CONVERGENCE ANALYSIS AND NUMERICAL SOLUTIONS OF THE FISHER'S AND BENJAMIN-BONO-MAHONY EQUATIONS BY OPERATOR SPLITTING METHOD

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ABSTRACT

CONVERGENCE ANALYSIS AND NUMERICAL SOLUTIONS OF THE FISHER'S AND BENJAMIN-BONO-MAHONY EQUATIONS BY OPERATOR SPLITTING METHOD

This thesis is concerned with the operator splitting method for the Fisher's and Benjamin-Bono-Mahony type equations. We show that the correct convergence rates in $H^s(\mathbb{R})$ space for Lie- Trotter and Strang splitting method which are obtained for these equations. In the proofs, the new framework originally introduced in (Holden, Lubich, and Risebro, 2013) is used.

Numerical quadratures and Peano Kernel theorem, which is followed by the differentiation in Banach space are discussed In addition, we discuss the Sobolev space $H^s(\mathbb{R})$ and give several properties of this space. With the help of these subjects, we derive error bounds for the first and second order splitting methods. Finally, we numerically check the convergence rates for the time step Δt.

ÖZET

FİSHER VE BENJAMİN-BONO-MAHONY DENKLEMLERİNİN OPERATÖR AYIRMA METODU İLE YAKINSAKLIK ANALİZİ VE NÜMERİK ÇÖZÜMLERİ

Bu tez Fisher ve Benjamin-Bono-Mahony tipindeki denklemlere uygulanan operatör ayırma methoduyla ilgilidir. Bu denklemlere uygulanan Lie-Trotter ve Strang ayırma methodları için $H^s(\mathbb{R})$ uzayında yakınsaklık oranlarını gösterdik. Kanıtlarda, ilk olarak (Holden, Lubich, and Risebro, 2013) de yapılan yeni yöntem kullanıldı.

Nümerik kuadratürler, Peano Kernel teoremi ve Banach uzaylarında türev ele alındı. Ayrıca, $H^s(\mathbb{R})$ Sobolev uzaylarından bahsettik ve birkaç özelliğini verdik. Bu konular yardımıyla birinci ve ikinci mertebeden ayırma methodlarının hata sınırları elde edildi. Son olarak, Δt zaman aralığında nümerik olarak yakınsaklık oranlarını kontrol ettik.

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CHAPTER 1

INTRODUCTION

Partial differential equations have become staggeringly successful as models of physical phenomena. With the rapid increase in computing power in recent years, such models have permeated virtually every physical and engineering problem. The phenomena modeled by partial differential equations become increasingly complicated, and so do the partial differential equations themselves. Often, one wishes a model to capture different aspects of a situation, for instance both convective transport and dispersive oscillations on a small scale. These different aspects of the model are then reflected in a partial differential equation, which may contain terms that are mathematically greatly different, making these models hard to analyze, both theoretically and numerically.

A computational scientist is therefore often faced with new and complex equations for which an efficient solution method must be developed. If one is lucky, the equation is of a well-known type, and it is quite easy to find efficient methods that are simple to implement. In most cases, however, one is not so lucky; good methods may be hard to find, and even good methods may be hard to implement.

A strategy to deal with complicated problems is to ”divide and conquer”. In the context of equations of evolution type, a rather successful approach in this spirit has been *operator splitting*.

The idea behind this type of approach is that the overall evolution operator is formally written as a sum of evolution operators for each term in the model. In other words, one splits the model into a set of sub-equations, where each sub-equation is of a type for which simpler and more practical algorithms are available. The overall numerical method is then formed by picking an appropriate numerical scheme for each sub-equation and piecing the schemes together by operator splitting.

The main focus in this text is to apply the operator splitting method on two equations; the Fisher’s equation and the Benjamin-Bono-Mahony type equation, and we will study the operator splitting method from an analytical and numerical point of view. Before we start with the mathematical treatment, we give a brief historical background for the two equations.

We consider the nonlinear Fisher’s reaction diffusion equation

$$\frac{\partial u}{\partial t} = \alpha \frac{\partial^2 u}{\partial x^2} + \beta u(1-u) \quad x \in (-\infty, \infty), \; t \geq 0 \tag{1.1}$$

where α is the diffusion coefficient, β is the reactive factor, t is the time, x is the distance and $u(x,t)$ is the population density.The coefficient α is a non-negative constant and β is a real parameter. The analytical properties and subsequent computation for minimum wave speed have been easily interpreted by removing the explicit dependence on coefficients and α, β in (1.1) by a suitable recalling of x and t. After recalling the time $t^* = \beta t$ and space $x^* = (\beta/\alpha)^{1/2}x$, and dropping the asterisk notation, equation (1.1) becomes

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + u(1-u) \tag{1.2}$$

Equation (1.2) may be transformed into an ordinary differential equation by substituting $u = u(z) = u(x - ct)$. Kolmogorov (Kolmogorov,Petrovskii and Piskunov, 1937) showed that with appropriate initial and boundary conditions, there exists a travelling wave solution to equation (1.2) of wave speed c for every $c \geq 2$.

Equation (1.2), energy released by non-linear term balances energy consumed by diffusion, resulting in traveling waves or fronts.Traveling wave fronts have important applications in chemistry, biology and medicine. Such wave fronts were first studied by Fisher in 1930s by studying the equation (1.2).

The Fisher's equation (1.2) occurs in chemical kinetics and neutron population in a nuclear reaction. Moreover, the same equation also occurs in logistic population growth models, flame propagation, neurophysiology, autocatalytic chemical reactions, and branching Brownian motion processes.

In the past several decades, there has been great activity in developing numerical and analytical methods for the Fisher's equation. One of the first numerical solutions was described by Gazdag and Canosa (Gazdag and Canosa, 1974) with a pseudo-spectral approach. Ablowitz and Zepetella (Ablowitz and Zeppetella, 1979) established an explicit solution of Fisher's equation for a special wave speed. Twizell et al. (Twizell, Wang and Price, 1990) and Parekh and Puri (Parekh and Puri, 1990) demonstrated implicit and explicit finite-differences algorithms to discuss the numerical study of Fisher's equation. Tang and Weber proposed a Galerkin finite element method for solving Fisher's equation. Mickens (Mickens, 1994) introduced a best finite difference scheme for Fisher's equation. Mavoungou and Cherruault (Mavoungou and Cherruault, 1994) depicted a numerical study of Fisher's equation by Adomian's method. Qiu and Sloan (Qiu and Sloan, 1998) used a moving mesh method for numerical solution of Fisher's equation. Al-Khaled (Khaled, 2001) proposed the sinc collocation method for Fisher's equation. Zhao and Wei (Zhao and Wei, 2003) presented a comparison of the discrete singular convolution and three other numerical schemes for solving Fisher's equation. Wazwaz and Gorguis (Wazwaz and Gorguis, 2004) gave the exact solutions to Fisher's equation and to a nonlinear diffusion equation of the Fisher type by employing the Adomian decomposition method. Olmos and Shizgal (Olmos and Shizgal, 2006) constructed the numerical solutions to Fisher's equation using a pseudo-spectral approach. Mittal and Kumar and El-Azab (El-Azab, 2007) contrived Fisher's equation by applying the wavelet Galerkin method. Recently, Sahin et al. (Sahin,Dag and Saka, 2008) applied the Bspline Galerkin approach to find numerical solution of Fisher's equation. Cattani and Kudreyko (Cattani and Kudreyko, 2008) developed multiscale analysis of the Fisher equation. Mittal and Jiwari (Mittal and Jiwari, 2009) developed a numerical study of Fisher's equation by using differential quadrature method. Mittal and Arora (Mittal and Arora, 2010) presented efficient numerical solution of Fisher's equation by using B-spline collocation method. Mittal and Jain (Mittal and Jain, 2013) constructed the numerical solution of Fisher's equation by using modified cubic B-spline collocation method.

We also study the Benjamin-Bona-Mahony type equations

$$u_t = (1 - \partial_x^2)^{-1} P(\partial_x) u + \frac{1}{2}(1 - \partial_x^2)^{-1} \partial_x (u^2) \tag{1.3}$$

This is a class of dispersive nonlinear wave equations including various important cases, e.g.,

$$\begin{aligned} P(\xi) &= \xi^2, \text{ the BBM equation}, \\ P(\xi) &= \xi^3, \text{ the KdV-BBM equation}, \end{aligned}$$

Benjamin, Bona and Mahoney (Benjamin,Bona and Mahoney, 1972) advocated that the PDE modeled the same physical phenomena equally well as the KdV equation, given the same assumptions and approximations that were originally used by Korteweg and de Vries (Korteweg and de Vries, 1985). This PDE of Benjamin et al. (Benjamin,Bona and Mahoney, 1972) is now often called the BBM equation.It is used as an alternative to the KdV equation which describes unidirectional propagation of weakly long dispersive waves. As a model that characterizes long waves in nonlinear dispersive media, the BBM equation, like KdV equation, was formally derived to describe an approximation for surface water waves in a uniform channel. The equation covers not only the surface waves of long wave-length in liquids, but also hydromagnetic waves in cold plasma, acoustic waves in anharmonic crystals, and acoustic gravity waves in compressible fluids. Many researchers are attracted by the wide applica- bility of the BBM equation.In (Khaled, 2001), Khaled-Momani-Alawneh implemented the Adomian's decomposition method for obtaining explicit and numerical solutions of the BBM equation. By applying the classical Lie method of infinitesimals Bruzón and

Gandarias (Bruzón and Gandarias, 2008) obtained, for a generalization of a family of BBM equations, many exact solutions expressed by various single and combined nondegenerative Jacobi elliptic functions. Tari and Ganji, (Tari and Ganji, 2007), have applied two methods for solving nonlinear differential equations known as variational iteration and homotopy perturbation methods in order to derive approximate explicit solutions for BBM. El-Wakil-Abdou-Hendi (El-Wakil, Abdou and Hendi, 2008) used the exp-function method with the aid of symbolic computational system to obtain the generalized solitary solutions and periodic solutions.

A wide range of numerical methods have been employed to compute approximate solutions to dispersive wave equations of KdV-BBM type: finite volume methods (Dutkyh,Katsaounis and Mitsotakis , 2013), finite difference schemes (Bona, Pritchard, and Scott, 1985), (Wei,Kirby, Grilli, and Subramanya, 1995), finite element methods (Bona,Dougalis, and Mitsotakis, 2007), (Mitsotakis, 2009), (Avilez-Valente and Seabra-Santos, 2009) and spectral methods (Ozkan-Haller and Kirby, 1997), (Pelloni and Dougalis, 2001), (Dutykh and Dias, 2007) , (Nguyen and Dias, 2008) .

We now describe each chapter of the thesis:

- In Chapter 2, we give some basic concepts. This chapter consist of three main sections. In the first section, we introduce the Sobolev space and give lemmas of for instance Banach-algebra property for the standard Sobolev spaces. In the second one, we investigate the differentiability of general operators in Banach spaces. Lastly, we present the error term for one and two dimensional quadrature formulas.

- In Chapter 3, we introduce the operator splitting method and give proofs of the convergence rates for the Fisrher's and BBM type equations.

- In Chapter 4, we present numerical experiments with the use of the operator splitting method and confirm the theoretical results.

- In the conclusion we summarize the main results in the thesis.

CHAPTER 2

BASIC CONCEPTS

In this chapter, we investigate some concepts for the estimation which follows the convergence analysis of the operator splitting method. The proofs of lemmas in this chapter is clearly given in Appendix A

2.1. Sobolev Spaces

We interested in the Sobolev spaces which forms a Hilbert space.These spaces are denoted as $H^s(\mathbb{R}) = W^{s,2}(\mathbb{R})$, where s is integer. The inner product and norm are defined as

$$(u,v)_{H^s} = \sum_{j=0}^{s} \int_{\mathbb{R}} \partial_x^j u(x) \partial_x^j v(x) dx \text{ and } \|u\|_{H^s} = \sqrt{(u,u)_{H^s}}. \tag{2.1}$$

We see that $H^s(\mathbb{R})$ contains all functions which has weak derivatives up to order s in $L^2(\mathbb{R})$, and we remark that $L^2(\mathbb{R}) = H^0(\mathbb{R})$. Consider $H^s(\mathbb{R})$ defined when s is a positive integer, with inner product and norm as in (2.1). From the definition, we observe that $H^r(\mathbb{R})$ is continously imbedded in $H^s(\mathbb{R})$ for $r > s$, which results in that the respective norms are comparable in the following way

$$\|u\|_{H^s} \leq C\|u\|_{H^r}, \tag{2.2}$$

for u in $H^r(\mathbb{R})$. We first show that $H^s(\mathbb{R})$ is imbedded in $L^\infty(\mathbb{R})$ for $s \geq 1$.

Lemma 2.1 *If u is in $H^s(\mathbb{R})$ for $s \geq 1$, then u is in $L^\infty(\mathbb{R})$. Moreover,*

$$\|u\|_{L^\infty} \leq \frac{1}{\sqrt{2}}\|u\|_{H^1} \leq C_s\|u\|_{H^s}, \tag{2.3}$$

where C_s depends only on s.

Proof See Appendix A. □

Lemma 2.2 *The space $H^s(\mathbb{R})$ is a Banach algebra for $s \geq 1$. In particular, if u, v are in $H^s(\mathbb{R})$ for $s \geq 1$, then*

$$\|uv\|_{H^s} \leq C_s \|u\|_{H^s} \|u\|_{H^s},$$

where where C_s depends only on s.

Proof See Appendix A. □

Lemma 2.3 *Let $u,\, v \in L^2(\mathbb{R})$. Then $\|\partial_x(1-\partial_x^2)^{-1}(uv)\|_{L^2} \leq \|u\|_{L^2}\|v\|_{L^2}$.*

Proof See (Stanislavova, 2005) □

Lemma 2.4 *Let $u, v \in H^s(\mathbb{R})$ Then $\|(1-\partial_x^2)^{-1}\partial_x(uv)\|_{H^s} \leq K\|u\|_{H^s}\|v\|_{H^s}$.*

Proof See Appendix A. □

2.2. Fréchet Derivative

Recall the definition of derivative for real valued functions:

$$f'(x) := \lim_{h \to 0} \frac{f(x+h) - f(x)}{h},$$

if the limit exists. Writing

$$\psi(h) = \begin{cases} \frac{f(x+h)-f(x)}{h} - f'(x) & h \neq 0 \\ 0 & h = 0 \end{cases}$$

we see that the definition of derivative implies that $\psi(h)$ is a continuous function at 0, while it is clearly a continuous function elsewhere (as the differentiable function f is continuous). Moreover we have the equation

$$f(x+h) = f(x) + f'(x)h + h\psi(h) \tag{2.4}$$

We can now generalize this idea to obtain a more general definition of derivative. Notice that we need that domain and range of f are normed vector spaces, otherwise we can't add (if we don't have a vector space) or talk about continuity (if we don't have norms). The derivative defined in this way, the usual definition on general vector spaces, is called Fréchet derivative.

Definition 2.1 *Let X and Y be normed vector spaces, and $U \subset X$ open, $f : U \to Y$. We say f is differentiable at $x \in U$ if there exists a bounded linear map $Df(x) \in L(X, Y)$*[1] *and a continuous function $\psi : V \to Y$, where V is an open neighbourhood of $0 \in X$, with $\psi(0) = 0$, such that*

$$f(x + h) = f(x) + (Df(x))h + \|h\|\psi(h)$$

for all $h \in V$. (Note V must be chosen such that $x + V = x + v|v \in V \subset U$.)

We can also relate this to the original limit definition of f' by the following lemma.

Lemma 2.5 *If $f : U \to Y$ differentiable at x, then for all $h \in X$ we have*

$$Df(x)h = \lim_{t\to 0} \frac{f(x + th) - f(x)}{t},$$

where t is chosen in $\mathbb{R}$.

Proof See Appendix A. □

2.2.1. Higher Derivative

Let $f \in C(U, Y)$ be differentiable in the open set $U \subset X$ and consider $f' : U \to L(X, Y)$

Definition 2.2 *Let $u \in U$: f is twice (Fréchet-) differentiable at u. The second (Fréchet) differential of f at u is defined as*

$$d^2 f(u) = df'(u). \tag{2.5}$$

[1] *the space of linear continuous maps from X to Y*

If f is twice differentiable at all points of U we say that f is twice differentiable in U According to the above definition $d^2 f(u)$ is a linear continuous map from X to $L(x, Y)$:

$$d^2 f(u) \in L(X, L(X, Y)). \tag{2.6}$$

It is convenient to see $d^2 f(u)$ as a bilinear map on X. For this, let $L_2(X, Y)$ denote the space of continuous bilinear maps from $X \times X \to Y$. To any $A \in L(X, L(X, Y))$ we can associate $\Phi_A \in L_2(X, Y)$ given by $\Phi_A(u_1, u_2) = [A(u_1)](u_2)$. Conversely, given $\Phi \in L_2(X, Y)$ and $h \in X, \Phi(h, .) : k \to \Phi(h, k)$ is a continuous linear map from X to Y; hence to any $\Phi \in L_2(X, Y)$ is associated the linear application $X \to L(X, Y)$,

$$\Phi : h \to \Phi(h, .) \in L(X, Y) \tag{2.7}$$

It is easy to see that in this way we define an isomorphism between $L(X, L(X, Y))$ and $L_2(X, Y)$. Actually, such an isomorphism is an isometry because there results

$$\|\Phi\|_{L(X,L(X,Y))} = \sup_{\|h\| \leq 1} \|\Phi(h)\|_{L(X,Y)} \tag{2.8}$$

$$= \sup_{\|h\| \leq 1} \sup_{\|k\| \leq 1} \|\Phi(h, k)\| = \|\Phi(h, k)\|_{L_2(X,Y)} \tag{2.9}$$

In the following we will use the same symbol $d^2 f(u)$ to denote the continuous bilinear map obtained by the preceding isometry. The value of $d^2 f(u)$ at a pair (h, k) will be denoted by

$$d^2 f(u)[h, k]. \tag{2.10}$$

If f is twice differentiable in U, the second (Fréchet) derivative of f is the map $f'' : U \to L_2(X, Y)$,

$$f'' : u \to d^2 f(u). \tag{2.11}$$

If f'' is continuous from U to $L_2(X, Y)$ we say that $f \in C^2(U, Y)$.

To define $(n + 1)-$th derivatives $(n \geq 2)$ we can proceed by induction. Given f :

$U \to Y$, let f be n times differentiable in U. The nth differential at a point $x \in U$ will be identified with a continuous n-linear map from $X \times X \times X \times ... \times X$ ($n$times) to Y (recall that, as before , there is an isometry between $L(X, ..., L(X, Y)$ and $L_n(X, Y)$).
Let $f^{(n)} : U \to L_n(X, Y)$

$$f^{(n)} : u \to d^n f(u).$$

The $(n+1)-$th differential at u will be defined as the differential of $f^{(n)}$, namely

$$d^{n+1} f(u) = df^n(u) \in L(X, L_n(X, Y)) \approx L_n(X, Y).$$

We will say that $f \in C^n(U, Y)$ if f is n times (Fréchet) differentiable in U and the nth derivative f^n is continuous from U to $L_n(X, Y)$. The value of $d^n f(u)$ at $(h_1, ..., h_n)$ will be denoted by

$$d^n f(u)[h_1, ..., h_n].$$

If $h = h_1 = ... = h_n$ we will write for short $d^n f(u)[h]^n$.

2.2.2. Taylor's Formula

Let $f \in C^n(Q, Y)$ and let $u, u + v \in Q$ be such that the interval $[u, u + v] \subset Q$. Then, Taylor's formula for Fréchet differentiable maps is that

$$f(u + v) = f(u) + df(u)[v] + ... + \frac{1}{(n-1)!} \int_0^1 (1-t)^{n-1} d^{(n)} f(u + tv) dt[v]^n.$$

The last integral can be written as

$$\frac{1}{(n-1)!}\int_0^1 (1-t)^{n-1} d^{(n)} f(u+tv)dt[v]^n = \frac{1}{n!} d^{(n)} f(u)[v]^n + \varepsilon(u,v)[v]^n$$

where

$$\varepsilon(u,v) = \frac{1}{(n-1)!}\int_0^1 (1-t)^{n-1}[d^n f(u+tv) - d^n f(u)]dt \to 0 \ as \ v \to 0.$$

2.3. Numerical Quadrature

Quadrature refers to any method for numerically approximating the value of the definite integrals

$$\int_a^b f(x)dx \text{ and } \int_a^b \int_c^d f(x,y)dydx,$$

The main focus in this section is the error term for one and two dimensional quadrature formulas, and we start with a treatment of former. We use Peano kernel theorem to find error formulas for two one dimensional quadrature formulas and also we find a error formula for a two dimensional quadrature formula.

2.3.1. One Dimensional Quadratures

The quadrature formula yield an approximation to the integral, and can in general be given as

$$I(f) = \sum_{k=0}^{m_0} w_{k0} f(x_{k0}) + \sum_{k=0}^{m_1} w_{k1} f'(x_{k1}) + ... + \sum_{k=0}^{m_n} w_{kn} f^{(n}(x_{kn}), \tag{2.12}$$

where w_{ki} are the weights, $\{m_i\}_i \subset \mathbb{N}$, $\{x_{ki}\}_{k,i} \subset [a,b]$ and f is in $C^n[a,b]$.The error function for this quadrature rule is given as

$$E(f) = I(f) - \int_a^b f(x)dx, \tag{2.13}$$

and tells us of how good (2.12) approximates the integral. Several different forms for (2.13) exist, and we focus on the Peano kernel form, which we present below.

