HOMOLOGICAL OBJECTS OF PROPER CLASSES GENERATED BY SIMPLE MODULES

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ABSTRACT

HOMOLOGICAL OBJECTS OF PROPER CLASSES GENERATED BY SIMPLE MODULES

The main purpose of this thesis is to study some classes of modules determined by neat, coneat and s-pure submodules. A right R-module M is called neat-flat (resp. coneat-flat) if the kernel of any epimorphism $Y \to M \to 0$ is neat (resp. coneat) in Y. A right R-module M is said to be absolutely s-pure if it is s-pure in every extension of it. If R is a commutative Noetherian ring, then R is C-ring if and only if coneat-flat modules are flat. A commutative ring R is perfect if and only if coneat-flat modules are projective. R is a right $\sum$-CS ring if and only if neat-flat right R-modules are projective. R is a right Kasch ring if and only if injective right R-modules are neat-flat if and only if the injective hull of every simple right R-module is neat-flat. If R is a right N-ring, then R is right $\sum$-CS ring if and only if pure-injective neat-flat right R-modules are projective if and only if absolutely s-pure left R-modules are injective and R is right perfect. A domain R is Dedekind if and only if absolutely s-pure modules are injective. It is proven that, for a commutative Noetherian ring R, (1) neat-flat modules are flat if and only if absolutely s-pure modules are absolutely pure if and only if $R \cong A \times B$, wherein A is QF-ring and B is hereditary; (2) neat-flat modules are absolutely s-pure if and only if absolutely s-pure modules are neat-flat if and only if $R \cong A \times B$, wherein A is QF-ring and B is Artinian with $J^2(B) = 0$.

ÖZET

BASİT MODÜLLER İLE ÜRETİLEN ÖZ SINIFLARIN HOMOLOJİK NESNELERİ

Bu tezde temel olarak düzenli, eşdüzenli ve s-saf altmodüller yardımıyla tanımlanan bazı modül sınıflarının çalışılması amaçlanmaktadır. Bir M sağ R-modülü için, her $Y \to M \to 0$ epimorfizmasının çekirdeği Y'de düzenli (sırasıyla eşdüzenli) ise, M'ye düzenli-düz (sırasıyla eşdüzenli-düz) denir. Her genişlemesi s-saf olan M sağ R-modülüne, mutlak s-saf denir. R değişmeli Noetherian halkası ise, R bir C-halkasıdır ancak ve ancak eşdüzenli-düz modüller düzdür. Bir değişmeli R halkası mükemmeldir ancak ve ancak eşdüzenli-düz modüller projektiftir. R bir sağ Kasch halkasıdır ancak ve ancak injektif sağ R-modüller düzenli-düzdür ancak ve ancak her basit sağ R-modülünün injektif kapanışı düzenli-düzdür. Eğer R bir N-halkası ise, R sağ $\sum$-CS halkasıdır ancak ve ancak saf-injektif düzenli-düz sağ R-modüller projektiftir ancak ve ancak mutlak s-saf sol R-modüller injektiftir ve R bir sağ mükemmel halkadır. R tamlık bölgesi Dedekind'dir ancak ve ancak mutlak s-saf modüller injektiftir. Ayrıca değişmeli Noetherian R halkası üzerinde şunlar ispatlanmıştır: (1) düzenli-düz modüller düzdür ancak ve ancak mutlak s-saf modüller mutlak saftır ancak ve ancak $R \cong A \times B$, A QF-halkasıdır ve B kalıtsal halkasıdır; (2)düzenli-düz modüller mutlak s-saftır ancak ve ancak mutlak s-saf modüller düzenli-düzdür ancak ve ancak $R \cong A \times B$, A QF-halkasıdır ve B, $J^2(B) = 0$ olan bir Artinian halkasıdır.

This thesis is dedicated to all Child Laborers ...

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LIST OF ABBREVIATIONS

R	an associative ring with unit unless otherwise stated
$\mathbb{Z}$,	the ring of integers
$\mathbb{Q}$	the field of rational numbers
$R\text{-}\mathcal{M}od$	the category of *left* R-modules
$\mathcal{M}od\text{-}R$	the category of *right* R-modules
$\mathcal{A}b = \mathbb{Z}\text{-}\mathcal{M}od$	the category of abelian groups ($\mathbb{Z}$-modules)
$\mathrm{Hom}_R(M, N)$	all R-module homomorphisms from M to N
$\mathrm{Ker}\, f$	the kernel of the map f
$\mathrm{Im}\, f$	the image of the map f
M^+	the character module $\mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})$
M^*	the dual module $\mathrm{Hom}_R(M, R)$
$\mathrm{Soc}\, M$	the socle of the R-module M
$\mathrm{Rad}\, M$	the radical of the R-module M
$\mathcal{P}$	a proper class of R-modules
$\pi(\mathcal{P})$	all $\mathcal{P}$-projective modules
$\pi^{-1}(\mathcal{M})$	the proper class of R-modules projectively generated by a class $\mathcal{M}$ of R-modules
$\iota(\mathcal{P})$	all $\mathcal{P}$-injective modules
$\iota^{-1}(\mathcal{M})$	the proper class of R-modules injectively generated by a class $\mathcal{M}$ of R-modules
$\mathrm{Ext}_R(C, A) = \mathrm{Ext}^1_R(C, A)$	set of all equivalence classes of short exact sequences starting with the R-module A and ending with the R-module C
$\mathcal{A}$	an abelian category (like $R\text{-}\mathcal{M}od$ or $\mathbb{Z}\text{-}\mathcal{M}od = \mathcal{A}b$)
	For a suitable abelian category $\mathcal{A}$ like $R\text{-}\mathcal{M}od$ or $\mathbb{Z}\text{-}\mathcal{M}od$, the following classes are defined:
$\mathcal{S}plit_{\mathcal{A}}$	the smallest proper class consisting of *only splitting* short exact sequences in the abelian category $\mathcal{A}$
$\mathcal{A}bs$	the largest proper class consisting of *all* short exact sequences in the category $R\text{-}\mathcal{M}od$
$\cong$	isomorphic
$\leq$	submodule
$\ll$	small (=superfluous) submodule

CHAPTER 1

INTRODUCTION

Throughout this work, we shall assume that all rings are associative with identity. All modules are unitary left modules, unless otherwise stated. Let R be any ring. A submodule K of an R-module M is called closed (in M) provided that K has no proper essential extension in M. Moreover, if L is any submodule of M then there exists, by Zorn's Lemma, a submodule K of M maximal with respect to the property that L is an essential submodule of K, and in this case K is a closed submodule of M. A module M is called an extending module if every closed submodule is a direct summand, and in this case every submodule of M is essential in a direct summand of M. For the properties of closed submodules and extending modules, see Dung and Wisbauer (1994).

A submodule K of an R-module M is called pure provided for every (finitely presented) right R-module U, the induced homomorphism $U \otimes_R K \to U \otimes_R M$ of Abelian groups is a monomorphism. When R is a Dedekind domain (more generally a Prüfer domain), a submodule K of an R-module M is pure if and only if $K \cap aM = aK$ for all $a \in R$. Inspired by this characterization of pure submodules over Dedekind domains, Honda (1956) introduced neat subgroups in order to characterize the closed subgroup in abelian groups. Namely, a subgroup A of an Abelian group B is called neat in B if $Ap = A \cap Bp$ for every prime p. It is easy to see that a closed exact sequence of Abelian groups can also be defined in terms of either of the following homological properties (p denotes primes): $0 \to A \xrightarrow{\iota} B \to C \to 0$ is a closed exact sequence of abelian groups i.e. $\iota(A)$ is closed in B if and only if

(a) $\iota(A)p = \iota(A) \cap Bp$ for all p, i.e. $\iota(A)$ is neat in B;

(b) the sequence $0 \to \mathrm{Hom}(\mathbb{Z}/p\mathbb{Z}, A) \to \mathrm{Hom}(\mathbb{Z}/p\mathbb{Z}, B) \to \mathrm{Hom}(\mathbb{Z}/p\mathbb{Z}, C) \to 0$ is exact for all p;

(c) the sequence $0 \to \mathbb{Z}/p\mathbb{Z} \otimes A \to \mathbb{Z}/p\mathbb{Z} \otimes B \to \mathbb{Z}/p\mathbb{Z} \otimes C \to 0$ is exact for all p;

(d) the sequence $0 \to \mathrm{Hom}(C, \mathbb{Z}/p\mathbb{Z}) \to \mathrm{Hom}(B, \mathbb{Z}/p\mathbb{Z}) \to \mathrm{Hom}(A, \mathbb{Z}/p\mathbb{Z}) \to 0$ is exact for all p.

The definition of closed subgroup can be extended to arbitrary rings R either via (a), (b), (c) or (d).

Neatness over arbitrary associative rings in the sense of (a) has been considered by Mermut et al. (2009), a submodule A of an R-module B is called P-pure if $PA = A \cap PB$ for every maximal right ideal P of R. Generalization in the sense of (b) was discussed by Renault (1964), a submodule A of an R-module B is called neat if $\mathrm{Hom}(S, B) \to \mathrm{Hom}(S, B/A) \to 0$ is surjective for each simple R-module S. In the sense of (c) it has been considered by Crivei (2005), a submodule A of an R-module B is called s-pure if the map $S \otimes A \to S \otimes B$ is monic for each simple right R-module S. In the sense of (d) it was discussed by Fuchs Fuchs (2012), a submodule A of an R-module B is called coneat if $\mathrm{Hom}(B, S) \to \mathrm{Hom}(B/A, S) \to 0$ is surjective for each simple R-module S.

Let A be a submodule of a left R-module B. For a right ideal I of R, $A \cap IB = IA$ if and only if the map $R/I \otimes A \to R/I \otimes B$ is monic, (Skljarenko, 1978, Lemma 6.1). This result can be used to show that P-pure submodules and s-pure submodules coincide. P-pure submodules coincide with coneat submodules over commutative rings by (Fuchs, 2012, Proposition 3.1). Closed submodules are neat ((Stenström, 1967, Proposition 5)). Neat submodules of each R-module are closed if and only if R is left C-ring i.e. for every proper essential left ideal I of R, the module R/I has a simple submodule (Generalov, 1978, Theorem 5). As one may see from (Fuchs, 2012, Example 3.2,3.3), neat submodules and coneat submodules are not only inequivalent, but even incomparable.

In Chapter 3 the proper class *Coneat* which is determined by coneat submodules is investigated. Some homological objects which are related with the proper class *Coneat* are introduced and investigated. We call M *coneat-flat* if the kernel of any epimorphism $Y \to M \to 0$ is coneat in Y. Several characterizations of coneat-flat modules are given. Some known results are generalized, and relations between coneat-flat modules and flat, m-injective, absolutely pure and projective modules are studied. It is shown that a submodule N of a right R-module M is coneat if and only if for every maximal submodule K of N, N/K is a direct summand of M/K. A ring R is a right V-ring if and only if submodules of right R-modules are coneat. R is right small if and only if absolutely coneat right modules are precisely those modules M such that $M = \mathrm{Rad}(M)$. We prove that, a module M is coneat-flat if and only if $M \cong P/N$, where P is a projective R-module and N is a coneat submodule of P. Over commutative rings, an R-module M is coneat-flat if and only if and only if M^+ is m-injective. R is a right V-ring if and only if every right R-module is coneat-flat. We prove that, if R is a left C-ring, then a right R-module M is flat if and only if $\mathrm{Tor}_1^R(M, S) = 0$ for each simple left R-module S. If R is a commutative C-ring, then coneat-flat modules are only the flat modules, and the converse holds when R is Noetherian. R is a left N-ring (i.e. maximal left ideals are finitely generated) if and only if every absolutely pure module is m-injective. A ring R is left Artinian if and only if m-injective left R-modules are precisely those modules M with M^+ is projective. We consider the projectivity of coneat-flat modules. We show that, if R is right perfect then every coneat-flat R-module is projective, the converse hold if R is commutative. Finitely presented coneat-flat modules are projective, over semiperfect rings and over commutative rings.

In Chapter 4 some homological objects which are related with the proper classes $s-\mathcal{P}ure$ and $\mathcal{N}eat$ are investigated. Namely, we call M is *neat-flat* if the kernel of any epimorphism $Y \to M$ is neat in Y, i.e. the induced map $\mathrm{Hom}(S, Y) \to \mathrm{Hom}(S, M)$ is surjective for any simple R-module S. Similar to a well known notion of absolutely pure (or FP-injective) module, a right R-module M is said to be absolutely s-pure if it is s-pure in every extension of it. Projective modules, weakly-flat modules and nonsingular modules are neat-flat. In Mao (2007), the author introduced *simple-projective* modules to in order to characterize the rings whose simple modules have projective (pre)envelope. An R-module M is called simple-projective if for any simple right R-module N, every homomorphism $f : N \to M$ factors through a finitely generated free right R-module F. First we show that an R-module M is neat-flat if and only if M is simple-projective. Next, we give the main properties of the class of neat-flat R-modules. The right socle of R is zero if and only if neat-flat modules coincide with the modules that have zero socle. A ring R is right C-ring if and only if neat-flat modules are weakly-flat. We also investigate the rings over which neat-flat modules are projective. Namely, we prove that, (1) every neat-flat module is projective if and only if R is a right $\sum$-CS ring; (2) every finitely generated neat-flat module is projective if and only if R is a right C-ring and every finitely generated free right R-module is extending ; (3) every cyclic right R-module is projective if and only if R is right CS and right C-ring. It is shown that, over a commutative Noetherian ring R, (1) every neat-flat module is flat if and only if every absolutely s-pure module is injective if and only if $R \cong A \times B$, wherein A is QF-ring and B is hereditary; (2) every neat-flat module is absolutely s-pure if and only if every aabsolutely s-pure module is neat-flat if and only if every neat-flat module is weakly-injective if and only if every absolutely s-pure module is weakly-flat if and only if $R \cong A \times B$, wherein A is QF-ring and B is Artinian with $J^2(B) = 0$. Localization of neat exact sequences and neat-flat modules are investigated. It is shown that, over a commutative N-ring R, (1) a short exact sequence $0 \to A \to B \to C \to 0$ is neat if and only if $0 \to A_P \to B_P \to C_P \to 0$ is neat exact for each maximal ideal P of R; (2) a module M is neat-flat if and only if, for all maximal ideals P of R, M_P is neat-flat R_P-module. Some connections between absolutely s-pure and neat-flat modules are established. For a right N-ring, we prove that a left R-module M is absolutely s-pure if and only if $\mathrm{Ext}^1_R(Tr(S), M) = 0$ for each simple right R-module S; a right R-module M is neat-flat if and only if M^+ is absolutely s-pure. For a commutative nonsingular ring, we prove that every absolutely s-pure module is injective if and only if R is hereditary and Noetherian. In particular, a domain R is Dedekind if and only if every absolutely s-pure module is injective. A ring R is right Kasch if and only if the injective hull of every simple right R-module is neat-flat if and only if for every free left R-module F, F^+ is neat-flat. For a right N-ring, we show that, ${}_RR$ is absolutely s-pure if and only if R is a right Kasch ring; R is a right $\sum$-CS ring if and only if every pure-injective neat-flat right R-module is projective if and only if every absolutely s-pure left R-module is injective and R is right perfect. The last section is devoted for the study of enveloping and covering properties of absolutely s-pure and neat-flat modules. For a right N-ring R, we show that every quotient of any injective left R-module is absolutely s-pure if and only if every left R-module has a monic absolutely s-pure cover if and only if R is a right PS ring; R is a right Kasch ring if and only if every left R-module has an epic absolutely s-pure cover. For a commutative ring R, we show that, every simple R-module has an injective cover if R is coherent ring; every simple R-module has a monic injective cover if and only if every R-module has a monic absolutely s-pure cover if and only if a simple R-module S is either injective or $\mathrm{Hom}(E, S) = 0$ for every injective R-module E.

CHAPTER 2

PRELIMINARIES

Throughout this thesis, by a *ring* we mean an associative ring with unity; R will denote such a general ring, unless otherwise stated. So, *if nothing is said about R in the statement of a theorem, proposition, etc., then that means R is just an arbitrary ring.* We consider unital left R-modules; R-module will mean *left* R-module. $R\text{-}\mathcal{M}od$ denotes the category of all *left* R-modules. $\mathcal{M}od\text{-}R$ denotes the category of *right* R-modules. $\mathbb{Z}$ denotes the ring of integers. $\mathcal{A}b$, or $\mathbb{Z}\text{-}\mathcal{M}od$, denotes the category of Abelian groups ($\mathbb{Z}$-modules). *Group* will mean *Abelian group* only. *Integral domain*, or shortly *domain*, will mean a nonzero ring without zero divisors, *not* necessarily commutative. Also a Dedekind domain is assumed to be commutative as usual.

All definitions not given here can be found in (Anderson and Fuller, 1992), (Wisbauer, 1991), (Dung and Wisbauer, 1994), (Clark et al., 2006) and (Fuchs, 1970).

We do not delve into the details of definitions of every term in modules, rings and homological algebra. Essentially, we accept fundamentals of module theory, categories, pullback and pushout, the Hom and tensor ($\otimes$) functors, projective modules, injective modules, flat modules, homology functor, projective and injective resolutions, derived functors, the functor $\text{Ext}_R = \text{Ext}_R^1 : R\text{-}\mathcal{M}od \times R\text{-}\mathcal{M}od \longrightarrow \mathcal{A}b$ are known.

For more details in homological algebra see the books (Rotman, 1979), (Enochs and Jenda, 2000) and (Mac Lane, 1995). For modules and rings see the books (Anderson and Fuller, 1992), (Lam, 2001). For Abelian groups, see (Fuchs, 1970). For relative homological algebra, our main references are the books (Mac Lane, 1995) and (Enochs and Jenda, 2000) and the article (Skljarenko, 1978). We will explain most of the terms and summarize the necessary concepts.

2.1. Proper Classes

In this section we introduce the main subject of this work, namely proper classes. Proper classes were introduced by Buchsbaum in (Buchsbaum, 1959). They have been extensively studied in different contexts. We refer to (Mac Lane, 1995), (Mišina and Skornjakov, 1969), (Skljarenko, 1978), (Mermut, 2004) for complete surveys and further reading.

Let $\mathcal{P}$ be a class of short exact sequences of R-modules and R-module homomorphisms. If a short exact sequence

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0 \tag{2.1}$$

belongs to $\mathcal{P}$, then f is said to be a *$\mathcal{P}$-monomorphism* and g is said to be a *$\mathcal{P}$-epimorphism* (both are said to be *$\mathcal{P}$-proper* and the short exact sequence is said to be a *$\mathcal{P}$-proper* short exact sequence). The class $\mathcal{P}$ is said to be *proper* (in the sense of Buchsbaum) if it satisfies the following conditions (see (Buchsbaum, 1959), (Mac Lane, 1995, Ch. 12, §4), (Skljarenko, 1978)):

P-1) If a short exact sequence $\mathbb{E}$ is in $\mathcal{P}$, then $\mathcal{P}$ contains every short exact sequence isomorphic to $\mathbb{E}$.

P-2) $\mathcal{P}$ contains all splitting short exact sequences.

P-3) The composite of two $\mathcal{P}$-monomorphisms is a $\mathcal{P}$-monomorphism if this composite is defined.

P-3') The composite of two $\mathcal{P}$-epimorphisms is a $\mathcal{P}$-epimorphism if this composite is defined.

P-4) If g and f are monomorphisms, and $g \circ f$ is a $\mathcal{P}$-monomorphism, then f is a $\mathcal{P}$-monomorphism.

P-4') If g and f are epimorphisms, and $g \circ f$ is a $\mathcal{P}$-epimorphism, then g is a $\mathcal{P}$-epimorphism.

One of the most important examples of proper classes in Abelian groups is $\mathcal{P}ure_{\mathbb{Z}\text{-}\mathcal{M}od}$. It is the class of all short exact sequences (2.1) of Abelian groups and Abelian group homomorphisms such that $\text{Im}(f)$ is a pure subgroup of B, where a subgroup A of a group B is *pure* in B if $A \cap nB = nA$ for all integers n (see (Fuchs, 1970, §26-30) for the important notion of purity in Abelian groups).

