theory of nuclear disintegration. *Nature 122*, 805–806.

Gamow, G. (1931). *Constitution of atomic nuclei and radioactivity*. Oxford.

Gompertz, B. (1825). On the nature of the function expressive of the law of human mortality, and on a new mode of determining the value of life contingencies. *Philosophical transactions of the Royal Society of London*, 513–583.

Griffiths, D. J. (2005). *Introduction to quantum mechanics*. Pearson Education India.

Grossmann, F. (2000). Semiclassical real-time tunneling by multiple spawning of classical trajectories. *Physical review letters 85*(5), 903.

Haessler, S., J. Caillat, W. Boutu, C. Giovanetti-Teixeira, T. Ruchon, T. Auguste, Z. Diveki, P. Breger, A. Maquet, B. Carré, et al. (2010). Attosecond imaging of molecular electronic wavepackets. *Nature Physics 6*(3), 200–206.

Hartle, J. B. and S. W. Hawking (1983). Wave function of the universe. *Physical Review*

D 28(12), 2960.

Hauge, E. and J. Støvneng (1989). Tunneling times: a critical review. *Reviews of Modern Physics 61*(4), 917.

Keldysh, L. (1965). Ionization in the field of a strong electromagnetic wave. *Sov. Phys. JETP 20*(5), 1307–1314.

Kramers, H. A. (1926). Wellenmechanik und halbzahlige quantisierung. *Zeitschrift für Physik 39*(10-11), 828–840.

Lambert, N., Y.-N. Chen, Y.-C. Cheng, C.-M. Li, G.-Y. Chen, and F. Nori (2013). Quantum biology. *Nature Physics 9*(1), 10–18.

Landauer, R. (1989). Barrier traversal time. *Nature 341*, 567–568.

Landauer, R. and T. Martin (1994). Barrier interaction time in tunneling. *Reviews of Modern Physics 66*(1), 217.

Landsman, A., M. Weger, J. Maurer, R. Boge, A. Ludwig, S. Heuser, C. Cirelli, L. Gallmann, and U. Keller (2013). Tunneling time in ultrafast science is real and probabilistic. *arXiv preprint arXiv:1301.2766*.

Leavens, C. and G. Aers (1989). Dwell time and phase times for transmission and reflection. *Physical Review B 39*(2), 1202.

Lein, M. (2012). Atomic physics: Electrons get real. *Nature 485*(7398), 313–314.

Low, F. E. and P. F. Mende (1991). A note on the tunneling time problem. *Annals of Physics 210*(2), 380–387.

MacColl, L. (1932). Note on the transmission and reflection of wave packets by potential barriers. *Physical Review 40*(4), 621.

Maji, K., C. Mondal, and S. Bhattacharyya (2007). Tunnelling time and tunnelling dynamics. *International Reviews in Physical Chemistry 26*(4), 647–670.

Martin, T. (1996). On the path integral approach to the tunneling time. *International Journal of Modern Physics B 10*(27), 3747–3763.

Matsubara, T. (1955). A new approach to quantum-statistical mechanics. *Progress of theoretical physics 14*(4), 351–378.

McMahon, R. J. (2003). Chemical reactions involving quantum tunneling. *Science 299*(5608), 833–834.

Mugnai, D. and A. Ranfagni (1992). Complex classical trajectories in tunnelling: how instanton bounces can become real processes. *Il Nuovo Cimento D 14*(5), 541–551.

Olesen, L., M. Brandbyge, M. R. Sørensen, K. W. Jacobsen, E. Lægsgaard, I. Stensgaard, and F. Besenbacher (1996). Apparent barrier height in scanning tunneling microscopy revisited. *Physical review letters 76*(9), 1485.

Olkhovsky, V. S. and E. Recami (1992). Recent developments in the time analysis of tunneling processes. *Physics Reports 214*(6), 339–356.

Olkhovsky, V. S., E. Recami, and J. Jakiel (2001). Unified time analysis of photon and (nonrelativistic) particle tunnelling, and the superluminal group-velocity problem. *arXiv preprint quant-ph/0102007*.

Pincus, S. M., I. M. Gladstone, and R. A. Ehrenkranz (1991). A regularity statistic for medical data analysis. *Journal of clinical monitoring 7*(4), 335–345.

Pitarke, J., P. Echenique, and F. Flores (1989). Apparent barrier height for tunneling electrons in stm. *Surface Science 217*(1), 267–275.

Privitera, G., G. Salesi, V. S. Olkhovsky, and E. Recami (2004). Tunnelling times: an elementary introduction. *arXiv preprint quant-ph/0412146*.

Razavy, M. (2003). *Quantum theory of tunneling*, Volume 1222445599. World Scientific.