2.3.1.1. The Peano Kernel Theorem

$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)(x-a)^k}{k!} + \frac{1}{n!}\int_a^x f^{(n+1)}(t)(x-t)^n dt = p_n(x) + r_n(x), \tag{2.14}$$

which is valid since f is in $C^{n+1}([a,b])$. The error E is a linear operator, thus $E(f) = E(p_n + r_n) = E(p_n) + E(r_n)$, where $E(p_n) = 0$ because (2.12) integrates all p in $\mathbb{P}_n$ exactly.Thus by using (2.13),

$$\begin{aligned} E(f) = E(r_n) &= I(r_n) - \int_a^b r_n(x)dx \\ &= I(r_n) - \frac{1}{n!}\int_a^b \int_a^x f^{(n+1)}(t)(x-t)^n dt dx \\ &= I(r_n) - \frac{1}{n!}\int_a^b \int_a^b f^{(n+1)}(t)(x-t)_+^n dt dx \end{aligned}$$

where we have introduced

$$(x-t)_+^n = \begin{cases} (x-t)^n & if x \geq t, \\ 0 & if x \leq t. \end{cases}$$

Using (2.12) for r_n gives

$$\begin{aligned}E(r_n) &= I(\frac{1}{n!}\int_a^b f^{(n+1)}(t)(x-t)_+^n dtdx) - \frac{1}{n!}\int_a^b\int_a^b f^{(n+1)}(t)(x-t)_+^n dtdx \\ &= \frac{1}{n!}\sum_{k=0}^{m_0} w_{k0}\int_a^b f^{(n+1)}(t)(x-t)_+^n dt \\ &\quad +\frac{1}{n!}\sum_{k=0}^{m_1} w_{k1}\frac{d}{dx}[\int_a^b f^{(n+1)}(t)(x-t)_+^n dt] + ... \\ &\quad +\frac{1}{n!}\sum_{k=0}^{m_n} w_{kn}\frac{d^n}{dx^n}[\int_a^b f^{(n+1)}(t)(x-t)_+^n dt] \\ &\quad -\frac{1}{n!}\int_a^b\int_a^b f^{(n+1)}(t)(x-t)_+^n dtdx\end{aligned} \tag{2.15}$$

The integration and differentiation in the sums have to be interchanged, to yield what we want. Thus, the following needs a proof,

$$\frac{d^k}{dx^k}[\int_a^b f^{(n+1)}(t)(x-t)_+^n dt] = \int_a^b f^{(n+1)}(t)\frac{d^k}{dx^k}[(x-t)_+^n]dt \tag{2.16}$$

for $1 \leq k \leq n$. For $k \leq n$ this follows immediately since $(x-t)_+^n$ is $n-1$ times continuously differentiable. In particular, for $k = n-1$ we get

$$\frac{d^{n-1}}{dx^{n-1}}[\int_a^b f^{(n+1)}(t)(x-t)_+^n dt] = \int_a^b f^{(n+1)}(t)\frac{d^{n-1}}{dx^{n-1}}[(x-t)_+^n]dt \tag{2.17}$$

which by evaluating the differentiation on the right hand side leads to

$$\frac{d^{n-1}}{dx^{n-1}}[\int_a^b f^{(n+1)}(t)(x-t)_+^n dt] = n!\int_a^b f^{(n+1)}(t)(x-t)_+^n dt \tag{2.18}$$

$$n!\int_a^x f^{(n+1)}(t)(x-t)_+^n dt. \tag{2.19}$$

The integral on the right hand side is differentiable as a function of x, because the integrand is jointly continuous in x and t. Thus, by the fundamental theorem of calculus,

$$\frac{d}{dx}[\frac{d^{n-1}}{dx^{n-1}}[\int_a^b f^{(n+1)}(t)(x-t)_+^n dt]] = n!f^{n+1}(x)(x-x) + n!\int_a^x f^{n+1}(t)dt \quad (2.20)$$
$$= 0 + \int_a^x f^{n+1}(t)\frac{d^n}{dx^n}[(x-t)^n]dt$$
$$= \int_a^x f^{n+1}(t)\frac{d^n}{dx^n}[(x-t)_+^n]dt.$$

This proves that (2.16) holds for $k = n$. We return to (2.15), and interchange the two operators,

$$\begin{aligned} E(r_n) &= \frac{1}{n!}\sum_{k=0}^{m_0} w_{k0}\int_a^b f^{(n+1)}(t)(x_{k0}-t)_+^n dt \\ &+\frac{1}{n!}\sum_{k=0}^{m_1} w_{k1}\frac{d}{dx}[\int_a^b f^{(n+1)}(t)(x_{k1}-t)_+^n dt] \\ &+ ... + \frac{1}{n!}\sum_{k=0}^{m_n} w_{kn}\frac{d^n}{dx^n}[\int_a^b f^{(n+1)}(t)(x_{kn}-t)_+^n dt] \\ &-\frac{1}{n!}\int_a^b\int_a^b f^{(n+1)}(t)(x-t)_+^n dtdx \end{aligned}$$

From the continuity of the integrand $f^{(n+1)}(t)(x-t)_+^n$, we interchange the integration order of the double integral,

$$\frac{1}{n!}\int_a^b\int_a^b f^{(n+1)}(t)(x-t)_+^n dtdx = \frac{1}{n!}\int_a^b f^{(n+1)}(t)\int_a^b (x-t)_+^n dxdt.$$

Thus,

$$\begin{aligned} E(r_n) &= \frac{1}{n!}\int_a^b f^{(n+1)}(t)\Bigg(\sum_{k=0}^{m_0} w_{k0}(x_{k0}-t)_+^n + \sum_{k=0}^{m_1} w_{k1}\frac{d}{dx}(x_{k1}-t)_+^n \\ &+ ... + \sum_{k=0}^{m_n} w_{kn}\frac{d^n}{dx^n}(x_{kn}-t)_+^n\Bigg)dt - \frac{1}{n!}\int_a^b f^{(n+1)}(t)\int_a^b (x-t)_+^n dxdt \\ &= \frac{1}{n!}\int_a^b f^{(n+1)}(t)E_x((x-t)_+^n)dt. \end{aligned}$$

Hence, the error of the one dimensional quadrature rule has a compact and elegant form. This is the essential of the classic Peano kernel theorem, which we have proven above.

Theorem 2.1 *(Peano Kernel theorem) If f is in $C^{n+1}([a,b])$ and I is a quadrature rule given in (2.12) that integrates all p in $\mathbb{P}_n$ exactly, then*

$$E(f) = I(f) - \int_a^b f(x)dx = \frac{1}{n!}\int_a^b f^{n+1}(t)K(t)dt. \quad (2.21)$$

where $K(t) = E_x((x-t)_+^n)$ is the Peano kernel.

2.3.1.2. The Rectangle Rule

The simplest quadrature rule in one dimension is the (right corner)rectangle rule, which is given as

$$\int_0^h f(x)dx \approx h.f(h) \quad (2.22)$$

and the rule integrates all f in $\mathbb{P}_0$ exactly. The error is

$$E(f) = h.f(h) - \int_0^h f(x)dx,$$

and from Theorem (2.1) we get the Peano kernel for the rectangle rule, which is given as

$$K(t) = E_x((x-t)_+^0) = h(h-t)_+^0 - \int_0^h (x-t)_+^0 dx = h - \int_t^h dx = t \quad (2.23)$$

where we have used the fact that $(h-t)_+ = h-t$ since $h \geq t$ for all $t \in [0,h]$. Thus the error for the rectangle rule can be written as

$$E(f) = \int_0^h f'(t)K(t)dt. \quad (2.24)$$

In particular, $K(t)$ does not change signs on $[0,h]$, so we can apply the mean value theorem for integrals.

Hence,

$$E(f) = f'(\xi) \int_0^h K(t)dt.$$

for some $\xi \in [0, h]$. Thus, we conclude that the error is

$$E(f) = f'(\xi)h^2/2$$

2.3.1.3. The Midpoint Rule

Another simple quadrature rule is the midpoint rule, which integrates all f in $\mathbb{P}_1$ exactly. It is given as

$$\int_0^h f(x)dx \approx h.f(h/2) \tag{2.25}$$

Since (2.25) is exact for all p in $\mathbb{P}_1$, it is possible to obtain two Peano kernels. Similarly as above, the first order Peano kernel is found as

$$K_1(t) = E_x((x-t)_+^0) = h(\frac{h}{2}-t)_+^0 - \int_0^h (x-t)_+^0 dx = h - \int_t^h dx = t,$$

while the second order Peano kernel becomes

$$\begin{aligned} K_2(t) &= E_x((x-t)_+^1) = h(\frac{h}{2}-t)_+ - \int_0^h (x-t)_+ dx \\ &= h(\frac{h}{2}-t) - \int_t^h (x-t)dx = h(\frac{h}{2}-t) - \frac{(h-t)^2}{2} \end{aligned} \tag{2.26}$$

Hence, there exist two error formulas for the midpoint rule,

$$\begin{aligned} E_1(f) &= \int_0^h f'(t)K_1(t)dt = f'(\xi)\int_0^h K_1(t)dt, \\ E_2(f) &= \int_0^h f''(t)K_2(t)dt = f''(\xi)\int_0^h K_2(t)dt, \end{aligned}$$

Thus, we conclude that the errors are,

$$\begin{aligned} E_1(f) &= f'(\xi)h^2/2 \\ E_2(f) &= -f''(\xi)h^3/6, \end{aligned}$$

2.3.2. Two Dimensional Quadratures

We derive a two dimensional midpoint rule for the double integral

$$\int_a^b \int_c^d f(x,y)dydx,$$

The starting point is the two dimensional Taylor series expansion. In one dimension, the Taylor series expansion for $F(x)$ in $C^{n+1}([0,1])$ is given as

$$F(1) = F(0) + F'(0) + \frac{F''(0)}{2!} + \ldots + \frac{F^n(0)}{n!} + \frac{F^{n+1}(\xi)}{(n+1)!}, \tag{2.27}$$

for ξ in $[0,1]$. Define the parametrization of F as

$$F(t) = f(a+th, b+tk), \tag{2.28}$$

for some $f(x,y)$ and t in $[0,1]$. We assume that $f(x,y)$ has continuous partial derivatives up to order $n+1$ at all points in an open set containing the line segment joining the points (a,b) and $(a+h, b+k)$ in its domain. For simplicity we will only derive the Taylor's formula for

$n = 1$. The derivatives of $F(t)$ is given as

$$\begin{aligned} F'(t) &= hf_x(x+th, y+tk) + kf_y(x+th, y+tk), \\ F''(t) &= h^2 f_{xx}(x+th, y+tk) + 2khf_{xy}(x+th, y+tk) + k^2 f_{yy}(x+th, y+tk). \end{aligned}$$

Thus by using (2.27 and (2.28), we obtain

$$\begin{aligned} F(1) &= f(a,b) + hf_x(x+h, y+k) + kf_y(x+h, y+k) \\ &\quad + \frac{1}{2}(h^2 f_{xx}(x+\xi h, y+\xi k) + 2khf_{xy}(x+\xi h, y+\xi k) \\ &\quad + k^2 f_{yy}(x+\xi h, y+\xi k)) + \ldots \end{aligned}$$

where the dots involves higher order derivatives on f which not are of interests. Letting $h = x - a$ and $k = y - b$, we obtain the second order Taylor formula for $f(x,y)$,

$$\begin{aligned} f(x,y) &= f(a,b) + (x-a)f_x(a,b) + (y-b)f_y(a,b) \\ &\quad + \frac{1}{2}(h^2 f_{xx}(a+\xi(x-a), b+\xi(y-b)) + 2khf_{xy}(a+\xi(x-a), b+\xi(y-b)) \\ &\quad + k^2 f_{yy}(a+\xi(x-a), b+\xi(y-b))) \end{aligned} \tag{2.29}$$

Returning to the double integral, we obtain using (2.29),

$$\int_0^h \int_0^x f(x,y)dydx = \int_0^h \int_0^x f(x,y)dydx + R(f), \tag{2.30}$$

where

$$\begin{aligned} R(f) &= \int_0^h \int_0^x ((x-a)f_x(a,b) + (y-b)f_y(a,b))dydx \\ &= \frac{1}{2}\int_0^h \int_0^x (h^2 f_{xx}(a+\xi(x-a), b+\xi(y-b)) \\ &\quad + 2khf_{xy}(a+\xi(x-a), b+\xi(y-b)) \\ &\quad + k^2 f_{yy}(a+\xi(x-a), b+\xi(y-b)))dydx \end{aligned} \tag{2.31}$$

The integral on the right-hand-side in (2.30) is just the area of integration domain times the function itself. Thus, an approximation for the double integral is given as

$$\int_0^h \int_0^x f(x,y)dydx = \frac{h^2}{2} f(a,b) + R(f), \tag{2.32}$$

and the error is given as

$$E(f) = \int_0^h \int_0^x f(x,y)dydx - \frac{h^2}{2} f(a,b) = R(f),$$

which by (2.31) is bounded as

$$\begin{aligned}
|E(f)| &< \max_T |f_x| |\int_0^h \int_0^x (x-a)dydx| + \max_T |f_y| |\int_0^h \int_0^x (y-b)dydx| \\
&= \frac{h^2}{2} \max_T |f_{xx}| |\int_0^h \int_0^x (x-a)^2 dydx| \\
&\quad + kh \max_T |f_{xy}| |\int_0^h \int_0^x (x-a)(y-b)dydx| \\
&\quad + \frac{k^2}{2} \max_T |f_{yy}| |\int_0^h \int_0^x (y-b)^2 dydx|
\end{aligned}$$

where $T = (x,y) : 0 \le y \le x \le h$. By evaluating all of the above integrals, we get

$$\begin{aligned}
|E(f)| &< \max_T |f_x| |\frac{1}{3}h^3 - \frac{a}{2}h^2| + \max_T |f_y| |\frac{1}{6}h^3 - \frac{b}{2}h^2| \\
&= \frac{h^2}{2} \max_T |f_{xx}| |\frac{1}{4}h^4 - \frac{2a}{3}h^3 + \frac{a^2}{2}h^2| \\
&\quad + kh \max_T |f_{xy}| |\frac{1}{4}h^4 - \frac{2b}{3}h^3 - \frac{a}{3}h^3 + abh^2| \\
&\quad + \frac{k^2}{2} \max_T |f_{yy}| |\frac{1}{4}h^4 - \frac{b}{3}h^3 + \frac{b^2}{2}h^2|
\end{aligned} \tag{2.33}$$

which is the error bound for the two dimensional midpoint rule in (2.32).

CHAPTER 3

OPERATOR SPLITTING

Operator splitting methods help us reduce the complexity of the system and reduce the computational time. It can be regarded as a time-discretization method. We divide the time interval $[0, t_{end}]$ into N equal parts. With splitting it is possible to solve each subproblem with a suitable numerical method chosen to the corresponding operator for a small time steps Δt.

3.1. General Formulation

In this study, we use Lie-Trotter and Strang splitting methods, respectively, i.e.,

$$u_{n+1} = \Psi^{\Delta t}(u_n) = \Phi_B^{\Delta t} \circ \Phi_A^{\Delta t}(u_n), \;\; n = 0, 1, 2, \ldots . \tag{3.1}$$

$$u_{n+1} = \Psi^{\Delta t}(u_n) = \Phi_A^{\Delta t/2} \circ \Phi_B^{\Delta t} \circ \Phi_A^{\Delta t/2}(u_n), \;\; n = 0, 1, 2, \ldots . \tag{3.2}$$

3.2. Statement of the Problem

With the general formulation in Section 3.1 in mind, we formulate the problem which we shall delve into. Consider the initial value problem

$$u_t = Au + B(u) \;\; u|_{t=0} = u_0 \tag{3.3}$$

where x is in $\mathbb{R}$ and t is in the interval $[0, T]$ for a fixed time $T > 0$. We require that u_0 and $u(t)$ are in $H^s(\mathbb{R})$, where the order s is specified in details later. Applying the operator splitting method to (3.3), and splitting it into two subequations gives

$$v_t = Av \tag{3.4}$$

and

$$u_t = B(u) \tag{3.5}$$

where A is an unbounded linear operator and $B(u)$ is a bounded nonlinear operator.
In this study, our focus is on the Fisher's equation

$$\begin{aligned} u_t &= \beta\partial_x^2 u + \alpha u(1-u) \\ &= \beta P(\partial_x)u + \alpha u(1-u) \end{aligned} \tag{3.6}$$

and Benjamin-Bona-Mahony type equation

$$u_t = (1-\partial_x^2)^{-1}P(\partial_x)u + \frac{1}{2}(1-\partial_x^2)^{-1}\partial_x(u^2) \tag{3.7}$$

where α, β are in $\mathbb{R}$ and P is a polynomial of degree $l \geq 2.$ Thus, the polynomial P is linear with degree $l \geq 2$ such that the corresponding Sobolev norm of the solution do not increase. The formal requirement and proof are given as follows

Lemma 3.1 *Let P be a linear polynomial of degree $l \geq 2$ with constant coefficients, which satisfies*

$$ReP(i\xi) \leq 0 \ \textit{ for all } \ \xi \in \mathbb{R} \tag{3.8}$$

In addition, let m be a integer such that $m \geq l$, and assume v_0 is in $H^{m+l}(\mathbb{R})$ and the solution $\Phi^t(v_0) = v(t)$ of $v_t = P(\partial_x)v, \ v|_{t=0} = v_0$ is in $H^m(\mathbb{R})$ and satisfies

$$\int_{\mathbb{R}} (\partial_x^{j+l/2}v)^2 < \infty,$$

for all $j \leq m$ and l even. Then $\Phi^t(v_0)$ has a non-increasing norm in $H^m(\mathbb{R})$, in particular

$$\|\Phi_P^t(v_0)\|_{H^m} \leq \|v_0\|_{H^{m+1}}$$

Proof Using the assumptions, P is given as $P(x) = \sum_{\alpha=2}^{l} a_\alpha x^\alpha$, where a_α is in $\mathbb{R}$ for all α. Thus (3.1) becomes

$$v_t = a_l \partial_x^l v + a_{l-1}\partial_x^{l-1} v + ... + a_2 \partial_x^2 v.$$

The time evolution of $\Phi^t(v_0)$ is given as

$$\frac{1}{2}\frac{d}{dt}\|\Phi^t(v_0)\|_{H^m}^2 = (v, v_t)_{H^m} = \sum_{j=0}^{m} \int_{\mathbb{R}} \partial_x^j v(a_l \partial_x^l + a_{l-1}\partial_x^{l-1} + ... + a_2 \partial_x^2) v dx. \quad (3.9)$$

It is sufficient to estimate one general term in the above sum, say

$$\sum_{j=0}^{m} \int_{\mathbb{R}} \partial_x^j v a_l \partial_x^{j+l} v dx = a_l \sum_{j=0}^{m} \int_{\mathbb{R}} \partial_x^j v \partial_x^{j+l} v dx. \quad (3.10)$$

By partial integration the above equation turns into

$$\begin{aligned} a_l \sum_{j=0}^{m} \int_{\mathbb{R}} \partial_x^j v \partial_x^{j+l} v dx &= a_l \sum_{j=0}^{m} \left([\partial_x^j v \partial_x^{j+l-1} v]_{-\infty}^{\infty} - \int_{\mathbb{R}} \partial_x^{j+1} v \partial_x^{j+l-1} v dx \right) \\ &= -a_l \sum_{j=0}^{m} \int_{\mathbb{R}} \partial_x^{j+1} v \partial_x^{j+l-1} v dx, \end{aligned}$$

where we have used that the derivatives on v of order up to m decay to zero when $x \to \pm\infty$. Performing partial integration together with the decay property for the derivatives of v subsequently, we get if l even

$$a_l \sum_{j=0}^{m} \int_{\mathbb{R}} \partial_x^j v \partial_x^{j+l} v dx = a_l \sum_{j=0}^{m} (-1)^{l-1} \int_{\mathbb{R}} (\partial_x^{j+l/2} v)^2 dx = -a_l \sum_{j=0}^{m} \int_{\mathbb{R}} (\partial_x^{j+l/2} v)^2 dx,$$

By the property given in (3.8), the coefficient al is such that the right-hand-side of the above equation is negative, that is $a_l > 0$. We write this for simplicity as

$$a_l \sum_{j=0}^{m} \int_{\mathbb{R}} \partial_x^j v \partial_x^{j+l} v dx = -\sum_{j=0}^{m} \int_{\mathbb{R}} (\partial_x^{j+l/2} v)^2 dx = -\|\partial_x^{l/2} v\|_{H^m}. \quad (3.11)$$

If l is odd, we obtain by partial integration

$$
\begin{aligned}
a_l \sum_{j=0}^{m} \int_{\mathbb{R}} \partial_x^j v \partial_x^{j+l} v dx &= a_l \sum_{j=0}^{m} (-1)^l \int_{\mathbb{R}} \partial_x (\partial_x^{j+(l-1)/2} v)^2 dx \\
&= -a_l \sum_{j=0}^{m} [(\partial_x^{j+(l-1)/2} v)^2]_{\infty}^{\infty} dx = 0 \qquad (3.12)
\end{aligned}
$$

By using the estimates in (3.11) and (3.12), we get for (3.9),

$$
\frac{1}{2}\frac{d}{dt}\|\Phi^t(v_0)\|_{H^m}^2 = -C\|\partial_x^{j+l/2} v\|_{H^m} \leq 0 \qquad (3.13)
$$

where C is a constant. Solving the differential equation gives

$$
\|\Phi_P^t(v_0)\|_{H^m} \leq \|v_0\|_{H^{m+1}}
$$

□

The upcoming analysis relies on a well- posedness theory for (3.3) in $H^s(\mathbb{R})$. For simplicity ints for the refering, we list the well- posedness requirements for (3.3) in addition with the assumptions for $u0$ and $u(t)$, as hypotheses for arbitrary order $k \geq 0$ of $H^k(\mathbb{R})$, and specify for which k they should hold in details for the Lie-Troter and Strang splittings below.

Hypothesis 3.1 *(Local well-posedness). For a fixed time T, there exists $R > 0$such that for all u^0 in $H^k(\mathbb{R})$ with $\|u_0\| \leq R$, there exists a unique strong solution u in $C([0,T], H^k)$ of (3.3). In addition, for the initial data u_0 there exists a constant $K(R,T) < \infty$, such that*

$$
\|\tilde{u}(t) - u(t)\|_{H^k} \leq K(R,T)\|\tilde{u}_0 - u_0\|_{H^k}
$$

for two arbitrary solutions u and $\tilde{u}$, corresponding to two different initial data $\tilde{u}_0$ and u_0.

The requirement in (3.14) is the same as requering that u_0 is local Lipschitz continuous. The last hypothesis requires that the solution and the initial data are bounded in the Sobolev spaces.

Hypothesis 3.2 *(Boundedness). The solution $u(t)$ and the initial data u_0 of (3.3) are both in*

$H^k(\mathbb{R})$, *and are bounded as*

$$\|u(t)\|_{H^k} \leq R < \rho \text{ and } \|u_o\|_{H^k} \leq C < \infty, \tag{3.14}$$

for $0 \leq t \leq T$.