The smallest proper class of R-modules consists of only *splitting* short exact sequences of R-modules which we denote by $\mathcal{S}plit_{R\text{-}\mathcal{M}od}$. The largest proper class of R-modules consists of *all* short exact sequences of R-modules which we denote by $\mathcal{A}bs_{R\text{-}\mathcal{M}od}$.

For a proper class $\mathcal{P}$ of R-modules, call a submodule A of a module B a $\mathcal{P}$-submodule of B, if the inclusion monomorphism $i_A : A \to B$, $i_A(a) = a$, $a \in A$, is a $\mathcal{P}$-monomorphism.

2.1.1. Homological Objects of Proper Classes

In a proper class $\mathcal{P}$ in R-$\mathcal{M}od$, there need not be a $\mathcal{P}$-epimorphism from some $\mathcal{P}$-projective module to a given R-module A. For this reason, in general, it is not possible to define the functor $\text{Ext}^1_{\mathcal{P}}$ by using the derived functor of the functor Hom. However, the alternative definition of $\text{Ext}^1_{\mathcal{P}}$ may be used in this case.

For a proper class $\mathcal{P}$ and R-modules A, C, denote by $\text{Ext}^1_{\mathcal{P}}(C, A)$ or shortly by $\text{Ext}_{\mathcal{P}}(C, A)$, the equivalence classes of all short exact sequences in $\mathcal{P}$ which start with A and end with C. This turns out to be a subgroup of $\text{Ext}_R(C, A)$ and a bifunctor $\text{Ext}^1_{\mathcal{P}}$: $R\text{-}\mathcal{M}od \times R\text{-}\mathcal{M}od \longrightarrow \mathcal{A}b$ is obtained which is a subfunctor of Ext^1_R.

Take a short exact sequence

$$\mathbb{E} : 0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

of R-modules and R-module homomorphisms.

An R-module M is said to be *projective with respect to the short exact sequence* $\mathbb{E}$, or *with respect to the epimorphism g* if any of the following equivalent conditions holds:

1. every diagram

$$\begin{array}{ccccccccc} \mathbb{E}: & 0 & \longrightarrow & A & \xrightarrow{f} & B & \xrightarrow{g} & C & \longrightarrow 0 \\ & & & & & & \nwarrow^{\tilde{\gamma}} & \uparrow \gamma & \\ & & & & & & & M & \end{array}$$

   where the first row is $\mathbb{E}$ and $\gamma : M \longrightarrow C$ is an R-module homomorphism which can be embedded in a commutative diagram by choosing an R-module homomorphism $\tilde{\gamma} : M \longrightarrow B$; that is, for every homomorphism $\gamma : M \longrightarrow C$, there exits a homomorphism $\tilde{\gamma} : M \longrightarrow B$ such that $g \circ \tilde{\gamma} = \gamma$.

2. The sequence

$$\text{Hom}(M, \mathbb{E}) : \quad 0 \longrightarrow \text{Hom}(M, A) \xrightarrow{f_*} \text{Hom}(M, B) \xrightarrow{g_*} \text{Hom}(M, C) \longrightarrow 0$$

   is exact.

Dually, an R-module M is said to be *injective with respect to the short exact sequence* $\mathbb{E}$, or *with respect to the monomorphism* f if any of the following equivalent conditions holds:

1. every diagram

$$\begin{array}{rcccccccc} \mathbb{E}: & 0 & \longrightarrow & A & \xrightarrow{f} & B & \xrightarrow{g} & C & \longrightarrow 0 \\ & & & {\scriptstyle\alpha}\downarrow & \swarrow_{\tilde{\alpha}} & & & & \\ & & & M & & & & & \end{array}$$

   where the first row is $\mathbb{E}$ and $\alpha : A \longrightarrow M$ is an R-module homomorphism which can be embedded in a commutative diagram by choosing an R-module homomorphism $\tilde{\alpha} : B \longrightarrow M$; that is, for every homomorphism $\alpha : A \longrightarrow M$, there exists a homomorphism $\tilde{\alpha} : B \longrightarrow M$ such that $\tilde{\alpha} \circ f = \alpha$.

2. The sequence

$$\mathrm{Hom}(\mathbb{E}, M): \quad 0 \longrightarrow \mathrm{Hom}(C, M) \xrightarrow{g^*} \mathrm{Hom}(B, M) \xrightarrow{f^*} \mathrm{Hom}(A, M) \longrightarrow 0$$

   is exact.

An R-module M is said to be $\mathcal{P}$*-projective* [$\mathcal{P}$*-injective*] if it is projective [injective] with respect to all short exact sequences in $\mathcal{P}$. The relative projectiveness [injectiveness] of M is equivalent to the requirement that $\mathrm{Ext}^1_{\mathcal{P}}(M, B) = 0$, for any B [$\mathrm{Ext}^1_{\mathcal{P}}(A, M) = 0$, for any A]. Denote all $\mathcal{P}$-projective [$\mathcal{P}$-injective] modules by $\pi(\mathcal{P})$ [$\iota(\mathcal{P})$].

A class $\mathcal{P}$ of R-modules is said to have *enough projectives* if for every module A we can find a $\mathcal{P}$-epimorhism from some $\mathcal{P}$-projective module P to A. A class $\mathcal{P}$ of R-modules is said to have *enough injectives* if for every module B we can find a $\mathcal{P}$-monomorphism from B to some $\mathcal{P}$-injective module J. A proper class $\mathcal{P}$ of R-modules with enough projectives [enough injectives] is also said to be a *projective proper class* [resp. *injective proper class*].

Definition 2.1 *An R -module C is said to be $\mathcal{P}$-flat if every short exact sequence of R-*

modules and R-module homomorphisms of the form

$$\mathbb{E} : 0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

ending with C is in the proper class $\mathcal{P}$.

Definition 2.2 *An R-module A is said to be $\mathcal{P}$-divisible if every short exact sequence of R-modules and R-module homomorphisms of the form*

$$\mathbb{E} : 0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

starting with A is in the proper class $\mathcal{P}$.

Using the functor $\mathrm{Ext}_{\mathcal{P}}$, the $\mathcal{P}$-projectives, $\mathcal{P}$-injectives, $\mathcal{P}$-flats, $\mathcal{P}$-divisibles are simply described in terms of the subgroup $\mathrm{Ext}_{\mathcal{P}}(C, A) \leq \mathrm{Ext}_R(C, A)$ being 0 or the whole of $\mathrm{Ext}_R(C, A)$:

1. An R-module C is *$\mathcal{P}$-projective* if and only if
$$\mathrm{Ext}_{\mathcal{P}}(C, A) = 0 \text{ for all } R\text{-modules } A.$$

2. An R-module C is *$\mathcal{P}$-flat* if and only if
$$\mathrm{Ext}_{\mathcal{P}}(C, A) = \mathrm{Ext}_R(C, A) \text{ for all } R\text{-modules } A.$$

3. An R-module A is *$\mathcal{P}$-injective* if and only if
$$\mathrm{Ext}_{\mathcal{P}}(C, A) = 0 \text{ for all } R\text{-modules } C.$$

4. An R-module A is *$\mathcal{P}$-divisible* if and only if
$$\mathrm{Ext}_{\mathcal{P}}(C, A) = \mathrm{Ext}_R(C, A) \text{ for all } R\text{-modules } C.$$

Proposition 2.1 ((Mišina and Skornjakov, 1969), Propositions 1.9 and 1.14) *Let $0 \to M \to N \to K \to 0$ be a short exact sequence of R-modules. If M and K are $\mathcal{P}$-flat ($\mathcal{P}$-divisible), then N is $\mathcal{P}$-flat ($\mathcal{P}$-divisible).*

Proposition 2.2 ((Mišina and Skornjakov, 1969), Proposition 1.12) *An R-module M is $\mathcal{P}$-flat if and only if there is a $\mathcal{P}$-epimorphism from a projective R-module P to M.*

Corollary 2.1 ((Mišina and Skornjakov, 1969), Proposition 1.13) *If $0 \to A \to B \to C \to 0$ is a short exact sequence in a proper class $\mathcal{P}$ and B is $\mathcal{P}$-flat, then C is also $\mathcal{P}$-flat.*

Dually, for $\mathcal{P}$-divisible modules we have the following :

Proposition 2.3 ((Mišina and Skornjakov, 1969), Proposition 1.7) *An R-module N is $\mathcal{P}$-divisible if and only if there is $\mathcal{P}$-monomorphism from N to an injective module I.*

Corollary 2.2 ((Mišina and Skornjakov, 1969), Proposition 1.8) *If $0 \to A \to B \to C \to 0$ is a short exact sequence in a proper class $\mathcal{P}$ and B is $\mathcal{P}$-divisible, then A is also $\mathcal{P}$-divisible.*

2.1.2. Projectively Generated Proper Classes

For a given class $\mathcal{M}$ of modules, denote by $\pi^{-1}(\mathcal{M})$ the class of all short exact sequences $\mathbb{E}$ of R-modules and R-module homomorphisms such that $\text{Hom}(M, \mathbb{E})$ is exact for all $M \in \mathcal{M}$, that is,

$$\pi^{-1}(\mathcal{M}) = \{\mathbb{E} \in \mathcal{A}bs_{R\text{-}\mathcal{M}od} | \,\text{Hom}(M, \mathbb{E}) \text{ is exact for all } M \in \mathcal{M}\}.$$

$\pi^{-1}(\mathcal{M})$ is the largest proper class $\mathcal{P}$ for which each $M \in \mathcal{M}$ is $\mathcal{P}$-projective and it is called the proper class *projectively generated* by $\mathcal{M}$.

For a proper class $\mathcal{P}$, the *projective closure* of $\mathcal{P}$ is the proper class $\pi^{-1}(\pi(\mathcal{P}))$ which contains $\mathcal{P}$. If the projective closure of $\mathcal{P}$ is equal to itself, then it is said to be *projectively closed*, and that occurs if and only if it is projectively generated.

Proposition 2.4 *(Skljarenko, 1978, Proposition 1.1) Every projective proper class is projectively generated.*

Let $\mathcal{P}$ be a proper class of R-modules. Direct sums of $\mathcal{P}$-projective modules is $\mathcal{P}$-projective. Direct summand of an $\mathcal{P}$-projective module is $\mathcal{P}$-projective.

A proper class $\mathcal{P}$ is called $\prod$-*closed* if for *every* collection $\{\mathbb{E}_\lambda\}_{\lambda\in\Lambda}$ in $\mathcal{P}$, the product $\mathbb{E} = \prod_{\lambda\in\Lambda} \mathbb{E}_\lambda$ is in $\mathcal{P}$, too.

Proposition 2.5 *(Skljarenko, 1978, Proposition 1.2) Every projectively generated proper class is $\prod$-closed.*

A subclass $\mathcal{M}$ of a class $\overline{\mathcal{M}}$ of modules is called a *projective basis* for $\overline{\mathcal{M}}$ if every module in $\overline{\mathcal{M}}$ is a direct summand of a direct sum of modules in $\mathcal{M}$ and of free modules.

Proposition 2.6 *(Skljarenko, 1978, Proposition 2.1) If $\mathcal{M}$ is a set, then the proper class $\pi^{-1}(\mathcal{M})$ is projective, and $\mathcal{M}$ is a projective basis for the class of all $\mathcal{P}$-projective modules.*

Even when $\mathcal{M}$ is not a set but:

Proposition 2.7 *(Skljarenko, 1978, Proposition 2.3) If $\mathcal{M}$ is a class of modules closed under passage to factor modules, then the proper class $\pi^{-1}(\mathcal{M})$ is projective, and $\mathcal{M}$ is a projective basis for the class of all $\mathcal{P}$-projective modules.*

2.1.3. Injectively Generated Groper Classes

For a given class $\mathcal{M}$ of modules, denote by $\iota^{-1}(\mathcal{M})$ the class of all short exact sequences $\mathbb{E}$ of R-modules and R-module homomorphisms such that $\mathrm{Hom}(\mathbb{E}, M)$ is exact for all $M \in \mathcal{M}$, that is,

$$\iota^{-1}(\mathcal{M}) = \{\mathbb{E} \in \mathcal{A}bs_{R\text{-}\mathcal{M}od} | \, \mathrm{Hom}(\mathbb{E}, M) \text{ is exact for all } M \in \mathcal{M}\}.$$

$\iota^{-1}(\mathcal{M})$ is the largest proper class $\mathcal{P}$ for which each $M \in \mathcal{M}$ is $\mathcal{P}$-injective which is called the proper class *injectively generated* by $\mathcal{M}$. For a proper class $\mathcal{P}$, the *injective closure* of $\mathcal{P}$ is the proper class $\iota^{-1}(\iota(\mathcal{P}))$ which contains $\mathcal{P}$. If the injective closure of $\mathcal{P}$ is equal to itself, then it is said to be *injectively closed*, and that occurs if and only if it is injectively generated.

Proposition 2.8 *(Skljarenko, 1978, Proposition 3.1) Every injective proper class is injectively generated.*

Let $\mathcal{P}$ be a proper class of R-modules. Direct product of $\mathcal{P}$-injective modules is $\mathcal{P}$-injective. Direct summand of an $\mathcal{P}$-injective module is $\mathcal{P}$-injective.

A proper class $\mathcal{P}$ is called $\oplus$-*closed* if for *every* collection $\{\mathbb{E}_\lambda\}_{\lambda\in\Lambda}$ in $\mathcal{P}$, the direct sum $\mathbb{E} = \bigoplus_{\lambda\in\Lambda} \mathbb{E}_\lambda$ is in $\mathcal{P}$, too.

Proposition 2.9 *(Skljarenko, 1978, Proposition 1.2) Every injectively generated proper class is $\oplus$-closed.*

An injective module is called *elementary* if it coincides with the injective envelope of some *cyclic* submodule. Such modules form a set and every injective module can be embedded in a direct product of elementary injective modules (Skljarenko, 1978, Lemma 3.1).

A subclass $\mathcal{M}$ of a class $\overline{\mathcal{M}}$ of modules is called an *injective basis* for $\overline{\mathcal{M}}$ if every module in $\overline{\mathcal{M}}$ is a direct summand of a direct product of modules in $\mathcal{M}$ and of certain elementary injective modules.

Proposition 2.10 *(Skljarenko, 1978, Proposition 3.3) If $\mathcal{M}$ is a set, then the proper class $\iota^{-1}(\mathcal{M})$ is injective, and $\mathcal{M}$ is an injective basis for the class of all $\mathcal{P}$-injective modules.*

Even when $\mathcal{M}$ is not a set but:

Proposition 2.11 *(Skljarenko, 1978, Proposition 3.4) If $\mathcal{M}$ is a class of modules closed under taking submodules, then the proper class $\iota^{-1}(\mathcal{M})$ is injective, and $\mathcal{M}$ is an injective basis for the class of all $\mathcal{P}$-injective modules.*

2.1.4. Flatly Generated Proper Classes

When the ring R is *not* commutative, we must be careful with the sides for the tensor product analogues of projectives and injectives with respect to a proper class. Remember that by an R-module, we mean a *left* R-module.

Take a short exact sequence

$$\mathbb{E} : 0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

of R-modules and R-module homomorphisms. We say that a *right* R-module M is *flat with respect to the short exact sequence* $\mathbb{E}$, or *with respect to the monomorphism g* if

$$M \otimes \mathbb{E} : \quad 0 \longrightarrow M \otimes A \xrightarrow{1_m \otimes f} M \otimes B \xrightarrow{1_m \otimes g} M \otimes C \longrightarrow 0$$

is exact.

For a given class $\mathcal{M}$ of *right* R-modules, denote by $\tau^{-1}(\mathcal{M})$ the class of all short exact sequences $\mathbb{E}$ of R-modules and R-module homomorphisms such that $M \otimes \mathbb{E}$ is exact for all $M \in \mathcal{M}$:

$$\tau^{-1}(\mathcal{M}) = \{\mathbb{E} \in \mathcal{A}bs_{R\text{-}\mathcal{M}od} | M \otimes \mathbb{E} \text{ is exact for all } M \in \mathcal{M}\}.$$

$\tau^{-1}(\mathcal{M})$ is called the proper class *flatly generated* by the class $\mathcal{M}$ of *right* R-modules.

When the ring R is commutative, there is no need to mention the sides of the modules since a right R-module may also be considered as a left R-module and vice versa.

Theorem 2.1 *(Lam, 2001, Theorem 4.89) Let* $\mathbb{E} : 0 \to A \to B \to C \to 0$ *be a short exact sequence of left R-modules. Then the following are equivalent:*

(1) $\mathbb{E}$ *is the direct limit of the direct system of split exact sequences* $0 \to A_i \to B_i \to C_i \to 0$ $(i \in I)$*, where the* C_i*'s are finitely presented left R-modules.*

(2) The sequence $0 \to M \otimes A \to M \otimes B \to M \otimes C \to 0$ *is exact for any (finitely presented) right R-module M.*

(3) The sequence $0 \to \mathrm{Hom}(M, A) \to \mathrm{Hom}(M, B) \to \mathrm{Hom}(M, C) \to 0$ *is exact for any finitely presented left R-module M.*

The class of all short exact sequences $\mathbb{E}$ which satisfies one of the equivalent conditions of Theorem 2.1 is called the Cohn purity and is denoted by $\mathcal{P}ure$. Clearly, by Theorem 2.1, $\mathcal{P}ure = \pi^{-1}({}_R\mathcal{FP}) = \tau^{-1}(\mathcal{FP}_R) = \tau^{-1}(\mathcal{M}od\text{-}R)$, where $\mathcal{FP}_R$ (resp. ${}_R\mathcal{FP}$) is the class of all finitely presented right (resp. left) R-modules.

2.1.5. Auslander- Bridger Transpose

For an R-module M, the character module $\mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})$ is denoted by M^+, the dual module $\mathrm{Hom}_R(M, R)$ is denoted by M^*, and $\delta_M : M \to M^{**}$ stands for the evaluation map. M is said to be *torsionless* if δ_M is a monomorphism.

Let M be a *finitely presented* R-module, that is, $M \cong F/G$ for some *finitely generated free* R-module F and some *finitely generated* submodule G of F. So, we have a short exact sequence

$$0 \longrightarrow G \longrightarrow F \longrightarrow M \longrightarrow 0$$

Any short exact sequence

$$0 \longrightarrow H \longrightarrow F' \longrightarrow M \longrightarrow 0$$

where F' is a *finitely generated free* module and H is a *finitely generated* module is called a *free presentation* of M. An exact sequence

$$F_1 \longrightarrow F_0 \longrightarrow M \longrightarrow 0$$

where F_0 and F_1 are *finitely generated free* modules is also called a *free presentation* of M. If we apply the functor $\mathrm{Hom}_R(., R)$ to this presentation, we obtain the sequence

$$0 \to M^* \to F_0^* \to F_1^* \to Tr(M) \to 0$$

where $Tr(M)$ is the cokernel of the dual map $F_0^* \to F_1^*$. Note that $Tr(M)$ is a finitely presented right R-module. The right R-module $Tr(M)$ is called an *Auslander-Bridger Transpose* of the left R-module M.

Note that the correspondence $M \mapsto Tr(M)$ is not one-to-one, since it depends on the presentation of M. Also M can be interpreted as $Tr(Tr(M))$ (by taking a free presentation of F). See (Skljarenko, 1978, §5).

Proposition 2.12 *(Skljarenko, 1978, Corollary 5.1) For any finitely presented R-module M and any short exact sequence* $\mathbb{E}$ *of R-modules, the sequence* $\mathrm{Hom}(M, \mathbb{E})$ *is exact if and only if the sequence* $Tr(M) \otimes \mathbb{E}$ *is exact.*

Theorem 2.2 *(Skljarenko, 1978, Theorem 8.3) Let* $\mathcal{M}$ *be a set of finitely presented R-modules. Associate with each* $F \in \mathcal{M}$*, the right R-module* $Tr(F)$ *and let* $Tr(\mathcal{M})$ *be the set of all these* $Tr(F)$*. We may assume that* $Tr(Tr(\mathcal{M})) = \mathcal{M}$*. Then*

$$\pi^{-1}(\mathcal{M}) = \tau^{-1}(Tr(\mathcal{M})) \quad \textit{and} \quad \tau^{-1}(\mathcal{M}) = \pi^{-1}(Tr(\mathcal{M}))$$

Proposition 2.13 *(Skljarenko, 1978, Lemma 5.1) For any short exact sequence* $\mathbb{E}$ *of R-modules, any* right *R-module M, the sequence* $M \otimes \mathbb{E}$ *is exact if and only if the sequence*

$\mathrm{Hom}(M, \mathbb{E}^+)$ *is exact.*

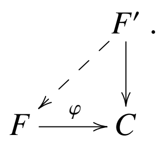
A proper class $\mathcal{P}$ is said to be *inductively closed* if for every direct system $\{\mathbb{E}_i (i \in I); \pi_i^j (i \leq j)\}$ in $\mathcal{P}$, the direct limit $\mathbb{E} = \varinjlim \mathbb{E}_i$ is also in $\mathcal{P}$ (see (Fedin, 1983) and (Skljarenko, 1978, §8)). As in (Fedin, 1983), for a proper class $\mathcal{P}$, denote by $\widetilde{\mathcal{P}}$, the smallest inductively closed proper class containing $\mathcal{P}$; it is called the *inductive closure* of $\mathcal{P}$.