Renyi, A. (1961). Proceedings of the 4th berkeley symposium on mathematics and statistical probability.

Rybachenko, V. (1967). On particle penetration through a barrier. *Soviet Journal of Nuclear Physics 5*, 484.

Sabbar, M., S. Heuser, R. Boge, M. Lucchini, L. Gallmann, C. Cirelli, and U. Keller (2014).

Attochirp-corrected photo ionization time delays using coincidence attosecond streaking. *arXiv preprint arXiv:1407.6623*.

Shafir, D., H. Soifer, B. D. Bruner, M. Dagan, Y. Mairesse, S. Patchkovskii, M. Y. Ivanov, O. Smirnova, and N. Dudovich (2012). Resolving the time when an electron exits a tunnelling barrier. *Nature 485*(7398), 343–346.

Smirnova, O., Y. Mairesse, S. Patchkovskii, N. Dudovich, D. Villeneuve, P. Corkum, and M. Y. Ivanov (2009). High harmonic interferometry of multi-electron dynamics in molecules. *Nature 460*(7258), 972–977.

Smith, F. T. (1960). Lifetime matrix in collision theory. *Physical Review 118*(1), 349.

Sokolovski, D. and L. Baskin (1987). Traversal time in quantum scattering. *Physical Review A 36*(10), 4604.

Sokolovski, D., S. Brouard, and J. Connor (1994). Traversal-time wave-function analysis of resonance and nonresonance tunneling. *Physical Review A 50*(2), 1240.

Sokolovski, D. and J. Connor (1991). Negative probability and the distributions of dwell, transmission, and reflection times for quantum tunneling. *Physical Review A 44*(3), 1500.

Sokolovski, D. and J. Connor (1993). Quantum interference and determination of the traversal time. *Physical Review A 47*(6), 4677.

Steinberg, A. M., P. G. Kwiat, and R. Y. Chiao (1993). Measurement of the single-photon tunneling time. *Physical Review Letters 71*(5), 708.

Stenger, V. J. (2006). *The comprehensible cosmos: where do the laws of physics come from?* Prometheus Books.

Torlina, L., F. Morales, J. Kaushal, H. G. Muller, I. Ivanov, A. Kheifets, A. Zielinski, A. Scrinzi, M. Ivanov, and O. Smirnova (2014). Interpreting attoclock measurements of tunnelling times. *arXiv preprint arXiv:1402.5620*.

Tsallis, C. (1988). Possible generalization of boltzmann-gibbs statistics. *Journal of statistical physics 52*(1-2), 479–487.

Uiberacker, M., T. Uphues, M. Schultze, A. J. Verhoef, V. Yakovlev, M. F. Kling, J. Rauschenberger, N. M. Kabachnik, H. Schröder, M. Lezius, et al. (2007). Attosecond real-time observation of electron tunnelling in atoms. *Nature 446*(7136), 627–632.

Wentzel, G. (1926). Eine verallgemeinerung der quantenbedingungen für die zwecke der wellenmechanik. *Zeitschrift für Physik 38*(6-7), 518–529.

Wigner, E. P. (1955). Lower limit for the energy derivative of the scattering phase shift. *Physical Review 98*(1), 145.

Winful, H. G. (2006). Tunneling time, the hartman effect, and superluminality: A proposed resolution of an old paradox. *Physics Reports 436*(1), 1–69.

Wolfram Research, I. (2010). *Mathematica* (Version 8.0 ed.). Wolfram Research, Inc.

Yamada, N. (2004). Unified derivation of tunneling times from decoherence functionals. *Physical review letters 93*(17), 170401.

Zeh, H. D. (2007). *The physical basis of the direction of time*. Springer.

APPENDIX A

TRANSMISSION COEFFICIENT IN WKB APPROACH

According to Section.1.1.2, we have found that wavefunction is of the form (1.33). However, we have mentioned that, solution is a linear combination which we didn't consider the whole solution in that section. Now, to calculate transmission coefficient appropriately [1], we have to write down solution for the wavefunction $\psi(x)$ for all regions. By looking at Fig.1.2, we separate solution for every region. Therefore, similar to the simple rectangular potential barrier case in Section.(1.1.1), we write

$$\psi(x) = \begin{cases} \frac{A}{\sqrt{p(x)}} e^{\frac{i}{\hbar}\int_{x_1}^{x} p(x)dx} + \frac{R}{\sqrt{p(x)}} e^{-\frac{i}{\hbar}\int_{x_1}^{x} p(x)dx}, & x \leq x_1 \\ \frac{C}{\sqrt{q(x)}} e^{\frac{1}{\hbar}\int_{x_{1,2}}^{x} q(x)dx} + \frac{D}{\sqrt{q(x)}} e^{-\frac{1}{\hbar}\int_{x_{1,2}}^{x} q(x)dx}, & x_1 \leq x \leq x_2 \\ \frac{T}{\sqrt{p(x)}} e^{\frac{i}{\hbar}\int_{x_2}^{x} p(x)dx}, & x \geq x_2 \end{cases} \tag{A.1}$$

