STRONGLY NONCOSINGULAR MODULES

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ABSTRACT

STRONGLY NONCOSINGULAR MODULES

The main purpose of this thesis is to investigate the notion of *strongly noncosingular* modules. We call a right R-module M *strongly noncosingular* if for every nonzero right R-module N and every nonzero homomorphism $f : M \to N$, $Im(f)$ is not a cosingular (or Rad-small) submodule of N in the sense of Harada. It is proven that (1) A right R-module M is strongly noncosingular if and only if M is coatomic and noncosingular; (2) a right perfect ring R is Artinian hereditary serial if and only if the class of injective right R-modules coincides with the class of (strongly) noncosingular right R-modules; (3) a right hereditary ring R is Max-ring if and only if absolutely coneat right R-modules are strongly noncosingular; (4) a commutative ring R is semisimple if and only if the class of injective R-modules coincides with the class of strongly noncosingular R-modules.

ÖZET

GÜÇLÜ DUAL TEKİL OLMAYAN MODÜLLER

Bu tezde, temel olarak güçlü dual tekil olmayan R-modüllerin yapısının çalışılması amaçlanmaktadır. Bir M sağ R-modülünün, sıfırdan farklı her N sağ R-modülü ve sıfırdan farklı her $f : M \to N$ homomorfizması için, $Gor(f)$ Harada anlamında N'nin eş-tekil sağ alt modülü değilse, M'ye *güçlü dual tekil olmayan* sağ R-modül denir. Bir R halkası için şunlar ispatlanmıştır: (1) Bir M sağ R-modülü güçlü dual tekildir ancak ve ancak M koatomik ve dual tekil olmayan modüldür. (2) Bir R sağ tam halkası Artin kalıtsal sıralıdır ancak ve ancak injektif sağ R-modüllerin sınıfı ve (güçlü) dual tekil olmayan sağ R-modüllerin sınıfı çakışır. (3) Bir R sağ kalıtsal halkası Max-halka'dır ancak ve ancak mutlak eşdüzenli sağ R-modüller güçlü dual tekildir. (4) Bir R değişmeli halkası yarı basittir ancak ve ancak injektif R-modüller sınıfı ve güçlü dual tekil olmayan R-modüller sınıfı çakışır.

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LIST OF ABBREVIATIONS

R	an associative ring with unit unless otherwise stated
$R - module$	right R- module
$\oplus_{i \in I} M_i$	direct sum of R- modules M_i
$\prod_{i \in I} M_i$	direct product of R- modules M_i
$\mathbb{Z}$	the ring of integers
$\mathbb{Q}$	the field of rational numbers
${}_R Ann(X)$	left annihilator of the set X
$Ann_R(X)$	right annihilator of the set X
$\subseteq$	submodule
$\subset$	proper submodule
$\ll$	small (or superfluous) submodule
$\trianglelefteq$	essential submodule
$Kerf$	the kernel of the map f
Imf	the image of the map f
$\cong$	isomorphic
$Hom(M, N)$	all R-module homomorphisms from M to N
$End(M)$	the endomorphism ring of a module M
$E(M)$	the injective envelope (hull) of a module M
$Soc(M)$	the socle of the R-module M
$Rad(M)$	the radical of the R-module M
$J(R)$	Jacobson radical of the ring R
Ω	the set of all maximal ideals of a ring
$Z(M_R)$	$\{x \in M \mid xI = 0 \ for \ some \ I \trianglelefteq R_R\}$
$Z^*(M)$	$\{m \in M \mid mR$ is a small module $\}$

CHAPTER 1

INTRODUCTION

Throughout this thesis, the rings that we consider are associative with an identity element and all modules are unitary right modules. Let M be an R-module. A submodule N of M is called small in M, denoted as $N \ll M$, if $N + K = M$ implies $K = M$ for any submodule K of M. Leonard defines a module M to be small if it is a small submodule of some R-module and he shows that M is small if and only if M is small in its injective hull (Leonard, 1966). The submodule of M is defined by Rayar as: $Z^*(M) = \{m \in M \mid Rm$ is a small module $\}$ (Rayar, 1971). Since $Rad(M)$ is the union of all small submodules of M, $Rad(M) \subseteq Z^*(M)$. We see that $Z^*(M) = M \cap Rad(E(M))$ and $Z^*(E) = Rad(E)$ for any injective module E. The functor $Z^*(M)$ also appear in (Ozcan, 2002). As the dual notion of singular (nonsingular), M is called cosingular (noncosingular) if $Z^*(M) = M$ ($Z^*(M) = 0$). In a series of papers, Özcan developed much of the properties of the functor $Z^*(M)$ and cosingular modules. For convenience in concepts, the cosingular R-modules are called Rad-small in this thesis.

Following (Talebi and Vanaja, 2002), a module M is called noncosingular if for every nonzero R-module N and every nonzero homomorphism $f : M \to N$, $Im(f)$ is not a small submodule of N. An R-module M is noncosingular if and only if every homomorphic image of M is weakly injective (Zöschinger, 2006). Recently, there is a significant interest to noncosingular R-modules, see (Kalati and Tütüncü, 2013), (Tribak, 2014), (Tütüncü et al. , 2014), (Tütüncü, Tribak, 2009), (Zöschinger, 2006).

Motivated by the noncosingular modules, in this thesis, we introduce the concept of strongly noncosingular modules. An R-module M is called *strongly noncosingular* if for every nonzero module N and every nonzero homomorphism $f : M \to N$, $Im(f)$ is not a Rad-small submodule of N. Clearly, since small modules are Rad-small, strongly noncosingular R-modules are noncosingular, but the converse is not true in general (see Example 3.1). Our aim is to work on the concept of strongly noncosingular modules and investigate the rings and modules that can be characterized via these modules.

In the second chapter of this dissertation, we give the definitions of some basic notions and investigate some of their properties which are useful tools for our further studies.

In chapter 3 we present some properties of strongly noncosingular R-modules. We also prove that an R-module M is strongly noncosingular if and only if M is coatomic and every simple homomorphic image of M is injective if and only if M is coatomic and noncosingular. It is known that the class of projective R-modules coincides with the class of nonsingular R-modules if and only if R is Artinian hereditary serial (Chatters and Khuri, 1980). Dually, it is shown that a right perfect ring R is Artinian hereditary serial if and only if the class of injective R-modules coincides with the class of (strongly) noncosingular R-modules. A right hereditary ring R is Max-ring if and only if absolutely coneat R-modules are strongly noncosingular. For a semilocal right Kasch ring, we show that, an R-module M is strongly noncosingular if and only if M is semisimple injective.

Chapter 4 deals with the structure of strongly noncosingular R-modules on commutative rings. We show that strongly noncosingular R-modules are exactly the semisimple injective modules on commutative noetherian rings. A commutative ring R is semisimple if and only if the class of injective modules coincides with the class of strongly noncosingular R-modules.

CHAPTER 2

RINGS AND MODULES

In this chapter, we shall give some basic notions and their properties which will be frequently used. The basic notions and all definitions not given here can be found in any standart text of Ring and Module theory (e.g. (Anderson and Fuller, 1992), (Wisbauer, 1991), (Clark et al., 2006) and (Lam, 1999)).

2.1. Rings and Their Homomorphisms

Definition 2.1 *A ring is a set R with two binary operations $+$ and $.$, called addition and multiplication, respectively, such that the following properties are satisfied:*

(1) *Addition is associative: For all $r, s, t, \in R$ we have $r + (s + t) = (r + s) + t$.*

(2) *Addition is commutative: For all $r, s, \in R, r + s = s + r$*

(3) *There is an element denoted by 0_R such that $r + 0_R = 0_R + r = r, \forall r \in R$. 0_R is called the zero element of the ring.*

(4) *Every element has an additive inverse, that is, for every $r \in R$ there is an element $-r \in R$ such that $r + (-r) = (-r) + r = 0_R$.*

(5) *Multiplication is associative: For every $r, s, t \in R$ we have $r.(s.t) = (r.s).t$.*

(6) *The left and right distributive laws hold: For all $r, s, t \in R, r.(s + t) = r.s + r.t$ and $(r + s).t = r.t + s.t$.*

Note on notation: For simplicity, we will denote $a.b$ by just ab, as long as there is no chance of ambiguity. Also, 0_R will be written as 0.

