# VIBRATION ANALYSIS OF LAMINATED COMPOSITE CIRCULAR PLATES WITH RADIAL SLOTS 

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## ABSTRACT <br> VIBRATION ANALYSIS OF LAMINATED COMPOSITE CIRCULAR PLATES WITH RADIAL SLOTS

Vibration characteristics of laminated composite annular circular plates with radial slots are studied by Finite Element Method (FEM). As theoretical background, vibration analysis of orthotropic annular circular plates, mechanics of laminated composites and finite element modeling are summarized. Laminated composite annular circular plates with radial slots are introduced. The APDL program in ANSYS is developed for the titled problem and verified by the available literature for the annular circular plate. Then, the effects of lamination parameters on natural frequencies are investigated.

## ÖZET

## RADYAL KANALLI DAIRESEL KOMPOZIT PLAKALARIN TITREŞIM ANALIZI

Radyal kanallı halkasal tabakalı kompozit plakların titreşim karakteristikleri Sonlu Elemanlar Yöntemi (SEY) ile çalışılmıştır. Teorik altyapı olarak, isotropic halkasal plakların titreşim analizi, tabakalı kompozitlerin mekaniği ve sonlu elemen modellemesi özetlenmiştir. Başlıktaki problem için ANSYS de APDL programı geliştirilmiş ve literatürde olan halkasal plak sonuçları ile doğrulanmıştır. Daha sonra, kompozit tabaka parametrelerinin doğal frekanslara etkileri araştırılmıştır.

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## LIST OF SYMBOLS

| [A], [B], [D] | Extensional, Coupling, and Bending stiffness matrices, respectively |
| :---: | :---: |
| D | Flexural rigidity |
| E | Young modulus of isotropic material |
| $E_{1}$ | Longitudinal elastic modulus |
| $E_{2}$ | Transverse elastic modulus |
| $f$ | Natural frequency (Hz) |
| $G_{12}$ | Shear modulus |
| $h$ | Thickness of the plate |
| $\sigma_{1}{ }^{\text {T ult }}$ | Ultimate longitudinal tensile strength |
| $\sigma_{1}{ }_{\text {ult }}$ | Ultimate longitudinal compressive strength |
| $\sigma_{2}{ }^{\text {ult }}$ | Ultimate transverse tensile strength |
| $\sigma_{2}{ }_{\text {ult }}$ | Ultimate transverse compressive strength |
| $\tau_{12}$ | Ultimate in-plane shear strength |
| $\sigma_{x}, \sigma_{y}$ | Normal stress components for lamina in cartesian coordinate system |
| $\tau_{x y}$ | Shear stress component for lamina in cartesian coordinate system |
| $\rho$ | Density (kg/m ${ }^{3}$ ) |
| $\lambda$ | Non-dimentional frequency parameters |
| $\mu$ | Poission ratio of isotropic material |
| $J_{n}, Y_{n}$ | Bessel functions of first and second kind |
| $I_{n}, K_{n}$ Modified Bessel functions of first and second kind |  |
| $\varepsilon_{x}, \varepsilon_{y}$ | Normal strain components for lamina in cartesian coordinate system |
| $\gamma_{x y}$ | Shear strain component for lamina in cartesian coordinate system |
| $\varepsilon_{x}^{0}, \varepsilon_{y}^{0}, \gamma_{x y}^{0}$ | Midplane shear strains in cartesian coordinate system |
| $\varepsilon_{r}^{0}, \varepsilon_{\theta}^{0}, \gamma_{r \theta}^{0}$ | Midplane shear strains in polar coordinate system |
| $\kappa_{x}, \kappa_{y}, \kappa_{x y}$ | Midplane curvatures in cartesian coordinate system |
| $\kappa_{r}, \kappa_{\theta}, \kappa_{r \theta}$ | Midplane curvatures in polar coordinate system |
| $m$ | Number of nodal circle |
| $n$ | Number of nodal diameter |

$M_{x}, M_{y}, M_{x y}$ Resultant moment components in cartesian coordinate system
$M_{r}, M_{\theta}, M_{r \theta}$ Resultant moment components in polar coordinate system
$N_{x}, N_{y}, N_{x y}$ Resultant force components in cartesian coordinate system
$N_{r}, N_{\theta}, N_{r \theta}$ Resultant force components in cartesian polar system
$[M],[K] \quad$ Mass and elastic stiffness matrices
$\left[\bar{Q}_{i j}\right] \quad$ Reduced transfer matrix
$\left[Q_{i j}\right] \quad$ Stiffness coefficient matrix
$u, v, w \quad$ Displacement in $\mathrm{x}, \mathrm{y}, \mathrm{z}$ directions respectively
$U_{1}, U_{2}, U_{3}, U_{, 4}$ Invariants for reduced transfer matrix
$V_{f} \quad$ Fiber volume fraction
$v_{12} \quad$ Major Poisson's ratio
$(\cdot) \quad$ Differentiation with respect to time

## CHAPTER 1

## GENERAL INTRODUCTION

Composite laminated annular circular plates are used in a wide range of engineering applications such as saw blades, disk brakes of bikes etc. Nowadays, engineers are very interested in composite materials since they have lower weight, more stiffness and strength.

There are a lot of studies on the vibration of annular and circular plates in literature. First of all, Southwell (1922) investigated natural frequency for annular plate vibration in a vacuum atmosphere as a pioneer .Conway (1948) developed the static analysis of radially tapered disc springs. Timoshenko and Krieges (1959) proposed an exact solution for Conway's (1948) study. Conway et al (1964) presented vibration of tapered bars and circular plates. Vogel and Skinner (1965) studied on vibration of circular plates with different boundary conditions.

Ramaiah and Vijayakumar (1973) investigated natural frequencies of polar orthotropic annular plates. Also, Narita (1984) found the natural frequencies of completely free annular and circular plates which have polar orthotropy. Chen and Juang (1987) studied on axisymmetric vibration of bimodulus thick circular and annular plates. Narita and Leissa (1992) applied Ritz method to study frequencies and mode shapes of cantilevered laminated composite rectangular plates.

Viswanathan et al (2009) presented asymmetric free vibrations of laminated annular cross ply circular plates including the effects of shear deformation and rotary inertia by using spline method.

In this study, vibration characteristics of laminated composite annular circular plates with and without radial slots are studied by Finite Element Method (FEM). The APDL program in ANSYS is developed for the titled problem and verified by the available literature for the annular circular plate. Then, the effects lamination parameters on natural frequencies are investigated.

## CHAPTER 2

## THEORETICAL BACKGROUND

### 2.1. Introduction

This chapter is presented to describe the problem, to introduce the geometry of the circular plates with radial slots, to summarize the vibration analysis of circular plate and mechanics of laminated composites, to model the system by finite elements and finally to review the vibration analysis of the system.

The detailed information about the circular plates with radial slots can be found in numerious textbooks. Althougth, the aforementioned textbooks present the fundamental concepts to the readers, some important concepts are summarized in this chapter.