where $p(x) = \sqrt{2m(E - V(x))}$. One immediately notices that, both $p(x)$ and $q(x)$ diverge at the turning points. Therefore, boundary conditions containing these points become problematic, and without using the boundary conditions, finding transmission and reflection coefficients are not possible. Solution is, we have to analyse neighbourhood of these turning points with approximating the change of the the potential as linear.

We have two cases here, around x_1 which $V'(x_1) = \frac{\partial V(x_1)}{\partial x} > 0$, and around x_2 which $V'(x_2) = \frac{\partial V(x_2)}{\partial x} < 0$. For the case of $V'(x_1) > 0$, we write potential as

$$V(x) \approx V(x_1) + V'(x_1)(x - x_1) \tag{A.2}$$

[1]Further details of this calculation of the transmission coefficient can be found in, Marcus A. M. de Aguiar, "The WKB Approximation"(lecture, IFGW - UNICAMP, November 12, 2013). http://sites.ifi.unicamp.br/aguiar/files/2014/10/wkb1.pdf

and we know that $V(x_1) = E$ since x_1 is turning point, therefore

$$V(x) \approx E + V'(x_1)(x - x_1) \tag{A.3}$$

if we put (A.3) into the time-independent Schroedinger equation,

$$\left(\frac{\hat{p}^2}{2m} + E + V'(x_1)(x - x_1)\right)\psi(x) = E\psi(x) \tag{A.4}$$

which is then

$$\left(\frac{\hat{p}^2}{2m} + V'(x_1)x\right)\psi(x) = V'(x_1)x_1\psi(x) \tag{A.5}$$

and we have to solve this equation. Since $\hat{p}^2$ term contains second order differentiation, first, we turn this position representation to the momentum representation $\psi(x) \to \psi(p)$ to make it easier, and after finding the solution in terms of momentum wave-function, we use Fourier transformation to get wave-function in position representation. So, according to the momentum representation which $\hat{x} = i\hbar\frac{\partial}{\partial p}$, $\hat{p} = p$, (A.5) becomes

$$\frac{p^2}{2m}\psi(p) + i\hbar V'(x_1)\frac{\partial\psi(p)}{\partial p} = V'(x_1)x_1\psi(p) \tag{A.6}$$

and rearranging this equation

$$\frac{\psi'(p)}{\psi(p)} = i\left(\frac{p^2 - 2mV'(x_1)x_1}{2m\hbar V'(x_1)}\right) \tag{A.7}$$

where $\psi'(p) = \frac{\partial\psi(p)}{\partial p}$. And, solution of this first order differential equation is simply,

$$\psi(p) = Ce^{\frac{ip}{6m\hbar V'(x_1)}(p^2 - 6mx_1V'(x_1))} \tag{A.8}$$

we find momentum wavefunction, and therefore, position wavefunction is

$$\psi(x) = \frac{C}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} e^{\frac{ip}{6m\hbar V'(x_1)}(p^2 - 6mx_1V'(x_1))} e^{\frac{ipx}{\hbar}} dp \tag{A.9}$$

from here, since cosine term is symmetric, and sinus term is antisymmetric between $-\infty$ and ∞ according to the point zero, we write the exponential function above only depending on the cosine term with limits from 0 to ∞. Therefore, if we set $\frac{2C}{\sqrt{2\pi\hbar}} \to C'$, then

$$\psi(x) = C' \int_{0}^{\infty} \cos\left(\frac{p^3}{6m\hbar V'(x_1)} + \frac{p(x - x_1)}{\hbar}\right) dp \tag{A.10}$$

to see what this integral leads to, we should define variables

$$j = p\left(\frac{1}{2m\hbar V'(x_1)}\right)^{\frac{1}{3}} \tag{A.11}$$

and

$$z = (x_1 - x)\left(\frac{2mV'(x_1)}{\hbar^2}\right)^{\frac{1}{3}} \tag{A.12}$$

then, (A.10) becomes

$$\psi(x) = C'' \int_{0}^{\infty} \cos\left(\frac{j^3}{3} - jz\right) dj \tag{A.13}$$

where $C' \to C''$ due to emerging constants while $dp \to (2m\hbar V'(x_1))^{1/3} dj$. Notice that, Eq.(A.13) is nothing but the

$$\psi(x) = D\frac{1}{\pi} \int_{0}^{\infty} \cos\left(\frac{j^3}{3} - jz\right) dj \equiv DAi(-z) \tag{A.14}$$