Since tensor product and direct limit commute, a flatly generated proper class is inductively closed; moreover:

Theorem 2.3 *(Skljarenko, 1978, Theorem 8.1) For a given class $\mathcal{M}$ of* right *R-modules, the proper class $\tau^{-1}(\mathcal{M})$ is inductively closed. It is injectively generated, and if $\mathcal{M}$ is a set, then it is an injective proper class. A short exact sequence $\mathbb{E}$ belongs to $\tau^{-1}(\mathcal{M})$ if and only if $\mathbb{E}^+ \in \pi^{-1}(\mathcal{M})$.*

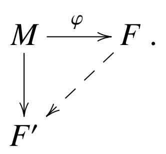
2.2. Covers and Envelopes

Given a class $\mathscr{F}$ of objects in an Abelian category $\mathcal{A}b$, recall from (Enochs, 1981) that, an $\mathscr{F}$-precover of an object C is a morphism $\varphi : F \longrightarrow C$ with $F \in \mathscr{F}$ such that $\mathrm{Hom}_{\mathcal{A}b}(F', F) \longrightarrow \mathrm{Hom}_{\mathcal{A}b}(F', C) \longrightarrow 0$ is exact for every $F' \in \mathscr{F}$, that is, the following diagram commutes:



If, moreover, every morphism $f : F \longrightarrow F$ such that $\varphi f = \varphi$ is an automorphism, then φ is said to be an $\mathscr{F}$-cover.

Dually, an *$\mathscr{F}$-preenvelope* of M is a morphism $\varphi : M \longrightarrow F$ with $F \in \mathscr{F}$ such that $\mathrm{Hom}_{\mathcal{A}b}(F, F') \longrightarrow \mathrm{Hom}_{\mathcal{A}b}(M, F')$ is surjective for every $F' \in \mathscr{F}$, that is, the following diagram commutes:



An $\mathscr{F}$-preenvelope φ is said to be an *$\mathscr{F}$-envelope* if every endomorphism $f : F \longrightarrow F$ such that $f\varphi = \varphi$ is an automorphism.

So, for instance, if we take $\mathscr{F}$ to be the class of all flat modules, then a flat cover of a module will be an $\mathscr{F}$-cover.

The study of covers and envelopes started in 1953, when Eckman and Schopf proved that each module over an associative ring has an injective envelope (Eckmann and Schopf, 1953). On the other hand, Bass characterized rings over which every module has a projective cover: perfect rings (Bass, 1960). Enochs studied torsion free covers and proved the existence of torsion free covers of modules over a commutative domain (Enochs, 1963). In the arguments after (Enochs and Jenda, 2000, Definition 5.1.1), it has been pointed out that torsion free covers and $\mathscr{F}$-covers coincide over a commutative domain R, where $\mathscr{F}$ is the class of torsion free R-modules. Moreover, in 1981, Enochs conjectured that every module over an associative ring admits a flat cover (Enochs, 1981). This is known as the "flat cover conjecture". In the same paper, he noticed the categorical version of injective cover, and then gave a general definition of covers and envelopes in terms of commutative diagrams, for a given class of modules. Independently, this definition of covers and envelopes were given by Auslander and Smalø in terms of *minimal left and right approximations* (Auslander and Smalø, 1980). Enochs gave the general definition for a class of modules over arbitrary rings while Auslander and Smalø considered finitely generated modules over finite dimensional algebras. The main idea for studying covers and envelopes is to use certain aspects of a special class of modules, or more generally objects to study entire category. Because, once we understand the structure of a class of objects, we may approximate arbitrary objects by the objects from this class. In 2001, once the "flat cover conjecture" has been proved in (Bican et al., 2001), in a natural way, flat covers and covers by more general classes of objects have been studied in more general settings than that of modules.

The proofs of the following elementary properties of $\mathcal{F}$-covers and $\mathcal{F}$-envelopes can be found, for example, in (Xu, 1996, §1.2) for module categories, but the same argument of the proofs carry over to Abelian categories. Suppose that $\mathcal{F}$ is closed under isomorphisms, direct summands and under finite direct sums.

If an $\mathcal{F}$-cover exists, then it is unique up to isomorphism:

Proposition 2.14 *If $\varphi_1 : F_1 \longrightarrow M$ and $\varphi_2 : F_2 \longrightarrow M$ are two different $\mathcal{F}$-covers of an object M, then $F_1 \cong F_2$.*

Also, $\mathcal{F}$-covers are direct summands of $\mathcal{F}$-precovers:

Proposition 2.15 *Suppose that an object M admits an $\mathcal{F}$-cover, and that $\varphi : F \longrightarrow M$ is an $\mathcal{F}$-precover. Then $F = F_1 \oplus K$ for subobjects F_1 and K of F such that the restriction $\varphi\,|_{F_1} : F_1 \longrightarrow M$ is an $\mathcal{F}$-cover of M and $K \subseteq \mathrm{Ker}(\varphi)$.*

We have the dual results for $\mathcal{F}$-envelopes, that is, if an $\mathcal{F}$-envelope exists then it is unique up to isomorphism, and $\mathcal{F}$-envelopes are direct summand of $\mathcal{F}$-preenvelopes.

The following result is useful while proving whether a class of modules is (pre)enveloping or (pre)covering.

Lemma 2.1 (1) *(Rada and Saorin, 1998, Corollary 3.5(c)) If a class $\mathcal{M}$ of modules over a ring is closed under pure submodules, then $\mathcal{M}$ is preenveloping if and only if it is closed under direct products.*

(2) *(Holm and Jørgensen, 2008, Theorem 2.5)If a class $\mathcal{M}$ of modules over a ring is closed under pure quotients, then $\mathcal{M}$ is precovering if and only if it is covering if and only if it is closed under direct sums.*

2.3. Closed Submodules

In this section we will give definitions and some properties of the proper classes generated by simple modules, which are generalizations of closed submodules in different ways.

2.3.1. Complement Submodules

Let B be an R-module. A submodule A of B is called a *complement of K in B* and K is said to *have a complement in B* if $K \cap A = 0$ and A is *maximal* with respect to this property (that is there is *no* submodule $\tilde{A}$ of B such that $\tilde{A} \supsetneq A$ but still $K \cap \tilde{A} = 0$). By Zorn's Lemma, it is seen that K has always a complement in B. In fact, by Zorn's Lemma, we know that if we have a submodule A' of B such that $A' \cap K = 0$, then there exists a complement A of K in B such that $A \supseteq A'$. See the monograph (Dung and Wisbauer, 1994) for a survey of results in the related concepts. A submodule A of a module B is said to be a *complement in B* if A is a complement of some submodule of B; shortly, we also say that A is a *complement submodule of B* in this case and denote this by $A \leq_c B$.

2.3.2. Closed and Neat Submodules

A submodule A of a module B is said to be *closed in B* if A has *no* proper essential extension in B, that is, there exists *no* submodule $\tilde{A}$ of B such that $A \subsetneq \tilde{A}$ and A is *essential* in $\tilde{A}$ (which is denoted by $A \trianglelefteq \tilde{A}$ and meaning that for every *nonzero* submodule X of $\tilde{A}$, we have $A \cap X \neq 0$). We also say in this case that A is a *closed submodule* and it is known that closed submodules and complement submodules in a module coincide (see (Dung and Wisbauer, 1994, §1)).

In order to characterize the closed subgroup in Abelian groups, in (Honda, 1956) introduced the concept of neat subgroups and proved several reminiscent of purity.

Definition 2.3 *A subgroup A of an Abelian group B is called neat in B if $pA = A \cap pB$ for every prime p.*

For a proof of the following result we refer to (Mermut, 2004, Theorem 4.1.1).

Theorem 2.4 *The following are equivalent for a subgroup A of an Abelian group B.*

(1) A is closed in B;

(2) A is neat in B;

(3) the sequence $0 \to \mathrm{Hom}(\mathbb{Z}/p\mathbb{Z}, A) \to \mathrm{Hom}(\mathbb{Z}/p\mathbb{Z}, B) \to \mathrm{Hom}(\mathbb{Z}/p\mathbb{Z}, B/A) \to 0$ *is exact for all primes p;*

(4) the sequence $0 \to \mathbb{Z}/p\mathbb{Z} \otimes A \to \mathbb{Z}/p\mathbb{Z} \otimes B \to \mathbb{Z}/p\mathbb{Z} \otimes B/A \to 0$ *is exact for all primes p;*

(5) the sequence $0 \to \mathrm{Hom}(B/A, \mathbb{Z}/p\mathbb{Z}) \to \mathrm{Hom}(B, \mathbb{Z}/p\mathbb{Z}) \to \mathrm{Hom}(A, \mathbb{Z}/p\mathbb{Z}) \to 0$ *is exact for all primes p.*

The definition of neatness can be extended to arbitrary rings R via (3) if we replace the groups $\mathbb{Z}/p\mathbb{Z}$ by simple R-modules S, (see (Renault, 1964)). Namely, a submodule N of R-module M is called *neat* in M if, for every simple R-module S, every homomorphism $f : S \to M/N$ can be lifted to a homomorphism $g : S \to M$. Equivalently, N is neat in M if and only if $\mathrm{Hom}(S, g) : \mathrm{Hom}(S, M) \to \mathrm{Hom}(S, M/N)$ is an epimorphism for every simple R-module S. Neat submodules have recently been studied in (Fuchs, 2012), (Crivei, 2014).

The class *Closed* consists of all short exact sequences

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0 \tag{2.2}$$

in R-$\mathcal{M}od$ such that A is a closed submodule of B. The class $\mathcal{N}eat$ consists of all short exact sequences (2.4) in R-$\mathcal{M}od$ such that every simple module is relative projective for it, where a left R-module S is called *simple* if it has no submodule except 0 and S, denoted by

$$\mathcal{N}eat = \pi^{-1}(\{S \in R\text{-}\mathcal{M}od | \text{S simple}\}).$$

Theorem 2.5 *(Stenström, 1967, Propositons 4-6) Let $\mathcal{A}$ be an Abelian category in which every object has an injective envelope. Then:*

1. *$Closed_{\mathcal{A}}$ and $\mathcal{N}eat_{\mathcal{A}}$ form proper classes.*
2. *$Closed_{\mathcal{A}} \subseteq \mathcal{N}eat_{\mathcal{A}}$.*

2.3.3. *s*-pure Submodules

Let A be a submodule of an R-module B. A is said to be s-pure submodule of B if the map $S \otimes A \rightarrow S \otimes B$ is a monic for every simple right R-module S. s-pure submodules was discussed by (Crivei, 2005), as well as by (Mermut et al., 2009). The definition corresponding to Honda's: $PA = A \cap PB$ for all maximal ideals P was adopted by (Mermut et al., 2009) where it was called P-purity.

The class $s-\mathcal{P}ure$ consists of all short exact sequences

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0 \tag{2.3}$$

in R-$\mathcal{M}od$ such that $f(A)$ is s-pure submodule of B. The class $s-\mathcal{P}ure$ is a flatly generated proper class by the set of all representative simple modules i.e., $s-\mathcal{P}ure = \tau^{-1}(\mathcal{S})$, where $\mathcal{S}$ is the set of all representative simple modules.

Proposition 2.16 *(Skljarenko, 1978, Lemma 6.1) Let A be a submodule of an R-module B and $i_A : A \hookrightarrow B$ be the inclusion map. For a* right *ideal I of R, $A \cap IB = IA$ if and only if*

$$R/I \otimes A \xrightarrow{1_{R/I} \otimes i_A} R/I \otimes B$$

is monic.

As a corollary of Proposition 2.16 we have:

Corollary 2.3 *Let B be an R-module and $A \leq B$. The following are equivalent*

(1) A is an s-pure submodule of B.

(2) $IA = A \cap IB$ for each maximal right ideal I of R.

2.3.4. Coneat Submodules

Dual of neat submodules, we say that a submodule N of an R-module M is *coneat* in M if $\mathrm{Hom}(M, S) \to \mathrm{Hom}(N, S) \to 0$ is epic for every simple R-module S. The notions of neat s-pure and coneat coincide over the ring of integers by Theorem 2.4. By (Fuchs, 2012, Theorem 5.2), the commutative domains over which neat and coneat submodules coincide are exactly the domains with finitely generated maximal ideals (i.e. N-domains). This result recently extended to certain commutative rings by Crivei (2014).

The class *Coneat* consists of all short exact sequences

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0 \qquad (2.4)$$

in R-$\mathcal{M}od$ such that $f(A)$ is coneat submodule of B. The class *Coneat* is an injectively generated proper class by the set of all representative simple modules i.e., $Coneat = \iota^{-1}(\mathcal{S})$ where $\mathcal{S}$ is the set of all representative simple modules.

For a proof of the following we refer to (Fuchs, 2012, Proposition 3.1) or (Mermut et al., 2009, Corollary 2.5).

Theorem 2.6 *Let R be a commutative ring. Then $s - \mathcal{P}ure = Coneat$.*

An integral domain R is said to be an N-domain if for R-modules, the concepts neatness and coneatness coincide (see (Fuchs, 2012)).

Theorem 2.7 *(Fuchs, 2012, Theorem 5.2) Let R be a commutative domain. R is an N-domain if and only if all the maximal ideals of R are (finitely generated) projective modules, i.e. they are invertible.*

S. Crivei proved that neat and coneat submodules of a module coincide when R is a commutative ring such that every maximal ideal is principal, see (Crivei, 2014, Theorem 2.1).

2.3.5. Coclosed Submodules

A submodule K of M is called *small* in M (denoted by $K \ll M$) if $M \neq K + T$ for every proper submodule T of M. A submodule $L \leq M$ is called *coclosed in M* if $L/N \ll M/N$ implies $L = N$, for every $N \leq L$. For more detailed about *coclosed submodules*, see (Clark et al., 2006, §3).

Proposition 2.17 *(Clark et al., 2006, 3.7) Let $K \leq L \leq M$ be submodules. Then the following hold.*

(1) If L is coclosed in M then L/K is coclosed in M/K.

(2) If $K \ll L$ and L/K is coclosed in M/K then L is coclosed in M.

(3) If $L \leq M$ is coclosed, then $K \ll M$ implies $K \ll L$; hence $\mathrm{Rad}(L) = L \cap \mathrm{Rad}(M)$.

(4) If $f : M \to N$ is a small epimorphism and L is coclosed in M, then $f(L)$ is coclosed in N.

(5) If K is coclosed in M, then K is coclosed in L and the converse is true if L is coclosed in M.

Proposition 2.18 *(Zöschinger, 2006, Lemma A.4) Let $K \leq L \leq M$ be submodules of M. If K is coclosed in M and L/K is coclosed in M/K, then L is coclosed in M.*

The class *Coclosed* consists of all short exact sequences

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0 \tag{2.5}$$

in R-$\mathcal{M}od$ such that $f(A)$ is coclosed submodule of B. The class $Coclosed$ is a proper class in the sense of Buchsbaum by Proposition 3.2 and Proposition **??**.

2.4. Modules Generated by Simple Modules

In this section definitions and some properties of modules which are generated by simple modules are given.

2.4.1. Weakly-injective Modules and Weakly-flat Modules

Definition 2.4 *A right R-module M is said to be weakly-injective if, for every extension $M \leq X$, M is coclosed in X.*

Weakly-injective modules introduced in (Zöschinger, 2006). By Definition 2.2, we have:

Proposition 2.19 *Let M be a right R-module. Then M is weakly-injective if and only if M is $Coclosed$-divisible.*

Weakly-injective modules studied recently in (Zöschinger, 2008) and (Zöschinger, 2011).

Definition 2.5 *A right R-module M is said to be weakly-flat if the kernel of any epimorphism $Y \to M \to 0$ is closed in Y.*

Weakly-flat modules introduced in (Zöschinger, 2013). By Definition 2.1, we have:

Proposition 2.20 *Let M be a right R-module. Then M is weakly-flat if and only if M is $Closed$-flat.*

2.4.2. m-injective Modules

Definition 2.6 *A right R-module M is said to be m-injective if, for any maximal right ideal I of R, any homomorphism $I \to M$ can be extended to a homomorphism $R \to M$.*

Proposition 2.21 *The following are equivalent for a right R-module M:*

(1) M is right m-injective.

(2) M is a neat submodule of an m-injective R-module.

(3) M is a neat submodule of every module containing it.

(4) $\mathrm{Ext}_R(S, M) = 0$ *for every simple right R-module S .*

Proof (1) ⇔ (4) Let I be a right ideal of R. Then applying $\mathrm{Hom}(-, M)$ to the short exact sequence $0 \to I \xrightarrow{i} R \to R/I \to 0$, we get $0 \to \mathrm{Hom}(R/I, M) \to \mathrm{Hom}(R, M) \xrightarrow{i^*} \mathrm{Hom}(I, M) \to \mathrm{Ext}^1(R/I, M) \to \mathrm{Ext}^1(R, M) = 0$. Then i^* is epic if and only if $\mathrm{Ext}^1(R/I, M) = 0$.

(2) ⇔ (3) By (Crivei, 2014, Theorem 3.3).

(3) ⇔ (4) By (Crivei, 2014, Theorem 3.4.). □

By Proposition 2.21 and Definition 2.2, we have:

Proposition 2.22 *Let M be a right R-module. Then M is weakly-injective if and only if M is* $\mathcal{N}$*eat-divisible.*

In (Crivei, 2014), a right R-module M is called absolutely neat if M is a neat submodule of any module containing it. By Proposition 2.21, a right R-module M is absolutely neat if and only if M is m-injective. Note that, m-injective modules are called max-injective in (Wang and Zhao, 2005) and (Xiang, 2010).

Theorem 2.8 *(Crivei, 1998, Theorem 3) Let E be a non-zero injective R-module and let* $0 \neq D \trianglelefteq E$. *Then D is m-injective if and only if* $\mathrm{Soc}(E/D) = 0$.

Definition 2.7 *A ring R is called a right C-ring if R/E has non-zero socle for every proper essential right ideal E of R.*

Remark 2.1 *The notion of* C*-ring has been introduced by (Renault, 1964). Left perfect rings and right semiartinian rings are examples of right C-rings. A commutative domain R is called almost perfect if R/I is a perfect ring for each nonzero ideal I of R. It is clear that almost perfect domains are C-rings. In (Salce, 2011), the authors prove that if R is an almost perfect domain then an R-module M is injective if and only if M is m-injective. Actually, one of the characterizations of right C-rings is the following: R is a right C-ring if and only if every m-injective right R-module is injective (see, (Smith, 1981, Lemma 4)).*

Theorem 2.9 *(Generalov, 1978, Theorem 5) For a ring R,*

$$\mathit{Closed}_{R\text{-}\mathcal{M}od} = \mathcal{N}eat_{R\text{-}\mathcal{M}od} \qquad \text{if and only if} \qquad R \text{ is a left } C\text{-ring.}$$

CHAPTER 3

ON CONEAT SUBMODULES

In this section several characterizations and some properties of coneat submodules are given. Recall that a submodule N of an R-module M is called *coneat* if for every simple R-module S, any homomorphism $N \to S$ can be extended to a homomorphism $M \to S$.

3.1. Characterization and Closure Properties of Coneat Submodules

Coneat submodules can be characterized as follows.

Proposition 3.1 *For a submodule $N \leq M$ the following are equivalent.*

(1) N is a coneat submodule of M.