### 2.2. Description of the Problem

The problem is finding the natural frequencies and mode shapes of circular plates with radial slots with different lamination parameters. After validating the finite element model for vibration analysis, vibration characteristics of the laminated circular plates with radial slots are studied.(Figure 2.1)

### 2.3. Geometry of the Circular Plates with Radial Slots

Geometry of the composite laminated annular circular plates with radial slots is shown in Figure 2.1. In this figure, notations related to geometry of it are also shown. It can seen from Figure 2.1 that $R_{a}$ is the inner radius of the disk, $R_{b}$ is the outer radius of the disk, $R_{c}$ is the inner radius of the slot, $h$ is the thickness of the plate, $h_{k}$ is the thickness of the $k^{\text {th }}$ layer and $z_{k}$ is the distance from the mid-plane of the disk to the $k^{\text {th }}$ layer.


Figure 2.1. Section view and Geometry of the circular plates with radial slots.

### 2.4. Vibration Analysis of Isotropic Circular Plates

An isotropic annular plate of constant thickness $t$, inner radius $R_{a}$ and outer radius $R_{b}$ is considered. The equations of motion is given as (Amabili 2008),

$$
\begin{equation*}
D \nabla^{4} w+\rho h \ddot{w}=0 \tag{2.1}
\end{equation*}
$$

where

$$
D=\frac{E h^{3}}{12\left(1-v^{2}\right)}
$$

$$
\nabla^{4}=\left[\partial^{2} / \partial r^{2}+(1 / r)(\partial / \partial r)+\partial^{2} /\left(r^{2} \partial \theta^{2}\right)\right]^{2}
$$

In the case of axisymmetric boundary conditions, by using the separation of variables, the solution of the Equation 2.1 takes the following form:

$$
\begin{equation*}
w(r, \theta, t)=\sum^{\infty} \sum^{\infty} W_{m, n}(r) \cos (n \theta) e^{i \omega_{m, n} t} \tag{2.2}
\end{equation*}
$$

where

$$
\begin{align*}
W_{m, n}(r) & =c_{m, n} J_{n}\left(\frac{\lambda_{m, n} r}{R_{b}}\right)+d_{m, n} Y_{n}\left(\frac{\lambda_{m, n} r}{R_{b}}\right) \\
& +e_{m, n} I_{n}\left(\frac{\lambda_{m, n} r}{R_{b}}\right)+g_{m, n} K_{n}\left(\frac{\lambda_{m, n} r}{R_{b}}\right) \tag{2.3}
\end{align*}
$$

in which $m$ and $n$ represent the number of nodal circles and diameters, respectively; $c_{m, n}, d_{m, n}, e_{m, n}$ and $g_{m, n}$ are the mode shape coefficients, which are determined by the boundary conditions; $J_{n}$ and $Y_{n}$ are the Bessel functions of first and second kind, $I_{n}$ and $K_{n}$ are the modified Bessel functions of first and second kind, respectively, and $\lambda_{\mathrm{m}, \mathrm{n}}$ is the frequency parameter, which is also determined by the boundary conditions.

The frequency parameter $\lambda_{m, n}$ is related to the circular natural frequency $\omega_{m, n}$ by

$$
\begin{equation*}
\omega_{m, n}=\frac{\lambda_{m, n}^{2}}{R_{b}{ }^{2}} \sqrt{\frac{D}{\rho h}} \tag{2.4}
\end{equation*}
$$

Mode shapes of circular plate depending on $(m, n)$ are shown in Figure 2.2.


Figure 2.2. Mode shapes of clamped circular plate depending on $(m, n)$

### 2.5. Mechanics of Laminated Composites

### 2.5.1. Macromechanical Analysis of a Lamina in cartesian coordinates

In order to explain the stress-strain relationship for an angle lamina in $x-y$ coordinate system, Figure 2.3 is considered.


Figure 2.3. Local and global axes of an angle lamina.

The stress-strain relationship in $x$ - $y$ coordinate system is (Kaw 2006)

$$
\left\{\begin{array}{c}
\sigma_{x}  \tag{2.5}\\
\sigma_{y} \\
\tau_{x y}
\end{array}\right\}=\left[\begin{array}{lll}
\bar{Q}_{11} & \bar{Q}_{12} & \bar{Q}_{16} \\
\bar{Q}_{12} & \bar{Q}_{22} & \bar{Q}_{26} \\
\bar{Q}_{16} & \bar{Q}_{26} & \bar{Q}_{66}
\end{array}\right]\left\{\begin{array}{c}
\varepsilon_{x} \\
\varepsilon_{y} \\
\gamma_{x y}
\end{array}\right\}
$$

where $\bar{Q}_{i j}$ are called the elements of the transformed reduced stiffness matrix $[\bar{Q}]$ and are given by

$$
\begin{align*}
& \bar{Q}_{11}=Q_{11} c^{4}+Q_{22} s^{4}+2\left(Q_{12}+2 Q_{66}\right) s^{2} c^{2}  \tag{2.6}\\
& \bar{Q}_{12}=\left(Q_{11}+Q_{22}-4 Q_{66}\right) s^{2} c^{2}+Q_{12}\left(c^{4}+s^{2}\right)  \tag{2.7}\\
& \bar{Q}_{22}=Q_{11} s^{4}+Q_{22} c^{4}+2\left(Q_{12}+2 Q_{66}\right) s^{2} c^{2}  \tag{2.8}\\
& \bar{Q}_{16}=\left(Q_{11}-Q_{12}-2 Q_{66}\right) c^{3} s-\left(Q_{22}-Q_{12}-2 Q_{66}\right) s^{3} c \tag{2.9}
\end{align*}
$$

$$
\begin{align*}
& \bar{Q}_{26}=\left(Q_{11}-Q_{12}-2 Q_{66}\right) c s^{3}-\left(Q_{22}-Q_{12}-2 Q_{66}\right) s c^{3}  \tag{2.10}\\
& \bar{Q}_{66}=\left(Q_{11}+Q_{22}-2 Q_{12}-2 Q_{66}\right) c^{2} s^{2}+Q_{66}\left(s^{4}+c^{4}\right) \tag{2.11}
\end{align*}
$$

in which $c=\operatorname{Cos}(\theta)$ and $s=\operatorname{Sin}(\theta)$ The stiffness coefficients $Q_{i j}$ are related to the engineering constants and given as:

$$
\begin{align*}
& Q_{11}=\frac{E_{1}}{1-v_{21} v_{12}}  \tag{2.12}\\
& Q_{12}=\frac{v_{12} E_{2}}{1-v_{21} v_{12}}  \tag{2.13}\\
& Q_{22}=\frac{E_{2}}{1-v_{21} v_{12}}  \tag{2.14}\\
& Q_{66}=G_{12} \tag{2.15}
\end{align*}
$$