where $Ai(-z)$ is the airy function. Since $z = 0$ at the turning points $x_1 = x$, we are not able to benefit from this result. Therefore, assuming $\hbar^2/mV'(x_1) \ll 1$, it makes z to become very large when move away from the turning points. By doing this, on find approximate wavefunctions $\psi(x)$ representing the WKB approximation. Since z becomes very large, integral in Eq.(A.13) starts to oscillate very rapidly. Hence, we need to find stationary behaviour of $\frac{\partial}{\partial j}(\frac{j^3}{3} - jz) = 0$ that is hiding inside this cosine function which makes some constant contribution to the integral. Following this assumption, and using the integral representation of the airy function, we get

$$Ai(-z) = \begin{cases} \sqrt{\frac{\pi}{\sqrt{z}}} \cos\left(\frac{2z^{3/2}}{3} - \pi/4\right), & x < x_1, \text{hence } z > 0 \\ \sqrt{\frac{\pi}{4\sqrt{-z}}} e^{-\frac{2}{3}(-z)^{3/2}}, & x > x_1, \text{hence } z < 0 \end{cases} \tag{A.15}$$

now, we have to arrange them in terms of $q(x)$ and $p(x)$ to make these to become most appropriate form for the WKB approach. We achieve this by

$$\int_{x_1}^{x} q(x')dx' = \int_{x_1}^{x} \sqrt{2m(V(x) - E)}dx' \tag{A.16}$$

using (A.3)

$$\int_{x_1}^{x} q(x')dx' \approx \int_{x_1}^{x} \sqrt{2mV'(x_1)(x - x_1)}dx' \tag{A.17}$$

this integration then results as

$$\int_{x_1}^{x} q(x')dx' \approx \frac{2}{3}(x - x_1)^{3/2}\sqrt{2mV'(x_1)} = \hbar\frac{2}{3}(-z)^{3/2} \tag{A.18}$$

with the same way,

$$\int_{x_{1,2}}^{x} p(x')dx' \approx -\frac{2}{3}(x_1 - x)^{3/2}\sqrt{2mV'(x_1)} = -\hbar\frac{2}{3}z^{3/2} \tag{A.19}$$

using these and

$$z^{1/2} = \frac{p(x)}{(2m\hbar V'(x_1))^{1/3}} \tag{A.20}$$

$$(-z)^{1/2} = \frac{q(x)}{(2m\hbar V'(x_1))^{1/3}} \tag{A.21}$$

then, what we get for this case of $V'(x_1) > 0$ is

$$\psi(x) \approx \begin{cases} \frac{1}{\sqrt{p(x)}} \cos\left(\frac{1}{\hbar}\int_{x_1}^{x} p(x')dx' + \pi/4\right), & x < x_1 \\ \frac{1}{2\sqrt{q(x)}} e^{-\frac{1}{\hbar}\int_{x_1}^{x} q(x')dx'}, & x > x_1 \end{cases} \tag{A.22}$$

Now, for the case of $V'(x_2) < 0$, again following the same steps that we have already done above, we immediately get

$$\psi(x) \approx \begin{cases} \frac{1}{\sqrt{p(x)}} \cos\left(\frac{1}{\hbar}\int_{x_2}^{x} p(x')dx' - \pi/4\right), & x > x_2 \\ \frac{1}{2\sqrt{q(x)}} e^{\frac{1}{\hbar}\int_{x_2}^{x} q(x')dx'}, & x < x_2 \end{cases} \tag{A.23}$$

However, these $\psi(x)$ solutions around the turning points x_1 and x_2 in terms of the approximated Airy function $Ai(-z)$ cannot be enough alone for the case of tunneling. We will explain why this is so, but now, we have to understand what is necessary to get full solution. By looking at the equation in (A.5), if we continue that equation on the position representation, we get

$$-\frac{\hbar^2}{2m}\frac{\partial^2\psi(x)}{\partial x^2} + V'(x_t)(x - x_t)\psi(x) = 0 \tag{A.24}$$

where x_t represents the turning points and using the definition of z in (A.12) while, we get

$$\frac{\partial^2\psi(x)}{\partial z^2} + z\psi(x) = 0 \tag{A.25}$$

and this is nothing but the differential form of the Airy functions. As you notice that, this is a second order differential equation, and we have focused on only one solution which is (A.14). The other solution is known as in the form

$$\psi(x) = \int_0^\infty \left(e^{-j^3/3 - zj} + \sin j^3/3 - jz\right)dj \equiv Bi(-z) \tag{A.26}$$