Definition 2.2 *A ring R is called commutative if $rs = sr$ for every $r, s \in R$. Also, a ring in which there is a multiplicative identity 1_R such that $1_R r = r1_R = r$ for all $r \in R$ is called a ring with identity. This multiplicative identity is called unity. We will denote the unity of a ring by 1 unless there is no ambiguity.*

Some rings satisfy certain multiplicative properties. Namely, a commutative ring R is called a field if every nonzero element has a multiplicative inverse, that is, for every $r \in R$, there exists $s \in R$ such that $rs = 1$. Also, R is called an integral domain if it has no divisors of zero, which means that, whenever $rs = 0$ for some $r, s \in R$ then either $r = 0$ or $s = 0$. Throughout our work, by a ring, we will always mean a ring with identity.

Definition 2.3 *A subset S of a ring R is called a subring if it is a ring with the operations of R, and $1_R = 1_S$ in case R has identity.*

A list of some examples of rings is:

(1) The set $\mathbb{Z}$ of integers is a commutative ring with usual addition and multiplication.

(2) The set of complex numbers is a field .

(3) For $n \geq 2$ the set $Mn(R)$ of all $n \times n$ matrices with coefficients in a ring R is a noncommutative ring with matrix addition and multiplication.

After these definitions and examples, we give the necessary and sufficient conditions to be a subring:

Proposition 2.1 *The Subring Criterion. Let$R$ be a ring and S be a subset of R. Then S is a subring of R if and only if for every $a, b \in S$:*

(i) $a - b \in S$;

(ii) $ab \in S$.

Now we can give the definiton of a ring homomorphism:

Definition 2.4 *Let R, S be rings. The mapping $f : R \to S$ is called a ring homomorphism if it satisfies the following:*

(i) $f(a + b) = f(a) + f(b)$ *for all* $a, b \in R$;

(ii) $f(ab) = f(a)f(b)$, *for all* $a, b \in R$;

(iii) $f(1_R) = 1_S$.

Special names are given to homomorphisms which satisfy certain properties. An onto homomorphism is called an epimorphism, and a one-to-one homomorphism is called a monomorphism. A one-to-one and onto ring homomorphism is called an isomorphism. If there is an isomorphism between two rings R and S, we say that R and S are isomorphic and denote it by $R \cong S$.

2.2. Ideals and Factor Rings

We go on developing the necessary tools for our work. Usage of ideals to develop ring theory is of great importance. In this section we will give the fundamental properties of ideals.

Definition 2.5 *Let R be a ring. We say that the subset I of R is a left ideal of R if the following are satisfied:*

(i) $I \neq \emptyset$*;*

(ii) *whenever $a, b, \in I$, then $a + b \in I$;*

(iii) *whenever $a \in I$ and $r \in R$, then $ra \in I$, also.*

Similarly a right ideal of a ring can be defined by changing the left multiplication in the definition with right multiplication. If I is both left and right ideal, we say that I is a two sided ideal. Clearly, for a commutative ring, left and right ideals coincide. By an ideal we will always mean a two sided ideal.

The kernel of a homomorphism $f : R \to S$ is the set

$$Kerf := \{r \in R : f(r) = 0\}.$$

The kernel of a homomorphism is an ideal of its domain. We can tell f is a monomorphism if and only if $Kerf = 0$ (see, (Anderson and Fuller, 1992)).

Suppose that I is a proper ideal of a ring R. The relation defined by

$$a \equiv b(modI) \Leftrightarrow a - b \in I$$

determines an equivalence relation on R. The congruence class of an element a is defined by $a + I = \{a + x : x \in I\}$ and is called a coset of the element a, and the set R/I of all cosets of I is a ring with operations defined by

$$(a + I) + (b + I) = (a + b) + I \ \ and \ \ (a + I)(b + I) = ab + I.$$

Additive and multiplicative identities are $0 + I$ and $1 + I$.

The ring R/I is called the factor ring of R modulo I. Further, the map $\sigma : R \to R/I$ defined by $r \mapsto r + I$ is an epimorphism with kernel I, is called the natural or canonical epimorphism.

Definition 2.6 *We say that an ideal M of a ring R is a maximal ideal, if*

(i) $M \subsetneqq R$*, and*

(ii) $M \subsetneqq I \subseteq R$ *implies that* $I = R$ *for every ideal* I *of* R*.*

From now on, we give the results on modules necessary for our work. Briefly, an R-module can be considered as the generalization of the notion of vector space in the sense scalars are allowed to be taken from a ring R instead of a field.

2.3. Modules, Submodules, Factor Modules and Module Homomorphisms

Although modules are in fact considered as a pair (M, λ), where M is an additive abelian group and λ is a map from R to the set of endomorphisms of M, we find the following definition more common and simple:

Definition 2.7 *Let* R *be a ring (with unity 1). A right* R*-module is an additive abelian group* M *together with a mapping* $M \times R \to M$*, which we call a scalar multiplication, denoted by*

$$(m, r) \mapsto mr$$

such that the following properties hold: for all $m, n \in M$ *and* $r, s \in R$*;*

(1) $(m + n)r = mr + nr$*,*

(2) $m(r + s) = mr + ms$*,*

(3) $m(rs) = (mr)s$*.*

If, in addition, for every $m \in M$ *we have* $m1 = m$*, then* M *is called a unitary right* R*-module. If* M *is a right* R*-module, we denote it by* M_R*.*

Note that one can obtain the left R-module definition by applying the scalar multiplications from the left. For commutative rings, two notions of left and right R-module coincide. In our work, all modules will be unitary right(left) R-modules. To simplify terminology, the expression ”R-module” or ”module” will mean right R-module.

Example 2.1 *Here is a list of some elementary examples of modules:*

(1) *As we indicated at the beginning, every vector space over a field F is an F-module.*

(2) *Every abelian group is a $\mathbb{Z}$-module, where $\mathbb{Z}$ is the set of integers. Hence, abelian groups can be generalized via module theory.*

(3) *Every ring R is a module over itself.*

A submodule of an R-module M is a subgroup N of M which is closed under scalar multiplication, i.e., $nr \in N$ for all $r \in R, n \in N$. Clearly, the 0 and the module M itself are submodules of M. They are called trivial submodules of M. A nonzero right R-module S that has only 0 and S for its submodules is said to be a simple module. The set of all submodules of a right R-module M is partially ordered by $\subseteq$, that is, by inclusion. Under this ordering, a minimal submodule of M is just a simple submodule of M. We call a submodule N of M a proper submodule of M if $N \subsetneqq M$. A proper submodule N of M is said to be a maximal submodule of M if whenever N' is a submodule of M such that $N \subseteq N' \subseteq M$, either $N = N'$ or $N' = M$. When a ring R is considered as a right module over itself, its submodules are precisely the right ideals of R. Clearly, A is a minimal right ideal of R if and only if A_R is a simple R-module.

Given any two R-modules M_1, M_2, we can always produce a new module, which we call the sum of M_1, M_2, containing both M_1 and M_2. This is done by defining

$$M_1 + M_2 = \{m_1 + m_2 : m_1 \in M_1, m_2 \in M_2\}.$$

Also, for an infinite family $\{M_\lambda\}_{\lambda\in\Lambda}$ of submodules of M, we define the sum as $\sum_{\lambda\in\Lambda} M_\lambda = \{\sum_{k=1}^{r} m_{\lambda k} : r \in \mathbb{N}$ and for $k = 1, 2, ..., r, \lambda_{k\in\Lambda}, m_{\lambda_k} \in M_{\lambda_k}\}$.
This is a submodule of M and so is the intersection $\bigcap_{\lambda\in\Lambda} M_\lambda$. It is worth noting that $\bigcap_{\lambda\in\Lambda} M_\lambda$ is the largest submodule of M which is contained in all M_λ, and $\sum_{\lambda\in\Lambda} M_\lambda$ is the smallest submodule which contains all M_λ. ((Wisbauer, 1991), §6.2).

Proposition 2.2 *Modular law. ((Wisbauer, 1991), §6.2). If H, K, L are submodules of an R-module M and $K \subset H$, then*

$$H \cap (K + L) = K + (H \cap L).$$

Now we define the module homomorphisms:

Definition 2.8 *Let R be a ring and M, N R-modules. A function $f : M \to N$ is called an R-homomorphism if, for all $m_1, m_2 \in M$ and for all $r \in R$,*

(i) $f(m_1 + m_2) = f(m_1) + f(m_2)$;

(ii) $f(m_1 r) = f(m_1)r$.