Since the $\bar{Q}_{i j}$ presented above do not allow a direct study of the effect of the angle of the lamina on the $\bar{Q}_{i j}$, they can be written in invariant form as

$$
\begin{align*}
& \bar{Q}_{11}=U_{1}+U_{2} \operatorname{Cos} 2 \theta+U_{3} \operatorname{Cos} 4 \theta  \tag{2.16}\\
& \bar{Q}_{12}=U_{4}-U_{3} \operatorname{Cos} 4 \theta  \tag{2.17}\\
& \bar{Q}_{22}=U_{1}-U_{2} \operatorname{Cos} 2 \theta+U_{3} \operatorname{Cos} 4 \theta  \tag{2.18}\\
& Q_{16}=\frac{U_{2}}{2} \operatorname{Sin} 2 \theta+U_{3} \operatorname{Sin} 4 \theta  \tag{2.19}\\
& Q_{26}=\frac{U_{2}}{2} \operatorname{Sin} 2 \theta-U_{3} \operatorname{Sin} 4 \theta \tag{2.20}
\end{align*}
$$

$$
\begin{equation*}
\bar{Q}_{66}=\frac{1}{2}\left(U_{1}-U_{4}\right)-U_{3} \operatorname{Cos} 4 \theta \tag{2.21}
\end{equation*}
$$

$U_{1}, U_{2}, U_{3}, U_{, 4}$ are the four invariants and are combinationsof the $Q_{\mathrm{ij}}$, in which

$$
\begin{align*}
& U_{1}=\frac{1}{8}\left(3 Q_{11}+3 Q_{22}+2 Q_{12}+4 Q_{66}\right)  \tag{2.22}\\
& U_{2}=\frac{1}{2}\left(Q_{11}-Q_{22}\right)  \tag{2.23}\\
& U_{3}=\frac{1}{8}\left(3 Q_{11}+Q_{22}-2 Q_{12}-4 Q_{66}\right)  \tag{2.24}\\
& U_{4}=\frac{1}{8}\left(Q_{11}+Q_{22}+6 Q_{12}-4 Q_{66}\right) \tag{2.25}
\end{align*}
$$

### 2.5.2. Macromechanical Analysis of Laminates in Cartesian Coordinates

A laminate is made of a group of single layers bonded to each other as shown in Figure 2.4. Special notations are used for the laminate code. Some laminate codes are illustrated in Figure 2.5.a-d.


Figure 2.4. Schematic of a laminate.

| 0 |
| :---: |
| -45 |
| 90 |
| 60 |
| 30 |

(a)

| 0 |
| :---: |
| -45 |
| 60 |
| 60 |
| -45 |
| 0 |

(c)

| 0 |
| :---: |
| -45 |
| 90 |
| 90 |
| 60 |
| 0 |

(b)

| 0 |
| :---: |
| -45 |
| 60 |
| -45 |
| 0 |

(d)

Figure 2.5. Laminate code examples
[0/-45/90/60/30] denotes the code for the laminate shown in Figure 2.5.a. It consists of five plies, each of which has a different angle to the reference $x$-axis. A slash separates each lamina. The code also implies that each ply is made of the same material and is of the same thickness.
[0/-45/902/60/0] denotes the laminate shown in Figure 2.5.b, which consists of six plies. Because two $90^{\circ}$ plies are adjacent to each other, $90_{2}$ denote them, where the subscript 2 is the number of adjacent plies of the same angle.
$[0 /-45 / 60]_{s}$ denotes the laminate consisting six plies as shown in Figure 2.5.c. The plies above the midplane are of the same orientation, material, and thickness as the plies below the midplane, so this is a symmetric laminate. The top three plies are written in the code, and the subscript s outside the brackets represents that the three plies are repeated in the reverse order.
$[0 /-45 / \overline{6} \overline{0}]_{s}$ denotes this laminate shown in Figure 2.5.d, which consists of five plies. The number of plies is odd and symmetry exists at the midsurface; therefore, the $60^{\circ}$ ply is denoted with a bar on the top (Kaw 2006).

In this section, the classical lamination theory for a plate under the in-plane loads and moments directions shown in Figure 2.6 is presented.

$N_{x}, N_{y}=$ normal force per unit length, $N_{x y}=$ sher force per unit length $M_{x}, M_{y}=$ bending moments per unit length, $M_{x y}=$ twsiting moments per unit length

Figure 2.6. Resultant forces and moments on a laminate.

The strain-displacement equations can be written in matrix form as:

$$
\left\{\begin{array}{c}
\varepsilon_{x}  \tag{2.26}\\
\varepsilon_{y} \\
\gamma_{x y}
\end{array}\right\}=\left\{\begin{array}{c}
\varepsilon_{x}^{0} \\
\varepsilon_{y}^{0} \\
\gamma_{x y}^{0}
\end{array}\right\}+z\left\{\begin{array}{c}
\kappa_{x} \\
\kappa_{y} \\
\kappa_{x y}
\end{array}\right\}
$$

where the midplane strains and curvatures are given as

$$
\begin{align*}
& \left\{\begin{array}{c}
\varepsilon_{x}^{0} \\
\varepsilon_{Y}^{0} \\
\gamma_{x y}^{0}
\end{array}\right\}=\left\{\begin{array}{c}
\frac{\partial u_{0}}{\partial x} \\
\frac{\partial v_{0}}{\partial y} \\
\frac{\partial u_{0}}{\partial y}+\frac{\partial v_{0}}{\partial x}
\end{array}\right\}  \tag{2.27}\\
& \left\{\begin{array}{l}
\kappa_{x} \\
\kappa_{y} \\
\kappa_{x y}
\end{array}\right\}=\left\{\begin{array}{c}
\frac{\partial^{2} w_{0}}{\partial x^{2}} \\
\frac{\partial^{2} w_{0}}{\partial y^{2}} \\
-2 \frac{\partial^{2} w_{0}}{\partial x \partial y}
\end{array}\right\} \tag{2.28}
\end{align*}
$$

Let us consider a laminate made of $n$ plies shown in Figure 2.7. Each ply has a thickness of $t_{k}$. Then the thickness of the laminate $h$ is

$$
\begin{equation*}
h=\sum_{k=1}^{n} t_{k} \tag{2.29}
\end{equation*}
$$



Figure 2.7. Coordinate locations of plies in a laminate.