Using this solution, and again following the same steps which we have done while dealing with $Ai(-z)$ above, we would get for $Bi(-z)$ in the case of $V'(x_1) > 0$

$$\psi(x) \approx \begin{cases} \frac{1}{\sqrt{p(x)}} \cos\left(\frac{1}{\hbar}\int_{x_1}^{x} p(x')dx' - \pi/4\right), & x < x_1 \\ \frac{1}{\sqrt{q(x)}} e^{\frac{1}{\hbar}\int_{x_1}^{x} q(x')dx'}, & x > x_1 \end{cases} \tag{A.27}$$

and for in the case of $V'(x_2) < 0$,

$$\psi(x) \approx \begin{cases} \frac{1}{\sqrt{p(x)}} \cos\left(\frac{1}{\hbar}\int_{x_2}^{x} p(x')dx' + \pi/4\right), & x > x_2 \\ \frac{1}{\sqrt{q(x)}} e^{-\frac{1}{\hbar}\int_{x_2}^{x} q(x')dx'}, & x < x_2 \end{cases} \tag{A.28}$$

since we are dealing with the tunneling case, barrier region remains finite, and therefore, resulting $\psi(x)$ from using $Bi(-z)$ cannot diverge as $x \to \pm\infty$. However, unlike the case of tunneling, if it is the case of boundary states, then one lets $x \to \pm\infty$ which diverges the corresponding $\psi(x)$. This property of $Bi(-z)$ in boundary state problem, forces one to set the coefficient corresponding this function to zero, then $Bi(-z)$ vanishes. As a result, we conclude that, in the tunneling case, it is appropriate to use also $Bi(-z)$.
Now, we know that, solution for the wave-function $\psi(x)$ is (A.1), and we have found that, to find these coefficients, solution must include a linear combination of the two independent solutions $Ai(-z)$ and $Bi(-z)$ for all regions: region I, region II, and region III. Therefore, for the case of $V'(x_2) < 0$,

$$\psi(x) \approx KAi(-z) + LBi(-z) \tag{A.29}$$

is the expected solution. For region III, and then taking the $x > x_2$ solutions both from the $Ai(-z)$ and $Bi(-z)$, we get

$$\psi(x) = \frac{K}{\sqrt{p(x)}} \cos\left(\frac{1}{\hbar}\int_{x_2}^{x} p(x')dx' - \pi/4\right) + \frac{L}{\sqrt{p(x)}} \cos\left(\frac{1}{\hbar}\int_{x_2}^{x} p(x')dx' + \pi/4\right) \tag{A.30}$$

if we set$L = iK$, then combining this linear combination results as

$$\psi(x) \approx \frac{K}{\sqrt{p(x)}} e^{\frac{i}{\hbar}\int_{x_2}^{x} p(x')dx' - \pi/4} \tag{A.31}$$

and we have to compare this solution above with the $x \geq x_2$ case in (A.1), what we have found is that

$$T = Ke^{-i\frac{\pi}{4}} \tag{A.32}$$

For the region II, which we have to consider $x < x_2$ in this case, and hence, again by looking at solutions both from $Ai(-z)$ and $Bi(-z)$, we have

$$\psi(x) \approx \frac{K}{2\sqrt{q(x)}} e^{\frac{1}{\hbar}\int_{x_2}^{x} q(x')dx'} + \frac{L}{\sqrt{q(x)}} e^{-\frac{1}{\hbar}\int_{x_2}^{x} q(x')dx'} \tag{A.33}$$

again if we set $L = iK$, then result becomes

$$\psi(x) \approx K\left(\frac{1}{2\sqrt{q(x)}} e^{\frac{1}{\hbar}\int_{x_2}^{x} q(x')dx'} + \frac{i}{\sqrt{q(x)}} e^{-\frac{1}{\hbar}\int_{x_2}^{x} q(x')dx'}\right) \tag{A.34}$$

and when we compare this solution with $x_1 \leq x \leq x_2$ in (A.1), then using (A.32), what we obtain is

$$C = \frac{K}{2} = \frac{T}{2} e^{i\frac{\pi}{4}} \tag{A.35}$$

and

$$D = iK = iTe^{i\frac{\pi}{4}} \tag{A.36}$$

we have done our analysis for he case of $V'(x_2) < 0$. Now, we should investigate the $V'(x_1) > 0$ case, which solution of the form is expected now

$$\psi(x) \approx NAi(-z) + MBi(-z) \tag{A.37}$$

then for the region II, we have to consider now $x > x_1$ cases for both $Ai(-z)$ and $Bi(-z)$ which

$$\psi(x) \approx \frac{N}{2\sqrt{q(x)}} e^{-\frac{1}{\hbar}\int_{x_1}^{x} q(x')dx'} + \frac{M}{\sqrt{q(x)}} e^{\frac{1}{\hbar}\int_{x_1}^{x} q(x')dx'} \tag{A.38}$$