We see no need to list the definitions of R-epimorphism, R-monomorphism, and R-isomorphism since they are similar to the corresponding definitions for ring homomorphisms. For a module homomorphism $f : M \to N$, as one may expect, $Kerf$ is a submodule of M and Imf is a submodule of N. Note that we will just write homomorphism instead of R-homomorphism. The additive group of all the homomorphism from an R-module M to an R-module N is denoted by $Hom_R(M, N)$; R-endomorphisms on M is denoted by $End_R(M)$.

Definition 2.9 *Let M be an R-module and N be a submodule of M. Then the set of cosets*

$$M/N = \{x + N : x \in M\}.$$

is a right R-module if we define the addition and scalar multiplication as

$$(x + N) + (y + N) = (x + y) + N, (x + N)r = xr + N.$$

This new module is called the factor module of M modulo N. The map $\pi : M \to M/N$ defined by $m \mapsto (m + N)$ is an epimorphism called the natural or canonical epimorphism.

Theorem 2.1 *The Factor Theorem. ((Anderson and Fuller, 1992), Theorem 3.6) Let M, M' and N be R-modules and let $f : M \to N$ be an R-homomorphism. If $g : M \to M'$ is an epimorphism with $Ker(g) \leq Ker(f)$, then there exists a unique homomorphism $h : M' \to N$ such that $f = hg$.*
Moreover, $Kerh = g(Ker(f))$ and $Im(h) = Im(f)$, so that h is monic if and only if $Ker(g) = Ker(f)$ and h is epic if and only if f is epic.

It is wise to give the isomorphism theorems now:

Theorem 2.2 *Isomorphism Theorems. ((Anderson and Fuller, 1992), Corollary 3.7) Let M and N be R-modules.*

(1) *If $f : M \to N$ is an epimorphism with $Kerf = K$, then there is a unique isomorphism*

$$\eta : M/K \to N \text{ such that } \eta(m + K) = f(m)$$

for all $m \in M$.

(2) *If K and L are submodules of M such that $K \subseteq L$, then*

$$(M/K)/(L/K) \cong M/L$$

(3) *If H and K are submodules of M, then*

$$(H+K)/K \cong H/(H \cap K)$$

The next theorem characterizes the submodules of factor modules:

Theorem 2.3 *Correspondence Theorem. ((Anderson and Fuller, 1992), Proposition 2.9) Let T be a submodule of an R-module M. Then there is an isomorphism between the set of submodules of M/T and the set of submodules of M which contains T, that is, the submodules of M/T are precisely all factor modules N/T, where N is a submodule of M which contains T.*

Let M be a left R- module. Then for each subset X of M, *the (left) annihilator of X in R* is

$$_RAnn(X) = \{r \in R \mid rx = 0 \ for \ all \ x \in X\},$$

and for each $I \subseteq R$, *the (right) annihilator of I in M* is

$$Ann_R(I) = \{x \in M \mid rx = 0 \ for \ all \ r \in R)\}.$$

Proposition 2.3 *((Anderson and Fuller, 1992), Proposition 2.14) Let M be a left R-module and X be a subset of M. Then $_RAnn(X)$ is a left ideal of R. Moreover, if X is a submodule of M, then $_RAnn(X)$ is an ideal of R.*

2.4. Generating Sets, Finitely Generated Modules, and Maximal Submodules

Let M be a left R-module. A subset N of M is called a generating set of M if

$$M = RN = \{\textstyle\sum_{i=1}^{k} r_i n_i : k \in \mathbb{N}, \ and \ for \ i = 1, 2, ..., k, \ r_i \in R, \ n_i \in N\}.$$

If this is the case, we say that N generates M or that M is generated by N. If M has finite generating set, then we say that M is finitely generated. In particular, if M is generated by a single element, then M is called cyclic. In this case, $M = Ra$ for some element a of M.

Let N be a submodule of M. If the factor module M/N is finitely generated, then the submodule N of M is called a cofinite submodule of M. We will use the following properties without mentioning in our work. Proofs can be found in ((Wisbauer, 1991), §6.6):

Lemma 2.1 *Let $f : M \to N$ be a module homomorphism and L a generating set of M. Then*

(1) *$f(L)$ is a generating set of $Im(f)$, and*

(2) *if M is finitely generated, then $Im(f)$ is also finitely generated.*

Every ring R is a cyclic module over itself. Recall that a proper submodule N of a module M is called maximal if N is not contained in any proper submodule of M. That is, if $N \subsetneqq K$, then $K = M$. If an R module M is finitely generated, then M has a maximal submodule (see, (Anderson and Fuller, 1992), §10). Factor modules of finitely generated modules are also finitely generated. To see this, consider the natural epimorphism $\delta : M \to M/N$, where M is a finitely generated module and N is a submodule of M. Then by 2.1, it follows that M/N is also finitely generated.

Theorem 2.4 *((Anderson and Fuller, 1992), Theorem 10.4.) Let M be a right R-module. Then, M is finitely generated if and only if $M/Rad(M)$ is finitely generated and the natural epimorphism $M \to M/Rad(M) \to 0$ is small (i.e., $Rad(M) \ll M$).*

2.5. Direct Products and Direct Sums

While a considerable amount of module theory deals with decomposing a module into smaller parts, either by additive decompositions or residue class decomposition, we may also want to construct new modules from the modules we already have. As we have mentioned before, given any number of modules, we can create a larger module containing all of the given modules. This is possible with the so-called notion of products, and in this section we briefly give this notion.

Let $\{Mi : i \in I\}$ be a family of left R-modules, where I is a nonempty index set. Consider the set theoretic cartesian product $\prod_{i \in I} M_i$ of these modules. Then this product of the family $\{M_i\}$ becomes a left R-module in the following way: let $(m_i), (n_i) \in \prod_{i \in I} M_i, r \in R$. Addition is defined componentwise by $(m_i) + (n_i) = (m_i + n_i)$ and scalar multiplication is defined by: $r(m_i) = (rm_i)$. This componentwise addition and scalar multiplication makes sense because each M_i is a left R-module alone.

Definition 2.10 *We say that the element $(m_i)_{i \in I} \in \prod_{i \in I} M_i$ has finite support if the set $\{i \in I : m_i \neq 0\}$ is finite.*

Now consider the set of all elements of $\prod_{i \in I} M_i$ with finite support. This subset is actually a submodule of $\prod_{i \in I} M_i$ ((Anderson and Fuller, 1992), §6).

Definition 2.11 *Let $\{Mi : i \in I\}$ be a family of left R-modules, where I is a nonempty index set. Then the left R-module $\prod_{i \in I} M_i$ is called the direct product of the family $\{M_i : i \in I\}$. The submodule of all elements with finite support of $\prod_{i \in I} M_i$ is called the external direct sum* *of the family* $\{Mi : i \in I\}$*, denoted* $\coprod_{i\in I} M_i$*. External direct sum of the family* $\{Mi : i \in I\}$ *is also denoted by* $\bigoplus_{i\in I} M_i, \bigoplus_I M_i$*, or* $\bigoplus M_i$*. One can see that in case the index set* I *is finite, product and coproduct of the family* $\{Mi : i \in I\}$ *coincide. That is,* $\prod_I M_i = \coprod_I M_i$ *when I is finite. Also, if* $M_i = M$ *for all* $i \in I$ *then we write*

$$M^{(I)} = \oplus_I M$$

for the external direct sum of $cardI$ *copies of* M*.*

Now, we will give the definition of internal direct sum . This notion is a little different from that of products and coproducts. Here we deal with sums of submodules of a given module. So, let $\{Mi : i \in I\}$ be a family of submodules of a module M. M is said to be the internal direct sum of the family $\{Mi : i \in I\}$ if

(1) $M = \sum_{i\in I} M_i$, and

(2) $M_j \bigcap (\sum_{i\neq j} M_i) = \{0\}$ for all $j \in J$.

In this case, we write $M = \bigoplus_{i\in I} M_i$. This should not lead to any confusion that we use the same notation for external direct sums, because as we have said, in case of a direct sum, we have submodules of a module as summands, while in external direct sum we have any family of modules as summands. For simplicity, we will use the phrase direct sum for internal direct sum.