3.3. Convergence Analysis for the Fisher's Equation

3.3.1. Regularity results for Fisher's Equation

Here, we study on the regularization estimates. Let $u(t)$, $\tilde{u}(t)$ be solutions of (1.2) with initial conditions u_0, $\tilde{u}_0$ for $0 < t < T$ interval, then,

$$u(t) = e^{t\partial_x^2}u_0 + \int_0^t e^{(t-s)(\partial_x^2)}u(s)(1-u(s))ds.$$

and

$$\tilde{u}(t) = e^{t\partial_x^2}\tilde{u}_0 + \int_0^t e^{(t-s)(\partial_x^2)}\tilde{u}(s)(1-\tilde{u}(s))ds.$$

then subtracting implies

$$u(t) - \tilde{u}(t) = e^{t(\partial_x^2)}(u_0 - \tilde{u}_0) + \int_0^t e^{(t-s)(\partial_x^2)}(u(1-u) - \tilde{u}(1-\tilde{u}))ds. \tag{3.15}$$

Taking H^s norm

$$\|u(t) - \tilde{u}(t)\| \leq \|u_0 - \tilde{u}_0\| + \int_0^t \|e^{(t-s)(\partial_x^2)}(u(1-u) - \tilde{u}(1-\tilde{u}))\|ds.$$

$$
\begin{aligned}
\|u(t)-\tilde{u}(t)\|_{H^s} &\le \|u_0-\tilde{u}_0\|_{H^s}+\int_0^t \|(u(1-u)-\tilde{u}(1-\tilde{u}))\|_{H^s}ds. \\
&\le \|u_0-\tilde{u}_0\|_{H^s}+C\int_0^t \|(u-\tilde{u})\|\|u+\tilde{u}-1\|_{H^s}ds. \\
&\le \|u_0-\tilde{u}_0\|_{H^s}+C\int_0^t \|(u-\tilde{u})\|\max\{\|u\|_{H^s},\|\tilde{u}\|_{H^s},\|1\|_{H^s}\}ds. \\
&\le \|u_0-\tilde{u}_0\|_{H^s}+\int_0^t L\|(u-\tilde{u})\|ds.
\end{aligned}
$$

where $L = C\max\{\|u\|_{H^s},\|\tilde{u}\|_{H^s},\|1\|_{H^s}\}$. by Gronwall's lemma we obtain the bound

$$\|u(t)-\tilde{u}(t)\|_{H^s} \le e^{Lt}\|u_0-\tilde{u}_0\|$$

Lemma 3.2 *If $\|u_0\|_{H^s} \le M$ then there exists $\bar{t}(M) > 0$ such that $\|\Phi_B(u_0)\|_{H^s} \le 2M$ for $0 \le t \le \bar{t}(M)$.*

Proof Since the linear flow e^A preserves the H^s norm, we only need to compare the nonlinear part. The nonlinear part that is $w_t = w(1-w)$ can be solved exactly ,it is

$$w(t) = \frac{u_0e^t}{1-u_0+u_0e^t} \tag{3.16}$$

The solution of the nonlinear part is a function of initial condition. Hence we obtain the inequality $\|\Phi_B(u_0)\|_{H^s} \le 2M$. □

In the proofs of the convergence rates for (3.1) and (3.2) , we need to expand $\Phi_B(u_0)$ using Taylor series expansions of first and second order. Thus, $\Phi_B(u_0)$ needs to be continuous, such that the expansions are valid. The following lemma proves the sufficient continuity.

Lemma 3.3 *If $\|u_0\|_{H^s} \le M$ then there exists $\bar{t}$ depending on M such that the solution of Fisher's equation with initial data u_0, $w(t) = \Phi_B^t(u_0)$ satisfies*

$$w \in C^2([0,\bar{t}],H^s) \text{ and } w \in C^3([0,\bar{t}],H^s). \tag{3.17}$$

Proof Recall the Lemma 3.2, if $\|u_0\|_{H^s} \le M$ then $\|w(t)\|_{H^s} = \|\Phi_B(u_0)\|_{H^s} \le 2M$ for $t \in [0,\bar{t}]$ and we can define

$$\tilde{w}(t) = u_0 + tB(u_0) + \int_0^t (t-s)dB\big(w(s)\big)[B\big(w(s)\big)]ds, \tag{3.18}$$

where $dB\big(w\big)[B\big(w\big)] = w(1-w)(1-2w)$. Since $\tilde{w}_{tt} = dB\big(w\big)[B\big(w\big)] = B(w)_t$, $\tilde{w}_t(0) = B(u_0) = w_t(0)$ and $\tilde{w}(0) = u_0 = w(0)$, we have $\tilde{w} = w$. Now we have to show that $\tilde{w} \in C^2([0,\tilde{t}], H^s)$. Start with

$$\begin{aligned} \|\tilde{w}_{tt}\|_{H^s} &= \|dB\big(w\big)[B\big(w\big)]\|_{H^s} = \|w(1-w)(1-2w)\|_{H^s} \\ &\le \|w\|_{H^s}\|(1-w)\|_{H^s}\|(1-2w)\|_{H^s} \\ &\le K_1\|w\|_{H^s}\|w\|_{H^s}\|w\|_{H^s}. \end{aligned} \tag{3.19}$$

Hence $\|\tilde{w}_{tt}\|_{H^s} \le K_1\|w\|^3_{H^s}$ and Lemma 3.2 completes the proof.

For the second statement, define

$$\begin{aligned} \tilde{w}(t) &= u_0 + tB(u_0) + \frac{t^2}{2!}dB(u_0)[B(u_0)] \\ &\quad + \int_0^t \frac{(t-s)^2}{2!}\Big(d^2B\big(w(s)\big)[B\big(w(s)\big), B\big(w(s)\big)] \\ &\quad + dB\big(w(s)\big)\big[dB\big(w(s)\big)[B\big(w(s)\big)]\big]\Big)ds, \end{aligned}$$

where $d^2B(w)[B(w), B(w)] = w(1-w)(6w^2-6w+1)$ and $dB(w)\big[dB(w)[B(w)]\big] = w(1-w)(1-2w)^2$. Since $\tilde{w}_{ttt}(0) = d^2B(w)[B(w), B(w)] + dB(w)\big[dB(w)[B(w)]\big]$, $\tilde{w}_{tt}(0) = dB\big(u_0\big)[B\big(u_0\big)] = B(u_0)_t$, $\tilde{w}_t(0) = B(u_0) = w_t(0)$ and $\tilde{w}(0) = u_0 = w(0)$, we have $\tilde{w} = w$. Now we have to show that $\tilde{w} \in C^3([0,\tilde{t}], H^s)$. Start with

$$\begin{aligned} \|\tilde{w}_{ttt}\|_{H^s} &= \|d^2B(w)[B(w), B(w)] + dB(w)\big[dB(w)[B(w)]\big]\|_{H^s} \\ &\le \|-2w^2(1-w)^2\|_{H^s} \\ &\le K_2\|w\|^2_{H^s}\|(1-w)\|^2_{H^s} \\ &\le K\|w\|^4_{H^s} \end{aligned} \tag{3.20}$$

Hence $\|\tilde{w}_{ttt}\|_{H^s} \le K\|w\|^4_{H^s}$ and Lemma 3.2 completes the proof. □

3.3.2. Lie-Trotter Splitting

3.3.2.1. Stability in H^s space

Lemma 3.4 *Let u_1, $\tilde{u}_1$ be the Lie-Trotter splitting solutions satisfying Equation (3.1) with initial data $u_0, \tilde{u}_0$ in H^s. Then*

$$\|u_1 - \tilde{u}_1\|_{H^s} \leq e^{L\Delta t}\|u_0 - \tilde{u}_0\|_{H^s}, \tag{3.21}$$

where $L = K \max\{\|u_1\|_{H^s}, \|\tilde{u}_1\|_{H^s}, \|1\|_{H^s}\}$

Proof Since e^A preserves the H^s norm, we only need to compare nonlinearities in splitting solutions.

$$w(\Delta t) = \Phi_B^{\Delta t}, \quad w'(\Delta t) = B(u_0)$$

Hence,

$$\begin{aligned} u_{1t} &= u(1-u) \\ \tilde{u}_{1t} &= \tilde{u}(1-\tilde{u}) \\ (u_1 - \tilde{u}_1)_t &= (u(1-u) - \tilde{u}(1-\tilde{u})) \end{aligned}$$

integrate from 0 to t

$$u_1 - \tilde{u}_1 = u_0 - \tilde{u}_0 + \int_0^t (u(1-u) - \tilde{u}(1-\tilde{u}))ds \tag{3.22}$$

Taking H^s norm,

$$\begin{aligned} \|u_1 - \tilde{u}_1\|_H^s &\leq \|u_0 - \tilde{u}_0\|_H^s + \int_0^t \|(u(1-u) - \tilde{u}(1-\tilde{u}))\|_H^s ds \\ &\leq \|u_0 - \tilde{u}_0{}_-H^s| + K\int_0^t \|u - \tilde{u}\|_H^s \|(u + \tilde{u} + 1))\|_H^s ds \\ &\leq \|u_0 - \tilde{u}_0\|_H^s + L\int_0^t \|u - \tilde{u}\|_H^s ds \end{aligned}$$

where $L = K\max\{\|u_1\|_{H^s}, \|\tilde{u}_1\|_{H^s}, \|1\|_{H^s}\}$. By using Gronwall's lemma,we obtain the bound in equation 3.21 $\square$

3.3.2.2. Local error in H^s space

Lemma 3.5 *Let $s \geq 1$ be an integer and assumption (3.14) holds for $k = s+2$ for the solution $u(t) = \Phi^{\Delta t}_{A+B}(u_0)$ of (1.2). If the initial data u_0 is in $H^{s+2}(\mathbb{R})$, then the local error of the Lie-Trotter splitting (3.1) is bounded in $H^s(\mathbb{R})$ by*

$$\|\Psi^{\Delta t}(u_0) - \Phi^{\Delta t}(u_0)\|_{H^s} \leq C_1 \Delta t^2, \tag{3.23}$$

where C only depends on $\|u_0\|_{H^{s+2}}$.

Proof In the following proof, we follow similar way to (Lubich, 2008) and (Holden, Lubich, and Risebro, 2013). Fisher's equations are in the form

$$u_t = Au + B(u), \tag{3.24}$$

where $Au = (\partial_x^2)u$ and $B(u) = u(1-u)$. The exact solution is $u(t) = \Phi^t(u_0)$, from variation of constant formula, for $[0, \Delta t]$ interval, it can be written as

$$u(\Delta t) = e^{\Delta t A} u_0 + \int_0^{\Delta t} e^{(\Delta t - s)A} B\big(u(s)\big) ds. \tag{3.25}$$

This is similar to formula $\varphi(t) - \varphi(0) = \int_0^t \dot{\varphi}(s) ds$ when $\varphi(s) = e^{(\Delta t - s)A} u(s)$.

$$\varphi(\Delta t) = u(\Delta t) \;\; \varphi(0) = e^{\Delta t A} u_0$$
$$\varphi'(s) = -Ae^{(\Delta t - s)A} u(s) + e^{(\Delta t - s)A} \underbrace{u'(s)}_{Au+B(u)}$$

Second part of the Equation (3.25) can be written with similar formula by taking
$\varphi(\rho) = B(e^{(s-\rho)A} u(\rho))$, and then we have

$$\begin{aligned} B(\varphi(s)) - B(\varphi(0)) &= \int_0^s dB(\varphi(\rho))[\varphi'(\rho)]d\rho \\ B(e^{(s-t)A}u(t)) - B(e^{sA}u_0) &= \int_0^s dB(e^{(s-\rho)A}u(\rho))[e^{(s-\rho)A}B(u(\rho))]d\rho. \end{aligned} \tag{3.26}$$

Writing s instead of t, we obtain $B(u(s))$

$$B(u(s)) = B(e^{sA}u_0) + \int_0^s dB(e^{(s-\rho)A}u(\rho))[e^{(s-\rho)A}B(u(\rho))]d\rho. \tag{3.27}$$

After inserting Equation (3.27) into Equation (3.25), we get

$$u(\Delta t) = e^{\Delta tA}u_0 + \int_0^{\Delta t} e^{(\Delta t - A)}B(e^{sA}u_0)ds + E_1 \tag{3.28}$$

where

$$E_1 = \int_0^{\Delta t}\int_0^s e^{(\Delta t-s)A}dB(e^{(s-\rho)A}u(\rho))[e^{(s-\rho)A}B(u(\rho))]d\rho ds. \tag{3.29}$$

The Lie-Trotter splitting solution for $[0, \Delta t]$ interval can be written as

$$u_1 = \Psi^{\Delta t}(u_0) = \Phi_B^{\Delta t}(e^{\Delta tA}u_0), \tag{3.30}$$

We use the first-order Taylor expansion with integral remainder term in H^s,

$$\Phi_B^{\Delta t}(v) = v + \Delta tB(v) + \Delta t^2\int_0^1 (1-\theta)dB(\Phi_B^{\theta\Delta t}(v))[B(\Phi_B^{\theta\Delta t}(v))]d\theta. \tag{3.31}$$

This is justified for $v = e^{\Delta tA}u_0 \in H^s$.Therefore, we obtain

$$u_1 = e^{\Delta tA}u_0 + \Delta tB(e^{\Delta tA}u_0) + E_2 \tag{3.32}$$

with

$$E_2 = \Delta t^2 \int_0^1 (1-\theta) dB(\Phi_B^{\theta\Delta t}(e^{\Delta tA}u_0))[B(\Phi_B^{\theta\Delta t}(e^{\Delta tA}u_0))]d\theta \tag{3.33}$$

Thus, the error becomes,

$$u_1 - u(\Delta t) = \Delta t B(e^{\Delta tA}u_0) - \int_0^{\Delta t} e^{(\Delta t - A)} B(e^{sA}u_0)ds + (E_2 - E_1) \tag{3.34}$$

and hence the principal error term is the just the quadrature error of rectangle rule applied to the integral over $[0, \Delta t]$ of the function

$$h(s) = e^{(\Delta t - s)A} B(e^{sA}u_0) \tag{3.35}$$

we express the quadrature error in first-order Peano form,

$$\Delta t h(\Delta t) - \int_0^{\Delta t} h(s)ds = \Delta t^2 \int_0^1 \kappa(\theta) h'(\theta \Delta t) d\theta \tag{3.36}$$

where κ is bounded kernel. Here $h'(s) = -e^{(\Delta t - s)A}[A, B](e^{sA}u_0)$ with double Lie commutator

$$[A, B] = dA(v)[B(v)] - dB(v)[Av] \tag{3.37}$$

$$\begin{aligned} dA(v)[B(v)] &= \lim_{t \to 0} \frac{A(v + tB(v)) - A(v)}{t} \\ &= \lim_{t \to 0} \frac{(\partial_x^2)(v + tB(v)) - (\partial_x^2)v}{t} \\ &= \partial_x^2 (v(1-v)) \end{aligned}$$

$$
\begin{aligned}
dB(v)[Av] &= \lim_{t\to 0}\frac{B(v+tAv)-B(v)}{t} \\
&= \lim_{t\to 0}\frac{(v+tAv)(1-(v+tAv))-B(v)}{t} \\
&= Av(1-2v)
\end{aligned}
$$

We need to find bound in H^s for each of the terms. Lemma 2.2 will be helpfull to bound them. Now, start with the first term, then we get

$$
\begin{aligned}
\|dA(v)[B(v)]\| &= \|\partial_x^2(v(1-v))\|_{H^s} \\
&\leq \|(v(1-v))\|_{H^{s+2}} \\
&\leq C_1\|v\|_{H^{s+2}}^2
\end{aligned}
$$

The other term follows in a similar way i.e.

$$
\begin{aligned}
\|dB(v)[Av]\| &= \|\partial_x^2 v(1-2v)\|_{H^s} \\
&\leq \|v(1-2v)\|_{H^{s+2}} \\
&\leq C_2\|v\|_{H^{s+2}}^2
\end{aligned}
$$

Since e^{tA} does not increase the Sobolev norms, it follows that

$$
\|h'(s)\| \leq C\|e^{sA}u_0\|_{H^{s+2}}^2 \leq C\|u_0\|_{H^{s+2}}^2
$$

Thus, the integral (3.36) is bounded as

$$
\Delta t^2\int_0^1 \kappa(\theta)h'(\theta\Delta t)d\theta \leq C\|u_0\|_{H^{s+2}}^2\Delta t^2 \tag{3.38}
$$

We continue with the error bound for E_1 in (3.34).

$$
\|E_1\| \leq \int_0^{\Delta t}\int_0^s \|e^{(\Delta t-s)A}dB(e^{(s-\rho)A}u(\rho))[e^{(s-\rho)A}B(u(\rho))]\|_{H^s}d\rho ds. \tag{3.39}
$$

$$dB(e^{(s-\rho)A}u(\rho))[e^{(s-\rho)A}B(u(\rho))] = e^{(s-\rho)A}B(u(\rho))(1-2e^{(s-\rho)A}u(\rho))$$

Inserting the result into the equation (3.39), we get

$$\begin{aligned}
\|E_1\| &\leq \int_0^{\Delta t}\int_0^s \|e^{(\Delta t-s)A}(e^{(s-\rho)A}B(u(\rho))(1-2e^{(s-\rho)A}u(\rho)))\|_{H^s}d\rho ds \\
&\leq \int_0^{\Delta t}\int_0^s \|e^{(s-\rho)A}B(u(\rho))(1-2e^{(s-\rho)A}u(\rho))\|_{H^s}d\rho ds \\
&\leq C\int_0^{\Delta t}\int_0^s \|B(u(\rho)))\|_{H^s}\|1-2e^{(s-\rho)A}u(\rho)\|_{H^s}d\rho ds \\
&\leq C\int_0^{\Delta t}\int_0^s \|B(u(\rho)))\|_{H^s}\|u(\rho)\|_{H^s}d\rho ds \\
&\leq C\int_0^{\Delta t}\int_0^s \|u(\rho)\|^3_{H^s}d\rho ds \\
&\leq C\Delta t^2R^3 \qquad (3.40)
\end{aligned}$$

and the bound for E_2

$$\|E_2\| \leq \Delta t^2\int_0^1(1-\theta)\|dB(\Phi_B^{\theta\Delta t}(e^{\Delta tA}u_0))[B(\Phi_B^{\theta\Delta t}(e^{\Delta tA}u_0))]\|_{H^s}d\theta$$

$$dB(\Phi_B^{\theta\Delta t}(e^{\Delta tA}u_0))[B(\Phi_B^{\theta\Delta t}(e^{\Delta tA}u_0))] = (1-2\Phi_B^{\theta\Delta t}(e^{\Delta tA}u_0))B(\Phi_B^{\theta\Delta t}(e^{\Delta tA}u_0))$$

Then, we can rewrite the norm of E_2 as follows,

$$\begin{aligned}
\|E_2\| &\leq \Delta t^2\int_0^1(1-\theta)\|(1-2\Phi_B^{\theta\Delta t}(e^{\Delta tA}u_0))B(\Phi_B^{\theta\Delta t}(e^{\Delta tA}u_0))\|_{H^s}d\theta \\
&\leq \Delta t^2C\int_0^1 \|(1-2\Phi_B^{\theta\Delta t}(e^{\Delta tA}u_0))\|_{H^s}\|B(\Phi_B^{\theta\Delta t}(e^{\Delta tA}u_0))\|_{H^s}d\theta \\
&\leq \Delta t^2\int_0^1 C\|\Phi_B^{\theta\Delta t}(e^{\Delta tA}u_0)\|_{H^s}\|B(\Phi_B^{\theta\Delta t}(e^{\Delta tA}u_0))\|_{H^s}d\theta \\
&\leq \Delta t^2\int_0^1 C\|\Phi_B^{\theta\Delta t}(e^{\Delta tA}u_0)\|_{H^s}\|\Phi_B^{\theta\Delta t}(e^{\Delta tA}u_0)\|^2_{H^s}d\theta
\end{aligned}$$

For a sufficiently small Δt, Lemma 3.2 ensures that
$\|\Phi_B^{\theta\Delta t}(e^{\Delta tA}u_0)\|_{H^s} \leq \|\Phi_B^{\theta\Delta t}(e^{\Delta tA}u_0)\|_{H^{s+2}} \leq R$. Thus,

$$\|E_2\| \;\; \leq \;\; C\Delta t^2 R^3 \tag{3.41}$$

Hence, combining the estimates in (3.38),(3.40) and (3.41), we obtain the quadrature error is $O(\Delta t^2)$ in the H^s norm for $u_0 \in H^{s+2}$. □

3.3.2.3. Global error in H^s space

Theorem 3.1 *Suppose that the exact solution $u(\cdot, t)$ of Equation (1.2) is in H^{s+2} for $0 \leq t \leq T$. Then Lie Trotter splitting solution u_n given in Equation (3.1) has first order global error for $\Delta t < \bar{\Delta} t$ and $t_n = n\Delta t \leq T$,*

$$\|u_n - u(\cdot, t_n)\|_{H^s} \leq G\Delta t, \tag{3.42}$$

where G only depends on $\|u_0\|_{H^{s+2}}$, α and T.

Proof The Lady Windermere's fan is used in the proof. The stability estimate is given in Lemma 3.4 and local error is in Lemma 3.5. Let $u(t_n) = \Phi^{(n-k)\Delta t}(u(t_k))$ be the exact solution of Equation (1.2) at time t_n with initial data $u(t_k)$ at time t_k. Lie-Trotter splitting solution $u_n = \Psi^{\Delta t}(u_{n-1})$ is

$$u_n = \Phi_B^{\Delta t} \circ \Phi_A^{\Delta t}(u_{n-1}), \;\; n = 1, 2, \ldots. \tag{3.43}$$

Then we estimate

$$\begin{aligned} u_n - u(\cdot, t_n) \;\; &\leq \;\; \prod_{k=0}^{n-1} \Psi(\Delta t)(u_0 - u(t_0)) + \sum_{j=1}^{n} \prod_{k=j}^{n-1} \Phi(\Delta t) d_j \\ \text{where } d_j = (\Psi(\Delta t) - \Phi(\Delta t))u(t_{j-1}) & \\ &\leq \;\; \sum_{k=1}^{n-1} \Psi^{(n-k-1)\Delta t}(\Phi^{\Delta t}(u(t_k)) - u(t_{k+1})) \\ &\leq \;\; \sum_{k=1}^{n-1} \Psi^{(n-k-1)\Delta t}(\Phi^{\Delta t}(u(t_k)) - \Psi^{\Delta t}(u(t_k))) \end{aligned}$$

Taking H^s norm,

$$\begin{aligned}
\|u_n - u(\cdot,t_n)\|_{H^s} &\leq \sum_{k=0}^{n-1} \|\Psi^{(n-k-1)\Delta t}(\Phi^{\Delta t}(u(t_k)) - \Psi^{\Delta t}(u(t_k)))\|_{H^s} \\
&\leq \sum_{k=0}^{n-1} e^{L(n-k-1)\Delta t}\|\Phi^{\Delta t}(u(t_k)) - \Psi^{\Delta t}(u(t_k))\|_{H^s} \\
&\leq \sum_{k=0}^{n-1} e^{LT}C(\alpha)\Delta t^2 \\
&\leq ne^{LT}C(\alpha)\Delta t^2 \\
&\leq Te^{LT}C(\alpha)\Delta t
\end{aligned} \tag{3.44}$$

since $e^{L(n-k-1)\Delta t} \leq e^{LT}$ and $n\Delta t \leq T$. This completes the proof. □

3.3.3. Strang Splitting

3.3.3.1. Stability in H^s space

Lemma 3.6 *Let u_1, $\tilde{u}_1$ be the Strang splitting solutions satisfying Equation (3.2) with initial data $u_0, \tilde{u}_0$ in H^s. Then*

$$\|u_1 - \tilde{u}_1\|_{H^s} \leq e^{L\Delta t}\|u_0 - \tilde{u}_0\|_{H^s}, \tag{3.45}$$

where $L = K\max\{\|u_1\|_{H^s}, \|\tilde{u}_1\|_{H^s}, \|1\|_{H^s}\}$.