from here, to make this equation look similar to the solution above (A.34), we have to rearrange integrals, if we define

$$\Phi = \frac{1}{\hbar}\int_{x_1}^{x_2} q(x)dx \tag{A.39}$$

and then if we separate integrals

$$\int_{x_1}^{x} q(x')dx' = \int_{x_1}^{x_2} q(x)dx + \int_{x_2}^{x} q(x')dx' = \hbar\Phi + \int_{x_2}^{x} q(x')dx' \tag{A.40}$$

then using this property of integral in (A.38), we get

$$\psi(x) \approx \frac{N}{2\sqrt{q(x)}} e^{-\frac{1}{\hbar}(\hbar\Phi + \int_{x_2}^{x} q(x')dx')} + \frac{M}{\sqrt{q(x)}} e^{\frac{1}{\hbar}(\hbar\Phi + \int_{x_2}^{x} q(x')dx')} \tag{A.41}$$

then one gets

$$\psi(x) \approx \frac{Ne^{-\Phi}}{2\sqrt{q(x)}} e^{-\frac{1}{\hbar}\int_{x_2}^{x} q(x')dx'} + \frac{Me^{\Phi}}{\sqrt{q(x)}} e^{\frac{1}{\hbar}\int_{x_2}^{x} q(x')dx'} \tag{A.42}$$

then, we have to compare this equation with the (A.34), and (A.1). Comparing with (A.34) is necessary because unlike the other cases which there exists one to one correspondence between region I, and region III in terms of Airy solutions and WKB solutions (A.1) as we see above, here, for the region II, we have to understand two different solutions must represent one unique solution in this region. $V'(x_1) > 0$ where $x > x_1$, and $V'(x_2) < 0$ where $x < x_2$ cases are the solutions corresponding to same wave-function, therefore, according to this fact, since we have already set C in (A.1) corresponds to K which is the coefficient of $Ai(-z)$ in (A.29), and set D in (A.1) corresponds to iK which is the coefficient of $Bi(-z)$ in(A.29), we must compare C and D coefficients in (A.1) with the coefficients of $Ai(-z)$ which is N, and coefficient of $Bi(-z)$ which is M for this case now. This change of sign of the integral in this case ($V'(x_1) > 0$ where $x > x_1$) according to the previous case ($V'(x_2) < 0$ where $x < x_2$) are already encoded in their related airy functions $Ai(-z)$,and $Bi(-z)$, and therefore, instead of determining coefficients by looking at their integral signs as we did for previous cases, we must compare these coefficients by looking at their corresponding airy functions. Then, this statement leads to a solution for the coefficients

$$\frac{Ne^{-\Phi}}{2} = C = \frac{K}{2} = \frac{T}{2}e^{i\frac{\pi}{4}} \tag{A.43}$$

and

$$Me^{\Phi} = D = iK = iTe^{i\frac{\pi}{4}} \tag{A.44}$$

which then leaving N and M alone gives

$$N = Te^{i\frac{\pi}{4}+\Phi} \tag{A.45}$$

and

$$M = iTe^{i\frac{\pi}{4}-\Phi} \tag{A.46}$$

we are now in the last part of this calculation. We have to consider region I, which we need to look at the solutions of both $Ai(-z)$ and $Bi(-z)$ for the case of $x < x_1$. Therefore, again using the expected solution (A.37),

$$\psi(x) \approx \frac{N}{\sqrt{p(x)}} \cos\left(\frac{1}{\hbar}\int_{x_1}^{x} p(x')dx' + \pi/4\right) + \frac{M}{\sqrt{p(x)}} \cos\left(\frac{1}{\hbar}\int_{x_1}^{x} p(x')dx' - \pi/4\right) \tag{A.47}$$

writing these cosine functions in terms of their exponential equivalent form

$$\psi(x) \approx \frac{N}{\sqrt{p(x)}}\left(\frac{e^{i(\frac{1}{\hbar}\int_{x_1}^{x} p(x')dx'+\pi/4)} + e^{-i(\frac{1}{\hbar}\int_{x_1}^{x} p(x')dx'+\pi/4)}}{2}\right)+ \\ \frac{M}{\sqrt{p(x)}}\left(\frac{e^{i(\frac{1}{\hbar}\int_{x_1}^{x} p(x')dx'-\pi/4)} + e^{-i(\frac{1}{\hbar}\int_{x_1}^{x} p(x')dx'-\pi/4)}}{2}\right) \quad \text{(A.48)}$$