A submodule N of a module M is called a direct summand of M if there exists a submodule K of M such that $M = N \oplus K$. Trivially, for any module M, the zero submodule and M, itself are direct summands.

2.6. Exact Sequences

Let K, L, M be R-modules. Consider the sequence $K \xrightarrow{f} M \xrightarrow{g} L$ where f and g are module homomorphisms. We say that this sequence is exact at M if $Imf = Kerg$. Generally, a sequence of homomorphisms

$$\ldots \longrightarrow M_{n-1} \xrightarrow{f_n} M_n \xrightarrow{f_{n+1}} M_{n+1} \longrightarrow \ldots$$

is exact if it is exact at each M_i, that is, if $Im(f_n) = ker(f_{n+1})$ for all n. The next result follows from the definition:

Proposition 2.4 *((Wisbauer, 1991), §7.14) Let M, N be modules and $f : M \to N$ a homomorphism. Then*

(1) $0 \longrightarrow M \xrightarrow{f} N$ *is exact if and only if f is a monomorphism;*

(2) $M \xrightarrow{f} N \longrightarrow 0$ *is exact if and only if f is an epimorphism;*

(3) $0 \longrightarrow M \xrightarrow{f} N \longrightarrow 0$ *is exact if and only if f is an isomorphism.*

More generally, an exact sequence of the form

$$0 \longrightarrow K \xrightarrow{f} M \xrightarrow{g} N \longrightarrow 0$$

is called a short exact sequence. It can be derived from the above proposition that in such an exact sequence f is a monomorphism and g is an epimorphism. By isomorphism theorems, one can see that $K \cong Imf$ and that $M/Imf \cong N$. Thus, in an exact sequence generally K is regarded as a submodule of M and N is regarded as a factor module of M.

2.7. Injective Modules and Noetherian Rings

In this section, we will introduce injective modules and give some basic properties of these modules.

Definition 2.12 *An R-module M is said to be* injective *if for any monomorphism $g : A \to B$ of R-modules and any R-homomorphism $h : A \to M$, there exsists an R-homomorphism $h' : B \to M$ such that $h = h'g$:*

$$\begin{array}{ccccc} 0 & \longrightarrow & A & \xrightarrow{g} & B \\ & & \downarrow{\scriptstyle h} & \swarrow{\scriptstyle h'} & \\ & & M & & \end{array}$$

Proposition 2.5 *(Anderson and Fuller, 1992)*

(1) *If $(E_k)_{k \in K}$ is a family of injective right R-modules, then $\prod_{k \in K} E_k$ is also injective right R-module;*

(2) *Every direct summand of an injective right R-module E is injective;*

(3) *A finite direct sum of injective right R-modules is injective;*

(4) *If M is an injective submodule of R-module N, then M is a direct summand of N.*

It is not true that every direct sum of injective modules is injective. Before we see that all such direct sums are injective, we shall give the definition of noetherian ring first.

Definition 2.13 *A ring R is said to be noetherian if it satisfies the following three equivalent conditions:*

(1) *Every non-empty set of ideals in R has a maximal element;*

(2) *Every ascending chain of ideals in R is stationary;*

(3) *Every ideal in R is finitely generated.*

Proposition 2.6 *((Anderson and Fuller, 1992), Proposition 18.13) For a ring R, the following are equivalent:*

(1) *Every direct sum of injective right R-modules is injective;*

(2) *R is a right noetherian ring.*

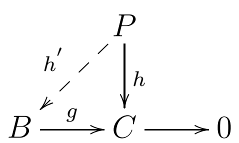
Definition 2.14 *A ring R is called an artinian ring if every descending chain of ideals in R is stationary.*

Definition 2.15 *Let M be an R-module. A monomorphism $f : M \to Q$ is called an injective hull of M if Q is injective and f is an essential monomorphism, i.e. $Im(f)$ is essential in Q. We denote the injective hull of a module M by E(M).*

2.8. Projective Modules and Perfect Rings

In this section we recall the definition of a projective module and give some basic properties of these modules.

Definition 2.16 *An R- module P is said to be* projective *if for any epimorphism of R-modules, say, $g : B \to C$, and any R-homomorphism $h : P \to C$, there exists an R-homomorphism $h^{'} : P \to B$ such that $h = gh^{'}$:*



Proposition 2.7 *((Lam, 1999), §2A) The following hold for a ring R.*

(1) *If* $\{M_\alpha\}_\triangle$ *is a family of* R*- modules, then* $\bigoplus_\triangle M_\alpha$ *is projective if and only if each* M_α *is projective.*

(2) *A direct summand of a projective* R*-module is projective.*

(3) *The ring* R *is a projective* R*-module.*

(4) *Every free* R*-module is projective.*

Proposition 2.8 *((Anderson and Fuller, 1992), Proposition 17.2) The following statements about a right* R*- module* P *are equivalent;*

(1) P *is projective ;*

(2) *Every epimorphism* ${}_RM \rightarrow_R P \rightarrow 0$ *splits;*

(3) P *is isomorphic to a direct summand of a free right* R*- module.*

Definition 2.17 *A ring* R *is* right hereditary *if every right ideal is projective.*

Semisimple rings are easily seen to be left and right hereditary via the equivalent definitions: all left and right ideals are summands of R, and hence are projective. Also, the ring R of $n \times n$ upper triangular matrices over a field K is both left and right hereditary.

Proposition 2.9 *((Anderson and Fuller, 1992)) For a ring* R *the following are equivalent;*

(1) R *is right hereditary;*

(2) *Every factor module of an injective right* R*-module is injective;*

(3) *Every submodule of a projective right* R*-module is projective.*

2.9. Flat Modules

In this section we do not delve into the details of definitions of every term in homolojical algebra. Essentially, we accept the Hom and Tensor ($\bigotimes$) functors are known. For more details on homological algebra see, (Rotman, 2009). The idea of flat modules plays a special role in many parts of ring theory. On the other hand, flat modules are natural generalizations of projective modules and they are related to injective modules via the formation of character modules.

Definition 2.18 *((Lam, 1999), Definition 4.0) A right module M_R is called flat if $0 \to M \otimes_R A \to M \otimes_R B$ is exact in the category of abelian groups whenever $0 \to A \to B$ is an exact sequence of left R-modules.*
We note that any projective R-module is flat, and the converse is false, in general.

Definition 2.19 *(Anderson and Fuller, 1992) A pair (P, π) is a projective cover of the module M_R in case P is a projective right R-module and $\pi : P \to M$ is a small epimorphism i.e. $Ker(\pi) \ll P$. A ring R is right perfect in case each of its right modules has a projective cover.*

Theorem 2.5 *Let R be a ring with jacobson radical J. Then the following statements are equivalent:*

(1) *R is right perfect;*

(2) *R/J is semisimple and for every right R-module M, $MJ \ll M$*

(3) *Every flat right R-module is projective.*

While flat modules are related to projective modules, there is also an interesting relationship between flat modules and injective modules, discovered by J. Lambek. This relationship is formulated by using the notion of character modules. For any right R-module P, the character module of P is defined to be

$$P^{'} := Hom_{\mathbb{Z}}(P, \mathbb{Q}/\mathbb{Z}).$$

This is a left R module via the action $(r, f) \mapsto rf$, where $(rf)(x) = f(xr)$ for $r \in R, f \in P^{'}$, and $x \in P$.

Theorem 2.6 *(Lambek) ((Lam, 1999), Theorem 4.9) A right R-module P is flat if and only if its character module $P^{'}$ is injective.*

Definition 2.20 *((Lam, 1999), Definition 4.25) A module P_R is said to be finitely presented if there exists an exact sequence $0 \to K \to F \to P \to 0$ where F is free of finite rank, and K is finitely generated. Equivalently, there exists an exact sequence $R^m \to R^n \to P \to 0$ with $m, n \in \mathbb{N}$.*

Proposition 2.10 *((Lam, 1999), Proposition 4.29) A ring R is right noetherian if and only if all finitely generated (cyclic) right R-modules are finitely presented.*

Theorem 2.7 *((Lam, 1999), Theorem 4.30) Let P_R be a finitely presented module over any ring R. Then P is flat if and only if it is projective.*

2.10. Socle and Radical of a Module

Definition 2.21 *A submodule N of an R-module M is said to be an* essential (or a large) *submodule of M, written $N \trianglelefteq M$, if $N \cap N' \neq 0$ for each nonzero submodule N' of M. If N is an essential submodule of M, then M is referred to as an* essential extension *of N.*

Definition 2.22 *Let M be an R- module. The submodule*

$$Z(M_R) = \{x \in M \,|\, xI = 0 \ for \ some \ I \trianglelefteq R_R\}$$

is called the singular submodule of M. An R-module M is said to be singular (nonsingular) if $Z(M) = M$ ($Z(M) = 0$).