The forces and moments in the plate having thickness $h$ are written as

$$
\begin{align*}
& {\left[\begin{array}{l}
N_{x} \\
N_{y} \\
N_{x y}
\end{array}\right]=\int_{-h / 2}^{h / 2}\left[\begin{array}{c}
\sigma_{x} \\
\sigma_{y} \\
\tau_{x y}
\end{array}\right] d z}  \tag{2.30}\\
& {\left[\begin{array}{l}
M_{x} \\
M_{y} \\
M_{x y}
\end{array}\right]=\int_{-h / 2}^{h / 2}\left[\begin{array}{c}
\sigma_{x} \\
\sigma_{y} \\
\tau_{x y}
\end{array}\right] z d z} \tag{2.31}
\end{align*}
$$

Since the laminate made of $n$ plies, forces and moments in each lamina are summed after integrating the stresses for each lamina to give the resultant forces and moments in the laminate as

$$
\begin{align*}
& {\left[\begin{array}{c}
N_{x} \\
N_{y} \\
N_{x y}
\end{array}\right]=\sum_{k=1}^{n} \int_{h k-1}^{h k}\left[\begin{array}{c}
\sigma_{x} \\
\sigma_{y} \\
\tau_{x y}
\end{array}\right]_{k} d z}  \tag{2.32}\\
& {\left[\begin{array}{l}
M_{x} \\
M_{y} \\
M_{x y}
\end{array}\right]=\sum_{k=1}^{n} \int_{h k-1}^{n k}\left[\begin{array}{c}
\sigma_{x} \\
\sigma_{y} \\
\tau_{x y}
\end{array}\right]_{k} z d z} \tag{2.33}
\end{align*}
$$

Now, the resultant forces and moments are written in terms of the midplane strains and curvatures by using the stress-strain relationship as

$$
\begin{align*}
& {\left[\begin{array}{l}
N_{x} \\
N_{y} \\
N_{x y}
\end{array}\right]=\left[\begin{array}{lll}
A_{11} & A_{12} & A_{16} \\
A_{12} & A_{22} & A_{26} \\
A_{16} & A_{26} & A_{66}
\end{array}\right]\left[\begin{array}{l}
\varepsilon_{x}^{0} \\
\varepsilon_{Y}^{0} \\
\gamma_{x y}^{0}
\end{array}\right]+\left[\begin{array}{lll}
B_{11} & B_{12} & B_{16} \\
B_{12} & B_{22} & B_{26} \\
B_{16} & B_{26} & B_{66}
\end{array}\right]\left[\begin{array}{c}
\kappa_{x} \\
\kappa_{y} \\
\kappa_{x y}
\end{array}\right]}  \tag{2.34}\\
& {\left[\begin{array}{c}
M_{x} \\
M_{y} \\
M_{x y}
\end{array}\right]=\left[\begin{array}{lll}
B_{11} & B_{12} & B_{16} \\
B_{12} & B_{22} & B_{26} \\
B_{16} & B_{26} & B_{66}
\end{array}\right]\left[\begin{array}{l}
\varepsilon_{x}^{0} \\
\varepsilon_{Y}^{0} \\
\gamma_{x y}^{0}
\end{array}\right]+\left[\begin{array}{lll}
D_{11} & D_{12} & D_{16} \\
D_{12} & D_{22} & D_{26} \\
D_{16} & D_{26} & D_{66}
\end{array}\right]\left[\begin{array}{c}
\kappa_{x} \\
\kappa_{y} \\
\kappa_{x y}
\end{array}\right]} \tag{2.35}
\end{align*}
$$

where the $[A],[B]$, and $[D]$ matrices are called the extensional, coupling, and bending stiffness matrices, respectively. The matrix $[A]$ relates the resultant in-plane forces to the in-plane strains, and the matrix $[D]$ relates the resultant bending moments to the plate curvatures. The matrix $[B]$ couples the force and moment terms to the midplane strains and midplane curvatures. Elements of the $[A],[B]$, and $[D]$ are given as

$$
\begin{align*}
& A_{i j}=\sum_{k=1}^{n}\left[\left(\bar{Q}_{i j}\right)\right]_{k}\left(h_{k}-h_{k-1}\right), \quad i=1,2,6 ; \quad j=1,2,6  \tag{2.36}\\
& B_{i j}=\frac{1}{2} \sum_{k=1}^{n}\left[\left(\bar{Q}_{i j}\right)\right]_{k}\left(h_{k}^{2}-h_{k-1}^{2}\right), \quad i=1,2,6 ; \quad j=1,2,6  \tag{2.37}\\
& D_{i j}=\frac{1}{3} \sum_{k=1}^{n}\left[\left(\bar{Q}_{i j}\right)\right]_{k}\left(h_{k}^{3}-h_{k-1}^{3}\right), \quad i=1,2,6 ; \quad j=1,2,6 \tag{2.38}
\end{align*}
$$

### 2.5.3. Macromechanical Analysis of Laminates in Polar Coordinates

In order to understand the angle lamina in $r-\theta$ coordinate system, Figure 2.8 is provided from excellent textbook written by Qatu (2004).

It can be concluded from Figure 2.3 and Figure 2.8 that the equations presented in the previous two subsections are valid for the circular plates with circular orthotropy by replacing $r$ and $\theta$ to $x$ and $y$, respectively, except the midplane strains and curvatures equations given in Equations 2.27 and 2.28.


Figure 2.8 Circular plates with circular orthotropy (Source: Qatu 2004)

Therefore, the laminate constitutive equations are written as

$$
\begin{align*}
& {\left[\begin{array}{c}
N_{r} \\
N_{\theta} \\
N_{r \theta}
\end{array}\right]=\left[\begin{array}{lll}
A_{11} & A_{12} & A_{16} \\
A_{12} & A_{22} & A_{26} \\
A_{16} & A_{26} & A_{66}
\end{array}\right]\left[\begin{array}{c}
\varepsilon_{r}^{0} \\
\varepsilon_{\theta}^{0} \\
\gamma_{r \theta}^{0}
\end{array}\right]+\left[\begin{array}{lll}
B_{11} & B_{12} & B_{16} \\
B_{12} & B_{22} & B_{26} \\
B_{16} & B_{26} & B_{66}
\end{array}\right]\left[\begin{array}{c}
\kappa_{r} \\
\kappa_{\theta} \\
\kappa_{r \theta}
\end{array}\right]}  \tag{2.39}\\
& {\left[\begin{array}{c}
M_{r} \\
M_{\theta} \\
M_{r \theta}
\end{array}\right]=\left[\begin{array}{lll}
B_{11} & B_{12} & B_{16} \\
B_{12} & B_{22} & B_{26} \\
B_{16} & B_{26} & B_{66}
\end{array}\right]\left[\begin{array}{c}
\varepsilon_{r}^{0} \\
\varepsilon_{\theta}^{0} \\
\gamma_{r \theta}^{0}
\end{array}\right]+\left[\begin{array}{lll}
D_{11} & D_{12} & D_{16} \\
D_{12} & D_{22} & D_{26} \\
D_{16} & D_{26} & D_{66}
\end{array}\right]\left[\begin{array}{c}
\kappa_{r} \\
\kappa_{\theta} \\
\kappa_{r \theta}
\end{array}\right]} \tag{2.40}
\end{align*}
$$