then if we rearrange above,

$$\psi(x) \approx \frac{1}{\sqrt{p(x)}}\left(\frac{Ne^{i\frac{\pi}{4}} + Me^{-i\frac{\pi}{4}}}{2}\right)e^{\frac{i}{\hbar}\int_{x}^{x_1} p(x')dx'}+ \\ \frac{1}{\sqrt{p(x)}}\left(\frac{Ne^{-i\frac{\pi}{4}} + Me^{i\frac{\pi}{4}}}{2}\right)e^{-\frac{i}{\hbar}\int_{x}^{x_1} p(x')dx'} \quad \text{(A.49)}$$

comparing this result with the $x \leq x_1$ case in (A.1), what we get for the coefficient A is

$$A = \frac{Ne^{i\frac{\pi}{4}} + Me^{-i\frac{\pi}{4}}}{2} \quad \text{(A.50)}$$

then putting (A.45) and (A.46), result becomes

$$A = \frac{Te^{i\frac{\pi}{4}+\Phi}e^{i\frac{\pi}{4}} + iTe^{i\frac{\pi}{4}-\Phi}e^{-i\frac{\pi}{4}}}{2} = \frac{Te^{i\frac{\pi}{2}+\Phi} + iTe^{-\Phi}}{2} \quad \text{(A.51)}$$

and if we write $e^{\frac{i\pi}{2}} = i$, then

$$A = \frac{iTe^{\Phi} + iTe^{-\Phi}}{2} = iT\left(\frac{e^{\Phi} + e^{-\Phi}}{2}\right) = iT\cosh\Phi \quad \text{(A.52)}$$

using the definition of transmission coefficient, which is

$$T_{rans} = \frac{|T|^2}{|A|^2} = \frac{|T|^2}{(iT\cosh\Phi)(-iT\cosh\Phi)} = \frac{1}{\cosh^2\Phi} \quad \text{(A.53)}$$

APPENDIX B

EXPERIMENTAL SET-UP IN TERMS OF ATOMIC UNITS

B.1. Hamiltonian Representation

First we have to write the standard representation of the Hamiltonian

$$\hat{H} = -\frac{\hbar^2}{2m_e}\hat{\nabla}^2 - \frac{Ze^2}{4\pi\epsilon_0 x} - |e|Ex \tag{B.1}$$

where m_e is the mass of the electron, $\hat{\nabla}$ is the gradient operator, ϵ_0 is the vacuum permittivity, E is the applied electric field, and e is charge of the electron. From here, if we make a substitution to express above with dimensionless parameters such as

$$\xi = \frac{x}{a_0} \equiv \text{dimensionless} \tag{B.2}$$

$$\mathcal{E} = \frac{E}{|e|/(4\pi\epsilon_0 a_0^2)} \equiv \text{dimensionless} \tag{B.3}$$

where a_0 is the Bohr radius. If we rewrite the Hamiltonian again in terms of these dimensionless parameters

$$\hat{H} = -\frac{\hbar^2}{2m_e a_0^2}\hat{\nabla}_\xi^2 - \frac{e^2 Z}{4\pi\epsilon_0 a_0 \xi} - |e|a_0\frac{|e|}{4\pi\epsilon_0 a_0^2}\left(\frac{E}{|e|/(4\pi\epsilon_0 a_0^2)}\right)\xi \tag{B.4}$$

which is

$$\hat{H} = -\frac{\hbar^2}{2m_e a_0^2}\hat{\nabla}_\xi^2 - \frac{e^2 Z}{4\pi\epsilon_0 a_0}\left(\frac{Z}{\xi} + \mathcal{E}\xi\right) \tag{B.5}$$

now, to take all expression into one parenthesis, we have to do some algebra on $\frac{\hbar^2}{2m_e a_0^2}$ term above

$$\frac{e^2}{4\pi\epsilon_0} \times \left(\frac{1}{\frac{e^2}{4\pi\epsilon_0}} \times \frac{\hbar^2}{2m_e a_0^2}\right) = \frac{4\pi\epsilon_0\hbar^2}{m_e e^2} \times \frac{e^2}{(4\pi\epsilon_0)2a_0^2} \tag{B.6}$$

since first term of the result above is Bohr radius a_0, we immediately get

$$\frac{\hbar^2}{2m_e a_0^2} = \frac{e^2}{2(4\pi\epsilon_0 a_0)} \tag{B.7}$$

therefore, if we put above into the Hamiltonian (B.5), we get

$$\hat{H} = \frac{e^2}{4\pi\epsilon_0 a_0}\left(-\frac{1}{2}\hat{\nabla}_\xi^2 - (\frac{Z}{\xi} + \mathcal{E}\xi)\right) \tag{B.8}$$

here, $\frac{e^2}{4\pi\epsilon_0 a_0}$ is the Hartree energy, and it is $4.36 \times 10^{-18}J$ or $27.211eV$. This Hartree energy is considered as $1a.u.$. Therefore, we have just found the atomic unit representation of the Hamiltonian, which is

$$\hat{H} = -\frac{1}{2}\hat{\nabla}_\xi^2 - (\frac{Z}{\xi} + \mathcal{E}\xi) \tag{B.9}$$