(2) If $K \leq N$ with N/K finitely generated and $N/K \ll M/K$, then $K = N$.

(3) For any maximal submodule K of N, N/K is a direct summand of M/K.

(4) If K is a maximal submodule of N, then there exists a maximal submodule L of M such that $K = N \cap L$.

Proof

(1) $\Rightarrow$ (4) Let K be a maximal submodule of N and $\pi : N \to N/K$ the canonical epimorphism. By the hypothesis, there exists a homomorphism $f : M \to N/K$ such that $f|_N = \pi$. Then $\operatorname{Ker} f$ is a maximal submodule of M and $N + \operatorname{Ker} f = M$, so that $N \cap \operatorname{Ker} f$ is a maximal submodule of N. Then $\pi(N \cap \operatorname{Ker} f) = f(N \cap \operatorname{Ker} f) = 0$. Therefore $K = N \cap \operatorname{Ker} f$.

(3) $\Rightarrow$ (1) Let S be a simple right R-module and $f : N \to S$ a nonzero homomorphism. Since f is an epimorphism, without loss of generality we may assume that $S = N/K$ for some maximal submodule K of N, so that $\operatorname{Ker} f$ is a maximal submodule of N. Then, by (3), $M/\operatorname{Ker} f = (N/\operatorname{Ker} f) \oplus (L/\operatorname{Ker} f)$ for some $L \leq M$. Let $\tilde{f} : N/\operatorname{Ker} f \to N/K$ be the isomorphism induced by f. Consider the canonical epimorphisms $\pi : M \to M/\operatorname{Ker} f$ and $\pi' : M/\operatorname{Ker} f \to N/\operatorname{Ker} f$. Then the homomorphism $g = \tilde{f}\pi'\pi$ is the extension of f.

(2) $\Rightarrow$ (3) is clear.

(3) $\Rightarrow$ (2) Suppose N/K is finitely generated and $N/K \ll M/K$ for some proper submodule $K \leq N$. Then there is a maximal submodule T of N such that $K \leq T$ and $N/T \ll M/T$, because N/T is the image of N/K under the canonical epimorphism $f : M/K \to M/T$, a contradiction.

(3) $\Leftrightarrow$ (4) is straight forward.

□

Properties of coclosed modules in Proposition 2.17 are adapted to coneat submodules as follows. The proof is omitted.

Proposition 3.2 *Let $K \leq L \leq M$ be submodules. Then the following hold.*

(1) If L is coneat in M, then L/K is coneat in M/K.

(2) If $K \leq \mathrm{Rad}(L)$ and L/K is coneat in M/K, then L is coneat in M.

(3) If $L \leq M$ is coneat, then $K \leq \mathrm{Rad}(M)$ implies $K \leq \mathrm{Rad}(L)$; hence $\mathrm{Rad}(L) = L \cap \mathrm{Rad}(M)$.

(4) If $f : M \to N$ is a small epimorphism and L is coneat in M, then $f(L)$ is coneat in N.

(6) If K is coneat in M, then K is coneat in L, and the converse is true if L is coneat in M.

The proof of (Zöschinger, 2006, Lemma A.4) can be adapted to prove the following.

Proposition 3.3 *Let $K \leq L \leq M$ be submodules of M. If K is coneat in M and L/K is coneat in M/K, then L is coneat in M.*

Proof Suppose X is a submodule of L such that L/X finitely generated and L/X is small in M/X. Firstly we will prove that $K/(K \cap X)$ is small in $M/(K \cap X)$.

Assume the contrary. Then there is an R-module W such that

$$K \cap X \leq W \text{ and } W + K = M. \qquad (*)$$

Suppose $L/[K + (W \cap X)]$ is not small in $M/[K + (W \cap X)]$. Then there is an R-module Z such that $K + (W \cap X) \leq Z$ and $Z + L = M$. Since $K \leq Z$, $Z = Z \cap W + K$

by (*), and so $M = Z \cap W + L$. By smallness of L/X in M/X, $Z \cap W + X = M$. Now $W = Z \cap W + X \cap W$, and $W \leq Z$. Finally, since $Z + W = M$, $Z = M$. Recall that L/K is coneat in M/K and $L/[K + (W \cap X)]$ is epimorphic image of the finitely generated module L/X. Hence, $L = K + W \cap X$ by Proposition 3.1(2). By modular law, $X = K \cap X + W \cap X$, and $X \leq W$. Then $K + X = L$. Since L/X is small in M/X, $W = M$ by (*). By our assumption, K is coneat in M, and hence $K = K \cap X$ and $K \leq X$. Since L/X is an epimorphic image of L/K and L/K is coneat in M/K, $L = X$ by Proposition 3.1(2), again. □

An exact sequence $0 \to A \xrightarrow{f} B \to C$ is said to be *coneat exact* if $f(A)$ is a coneat submodule of B. A monomorphism $f : A \to B$ is said to be a coneat monomorphism, if the short exact sequence $0 \to A \xrightarrow{f} B \to B/f(A)$ is coneat exact.

Theorem 3.1 *Let R be a commutative ring and $f : N \to M$ a monomorphism. The following are equivalent.*

(1) $f(N)$ is a coneat submodule of M.

(2) $S \otimes_R N \xrightarrow{1_S \otimes f} S \otimes_R M$ is a monomorphism for each simple R-module S.

(3) $mf(N) = f(N) \cap mM$ for each maximal ideal m of R.

Proof

(1) ⇔ (2) By Theorem 2.6.

(2) ⇔ (3) Follows by Proposition 2.16. □

Remark 3.1 *If N is a pure submodule of M, then $IN = N \cap IM$ for every left ideal of R (see, (Lam, 2001, Corollary 4.92)). Therefore, over commutative rings, every pure submodule is coneat by Theorem 3.1(3). This fact will be used in the sequel.*

Corollary 3.1 *Let R be a commutative ring. The following are equivalent.*

(1) $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ is coneat exact.

(2) $0 \to C^+ \xrightarrow{g_*} B^+ \xrightarrow{f_*} A^+ \to 0$ is neat exact.

Proof By Theorem 3.1(2) and the adjoint isomorphism $(M \otimes N)^+ \cong \mathrm{Hom}(M, N^+)$. □

If A is a pure submodule of B, then B is a pure essential extension of A if there are no nonzero submodules $S \subseteq B$, with $S \cap A = 0$ and the image of A pure in $B \cap S$, Warfield (1969). The definition of pure essential extension is adapted to coneat submodules as follows.

Definition 3.1 *Let M be an R-module and N a coneat submodule of M. We call M an essential coneat-extension of N if there is no nonzero submodule K of M such that $K \cap N = 0$ and $(K + N)/K$ is a coneat submodule of M/K.*

Lemma 3.1 *Let N be a coneat submodule of an R-module M. Then M is an essential coneat-extension of N if and only if for any homomorphism $\varphi : M \to L$ such that $\varphi \mid_N$ is a coneat monomomorphism, it follows that φ is injective.*

Proof Suppose that M is an essential coneat-extension of N. Let $\varphi : M \to L$ be a homomorphism such that $\varphi \mid_N$ is an coneat monomomorphism. Let $K = \operatorname{Ker}\varphi$. From the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & (K+N)/K & \longrightarrow & M/K & \longrightarrow & M/(K+N) & \longrightarrow & 0 \\ & & \| & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & N & \xrightarrow{\varphi|_N} & L & \longrightarrow & N/L & \longrightarrow & 0 \end{array}$$

we deduce that $(K + N)/K$ is a coneat submodule of M/K since coneat exact sequences are preserved under pullback diagrams. Therefore $K = 0$.

Conversely, suppose that there is a nonzero submodule K of M such that $K \cap N = 0$ and $(K + N)/K$ is a coneat submodule of M/K. Since $N \cong (K + N)/K$, $\varphi \mid_N$ is a coneat monomomorphism. Then φ is injective, and $K = 0$. This contradicts with our assumption. So, M is an essential coneat-extension of N. □

The proof of the following lemma is similar to the proof of (Divaani-Aazar et al., 2006, Lemma 3.3). We include it for completeness.

Lemma 3.2 *Let R be a commutative ring and N a coneat submodule of an R-module M. Then there exists a submodule K of M such that*

(i) $K \cap N = 0$,

(ii) $(K + N)/K$ *is a coneat submodule of* M/K, *and*

K is maximal with respect to inclusion among all submodules of M which satisfies the conditions (i) and (ii). In particular, M/K is an essential coneat-extension of $(K + N)/K$.

Proof Let Ω denote the class of all submodules of M which satisfies the conditions (i) and (ii). Then Ω is not empty, because $0 \in \Omega$. Consider any chain of submodules $(K_\lambda)_\Lambda$ and let $K = \bigcup_{K_\lambda \in \Lambda} K_\lambda$. Clearly, $K \cap N = 0$. By Theorem 3.1, $m(\frac{K_\lambda + N}{K_\lambda}) = (\frac{K_\lambda + N}{K_\lambda}) \cap m(\frac{M}{K_\lambda})$ for each maximal ideal m of R and all $\lambda \in \Lambda$. Then $(K_\lambda + mM) \cap (K_\lambda + N) = mN + K_\lambda$ for each maximal ideal m of R and all $\lambda \in \Lambda$, and so $(K + mM) \cap (K + N) \subseteq (mN + K)$. Again by Theorem 3.1, $(K + N)/K$ is an coneat submodule of M/K, and the conclusion follows by Zorn's lemma.

For the last assertion, suppose there is a submodule L/K of M/K such that $(L/K) \cap ((N+K)/K) = 0$ and $(L+N)/L$ is a coneat submodule of M/L. Then $L \cap N \subseteq L \cap (N+K) = K$, and so $L \cap N \subseteq K \cap N = 0$. Thus $L \in \Omega$ and so $L = K$, by the assumption on K. Therefore, M/K is an essential coneat-extension of $(K + N)/K$, as required. □

We say that an R-module M has property (E) if for every proper submodules $K \lneqq L \lneqq M$, there is a maximal submodule N of M such that $K \leq N$ and $L \nleq N$. Note that the property (E) is closed under taking factor modules. A module M is called *cosemisimple*(or V-module) if $\mathrm{Rad}(M/N) = 0$ for every proper submodule N of M, equivalently every proper submodule of M is an intersection of maximal submodules of M; see (Wisbauer, 1991, §23). In (Clark et al., 2006, 3.8), it was shown that a module M is cosemisimple if and only if every submodule of M is coclosed in M; this also holds if we replace coclosed by the weaker coneat condition.

Proposition 3.4 *For a module M the following are equivalent.*

(1) M is cosemisimple,

(2) M has property (E),

(3) Every submodule of M is coneat.

Proof (1) $\Rightarrow$ (2) Let K and L be proper submodules of M such that $K \leq L$. Suppose any maximal submodule containing K also contains L, then $L/K \leq \mathrm{Rad}(M/K) = 0$, and so $K = L$, a contradiction. Therefore there exists a maximal submodule of M which contains K but does not contain L.

(2) $\Rightarrow$ (3) Let $L \leq M$ and K a maximal submodule of L. By hypothesis, there is a maximal submodule N of M such that L/K does not contained in the maximal submodule N/K of M/K. So L/K is not contained in $\mathrm{Rad}(M/K)$. Hence L/K is not small in M/K. Therefore L is a coneat submodule of M.

(3) $\Rightarrow$ (1) Let $0 \neq m + N \in \mathrm{Rad}(M/N)$, then $(Rm + N)/N \ll M/N$. Since $Rm + N$ is coneat, we have $Rm + N = N$. That is $m \in N$. So $\mathrm{Rad}(M/N) = 0$. Hence M is cosemisimple. □

Corollary 3.2 *R is a right V-ring if and only if for every right R-module M, every submodule of M is coneat in M.*

In (Crivei, 2014), a right R-module M is called absolutely coneat if M is a coneat submodule of any module containing it. Let M be an R-module with $\operatorname{Rad} M = M$. It is easy to see that $\operatorname{Hom}(M, S) = 0$ for each simple module. Hence

Corollary 3.3 *Let M be a right R-module with* $\operatorname{Rad}(M) = M$. *Then M is absolutely coneat.*

A ring R is said to be right *small* if $R_R \ll E(R_R)$. A ring R is small if and only if $E = \operatorname{Rad}(E)$ for every injective R-module E (see, (Lomp, 2000, Proposition 3.3)).

Proposition 3.5 *The following statements are equivalent for a ring R.*

(1) R is a right small ring.

(2) Absolutely coneat right R-modules are precisely those modules N such that $\operatorname{Rad}(N) = N$.

Proof (1) $\Rightarrow$ (2) Let E be the injective hull of N. Then $\operatorname{Rad}(E) = E$ as R is a small ring. Suppose N is coneat in E. So $\operatorname{Rad}(N) = N \cap \operatorname{Rad}(E) = N$ by Proposition 3.2(3). The rest of (2) by Corollary 3.3.

(2) $\Rightarrow$ (1) Every injective right R-module E is absolutely coneat. Then (2) implies $\operatorname{Rad}(E) = E$, and so R is a small ring. □

Coclosed submodules are coneat by Proposition 3.1. The following example shows that in general a coneat submodule need not be coclosed.

Example 3.1 *Let R be a valuation domain with a maximal ideal P, which is not finitely generated. Then* $\operatorname{Rad}(P) = P^2 = P$, *and so P is a coneat submodule of R by Corollary 3.3. On the other hand, P is not closed in R since $P \ll R$.*

Let R be a ring and M a nonzero R-module. M is called *coatomic* if every proper submodule N of M is contained in a maximal submodule of M, i.e. $\operatorname{Rad}(M/N) \neq 0$.

Proposition 3.6 *Let M be a module and N a coatomic submodule of M. Then N is coneat in M if and only if it is coclosed in M.*

Proof Suppose N is coneat and $N/X \ll M/X$ for some proper submodule $X \leq N$. Since N is coatomic, X is contained in a maximal submodule, say K, of N. Then $N/K \ll M/K$, and this contradicts with the fact that N is coneat. Hence N is coclosed. The converse implication is obvious. □

In (Zöschinger, 1980), a ring R is called right *K-ring* if every non-zero small right R-module is coatomic. Dedekind domains and right max rings (i.e. every nonzero right R-module has a maximal submodule) are right K-rings.

Theorem 3.2 *R is a right K-ring if and only if coneat submodules of any right R-module are coclosed.*

Proof For the necessity, let M be a non-zero small module and suppose M/K has no maximal submodule, i.e., $\mathrm{Rad}(M/K) = M/K$ for some proper submodule K of M. Then M/K is small and coneat submodule in $E(M/K)$. Hence M/K is coclosed in $E(M/K)$ by (1). This gives a contradiction since coclosed submodules are not small. Consequently, K is contained in a maximal submodule of M, and so M is coatomic.

For the sufficiency, suppose the contrary that there is a module M and a submodule N of M which is coneat but not coclosed. Then there is a proper submodule K of N such that $N/K \ll M/K$. By Proposition 3.2(1), N/K is a coneat submodule of M/K. Then N/K is coatomic by the hypothesis, and so N/K is coclosed by Proposition 3.6, a contradiction. □

3.2. Coneat-Injective Modules

In this section we investigate coneat-injective modules, and we show that every R-module possesses, up to isomorphism, unique coneat-injective envelope on commutative rings. We also give a characterization of coneat-injective modules in terms of coneat submodules. Let $\mathcal{S}$ be the set of all representatives of simple R-modules.

Definition 3.2 *An R-module D is called coneat-injective if for any coneat exact sequence $0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$, the induced sequence $0 \longrightarrow \mathrm{Hom}_R(C, D) \longrightarrow \mathrm{Hom}_R(B, D) \longrightarrow \mathrm{Hom}_R(A, D) \longrightarrow 0$ is exact.*

Clearly, injective modules are coneat-injective, and simple modules are coneat-injective by Proposition 3.1.

The following lemma can be proved by using similar arguments that for injective modules.

Lemma 3.3 *Let $\{D_i\}_{i \in I}$ be a class of R-modules. Then $\prod_{i \in I} D_i$ is a coneat injective R-module if and only if D_i is coneat injective for all $i \in I$.*

Lemma 3.4 *For any R-module M, there is an extension D of M such that D is coneat-injective and M is a coneat submodule of D.*

Proof Let $\{S_j\}_{j\in J}$ be a set of representatives of the simple R-modules. Consider the R-module

$$D = \prod_{m\in M} E(Rm) \otimes \prod_{j\in J} (\prod_{\phi_j} S_j)$$

where $E(Rm)$ is the injective hull of Rm for each $m \in M$, while ϕ_j runs over the non-zero elements of $\mathrm{Hom}_R(M, S_j)$. By Lemma 3.3, D is a coneat-injective module. The map $\phi : M \to D$ is defined by mapping $m \in M$ to $m \in Rm$ and acting on S_j as ϕ_j. It is obvious that ϕ is injective. Therefore, M is coneat submodule of D since by construction all the simple R-modules have the injective property with respect to ϕ. □

Theorem 3.3 *Let R be a ring and D an R-module. The following are equivalent.*

(1) D is a coneat-injective module.

(2) For any coneat monomorphism $f : A \to B$, every homomorphism from A to D can be extended to a homomorphism from B to D.

(3) D is a direct summand of every R-module L such that D is a coneat submodule of L.

(4) D is isomorphic to a direct summand of a direct product of modules in $\mathcal{S}$ and injective hull of some cyclic modules.

Proof (1) ⇒ (2) ⇒ (3) are clear.

(3) ⇒ (1) By Lemma 3.4, there exists a coneat-injective extension L of D. Therefore, D is a direct summand of L, and so it is coneat-injective, by Lemma 3.3.

(3) ⇒ (4) In view of the proof of Lemma 3.4, D is a coneat submodule of H. Then by (3), D is isomorphic to a direct summand of a direct product of modules in $\mathcal{S}$ and injective hull of some cyclic modules.

(4) ⇒ (1) By Lemma 3.3. □

Corollary 3.4 *Let R be a commutative ring. Then the following are equivalent for an R-module D.*

(1) D is coneat-injective.

(2) D has no proper essential coneat-extension.

Proof

(1) $\Rightarrow$ (2) Let M be an essential coneat-extension of D. Then there is a submodule L of M such that $M = L + D$ and $L \cap D = 0$. Since M is essential coneat-extension of D and $(L + D)/L = M/L$, we deduce that $L = 0$, and so $M = D$.

(2) $\Rightarrow$ (1) Suppose L is a coneat-extension of D. It may be assumed that L is a proper coneat-extension of D. By Lemma 3.2, there is a submodule K of L such that L/K is an essential coneat-extension of $(D + K)/K$ and that $D \cap K = 0$. But D has no proper essential coneat-extension, so $D + K = L$, from which it follows that $L = D \oplus K$.

□

The following definition given in Enochs and Jenda (2000) in order to prove existness of pure-injective (pre)envelope.

Definition 3.3 (*a*) *A pair ($\mathcal{A}$, $\mathcal{I}$), where $\mathcal{A}$ is a class of morphism between R-modules and $\mathcal{I}$ is a class of R-modules, is called an injective structure on the category of right R-modules if it satisfies all of the following*

(i) $I \in \mathcal{I}$ if and only if $\mathrm{Hom}_R(N, I) \to \mathrm{Hom}_R(M, I) \to 0$ is exact for all $M \to N \in \mathcal{A}$.

(ii) $M \to N \in \mathcal{A}$ if and only if $\mathrm{Hom}_R(N, I) \to \mathrm{Hom}_R(M, I) \to 0$ is exact for all $I \in \mathcal{I}$.

(iii) Every right R-modules has an $\mathcal{I}$-preenvelope.

(*b*) *If $\mathcal{G}$ is a class of right R-modules, then it is called that the pair ($\mathcal{A}$, $\mathcal{I}$) is determined by $\mathcal{G}$ if $M \to N \in \mathcal{A}$ if and only if $0 \to G \otimes M \to G \otimes N$ is exact for all $G \in \mathcal{G}$.*

Theorem 3.4 *Let a pair ($\mathcal{A}$, $\mathcal{E}$), where $\mathcal{A}$ is a class of coneat monomorphism between R-modules and $\mathcal{E}$ is a class of coneat-injective R-modules. Then the pair ($\mathcal{A}$, $\mathcal{E}$) is an injective structure on the category of left R-modules.*

Proof (*i*) and (*ii*) hold by Theorem 3.3 and the definition of coneat-injective modules.