Proof Since e^A preserves the H^s norm, we only need to compare nonlinearities in splitting solutions.

$$w(\Delta t) = \Phi_B^{\Delta t}, \quad w'(\Delta t) = B(u_0)$$

Hence,

$$\begin{aligned} u_{1t} &= u(1-u) \\ \tilde{u}_{1t} &= \tilde{u}(1-\tilde{u}) \\ (u_1 - \tilde{u}_1)_t &= (u(1-u) - \tilde{u}(1-\tilde{u})) \end{aligned}$$

integrate from 0 to t

$$u_1 - \tilde{u}_1 = u_0 - \tilde{u}_0 + \int_0^t (u(1-u) - \tilde{u}(1-\tilde{u}))ds \tag{3.46}$$

Taking H^s norm,

$$\begin{aligned} \|u_1 - \tilde{u}_1\|_H^s &\leq \|u_0 - \tilde{u}_0\|_{H^s} + \int_0^t \|(u(1-u) - \tilde{u}(1-\tilde{u}))\|_{H^s} ds \\ &\leq \|u_0 - \tilde{u}_0 \_H^s| + K \int_0^t \|u - \tilde{u}\|_{H^s} \|(u + \tilde{u} - 1))\|_{H^s} ds \end{aligned}$$

he nonlinear term has Lipshitz constant L which is bounded by Lemma 3.2. Finally Gronwall's Lemma implies the bound in Equation (3.45).

□

3.3.3.2. Local error in H^s space

Lemma 3.7 *Let $s \geq 1$ be an integer and assumption (3.14) holds for $k = s + 4$ for the solution $u(t) = \Phi^{\Delta t}_{A+B}(u_0)$ of (1.2). If the initial data u_0 is in $H^{s+4}(\mathbb{R})$, then the local error of the Strang splitting (3.2) is bounded in $H^s(\mathbb{R})$ by*

$$\|\Psi^{\Delta t}(u_0) - \Phi^{\Delta t}(u_0)\|_{H^s} \leq C\Delta t^3, \tag{3.47}$$

where C only depends on $\|u_0\|_{H^{s+4}}$.

Proof In the following proof, we follow similar way to (Lubich, 2008) and (Holden, Lubich, and Risebro, 2013). Fisher's equation is in the form

$$u_t = Au + B(u), \tag{3.48}$$

where $Au = (\partial_x^2)u$ and $B(u) = u(1-u)$. The exact solution is $u(t) = \Phi^t(u_0)$, from variation of constant formula, for $[0, \Delta t]$ interval, it can be written as

$$u(\Delta t) = e^{\Delta t A}u_0 + \int_0^{\Delta t} e^{(\Delta t - s)A} B\big(u(s)\big)ds. \tag{3.49}$$

This is similar to formula $\varphi(t) - \varphi(0) = \int_0^t \dot{\varphi}(s)ds$ when $\varphi(s) = e^{(t-s)A}u(s)$.
Second part of the Equation (3.49) can be written with similar formula by taking
$\varphi(\rho) = B(e^{(s-\rho)A}u(\rho))$, and then we have

$$B(u(s)) = B(e^{sA}u_0) + \int_0^s dB(e^{(s-\rho)A}u(\rho))[e^{(s-\rho)A}B(u(\rho))]d\rho. \tag{3.50}$$

After inserting Equation (3.50) into Equation (3.49), we get

$$u(\Delta t) = e^{\Delta t A}u_0 + \int_0^{\Delta t} e^{(\Delta t - s)A}B(e^{sA}u_0)ds + E_1 \tag{3.51}$$

where

$$E_1 = \int_0^{\Delta t}\int_0^s e^{(\Delta t - s)A}dB(e^{(s-\rho)A}u(\rho))[e^{(s-\rho)A}B(u(\rho))]d\rho ds. \tag{3.52}$$

The Strang splitting solution for $[0, \Delta t]$ interval can be written as

$$u_1 = \Psi^{\Delta t}(u_0) = e^{\frac{\Delta t A}{2}}\Phi_B^{\Delta t}(e^{\frac{\Delta t A}{2}}u_0), \tag{3.53}$$

and Taylor series expansion can be used for nonlinear part such as

$$\begin{aligned}\Phi_B^{\Delta t}(v) &= v + \Delta t B(v) + \frac{1}{2}\Delta t^2 dB(v)[B(v)] \\ &+\Delta t^3 \int_0^1 \frac{1}{2}(1-\theta)^2 (d^2B(\Phi_B^{\theta\Delta t}(v))[B(\Phi_B^{\theta\Delta t}(v)), B(\Phi_B^{\theta\Delta t}(v))] \\ &+dB(\Phi_B^{\theta\Delta t}(v))[dB(\Phi_B^{\theta\Delta t}(v))[B(\Phi_B^{\theta\Delta t}(v))]])d\theta \end{aligned} \tag{3.54}$$

where $v = e^{\frac{\Delta t A}{2}} u_0$. The simplified integral remainder term is

$$\Delta t^3 \int_0^1 \frac{1}{2}(1-\theta)^2 (d^2B(B,B) + dBdBB)(\Phi_B^{\theta\Delta t}(v))d\theta. \tag{3.55}$$

Hence the Strang solution is

$$\begin{aligned} u_1 &= e^{\Delta t A} u_0 + \Delta t e^{\frac{\Delta t A}{2}} B(e^{\frac{\Delta t A}{2}} u_0) \\ &+\frac{1}{2}\Delta t^2 e^{\frac{\Delta t A}{2}} dB(e^{\frac{\Delta t A}{2}} u_0)[B(e^{\frac{\Delta t A}{2}} u_0)] \\ &+\Delta t^3 \int_0^1 \frac{1}{2}(1-\theta)^2 e^{\frac{\Delta t A}{2}} (d^2B(B,B) + dBdBB)(\Phi_B^{\theta\Delta t}(v))d\theta, \end{aligned} \tag{3.56}$$

or it can be written as

$$u_1 = e^{\Delta t A} u_0 + \Delta t e^{\frac{\Delta t A}{2}} B(e^{\frac{\Delta t A}{2}} u_0) + E_2. \tag{3.57}$$

where

$$E_2 = \Delta t^3 \int_0^1 \frac{1}{2}(1-\theta)^2 e^{\frac{\Delta t A}{2}} (d^2B(B,B) + dBdBB)(\Phi_B^{\theta\Delta t}(v))d\theta$$

In order to calculate the local error we need to subtract Equation (3.51) from Equation (3.57). But before subtracting, we need to make some arrangements on E_1. Let

$$G(v) = G_{s,\sigma}(v) = dB(e^{(s-\sigma)A} v)[e^{(s-\sigma)A} B(v)], \tag{3.58}$$

Then the exact solution after one step is

$$u(\Delta t) = e^{\Delta t A}u_0 + \int_0^{\Delta t} e^{(\Delta t - s)A}B(e^{sA}u_0)ds + \int_0^{\Delta t}\int_0^s e^{(\Delta t - s)A}(G(u(\sigma)))d\sigma ds$$

Using the integral formula, we obtain

$$G(u(\sigma)) = G(e^{\sigma A}u_0) + \int_0^{\sigma} dG(e^{(s-\sigma)A}u(\tau))[e^{(s-\sigma)A}B(u(\tau))]d\tau, \tag{3.59}$$

and where the integrand is calculated as,

$$\begin{aligned} dG(v)[w] &= d^2B(e^{(s-\sigma)A}v)[e^{(s-\sigma)A}w, e^{(s-\sigma)A}B(v)] \\ &\quad + dB(e^{(s-\sigma)A}v)[e^{(s-\sigma)A}dB(v)[w]]. \end{aligned} \tag{3.60}$$

Then we obtain

$$\begin{aligned} E_1 &= \int_0^{\Delta t}\int_0^s e^{(\Delta t - s)A}dB(e^{sA}u_0)[e^{(s-\sigma)A}B(e^{sA}u_0)]d\sigma ds \\ &\quad + \int_0^{\Delta t}\int_0^s\int_0^{\sigma} dG_{s,\sigma}(e^{(\sigma-\tau)A}u(\tau))[e^{(\sigma-\tau)A}B(u(\tau))]d\tau d\sigma ds. \end{aligned} \tag{3.61}$$

Hence the local error is read as

$$\begin{aligned} u_1 - u(\Delta t) &= \Delta t e^{\frac{\Delta t A}{2}}B(e^{\frac{\Delta t A}{2}}u_0) - \int_0^{\Delta t} e^{(\Delta t - s)A}B(e^{sA}u_0)ds \\ &\quad + \frac{\Delta t^2}{2}e^{\frac{\Delta t A}{2}}dB(e^{\frac{\Delta t A}{2}}u_0)[B(e^{\frac{\Delta t A}{2}}u_0)] \\ &\quad - \int_0^{\Delta t}\int_0^s e^{(\Delta t - s)A}dB(e^{sA}u_0)(e^{(s-\sigma)A}B(e^{sA}u_0))d\sigma ds \\ &\quad + \Delta t^3\int_0^1 \frac{1}{2}(1-\theta)^2 e^{\frac{\Delta t A}{2}}(d^2B(B,B) + dBdBB)(\Phi_B^{\theta\Delta t}(v))d\theta \\ &\quad - \int_0^{\Delta t}\int_0^s\int_0^{\sigma} dG_{s,\sigma}(e^{(\sigma-s)A}u(\tau))[e^{(\sigma-\tau)A}B(u(\tau))]d\tau d\sigma ds. \end{aligned} \tag{3.62}$$

In order to show local error is in H^s space, we rearrange the terms of Equation (3.62). The difference of first two terms gives the quadrature error of midpoint rule over $[0, \Delta t]$ interval and can be expressed in second order Peano form such as

$$\Delta t h(\frac{\Delta t}{2}) - \int_0^{\Delta t} h(s)ds = \Delta t^3 \int_0^{\Delta t} \kappa(t)h''(t)dt = \Delta t^3 \int_0^1 \kappa(\theta)h''(\theta\Delta t)d\theta \qquad (3.63)$$

where κ is bounded kernel. Here $h''(s) = e^{(\Delta t - s)A}[A,[A,B]](e^{sA}u_0)$ with double Lie commutator

$$\begin{aligned}
[A,[A,B]](v) &= [A, dAB(v) - dB(v)A(v)](v) \\
&= dA^2B(v) - dAdB(v)A(v) - d^2AB(v)A(v) \\
&\quad -dAdB(v)A(v) + d^2B(A(v))^2 + dB(v)dAA(v) \\
&= A^2(B(v)) - 2A(dB(v)A(v)) + d^2B(A(v))^2 + dB(v)A^2(v)
\end{aligned}$$

since $d^2A = 0$. We need to find bounds in H^s for each of the terms. Lemma 2.2 will be helpfull to bound them.

Now, start with the first term, then we get

$$\begin{aligned}
\|A^2(B(v))\|_{H^s} &= \|((\partial_x^2)^2)v(1-v)\|_{H^s} \\
&\leq \|v(1-v)\|_{H^{s+4}} \\
&\leq K_1\|v\|^2_{H^{s+4}}.
\end{aligned}$$

The other terms follow in a similar way, i.e.

$$\begin{aligned}
\|A(dB(v)[A(v)])\|_{H^s} &= \|\partial_x^2((1-2v)(\partial_x^2)v)\|_{H^s} \\
&\leq K_2\|(1-2v)\|_{H^{s+2}}\|(\partial_x^2 v)\|_{H^{s+2}} \\
&\leq K_2\|v\|^2_{H^{s+4}}
\end{aligned}$$

$$\begin{aligned}
\|d^2B(v)[A(v)]^2\|_{H^s} &= \|-2(\partial_x^2 v)^2\|_{H^s} \\
&\leq K_3\|v\|^2_{H^{s+4}}
\end{aligned}$$

$$\begin{aligned}\|dB(v)[A^2(v)]\|_{H^s} &= \|(1-2v)(\partial_x^2)^2 v\|_{H^s} \\ &\leq K_4\|v\|_{H^s}\|(\partial_x^2)^2 v\|_{H^s} \\ &\leq K_4\|v\|^2_{H^{s+4}}.\end{aligned}$$

The difference of third and fourth terms of Equation (3.62) is the quadrature error of a first order two dimensional quadrature formula, which is bounded by

$$\|\frac{1}{2}\Delta t^2 g(\frac{1}{2}\Delta t, \frac{1}{2}\Delta t) - \int_0^{\Delta t}\int_0^s g(s,\sigma)d\sigma ds\|_{H^s} \leq K\Delta t^3(max\|\frac{\partial g}{\partial s}\|_{H^s} + max\|\frac{\partial g}{\partial \sigma}\|_{H^s}),$$

where the maxima are taken over triangle $\{(\sigma, s) : 0 \leq \sigma \leq s \leq \Delta t\}$. In order to estimate derivatives we write

$$g(s,\sigma) = e^{(\Delta t - s)A} dB(v(s))w(s,\sigma),$$

and

$$v(s) = e^{sA}u_0$$

and

$$w(s,\sigma) = e^{(s-\sigma)A}B(v(\sigma)).$$

We need to calculate each of the derivative terms. Start with the first derivative

$$\frac{\partial g}{\partial s} = e^{(\Delta t - s)A}(-AdB(v(s))[w(s,\sigma)]) + d^2B(v)[Av(s), w(s,\sigma)] + dB(v(s))[Aw(s,\sigma)]).$$

Since the flow of A does not increase in H^s, we only evaluate the following terms $-AdB(v(s))[w(s,\sigma)] + d^2B(v(s))[Av(s), w(s,\sigma)] + dB(v(s))[Aw(s,\sigma)]$, i.e.

$$\begin{aligned}\|AdB(v(s))[w(s,\sigma)]\|_{H^s} &= \|\partial_x^2((1-2v(s))w(s,\sigma))\|_{H^s} \\ &\leq \|(1-2v(s))w(s,\sigma))\|_{H^{s+2}} \\ &\leq K_1\|v(s)\|_{H^{s+2}}\|w(s,\sigma)\|_{H^{s+2}} \\ &\leq K_2\|v(s)\|^3_{H^{s+2}}, \qquad (3.64)\end{aligned}$$

$$\begin{aligned}\|d^2B(v(s))[Av(s),w(s,\sigma)]\|_{H^s} &= \|-2\partial_x^2 v(s)w(s,\sigma)\|_{H^s}\\ &\le K_3\|(v(s)\|_{H^{s+2}}\|w(s,\sigma)\|_{H^s}\\ &\le K_3\|v(s)\|_{H^{s+2}}\|w(s,\sigma)\|_{H^{s+2}}\\ &\le K_3\|v(s)\|^3_{H^{s+2}}, \end{aligned} \tag{3.65}$$

$$\begin{aligned}\|dB(v(s))[Aw(s,\sigma)]\|_{H^s} &= \|(1-2v(s))(\partial_x^2 w(s,\sigma))\|_{H^s}\\ &\le K_4\|1-2v(s)\|_{H^s}\|w(s,\sigma)\|_{H^{s+2}}\\ &\le K_4\|v(s)\|_{H^{s+2}}\|v(s)\|^2_{H^s+2}\\ &\le K_4\|v(s)\|^3_{H^{s+2}}, \end{aligned} \tag{3.66}$$

Using the bound of v,

$$\|v(s)\|_{H^{s+2}} = \|e^{sA}u_0\|H^{s+2} \le \|u_0\|H^{s+2}$$

Then,

$$\|\frac{\partial g}{\partial s}\| \le C.\|u_0\|H^{s+2}$$

For the second derivative term, we obtain

$$\frac{\partial g}{\partial \sigma} = e^{(\Delta t-s)A}dB(v(s))[e^{(s-\sigma)A}(-AB(v(s))+dB(v(\sigma))Av(\sigma))], \tag{3.67}$$

and each of the terms can be bounded as

$$\begin{aligned}\|dB(v(s))[e^{(s-\sigma)A}AB(v(s))]\|_{H^s} &= \|(1-2v(s))e^{(s-\sigma)A}\partial_x^2(v(s)(1-v(s)))\|_{H^s}\\ &\le K_5\|1-2v(s)\|_{H^s+2}\|v(s)(1-v(s))\|_{H^{s+2}}\\ &\le K_5\|v(s)\|^3_{H^{s+2}}, \end{aligned} \tag{3.68}$$

$$\begin{aligned}\|dB(v(s))[e^{(s-\sigma)A}(dB(v(\sigma))Av(\sigma))]\|_{H^s} &= \|(1-2v(s))e^{(s-\sigma)A}(1-2v(\sigma))\partial_x^2 v(\sigma)\|_{H^s} \\ &\le K_6\|v(s)\|^3_{H^{s+2}}. \qquad (3.69)\end{aligned}$$

Then, we get by using bounds

$$\|\frac{1}{2}\Delta t^2 g(\frac{1}{2}\Delta t, \frac{1}{2}\Delta t) - \int_0^{\Delta t}\int_0^s g(s,\sigma)d\sigma ds\|_{H^s} \le C(\Delta t)^3\|u_0\|^3_{H^{s+2}} \qquad (3.70)$$

We also have to bound fifth and the sixth terms of the local error Equation (3.62). It is known from Equation (3.20) i.e.

$$\|e^{\frac{\Delta t A}{2}}(d^2B(B,B) + dBdBB)(\Phi_B^{\theta\Delta t}(v))\|_{H^s} \le K\|\Phi_B^{\theta\Delta t}(v)\|^4_{H^s}. \qquad (3.71)$$

The bound for sixth term yields

$$\begin{aligned}\|dG_{s,\sigma}(v)[w)]\|_{H^s} &\le \|d^2B(e^{(s-\sigma)A}v)[e^{(s-\sigma)A}w, e^{(s-\sigma)A}B(v)]\|_{H^s} \\ &+ \|dB(e^{(s-\sigma)A}v)[e^{(s-\sigma)A}(dB(v)[w])]\|_{H^s}\end{aligned}$$

where we have redefined

$$v = e^{(s-\sigma)A}u(\tau) \text{ and } w = e^{(\sigma-\tau)s}B(u(\tau))$$

$$\begin{aligned}\|dG_{s,\sigma}(v)[w]\|_{H^s} &\le \|2.e^{2(s-\sigma)A}b(v)w\|_{H^s} + \|(1-2e^{(s-\sigma)A})e^{(s-\sigma)A}(1-2v)w\|_{H^s} \\ &\le K\|v(s)\|^4_{H^s} \qquad (3.72)\end{aligned}$$

This completes the proof. □

3.3.3.3. Global error in H^s space

Theorem 3.2 *Suppose that the exact solution $u(\cdot,t)$ of Equation (1.2) is in H^{s+4} for $0 \le t \le T$. Then Strang splitting solution u_n given in Equation (3.2) has second order global error for $\Delta t < \bar{\Delta} t$ and $t_n = n\Delta t \le T$,*

$$\|u_n - u(\cdot,t_n)\|_{H^s} \le G\Delta t^2, \tag{3.73}$$

where G only depends on $\|u_0\|_{H^{s+2d-4}}$, α and T.

Proof The Lady Windermere's fan is used in the proof. The stability estimate is given in Lemma 3.6 and local error is in Lemma 3.7. Let $u(t_n) = \Phi^{(n-k)\Delta t}(u(t_k))$ be the exact solution of Equation (1.2) at time t_n with initial data $u(t_k)$ at time t_k. Strang splitting solution $u_n = \Psi^{\Delta t}(u_{n-1})$ is

$$u_n = \Phi_B^{\Delta t} \circ \Phi_A^{\Delta t}(u_{n-1}), \;\; n = 1, 2, \ldots. \tag{3.74}$$

Then we estimate

$$\begin{aligned} u_n - u(\cdot,t_n) &\le \prod_{k=0}^{n-1} \Psi(\Delta t)(u_0 - u(t_0)) + \sum_{j=1}^{n} \prod_{k=j}^{n-1} \Phi(\Delta t) d_j \\ \text{where } d_j = (\Psi(\Delta t) - \Phi(\Delta t))u(t_{j-1}) & \\ &\le \sum_{k=1}^{n-1} \Psi^{(n-k-1)\Delta t}(\Phi^{\Delta t}(u(t_k)) - u(t_{k+1})) \\ &\le \sum_{k=1}^{n-1} \Psi^{(n-k-1)\Delta t}(\Phi^{\Delta t}(u(t_k)) - \Psi^{\Delta t}(u(t_k))) \end{aligned}$$

Taking H^s norm,

$$\begin{aligned} \|u_n - u(\cdot,t_n)\|_{H^s} &\le \sum_{k=0}^{n-1} \|\Psi^{(n-k-1)\Delta t}(\Phi^{\Delta t}(u(t_k)) - \Psi^{\Delta t}(u(t_k)))\|_{H^s} \\ &\le \sum_{k=0}^{n-1} e^{L(n-k-1)\Delta t} \|\Phi^{\Delta t}(u(t_k)) - \Psi^{\Delta t}(u(t_k))\|_{H^s} \\ &\le \sum_{k=0}^{n-1} e^{LT} C(\alpha)\Delta t^2 \\ &\le n e^{LT} C(\alpha) \Delta t^2 \\ &\le T e^{LT} C(\alpha) \Delta t \end{aligned} \tag{3.75}$$

since $e^{L(n-k-1)\Delta t} \leq e^{LT}$ and $n\Delta t \leq T$. This completes the proof

□

3.4. Convergence Analysis for the BBM Type Equations

3.4.1. Regularity results for BBM type Equations

Here, we study on the regularization estimates. Let $u(t)$, $\tilde{u}(t)$ be solutions of (1.3) with initial conditions u_0, $\tilde{u}_0$ for $0 < t < T$ interval, then,

$$u(t) = e^{t(1-\partial_x^2)^{-1}P(\partial_x)}u_0 + \frac{1}{2}\int_0^t e^{(s-t)(1-\partial_x^2)^{-1}P(\partial_x)}(1-\partial_x^2)^{-1}\partial_x(u^2(s))ds.$$

and

$$\tilde{u}(t) = e^{t(1-\partial_x^2)^{-1}P(\partial_x)}\tilde{u}_0 + \frac{1}{2}\int_0^t e^{(s-t)(1-\partial_x^2)^{-1}P(\partial_x)}(1-\partial_x^2)^{-1}\partial_x(\tilde{u}^2(s))ds.$$

then subtracting implies

$$\begin{aligned} u(t) - \tilde{u}(t) &= e^{t(1-\partial_x^2)^{-1}P(\partial_x)}(u_0 - \tilde{u}_0) \\ &\quad + \frac{1}{2}\int_0^t e^{(s-t)(1-\partial_x^2)^{-1}P(\partial_x)}(1-\partial_x^2)^{-1}\partial_x(u^2(s) - \tilde{u}^2(s))ds. \end{aligned}$$

Taking H^s norm

$$\begin{aligned} \|u(t) - \tilde{u}(t)\|_{H^s} &\leq \|e^{t(1-\partial_x^2)^{-1}P(\partial_x)}(u_0 - \tilde{u}_0)\|_{H^s} \\ &+ \frac{1}{2}\int_0^t \|e^{(s-t)(1-\partial_x^2)^{-1}P(\partial_x)}(1-\partial_x^2)^{-1}\partial_x(u^2(s) - \tilde{u}^2(s))\|_{H^s}ds. \end{aligned}$$

Since the linear flow $e^{t(1-\partial_x^2)^{-1}P(\partial_x)}$ is preserved in H^s space, we have

$$\begin{aligned}
\|u(t)-\tilde{u}(t)\|_{H^s} &\leq \|(u_0-\tilde{u}_0)\|_{H^s} + \frac{1}{2}\int_0^t \|(1-\partial_x^2)^{-1}\partial_x(u^2(s)-\tilde{u}^2(s))\|_{H^s} ds. \\
&\leq \|(u_0-\tilde{u}_0)\|_{H^s} \\
&\quad + \frac{1}{2}\int_0^t \|(1-\partial_x^2)^{-1}\partial_x(u(s)-\tilde{u}(s)(u(s)+\tilde{u}(s)\|_{H^s} ds
\end{aligned}$$

Then,

$$\begin{aligned}
\|u(t)-\tilde{u}(t)\|_{H^s} &\leq \|(u_0-\tilde{u}_0)\|_{H^s} + \frac{K}{2}\int_0^t \|(u(s)-\tilde{u}(s)\|_{H^s}\|(u(s)+\tilde{u}(s)\|_{H^s} ds \\
&\leq \|(u_0-\tilde{u}_0)\|_{H^s} \\
&\quad + \frac{K}{2}\int_0^t \max \|u(s)\|_{H^s}, \|\tilde{u}(s)\|_{H^s}\|(u(s)-\tilde{u}(s)\|_{H^s} ds
\end{aligned}$$

by Gronwall's lemma we obtain the bound

$$\|u(t)-\tilde{u}(t)\| \leq e^{Lt}\|u_0-\tilde{u}_0\|$$

where $L = \frac{K}{2}\max \|u(s)\|_{H^s}, \|\tilde{u}(s)\|_{H^s}$.