B.2. Turning Points

To calculate turning points which make momentum to vanish $\hat{p} = 0$, first

$$\hat{H}\psi = -E_I\psi \tag{B.10}$$

where E_I is the ionization energy of the He atom. However, we have to make E_I dimensionless, using the Hamiltonian (B.8),

$$\hat{H} = \frac{e^2}{4\pi\epsilon_0 a_0}\left(-\frac{1}{2}\hat{\nabla}_\xi^2 - (\frac{Z}{\xi} + \mathcal{E}\xi)\right)\psi = \frac{e^2}{4\pi\epsilon_0 a_0}\left(-\frac{E_I}{\frac{e^2}{4\pi\epsilon_0 a_0}}\right)\psi \tag{B.11}$$

dividing and multiplying E_I with $\frac{e^2}{4\pi\epsilon_0 a_0}$ gives us dimensionless $\hat{E}_I$ which

$$\hat{E}_I = \frac{E_I}{\frac{e^2}{4\pi\epsilon_0 a_0}} \tag{B.12}$$

and since first ionization energy of the He is in SI units $E_{I_{SI}} = 24.59eV$, we immediately get $\hat{E}_I = 0.8941$ in terms of Hartree energy. From here, using the equality in (B.11),

$$\frac{\hat{p}^2}{2} - (\frac{Z}{\xi} + \mathcal{E}\xi) = -\hat{E}_I \tag{B.13}$$

then what we get for the momentum in terms of atomic units is

$$\hat{p} = \sqrt{2\left|\frac{\hat{p}^2}{2} - (\frac{Z}{\xi} + \mathcal{E}\xi - \hat{E}_I)\right|} \tag{B.14}$$

hence, one finds turning points as

$$\frac{Z}{\xi} + \mathcal{E}\xi - \hat{E}_I = 0 \tag{B.15}$$

if we multiply this equation with ξ and then solve this second order polynomial, we get turning points as

$$\xi_{L,R} = \frac{\hat{E}_I \pm \sqrt{\hat{E}_I^{\,2} - 4\mathcal{E}Z}}{2\mathcal{E}} \tag{B.16}$$

where $Z = 2$ for He atom.

B.3. Laser Intensity

Laser intensity is

$$\text{Laser Intensity} = L_I = \frac{1}{2}\epsilon_0 c E^2 \tag{B.17}$$

where c is the speed of light. Using the dimensionless form of the applied electric field in (B.3), then we get

$$L_I = \frac{1}{2}\epsilon_0 c \left(\frac{e}{4\pi\epsilon_0 a_0^2}\right)^2 \mathcal{E}^2 \tag{B.18}$$

if we divide and multiply the e term in parenthesis with $4\pi\epsilon_0\hbar c$,

$$L_I = \frac{1}{2}\epsilon_0 c \left(\frac{\frac{e^2}{4\pi\epsilon_0\hbar c}4\pi\epsilon_0\hbar c}{(4\pi\epsilon_0 a_0^2)^2}\right)\mathcal{E}^2 \tag{B.19}$$

then, using the definition of fine structure constant $\alpha = \frac{e^2}{4\pi\epsilon_0\hbar c} \approx \frac{1}{137}$, we get

$$L_I = \frac{1}{2}\alpha\frac{1}{4\pi}\frac{\hbar c^2}{a_0^4} \tag{B.20}$$

if we put their numerical values, what we get for the laser intensity in atomic units $1\mathrm{a.u.}$ is

$$L_I = 3.5152 \times 10^{20}\mathrm{W/m^2} \cdot \mathcal{E}^2 = 3.5152 \times 10^{16}\mathrm{W/cm^2} \cdot \mathcal{E}^2 \tag{B.21}$$

we know that, in this experiment, laser intensity range is given as

$$L_{I_{exp}} = (0.73 - 7.5) \times 10^{14}\mathrm{W/cm^2} \tag{B.22}$$

therefore, in terms of atomic units, this interval must give

$$(0.73 - 7.5) \times 10^{14}\mathrm{W/cm^2} = 3.5152 \times 10^{16}\mathrm{W/cm^2} \cdot \mathcal{E}^2 \tag{B.23}$$

as a result, peak electric field interval is found in terms of atomic units as

$$\mathcal{E} = (0.0456 - 0.1461)\mathrm{a.u.} \tag{B.24}$$