For (*iii*), let M be an arbitrary module. By Lemma 3.4, there is an extension D of M such that D is coneat-injective and it contains M as a coneat submodule. We will show that D is a coneat-injective preenvelope of M. Let $\varphi : M \to D'$ be a homomorphism with D' coneat-injective. Then we have the pushout (commutative) diagram

$$\begin{array}{ccccccccc} & 0 & \longrightarrow & M & \xrightarrow{\iota} & D & \longrightarrow & D/M & \longrightarrow & 0 \\ & & & \downarrow{\scriptstyle \varphi} & & \downarrow{\scriptstyle \varphi^*} & & \downarrow & & \\ E: & 0 & \longrightarrow & D' & \xrightarrow{\iota^*} & X & \longrightarrow & X/D' & \longrightarrow & 0 \end{array}$$

Since ι is a coneat monomorphism and coneat monomorphism is closed under push-out diagrams, ι^* is also a coneat monomorphism. Then, by Theorem 3.3, the sequence E splits, so there is a homomorphism $(\iota^*)^{-1}$ such that $(\iota^*)^{-1} \circ \iota^* = 1_{D'}$. Therefore, $(\iota^*)^{-1} \circ \varphi^* \circ \iota = \varphi$. □

Corollary 3.5 *If R is a commutative ring, then every R-module possesses, up to isomorphism, a unique coneat-injective envelope.*

Proof By Theorem 3.4, the pair $(\mathcal{A}, \mathcal{E})$ is an injective structure. By Theorem 3.1, $(\mathcal{A}, \mathcal{E})$ is determined by $\mathcal{S}$. Therefore every R-module has a coneat-injective envelope by (Enochs and Jenda, 2000, Theorem 6.6.4). □

Definition 3.4 *(i) Let N be an R-module. A coneat-essential extension M of N is said to be maximal if there is no proper extension of M which is a coneat-essential extension of N.*

(ii) Let M be a coneat submodule of a coneat-injective R-module D. We say that D is a minimal coneat-injective extension of M if there is no proper coneat-injective submodule of D containing M.

One can adapt the arguments in (Divaani-Aazar et al., 2006) to prove the following result.

Theorem 3.5 *Let R be a commutative ring. Let D be an R-module and M a submodule of D. The following are equivalent.*

(1) D is a coneat-injective envelope of M.

(2) D is a maximal coneat-essential extension of M.

(3) D is an essential coneat-extension of M which is coneat-injective.

(4) D is a minimal coneat-injective extension of M.

3.3. Coneat-Flat Modules

It is well known that a right R-module M is flat if and only if any short exact sequence of the form $0 \to K \xrightarrow{f} N \to M \to 0$ is pure exact, i.e. $f(K)$ is a pure submodule of N. It is natural to ask for which right R-modules P, any short exact sequence ending with P is coneat exact? In this section several characterizations of such modules are given.

M is called coneat-flat if the kernel of any epimorphism $Y \to M \to 0$ is a coneat submodule of Y. Clearly, projective modules are coneat-flat but the converse need not be true in general (see, Theorem 3.10). In the following theorem we give the relation between the coneat-flat modules and coneat submodules.

Theorem 3.6 *The following are equivalent for an R-module M:*

(1) M is coneat-flat.

(2) $\mathrm{Ext}^1_R(M, S) = 0$ *for each simple R-module* S.

(3) There is a coneat exact sequence $0 \to K \to L \to M \to 0$ *with L projective.*

(4) There is a coneat exact sequence $0 \to K \to L \to M \to 0$ *with L coneat-flat.*

Proof (1) $\Rightarrow$ (2) Let $\mathbb{E} : 0 \to S \xrightarrow{\alpha} L \to M \to 0$ be a short exact sequence with S simple R-module. Since M is coneat-flat, S is coneat in L, and there is a homomorphism $\beta : L \to S$ such that the following diagram is commutative.

$$\begin{array}{lccccccccc} \mathbb{E}: & 0 & \longrightarrow & S & \xrightarrow{\alpha} & L & \longrightarrow & P & \longrightarrow & 0 \\ & & & {\scriptstyle 1_S}\downarrow & \swarrow{\scriptstyle \beta} & & & & & \\ & & & S & & & & & & \end{array} \tag{3.1}$$

Then $1_S = \beta\alpha$, and so the sequence $\mathbb{E}$ splits. Hence $\mathrm{Ext}^1_R(M, S) = 0$.

(2) $\Rightarrow$ (3) Assuming (2). There is a short exact sequence $\mathbb{E} : 0 \to C \to F \to M \to 0$ with F free R-module. Applying $\mathrm{Hom}_R(-, S)$, we obtain the exact sequence $0 \to \mathrm{Hom}_R(M, S) \to \mathrm{Hom}_R(F, S) \to \mathrm{Hom}_R(C, S) \to \mathrm{Ext}^1_R(M, S) = 0$, that is $\mathrm{Hom}_R(\mathbb{E}, S)$ is exact for every simple R-module S, and so $\mathbb{E}$ is coneat exact.

(3) $\Rightarrow$ (4) is obvious.

(4) ⇒ (1) Let $s : B \to M$ be any epimorphism. Consider the following commutative diagram.

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & K & = & K & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \operatorname{Ker} s & \longrightarrow & X & \xrightarrow{\alpha} & L & \longrightarrow & 0 \\
 & & \| & & \downarrow{\scriptstyle t} & & \downarrow{\scriptstyle \beta} & & \\
0 & \longrightarrow & \operatorname{Ker} s & \longrightarrow & B & \xrightarrow{s} & M & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array} \tag{3.2}$$

$\beta\alpha = st$ is coneat epimorphism, i.e. Ker(st) is a coneat submodule of X, by Proposition 3.3. Then s is coneat epimorphism by Proposition 3.2(1). This completes the proof. □

Corollary 3.6 *The class of coneat-flat modules is closed under extensions, direct sums, direct summands and coneat quotients. In particular, coneat-flat modules are closed under pure quotients over commutative rings.*

Proof Coneat-flat modules are closed under extensions, direct sums, direct summands and coneat quotients by Theorem 3.6, and under pure quotients by Remark 3.1 and Theorem 3.6. □

Proposition 3.7 *Let R be a commutative ring and M an R-module. Then M is coneat-flat if and only if* $\operatorname{Tor}_1^R(M, S) = 0$ *for each simple R-module S.*

Proof Let $0 \to K \xrightarrow{i} F \to M \to 0$ be a short exact sequence with F projective. Applying $- \otimes S$, we get

$$0 = \operatorname{Tor}_1^R(F, S) \to \operatorname{Tor}_1^R(M, S) \to K \otimes S \xrightarrow{i \otimes 1_S} F \otimes S \to M \otimes S \to 0.$$

Then $i \otimes 1_S$ is a monomorphism if and only if $\operatorname{Tor}_1^R(M, S) = 0$. Now the proof is clear by Theorem 3.1 and Theorem 3.6. □

Proposition 3.8 *Let R be a commutative ring. An R-module M is coneat-flat if and only if M^+ is m-injective.*

Proof Let S be a simple R-module. We have the standard isomorphism

$$\mathrm{Ext}^1_R(S, M^+) \cong \mathrm{Tor}^R_1(M, S)^+$$

Now, the proof is immediate by Proposition 3.7 and Proposition 2.21. □

Corollary 3.7 *Let R be a commutative ring. The class of coneat-flat modules is closed under pure submodules.*

Proof Let $0 \to A \to B \to C \to 0$ be a pure exact sequence of R-modules with B coneat-flat. Then the short exact sequence $0 \to C^+ \to B^+ \to A^+ \to 0$ splits. By Proposition 3.8 the module B^+ is m-injective, and so A^+ is m-injective. Then A is coneat-flat by Proposition 3.8, again. □

Proposition 3.9 *A ring R is a right V-ring if and only if every right R-module is coneat-flat.*

Proof Suppose R is a right V-ring. Then every simple right R-module S is injective, so that $\mathrm{Ext}^1_R(M, S) = 0$ for every R-module M. Hence M is coneat-flat. For the sufficiency, let S be a simple R-module and E an injective module containing S. By the hypothesis E/S is coneat-flat. Hence the sequence $0 \to S \to E \to E/S \to 0$ splits by Theorem 3.6, and so S is injective. □

3.3.1. When Coneat-flat Modules are Flat

In this subsection we study the flatness of coneat-flat modules and the character of coneat-flat modules. A right R-module M is called *cotorsion* if $\mathrm{Ext}^1_R(F, M) = 0$ for any flat R-module F.

Lemma 3.5 *Let R be a ring and S a simple R-module. If R is commutative or semilocal, then S is cotorsion.*

Proof First suppose R is commutative and let $I = Ann_R(S)$. Then clearly S is an R/I-module. Since R/I is simple, S is cotorsion as an R/I-module, so that S is a cotorsion R-module by (Xu, 1996, Proposition 3.3.3). If R is semilocal, then $J(R).S = 0$, and so S

is an $R/J(R)$-module. As R is semilocal, $R/J(R)$ is semisimple, and so S is a cotorsion $R/J(R)$-module. Now, S is a cotorsion R-module by (Xu, 1996, Proposition 3.3.3), again. □

Corollary 3.8 *Suppose R is commutative or semilocal. Then every flat module is coneat-flat.*

Proof Let S be a simple R-module. Then S is a cotorsion module by Lemma 3.5. Therefore $\text{Ext}(M, S) = 0$, and so M is coneat-flat by Theorem 3.6. □

For a right flat module we prove the following.

Proposition 3.10 *Let R be a left C-ring. A right R-module M is flat if and only if* $\text{Tor}_1^R(M, S) = 0$ *for each simple left R-modules S.*

Proof Necessity is clear. For the sufficiency assume that $\text{Tor}_1^R(M, S) = 0$ for each simple left R-module S. Then $0 = \text{Tor}_1^R(M, S)^+ \cong \text{Ext}_R^1(S, M^+)$ implies that M^+ is m-injective by Theorem 2.21. Therefore M^+ is injective because R is a left C-ring. Hence M is flat by (Enochs and Jenda, 2000, Theorem 3.2.10). □

Theorem 3.7 *The following are equivalent for a commutative ring R.*

(1) Every coneat-flat module is flat.

(2) Flat modules are precisely those modules M satisfying $\text{Ext}_R^1(M, \prod_{i\in I} S_i) = 0$*, where the* S_i*'s are all the non-isomorphic simple modules.*

Proof

(1) ⇒ (2) By Lemma 3.5, simple modules are cotorsion. Then $\prod_{i\in I} S_i$ is cotorsion since cotorsion modules are closed under direct products. Hence, if M is flat then $\text{Ext}_R^1(M, \prod_{i\in I} S_i) = 0$. Conversely, suppose $\text{Ext}_R^1(M, \prod_{i\in I} S_i) = 0$. Then $\text{Ext}_R^1(M, S_i) = 0$ for each $i \in I$, so that M is coneat-flat by Theorem 3.6. Hence M is flat by (1).

(2) ⇒ (1) Suppose M is coneat-flat. Then $\text{Ext}_R^1(M, S) = 0$ for each simple R-module S. So that $\text{Ext}_R^1(M, \prod_{i\in I} S_i) = 0$ for any index set I and simple R-modules S_i. Hence M is flat by (2). □

In (Fuchs, 2012), a commutative domain R is called an N-domain if every maximal ideal of R is projective (finitely generated). A ring R is called a *right N-ring* if every maximal right ideal of R is finitely generated.

Proposition 3.11 *Let R be a commutative N-ring and M an arbitrary R-module. Then the following hold.*

(1) M is m-injective if and only if M^+ is coneat-flat.

(2) M is m-injective if and only if M^{++} is m-injective.

(3) M is coneat-flat if and only if M^{++} is coneat-flat.

(4) Any direct product of coneat-flat modules is coneat-flat.

(5) Any direct product of copies of R is coneat-flat.

(6) The class of m-injective modules is closed under pure quotients.

Proof

(1) (1) An R-module M is m-injective module if and only if M^+ is coneat-flat by (Rotman, 1979, Theorem 9.51) since R is an N-ring

(2) M is m-injective if and only if M^+ is coneat-flat by (1), and M^+ is coneat-flat if and only if M^{++} is m-injective by Proposition 3.8.

(3) If M is coneat-flat, then M^+ is m-injective by Proposition 3.8. So M^{+++} is m-injective by (2), and hence M^{++} is coneat-flat. Conversely, if M^{++} is coneat-flat, then M is coneat-flat by Corollary 3.7 since M is a pure submodule of M^{++}.

(4) Let $(M_i)_{i\in J}$ be a family of coneat-flat R-modules. Since the class of coneat-flat modules is closed under direct sums, $\bigoplus_{i\in J} M_i$ is coneat-flat. So $(\bigoplus M_i)^{++} \cong (\prod M_i^+)^+$ is coneat-flat by (3). Since $\oplus_{i\in J} M_i^+$ is a pure submodule of $\prod_{i\in J} M_i^+$, $(\oplus_{i\in J} M_i^+)^+$ is a direct summand of $(\prod_{i\in J} M_i^+)^+$, and so $(\oplus_{i\in J} M_i^+)^+ \cong \prod_{i\in J} M_i^{++}$ is coneat-flat. Since coneat-flat modules are closed under pure submodules and $\prod_{i\in J} M_i$ is a pure submodule of $\prod_{i\in J} M_i^{++}$, the module $\prod_{i\in J} M_i$ is coneat-flat.

(5) By (4).

(6) Take any pure exact sequence $0 \to A \to B \to C \to 0$ with B m-injective. Then we have a split exact sequence $0 \to C^+ \to B^+ \to A^+ \to 0$. By (1), B^+ is coneat-flat, and so C^+ is coneat-flat. Then C is m-injective by (1), again.

□

An R-module M is called *absolutely pure* if it is pure in every module containing it as a submodule. It is well known that a ring R is left Noetherian if and only if every absolutely pure left R-module is injective.

Proposition 3.12 *R is a left N-ring if and only if every absolutely pure left R-module is m-injective.*

Proof ($\Rightarrow$) Let M be an absolutely pure left R-module. Since R is a left N-ring, $\mathrm{Ext}(S, M) = 0$ for each simple left R-module S, that is, M is m-injective.

($\Leftarrow$) Let S be a simple left R-module. Then $\mathrm{Ext}^1_R(S, M) = 0$ for each absolutely pure left R-module M by the assumption. Then S is finitely presented by (Enochs, 1976, Proposition).

□

An R-module E is called pure-injective if it is injective with respect to all pure short exact sequences.

Theorem 3.8 *Let R be a ring. The following statements are equivalent.*

(1) (a) M is a flat right R-module if and only if $\mathrm{Tor}^R_1(M, S) = 0$ for each simple left R-module S,

(b) R is a left N-ring.

(2) M is an m-injective left R-module if and only if M^+ is flat.

(3) M is an m-injective left R-module if and only if M is an absolutely pure left R-module.

Proof (1) $\Rightarrow$ (2) Let M be a left R-module and S a simple left R-module. Suppose M is m-injective. Then $0 = \mathrm{Ext}^1_R(S, M)^+ \cong \mathrm{Tor}^R_1(M^+, S)$ by (Rotman, 1979, Theorem 9.51), and so M^+ is flat by (1). Conversely, suppose M^+ is flat. Then M^{++} is injective by (Rotman, 1979, Theorem 3.52), and so M is absolutely pure since M is pure in M^{++}. Therefore M is m-injective by Proposition 3.12.

(2) $\Rightarrow$ (3) Firstly, we shall prove that a right R-module M is flat if and only if M^{++} is flat. Then R is left coherent by (Cheatham and Stone, 1981, Theorem 1). Suppose M is a flat right R-module. Then M^+ is (m-)injective, and so M^{++} is flat by (2). Now, conversely suppose M^{++} is a flat right R-module. Then M is flat since M is pure submodule of M^{++} and flat modules closed under pure submodules.

Let M be a left R-module. Then M^+ is flat if and only if M is absolutely pure by (Cheatham and Stone, 1981, Theorem 1) since R is left coherent. Hence the rest of (3) follows by (2).

(3) $\Rightarrow$ (1) Suppose $\mathrm{Tor}_1^R(M, S) = 0$ for each simple left R-module S. Then $\mathrm{Ext}_R^1(S, M^+) = 0$, and so M^+ is m-injective. Then M^+ is absolutely pure by (3). Therefore M^+ is injective since it is pure-injective. Thus M is flat. This proves (a), and (b) follows by Proposition 3.12. □

In general, coneat-flat modules need not be flat. For example, let $M = R/P$ where R and P are as in Example 3.1. Then M is coneat-flat by Theorem 3.6, but M is not flat because it is torsion.

Proposition 3.13 *Let R be a commmutative ring. Consider the following statements.*

(1) R is C-ring.

(2) Coneat-flat R-modules are flat.

Then (1) $\Rightarrow$ (2). *If R is Noetherian, then* (2) $\Rightarrow$ (1).

Proof (1) $\Rightarrow$ (2) By Proposition 3.7 and Proposition 3.10.

(2) $\Rightarrow$ (1) Let M be an m-injective R-module. Then M^+ is flat by the hypothesis and Theorem 3.8. As R is Noetherian, M is injective by (Cheatham and Stone, 1981, Theorem 2). Hence R is a C-ring. □

It is easy to see that a left N-ring and left semiartian ring is left Noetherian. The following is a slight generalization of this fact.

Corollary 3.9 *If R is left N-ring and left C-ring, then R is left Noetherian.*

Proof By Proposition 3.10 and Theorem 3.8, a left R-module M is m-injective if and only if it is absolutely pure. So every absolutely pure left module is injective. Hence R is left Noetherian. □

Note that Corollary 3.9 generalizes (Crivei, 2014, Theorem 4.1 (ii) $\Rightarrow$ (i)).

In (Cheatham and Stone, 1981, Theorem 4), the authors proves that, R is left Artinian if and only if a left module M is injective exactly when M^+ is projective. We show that, this result still holds if we replace m-injective by injective.

Theorem 3.9 *Let R be a ring. The following are equivalent.*

(1) R is left Artinian.

(2) A left R-module M is m-injective if and only if M^+ *is projective.*

Proof (1) $\Rightarrow$ (2) R is left C-ring by (1), and so m-injective modules are injective. Now, (2) follows by (Cheatham and Stone, 1981, Theorem 4).

(2) $\Rightarrow$ (1) Firstly, we show that a left R-module M is m-injective if and only if M is absolutely pure.

Let M be an absolutely pure left R-module. Consider the pure exact sequence $0 \to M \to E(M) \to E(M)/M \to 0$. Then the short exact sequence $0 \to (E(M)/M)^+ \to E(M)^+ \to M^+ \to 0$ splits. Then $E(M)^+$ is projective, and hence M^+ is projective. By (2), M is m-injective. Conversely, let M be an m-injective left R-module. Since M is pure in M^{++} and M^{++} is injective, M is absolutely pure.

Then a left R-module M is m-injective if and only if M is absolutely pure if and only if M^+ is projective. By (Cheatham and Stone, 1981, Theorem 3), R is right perfect, and so it is a left C-ring i.e., m-injective left R-modules are injective. Hence R is left Artinian by (Cheatham and Stone, 1981, Theorem 4) and (2). □

3.3.2. When Coneat-flat Modules are Projective

In this section, we shall consider when coneat-flat modules are projective. We begin with the following result.

Theorem 3.10 *Consider the following statements.*

(1) R is a right perfect ring.

(2) Every coneat-flat right R-module is projective.

Then (1) $\Rightarrow$ (2). *If R is either commutative or semilocal, then* (2) $\Rightarrow$ (1).

Proof

(1) $\Rightarrow$ (2) Let P be a coneat-flat module. Consider a short exact sequence $0 \to K \to F \to P \to 0$ with F free module. Since R is perfect, F is supplemented by (Wisbauer, 1991, 43.9). So K has a supplement in F, that is, $K + N = F$ and $A \cap N \ll N$ for some submodule N of F. On the other hand, K is coatomic as R is a perfect ring. Then K is a coclosed submodule of F by Proposition 3.6, so that $K \cap N \ll K$. Hence K and N are mutual supplements, and so $K \oplus N = F$ by (Wisbauer, 1991, 41.15). Therefore $N \cong F/K \cong P$ is projective.