Lemma 3.8 *If* $\|u_0\|_{H^s} \leq M$ *then there exists* $\bar{t}(M) > 0$ *such that* $\|\Phi_B(u_0)\|_{H^s} \leq 2M$ *for* $0 \leq t \leq \bar{t}(M)$.

Proof Since the linear flow $e^{(1-\partial_x^2)^{-1}(P(\partial_x)}$ preserves the H^s Sobolev norm, we only need to compare the nonlinear part. We find that $w(t) = \Phi_B^t(u_0)$ satisfies following equality (see (Holden, Lubich, and Risebro, 2013), (Holden,Lubich,Risebro and Tao, 2011))

$$\begin{aligned}
\|w\|_{H^s}\frac{d}{dt}\|w\|_{H^s} &= \frac{1}{2}\frac{d}{dt}\|\Phi_B^t(u_0)\|_{H^s}^2 = (w, w_t)_{H^s} \\
&= \sum_{j=0}^{s}\int \partial_x^j w \partial_x^j (\frac{1}{2}(1-\partial_x^2)^{-1}\partial_x(w^2))dx \\
&= \frac{1}{2}\sum_{j=0}^{s}\int \partial_x^j w(1-\partial_x^2)^{-1}\partial_x(\sum_{k=0}^{j}\binom{s}{k}\partial_x^k w \partial_x^{j-k} w)dx.
\end{aligned}$$

For $j \leq s$, each of terms can be bounded by

$$
\begin{aligned}
&|\int \partial_x^j w(1-\partial_x^2)^{-1}\partial_x(\sum_{k=0}^{j}\binom{j}{k}\partial_x^k w\partial_x^{j-k}w)dx| \\
&\leq \|\partial_x^j w\|_{L^2}\|(1-\partial_x^2)^{-1}\partial_x(\sum_{k=0}^{j}\partial_x^k w\partial_x^{j-k}w)\|_{L^2} \\
&\leq K_1\|w\|_{H^j}\|(1-\partial_x^2)^{-1}\partial_x(w\partial_x^j w+\ldots)\|_{L^2} \\
&\leq K\|w\|_{H^j}\|w\|_{H^j}\|w\|_{H^j} \\
&\leq K\|w\|_{H^s}^3. \qquad (3.76)
\end{aligned}
$$

Hence we obtain following inequality

$$
\frac{d}{dt}\|w\|_{H^s} \leq K\|w\|_{H^s}^2,
$$

whose result follows by comparing with the differential equation $y' = cy^2$.

$$
y' = cy^2 \;\; y(0) = M
$$

then the solution,

$$
y = \frac{M}{1-kMt}.
$$

Thus,

$$
y = \|\Phi_B(u_0)\|_{H^s} \leq 2M
$$

□

In the proofs of the convergence rates for (3.1), we need to expand $\Phi_B(u_0)$ using Taylor series expansions of first order. Thus, $\Phi_B(u_0)$ needs to be continuous, such that the expansions are valid. The following lemma proves the sufficient continuity.

Lemma 3.9 *If $\|u_0\|_{H^s} \leq M$ then there exists $\bar{t}$ depending on M such that the solution of*

KdV-BBM type equation with initial data u_0, $w(t) = \Phi_B^t(u_0)$ *satisfies*

$$w \in C^2([0, \bar{t}], H^s). \tag{3.77}$$

Proof Recall the Lemma 3.8, if $\|u_0\|_{H^s} \leq M$ then $\|w(t)\|_{H^s} = \|\Phi_B(u_0)\|_{H^s} \leq 2M$ for $t \in [0, \bar{t}]$ and we can define

$$\tilde{w}(t) = u_0 + tB(u_0) + \int_0^t (t-s) dB\big(w(s)\big)[B\big(w(s)\big)]ds, \tag{3.78}$$

where $dB\big(w\big)[B\big(w\big)] = \frac{1}{2}(1-\partial_x^2)^{-1}\partial_x\big(w(1-\partial_x^2)^{-1}\partial_x(w^2)\big)$. Since $\tilde{w}_{tt} = dB\big(w\big)[B\big(w\big)] = B(w)_t$, $\tilde{w}_t(0) = B(u_0) = w_t(0)$ and $\tilde{w}(0) = u_0 = w(0)$, we have $\tilde{w} = w$. Now we have to show that $\tilde{w} \in C^2([0, \tilde{t}], H^s)$. Start with

$$\begin{aligned} \|\tilde{w}_{tt}\|_{H^s} &= \|dB\big(w\big)[B\big(w\big)]\|_{H^s} = \frac{1}{2}\|(1-\partial_x^2)^{-1}\partial_x\big(w(1-\partial_x^2)^{-1}\partial_x(w^2)\big)\|_{H^s} \\ &\leq \frac{K_1}{2}\|w\|_{H^s}\|(1-\partial_x^2)^{-1}\partial_x(w^2)\|_{H^s} \\ &\leq \frac{K_2}{2}\|w\|_{H^s}\|w\|_{H^s}\|w\|_{H^s}. \end{aligned} \tag{3.79}$$

Hence $\|\tilde{w}_{tt}\|_{H^s} \leq \frac{K_2}{2}\|w\|_{H^s}^3$ and Lemma 3.8 completes the proof. □

3.4.2. Lie-Trotter Splitting

3.4.2.1. Stability in H^s space

Lemma 3.10 *Let* u_1, $\tilde{u}_1$ *be the Lie-Trotter splitting solutions satisfying Equation (1.3) with initial data* $u_0, \tilde{u}_0$ *in* H^s*. Then*

$$\|u_1 - \tilde{u}_1\|_{H^s} \leq e^{L\Delta t}\|u_0 - \tilde{u}_0\|_{H^s}, \tag{3.80}$$

where $L = \frac{K}{2}\max\{\|u_1\|_{H^s}, \|\tilde{u}_1\|_{H^s}\}$.

Proof Since $e^{\Delta t(1-\partial_x^2)^{-1}P(\partial_x)}$ preserves the H^s norm, we only need to compare nonlinearities in Lie-Trotter splitting solutions.

$$u_1 t = \frac{1}{2}(1-\partial_x^2)^{-1}\partial_x(u_1^2)$$
$$\tilde{u}_1 t = \frac{1}{2}(1-\partial_x^2)^{-1}\partial_x(\tilde{u}_1^2)$$

Then

$$(u_1-\tilde{u}_1)_t = \frac{1}{2}(1-\partial_x^2)^{-1}\partial_x(u_1^2-\tilde{u}_1^2)$$

integrating from 0 to t

$$u_1-\tilde{u}_1 = u_0-\tilde{u}_0+\frac{1}{2}\int_0^t(1-\partial_x^2)^{-1}\partial_x(u_1-\tilde{u}_1)(u_1+\tilde{u}_1)ds$$

Taking norm,

$$\begin{aligned}\|u_1-\tilde{u}_1\|_{H^s} &\leq \|u_0-\tilde{u}_0\|_{H^s}+\frac{K}{2}\int_0^t \max\{\|u_1\|_{H^s},\|\tilde{u}_1\|_{H^s}\}\|u_1^-\tilde{u}_1)\|ds \\ \|u_1-\tilde{u}_1\|_{H^s} &\leq \|u_0-\tilde{u}_0\|_{H^s}+\int_0^t L\|u_1-\tilde{u}_1)\|ds\end{aligned}$$

The nonlinear term has Lipshitz constant L which is bounded by Lemma 3.8. Finally Gronwall's Lemma implies the bound in Equation (3.80).

$$\|u_1-\tilde{u}_1\|_{H^s} \leq e^{L\Delta t}\|u_0-\tilde{u}_0\|_{H^s} \tag{3.81}$$

□

3.4.2.2. Local error in H^s space

Lemma 3.11 *Let $s \geq 1$ be an integer and assumption (3.14) holds for $k = s + d - 2$ for the solution $u(t) = \Phi_{A+B}^{\Delta t}(u_0)$ of (1.3). If the initial data u_0 is in $H^{s+2}(\mathbb{R})$, then the local error of the Lie-Trotter splitting (3.1) is bounded in $H^s(\mathbb{R})$ by*

$$\|\Psi^{\Delta t}(u_0) - \Phi^{\Delta t}(u_0)\|_{H^s} \leq C\Delta t^2, \tag{3.82}$$

where C only depends on $\|u_0\|_{H^{s+d-2}}$.

Proof In the following proof, we follow similar way to (Lubich, 2008) and (Holden, Lubich, and Risebro, 2013). BBM type equations are in the form

$$u_t = Au + B(u), \tag{3.83}$$

where $Au = (1 - \partial_x^2)^{-1}P(\partial_x)u$ and $B(u) = \frac{1}{2}(1 - \partial_x^2)^{-1}\partial_x(u^2)$. The exact solution is $u(t) = \Phi^t(u_0)$, from variation of constant formula, for $[0, \Delta t]$ interval, it can be written as

$$u(\Delta t) = e^{\Delta t A}u_0 + \int_0^{\Delta t} e^{(\Delta t - s)A} B\big(u(s)\big)ds. \tag{3.84}$$

This is similar to formula $\varphi(t) - \varphi(0) = \int_0^t \dot{\varphi}(s)ds$ when $\varphi(s) = e^{(t-s)A}u(s)$.

$$\varphi(\Delta t) = e^{(\Delta t - s)A}u(s) \;\; \varphi(0) = e^{\Delta t A}u_0$$
$$\varphi'(s) = -Ae^{(\Delta t - s)A}u(s) + e^{(\Delta t - s)A} \underbrace{u'(s)}_{Au+B(u)}$$

Second part of the Equation (3.84) can be written with similar formula by taking $\varphi(\rho) = B(e^{(s-\rho)A}u(\rho))$, and then we have

$$B(e^{(s-t)A}u(t)) - B(e^{sA}u_0) = \int_0^s dB(e^{(s-\rho)A}u(\rho))[e^{(s-\rho)A}B(u(\rho))]d\rho. \tag{3.85}$$

Writing s instead of t, we obtain $B(u(s))$

$$B(u(s)) = B(e^{sA}u_0) + \int_0^s dB(e^{(s-\rho)A}u(\rho))[e^{(s-\rho)A}B(u(\rho))]d\rho. \tag{3.86}$$

After inserting Equation (3.86) into Equation (3.84), we get

$$u(\Delta t) = e^{\Delta tA}u_0 + \int_0^{\Delta t} e^{(\Delta t - A)}B(e^{sA}u_0)ds + E_1 \tag{3.87}$$

where

$$E_1 = \int_0^{\Delta t}\int_0^s e^{(\Delta t-s)A}dB(e^{(s-\rho)A}u(\rho))[e^{(s-\rho)A}B(u(\rho))]d\rho ds. \tag{3.88}$$

The Lie-Trotter splitting solution for $[0, \Delta t]$ interval can be written as

$$u_1 = \Psi^{\Delta t}(u_0) = \Phi_B^{\Delta t}(e^{\Delta tA}u_0), \tag{3.89}$$

We use the first-order Taylor expansion with integral remainder term in H^s,

$$\Phi_B^{\Delta t}(v) = v + \Delta tB(v) + \Delta t^2\int_0^1 (1-\theta)dB(\Phi_B^{\theta\Delta t}(v))[B(\Phi_B^{\theta\Delta t}(v))]d\theta. \tag{3.90}$$

This is justified for $v = e^{\Delta tA}u_0 \in H^s$. Therefore, we obtain

$$u_1 = e^{\Delta tA}u_0 + \Delta tB(e^{\Delta tA}u_0) + E_2 \tag{3.91}$$

with

$$E_2 = \Delta t^2\int_0^1 (1-\theta)dB(\Phi_B^{\theta\Delta t}(e^{\Delta tA}u_0))[B(\Phi_B^{\theta\Delta t}(e^{\Delta tA}u_0))]d\theta \tag{3.92}$$

Thus, the error becomes,

$$u_1 - u(\Delta t) = \Delta t B(e^{\Delta t A} u_0) - \int_0^{\Delta t} e^{(\Delta t - A)} B(e^{sA} u_0) ds + (E_2 - E_1) \tag{3.93}$$

and hence the principal error term is the just the quadrature error of the rectangular rule applied to the integral over $[0, \Delta t]$ of the function

$$h(s) = e^{(\Delta t - A)} B(e^{sA} u_0) \tag{3.94}$$

we express the quadrature error in first-order Peano form,

$$\Delta t h(\Delta t) - \int_0^{\Delta t} h(s) ds = \Delta t^2 \int_0^1 \kappa(\theta) h'(\theta \Delta t) d\theta \tag{3.95}$$

where κ is bounded kernel. Here $h'(s) = -e^{(\Delta t - s)A}[A, B](e^{sA} u_0)$ with double Lie commutator

$$[A, B] = dA(v)[B(v)] - dB(v)[Av] \tag{3.96}$$

$$\begin{aligned} dA(v)[B(v)] &= \lim_{t \to 0} \frac{A(v + tB(v)) - A(v)}{t} \\ &= \lim_{t \to 0} \frac{(1 - \partial_x^2)^{-1} P(\partial_x)(v + tB(v)) - (1 - \partial_x^2)^{-1} P(\partial_x) v}{t} \\ &= \frac{1}{2}(1 - \partial_x^2)^{-1} P(\partial_x)(1 - \partial_x^2)^{-1} \partial_x(v^2) \end{aligned}$$

$$\begin{aligned} dB(v)[Av] &= \lim_{t \to 0} \frac{B(v + tAv) - B(v)}{t} \\ &= \lim_{t \to 0} \frac{\frac{1}{2}(1 - \partial_x^2)^{-1} \partial_x (v + tAv)^2 - \frac{1}{2}(1 - \partial_x^2)^{-1} \partial_x(v^2)}{t} \\ &= (1 - \partial_x^2)^{-1} \partial_x (v(1 - \partial_x^2)^{-1} P(\partial_x) v) \end{aligned}$$

We need to find in H^s for each of the terms. Lemma 2.3 and Lemma 2.4 will be helpfull to bound them. Now, start with the first term, then we get

$$\begin{aligned}
\|dA(v)[B(v)]\| &= \|\frac{1}{2}(1-\partial_x^2)^{-1}P(\partial_x)(1-\partial_x^2)^{-1}\partial_x(v^2)\|_{H^s} \\
&\leq \frac{1}{2}\|(1-\partial_x^2)^{-1}\partial_x(v^2)\|_{H^{s+d-2}} \\
&\leq C_1\|v\|^2_{H^{s+d-2}}
\end{aligned}$$

$$\begin{aligned}
\|dB(v)[Av]\| &= \|(1-\partial_x^2)^{-1}\partial_x(v(1-\partial_x^2)^{-1}P(\partial_x)v)\|_{H^s} \\
&\leq \|v\|_{H^s}\|(1-\partial_x^2)^{-1}P(\partial_x)v\|_{H^s} \\
&\leq C_2\|v\|^2_{H^{s+d-2}}
\end{aligned}$$

Since e^{tA} does not increase the Sobolev norms, it follows that

$$\|h'(s)\| \leq C\|u_0\|^2_{H^{s+d-2}}$$

$$\|E_1\| \leq \int_0^{\Delta t}\int_0^s \|e^{(\Delta t-s)A}dB(e^{(s-\rho)A}u(\rho))[e^{(s-\rho)A}B(u(\rho))]\|_{H^s}d\rho ds. \tag{3.97}$$

$$dB(e^{(s-\rho)A}u(\rho))[e^{(s-\rho)A}B(u(\rho))] = (1-\partial_x^2)^{-1}\partial_x(e^{2(s-\rho)A}u(\rho)B(u(\rho)))$$

Inserting the result into the equation (3.97), we get

$$\begin{aligned}
\|E_1\| &\leq \int_0^{\Delta t}\int_0^s \|e^{(\Delta t-s)A}(1-\partial_x^2)^{-1}\partial_x(e^{2(s-\rho)A}u(\rho)B(u(\rho)))\|_{H^s}d\rho ds \\
&\leq \int_0^{\Delta t}\int_0^s \|e^{(\Delta t-s)A}(1-\partial_x^2)^{-1}\partial_x(e^{2(s-\rho)A}u(\rho)\frac{1}{2}(1-\partial_x^2)^{-1}\partial_x(u(\rho)^2))\|_{H^s}d\rho ds \\
&\leq \int_0^{\Delta t}\int_0^s \|(1-\partial_x^2)^{-1}\partial_x(e^{2(s-\rho)A}u(\rho)(1-\partial_x^2)^{-1}\partial_x(u(\rho)^2))\|_{H^s}d\rho ds \\
&\leq C\int_0^{\Delta t}\int_0^s \|u(\rho)(1-\partial_x^2)^{-1}\partial_x(u(\rho)^2))\|_{H^s}d\rho ds \\
&\leq C\int_0^{\Delta t}\int_0^s \|u(\rho)\|_{H^s}^3 d\rho ds \\
&\leq C\Delta t^2 R^3
\end{aligned}$$

and

$$E_2 \leq \Delta t^2 \int_0^1 (1-\theta)\|dB(\Phi_B^{\theta\Delta t}(e^{\Delta tA}u_0))[B(\Phi_B^{\theta\Delta t}(e^{\Delta tA}u_0))]\|_{H^s}d\theta$$

$$\begin{aligned}
&dB(\Phi_B^{\theta\Delta t}(e^{\Delta tA}u_0))[B(\Phi_B^{\theta\Delta t}(e^{\Delta tA}u_0))] \\
&= (1-\partial_x^2)^{-1}\partial_x(\Phi_B^{\theta\Delta t}(e^{\Delta tA}u_0)B(\Phi_B^{\theta\Delta t}(e^{\Delta tA}u_0)))
\end{aligned}$$

Then, we can rewrite the norm of E_2 as follows,

$$\begin{aligned}
E_2 &\leq \Delta t^2 \int_0^1 (1-\theta)\|(1-\partial_x^2)^{-1}\partial_x(\Phi_B^{\theta\Delta t}(e^{\Delta tA}u_0)B(\Phi_B^{\theta\Delta t}(e^{\Delta tA}u_0)))\|_{H^s}d\theta \\
&\leq \Delta t^2 \int_0^1 (1-\theta)\|(\Phi_B^{\theta\Delta t}(e^{\Delta tA}u_0)B(\Phi_B^{\theta\Delta t}(e^{\Delta tA}u_0)))\|_{H^s}d\theta \\
&\leq \Delta t^2 \int_0^1 (1-\theta)\|(\Phi_B^{\theta\Delta t}(e^{\Delta tA}u_0)\|_{H^s}\|(1-\partial_x^2)^{-1}\partial_x(\Phi_B^{\theta\Delta t}(e^{\Delta tA}u_0)))^2\|_{H^s}d\theta \\
&\leq C\Delta t^2 R^3
\end{aligned}$$

Hence the quadrature error is $O(\Delta t^2)$ in the H^s norm for $u_0 \in H^{s+d-2}$. □

3.4.2.3. Global error in H^s space

Theorem 3.3 *Suppose that the exact solution $u(\cdot,t)$ of Equation (1.3) is in H^{s+d-2} for $0 \leq t \leq T$. Then Lie-Trotter splitting solution u_n given in Equation (3.1) has second order global error for $\Delta t < \bar{\Delta} t$ and $t_n = n\Delta t \leq T$,*

$$\|u_n - u(\cdot,t_n)\|_{H^s} \leq G\Delta t^2, \tag{3.98}$$

where G only depends on $\|u_0\|_{H^{s+2d-4}}$, α and T.