We observe that the ring R is a nonsingular right R module if and only if $Z(R_R) = 0$, and in this case R is called a right nonsingular ring. Likewise, we say that R is a left nonsingular ring if $Z({}_RR) = 0$. Right and left nonsingular rings are not equivalent (see, (Goodearl, 1976)).

Let M be an R-module and $N \leq M$. M/N is singular whenever $N \trianglelefteq M$. The converse of this can easily fail; for example, let $M = \mathbb{Z}/2\mathbb{Z}$ and $N = 0$. M/N is a singular $\mathbb{Z}$ module, but N is not an essential submodule of M.

In the following definition, dual definitions for essential submodules and essential extension are introduced.

Definition 2.23 *N is called* superfluous *or* small *in M, written $N \ll M$, if, for every submodule $L \subseteq M$, the equality $N + L = M$ implies $L = M$. A module N is a small cover of a module M if there exists an epimorphism $f : N \to M$ such that $Ker(f) \ll N$.*

Let M be an R-module. The *jacobson radical* of M is defined by

$$Rad(M) = \bigcap\{K \subseteq M \mid K\ is\ a\ maximal\ submodule\ in\ M\}$$

$$= \sum\{L \subseteq M \mid L\ is\ a\ small\ submodule\ in\ M\}$$

and the *socle* of M is defined by

$$Soc(M) = \sum\{K \subseteq M \mid K\ is\ a\ minimal\ submodule\ in\ M\}$$

$$= \bigcap\{L \subseteq M \mid L\ is\ an\ essential\ submodule\ in\ M\}.$$

If M has no simple submodule, then we set $Soc(M) = 0$. Also, if M has no maximal submodule we set $Rad(M) = M$. The jacobson radical of a ring R is denoted by $J(R)$. The right socle of a ring is $S = Soc(R_R)$ and left socle is $S' = Soc({}_RR)$, and they are ideals of R. They need not to be equal for example; if R is the ring of 2×2 upper triangular matrices over a field, then $S \neq S'$.

Corollary 2.1 *((Anderson and Fuller, 1992), Corollary 15.4) If R is a ring, then*

$$Rad({}_RR) = Rad(R_R).$$

The Jacobson radical of a ring is $J(R) = Rad(R_R)$ and it is an ideal.

Corollary 2.2 *((Anderson and Fuller, 1992), Corollary 15.5) If R is a ring, then $J(R)$ is the annihilator in R of the class of simple right (left) R-modules.*

Corollary 2.3 *((Anderson and Fuller, 1992), Corollary 15.6) If I is an ideal of a ring R, and if $J(R/I) = 0$, then $J(R) \subseteq I$.*

Proof If $J(R/I) = 0$, then the intersection of the maximal right ideals of R containing I is exactly R. It follows that $J(R)$, the intersection of the maximal right ideals of R , is contained in I. □

Corollary 2.4 *((Anderson and Fuller, 1992), Corollary 15.8) If R and R' are rings and if $\phi : R \to R'$ is a surjective ring homomorphism, then $\phi(J(R)) \subseteq J(R')$. Moreover, if $ker\phi \subseteq J(R)$, then $\phi(J(R)) = J(R')$. In particular, $J(R/J(R)) = 0$.*

2.11. Semisimple Modules and Rings

Recall that an R-module M is called simple if $M \neq 0$ and it has no non-trivial submodules.

Proposition 2.11 *((Anderson and Fuller, 1992), Theorem 9.6.) A right R-module T is simple if and only if $T \cong R/I$ for some maximal right ideal I of R.*

Let $(T_\alpha)_{\alpha \in A}$ be an indexed set of simple submodules of M. If M is the direct sum of this set, then

$$M = \bigoplus_A T_\alpha$$

is a semisimple decomposition of M. A module M is said to be *semisimple* in case it has a semisimple decomposition.

Theorem 2.8 *((Anderson and Fuller, 1992), Theorem 9.6) For a right R-module M the following statements are equivalent:*

(1) *M is semisimple;*

(2) *M is generated by simple modules;*

(3) *M is the sum of some set of simple submodules;*

(4) *M is the sum of its simple submodules;*

(5) *Every submodule of M is a direct summand;*

(6) *Every short exact sequence*

$$0 \to K \to M \to N \to 0$$

of right R-modules splits.

Corollary 2.5 *((Lam, 1991), §2) For a right R-module M, the following hold.*

(1) *Every submodule of a semisimple module M is semisimple.*

(2) *Every epimorphic image of a semisimple module M is semisimple.*

Theorem 2.9 *((Lam, 1991), Theorem 2.5) For a ring R, the following are equivalent:*

(1) *R is right semisimple;*

(2) *All short exact sequences of right R-modules split;*

(3) *All finitely generated right R-modules are semisimple;*

(4) *All cyclic right R-modules are semisimple;*

(5) *All right R-modules are semisimple.*

Corollary 2.6 *((Lam, 1991), Corollary 2.6) A right semisimple ring R is both right noetherian and right artinian.*

Theorem 2.10 *((Lam, 1991), Theorem 2.8 and 2.9) The following conditions on a ring R are equivalent:*

(1) *R is right semisimple;*

(2) *All right R-modules are projective;*

(3) *All right R-modules are injective;*

(4) *All finitely generated right R modules are injective;*

(5) *All cyclic right R modules are injective.*

2.12. Weakly Injective Modules

Dualizing essential extensions and submodules, we are led to the following notion, which is called coessential extension.

Definition 2.24 *((Clark et al., 2006), Definition 3.1) Suppose that $0 \subseteq A \subseteq B \subseteq N$. Then, of course, A is an essential submodule of B if $A/0 \trianglelefteq B/0$. Dually, we say that A is a coessential submodule of B in N (denoted by $A \hookrightarrow^{ce} B$ in N) if $B/A \ll N/A$.*

Note that this is equivalent to saying that N/A is a small cover of N/B and that, trivially, $B \ll N$ if and only if $0 \hookrightarrow^{ce} B$. It is easy to see that $A \hookrightarrow^{ce} B$ in N if and only if $B + X = N$ implies $A + X = N$.

Definition 2.25 *((Clark et al., 2006), Definition 3.6) A submodule A of N is said to be coclosed in N (denoted by $A \hookrightarrow^{cc} N$) if it has no proper coessential submodule in N.*

Definition 2.26 *(Zöschinger, 2006) A module M is called weakly injective if for every extension N of M, M is coclosed in N.*

Definition 2.27 *(Oshiro, 1984) An R-module M is said to be lifting module if, for any submodule A of M, there exist a direct summand B of M such that B is a coessential submodule of A in M.*

These are the final results of this chapter. We close this chapter here, because we have developed enough terminology and theory to follow the third chapter of our work. We will give new terminology and theory as we proceed in chapter three.

CHAPTER 3

STRONGLY NONCOSINGULAR MODULES

In this chapter, we introduce the concept of strongly noncosingular modules and deal with their relations with some other modules. We start with characteristics of Rad-small modules.

3.1. Cosingular (Rad-small) and Noncosingular Modules

Before giving the definitions, let us talk about the motivating idea for the Rad-small modules. Leonard defines a module M to be small if it is a small submodule of some R-module, and he shows that M is small if and only if M is small in its injective hull (Leonard, 1966).

Theorem 3.1 *((Leonard, 1966), Theorem 2) Submodules, quotient modules and finite direct sums of small modules are small.*

Proposition 3.1 *((Clark et al., 2006), Proposition 8.2(3)) Any simple module is either small or injective.*

Now, we can give the fundamental properties of the functor $Z^*(.)$ that was defined by (Rayar, 1971) first. Let M be an R module. The submodule of M is then defined as:

$$Z^*(M) = \{m \in M \mid mR \text{ is a small module } \}.$$

Since $Rad(M)$ is the union of all small submodules of M, $Rad(M) \subseteq Z^*(M)$. We see that $Z^*(E) = Rad(E)$ for any injective module E, and $Z^*(M) = M \cap Rad(E(M))$.