(2) $\Rightarrow$ (1) Let M be a flat module. By Corollary 3.8, M is coneat-flat, and so M is projective by (2). Hence R is a perfect ring. □

The following is an immediate consequence of Theorem 3.10.

Corollary 3.10 *Let R be a commutative perfect ring. Then an R-module P is projective if and only if* $\mathrm{Ext}(P, S) = 0$ *for every simple R-module S.*

An epimorphism $f : N \to M$ is said to be a *small cover* of M if $\operatorname{Ker} f \ll N$. Moreover, if N is projective, then f is called a *projective cover*.

Proposition 3.14 *Let R be a ring and M a right R-module with a projective cover $f : P \to M$. Set $K = \operatorname{Ker} f$. Then M is a coneat-flat module if and only if $\operatorname{Rad}(K) = K$.*

Proof ($\Rightarrow$) Assume $\operatorname{Rad}(K) \neq K$. Then K has a maximal submodule, say A. By Proposition 3.1, there exists a maximal submodule L of P such that $A = K \cap L$. Then $K \leq \operatorname{Rad} P$ implies $K = K \cap \operatorname{Rad}(P) \leq K \cap L = A$. Contradiction. Hence (2) holds.

($\Leftarrow$) By Corollary 3.3 and Theorem 3.6. □

Corollary 3.11 *Let R be a semiperfect ring. Then finitely presented coneat-flat modules are projective.*

Lemma 3.6 *Let R be a commutative ring and M a coneat-flat R-module. Then, for all maximal ideals m of R, M_m is a coneat-flat R_m-module.*

Proof

Since M is a coneat-flat R-module, there is a short exact sequence $0 \to K \to F \to M \to 0$ where K is coneat submodule of F with F is a projective R-module by Theorem 3.6. By exactness of localization, for all maximal ideals m of R, the sequence $0 \to K_m \to F_m \to M_m \to 0$ is exact. Since $mK = K \cap mF$ for all maximal ideals m of R, we have $m_m K_m = K_m \cap m_m F_m$. Therefore M_m is a coneat-flat R_m-module by Theorem 3.1.

□

Corollary 3.12 *Let R be a commutative ring. Then a finitely presented R-module M is coneat-flat if and only if it is projective.*

Proof Sufficiency is clear. For the necessity, suppose M is coneat-flat. Let m be a maximal ideal of R. Then M_m is a coneat-flat R_m-module by Lemma 3.6. So that M_m is projective (and so flat) over R_m by Corollary 3.11. Then M is flat by (Lam, 2001, page 160, Exercise 14). Therefore M is projective by (Lam, 2001, Theorem 4.30). □

CHAPTER 4

ABSOLUTELY S-PURE MODULES AND NEAT-FLAT MODULES

In this chapter we studied neat-flat modules and absolutely s-pure modules. A right R-module N is said to be *neat-flat* if for any epimorphism $M \to N$, the induced map $\mathrm{Hom}(S, M) \to \mathrm{Hom}(S, N)$ is surjective for any simple right R-module S, that is, every short exact sequence ending with M is a neat-exact sequence. A right R-module M is said to be absolutely s-pure if it is s-pure in every extension of it, i.e. M is s-$\mathcal{P}ure$-divisible . One of the importance of absolutely s-pure (resp. neat-flat) modules is the fact that they are homological objects of the proper class s-$\mathcal{P}ure$ (resp. $\mathcal{N}eat$). From another point of view, absolutely s-pure and neat-flat modules are similar to that of absolutely pure modules and flat modules which are $\mathcal{P}ure$-divisible and $\mathcal{P}ure$-flat, respectively. In this regard, it is of interest to investigate the connection between these modules and the rings that are characterized via absolutely s-pure and neat-flat modules. The rings whose simple right modules have a projective preenvelope are characterized by using simple-projective modules in (Mao, 2007). At this point, it is natural to consider the rings whose simple right R-modules have an injective cover.

Some of the results given in this chapter can also be found in our recently published paper Büyükaşık and Durğun (2014).

4.1. Absolutely s-Pure Modules

In this section we give some closure properties of absolutely s-pure modules.

Remark 4.1 *If A is a pure submodule of a right R-module B, then $AI = A \cap BI$ for every left ideal I of R (see, (Lam, 2001, Corollary 4.92)). Therefore, pure submodules are s-pure by Proposition 2.3.*

The class of s-pure short exact sequences form a proper class by (Buchsbaum, 1959). The following characterization of absolutely s-pure modules is obtained by Proposition 2.3 and Corollary 2.2.

Lemma 4.1 *The following are equivalent for a right R-module M.*

(1) M is absolutely s-pure.

(2) M is s-pure in E(M).

(3) There is an injective module I containing M such that M is s-pure in I.

(4) There is an absolutely s-pure module I such that M is s-pure in I.

Now, we give another characterization of absolutely s-pure modules which will be used in the sequel.

Lemma 4.2 *The following are equivalent for a right R-module M.*

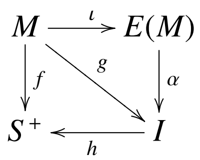
(1) M is absolutely s-pure.

(2) For any simple left R-module S, any homomorphism $f : M \to S^+$ factors through an injective right R-module.

Proof Let S be a simple left R-module and $f : M \to S^+$ a homomorphism. Let $E(M)$ be the injective hull of M and $\iota : M \to E(M)$ the inclusion map. Then the exactness of $0 \to M \otimes S \overset{\iota\otimes 1_S}{\to} E(M) \otimes S$ implies the exactness of $\mathrm{Hom}(E(M), S^+) \overset{\iota_*}{\to} \mathrm{Hom}(M, S^+) \to 0$ and vice versa by (Rotman, 1979, Theorem 2.11, Lemma 3.51).

Now assume (1). Then there is a homomorphism $g \in \mathrm{Hom}(E(M), S^+)$ such that $f = g\iota$, and this proves (2).

Assume (2). Then there exists an injective right R-module I, $g : M \to I$ and $h : I \to S^+$ such that $f = hg$. Since I is injective, there is a homomorphism $\alpha : E(M) \to I$ such that $\alpha\iota = g$. So we have the following commutative square



Then $f = h\alpha\iota$, and so M_R is absolutely s-pure by Lemma 4.1. □

Remark 4.2 *Note that if R is a commutative ring and E an injective cogenerator in $\mathcal{M}od$-R, then* $\mathrm{Hom}(S, E) \cong S$*. Hence Lemma 4.2 also holds if we replace S^+ with S. Therefore, by Theorem 3.1, the concepts absolutely s-pure and absolutely coneat are the same.*

Proposition 4.1 *The class of absolutely s-pure right R-modules is closed under extensions, direct sums, pure submodules, and direct summands.*

Proof Absolutely s-pure right R-modules are closed under extensions by Proposition 2.1, and also under direct sums and direct summands by properties of the tensor product. Since every pure exact sequence is s-pure exact, pure submodules of absolutely s-pure modules are absolutely s-pure by Lemma 4.1(4). □

A ring R is called *left SF-ring* if every simple left R-module is flat. A commutative ring R is SF-ring if and only if R is a regular ring. The question, whether a left SF-ring is regular or not is still open. The following is clear by the definitions.

Proposition 4.2 *Every right R-module is absolutely s-pure if and only if R is a left SF-ring.*

4.2. Neat-Flat Modules

In this section motivated by the relation between m-injective modules and neat submodules, we investigate the modules M, for which any short exact sequence ending with M is neat-exact. Namely, we call M neat-flat if the kernel of any epimorphism $Y \to M \to 0$ is neat in Y. The following lemma is obtained by Proposition 2.2 and Corollary 2.1.

Lemma 4.3 *The following are equivalent for a right R-module M.*

(1) M is neat-flat.

(2) Every exact sequence $0 \to A \to B \to M \to 0$ *is neat exact.*

(3) There exists a neat exact sequence $0 \to K \to F \to M \to 0$ *with F projective.*

(4) There exists a neat exact sequence $0 \to K \to F \to M \to 0$ *with F neat-flat.*

In (Mao, 2007), a right R-module M is called *simple-projective* if for any simple right R-module N, every homomorphism $f : N \to M$ factors through a finitely generated free right R-module F, that is, there exist homomorphisms $g : N \to F$ and $h : F \to M$ such that $f = hg$. Simple-projective modules and a generalization of these modules have been studied in (Mao, 2007) and (Parra and Rada, 2011), respectively. By using simple-projective modules, the authors, characterize the rings whose simple (resp. finitely generated) right modules have projective (pre)envelope in the sense of (Xu, 1996). Clearly, projective modules and modules with $\mathrm{Soc}(M) = 0$ are simple-projective. Also, a simple right R-module is simple-projective if and only if it is projective. Hence, R is a semisimple Artinian ring if and only if every right R-module is simple-projective (see, (Mao, 2007, Remark 2.2.)).

Theorem 4.1 *Let R be a ring and M an R-module. Then M is simple-projective if and only if M is neat-flat.*

Proof Suppose M is simple-projective and $s : R^{(I)} \to M$ is an epimorphism. Let S be a simple right R-module and $f : S \to M$ a homomorphism. As M is simple-projective; f factors through a finitely generated free module i.e. there are homomorphisms $h : S \to R^n$ and $g : R^n \to M$ such that $f = gh$. Since R^n is projective, there is a homomorphism $t : R^n \to R^{(I)}$ such that $g = st$. We get the following diagram

$$\begin{array}{ccc} R^n & \xleftarrow{\ h\ } & S \\ \downarrow t & \searrow^{g} & \downarrow f \\ R^{(I)} & \xrightarrow{\ s\ } & M \end{array}$$

Then $f = gh = sth$, and so the induced map $\mathrm{Hom}(S, R^{(I)}) \to \mathrm{Hom}(S, M) \to 0$ is surjective. Therefore the sequence $0 \to \mathrm{Ker}\, s \to R^{(I)} \xrightarrow{s} M \to 0$ is neat exact. Hence M is neat-flat by Lemma 4.3(3).

Conversely, let M be a neat-flat module. Then there is a neat exact sequence $0 \to K \to F \xrightarrow{g} M \to 0$ with F free by Lemma 4.3. Let S be a simple module and $f : S \to M$ any homomorphism. Then there is a homomorphism $h : S \to F$ such that $f = gh$. As S is finitely generated, $h(S) \subseteq H$ for some finitely generated free submodule of F. Then we get $f = gh = (gi)h'$, where $i : H \to F$ is the inclusion and $h' : S \to H$ is the homomorphism defined as $h'(x) = h(x)$ for each $x \in S$. Therefore f factors through H, and so M is simple projective. □

From the proof of the above lemma we have the following.

Corollary 4.1 *If M is a neat flat right R-module, then any simple submodule of M is isomorphic to a right ideal of R. Every simple submodule of an R-module M embeds in R.*

Let M be a module with $\mathrm{Soc}(M) = 0$. Then $\mathrm{Hom}(S, M) = 0$ for any simple right R-module S, and so M is neat-flat. Corollary 4.1 yields the following.

Proposition 4.3 *Let R be a ring and M any R-module. The following are equivalent:*

(1) $\mathrm{Soc}(R_R) = 0$.

(2) M is a neat-flat right R-module if and only if $\mathrm{Soc}(M) = 0$.

Proof (1) $\Rightarrow$ (2) Suppose M is a neat-flat right R-module. Then there is a neat exact sequence $0 \to K \to P \to M \to 0$ with P projective by Lemma 4.3. Then the sequence $\mathrm{Hom}_R(S, P) \to \mathrm{Hom}_R(S, M) \to 0$ is exact for any simple right R-module S. We have $\mathrm{Soc}(P) = 0$ by (1). Then $\mathrm{Hom}_R(S, P) = 0$, and so $\mathrm{Soc}(M) = 0$. The converse is clear.

(2) $\Rightarrow$ (1) Since every projective module is neat-flat, $\mathrm{Soc}(R_R) = 0$ by (2). □

Lemma 4.1 and (Mao, 2007, Proposition 2.4) yield the following.

Proposition 4.4 *The class of neat-flat right R-modules is closed under extensions, direct sums, pure submodules, and direct summands.*

Recall that a ring R is called right C-ring if $\mathrm{Soc}(M) \neq 0$ for every (cyclic) singular R-module M. Note that weakly-flat R-modules are neat-flat since closed submodules are neat. The converse is true exactly for C-rings.

Proposition 4.5 *A ring R is right C-ring if and only if neat-flat modules are weakly-flat.*

Proof Necessity is clear. For the sufficiency suppose an R-module M is m-injective. We claim that M is injective. Consider the exact sequence $0 \to M \hookrightarrow E(M) \to E(M)/M \to 0$. By Theorem 2.8, $\mathrm{Soc}(E(M)/M) = 0$. Now $E(M)/M$ is neat-flat and M is closed in $E(M)$. Then M is injective, and R is right C-ring by (Smith, 1981, Lemma 4). □

For a module A, the *singular* submodule $Z(A)$ consists of all elements $a \in A$, such that the annihilator left ideal $(0 : a) = \{r \in R; ra = 0\}$ of which is essential in R. A module A is said to be singular if $Z(A) = A$ and nonsingular if $Z(A) = 0$. If a right R-module M is nonsingular, then for every exact sequence of the form $0 \to K \xrightarrow{f} N \to M \to 0$ we have $f(K)$ is a closed submodule of N by (Sandomierski, 1968, Lemma 2.3). Every nonsingular module is weakly-flat, and the converse is true exactly for nonsingular rings, (see (Sandomierski, 1968, Lemma 2.3)). We get the following result by Corollary 4.5.

Corollary 4.2 *A ring R is right C-ring and right nonsingular if and only if neat-flat modules are nonsingular.*

Proposition 4.6 *Let R be a right C-ring. An R-module M is neat-flat (weakly-flat) if and only if* $\mathrm{Soc}(M) = M\,\mathrm{Soc}(R_R)$.

Proof Necessity is clear. For the sufficiency suppose $M \cong F/K$ for some free module F and a submodule K of F. Assume K is not closed in F. Then there is a submodule T of F containing K essentially. Now $\mathrm{Soc}(T/K) \neq 0$ since T/K is singular and R is right C-ring. Let A be a complement of K in F. Then $A \oplus K$ is essential in F, and so $\mathrm{Soc}(F) = \mathrm{Soc}(A) \oplus \mathrm{Soc}(K)$. We get $\mathrm{Soc}(\frac{F}{K}) = (\frac{F}{K})\,\mathrm{Soc}(R_R) = \frac{\mathrm{Soc}(F)+K}{K} = \frac{\mathrm{Soc}(A)+K}{K}$. Therefore $\frac{T}{K} \cap \frac{\mathrm{Soc}(A)+K}{K} \neq 0$, and this implies $A \cap K \neq 0$, a contradiction. Hence K is a closed submodule of F, and so M is weakly-flat. □

A module M is said to be an extending module or a CS-module if every closed submodule of M is a direct summand of M. R is a right CS ring if R_R is CS. M is called (countably) $\sum$-CS module if every direct sum of (countably many) copies of M is CS, (see, for example, (Dung and Wisbauer, 1994)). The $\sum$-CS rings were first introduced and termed as co-H-rings in (Oshiro, 1984).

Theorem 4.2 *Let R be a ring. The following are equivalent.*

(1) Every neat-flat right R-module is projective.

(2) R is a right $\sum$-CS ring.

Proof (1) ⇒ (2) Let P be a projective R-module and N a closed submodule of P. Then N is a neat submodule of P. So P/N is neat-flat by Lemma 4.3, and hence P/N is projective by (1). Therefore the sequence $0 \to N \to P \to P/N \to 0$ splits, and so N is a direct summand of P. Hence R is a $\sum$-CS ring.

(2) ⇒ (1) Every right $\sum$-CS ring is both right and left perfect by (Oshiro, 1984, Theorem 3.18). Hence, R is a right C-ring by (Anderson and Fuller, 1992, Theorem 28.4). Let M be a neat-flat right R-module. Then there is a neat exact sequence $\mathbb{E}: \ 0 \to K \hookrightarrow P \to \ M \to 0$ with P projective by Lemma 4.3. Since R is right C-ring, K is closed in P by Theorem 2.9. Hence the sequence $\mathbb{E}$ splits by (2), and so M is projective. □

Theorem 4.3 *Let R be a ring. The following are equivalent.*

(1) Every finitely generated neat-flat right R-module is projective.

(2) R is a right C-ring and every finitely generated free right R-module is extending.

Proof (1) ⇒ (2) Let I be an essential right ideal of R with $\mathrm{Soc}(R/I) = 0$. Then $\mathrm{Hom}(S, R/I) = 0$ for each simple right R-module S and hence I is neat ideal of R. So R/I is neat-flat by Lemma 4.3. But it is projective by (1), and so I is direct summand of R. This is contradict with essentiality of I in R. So R is a right C-ring.
Let F be a finitely generated free right R-module and K a closed submodule of F. Since every closed submodule is neat, F/K is neat-flat by Lemma 4.3. Then F/K is projective by (1), and so K is a direct summand of F.

(2) $\Rightarrow$ (1) Let M be a finitely generated neat-flat right R-module. Then there is an exact sequence $0 \rightarrow \text{Ker}(f) \hookrightarrow F \rightarrow M \rightarrow 0$ with F finitely generated free right R-module. By Lemma 4.3 Ker(f) is a neat submodule of F. Since R is C-ring, Ker(f) is a closed submodule of F by Theorem 2.9. Then $0 \rightarrow \text{Ker}(f) \hookrightarrow F \rightarrow M \rightarrow 0$ is a split exact sequence. So M is projective. □

Following the proof of Theorem 4.3, we obtain the following corollary.

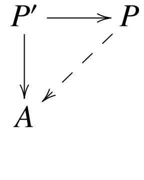
Corollary 4.3 *Every cyclic neat-flat right R-module is projective if and only if R is both right CS and right C-ring.*

Remark 4.3 *Let M be a right R-module. Then the socle series* $\{S_\alpha\}$ *of M is defined as:* $S_1 = \text{Soc}(M)$, $S_\alpha / S_{\alpha-1} = \text{Soc}(M/S_{\alpha-1})$, *and for a limit ordinal* α, $S_\alpha = \cup_{\beta<\alpha} S_\beta$. *Put* $S = \cup\{S_\alpha\}$. *Then, by construction M/S has zero socle. M is semiartinian (i.e. every proper factor of M has a simple module) if and only if S = M (see, for example, (Dung and Wisbauer, 1994)).*

From the proof of Theorem 4.2, we see the condition that every free right R-module is extending implies R is a right C-ring. In the following example we show that if every finitely generated free right R-module is extending then R need not be right C-ring. Hence the right C-ring condition in Theorem 4.3 is necessary.

Example 4.1 *Let R be the ring of all linear transformations (written on the left) of an infinite dimensional vector space over a division ring. Then R is prime, regular, right self-injective and* $\text{Soc}(R_R) \neq 0$ *by (Goodearl, 1979, Theorem 9.12). As R is a prime ring,* $\text{Soc}(R_R)$ *is an essential ideal of* R_R. *Let S be as in Remark 4.3, for M = R. Then* $S \neq R$, *by (Clark and Smith, 1996, Lemma 1(2)). Since R/S has zero socle, S is a neat submodule of* R_R. *On the other hand, S is not a closed submodule of R, otherwise S would be a direct summand of R because R is right self injective (i.e. extending). Therefore R is not a right C-ring. Also, as R is right self injective* R^n *is injective, and so extending for every* $n \geq 1$.

Following (Megibben, 1970), an R-module A is absolutely pure if and only if every diagram



with P' finitely generated and P projective can be completed to a commutative diagram.

Recall that, a ring R is *right IF* if every injective right R-module is flat. It is known that R is a right IF ring if and only if every finitely presented right R-module is a submodule of a free module ((Colby, 1975, Theorem 1)). A ring R is called *right Kasch* if any simple right R-module embeds in R. Now, we consider when every injective right R-module is neat-flat.

Theorem 4.4 *For any ring R, the following conditions are equivalent.*

(1) R is a right Kasch ring.

(2) Every absolutely pure right R-module is neat-flat.