Proof The Lady Windermere's fan is used in the proof. The stability estimate is given in Lemma 3.10 and local error is in Lemma 3.11. Let $u(t_n) = \Phi^{(n-k)\Delta t}(u(t_k))$ be the exact solution of Equation (1.3) at time t_n with initial data $u(t_k)$ at time t_k. Lie-Trotter splitting solution $u_n = \Psi^{\Delta t}(u_{n-1})$ is

$$u_n = \Phi_B^{\Delta t} \circ \Phi_A^{\Delta t}(u_{n-1}), \;\; n = 1, 2, \ldots. \tag{3.99}$$

Then we estimate

$$u_n = \Phi_B^{\Delta t} \circ \Phi_A^{\Delta t}(u_{n-1}), \;\; n = 1, 2, \ldots. \tag{3.100}$$

Then we estimate

$$\begin{aligned} u_n - u(\cdot,t_n) &\leq \prod_{k=0}^{n-1} \Psi(\Delta t)(u_0 - u(t_0)) + \sum_{j=1}^{n} \prod_{k=j}^{n-1} \Phi(\Delta t) d_j \\ \text{where } d_j = (\Psi(\Delta t) - \Phi(\Delta t)) u(t_{j-1}) & \\ &\leq \sum_{k=1}^{n-1} \Psi^{(n-k-1)\Delta t}(\Phi^{\Delta t}(u(t_k)) - u(t_{k+1})) \\ &\leq \sum_{k=1}^{n-1} \Psi^{(n-k-1)\Delta t}(\Phi^{\Delta t}(u(t_k)) - \Psi^{\Delta t}(u(t_k))) \end{aligned}$$

Taking H^s norm,

$$\begin{aligned}
\|u_n - u(\cdot,t_n)\|_{H^s} &\leq \sum_{k=0}^{n-1} \|\Psi^{(n-k-1)\Delta t}(\Phi^{\Delta t}(u(t_k)) - \Psi^{\Delta t}(u(t_k)))\|_{H^s} \\
&\leq \sum_{k=0}^{n-1} e^{L(n-k-1)\Delta t}\|\Phi^{\Delta t}(u(t_k)) - \Psi^{\Delta t}(u(t_k))\|_{H^s} \\
&\leq \sum_{k=0}^{n-1} e^{LT}C(\alpha)\Delta t^2 \\
&\leq ne^{LT}C(\alpha)\Delta t^2 \\
&\leq Te^{LT}C(\alpha)\Delta t \qquad (3.101)
\end{aligned}$$

since $e^{L(n-k-1)\Delta t} \leq e^{LT}$ and $n\Delta t \leq T$. This completes the proof □

CHAPTER 4

NUMERICAL COMPUTATION

In this chapter, we numerically investigate the operator splitting method of Lie Trotter and Strang types, given in (3.1) and (3.2) respectively. Since we have derived theoretical results for these two splitting methods for the split step size Δt, we naturally have the numerical convergence rates for Δt as the main focus throughout this entire chapter. The theoretical results are valid for several equations, but we only study two equations in details; that is, we use the Fisher's equation (1.1) and the Benjamin-Bona-Mahony type equations (1.3) equations as test equations. All simulations are run by programs written in Matlab programming language.

4.1. The Fisher's Equation

Our test problem is the following equation with initial condition :

$$u_t = \alpha \frac{\partial^2 u}{\partial x^2} + \beta u(1-u) \tag{4.1}$$

$$u(x,0) = sech^2(x) \tag{4.2}$$

and boundary conditions are

$$\lim_{t\to -\infty} u(x,t) = 0 \text{ and } \lim_{t\to \infty} u(x,t) = 0 \tag{4.3}$$

We performed a spacial discretization with lenght parameter $\Delta x = 0.5$ that is we divided $[-10, 10]$ into $N = 40$ parts of equal length. The spatial derivative is approximated with the finite difference scheme:

$$\partial_x^2 u(t,x_i) \approx \frac{u(t,x_{i+1}) - 2u(t,x_i) + u(t,x_{i-1})}{\Delta x^2}$$

Recall that when we apply the operator splitting method on (4.1),we obtain the two subequations

$$u_t = \alpha\frac{\partial^2 u}{\partial x^2}$$
$$v_t = \beta v(1-v)$$

which are solved subsequently for small time steps using Lie-Trotter and Strang splitting methods. We calculated the error and orders of these methods numerically. Also, We solved the full problem (4.1)-(4.3) with the fourth order Runge-Kutta method. This provides the reference solution for our study.

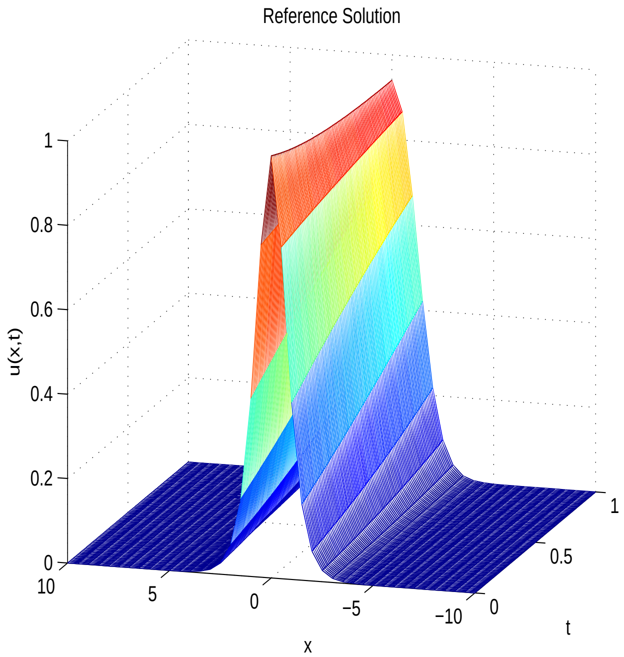

Figure 4.1. Reference solution generated by fourth order Runge-Kutta for $x \in [-10, 10]$ and $t \in [0, 1]$.

Figure 4.1 is showing the reference solution of (4.1)-(4.3) for $\alpha = 0.1$, $\beta = 1$ with time step $\Delta t = 0.01$. At the very beginning , near $x = 0$ $u_{xx} < 0$ with a large absolute value,but the reaction term $u(1-u)$is quite small, that is, the effect of diffusion dominates over the effect of reaction, so peak goes down and gets flatter.

The contour plots of the numerical solutions at different time t for $\alpha = 0.1$, $\beta = 1$ are shown in Figures 4.2, 4.3 and 4.4. Figure 4.2 is showing the numerical solutions of u for $t = 0$ to 1 with the step size $\Delta t = 0.01$. Figure 4.3 is for a short period of time, showing the results for $t = 0$ to 0.2 with the step size $\Delta t = 0.05$. Figure 4.4 is for the period of time, showing the results for $t = 0$ to 5 with the step size $\Delta t = 0.5$.

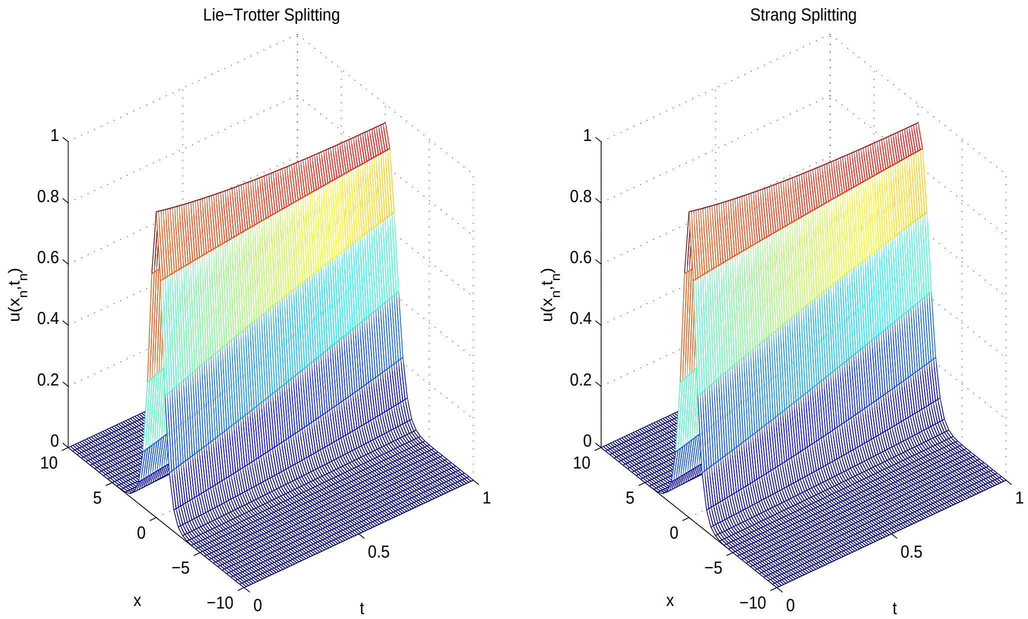

Figure 4.2. Numerical solutions of (4.1)-(4.3) for $\alpha = 0.1, \beta = 1$.

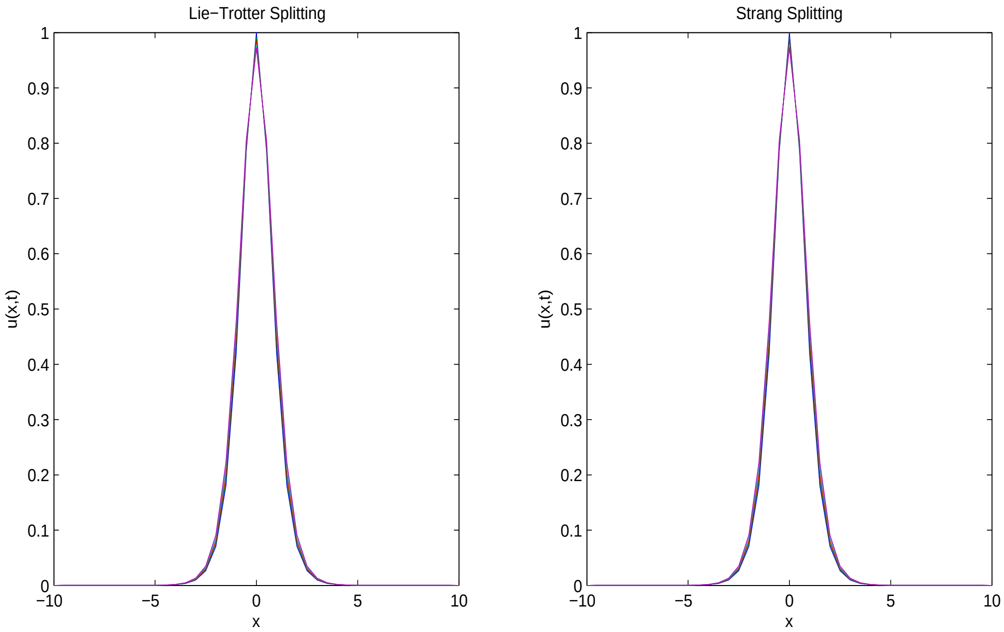

Figure 4.3. Approximate solutions at $t = 0$ to $t = 0.2$ with step size $\Delta t = 0.05$.

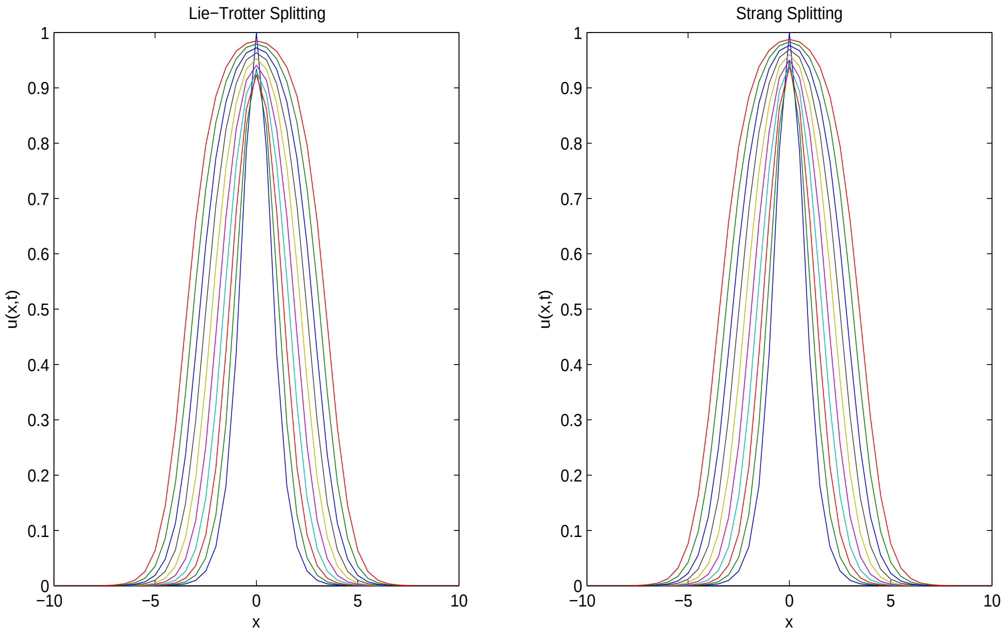

Figure 4.4. Approximate solutions at $t = 0$ to $t = 5$ with step size $\Delta t = 0.5$.

We obtain the following error tables using Lie-Troter and Strang splitting methods for the different time steps and coefficients. Also, numerical convergence rates for Δt for the operator splitting solutions of Lie-Trotter and Strang type are obtained.

			$Lie-Trotter$		
$time\ step$	α	β	L_1	L_2	L_∞
0.05	1	1	0.0176	0.0041	0.0013
0.01	1	1	0.0037	$8.3194e-004$	$2.5663e-004$
0.002	1	1	$7.3911e-004$	$1.6726e-004$	$5.1509e-005$
0.0004	1	1	$1.4808e-004$	$3.3486e-005$	$1.0309e-005$
0.05	1	0.1	0.0015	$3.3484e-004$	$1.0627e-004$
0.01	1	0.1	$3.0797e-004$	$6.8867e-005$	$2.2184e-005$
0.002	1	0.1	$6.1914e-005$	$1.3853e-005$	$4.4754e-006$
0.0004	1	0.1	$1.2395e-005$	$2.7739e-006$	$8.9664e-007$
0.05	1	0.01	$1.4705e-004$	$3.3034e-005$	$1.0708e-005$
0.01	1	0.01	$3.0094e-005$	$6.7796e-006$	$2.2163e-006$
0.002	1	0.01	$6.0498e-006$	$1.3641e-006$	$4.4722e-007$
0.0004	1	0.01	$1.2112e-006$	$2.7315e-007$	$8.9602e-008$
0.05	0.1	1	0.0065	0.0019	$8.8893e-004$
0.01	0.1	1	0.0013	$3.8520e-004$	$1.7955e-004$
0.002	0.1	1	$2.5501e-004$	$7.7050e-005$	$3.6052e-005$
0.0004	0.1	1	$5.0978e-005$	$1.5411e-005$	$7.2161e-006$
0.05	0.01	1	$9.0343e-004$	$2.8595e-004$	$1.4737e-004$
0.01	0.01	1	$1.6924e-004$	$5.5897e-005$	$2.8317e-005$
0.002	0.01	1	$3.3378e-005$	$1.1229e-005$	$5.8249e-006$
0.0004	0.01	1	$6.6567e-006$	$2.2486e-006$	$1.1714e-006$
0.05	0.01	0.1	$9.4244e-005$	$3.9109e-005$	$2.8664e-005$
0.01	0.01	0.1	$1.8813e-005$	$7.8003e-006$	$5.6971e-006$
0.002	0.01	0.1	$3.7611e-006$	$1.5592e-006$	$1.1380e-006$
0.0004	0.01	0.1	$7.5217e-007$	$3.1181e-007$	$2.2755e-007$
0.05	0.1	0.02	$1.1933e-004$	$4.4275e-005$	$2.8879e-005$
0.01	0.1	0.02	$2.3737e-005$	$8.7628e-006$	$5.6662e-006$
0.002	0.1	0.02	$4.7425e-006$	$1.7489e-006$	$1.1289e-006$
0.0004	0.1	0.02	$9.4829e-007$	$3.4964e-007$	$2.2561e-007$
0.05	0.1	0.01	$5.9624e-005$	$2.2157e-005$	$1.4500e-005$
0.01	0.1	0.01	$1.1859e-005$	$4.3852e-006$	$2.8454e-006$
0.002	0.1	0.01	$2.3693e-006$	$8.7522e-007$	$5.6694e-007$
0.0004	0.1	0.01	$4.7375e-007$	$1.7497e-007$	$1.1330e-007$

Table 4.1. Estimated errors using L_1, L_2 and L_∞ norm at $t = 1$.

Table 4.1 shows the errors of the Lie-Trotter splitting in L_1, L_2 and L_∞ norm for the different time steps and the different α, β parameters. We can clearly see how the errors changes when time steps are getting smaller.

			Lie − Trotter		
time step	α	β	L_1	L_2	L_∞
0.05	1	1			
0.01	1	1	0.9748	0.9843	0.9952
0.002	1	1	0.9945	0.9968	0.9978
0.0004	1	1	0.9989	0.9994	0.9996
0.05	1	0.1			
0.01	1	0.1	0.9838	0.9826	0.9734
0.002	1	0.1	0.9968	0.9964	0.9946
0.0004	1	0.1	0.9994	0.9993	0.9989
0.05	1	0.01			
0.01	1	0.01	0.9857	0.9840	0.9787
0.002	1	0.01	0.9968	0.9963	0.9945
0.0004	1	0.01	0.9994	0.9992	0.9989
0.05	0.1	1			
0.01	0.1	1	1.0071	1.006	0.9939
0.002	0.1	1	1.0014	0.9999	0.9976
0.0004	0.1	1	1.0003	1.0000	0.9995
0.05	0.01	1			
0.01	0.01	1	1.0407	1.0142	1.0249
0.002	0.01	1	1.0087	0.9973	0.9837
0.0004	0.01	1	1.0018	0.9992	0.9966
0.05	0.01	0.1			
0.01	0.01	0.1	1.0012	1.0017	1.0039
0.002	0.01	0.1	1.0002	1.0003	1.0008
0.0004	0.01	0.1	1.0000	1.0001	1.0002
0.05	0.1	0.02			
0.01	0.1	0.02	1.0034	1.0065	1.0119
0.002	0.1	0.02	1.0007	1.0013	1.0024
0.0004	0.1	0.02	1.0001	1.0003	1.0005
0.05	0.1	0.01			
0.01	0.1	0.01	1.0034	1.0065	1.0118
0.002	0.1	0.01	1.0007	1.0013	1.0023
0.0004	0.1	0.01	1.0001	1.0003	1.0005

Table 4.2. Numerical convergence rate for Δt for the operator splitting solution of Lie-Trotter type.

Table 4.2 shows that the expected order is confirmed, that is , Lie-Trotter splitting converges as $\mathcal{O}(\Delta t)$.

Table 4.3 shows the L_1, L_2 and L_∞ errors of the Strang splitting at $t = 1$ for various Δt.

			Strang		
time step	α	β	L_1	L_2	L_∞
0.05	1	1	2.3201e − 004	5.5705e − 005	1.9483e − 005
0.01	1	1	9.5936e − 006	2.3018e − 006	8.0183e − 007
0.002	1	1	3.8611e − 007	9.2644e − 008	3.2271e − 008
0.0004	1	1	1.5463e − 008	3.7103e − 009	1.2925e − 009
0.05	1	0.1	3.6305e − 005	1.0679e − 005	4.4861e − 006
0.01	1	0.1	1.4506e − 006	4.2760e − 007	1.8843e − 007
0.002	1	0.1	5.8022e − 008	1.7105e − 008	7.5495e − 009
0.0004	1	0.1	2.3210e − 009	6.8425e − 010	3.0201e − 010
0.05	1	0.01	3.6929e − 006	1.1398e − 006	4.8450e − 007
0.01	1	0.01	1.4705e − 007	4.3346e − 008	1.8818e − 008
0.002	1	0.01	5.8820e − 009	1.7354e − 009	7.6532e − 010
0.0004	1	0.01	2.3540e − 010	6.9460e − 011	3.0647e − 011
0.05	0.1	1	3.4909e − 004	9.7902e − 005	3.9164e − 005
0.01	0.1	1	1.4138e − 005	3.9735e − 006	1.6049e − 006
0.002	0.1	1	5.6692e − 007	1.5941e − 007	6.4509e − 008
0.0004	0.1	1	2.2690e − 008	6.3807e − 009	2.5830e − 009
0.05	0.01	1	3.3454e − 004	9.9235e − 005	3.9989e − 005
0.01	0.01	1	1.3568e − 005	4.0326e − 006	1.6422e − 006
0.002	0.01	1	5.4425e − 007	1.6182e − 007	6.6034e − 008
0.0004	0.01	1	2.1780e − 008	6.4766e − 009	2.6441e − 009
0.05	0.01	0.1	2.6366e − 007	8.1582e − 008	3.5760e − 008
0.01	0.01	0.1	1.0559e − 008	3.2667e − 009	1.4337e − 009
0.002	0.01	0.1	4.2214e − 010	1.3056e − 010	5.7349e − 011
0.0004	0.01	0.1	1.5957e − 011	4.8439e − 012	2.1954e − 012
0.05	0.1	0.02	2.8135e − 007	1.1677e − 007	7.7835e − 008
0.01	0.1	0.02	1.1254e − 008	4.6821e − 009	3.1686e − 009
0.002	0.1	0.02	4.5013e − 010	1.8729e − 010	1.2684e − 010
0.0004	0.1	0.02	1.5923e − 011	6.7225e − 012	4.6355e − 012
0.05	0.1	0.01	1.4199e − 007	5.8583e − 008	3.8461e − 008
0.01	0.1	0.01	5.6796e − 009	2.3538e − 009	1.5910e − 009
0.002	0.1	0.01	2.2715e − 010	9.4159e − 011	6.3731e − 011
0.0004	0.1	0.01	7.0005e − 012	2.9960e − 012	2.1060e − 012

Table 4.3. Estimated errors using L_1, L_2 and L_∞ norm at $t = 1$.

			Strang		
time step	α	β	L_1	L_2	L_∞
0.05	1	1			
0.01	1	1	1.9794	1.9798	1.9823
0.002	1	1	1.9962	1.9962	1.9962
0.0004	1	1	1.9992	1.9992	1.9992
0.05	1	0.1			
0.01	1	0.1	2.0007	1.9993	1.9697
0.002	1	0.1	2.0000	1.9990	1.9990
0.0004	1	0.1	2.0000	2.0000	1.9999
0.05	1	0.01			
0.01	1	0.01	2.0028	2.0314	2.0183
0.002	1	0.01	2.0000	1.9994	1.9897
0.0004	1	0.01	1.9997	1.9996	1.9993
0.05	0.1	1			
0.01	0.1	1	1.9923	1.9910	1.9850
0.002	0.1	1	1.9984	1.9982	1.9970
0.0004	0.1	1	1.9996	1.9996	1.9994
0.05	0.01	1			
0.01	0.01	1	1.9914	1.9902	1.9837
0.002	0.01	1	1.9983	1.9980	1.9967
0.0004	0.01	1	1.9997	1.9996	1.9994
0.05	0.01	0.1			
0.01	0.01	0.1	1.9993	1.9994	1.9986
0.002	0.01	0.1	2.0003	2.0005	2.0000
0.0004	0.01	0.1	2.0351	2.0467	2.0273
0.05	0.1	0.02			
0.01	0.1	0.02	2.0000	1.9985	1.9891
0.002	0.1	0.02	2.0001	2.0000	1.9995
0.0004	0.1	0.02	2.0764	2.0673	2.0561
0.05	0.1	0.01			
0.01	0.1	0.01	2.0000	1.9972	1.9791
0.002	0.1	0.01	2.0001	1.9991	1.9991
0.0004	0.1	0.01	2.1620	2.1422	2.1187

Table 4.4. Numerical convergence rate for Δt for the operator splitting solution of Strang type.