Definition 3.1 *Let R be a ring and M an R-module. $Z^*(M)$ is called cosingular submodule of M. As the dual notion of singular (nonsingular), M is called cosingular (noncosingular) if $Z^*(M) = M$ ($Z^*(M) = 0$). R is called right cosingular if the (right) R-module R is cosingular.*

For convenience in concepts, the cosingular R-modules are called Rad-small in this thesis. Clearly, small modules are Rad-small.

Note that if M is a vector space over the rational numbers $\mathbb{Q}$, then M is a semisimple injective $\mathbb{Q}$-module; hence $Z^*(M_{\mathbb{Q}}) = Rad(M_{\mathbb{Q}}) = 0$. However, M is also a module over the integers $\mathbb{Z}$, and as such is torsion-free injective, so that $Z^*(M_{\mathbb{Z}}) = M$. Thus, $Z^*(M)$ depends on which ring R one is considering.

Lemma 3.1 *((Ozcan, 2002), Lemma 2.1) Let R be a ring and let $f : M \to N$ be a homomorphism of R-modules M,N. Then $f(Z^*(M)) \subseteq Z^*(N)$.*

Lemma 3.2 *((Ozcan, 2002), Lemma 2.2) Let N be a submodule of an R-module M. Then $Z^*(N) = N \cap Z^*(M)$.*

Lemma 3.3 *((Ozcan, 2002), Lemma 2.3) Let $M_i (i \in I)$ be any collection of R-modules, and let $M = \oplus_{i \in I} M_i$. Then $Z^*(M) = \oplus_{i \in I} Z^*(M_i)$.*

Lemma 3.4 *((Ozcan, 2002), Lemma 2.4) Let R be a right Artinian ring with Jacobson radical J and let M be an R-module. Then $Z^*(M) = \{m \in M : mr_R(J) = 0\}$.*

Lemma 3.5 *((Ozcan, 2002), Lemma 2.6) For any ring R, the class of Rad-small R-modules is closed under submodules, homomorphic images and direct sums but not (in general) under essential extensions or extensions.*

Proof The class of Rad-small R-modules is closed under submodules by Lemma 3.2, under homomorphic images by Lemma 3.1 and under direct sums by Lemma 3.3.
Let F be a field, and let $R = \{\begin{pmatrix} a & b \\ 0 & a \end{pmatrix} : a, b \in F\}$. Then R is a commutative Artinian ring with Jacobson radical $J = \begin{pmatrix} 0 & F \\ 0 & 0 \end{pmatrix}$. Note that $r_R(J) = J$ and that J is an essential ideal of R. By Lemma 3.4, the R module J is Rad-small but its essential extension R_R is not. Moreover, J and R/J are both Rad-small by Lemma 3.4, but the R-module R is not. □

Definition 3.2 *(Talebi and Vanaja, 2002) A module M is called noncosingular if for every nonzero module N and every nonzero homomorphism $f : M \to N$, $Im(f)$ is not a small submodule of N.*

Proposition 3.2 *((Talebi and Vanaja, 2002), Proposition 2.4) The class of all noncosingular modules is closed under homomorphic images, direct sums, extensions, small covers and coclosed submodules.*

3.2. Strongly Noncosingular Modules

Motivated by the noncosingular modules, we introduce the concept of strongly noncosingular R-module. We start with the following definition.

Definition 3.3 *An R-module M is called* strongly noncosingular *if for every nonzero R-module N and every nonzero homomorphism $f : M \to N$, $Im(f)$ is not a Rad-small submodule of N, i.e. M has no nonzero Rad-small homomorphic image.*

After this definition, we will give some remarks for strongly noncosingular modules, and we will see an example of a module which is noncosingular but not strongly noncosingular, so our definition will make sense.

Remark 3.1 *(1) Simple injective R-modules are obviously strongly noncosingular.*
(2) Let R be a division ring (e.g. the rational numbers $\mathbb{Q}$). An R-module M is a vector space, and so it is a semisimple injective R-module. Therefore, it is strongly noncosingular.
(3) Let R be a right hereditary ring. Finitely generated injective R-modules are strongly noncosingular. Let M be a finitely generated injective R-module. Suppose that $Z^*(M/N)$ $=M/N$ for a submodule N of M. Since R is a right hereditary ring, by Proposition 2.9, M/N is injective, hence $Rad(M/N) = Z^*(M/N) = M/N$. Since M/N is finitely generated, by Theorem 2.4, $Rad(M/N) \ll M/N$, a contradiction. Thus, finitely generated injective R-modules are strongly noncosingular.
(4) Strongly noncosingular R-modules are noncosingular since small modules are Rad-small. However, there exists a noncosingular R-module which is not strongly noncosingular (see Example 3.1).

We have the following proposition as we had for noncosingular modules.

Proposition 3.3 *The class of all strongly noncosingular R-modules is closed under homomorphic images, direct sums, direct summand, extensions, small covers.*

Proof (1) Let M be a strongly noncosingular R-module and N a submodule of M. Suppose that M/N is not a strongly noncosingular R-module. Then there is a nonzero homomorphism g from M/N to some R-module T with $Im(g) \subseteq Rad(T)$. Then $Im(g\pi) \subseteq Rad(T)$, where π is the canonical epimorphism $M \to M/N$. Since M is strongly noncosingular, $Im(g\pi) = 0$. Then $g = 0$, a contradiction.

(2) Assume that $(M_i)_{i \in I}$ is a class of strongly noncosingular R-modules. Let f be a homomorphism from $\bigoplus_{i \in I} M_i$ to some R-module N with $Imf \subseteq Rad(N)$. Then $Im(f\iota_i) \subseteq Rad(N)$ for the inclusion maps $\iota_i : M_i \to \bigoplus_{i \in I} M_i$ for every $i \in I$. Since M_i is strongly noncosingular R-module, $Im(f\iota_i) = 0$ for every $i \in I$. Then $f = 0$, and $\bigoplus_{i \in I} M_i$ is strongly noncosingular.

(3) Let $N \subseteq M$ be a direct summand of M and $p : M \to N$ the projection map. Let f be a homomorphism from N to some R-module T with $Im(f) \subseteq Rad(T)$. Then $Im(fp) \subseteq Rad(T)$ and, by the hypothesis, $fp(M) = 0$. Hence, $f = 0$, and N is strongly noncosingular.

(4) Let $0 \to A \overset{\alpha}{\to} B \overset{\beta}{\to} C \to 0$ be a short exact sequence, and suppose that A and C strongly noncosingular R-modules. Assume that there is a homomorphism f from B to some R-module T with $Im(f) \subseteq Rad(T)$. Then $Im(f\alpha) \subseteq Rad(T)$. Since A is strongly noncosingular R-module, $Im(f\alpha) = 0$. Then there is a homomorphism g from C to T such that $f = g\beta$ by Theorem 2.1. Therefore, $Im(g) \subseteq Rad(T)$ and, since C is strongly noncosingular R-module, $Im(g) = 0$. Thus, $f = 0$ and so, B is strongly noncosingular.

(5) Let B be a strongly noncosingular R-module, and let $f : A \to B$ be a small cover, i.e. f is an epimorphism and $Kerf \ll A$. Suppose that A is not a strongly noncosingular R-module. Then there is a submodule X of A such that A/X is Rad-small. $B/f(X)$ is Rad-small since it is homomorphic image of A/X. But $B/f(X)$ is strongly noncosingular by (1), hence $B/f(X) = 0$, and $B = f(X)$. Then $f^{-1}(B) = X + Kerf = A$, and so $X = A$ since $Kerf \ll A$. Hence A is strongly noncosingular. □

Before giving the following corollary, let us mention the supplements. A submodule N of M is called a *supplement* of K in M if N is minimal with respect to the property $M = K + N$, equivalently, $M = K + N$ and $K \cap N \ll N$ ((Wisbauer, 1991), p. 348).