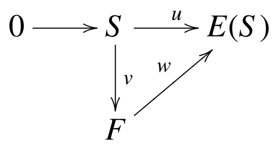
(3) Every injective right R-module is neat-flat.

(4) The injective hull of every simple right R-module is neat-flat.

Proof (1) $\Rightarrow$ (2) Let E be an absolutely pure right R-module and M a simple right R-module. Since R is a right Kasch ring, there is an embedding $\iota : M \to R$. Let $f : M \to E$ be a homomorphism. As E is absolutely pure, there is a homomorphism $g : R \to E$ such that $f = g\iota$. That is, E is simple-projective. Hence E is neat-flat by Theorem 4.1.

(2) $\Rightarrow$ (3) and (3) $\Rightarrow$ (4) are trivial.

(4) $\Rightarrow$ (1) Let S be a simple right R-module and $u : S \to E(S)$ the inclusion homomorphism. Since $E(S)$ is neat-flat, by Theorem 4.1, there is a finitely generated free module F and homomorphisms v, w such that the following diagram commutes,



Since wv is a monomorphism, v is a monomorphism, and so (1) holds. □

Corollary 4.4 *Let R be a ring. Then R is a right Kasch ring if and only if for every free left R-module F, F^+ is neat-flat.*

Proof Suppose R is a right Kasch ring and let F be a free left R-module. By (Rotman, 1979, Theorem 3.52), F^+ is an injective right R-module. Then F^+ is neat-flat by Theorem 4.4. Conversely, let M be any injective right R-module. There is a free left R-module F and an epimorphism $F \to M^+$ from which we obtain an exact sequence $0 \to M^{++} \to F^+$. Since F^+ is neat-flat and $M \leq M^{++}$, M is a direct summand of F^+, and so M is neat-flat. Hence R is a right Kasch ring by Theorem 4.4. □

A ring R is a QF ring if and only if every injective right R-module is projective if and only if every projective right R-module is injective. In (Oshiro, 1984) proved that R is a QF ring if and only if R is a right$\sum$-CS ring and $Z(R_R) = J(R)$, where $Z(R_R)$ is the center of R.

Corollary 4.5 *A ring R is right Kasch and right $\sum$-CS if and only if R is QF.*

Proof Necessity: By Theorem 4.2, every neat-flat right R-module is projective. Thus every injective right R-module is projective by Theorem 4.4, hence R is QF. Conversely, R is right Kasch by Theorem 4.4, and R is right $\sum$-CS by (Oshiro, 1984, Theorem 4.3). □

4.3. Rings Whose Simple Right R-Modules are Finitely Presented

Over a right N-ring every simple right R-module S and its transpose $Tr(S)$ are finitely presented. Using Theorem 2.2, we obtain the following characterization of neat-flat modules.

Theorem 4.5 *Let R be a right N-ring. Then M is a neat-flat right R-module if and only if* $\mathrm{Tor}_1^R(M, Tr(S)) = 0$ *for each simple right R-module S.*

Proof Let M be an R-module and $\mathbb{E} : 0 \to K \xrightarrow{f} F \to M \to 0$ a short exact sequence with F projective. Let S be simple right R-module. Tensoring $\mathbb{E}$ by $Tr(S)$ we get the exact sequence

$$0 = \mathrm{Tor}_1^R(F, Tr(S)) \to \mathrm{Tor}_1^R(M, Tr(S)) \to K \otimes Tr(S) \xrightarrow{f\otimes 1_{Tr(S)}} F \otimes Tr(S).$$

Now, suppose M is neat-flat. Then $\mathbb{E}$ is neat-exact by Lemma 4.3. So $f \otimes 1_{Tr(S)}$ is monic, by Theorem 2.2. Hence $\mathrm{Tor}_1^R(M, Tr(S)) = 0$.

Conversely, suppose $\mathrm{Tor}_1^R(M, Tr(S)) = 0$ for each simple right R-module S. Then the sequence $0 \to K \otimes Tr(S) \to F \otimes Tr(S)$ is exact, and so the sequence $0 \to K \to F \to M \to 0$ is neat-exact by Theorem 2.2. Then M is neat-flat by Lemma 4.3. □

Theorem 4.6 *Let R be a right N-ring. Then M is an absolutely s-pure left R-module if and only if* $\mathrm{Ext}_R^1(Tr(S), M) = 0$ *for each simple right R-module S.*

Proof ($\Rightarrow$) There is an s-pure exact sequence $\mathbb{E} : 0 \to M \to E(M) \xrightarrow{f} K \to 0$ by Lemma 4.1(3). Let S be a simple right R-module. Then the sequence

$$\mathrm{Hom}_R(Tr(S), E(M)) \xrightarrow{f^*} \mathrm{Hom}_R(Tr(S), K) \to \mathrm{Ext}^1(Tr(S), M) \to 0$$

is exact. Since the sequence $\mathbb{E}$ is s-pure exact and R is a right N-ring, f^* is an epimorphism by Theorem 2.2. So that $\mathrm{Ext}^1_R(Tr(S), M) = 0$.

($\Leftarrow$) Consider the exact sequence $\mathbb{E} : 0 \to M \to E(M) \to K \to 0$. Let S be a simple right R-module. Then $\mathrm{Hom}_R(Tr(S), E(M)) \to \mathrm{Hom}_R(Tr(S), K) \to 0$ is exact by the hypothesis, and so $\mathbb{E}$ is s-pure exact by Theorem 2.2. Then M is absolutely s-pure by Lemma 4.1(3). □

Remark 4.4 *(1) By (Rotman, 1979, Theorem 9.51),* $\mathrm{Tor}^R_1(B^+, A) \cong \mathrm{Ext}^1_R(A, B)^+$ *for any finitely presented left R-module B and a left R-module A.*

(2) *Let R be a right N-ring and M a right R-module. If K is a pure submodule of M, then* $\mathrm{Hom}(S, M) \to \mathrm{Hom}(S, M/K) \to 0$ *is an epimorphism for each simple right R-module S by (Facchini, 1998, 1.4. page 12). Hence K is a neat submodule of M, and in particular flat right R-modules are neat-flat.*

(3) *By (Enochs and Jenda, 2000, Proof of Proposition 5.3.9.), every right (left) R-module M is a pure submodule of the pure-injective right (left) R-module* M^{++}.

The following result will be used in the sequel.

Theorem 4.7 *(Cheatham and Stone, 1981, Theorem 1) The following statements are equivalent:*

(1) R is a right coherent ring.

(2) M_R *is FP-injective if and only if* M^+ *is a flat module.*

(3) M_R *is FP-injective if and only if* M^{++} *is an injective left R-module.*

(4) ${}_RM$ *is flat if and only if* M^{++} *is a flat left R-module.*

Proposition 4.7 *Let R be a right N-ring. Then the following are hold.*

(1) M is a neat-flat right R-module if and only if M^+ *is an absolutely s-pure R-module.*

(2) M is an absolutely s-pure left R-module if and only if M^+ *is a neat-flat R-module.*

(3) M is an absolutely s-pure left R-module if and only if M^{++} is an absolutely s-pure R-module.

(4) M is a neat-flat right R-module if and only if M^{++} is a neat-flat right R-module.

(5) The class of absolutely s-pure left R-modules is closed under direct products and pure quotients.

(6) The class of neat-flat right R-modules is closed under direct products and pure quotients.

Proof (1) This holds by Theorem 4.5 and Theorem 4.6, and the standard adjoint isomorphism $\mathrm{Ext}^1_R(Tr(S), M^+) \cong \mathrm{Tor}^R_1(M, Tr(S))^+$.

(2) Let M be a left R-module and S a simple right R-module. Then we have $\mathrm{Tor}^R_1(M^+, Tr(S)) = \mathrm{Ext}^1_R(Tr(S), M)^+$ by Remark 4.4(1). Hence, M is an absolutely s-pure left R-module if and only if M^+ is a neat-flat right R-module by Theorem 4.5 and Theorem 4.6.

(3) and (4) are clear by (1) and (2).

(5) Let $\{M_i\}_{i\in J}$ be a family of absolutely s-pure left R-modules and S a simple right R-module. Then $\mathrm{Ext}^1_R(Tr(S), \prod_{i\in J} M_i) \cong \prod_{i\in J} \mathrm{Ext}^1_R(Tr(S), M_i) = 0$ by Theorem 4.6. Hence $\prod_{i\in J} M_i$ is absolutely s-pure by Theorem 4.6, again.

Suppose M is an absolutely s-pure left R-module and N a pure submodule of M. Then the exact sequence $0 \to (M/N)^+ \to M^+ \to N^+ \to 0$ splits. By (2), M^+ is neat-flat, and so is $(M/N)^+$. Then M/N is absolutely s-pure by (2), again.

(6) Let $\{M_i\}_{i\in J}$ be a family of neat-flat right R-modules. Then $\bigoplus_{i\in J} M_i$ is neat-flat by Proposition 4.4. So $(\bigoplus_{i\in J} M_i)^{++} \cong (\prod_{i\in J} M_i^+)^+$ is neat-flat by (4). But $\bigoplus_{i\in J} M_i^+$ is a pure submodule of $\prod_{i\in J} M_i^+$, hence $(\prod_{i\in J} M_i^+)^+ \to (\bigoplus_{i\in J} M_i^+)^+ \to 0$ is a splitting epimorphism. Therefore $(\bigoplus_{i\in J} M_i^+)^+ \cong \prod_{i\in J} M_i^{++}$ is neat-flat. Since $\prod_{i\in J} M_i$ is a pure submodule of $\prod_{i\in J} M_i^{++}$, the module $\prod_{i\in J} M_i$ is neat-flat by Theorem 4.1 and Proposition 4.4.

Let N be a pure submodule of a neat-flat right R-module M, then the pure exact sequence $0 \to N \to M \to M/N \to 0$ induces the split exact sequence $0 \to (M/N)^+ \to M^+ \to N^+ \to 0$. Thus $(M/N)^+$ is absolutely s-pure, since M^+ is absolutely s-pure by (1). So M/N is neat-flat by (1), again.

□

Corollary 4.6 *If R is a right coherent and right Kasch ring, then ${}_RR$ is absolutely s-pure.*

Proof Every simple right R-module S embeds in R, hence S finitely presented. Thus R is a right N-ring. Since R is right Kasch, ${}_RR^+$ is neat-flat by Corollary 4.4. Then ${}_RR$ is absolutely s-pure by Proposition 4.7. □

In (Jensen and Simson, 1979, Proposition 1.4), authors proved that if R is a left coherent ring, then every pure-injective flat right R-module is projective if and only if R is right perfect.

Theorem 4.8 *Let R be a right N-ring. Then the following are equivalent.*

(1) R is a right $\sum$-CS ring.

(2) Every pure-injective neat-flat right R-module is projective.

(3) For any left R-module M, M is absolutely s-pure if and only if M^+ is projective.

(4) Every absolutely s-pure left R-module is injective and R is right perfect.

Proof (1) ⇒ (2) Follows by Theorem 4.2.

(2) ⇒ (3) Let M be a left R-module. By Proposition 4.7, M is absolutely s-pure if and only if M^+ is neat-flat. Since M^+ is pure-injective, (2) completes the proof of (3).

(3) ⇒ (1) Firstly, we show that R is left coherent and right perfect. Let F be an absolutely s-pure left R-module. Then F^{++} is injective by (Rotman, 1979, Theorem 3.52) since F^+ is projective by (3). Since the monomorphism $F \to F^{++}$ is pure and F^{++} is injective, F is absolutely pure. Then, the classes of absolutely s-pure R-modules and absolutely pure R-modules coincide. Hence F is absolutely pure if and only if F^+ is projective by (3). Then R is left coherent and right perfect by (Cheatham and Stone, 1981, Theorem 3).

Let M be a neat-flat right R-module. We claim that M is a flat right R-module. By Proposition 4.7 and (3), M^{++} is projective. Consider the pure exact sequence

$$0 \to M \to M^{++} \to M^{++}/M \to 0.$$

Since flat modules are closed under pure submodules, M is flat. By the first part of the proof M is projective since R is right perfect. Then R is a right $\sum$-CS ring by Theorem 4.2.

(3) ⇒ (4) In the proof of (3) ⇒ (1), we show that the classes of absolutely s-pure R-modules and absolutely pure R-modules coincide. Since the condition (3) implies R is a right $\sum$-CS ring, R is left Artinian by (Oshiro, 1989, Proposition 3.2). Hence R is right

perfect, and every absolutely s-pure left R-module is injective by (Cheatham and Stone, 1981, Theorem 2).

(4) $\Rightarrow$ (1) The condition (4) implies that every neat-flat right R-module is flat by Lemma 4.4. But R is right perfect, so neat-flat right R-modules are projective. Hence R is right $\sum$-CS by Theorem 4.2. □

Lemma 4.4 *The following are equivalent for a right N-ring R.*

(1) Every absolutely s-pure left R-module is absolutely pure.

(2) Every neat-flat right R-module is flat.

Proof (1) $\Rightarrow$ (2) Let M be a neat-flat right R-module. Then M^+ is absolutely s-pure by Proposition 4.7(1), and so M^+ is absolutely pure by (1). But M^+ is pure-injective, so it is injective. Hence M is flat by (Rotman, 1979, Theorem 3.52).

(2) $\Rightarrow$ (1) Let M be an absolutely s-pure left R-module. Then M^+ is neat-flat by Proposition 4.7(2), and so M^+ is flat by (2). Hence M^{++} is injective by (Rotman, 1979, Theorem 3.52). Then M is absolutely pure since M is a pure submodule of the injective module M^{++}. □

Proposition 4.8 *(Clark et al., 2006, 10.15(3)) A two-sided Noetherian hereditary ring is a left (and right) C-ring.*

Theorem 4.9 *Let R be a commutative ring. Consider the following statements.*

(1) Every neat-flat R-module is flat and R is nonsingular.

(2) R is semihereditary.

Then (1) $\Rightarrow$ (2)*. If R is Noetherian, then* (2) $\Rightarrow$ (1).

Proof (1) $\Rightarrow$ (2) Let M be a nonsingular module. Then, M is neat-flat and, by (1), M is flat. Therefore R is semihereditary by (Goodearl, 1972, Proposition 2.3).

(2) $\Rightarrow$ (1) First note that R is a C-ring by Proposition 4.8. Let M be a neat-flat R-module. Consider the sequence $0 \to H \to F \to M \to 0$ with F projective. Then H is neat in F by Lemma 4.3, and it is closed in F by Theorem 2.9. Since R is semihereditary, R is nonsingular. So F is nonsingular. Then M is nonsingular by (Sandomierski, 1968, Lemma 2.3). Now, M is flat by (Goodearl, 1972, Proposition 2.3). □

By (Megibben, 1970, Theorem 3), a ring R is right Noetherian if and only if every absolutely pure right R-module is injective. For absolutely s-pure modules, we have the following result.

Theorem 4.10 *The following are equivalent for a commutative ring R.*

(1) Every absolutely s-pure R-module is injective and R is nonsingular.

(2) R is a (semi)hereditary Noetherian ring.

Proof (1) $\Rightarrow$ (2) Note that each absolutely pure R-module is absolutely s-pure, and so injective by (1). Then R is Noetherian by (Megibben, 1970, Theorem 3). The rest of (2) follows by Lemma 4.4 and Theorem 4.9.

(2) $\Rightarrow$ (1) By Lemma 4.4 and Theorem 4.9, absolutely s-pure R-modules are absolutely pure. But R is Noetherian, so every absolutely s-pure R-module is injective by (Megibben, 1970, Theorem 3). □

Corollary 4.7 *The following are equivalent for a commutative domain R.*

(1) Every absolutely s-pure module is injective.

(2) R is a Dedekind domain.

Definition 4.1 *A right R-module M is called* max-flat *if* $\mathrm{Tor}^1_R(M, R/I) = 0$ *for every maximal left ideal I of R (see, Xiang (2010)).*

Note that a R-module M is max-flat if and only if M^+ is m-injective by the standard isomorphism $\mathrm{Ext}^1(S, M^+) \cong \mathrm{Tor}_1(M, S)^+$ for all simple left R-module S.

Using the similar arguments of (Xiang, 2010, Theorem 4.5), one can prove the following.

Lemma 4.5 *Let R be a right N-ring. The followings hold.*

(1) A right R-module M is m-injective if and only if M^+ is max-flat.

(2) A right R-module M is m-injective if and only if M^{++} is m-injective.

(3) A left R-module M is max-flat if and only if M^{++} is max-flat.

Proposition 4.9 *Assume that every neat-flat R-module is flat. Then the following hold.*

(1) Every m-injective R-module is FP-injective.

(2) For every right R-module M, M is max-flat if and only if M is flat.

Proof

(1) Let M be an m-injective R-module. By Theorem 2.8, $\mathrm{Soc}(E(M)/M) = 0$, and so $E(M)/M$ is neat-flat. Then $E(M)/M$ is flat by our hypothesis. Hence M is a pure submodule of $E(M)$, and so M is an FP-injective module.

(2) Assume M is a max-flat right R-module. Then M^+ is m-injective, and so it is FP-injective by (1). But M^+ pure-injective by (Enochs and Jenda, 2000, Proposition 5.3.7), so M^+ is injective. Then M is flat by (Rotman, 1979, Theorem 3.52). The converse statement is clear.

□

Proposition 4.10 *Let R be a ring. Consider the following statements.*

(1) R is right N-ring and every neat-flat right R-module is flat.

(2) A right R-module M is m-injective if and only if M^+ is flat.

(3) R is right coherent and a right R-module M is m-injective if and only if M is FP-injective, .

Then (1) ⇒ (2) ⇔ (3).

Proof (1) ⇒ (3) By Proposition 4.9(1), every m-injective R-module is FP-injective. On the other hand, every FP-injective R-module is m-injective since every simple right R-module is finitely presented by (1). Then, for every R-module M, M is FP-injective if and only if M is m-injective, if and only if M^+ is max-flat by Lemma 4.5(2), if and only if M^+ is a flat module by Proposition 4.9(2). Hence R is a right coherent ring by Theorem 4.7. This proves (3).

(2) ⇒ (3) Let M be a left R-module. We claim that M is a flat R-module if and only if M^{++} is a flat module. If M is flat, then M^+ is injective by (Rotman, 1979, Theorem 3.52), and so M^{++} is flat left R-module by (2). Conversely, if M^{++} is a flat module, then M is flat since M is a pure submodule of M^{++} by (Enochs and Jenda, 2000, Proof of Proposition 5.3.9.), and flat modules are closed under pure submodules (see, (Lam, 2001, Corollary 4.86)). So R is a right coherent ring by Theorem 4.7. The last part of (3) follows by (2) and Theorem 4.7 again.

(3) ⇒ (2) By Theorem 4.7. □

Proposition 4.11 *A finite direct product of left C-rings is also a left C-ring.*

Proof Assume R is a finite direct product of the left C-rings $R_1, R_2 \dots R_n$. We will show that $\mathrm{Soc}(R/I) \neq 0$ for each essential left ideal I of R. By assumption, $I = I_1 \times I_2 \times \dots \times I_n$, where $I_i \leq R_i$ for $i = 1, 2, \ldots, n$. Since I is essential ideal of R, I_i is essential ideal of R_i for $i = 1, 2, \ldots, n$. Then $\mathrm{Soc}(R_i/I_i) \neq 0$ for $i = 1, 2, \ldots, n$. $\mathrm{Soc}(R/I) \cong \prod_i^n \mathrm{Soc}(R_i/I_i) \neq 0$, as desired. □

A module M is called semiartinian if every nonzero homomorphic image of M contains a simple module. Set $Sa(M) := \Sigma_{M_i \in \Lambda} M_i$, where Λ is the class of all semiartinian submodules M_i of M. Then $M/Sa(M)$ is neat-flat for each R-module M since $\mathrm{Soc}(M/Sa(M)) = 0$, see (Kasch, 1982, pp-238).

Theorem 4.11 *Let R be a commutative Noetherian ring. The following are equivalent.*

(1) Every neat-flat module is flat.

(2) Every absolutely s-pure module is absolutely pure.

(3) $R \cong A \times B$, wherein A is QF-ring and B is hereditary.

Proof (1) $\Leftrightarrow$ (2) is by Lemma 4.4.