Table 4.4 indicate that the convergence rate of the Strang splitting solution. We can see that the Strang splitting converges as $\mathcal{O}(\Delta t^2)$.

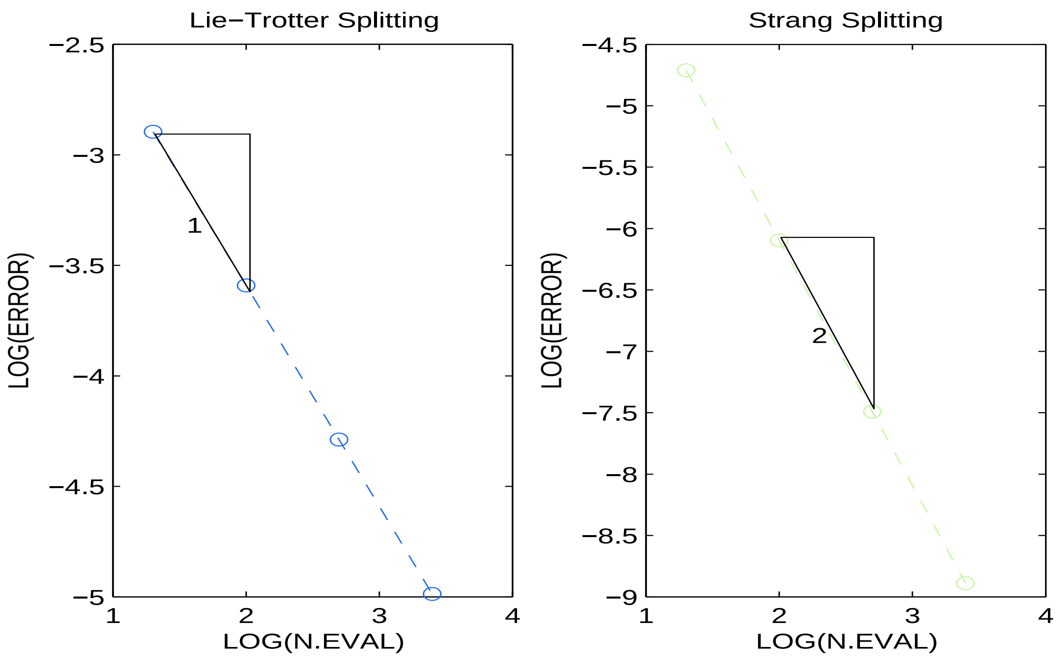

Figure 4.5. Order of the splitting methods for the Fisher's equation.

4.2. The Benjamin-Bona-Mahony Equation

As a last study we consider here a numerical solution of BBM

$$u_t - u_{xxt} - u_{xx} + u_x + uu_x = 0 \tag{4.4}$$

with the initial condition

$$u(x,0) = \frac{1}{2} + \frac{1}{4}\sin(x); \tag{4.5}$$

The numerical solution is obtained with the help of the fourier transform. So, we choose $N = 256$ and $x \in [0, 2\Pi]$.

Solving the full problem (4.4)-(4.5) with the fourth order Runge-Kutta method,we obtain the reference solution for our study. We need to make some arrangements to get this reference solution. Hence, we write the equation (4.4) as,

$$u_t = (1-\partial_x^2)^{-1}\partial_x^2 u - (1-\partial_x^2)^{-1}\partial_x u - (1-\partial_x^2)^{-1}\partial_x(\frac{u^2}{2}) \tag{4.6}$$

With Fourier transform

$$\hat{u}_t = -\frac{k^2+ik}{1+k^2}\hat{u} - \frac{ik}{2(1+k^2)}\hat{u}^2 \tag{4.7}$$

Now we multiply by $e^{\frac{k^2+ik}{1+k^2}t}$ to get

$$e^{\frac{k^2+ik}{1+k^2}t}\hat{u}_t = -e^{\frac{k^2+ik}{1+k^2}t}\frac{k^2+ik}{1+k^2}\hat{u} - e^{\frac{k^2+ik}{1+k^2}t}\frac{ik}{2(1+k^2)}\hat{u}^2 \tag{4.8}$$

If we define $\hat{U} = e^{\frac{k^2+ik}{1+k^2}t}\hat{u}$,with

$$\hat{U}_t = e^{\frac{k^2+ik}{1+k^2}t}\frac{k^2+ik}{1+k^2}\hat{u} + e^{\frac{k^2+ik}{1+k^2}t}\hat{u}_t \tag{4.9}$$

this becomes

$$\hat{U}_t = -\frac{ik}{2(1+k^2)}e^{\frac{k^2+ik}{1+k^2}t}\hat{u} \tag{4.10}$$

Working in Fourier space,we can discretize the problem in the form

$$\hat{U}_t = -\frac{ik}{2(1+k^2)}e^{\frac{k^2+ik}{1+k^2}t}\mathcal{F}((\mathcal{F}^{-1}(e^{-\frac{k^2+ik}{1+k^2}t}\hat{U}))^2) \tag{4.11}$$

where $\mathcal{F}$ is the Fourier transform operator. Then, we can apply the fourth order Runge-Kutta that is,

$$
\begin{aligned}
k_1 &= f(x_n, y_n) \\
k_2 &= f(x_n + \frac{h}{2}, y_n + \frac{h}{2}k_1) \\
k_3 &= f(x_n + \frac{h}{2}, y_n + \frac{h}{2}k_2) \\
k_4 &= f(x_n + h, y_n + hk_3) \\
y_{n+1} &= y_n + \frac{1}{6}h(k_1 + 2k_2 + 2k_3 + k_4)
\end{aligned}
\quad (4.12)
$$

where $y' = f(x, y)$.

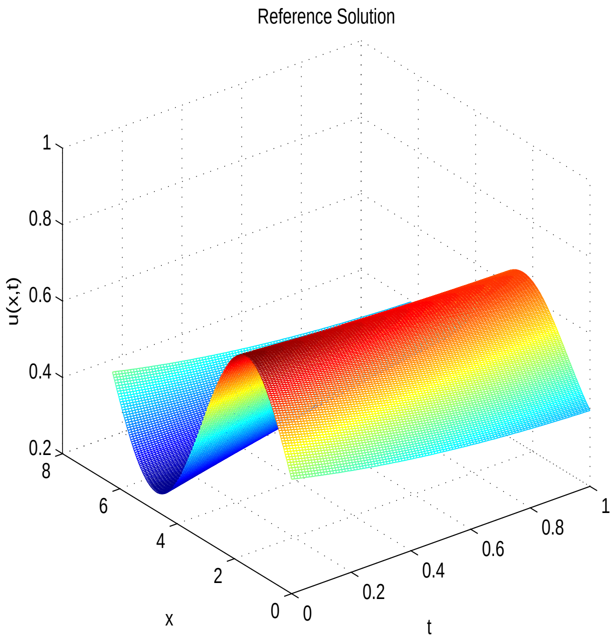

Figure 4.6. Reference solution generated by fourth order Runge-Kutta for $x \in [0, 2\Pi]$ and $t \in [0, 1]$.

Figure 4.6 shows the reference solution for the equation 4.4.

The contour plots of the numerical solutions at different time t for $\alpha = 0.1$, $\beta = 1$ are shown in Figures 4.7, 4.8 and 4.9. Figure 4.7 is showing the numerical solutions of u for $t = 0$ to 1 with the step size $\Delta t = 0.01$. Figure 4.8 is for a short period of time, showing the results for $t = 0$ to 0.2 with the step size $\Delta t = 0.05$.

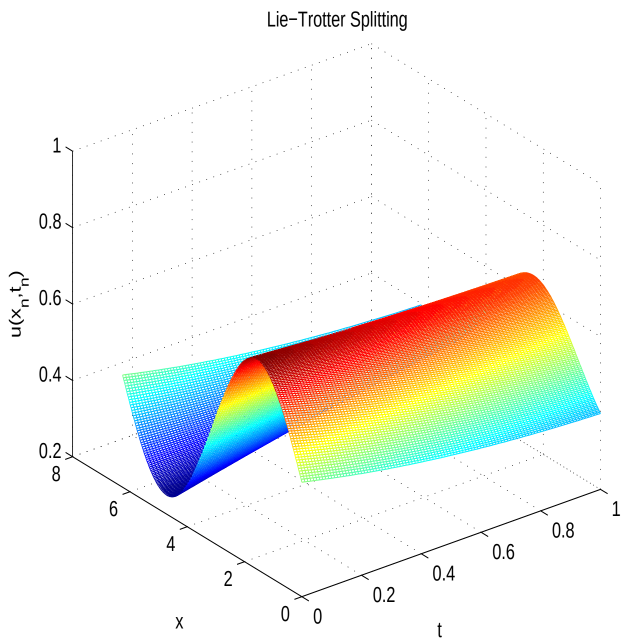

Figure 4.7. Numerical solution of BBM equation for $x \in [0, 2\Pi]$ and $t \in [0, 1]$.

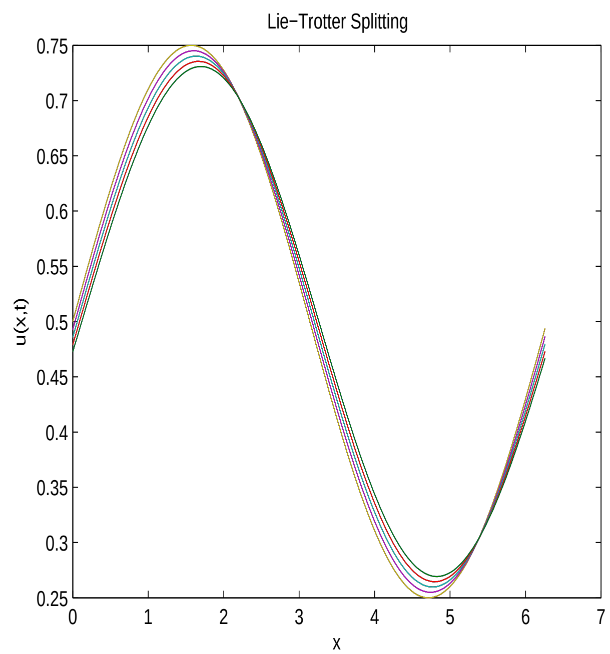

Figure 4.8. Approximate solution at $t = 0$ to $t = 0.2$ with step size $\Delta t = 0.05$.

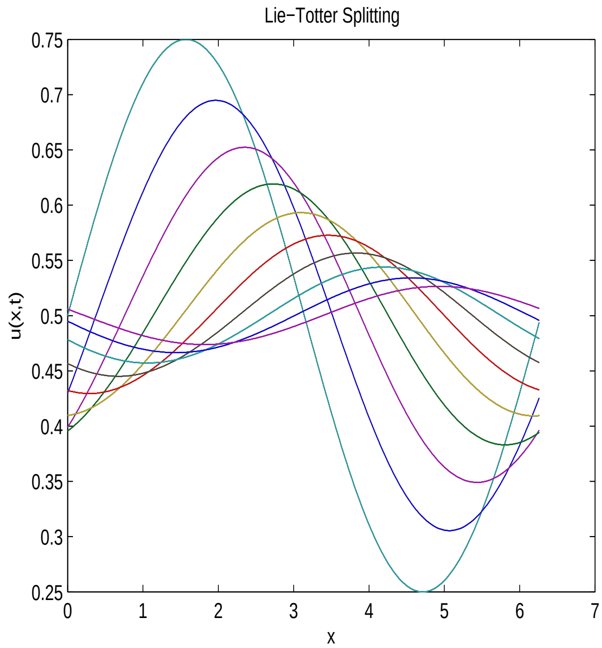

Figure 4.9. Approximate solution at $t = 0$ to $t = 5$ with step size $\Delta t = 0.5$.

Figure 4.9 is for the period of time, showing the results for $t = 0$ to 5 with the step size $\Delta t = 0.5$.

$time\ step$	$Lie - Trotter$		
	L_1	L_2	L_∞
0.5000	0.0626	$4.3463e-003$	$3.9425e-004$
0.3333	0.0419	$2.9076e-003$	$2.6428e-004$
0.2000	0.0252	$1.7505e-003$	$1.5937e-004$
0.1250	0.0158	$1.0964e-003$	$9.992e-005$
0.0833	0.0105	$7.3188e-004$	$6.674e-005$
0.0556	$7.0324e-003$	$4.8835e-004$	$4.455e-005$
0.0370	$4.6911e-003$	$3.2576e-004$	$2.972e-005$
0.0244	$3.0905e-003$	$2.1461e-004$	$1.959e-005$
0.0161	$2.0443e-003$	$1.4196e-004$	$1.296e-005$
0.0108	$1.3631e-003$	$9.4660e-005$	$8.64e-006$

Table 4.5. Estimated errors using L_1, L_2 and L_∞ norm at $t = 1$

In Table 4.5, we exhibite the L_1, L_2 and L_∞ errors of the Lie-Trotter splitting for the various time steps.Table 4.2 shows that we confirm the expected order for the Lie-Trotter splitting.

$time\ step$	$Lie - Trotter$		
	L_1	L_2	L_∞
0.5000			
0.3333	0.9917	0.9915	0.9865
0.2000	0.9933	0.9934	0.9900
0.1250	0.9954	0.9954	0.9933
0.0833	0.9968	0.9969	0.9955
0.0556	0.9978	0.9978	0.9969
0.0370	0.9985	0.9985	0.9979
0.0244	0.9990	0.9990	0.9986
0.0161	0.9993	0.9993	0.9991
0.0108	0.9996	0.9996	0.9994

Table 4.6. Numerical convergence rate for Δt for the operator splitting solution of Lie-Trotter type.

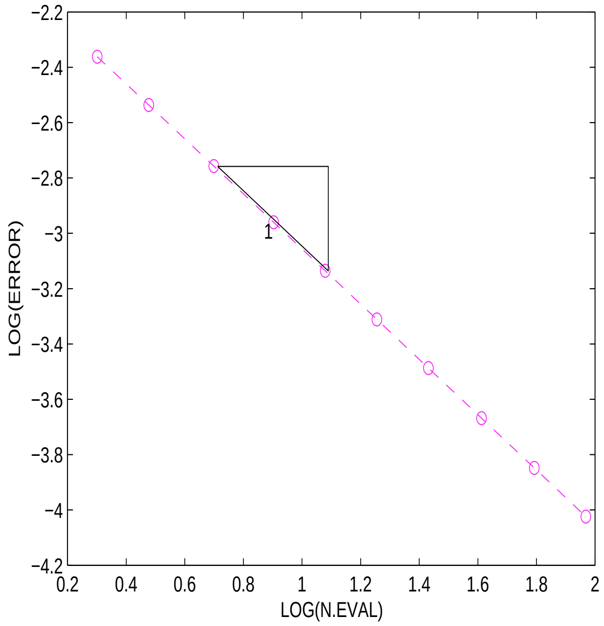

Figure 4.10. Order of the Lie-Trotter splitting for the BBM equation.

In Figure 4.10 shows that the expected order was confirmed for the Lie-Trotter splitting.

CHAPTER 5

CONCLUSIONS

In this thesis, we considered the operator splitting method of Lie-Trotter and Strang type for the Fisher's and Benjamin-Bona-Mahony equations. The main focus was to investigate the numerical convergence rates for Δt . We used the analytical approach presented in (Holden, Lubich, and Risebro, 2013) . This approach relied heavily on the differential theory of operators in Banach space and the error terms of one and two dimensional numerical quadratures.We adopted this idea to Fisher's and BBM equations for the splitting of Lie-Trotter and Strang types.

We tested how well operator splitting solved these equations. For space discretization, the Fisher's equation was discretized by the finite difference method and the BBM equation was discretized by the Fourier transform. We presented the errors of Lie-Trotter and Strang splitting measured by L_1, L_2 and L_∞ norms and investigated the numerical convergence rates for Δt numerically.

It was found that Lie-Trotter splitting converged as $\mathcal{O}(\Delta t)$ while Strang splitting converged as $\mathcal{O}(\Delta t^2)$. We compared them with the theoretical results.Thus, results give us expected order of the accuracy for the operator splitting methods are confirmed.

REFERENCES

A. Ambrosetti and G. Prodi, 1995: *A Primer of Nonlinear Analysis.* Cambridge UP, Cambridge

I.W. Sandberg ,1981: *Expansions for Nonlinear Systems.*

H. Holden, C. Lubich, and N. H. Risebro, 2013 : *Operator splitting for partial differential equations with Burger nonlinerity.* Mathematics of Computation, 82, 173-185.

H. Holden, C. Lubich, N. H. Risebro and T. Tao, 2011 : *Operator splitting for the KDV equation*. Mathematics of Computation, 80, 821-846.

C. Lubich, 2008 : *From quantum to classical moleculer dynamics: reduced models and numerical analysis.* European Math. Soc.

T. Jahnke, C. Lubich : *Error bounds for exponential operator splitting.* BIT,40,735-744

H. Holden, K.H. Karlsen, K. Lie, and N. H. Risebro, 2010: *Splitting Methods for Partial Differential Equations with Rough Solutions. Analysis and Matlab programs.* European Mathematical Society

E.B. Nilsen, 2011: *On operator splitting for the Viscous Burgers' and the Korteweg-de Vries equations.* Norwegian University of Science and Technology

L. N. Trefethen : *Spectral Methods in Matlab.* Society for Industrial and Applied Mathematics.

M. Stanislavova, 2005 : *On the global attractor for the damped Benjamin-Bona-Monoy equation*. Discrete and continuous dynamical systems, supplement volume, 824-832.

A. Kolmogorov, I. Petrovskii and N. Piskunov, 1937 : Moscow Univ. Bull. Math. 1, 1

J. Gazdag and J. Canosa, 1974 : *Numerical solution of Fisher's equation.* J. Appl. Probab. 11, 445-457

M. J. Ablowitz and A. Zeppetella, 1979 : *Explicit solution of Fisher's equation for a special wave speed.* Bull. Math. Biol. 41, 835-840

E.H. Twizell, Y. Wang and W.G. Price, 1990: *Chaos free numerical solutions of reaction-diffusion equations.* Proc. R. Soc. London Sci. A. 430, 541-576

N. Parekh, S. Puri, 1990 : *A new numerical scheme for the Fisher's equation.* J. Phys. A 23, L1085-L1091

S. Tang, R.O. Weber, 1991: *Numerical study of Fisher's equation by a Petrov- Galerkin finite element method.* J. Aus. Math. Soc. Sci. B 33, 27-38

R.E. Mickens, 1994: *A best finite-difference scheme for Fisher's equation.* Numer. Meth. Part. Diff. Eqns. 10, 581-585

T. Mavoungou and Y. Cherruault, 1994: *Numerical study of Fisher's equation by Adomian's method.* Math. Comput. Model. 19(1), 89-95

Y. Qiu and D. M. Sloan, 1998 : *Numerical solution of Fisher's equation using a moving mesh method.* J. Comput. Phys. 146, 726-746

K. Al-Khaled, 2001 : *Numerical study of Fisher's reaction-diffusion equation by the sinc collocation method.* J. Comput. Appl. Math. 137, 245-255 (2001)

S. Zhao, G.W. Wei, 2003 : *Comparison of the discrete singular convolution and three other numerical schemes for solving Fisher's equation.* SIAM Comput. 25(1), 127-147

A. M. Wazwaz and A. Gorguis, 2004 : *An analytic study of Fisher's equation by using Adomian decomposition method.* Appl. Math. Comput. 154, 609-620

D. Olmos and B. D. Shizgal, 2006 : *A pseudospectral method of solution of Fisher's equation.* J. Comput. Appl. Math. 193, 219-242

R. C. Mittal and S. Kumar, 2006 : *Numerical study of Fisher's equation by wavelet Galerkin method.* Int. J. Comput. Math. 83, 287-298

M.S. El-Azab, 2007 : *An approximation scheme for a nonlinear diffusion Fisher's equation.* Appl. Math. Comput. 186, 579-588

A. Sahin, I. Dag and B. Saka, 2008: *A B-spline algorithm for the numerical solution of Fisher's equation.* Kybernetes 37, 326-342

C. Cattani, A. Kudreyko, 2008: *Multiscale analysis of the Fisher equation.* In: (ed.)ICCSA 2008 Proceedings of the International Conference on Computational Science and its Applications, Part I. Lecture Notes in Computer Science, 5072nd edn, pp. 1171-1180. Springer, Berlin

R. C. Mittal and R. Jiwari, 2009 : *Numerical study of Fisher's equation by using differential quadrature method.* Int. J. Information and Systems Sciences 5(1), 143-160

R. C. Mittal and G. Arora, 2010 : *Efficient numerical solution of Fisher's equation by using B-spline method.* Int. J. Comput. Math. 87(13), 3039-3051

R. C. Mittal and R.K. Jain, 2013 : *Numerical solutions of nonlinear Fisher's reaction-diffusion equation with modified cubic B-spline collocation method* Mathematical Sciences, 7:12

T. Benjamin, J. Bona and J. Mahoney, 1972: *Model equations for long waves in nonlinear dispersive media.* Phil. Trans.Roy.Soc. Lond. A., 272: 47-78.

D. Korteweg and G. de Vries, 1985: *On the change of form of long waves advancing in a rectangular canal and a new typeof long stationary wave.* Phil. Mag., 39: 422-443.

K. Al-Khaled, S. Momani and A. Alawneh, 2005: *Approximate wave solutions for generalized Benjamin-Bona-Mahony-Burgers equations.* Appl. Math. Comput. 171, pp. 281-292.

M. Bruón and M. Gandarias, 2008 :*Travelling wave solutions for generalized benjamin-bona-mahony-burgers equations.* Int. J. Math. Models Methods Appl. Sci. 2,103-108

H. Tari and D. Ganji, 2007: *Approximate explicit solutions of nonlinear BBMB equations by Hes methods and comparison with the exact solution.* Phys. Lett. A 367, pp. 95-101.

SA. El-Wakil, MA. Abdou and A. Hendi, 2008: *New periodic wave solutions via Exp-function method.* Phys. Lett.A. 372,830-840

D. Dutkyh,T. D. Katsaounis and D. Mitsotakis , 2013: *Finitine Volume Methods for Unidirectional Dispersive Wave Models.* International Journal for Numerical Methods in Fluids, 71(6), 717-736.

J. L. Bona, V. A. Dougalis, and D. E. Mitsotakis, 2007 : *Numerical solution of KdV-KdV systems of Boussinesq equations: I. The numerical scheme and generalized solitary waves.* Mat. Comp. Simul., 74:214-228. 1

J. L. Bona, W. G. Pritchard, and L. R. Scott, 1985: *Numerical schemes for a model for nonlinear dispersive waves.* J. Comput. Phys., 60:167-186. 1

G. Wei, J. T. Kirby, S. T. Grilli, and R. Subramanya, 1995 :*A fully nonlinear Boussinesq model for surface waves. Part 1. Highly nonlinear unsteady waves.* J. Fluid Mech., 294:71-92. 1

P. Avilez-Valente and F. J. Seabra-Santos, 2009 :*A high-order Petrov-Galerkin finite element method for the classical Boussinesq wave model.* Int. J. Numer. Meth. Fluids, 59:969-1010. 1

D. Dutykh and F. Dias, 2007: *Dissipative Boussinesq equations.* C. R. Mecanique, 335:559-583.

D. E. Mitsotakis, 2009 :*Boussinesq systems in two space dimensions over a variable bottom for the generation and propagation of tsunami waves.* Math. Comp. Simul., 80:860-873.