Corollary 3.1 *Let M be a module and U and V submodules of M such that V is a supplement of U. Then V is a strongly noncosingular if and only if M/U is strongly noncosingular.*

Proof By the hypothesis, $M = U+V$, $U\cap V \ll V$ and $M/U \cong V/(U\cap V)$. Suppose that V is strongly noncosingular. Since strongly noncosingular R-modules closed under homomorphic image by Proposition 3.3, $M/U \cong V/(U \cap V)$ is strongly noncosingular. Conversely, assume that $M/U \cong V/(U \cap V)$ is strongly noncosingular. Since strongly noncosingular R-modules closed under small cover by Proposition 3.3 and $U \cap V \ll V$, V is strongly noncosingular. □

General properties of strongly noncosingular modules is the following:

Proposition 3.4 *Let M be a strongly noncosingular R-module. The following properties hold:*

(1) Every Rad-small submodule of M is small in M.

(2) Coclosed submodules of M are strongly noncosingular.

(3) $RadM \ll M$.

(4) $RadM = RadN \cap M$ for every extension N of M.

(5) $Rad(M) = Z^*(M)$.

Proof (1) Suppose that K is a Rad-small submodule of M and $K+L=M$ for a submodule L of M. Since $\frac{K}{K\cap L}$ is a homomorphic image of K, it is Rad-small by Proposition 3.5. But M is strongly noncosingular, and so $\frac{K}{K\cap L}=0$ by Proposition 3.3. Then $K\cap L=K$ and $L=M$. So $K\ll M$, as desired.

(2) Let A be a coclosed submodule of M. Suppose that A/X is a Rad-small R-module for a submodule X of A. Since M is strongly noncosingular, M/X is also strongly noncosingular by Proposition 3.3. Then, by (1), $A/X \ll M/X$. But A is coclosed submodule of M, so $X=A$ by Definition 2.25. This implies A is a strongly noncosingular R-module.

(3) and (4) follow by (1).

(5) $RadM \subseteq Z^*(M)$ is clear. Conversely, let $m \in Z^*(M)$. Then mR is a small module and, by (1), $mR \ll M$. Thus $m \in RadM$. □

Now we will give a nice characterization of strongly noncosingular modules. First we need characteristics of coatomic modules.

Definition 3.4 *(Zöschinger, 1980) Let M be an R-module. We say that M is a coatomic module if every proper submodule of M is contained in a maximal submodule of M, equivalently , for every submodule N of M, $Rad(M/N)=M/N$ implies $M/N=0$.*

Finitely generated and semisimple modules are coatomic.

Theorem 3.2 *Let M be an R-module. Then the following statements are equivalent:*

(1) M is strongly noncosingular;

(2) M is coatomic and every simple homomorphic image of M is injective;

(3) M is coatomic and noncosingular.

Proof Note that any simple module is either small or injective by Proposition 3.1.

$(1) \Rightarrow (2)$ Let N be a proper submodule of M. Suppose N is not contained in a maximal submodule of M. Then $M/N = Rad(M/N)$, and this implies M/N is Rad-small. But M is strongly noncosingular, and so $M/N=0$, a contradiction. By the given above, simple homomorphic image of a strongly noncosingular R-module is injective.

$(2) \Rightarrow (3)$ Let N be a proper submodule of M with M/N small module. If N is maximal submodule of M, then M/N is injective. Suppose that N is not a maximal submodule of M. By the assumption, M is coatomic, and hence there exists a maximal submodule K of M which contains N. Since small modules closed under homomorphic image by Theorem 3.1, $M/K \cong \frac{M/N}{K/N}$ is small. But M/K is injective by the assumption, hence $M/K=0$, a contradiction . Then M has no small homomorphic image, i.e. M is noncosingular.

$(3) \Rightarrow (1)$ Let N be a proper submodule of M with M/N a Rad-small R-module. By the assumption, M is coatomic, and hence there exists a maximal submodule K of M which contains N. M/K is injective since M is noncosingular. Rad-small modules closed under homomorphic image by Proposition 3.5, so $M/K \cong \frac{M/N}{K/N}$ is Rad-small. Then $Rad(M/K) = M/K$, and this contradicts the fact that M is coatomic. Hence, M is strongly noncosingular. □

It is clear that, by Theorem 3.2, a strongly noncosingular R-module exists if and only if there is a simple injective module.

Definition 3.5 *Let M be an R-module over a domain R. If $r \in R$ and $m \in M$, then we say that m is divisible by r if there is some $m' \in M$ with $m = m'r$. We say that M is a divisible module if each $m \in M$ is divisible by every nonzero $r \in R$.*

If R is a domain, then every injective R-module E is a divisible module (see, (Rotman, 2009), Lemma 3.33).

Proposition 3.5 *Let R be a domain which is not a division ring. Then there does not exist a strongly noncosingular R-module.*

Proof It is enough to show that there is no simple injective R-module. Assume that there exists a simple injective R-module, say S. Then S is divisible. Since S is simple, there exists a nonzero maximal ideal I of R such that $S \cong R/I$. Then $(R/I)r = 0$ for each $r \in I$. But this contradicts with the divisibility of S. Hence, there is no simple injective R-module. □

We give an example for a noncosingular R-module that fails to be strongly noncosingular.

Example 3.1 *Consider the ring $R = \{\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} | a, c \in \mathbb{Z}, b \in \mathbb{Q}\}$ and the R-module ${}_RM = \begin{pmatrix} 0 & \mathbb{Q} \\ 0 & 0 \end{pmatrix}$. The left R-module structure of M is completely determined by the left $\mathbb{Z}$-module structure of $\mathbb{Q}$. Then M is not coatomic since ${}_{\mathbb{Z}}\mathbb{Q}$ is not coatomic. But M is noncosingular since every nonzero homomorphic image of the $\mathbb{Z}$-module $\mathbb{Q}$ is not small.*

Definition 3.6 *(Tuganbaev, 2003) A ring R is called a right Max-ring if $Rad(M) \neq M$ for every R-module M. Equivalently, R is a right Max-ring if and only if every nonzero R-module is coatomic.*

A perfect ring R is a right Max-ring, and the converse is true if $R/J(R)$ is semisimple as a right R-module ((Anderson and Fuller, 1992), Theorem 28.4).

Theorem 3.2 yields the following.

Corollary 3.2 *Let R be a right Max-ring. An R-module M is strongly noncosingular if and only if it is noncosingular.*

Recall that an R-module M is called *weakly injective* if, for every extension N of M, M is coclosed in N.

Proposition 3.6 *Strongly noncosingular R-modules are weakly injective.*

Proof Let M be a strongly noncosingular R-module and $M \subseteq N$ any extension of M. Let L be a proper submodule of M. Since M/L is not Rad-small, M/L cannot be a small submodule of N/L. Hence, L is not coessential submodule of M in N, and by Definition 2.25, M is coclosed in N. So, M is weakly injective. □

The converse of Proposition 3.6 is not true, in general. In example 3.1, the R-module ${}_RM$ is injective, so weakly injective, but not strongly noncosingular.

Proposition 3.3, Theorem 3.2 and Proposition 3.6 yield the following corollary.

Corollary 3.3 *Let M be a coatomic module. Then the following statements are equivalent:*

(1) M is strongly noncosingular;

(2) Every homomorphic image of M is weakly injective;

(3) Every finitely generated quotient of M is weakly injective;

(4) Every cyclic quotient of M is weakly injective;

(5) Every simple quotient of M is injective.

Now, we shall present a standart result on simple injective modules and briefly discuss the notion of right V-rings.

An elemet a in a ring R is called *von Neumann regular* if $a \in aRa$. The ring R itself is called *von Neumann regular* if every $a \in R$ is von Neumann regular. A ring R is called a *right V-ring* if every simple R-module is injective. In the category of commutative rings, the V-rings are exactly the (commutative) von Neumann regular rings. For noncommutative rings, the situation is quite a bit more subtle. In the general case, right V-rings need not be von Neumann regular; also they need not be left V-rings.

Theorem 3.3 *((Lam, 1999), Theorem 3.75) For any ring R, the following are equivalent:*

(1) *R is a right V-ring;*

(2) *For any right R-module M, $Rad(M) = 0$.*

Any right semisimple ring is a V-ring. By the above theorem clearly, every R-module is coatomic if R is a right V-ring. Theorem 3.2 and Corollary 3.3 yield the following.

Corollary 3.4 *Let R be a ring. The following statements are equivalent:*

(1) R_R is strongly noncosingular.

(2) R is a right V-ring.

(3) Every quotient of R is weakly injective.

(4) Every R-module is strongly noncosingular.