The midplane strains and curvatures in Equation 2.39 and 2.40 are given by Qatu (2004) as

$$
\begin{gather*}
\left\{\begin{array}{c}
\varepsilon_{r}^{0} \\
\varepsilon_{\theta}^{0} \\
\gamma_{r \theta}^{0}
\end{array}\right\}=\left\{\begin{array}{c}
\frac{\partial u_{0}}{\partial r} \\
\frac{\partial v_{0}}{r \partial \theta}+\frac{u_{0}}{r} \\
\frac{\partial u_{0}}{r \partial \theta}+\frac{\partial v_{0}}{\partial r}-\frac{v_{0}}{r}
\end{array}\right\}  \tag{2.41}\\
\left\{\begin{array}{c}
\kappa_{r} \\
\kappa_{\theta} \\
\kappa_{r \theta}
\end{array}\right\}=\left\{\begin{array}{c}
-\frac{\partial^{2} w_{0}}{\partial r^{2}} \\
-\frac{\partial^{2} w_{0}}{r^{2} \partial \theta^{2}}-\frac{\partial w_{0}}{r \partial r} \\
-2 \frac{\partial^{2} w_{0}}{r \partial r \partial \theta}+\frac{\partial w_{0}}{r^{2} \partial \theta}
\end{array}\right\} \tag{2.42}
\end{gather*}
$$

### 2.5.4. Equations of Motions of Laminated Plates in Cartesian

## Coordinates

In this section, Classical Laminated Plate Theory (CLPT) is used for equation of motions. CLPT is an extension of classical plate theory (Reddy 2004) and based on that the transverse strains $\left(\gamma_{x z}, \gamma_{y z}, \varepsilon_{z z}\right)$ and consequently the transverse shear stresses are zero. Equations of motions are given by Reddy (2004) as follows:

$$
\begin{align*}
& \frac{\partial N_{x}}{\partial x}+\frac{\partial N_{x y}}{\partial y}=I_{0} \frac{\partial^{2} u_{0}}{\partial t^{2}}-I_{1} \frac{\partial^{2}}{\partial t^{2}}\left(\frac{\partial w_{0}}{\partial x}\right)  \tag{2.43}\\
& \frac{\partial N_{x y}}{\partial x}+\frac{\partial N_{y}}{\partial y}=I_{0} \frac{\partial^{2} v_{0}}{\partial t^{2}}-I_{1} \frac{\partial^{2}}{\partial t^{2}}\left(\frac{\partial w_{0}}{\partial y}\right)  \tag{2.44}\\
& \frac{\partial^{2} M_{x}}{\partial x^{2}}+2 \frac{\partial^{2} M_{x y}}{\partial y \partial x}+\frac{\partial^{2} M_{y}}{\partial y^{2}}+N\left(w_{0}\right)+q= \\
& I_{0} \frac{\partial^{2} w_{0}}{\partial t^{2}}-I_{2} \frac{\partial^{2}}{\partial t^{2}}\left(\frac{\partial^{2} w_{0}}{\partial x^{2}}+\frac{\partial^{2} w_{0}}{\partial y^{2}}\right)+I_{1} \frac{\partial^{2}}{\partial t^{2}}\left(\frac{\partial u_{0}}{\partial x}+\frac{\partial v_{0}}{\partial y}\right) \tag{2.45}
\end{align*}
$$

where $q$ is distributed load in $z$ direction, $I_{0}, I_{1}, I_{2}$ are the mass moment of inertia terms and given as

$$
\left[\begin{array}{l}
I_{0}  \tag{2.45}\\
I_{1} \\
I_{2}
\end{array}\right]=\sum_{k=1}^{n} \int_{n k-1}^{h k}\left[\begin{array}{c}
1 \\
z \\
z^{2}
\end{array}\right] \rho^{k} d z
$$

in which $\rho_{\mathrm{k}}$ is density of the material of $\mathrm{k}^{\text {th }}$ layer. Also, $N\left(w_{0}\right)$ in Equations 2.45 is given as

$$
\begin{equation*}
N\left(w_{0}\right)=\frac{\partial}{\partial x}\left(N_{x} \frac{\partial w_{0}}{\partial x}+N_{x y} \frac{\partial w_{0}}{\partial y}\right)+\frac{\partial}{\partial y}\left(N_{x y} \frac{\partial w_{0}}{\partial x}+N_{y} \frac{\partial w_{0}}{\partial y}\right) \tag{2.46}
\end{equation*}
$$

### 2.5.5. Equations of Motion for Laminated Plates in polar coordinates

Equations of motions are given by Qatu (2004) as follows:

$$
\begin{align*}
& \frac{\partial}{\partial r}\left(r N_{r}\right)+\frac{\partial}{\partial \theta}\left(N_{r \theta}\right)-N_{\theta}+r q_{r}=r\left(I_{0} \ddot{u}_{0}^{2}\right)  \tag{2.47}\\
& \frac{\partial}{\partial \theta}\left(N_{\theta}\right)+\frac{\partial}{\partial r}\left(r N_{r \theta}\right)+N_{r \theta}+r q_{\theta}=r\left(I_{0} \ddot{v}_{0}^{2}\right)  \tag{2.48}\\
& \frac{\partial}{\partial r}\left(r Q_{r}\right)+\frac{\partial}{\partial \theta}\left(Q_{\theta}\right)+r q_{n}=r\left(I_{0} \ddot{w}_{0}^{2}\right) \tag{2.49}
\end{align*}
$$

where $q_{r}, q_{\theta}$ and $q_{n}$ represent the distributed forces in the $r, \theta$ and $n$ directions, respectively. Also, $r Q_{r}$ and $r Q_{\theta}$ are given as

$$
\begin{align*}
& r Q_{r}=\frac{\partial}{\partial r}\left(r M_{r}\right)+\frac{\partial}{\partial \theta}\left(M_{r \theta}\right)-M_{\theta}  \tag{2.50}\\
& r Q_{\theta}=\frac{\partial}{\partial \theta}\left(M_{\theta}\right)+\frac{\partial}{\partial r}\left(r M_{r \theta}\right)+M_{r \theta} \tag{2.51}
\end{align*}
$$

### 2.6. Natural Frequencies by Finite Element Method

The free vibration equation of motion of a undamped multi-degree-of-freedom system is given by

$$
\begin{equation*}
[M]\{\ddot{x}(t)\}+([K])\{x(t)\}=0 \tag{2.52}
\end{equation*}
$$

where $[M]$ and $[K]$ are mass and elastic stiffness matrices, respectively. Also, $\{x(t)\}$ is displacement vector.