H. Y. Nguyen and F. Dias, 2008 :*A Boussinesq system for two-way propagation of interfacial waves.* Physica D, 237(18):2365-2389.

H. T. Ozkan-Haller and J. T. Kirby, 1997 :*A Fourier-Chebyshev collocation method for the shallow water equations including shoreline runup.* Applied Ocean Research, 19:21-34.

B. Pelloni and V. Dougalis, 2001 :*Numerical modelling of two-way propagation of non-linear dispersive waves.* Mat. Comp. Simul., 55:595-606.

APPENDIX A

THE PROOFS FOR BASIC CONCEPTS

Proof (Lemma 2.1) The proof uses the Cauchy-Schwarz inequality,$\|uv\|_{L^1} \leq \|u\|_{L^2}\|v\|_{L^2}$, and the Young inequality, $ab \leq a^2/2 + b^2/2$.
Let u be in $H^1(\mathbb{R})$. Then we get, using the above inequalities and the triangle inequality

$$\begin{aligned}
|u^2(y)| &= \Big|\int_{-\infty}^{y} \frac{1}{2}\partial_x(u(x)^2)dx - \int_{y}^{\infty} \frac{1}{2}\partial_x(u(x)^2)dx\Big| \\
&= \Big|\int_{-\infty}^{y} u(x)u'(x)dx - \int_{y}^{\infty} u(x)u'(x)dx\Big| \\
&\leq \int_{-\infty}^{y} |u(x)u'(x)|dx + \int_{y}^{\infty} |u(x)u'(x)|dx \\
&= \int_{-\infty}^{\infty} |u(x)u'(x)|dx \leq \|u\|_{L^2}\|u'\|_{L^2} \leq \frac{1}{2}(\|u\|_{L^2}^2 + \|u'\|_{L^2}^2) = \frac{1}{2}\|u\|_{H^1}^2
\end{aligned}$$

By taking the supremum and the square root, we get

$$\|u\|_{L^\infty} = \frac{1}{\sqrt{2}}\|u\|_{H^1}$$

The last inequality follows from the definition of the Sobolev norm, by adding the L^2 norm of the (weak) derivatives up to order s. □

Proof (Lemma 2.2) Since the Sobolev norm is a sum of (weak) derivatives of u and v, it is sufficient to show that for all $r \leq s$

$$\|\partial_x^r(uv)\|_{L^2} \leq C_s\|u\|_{H^s}\|u\|_{H^s}.$$

Consider $\partial_x^r(uv)$ and expand it using Leibniz rule

$$\partial_x^r(uv) = \sum_{j=0}^{r} \binom{r}{j} \partial_x^j u \partial_x^{r-j} v.$$

By the triangle inequality it is sufficient to look at one term in the above sum. Moreover, we need to be careful in the estimation of the term, since when we vary j and s we get different orders of the derivatives on u and v, which is not necessarily bounded in $H^s(\mathbb{R})$. However, we get for $r < s$ and $0 \leq j \leq r$

$$\begin{aligned}\|\partial_x^j u \partial_x^{r-j} v\|_{L^2}^2 &= \int_{-\infty}^{\infty} (\partial_x^j u)^2 (\partial_x^{r-j} v)^2 dx \leq \|\partial_x^j u\|_{L^\infty}^2 \int_{-\infty}^{\infty} (\partial_x^{r-j} v)^2 dx \\ &\leq C_s \|u\|_{H^{j+1}}^2 \|\partial_x^{r-j} v\|_{L^2}^2 \leq C_s \|u\|_{H^s}^2 \|v\|_{H^s}^2\end{aligned}$$

since $j+1 \leq r+1 \leq s$ and $r-j \leq s$. For $r = s$ and $0 \leq j < r$ we get, using same technique as above

$$\|\partial_x^j u \partial_x^{s-j} v\|_{L^2}^2 \leq C_s \|u\|_{H^s}^2 \|v\|_{H^s}^2. \tag{A.1}$$

we are left with one case; when $r = s = j$,

$$\|\partial_x^s u v\|_{L^2}^2 = \int_{-\infty}^{\infty} (\partial_x^s u)^2 (v)^2 dx \leq \|v\|_{L^\infty}^2 \|\partial_x^s u\|_{L^2}^2 \leq C_s \|u\|_{H^s}^2 \|v\|_{H^s}^2.$$

By taking the square root of the above estimates, and summing up all the derivatives, we get

$$\|uv\|_{H^s} \leq C_s \|u\|_{H^s} \|v\|_{H^s},$$

and the lemma is proven. □

Proof (Lemma 2.4) Lemma 2.4 is proved with the help of Lemma 2.3. Starting with the left hand side of lemma and applying H^s norm yields

$$\begin{aligned}
\|(1-\partial_x^2)^{-1}\partial_x(uv)\|_{H^s} &\leq \|(1-\partial_x^2)^{-1}\partial_x(\partial_x^s(uv))\|_{L^2} + \|(1-\partial_x^2)^{-1}\partial_x(\partial_x^{s-1}(uv))\|_{L^2} \\
&\quad + \|(1-\partial_x^2)^{-1}\partial_x(\partial_x^{s-2}(uv))\|_{L^2} + \ldots \\
&\quad + \|(1-\partial_x^2)^{-1}\partial_x(uv)\|_{L^2} \\
&\leq \|(1-\partial_x^2)^{-1}\partial_x(\sum_{k=0}^{s}\binom{s}{k}\partial_x^k u\partial_x^{s-k}v)\|_{L^2} \\
&\quad + \|(1-\partial_x^2)^{-1}\partial_x(\sum_{k=0}^{s-1}\binom{s-1}{k}\partial_x^k u\partial_x^{s-1-k}v)\|_{L^2} \\
&\quad + \|(1-\partial_x^2)^{-1}\partial_x(\sum_{k=0}^{s-2}\binom{s-2}{k}\partial_x^k u\partial_x^{s-2-k}v)\|_{L^2} + \ldots \\
&\quad + \|(1-\partial_x^2)^{-1}\partial_x(uv)\|_{L^2} \\
&\leq \|(1-\partial_x^2)^{-1}\partial_x(\binom{s}{0}u\partial_x^s v + \binom{s}{1}\partial_x u\partial_x^{s-1}v + \ldots + \partial_x^s uv\|_{L^2} \\
&\quad + \|(1-\partial_x^2)^{-1}\partial_x(\binom{s-1}{0}u\partial_x^{s-1}v + \ldots + \partial_x^{s-1}uv\|_{L^2} \\
&\quad + \ldots + \|(1-\partial_x^2)^{-1}\partial_x(uv)\|_{L^2} \\
&\leq \|(1-\partial_x^2)^{-1}\partial_x(u\partial_x^s v)\|_{L^2} \\
&\quad + \binom{s}{1}\|(1-\partial_x^2)^{-1}\partial_x(\partial_x u\partial_x^{s-1}v)\|_{L^2} \\
&\quad + \ldots + \|(1-\partial_x^2)^{-1}\partial_x(\partial_x^s uv)\|_{L^2} \\
&\quad + \|(1-\partial_x^2)^{-1}\partial_x(u\partial_x^{s-1}v)\|_{L^2} \\
&\quad + \binom{s-1}{1}\|(1-\partial_x^2)^{-1}\partial_x(\partial_x u\partial_x^{s-2}v)\|_{L^2} \\
&\quad + \ldots + \|(1-\partial_x^2)^{-1}\partial_x(\partial_x^{s-1}uv)\|_{L^2} \\
&\quad + \ldots + \|(1-\partial_x^2)^{-1}\partial_x(uv)\|_{L^2}
\end{aligned}$$

and applying Lemma 2.3, we obtain

$$\begin{aligned}
\|(1-\partial_x^2)^{-1}\partial_x(uv)\|_{H^s} &\leq \|u\|_{L^2}\|\partial_x^s v\|_{L^2} + \binom{s}{1}\|\partial_x u\|_{L^2}\|\partial_x^{s-1}v\|_{L^2} + \ldots \\
&\quad + \|\partial_x^s u\|_{L^2}\|v\|_{L^2} \\
&\quad + \|u\|_{L^2}\|\partial_x^{s-1}v\|_{L^2} \\
&\quad + \binom{s-1}{1}\|\partial_x u\|_{L^2}\|\partial_x^{s-2}v\|_{L^2} \\
&\quad + \ldots + \|\partial_x^{s-1}u\|_{L^2}\|v\|_{L^2} \\
&\quad + \ldots + \|u\|_{L^2}\|v\|_{L^2}
\end{aligned}$$

$$\begin{aligned}
&\leq \|u\|_{L^2}\|v\|_{H^s} + \binom{s}{1}\|u\|_{H^1}\|v\|_{H^{s-1}} + \ldots + \|u\|_{H^s}\|v\|_{L^2} \\
&\quad + \|u\|_{L^2}\|v\|_{H^{s-1}} + \binom{s-1}{1}\|u\|_{H^1}\|v\|_{H^{s-2}} + \ldots + \|u\|_{H^{s-1}}\|v\|_{L^2} \\
&\quad + \ldots + \|u\|_{L^2}\|v\|_{L^2} \\
&\leq K\|u\|_{H^s}\|v\|_{H^s}. \qquad \text{(A.2)}
\end{aligned}$$

Note that case $s = 0$, i.e. $H^0 = L^2$, implies Lemma 2.3. □

Proof (Lemma 2.5) Let us assume $t > 0$ (for $t < 0$ the argument is the same). By definition of derivative we have for t small enough (so $th \in V$)

$$f(x + th) = f(x) + (Df(x))(th) + \|th\|\psi(th),$$

or by rearranging and using linearity

$$(Df(x))h = \frac{f(x + th) - f(x)}{t} - \|h\|\psi(th)$$

Now we can take the limit as $t \to 0$ on both sides (as the left hand side is constant it has a limit) to get the desired result (note $\lim_{t\to 0} \|h\|\psi(th) = \|h\|\psi(0h) = 0$ as ψ is continuous.) □

APPENDIX B

MAT-LAB CODES

B.1. Codes for Fisher’s Equation

```
clear all
close all
h=0.5;
x1=-10;
x2=10;
N=(x2-x1)/h;
x=x1:h:x2;
t1=0;
t2=1;
a=1
b=1
for e=1:4
h1=0.01/(5ˆ(e-2));
N1=(t2-t1)/h1;
t=t1:h1:t2;
 A1=toeplitz([ -2 1 0 zeros(1,N-4)],[-2 1 0 zeros(1,N-4)]);
 A=(1/(h)ˆ2)*A1;%%u_xx,
 C=(1/(h)ˆ2)*[0;zeros(N-3,1);0];
g(:,1)=(sech(10*x).ˆ2)’;
 u(:,1)=g(2:N,1);
 %%runge kutta 4th order%%
 ex(:,1)=u(:,1);
 s(:,1)=u(:,1);
for i=1:N1
k_1 = f(a,b,A,C,ex(:,i));
k_2 = f(a,b,A,C,ex(:,i)+0.5*h1*k_1);
k_3 = f(a,b,A,C,ex(:,i)+0.5*h1*k_2);
```

```
k_4 = f(a,b,A,C,ex(:,i)+k_3*h1);
ex(:,i+1) = ex(:,i) + (1/6)*(k_1+2*k_2+2*k_3+k_4)*h1;
end

for j=1:N1 %lie
u(:,j+1)=expm(h1*a*A)*u(:,j);
v(:,j)=u(:,j+1);
v(:,j+1)=v(:,j)+h1/2*(f3(a,b,A,C,u(:,j))
 +f3(a,b,A,C,v(:,j)+h1*(f3(a,b,A,C,v(:,j)))));
u(:,j+1)=v(:,j+1);
end

for j=1:N1 %strang
s(:,j+1)=expm(h1/2*a*A)*s(:,j);
s1(:,j)=s(:,j+1);
s1(:,j+1)=s1(:,j)+h1/2*(f3(a,b,A,C,s1(:,j))
 +f3(a,b,A,C,s1(:,j)+h1*(f3(a,b,A,C,s1(:,j)))));
s2(:,j)=s1(:,j+1);
s2(:,j+1)=expm(h1/2*a*A)*s2(:,j);
s(:,j+1)=s2(:,j+1);
end

exact=[zeros(1,N1+1);ex;zeros(1,N1+1)];%exact solution
K=[zeros(1,N1+1);u;zeros(1,N1+1)];%solution of lie
S=[zeros(1,N1+1);s;zeros(1,N1+1)];%solution of strang
errorST=abs(S-exact);
errorlie=abs(K-exact);
A=norm(errorlie(:,N1+1),1);
B=norm(errorlie(:,N1+1),2);
C=norm(errorlie(:,N1+1),inf);
A1=norm(errorST(:,N1+1),1);
B1=norm(errorST(:,N1+1),2);
C1=norm(errorST(:,N1+1),inf);
disp(sprintf('h=%6.6f,h1=%6.6f,A1=%9.9f,B1=%9.9f,C1=%9.9f'
,h,h1,A1,B1,C1))
disp(sprintf('h=%6.6f,h1=%6.6f,A=%6.6f,B=%6.6f,C=%6.6f'
,h,h1,A,B,C))
```

```
errorA(e)=norm(errorlie(:,N1+1),1);
errorB(e)=norm(errorlie(:,N1+1),2);
errorC(e)=norm(errorlie(:,N1+1),inf);
errorA1(e)=norm(errorST(:,N1+1),1);
errorB1(e)=norm(errorST(:,N1+1),2);
errorC1(e)=norm(errorST(:,N1+1),inf);
dt(e)=h1;
if e>1
orderA(e)=(log(errorA(e)/errorA(e-1)))/
(log(dt(e)/dt(e-1)));
orderB(e)=(log(errorB(e)/errorB(e-1)))/
(log(dt(e)/dt(e-1)));
orderC(e)=(log(errorC(e)/errorC(e-1)))/
(log(dt(e)/dt(e-1)));
orderA1(e)=(log(errorA1(e)/errorA1(e-1)))/
(log(dt(e)/dt(e-1)));
orderB1(e)=(log(errorB1(e)/errorB1(e-1)))/
(log(dt(e)/dt(e-1)));
orderC1(e)=(log(errorC1(e)/errorC1(e-1)))/
(log(dt(e)/dt(e-1)));
else
orderA(e)=0;
orderB(e)=0;
orderC(e)=0;
orderA1(e)=0;
orderB1(e)=0;
orderC1(e)=0;
end
disp(sprintf('h1=%5.4f,A=%8.8f,orderA=%5.4f'
,dt(e),errorA(e),orderA(e)));
disp(sprintf('h1=%5.4f,B=%8.8f,orderB=%5.4f'
,dt(e),errorB(e),orderB(e)));
disp(sprintf('h1=%5.4f,C=%8.8f,orderC=%5.4f'
,dt(e),errorC(e),orderC(e)));
disp(sprintf('h1=%5.4f,A1=%8.8f,orderA1=%5.4f'
,dt(e),errorA1(e),orderA1(e)));
disp(sprintf('h1=%5.4f,B1=%8.8f,orderB1=%5.4f'
```

```
,dt(e),errorB1(e),orderB1(e)));
disp(sprintf('h1=%5.4f,C1=%8.8f,orderC1=%5.4f'
,dt(e),errorC1(e),orderC1(e)));
end
%% graph for the numerical solution.
[t,x]=meshgrid(t,x);
subplot(1,2,1); mesh(t,x,K)
subplot(1,2,2); mesh(t,x,S)
%%%%%%%%%%%%
function  j=f(a,b,A,C,v)
l=diag(v);
s=a*A*v+C+b*l*(1-v);
j=s;
%%%%%%%%%%%%
function  j3=f3(a,b,A,C,v)
l=diag(v);
s3=b*l*(1-v);
j3=s3;
```

B.2. Codes for BBM Equation

```
function mesh_bbm_split
%% Initial parameters
    t_0 = 0; %Initial time
    t_f = 1; %Final time
    M   = 100; % Time steps
    N   = 256; % Fourier modes
    k   = [0:N/2-1 N/2 -N/2+1:-1]; % Fourier modes
%%space grid with periodic boundary condition
    dx  = 2*pi/N;
    x   = 0:dx:2*pi-dx;
    x   = x';
%%Initial Data
     u0=1/2+1/4*sin(x);
    u0=u0';
alpha=1;
```

```
beta=1;
a=[ 1 ];
 b=[ 1 ];
%% Initialize variable
      err     = [];
      steps   = [];
      Uap     = [];
      Uap     = [Uap u0'];
      C       = [];
      dt      = (t_f-t_0)/M;
      NrSteps = round(t_f/dt);
      t       = t_0:dt:t_f;
      hvec    = [];
 %% Outer loop
U_now   = u0;
h       = dt;
hvec    = [hvec h];
uex(:,1) = u0;

for j=1:NrSteps,
      for s=1:length(a)
%B part (linear part)
U_now=fft(U_now);
U_now=exp((1./(1+k.^2)).*(-alpha*k.^2-1i*beta.*k).*b(s)*h)
.*U_now;
U_now=real(ifft(U_now));
%A part (Non-linear part)
if(a(s)~=0)
U_now=bbm_splitrk4(U_now,a(s)*h,a(s)*h/100,N,k);
end
end
U_now = real(U_now);
u  =  U_now;
Uap = [ Uap u' ];
steps= [steps NrSteps];
end
%% graph for the numerical solution.
```

```
[t,x] = meshgrid(t,x);
figure(1)
mesh(t,x,Uap)  ;
xlabel('t'); ylabel('x') ; zlabel('u')
```

Computing order;

```
function strFFF_split
%order check  for BBM equation..
%% Initial parameters
    t_0 = 0; %Initial time
    t_f = 1; %Final time
    M   = 100; % Time steps
    N   = 256; % Fourier modes
    k   = [0:N/2-1 0 -N/2+1:-1]; % Fourier modes
%%   space grid with periodic boundary condition
dx  = 2*pi/N;
x   = 0:dx:2*pi-dx;
x   = x';
%%  Initial Data
u0=1/2+1/4*sin(x);
u0=u0';
alpha=1;
beta=1;
Plotpoints=10;%For plotting errors at different number of
steps
%% Exact solution with integrating vector
[UEX]=bbm_specexact(alpha,beta,u0,t_0,t_f,M,N,k);
a=[ 1 ]; b=[ 1 ];
cost = 1; %number of function evaluation of the operator A
%% Initialize variable
      errL1     = [];
      errL2     = [];
      errLinf     = [];
      steps   = [];
      Uap     = [];
      C       = [];
      NrSteps = 1;
```

```
      hvec    = [];
%% Outer loop
for m = 1:Plotpoints
%% current stepsize
NrSteps = round(NrSteps*1.5);
U_now   = u0;
h       = t_f/NrSteps;
hvec    = [hvec h];
 %% ineer loop
for j=1:NrSteps,
for s=1:length(a)
%B part (linear part)
U_now=fft(U_now);
U_now=exp((1./(1+k.^2)).*(-alpha*k.^2-1i*beta.*k)*b(s)*h)
.*U_now;
U_now=real(ifft(U_now));
%A part (Non-linear part)
if(a(s)~=0)
U_now = bbm_splitrk4(U_now,a(s)*h,a(s)*h/100,N,k);
 end
 end
 end
 u   =  real( U_now);
 Uap = [ Uap u' ];
 steps= [steps NrSteps];
 C =Uap - UEX(:,ones(1,size(Uap')));
 errL2= sqrt(sum(abs(C).^2)); %L_2
 errL1= (sum(abs(C))); %L_1
 errLi= (max(abs(C))); %L_inf
if m>1
orderL1(m)=(log(errL1(m)/errL1(m-1)))/
(log(hvec(m)/hvec(m-1)));
orderL2(m)=(log(errL2(m)/errL2(m-1)))/
(log(hvec(m)/hvec(m-1)));
orderLi(m)=(log(errLi(m)/errLi(m-1)))/
(log(hvec(m)/hvec(m-1)));
else
```

```
orderL1(m)=0;
orderL2(m)=0;
orderLi(m)=0;
end
disp(sprintf('h=%5.4f, errorL1=%8.8f, orderL1=%5.4f',
hvec(m), errL1(m), orderL1(m)));
disp(sprintf('h=%5.4f, errorL2=%8.8f, orderL2=%5.4f',
hvec(m), errL2(m), orderL2(m)));
disp(sprintf('h=%5.4f, errorLi=%8.8f, orderLi=%5.4f',
hvec(m), errLi(m), orderLi(m)));
end
end
%%%%%%%%%%%
function [u]=bbm_splitrk4(u0,t_f,dt,N,k)
 % Solve Nonlinear part with spectral RK4 metod
 vv   = fft(u0);
 kk   = [0:N/2-1 0 -N/2+1:-1];
 nmax = round(t_f/dt);
for n = 1:nmax
    g  = -(1i/2*kk)./(1+k.^2);
    k1 = g.*fft(real(ifft(vv)).^2);
    k2 = g.*fft(real(ifft(vv+k1*dt/2)).^2);
    k3 = g.*fft(real(ifft(vv+dt/2*k2)).^2); % RK4
    k4 = g.*fft(real(ifft(vv+dt*k3)).^2);
    vv = vv+dt*(k1+2*k2+2*k3+k4)/6;
end
u=real(ifft(vv));
end
%%%%%%%%%%%
 function [u] = bbm_specexact(alpha,beta,u0,t_0,t_f,M,N,k)
    dt  = (t_f-t_0)/M; %Time steps
    u   = u0';
    v   = fft(u); %fast fourier transform of initial data
    k   = k';
    k2 = k.^2; %Coefficient of derivative in fourier domain
% Time stepping method for solving PDE
  tmax = t_f; % Final time
```

```
  nmax = round(tmax/dt); % Number of runge-kutta steps
  k1   = [0:N/2-1 0 -N/2+1:-1]'; % for odd derivative
  for n = 1:nmax
       g = -.5i*dt*k1./(1+k2);
       E = exp(-.5*dt*(alpha*k2+beta*1i*k1)./(1+k2));
       E2 = E.^2;
       a = g.*fft(real( ifft(     v    ) ).^2);
       b = g.*fft(real( ifft(E.*(v+a/2)) ).^2);% 4th-order
       c = g.*fft(real( ifft(E.*v + b/2) ).^2);% Runge-Kutta
       d = g.*fft(real( ifft(E2.*v+E.*c) ).^2);
       v = E2.*v + (E2.*a + 2*E.*(b+c) + d)/6;
  end
u = real(ifft(v));
end
```