(1) $\Rightarrow$ (3) By the assumption, $R/Sa(R)$ is projective and $Sa(R)$ is direct summand of R, i.e. $R \cong A \times B$, where A is semiartinian and $\mathrm{Soc}(B) = 0$. We can assume R is Artinian or $\mathrm{Soc}(R) = 0$. In the former case, every neat-flat modules are projective by the assumption, and hence R is QF-ring by Theorem 4.2 and (Oshiro, 1984, Theorem 4.4). In the later case, let I be an ideal of R. Since $\mathrm{Soc}(R) = 0$, $\mathrm{Soc}(I) = 0$ and, by Corollary 4.3, I is flat. But R is Noetherian, and so I is finitely generated. Therefore I is projective and R is hereditary.

(3) $\Rightarrow$ (1) Assume that $R \cong A \times B$, wherein A is QF-ring and B is Noetherian distributive. Let M be a neat-flat module. Since $M = MA \oplus MB$, MA is neat-flat A-module and MB is neat-flat B-module. By Theorem 4.2, MA is projective A-module. MB is flat B-module by Corollary 4.2 and (Goodearl, 1972, Proposition 2.3). Therefore M is flat.

□

Theorem 4.12 *Let R be a commutative Noetherian ring. The following are equivalent.*

(1) Every weakly-flat module is weakly-injective.

(2) Every weakly-injective module is weakly-flat.

(3) Every neat-flat module is absolutely s-pure.

(4) Every absolutely s-pure module is neat-flat.

(5) Every neat-flat module is weakly-injective.

(6) Every absolutely s-pure module is weakly-flat.

(7) $R \cong A \times B$, wherein A is QF-ring and B is Artinian with $J^2(B) = 0$.

Proof (1) ⇔ (2) ⇔ (7) (Zöschinger, 2013, Satz 3.8).

(5) ⇒ (3) and (6) ⇒ (4) are clear.

(3) ⇒ (4) Let M be an absolutely s-pure R-module. M^+ is neat-flat by Proposition 4.7. By (3), M^+ is neat-flat. Again by Proposition 4.7, M^{++} is neat-flat. Since M is pure submodule of M^{++}, M is neat-flat by Proposition 4.4.

(4) ⇒ (3) Let M be a neat-flat R-module. M^+ is absolutely s-pure by Proposition 4.7. By (4), M^+ is neat-flat. Again by Proposition 4.7, M^{++} is absolutely s-pure. Since M is pure submodule of M^{++}, M is absolutely s-pure by Proposition 4.7.

(7) ⇒ (5) A finite direct product of C-rings is also a left C-ring by Proposition 4.11, and so R is C-ring. Then neat-flat R-modules are weakly-flat and, by (Zöschinger, 2013, Satz 3.8), neat-flat R-modules are weakly-injective.

(3) ⇒ (7) We just show that every finitely generated weakly-flat R-module is weakly-injective. Let N be a finitely generated weakly-flat R-module and $N \leq M$ any extension of N. N is neat-flat, and absolutely s-pure by (3). Then $NI = N \cap MI$ for each maximal ideal I of R by Fuchs (2012). Since N is finitely generated, it is coatomic (i.e. every submodule $U \lneqq N$ lies in a maximal submodule of N). Hence N is coclosed in M by (Zöschinger, 2006, Lemma A.3(b)). Then N is weakly-injective.

The rest of proof by similar arguments as in proof of ($i' \Rightarrow iii$) of Satz 3.8 in Zöschinger (2013).

(4) ⇒ (6) By equality of (4) ⇔ (7), $R \cong A \times B$, wherein A is QF-ring and B artinian with $J^2(B) = 0$. R is C-ring by Proposition 4.11. Then neat-flat R-modules are weakly-flat. Therefore, the claim follows by (4). □

4.4. Localization of Neat-Flat Modules

In this section we shall consider localization of neat exact sequences and neat-flat modules on commutative N-rings.

A submodule A of B is neat in B if and only if the following holds: if for $b \in B$ and for a maximal ideal P we have $Pb \leq A$, then there is an element $a \in A$ such that $P(b - a) = 0$, (see (Fuchs, 2012, Lemma 2.1)).

We can also rephrase the definition in terms of systems of equations to make the resemblance to purity more transparent: if the maximal ideal P is generated by the elements r_i ($i \in I$), then we consider the system of equations $r_i x = a_i \in A, (i \in I)$ with the single unknown x and constants in A.

Lemma 4.6 *(Fuchs, 2012, Lemma 2.2) A is neat in B if and only if such systems are solvable in A, whenever they are solvable in B.*

Let R be a commutative ring and M a finitely presented R-module. It is well known that M is projective if and only if M_P is a free R_P-module for each prime ideal P of R, if and only if, M_P is a free R_P-module for each maximal ideal P of R.

Lemma 4.7 *Let R be a commutative N-ring. Then, a short exact sequence* $0 \to A \to B \to C \to 0$ *is neat if and only if* $0 \to A_P \to B_P \to C_P \to 0$ *is neat exact for each maximal ideal P of R.*

Proof ($\Rightarrow$) Assume that $0 \to A \xrightarrow{f} B \to C \to 0$ is neat exact sequence of R-modules and P is a maximal ideal of R. We show that the exact sequence

$$0 \to A_P \xrightarrow{f_P} B_P \to C_P \to 0$$

is neat exact of R_P-modules is neat. Assume that I is an index set, and

$$\frac{r_i}{s_i} x = \frac{f(a_i)}{s_i^{'}} \in f_P(A_P), r_i \in R_P, s_i, s_i^{'} \in R - P, a_i \in A, i \in I$$

is a system of equations which is solvable in B_P, i.e., $\frac{r_i}{s_i}\frac{b}{l} = \frac{f(a_i)}{s_i^{'}}$ for some $b \in B, l \in R - P$. Thus for each $i \in I$, there exists an element $t_i \in R - P$ such that $t_i r_i s_i^{'} b = t_i s_i l f(a_i) \in f(A)$. Now, consider the system of equations $t_i r_i s_i^{'} x = t_i s_i l f(a_i) \in f(A)$ which is also solvable in B. Since $f(A)$ is a neat submodule of B, by Lemma 4.6, there exists an $f(a) \in f(A)$ such that $t_i r_i s_i^{'} f(a) = t_i s_i l f(a_i)$ for each $i \in I$. Thus $\frac{r_i}{s_i}\frac{f(a)}{l} = \frac{f(a_i)}{s_i^{'}}$, i.e., the system of equations$\frac{r_i}{s_i} x = \frac{f(a_i)}{s_i^{'}}$ is solvable in $f_P(A_P)$. Therefore, by Lemma 4.6, $f_P(A_P)$ is a neat submodule of B_P.

($\Leftarrow$) Assume that $0 \to A \to B \to C \to 0$ is not neat exact sequence of R-modules but $0 \to A_P \to B_P \to C_P \to 0$ is neat exact for each maximal ideal P of R. Then there is a simple R-module $S = R/P$ where P is maximal ideal such that S has not the projective property with respect to all neat exact sequences. Therefore, by (2), the natural homomorphism

$$\mathrm{Hom}_{R_P}(S_P, B_P) \to \mathrm{Hom}_{R_P}(S_P, C_P)$$

is an epimorphism. Since S is finitely presented, we have the commutative diagram by (Rotman, 1979, Lemma 4.87),

$$\begin{array}{ccccccc} \mathrm{Hom}_{R_P}(S_P, B_P) & \longrightarrow & \mathrm{Hom}_{R_P}(S_P, C_P) & \longrightarrow & 0 & & (*) \\ \downarrow \cong & & \downarrow \cong & & & & \\ \mathrm{Hom}_R(S, B)_P & \longrightarrow & \mathrm{Hom}_R(S, C)_P & \longrightarrow & 0 & & (**) \end{array}$$

Since the $(*)$ row is exact, the $(**)$ row is also exact.

Note that for a maximal ideal $Q \neq P$, $S_Q = R_P \otimes_R S = 0$. Therefore, $\mathrm{Hom}_R(S, B)_Q = \mathrm{Hom}_R(S, C)_Q = 0$. Then $\mathrm{Hom}_R(S, B)_P \to \mathrm{Hom}_R(S, C)_P$ is an epimorphism for every maximal ideal P. Thus, by (Rotman, 1979, Lemma 4.90), $\mathrm{Hom}_R(S, B) \to \mathrm{Hom}_R(S, C)$ is a surjection. This contradict with our assumption, hence $0 \to A \to B \to C \to 0$ is neat exact sequence of R-modules. □

Corollary 4.8 *Let R be a commutative N-ring. A module M is neat-flat if and only if, for all maximal ideals P of R, M_P is neat-flat R_P-module.*

4.5. Absolutely *s*-Pure Covers and Neat-Flat Envelopes

It is known that a ring R is left Noetherian if and only if every left R-module M has an injective cover(see (Enochs and Jenda, 2000, Theorem 5.4.1). It is known that R is a left coherent right perfect ring if and only if every right R-module has a projective (pre)envelope, see (Asensio Mayor and Martínez Hernández, 1993) So the questions we ask are: What conditions on R imply that every left R-module M has an absolutely s-pure (pre)cover, and what conditions on R imply that every left R-module M has a neat-flat (pre)envelope?

Proposition 4.12 *Let R be a right N-ring. The following hold.*

(1) Every left R-module has an absolutely s-pure preenvelope.

(2) Every left R-module has an absolutely s-pure cover.

(3) Every right R-module has a neat-flat preenvelope.

(4) Every right R-module has a neat-flat cover.

Proof (1) Absolutely s-pure left R-modules are closed under pure submodules by Proposition 4.1. Then the claim follows by Proposition 4.7(5) and Lemma 2.1(1).

(2) Absolutely s-pure left R-modules are closed under direct sums and pure quotients by Proposition 4.1 and Proposition 4.7(5). Hence every R-module has an absolutely s-pure cover by Lemma 2.1(2).

(3) Neat-flat right R-modules are closed under direct product by Proposition 4.7(6) and pure submodules by Proposition 4.4. Then (3) follows by Lemma 2.1(1).

(4) Neat-flat right R-modules are closed under pure quotients by Proposition 4.7(6), and under direct sums by Proposition 4.4. Hence every module has a neat-flat cover by Lemma 2.1(2). □

A left R-module E is called *s-pure injective* if it is injective with respect to s-pure short exact sequences. Note that for each simple right R-module S, S^+ is an s-pure injective left R-module by the standard adjoint isomorphism.

Proposition 4.13 *Absolutely s-pure cover of an s-pure injective left R-module is injective.*

Proof Let M be an s-pure injective left R-module. Let $f : F \to M$ be an absolutely s-pure cover of M. By Lemma 4.1, there is an s-pure exact sequence $0 \to F \overset{i}{\to} E \to L \to 0$ with E injective. Since M is s-pure injective, there exists a homomorphism $g : E \to M$ such that $f = gi$. Since E is absolutely s-pure, there exists $\alpha : E \to F$ such that $g = f\alpha$. Therefore $f = gi = f\alpha i$, and so $\alpha i = 1_F$. It follows that F is isomorphic to a direct summand of E, and hence F is injective. □

Recall that every left R-module has an epic flat envelope if and only if R is a right semihereditary ring, (Rada and Saorin, 1998, Corolary 4.3). It is well known that R is a right semihereditary ring if and only if every right R-module has a monic absolutely pure cover if and only if every homomorphic image of an injective right R-module is absolutely pure, (see, (Rada and Saorin, 1998, Corolary 4.13) and (Crivei and Torrecillas, 2008, Corollary 3.8)). Next, we consider when every left R-module has a monic absolutely s-pure cover.

Theorem 4.13 *The following are equivalent for a ring R.*

(1) Every s-pure injective left R-module has a monic injective cover.

(2) Every quotient of any absolutely s-pure left R-module is absolutely s-pure.

(3) Every quotient of any injective left R-module is absolutely s-pure.

(4) Every left R-module has a monic absolutely s-pure cover.

When R is a right N-ring, these conditions are equivalent to:

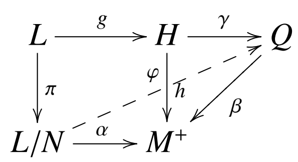
(5) Every submodule of any neat-flat right R-module is neat-flat.

(6) Every simple right R-module has an epic projective envelope.

(7) Every right R-module has an epic neat-flat envelope.

(8) For every simple right R-module S either $S^* = 0$ or S is projective (i.e. R is a right PS ring).

Proof (1) $\Rightarrow$ (2) Let L be an absolutely s-pure left R-module and $N \leq L$. Let M be a simple right R-module. For any homomorphism $\alpha : L/N \to M^+$, there exists an injective left R-module H, $g : L \to H$ and $h : H \to M^+$ such that $\alpha\pi = hg$ by Lemma 4.2, where $\pi : L \to L/N$ is the canonical epimorphism. By (1), M^+ has a monic injective cover $\beta : Q \to M^+$. Thus there exists $\gamma : H \to Q$ such that $h = \beta\gamma$, which implies that $\mathrm{Im}(\alpha) \subseteq \mathrm{Im}(\beta)$ and so there exists $\varphi : L/N \to Q$ such that $\beta\varphi = \alpha$.



That is, α factors trough the injective module Q. Therefore L/N is absolutely s-pure by Lemma 4.2.

(2) $\Rightarrow$ (4) Every pure quotient of any absolutely s-pure left R-module is absolutely s-pure by (2). By Proposition 4.1, absolutely s-pure left R-modules are also closed under direct sums. Now, the claim follows by (García R. and Torrecillas, 1994, Proposition 4).

(4) $\Rightarrow$ (1) by Proposition 4.13.

(2) $\Rightarrow$ (3) is obvious.

(3) $\Rightarrow$ (2) Suppose that N is a submodule of an absolutely s-pure left R-module L. Then there is an s-pure exact sequence $0 \to L \to E \to M \to 0$ with E injective by

Lemma 4.1. We have the pushout diagram

$$\begin{array}{ccccccccc} & 0 & \longrightarrow & L & \longrightarrow & E & \longrightarrow & M & \longrightarrow & 0 \\ & & & \downarrow & & \downarrow{\scriptstyle\gamma} & & \| & & \\ \mathbb{E}: & 0 & \longrightarrow & L/N & \longrightarrow & P & \longrightarrow & M & \longrightarrow & 0 \end{array}$$

Since s-pure exact sequences are closed under pushout, $\mathbb{E}$ is also s-pure exact. On the other hand, γ is an epimorphism, and so P is absolutely s-pure by (3). Therefore L/N is absolutely s-pure by Lemma 4.1.

(2) $\Rightarrow$ (5) Suppose N is a submodule of a neat-flat right R-module L. Then L^+ absolutely s-pure by Proposition 4.7. Clearly, N^+ is an epimorphic image of L^+, and so N^+ is absolutely s-pure by (2). Hence N is neat-flat by Proposition 4.7, again.

(5) $\Rightarrow$ (2) Suppose that N is a submodule of an absolutely s-pure left R-module L. We claim that L/N is absolutely s-pure. We have an exact sequence $0 \to (L/N)^+ \to L^+ \to N^+ \to 0$ where L^+ neat-flat since L is absolutely s-pure by Proposition 4.7. Then $(L/N)^+$ is neat-flat by (5), and so L/N absolutely s-pure by Proposition 4.7, again.

(5) $\Leftrightarrow$ (6) $\Leftrightarrow$ (7) $\Leftrightarrow$ (8) By Theorem 4.1 and (Mao, 2007, Theorem 3.7). □

Theorem 4.14 *Let R be a ring. Consider the following statements.*

(1) Every left R-module has a monic absolutely s-pure cover.

(2) Every simple left R-module has a monic injective cover.

(3) A simple left R-module S is either injective or $\mathrm{Hom}(E, S) = 0$ *for each injective left R-module E.*

Then (2) $\Leftrightarrow$ (3). *If R is commutative, then all these statements are equivalent.*

Proof First note that if R is commutative, then every simple R-module is s-pure injective by Proposition 2.3(3).

(1) $\Rightarrow$ (2) Since simple modules are s-pure injective, (2) follows by Theorem 4.13.

(2) $\Rightarrow$ (1) Similar to that proof of (1) $\Rightarrow$ (2) in Theorem 4.13, one can show that quotients of absolutely s-pure modules are absolutely s-pure. So, the claim follows by Theorem 4.13.

(2) $\Rightarrow$ (3) Let S be a simple left R-module. Suppose S is not injective. Then S has a monic injective cover $f : Q \to S$ by (2). Since S is simple and f is monic, $Q = 0$. Now, let E be an injective left R-module and $h \in \mathrm{Hom}(E, S)$. Then there is a homomorphism $g : E \to Q$ such that $h = fg = 0$. This proves (3).

(3) $\Rightarrow$ (2) Let S be a simple R-module. Then, by (3), S is either injective or $\mathrm{Hom}(E, S) = 0$ for each injective module E. If S is injective, then $1_S : S \to S$ is a monic injective cover of S. If $\mathrm{Hom}(E, S) = 0$ for each injective module E, then $0 \to S$ is a monic injective cover of S. □

Remark 4.5 *(1) For a left small ring R i.e.* $\mathrm{Rad}(E) = E$ *for every injective left R-module E, we have* $\mathrm{Hom}(E, S) = 0$ *for each simple left R-module S. If R is a left V-ring then every simple left R-module is injective. Hence, each simple left R-module has a monic injective cover over left small rings and over left V-rings.*

(2) Let R be a commutative semihereditary ring and S a simple R-module. Suppose $\mathrm{Hom}(E, S) \neq 0$ *for some injective R-module E. Then* $S \cong E/K$ *for some* $K \leq E$*, and so S is absolutely pure by (Crivei and Torrecillas, 2008, Corollary 3.8). But S is also s-pure injective by Proposition 2.3(3), so it is injective. Thus every simple R-module has a monic injective cover by Theorem 4.14(3).*

For a left coherent ring R, Mao and Ding Mao and Ding (2007) proved that, ${}_RR$ is absolutely pure if and only if every (finitely presented) left R-module has an epic absolutely pure cover (Mao and Ding, 2007, Corollary 3.2).

Theorem 4.15 *Let R be a right N-ring. Then the following are equivalent.*

(1) R is a right Kasch ring.

(2) Every left R-module has an epic absolutely s-pure cover.

(3) Every flat left R-module is absolutely s-pure.

(4) R is left absolutely s-pure.

Proof (1) $\Rightarrow$ (4) Suppose R is a right Kasch ring. Then $({}_RR)^+$ is neat-flat by Corollary 4.4, and so ${}_RR$ is absolutely s-pure by Proposition 4.7.

(4) $\Rightarrow$ (1) If ${}_RR$ is absolutely s-pure, then every free R-module F is absolutely s-pure. By Proposition 4.7, F^+ is neat-flat. Hence R is right Kasch by Corollary 4.4.

(1) $\Rightarrow$ (2) Since R is a right N-ring, every left R-module has an absolutely s-pure cover by Proposition 4.12. As R is a right Kasch ring, ${}_RR$ is absolutely s-pure by (4) $\Leftrightarrow$ (1). Hence any absolutely s-pure cover is epic by (Enochs and Jenda, 2000, pp-106).

(2) $\Rightarrow$ (3) Let F be a flat left R-module and $\varphi : M \to F$ be an epic absolutely s-pure cover of F. Then the pure exact sequence $0 \to \mathrm{Ker}(\varphi) \to M \to F \to 0$ induces the splitting exact sequence $0 \to F^+ \to M^+ \to (\mathrm{Ker}(\varphi))^+ \to 0$. Thus F^+ is neat-flat, since M^+ is neat-flat by Proposition 4.7. So F is absolutely s-pure by Proposition 4.7.

(3) $\Rightarrow$ (4) is trivial. □

We conclude the section with the following remark.

Remark 4.6 *Let R be a commutative ring. By Proposition 2.3(3), every simple module is s-pure injective. Hence absolutely s-pure cover of a simple module is injective by Proposition 4.13. Actually, by using the same method in the proof of Proposition 4.13 one can easily shows that absolutely pure cover of an s-pure injective left R-module is injective. Note that if R is left coherent then every left R-module has an absolutely pure cover by (Pinzon, 2008, Corollary 2.7). Hence, if R is a coherent ring, then every simple R-module has an injective cover. If R is coherent and absolutely pure, then every simple module has an epic injective cover by (Mao and Ding, 2007, Corollary 3.2).*

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