Finite Element Method is based on the interpolation functions which are properly selected for element geometry and nodal freedoms. For the present problem, the displacements are approximated over an element by Lagrange and Hermite interpolation functions as follows (Reddy 2004),

$$
\begin{align*}
& u(x, y, t)=\sum_{j=1}^{m} u_{j}(t) \psi_{j}(x, y) \\
& v(x, y,, t)=\sum_{j=1}^{m} v_{j}(t) \psi_{j}(x, y)  \tag{2.53}\\
& w(x, y, t)=\sum_{k=1}^{n} \Delta_{k}(t) \phi_{k}(x, y)
\end{align*}
$$

where $u_{j}(t)$ and $v_{j}(t)$ denote the values of $(u, v)$ at the $j$-th node of the Lagrange elements, $\Delta_{k}$ denote the values of $w$ and its derivatives with respect to x and y at the $k$-th node, and $\psi_{i}(x, y)$ and $\phi_{i}(x, y)$ are Lagrange and Hermite interpolation functions, respectively. Therefore, Equation 2.52 is reduced the following equation to find the natural frequencies and mode shapes of the system

$$
\left(\left[\begin{array}{ccc}
{\left[K^{11}\right]} & {\left[K^{12}\right]} & {\left[K^{13}\right]}  \tag{2.54}\\
{\left[K^{12}\right]^{T}} & {\left[K^{22}\right]} & {\left[K^{23}\right]} \\
{\left[K^{13}\right]^{T}} & {\left[K^{23}\right]^{T}} & {\left[K^{33}\right]}
\end{array}\right]+\omega^{2}\left[\begin{array}{ccc}
{\left[M^{11}\right]} & {[0]} & {\left[M^{13}\right]} \\
{[0]} & {\left[M^{22}\right]} & {\left[M^{23}\right]} \\
{\left[M^{13}\right]} & {\left[M^{23}\right]} & {\left[M^{33}\right]}
\end{array}\right]\right)\left\{\begin{array}{l}
\left\{u_{e}\right\} \\
\left\{v_{e}\right\} \\
\left\{\Delta_{e}\right\}
\end{array}\right\}=\left\{\begin{array}{l}
0 \\
0 \\
0
\end{array}\right\}
$$

where

$$
\begin{aligned}
K_{i j}^{11}= & A_{11} S_{i j}^{x x}+A_{16}\left(S_{i j}^{x x}+S_{i j}^{y x}\right)+A_{66} S_{i j}^{y y} \\
K_{i j}^{12}= & A_{12} S_{i j}^{x y}+A_{16} S_{i j}^{x x}+A_{26} S_{i j}^{y y}+A_{66} S_{i j}^{y x} \\
K_{i j}^{22}= & A_{66} S_{i j}^{x x}+A_{26}\left(S_{i j}^{x x}+S_{i j}^{y x}\right)+A_{22} S_{i j}^{y y} \\
K_{i k}^{13}= & -B_{11} R_{i k}^{x x x}-B_{12} R_{i k}^{x y y}-2 B_{26} R_{i k}^{x y} \\
& -B_{16} R_{i k}^{v x x}-B_{26} R_{i k}^{v y y}-2 B_{66} R_{i k}^{x y y} \\
K_{i k}^{23}= & -B_{16} R_{i k}^{x x x}-B_{26} R_{i k}^{x y y}-2 B_{66} R_{i k}^{x x y} \\
& -B_{12} R_{i k}^{v x x}-B_{22} R_{i k}^{v y y}-2 B_{26} R_{i k}^{x x y} \\
K_{k l}^{33}= & -D_{11} T_{k l}^{x x x}-D_{12}\left(T_{k l}^{x x y}+T_{k l}^{y y x}\right) \\
& +2 D_{16}\left(T_{k l}^{x x y}+T_{k l}^{x y x x}\right)+2 D_{26}\left(T_{k l}^{x y y}+T_{k l}^{y x y}\right) \\
& +4 D_{66} R_{k l}^{x y y}+D_{22} T_{k l}^{y y y y}
\end{aligned}
$$

in which

$$
\begin{align*}
& s_{i j}^{\xi \eta}=\int_{\Omega^{e}} \frac{\partial \psi_{i}^{e}}{\partial \xi} \frac{\partial \psi_{j}^{e}}{\partial \eta} d x d y \\
& s_{k l}^{\xi \eta}=\int_{\Omega^{e}} \frac{\partial \psi_{k}^{e}}{\partial \xi} \frac{\partial \psi_{l}^{e}}{\partial \eta} d x d y  \tag{2.56}\\
& R_{i k}^{\xi \eta \zeta}=\int_{\Omega^{e}} \frac{\partial \psi_{i}^{e}}{\partial \xi} \frac{\partial^{2} \psi_{k}^{e}}{\partial \eta \partial \zeta} d x d y \\
& T_{k l}^{\xi \eta \zeta \mu}=\int_{\Omega^{e}} \frac{\partial^{2} \psi_{k}^{e}}{\partial \xi \partial} \frac{\partial^{2} \psi_{l}^{e}}{\partial \zeta \partial \mu} d x d y
\end{align*}
$$

and

$$
\begin{align*}
& M_{i j}^{11}=\int_{\Omega^{e}} I_{i} \psi_{i}^{e} \psi_{j}^{e} d x d y \\
& M_{i j}^{22}=\int_{\Omega^{e}} I_{0} \psi_{i}^{e} \psi_{j}^{e} d x d y \\
& M_{i j}^{13}=-\int_{\Omega^{e}} I_{I} \psi_{i}^{e} \frac{\partial \psi_{k}^{e}}{\partial x} d x d y  \tag{2.57}\\
& M_{i j}^{23}=-\int_{\Omega^{e}} I_{1} \psi_{i}^{e} \frac{\partial \psi_{k}^{e}}{\partial y} d x d y \\
& M_{k l}^{33}=\int_{\Omega^{2}}\left[I_{0} \varphi_{k}^{e} \varphi_{l}^{e}+I_{2}\left(\frac{\partial \varphi_{k}^{e}}{\partial x} \frac{\partial \varphi_{l}^{e}}{\partial x}+\frac{\partial \varphi_{k}^{e}}{\partial y} \frac{\partial \varphi_{l}^{e}}{\partial y}\right)\right] d x d y
\end{align*}
$$

## CHAPTER 3

## NUMERICAL RESULTS AND DISCUSSIONS

### 3.1. Geometrical Models of the Annular Circular Plates

In this chapter, numerical analyses are carried out for the composite laminated annular circular plates with radial slots for different lay-up secuences in the order of $30^{\circ} / 0^{\circ} / 30^{\circ}, 45^{\circ} / 0^{\circ} / 45^{\circ}, 60^{\circ} / 0^{\circ} / 60^{\circ}, 75^{\circ} / 0^{\circ} / 75^{\circ}, 90^{\circ} / 0^{\circ} / 90^{\circ}$. The annular circular plates is shown in Figures 3.1.


Figure 3.1. Annular circular plate with radial slots

Geometrical data of the laminated annular plates have the following common parameters:

Inner radius of the laminated annular plate $R_{a}=60 \mathrm{~mm}$
Outer radius of the laminated annular plate $R_{b}=120 \mathrm{~mm}$
Inner radius of the slot of laminated annular plate $R_{c}=90 \mathrm{~mm}$
Total thickness of the laminated annular plate $t=3 \mathrm{~mm}$
Layer thickness of the laminated annular plate $t_{l}=1 \mathrm{~mm}$
Slot width of the laminated annular plate $b=t$

Other geometrical data for the each model is based on the following format:

## $\mathrm{L} \theta$-ss-r $r$

where $\mathrm{L},-, \mathrm{s}$ and r are used in the model name to identify the model, the $\theta, s$ and $r$ represent the top and bottom laminate angle, the number of slot $s$ and the ratio of inner slot radius to outer slot radius $r=R_{c} / R_{b}$. For example: L30-n2-r0.75 means a model with $30^{\circ}$ lamination angle, 2 slots and ratio $r=0.75$.

### 3.2. Finite Element Models

The physical models introduced in Section 3.1 are modelled in ANSYS by using SHELL99 element. S-Glass/ Epoxy (Scotchply 1002) is selected as plate materials. Material properties are given as follows (Kaw 2006):

- Fiber volume fraction $V_{f}=0.45$
- Longitudinal elastic modulus $E_{1}=38.6 \mathrm{GPa}$
- Transverse elastic modulus $E_{2}=8.27 \mathrm{GPa}$
- Major Poisson's ratio $v_{12}=0.26$
- Shear modulus $G_{12}=4.14 \mathrm{GPa}$
- Ultimate longitudinal tensile strength $\sigma_{1}{ }^{\mathrm{T}}{ }_{\text {ult }}=1062 \mathrm{MPa}$
- Ultimate longitudinal compressive strength $\sigma_{1}{ }^{\mathrm{C}}$ utt $=610 \mathrm{MPa}$
- Ultimate transverse tensile strength $\sigma_{2}{ }^{\mathrm{T}}{ }_{\text {ult }}=31 \mathrm{MPa}$
- Ultimate transverse compressive strength $\sigma_{2}{ }^{\mathrm{C}}$ ult $=118 \mathrm{MPa}$
- Ultimate in-plane shear strength $\tau_{12 \text { ult }}=72 \mathrm{Mpa}$
- Density $\rho=1.8 \quad 10^{9} \mathrm{~kg} / \mathrm{m}^{3}$

The annular circular plate with radial slots is free on outside and clamped on inside.
The proper numbers of elements for annular circular plates in the ANSYS program are determined by comparing the natural frequency results as follows:

- 10 elements in radial direction
- 72 elements in circumferential directions, Therefore total 7200 elements are used.


### 3.3. Validation of the Finite Element Model

The natural frequency parameters of the the isotropic annular plates obtained by using the aforementioned shell elements are compared with the exact results given by Leissa (1969). The results are given in Table 3.1. In order to demonstrate the selected element capabilities, the ratio=the finite element result to exact result ratio is provided in the same table.

Table 3.1. Comparison of natural frequency parameters $\lambda^{2}$

| $\lambda^{2}(\mathrm{~m}, \mathrm{n})$ | Exact | Shell99 | Ratio |
| :---: | :---: | :---: | :---: |
| $\lambda^{2}(0,0)$ | 13.0 | 13.0000 | 1.0000 |
| $\lambda^{2}(0,1)$ | 13.3 | 13.2784 | 0.9984 |
| $\lambda^{2}(0.2)$ | 14.7 | 14.7237 | 1.0016 |
| $\lambda^{2}(0.3)$ | 18.5 | 18.6055 | 1.0057 |
| $\lambda^{2}(1,0)$ | 85.1 | 84.0752 | 0.9881 |
| $\lambda^{2}(1,1)$ | 86.7 | 85.7338 | 0.9889 |
| $\lambda^{2}(1,2)$ | 91.7 | 90.7174 | 0.9893 |
| $\lambda^{2}(1,3)$ | 100.0 | 99.0536 | 0.9905 |

It can be conculded from Table 3.1 that, Shell99 element for the present problem which is vibration of isotropic plate gives good results.

In the reachable literature, there are no study on the vibration of laminated annular plate with numerical results. Therefore, the finite element model created in ANSYS is verified by isotropic plate studuy.

### 3.4. Natural Frequencies and Mode Shapes

The results found for the several cases are given in the Table 3.2 to Table 3.6.

Table 3.2. Natural frequencies for $30^{\circ} / 0^{\circ} / 30^{\circ}$

| $f(\mathrm{~m}, \mathrm{n})$ | L30-s0-r1 | L30-s2-r0.75 | L30-s4-r0.75 |
| :---: | :---: | :---: | :---: |
| $f(0,0)$ | 457.11 | 456.87 | 456.72 |
| $f(0,1)$ | 467.08 | 463.40 | 464.25 |
|  | 467.08 | 468.45 | 464.25 |
| $f(0.2)$ | 505.29 | 491.85 | 481.95 |
|  | 505.29 | 508.95 | 512.56 |
| $f(1,0)$ | 390.72 | 560.00 | 563.21 |
| $f(1,1)$ | 3029.9 | 3021.82 | 3053.21 |
|  | 3065.7 | 3060.98 | 3075.26 |
| $f(1,3)$ | 3173.5 | 3073.32 | 3075.26 |
|  | 3354.68 | 3133.58 |  |
|  | 3354.9 | 33302.7 | 3307.42 |

Table 3.3. Natural frequencies for $45^{\circ} / 0^{\circ} / 45^{\circ}$

| $f(\mathrm{~m}, \mathrm{n})$ | L45-s0-r1 | L45-s2-r0.75 | L45-s4-r0.75 |
| :---: | :---: | :---: | :---: |
| $f(0,0)$ | 400.73 | 398.97 | 397.63 |
| $f(0,1)$ | 413.89 | 406.7 | 406.86 |
|  | 413.89 | 414.7 | 406.86 |
| $f(0.2)$ | 463.56 | 442.76 | 427.49 |
|  | 463.56 | 466.97 | 470.33 |
| $f(1,0)$ | 571.03 | 527.41 | 529.83 |
|  | 2538.4 | 2479.69 | 529.83 |
| $f(1,2)$ | 2594.2 | 2486.58 | 2499.87 |
|  | 2758.32 | 2543.96 |  |
|  | 3025.8 | 2726.60 | 2543.96 |
|  | 3025.2 | 2771.35 | 2669.17 |
|  |  | 3013.37 | 2741.18 |

Table 3.4. Natural frequencies for $60^{\circ} / 0^{\circ} / 60^{\circ}$

| $f(\mathrm{~m}, \mathrm{n})$ | L60-s0-r1 | L60-s2-r0.75 | L60-s4-r0.75 |
| :---: | :---: | :---: | :---: |
| $f(0,0)$ | 358.00 | 353.10 | 394.64 |
| $f(0,1)$ | 367.27 | 357.26 | 356.60 |
|  | 367.27 | 367.50 | 356.60 |
| $f(0.2)$ | 416.04 | 393.12 | 375.02 |
| $f(0.3)$ | 546.04 | 419.08 | 421.95 |
| $f(1,0)$ | 543.63 | 491.53 | 490.99 |
| $f(1,1)$ | 2142.6 | 2114.70 | 490.99 |
|  | 2210.7 | 2165.88 | 2040.53 |
| $f(1,2)$ | 2410.2 | 2207.16 | 21117.03 |
| $f(1,3)$ | 2410.2 | 2399.92 | 2391.02 |
|  | 2729.6 | 2677.21 | 2465.04 |

Table 3.5. Natural frequencies for $75^{\circ} / 0^{\circ} / 75^{\circ}$

| $f(\mathrm{~m}, \mathrm{n})$ | L75-s0-r1 | L75-s2-r0.75 | L75-s4-r0.75 |
| :---: | :---: | :---: | :---: |
| $f(0,0)$ | 340.26 | 331.06 | 325.25 |
| $f(0,1)$ | 335.44 | 326.18 | 324.03 |
|  | 335.44 | 334.61 | 324.03 |
| $f(0.2)$ | 365.54 | 348.07 | 331.92 |
| $f(0.3)$ | 365.54 | 368.14 | 369.93 |
| $f(1,0)$ | 515.77 | 455.92 | 448.82 |
| $f(1,1)$ | 1955.78 | 1942.20 | 1913.37 |
|  | 2006.67 | 2004.43 | 1981.79 |
| $f(1,2)$ | 2165.37 | 2110.98 | 1981.79 |
| $f(1,3)$ | 2165.37 | 2155.4 | 2100.29 |
|  | 2446.26 | 2403.31 | 2150.42 |
|  | 2434.96 | 2336.96 |  |

Table 3.6. Natural frequencies for $90^{\circ} / 0^{\circ} / 90^{\circ}$

| $f(\mathrm{~m}, \mathrm{n})$ | L90-s0-r1 | L90-s2-r0.75 | L90-s4-r0.75 |
| :---: | :---: | :---: | :---: |
| $f(0,0)$ | 337.22 | 328.24 | 319.11 |
| $f(0,1)$ | $\begin{aligned} & \hline 323.92 \\ & 323.92 \\ & \hline \end{aligned}$ | $\begin{array}{r} \hline 316.45 \\ 322.04 \\ \hline \end{array}$ | $\begin{aligned} & \hline 312.90 \\ & 312.90 \\ & \hline \end{aligned}$ |
| $f(0.2)$ | $\begin{aligned} & 340.24 \\ & 340.24 \end{aligned}$ | $\begin{array}{r} 324.42 \\ 343.53 \\ \hline \end{array}$ | $\begin{array}{r} 313.52 \\ 344.04 \\ \hline \end{array}$ |
| $f(0.3)$ | $\begin{aligned} & \hline 504.84 \\ & 504.84 \\ & \hline \end{aligned}$ | $\begin{aligned} & \hline 439.71 \\ & 480.40 \\ & \hline \end{aligned}$ | $\begin{aligned} & \hline 428.19 \\ & 428.19 \\ & \hline \end{aligned}$ |
| $f(1,0)$ | 1916.82 | 1913.76 | 1883.07 |
| $f(1,1)$ | $\begin{aligned} & 1952.07 \\ & 1952.07 \end{aligned}$ | $\begin{aligned} & 1914.60 \\ & 1950.76 \end{aligned}$ | $\begin{aligned} & \hline 1942.11 \\ & 1942.11 \end{aligned}$ |
| $f(1,2)$ | $\begin{aligned} & 2071.09 \\ & 2071.09 \\ & \hline \end{aligned}$ | $\begin{array}{r} 2057.23 \\ 2058.22 \\ \hline \end{array}$ | $\begin{array}{r} 2049.43 \\ 2054.81 \\ \hline \end{array}$ |
| $f(1,3)$ | $\begin{aligned} & 2306.63 \\ & 2306.63 \end{aligned}$ | $\begin{aligned} & 2221.27 \\ & 2285.55 \end{aligned}$ | $\begin{aligned} & 2246.62 \\ & 2246.62 \end{aligned}$ |

The mode shapes of the annular plate with 4 slots and labeled by L75-s4-r0.75 are plotted in Figure 3.2 to 3.15.

It can be read from first column of the Table 3.2 to Table 3.6 which are correspond to unslotted annular plates that, the natural frequencies for $n \neq 0$ are repeated due to the axisymmetical geometry. On the other hand, the second and third column of the same tables presents the results for the non-axisymmetical geometries; therefore, there are no double frequencies.

Moreover, it can be seen from Figure 3.2 to 3.15 that, due to the nonaxisymmetical geometry of the annular plate with four slots, there are two different vibration frequencies for the same $(m, n)$, where $n \neq 0$. The difference between the same vibration mode ( $m, n$ ) shape is the orientation of the nodal diameters.

Finally, depending on the number of slots and the number of nodal diameters, the natural frequencies for $n \neq 0$ are repeated. For example, the annular plate with four radial slots have repeated natural frequencies for $n=1,3$, ., but the mode shape orientations are differents.


Figure 3.2. Vibration mode ( $\mathrm{m}=0, \mathrm{n}=0$ )


Figure 3.3. Vibration mode ( $\mathrm{m}=0, \mathrm{n}=1 \mathrm{a}$ )


Figure 3.4. Vibration mode ( $\mathrm{m}=0, \mathrm{n}=1 \mathrm{~b}$ )


Figure 3.5. Vibration mode $(m=0, n=2 a)$


Figure 3.6. Vibration mode $(\mathrm{m}=0, \mathrm{n}=2 \mathrm{~b})$


Figure 3.7. Vibration mode $(m=0, n=3 a)$


Figure 3.8. Vibration mode $(m=0, n=3 b)$


Figure 3.9. Vibration mode $(\mathrm{m}=1, \mathrm{n}=0)$


Figure 3.10. Vibration mode ( $\mathrm{m}=1, \mathrm{n}=1 \mathrm{a}$ )


Figure 3.11. Vibration mode $(\mathrm{m}=1, \mathrm{n}=1 \mathrm{~b})$


Figure 3.12. Vibration mode $(\mathrm{m}=1, \mathrm{n}=2 \mathrm{a})$


Figure 3.13. Vibration mode ( $\mathrm{m}=1, \mathrm{n}=2 \mathrm{~b}$ )


Figure 3.14. Vibration mode $(\mathrm{m}=1, \mathrm{n}=3 \mathrm{a})$


Figure 3.15 Vibration mode $(\mathrm{m}=1, \mathrm{n}=3 \mathrm{~b})$

## CHAPTER 4

## CONCLUSIONS

In this study vibration characteristics of laminated composite annular circular plates with and without radial slots are studied by Finite Element Method (FEM). The APDL program in ANSYS is developed for the titled problem and verified by the available literature for the annular circular plate. Then, the effects lamination parameters on natural frequencies are investigated. It can be seen that radial slots effects directly for natural frequencies and mode shapes which can shown in all tables. For example, in circullar plates without slots, the two frequencies results are equals in the same mode. However, in circullar plates with slots, these two frequencies are different from each other.

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