Proposition 3.7 *Injective modules are strongly noncosingular if and only if weakly injective modules are strongly noncosingular.*

Proof Let M be a weakly injective module. By the assumption, $E(M)$ is strongly noncosingular. Then M is strongly noncosingular by Proposition 3.4(2). The converse follows from the fact that injective modules are weakly injective. □

It is well known that a ring R is a right hereditary ring if and only if every homomorphic image of an injective R-module is injective (see Proposition 2.9). Semisimple rings are left and right hereditary.

Lemma 3.6 *Let R be a right hereditary ring. The following statements are equivalent:*

(1) Every weakly injective R-module is strongly noncosingular;

(2) Every injective R-module is strongly noncosingular;

(3) Every injective R-module is coatomic;

(4) Every weakly injective R-module is coatomic.

Proof $(1) \Leftrightarrow (2)$ is by Proposition 3.7, $(1) \Rightarrow (4)$ is by Theorem 3.2 and $(4) \Rightarrow (3)$ is clear. $(3) \Rightarrow (2)$ Let M be an injective module. Since R is right hereditary, every homomorphic image of M is injective, and M is strongly noncosingular by Theorem 3.2. □

It is clear that injective modules are noncosingular on hereditary rings. Corollary 3.2 and Lemma 3.6 yield the following.

Corollary 3.5 *Let R be a right hereditary Max-ring. Then every injective module is strongly noncosingular.*

Now we give an example to show that the converse of Corollary 3.5 is not true, in general.

Example 3.2 *Let K be any field, and let R be the commutative ring which is the direct product of a countably infinite number of copies of K, that is, $R = \prod_{i=1}^{\infty} K_i$, where $K_i = K$ for all $i \geq 1$. First, we show that R is a Von Neumann regular ring: By means of componentwise defined addition and similarly defined multiplication $(k_i)(k_i^{'}) = (k_i k_i^{'})$, R becomes a ring.*

For any $k_i \in R$, define $k_i' = k_i^{-1}$ if $k_i \neq 0$, and $k_i' = 0$ if $k_i = 0$. Then $k_i' \in R$ and $k_i k_i' k_i = k_i$. Therefore, R is Von Neumann regular, and so R is a V-ring. Hence every R-module is strongly noncosingular, so injective R-modules are strongly noncosingular. Now, we will show that R is not a semisimple ring. As we easily verify that $A := \coprod_{i=1}^{\infty} K_i$ is a proper two-sided ideal in $R = \prod_{i=1}^{\infty} K_i$ which is essential both in R_R and ${}_RR$. Consequently, A cannot be a direct summand in R_R (or in ${}_RR$). Hence, R is not semisimple. Also, since R is any direct product of fields, R is right and left self injective by Proposition 2.5. It is enough to show that R is not hereditary. Suppose R is right hereditary. Since R_R is injective, Proposition 2.9 implies that any quotient of R_R is also injective. This means that any cyclic right R module is injective, so by Theorem 2.9, R is semisimple, a contradiction. Consequently, R is not hereditary.

Definition 3.7 *(Fuchs, 2012) A submodule N of an R-module M is called coneat in M if for every simple R-module S, any homomorphism $\varphi : N \to S$ can be extended to a homomorphism $\theta : M \to S$.*

Definition 3.8 *(Crivei, 2014) An R module M is called absolutely coneat if M is coneat submodule of any module containing it.*

Proposition 3.8 *(Büyükaşık and Durğun, 2013) For a submodule $N \subseteq M$, the following are equivalent:*

(1) N is a coneat submodule of M;

(2) If $K \subseteq N$ with N/K finitely generated and $N/K \ll M/K$, then $K = N$;

(3) For any maximal submodule K of N, N/K is a direct summand of M/K;

(4) If K is a maximal submodule of N, then there exists a maximal submodule L of M such that $K = N \cap L$.

Recall that a submodule N of an R-module M is said to be *coclosed* in M if it has no proper coessential submodule in M, i.e. if for any $K \subseteq N$, $N/K \ll M/K$, then $K = N$. Then by Proposition 3.8, coclosed submodules are coneat. Thus, we may say that weakly injective modules are absolutely coneat. We have the following implications among the our concepts:

$$\text{strongly noncosingular} \Longrightarrow \text{noncosingular} \Longrightarrow \text{weakly injective} \Longrightarrow \text{absolutely coneat}$$

$$\Uparrow$$

$$\text{injective}$$

Proposition 3.9 *Let R be a right Max-ring. Then absolutely coneat modules are weakly injective.*

Proof Let A be an absolutely coneat module and M any extension of A. Suppose A is not coclosed submodule of M. Then for some submodule B of A , $A/B \ll M/B$. Since R is a right Max-ring, A is coatomic. Thus B is contained in a maximal submodule, say K, of A. Then $A/K \ll M/K$, and this contradicts with the fact that A is coneat in M. Hence, A is weakly injective. □

Proposition 3.10 *Let R be a ring. If absolutely coneat modules are strongly noncosingular, then R is a right Max-ring.*

Proof Let M be an R-module with $Rad(M) = M$. It is easy to see that $Hom(M, S) = 0$ for each simple module S. Hence, M is absolutely coneat and, by the assumption, M is strongly noncosingular. But, M is Rad-small, and so $M = 0$. Then, R is a right Max-ring. □

Corollary 3.6 *Let R be a right hereditary ring. R is right Max-ring if and only if absolutely coneat modules are strongly noncosingular.*

Proof By Proposition 3.9, absolutely coneat modules are weakly injective. Therefore, absolutely coneat modules are strongly noncosingular by Corollary 3.5 and Proposition 3.7. The converse follows by Proposition 3.10. □

Recall that An R-module M is said to be an *extending module* if, for any submodule A of M, there exists a direct summand B of M such that B is an essential extension of A. Dually, an R-module M is said to be a *lifting module* if, for any submodule A of M, there exists a direct summand B of M such that B is a coessential submodule of A in M. A ring R is called a *right co-H-ring* if every projective R-module is an extending module. A ring R is called a *right H-ring* if every injective right R-module is lifting (see, (Oshiro, 1984)).

Theorem 3.4 *((Oshiro, 1984), Theorem I) The following conditions are equivalent for a ring R:*

(1) Every injective R-module is a lifting module;

(2) R is a right Artinian ring, and every non-small R-module contains a non-zero injective submodule;

(3) R is a right perfect ring, and the family of all injective R-modules is closed under taking small covers;

(4) Every R-module is expressed as a direct sum of an injective module and a small module.

Theorem 3.5 *((Oshiro, 1984), Theorem 4.6) If R is a right nonsingular ring, then the following conditions are equivalent :*

(1) R is a right H-ring;

(2) R is a right co-H-ring.

Corollary 3.7 *(Oshiro, 1984) Let R be a ring. If R is a right co-H-ring, then every nonsingular R-module is projective. The converse also holds when R is a right nonsingular right co-H-ring.*

Theorem 3.6 *((Chatters and Khuri, 1980), Theorem 4.2) Let R be a right nonsingular ring. Then the following are equivalent:*

(1) Every nonsingular right R-module is projective;

(2) R is Artinian hereditary serial.

Right nonsingular rings are a very broad class, including right (semi)hereditary rings, von Neumann regular rings, domains and semisimple rings.
Artinian hereditary serial rings are right (left) H-rings by (Theorem 3.6, Corollary 3.7 and Theorem 3.5).

Theorem 3.7 *Let R be a right perfect ring. The following statements are equivalent:*

(1) The class of injective modules coincides with the class of (strongly) noncosingular R-modules.

(2) R is Artinian hereditary serial.

Proof $(1) \Rightarrow (2)$ Since strongly noncosingular R-modules closed under homomorphic images, every homomorphic image of an injective module is injective by the assumption. R is right hereditary ring by Proposition 2.9. Under the assumption, injective R-modules are closed under small covers by Proposition 3.3. Then R is right H-ring by Theorem 3.4. Since R is a right hereditary ring, R is right nonsingular. So, R is right co-H-ring by Theorem 3.5. Hence, every nonsingular R-module is projective by Corollary 3.7. Hence, R is Artinian serial by Theorem 3.6.
$(2) \Rightarrow (1)$ By Corollary 3.5, injective modules are strongly noncosingular. Let M be a strongly noncosingular R-module. Since Artinian hereditary serial ring R is right H-ring by (Oshiro, 1984), M has a decomposition $M = M_1 \oplus M_2$, where M_1 is injective and M_2 is small by Theorem 3.4. By Proposition 3.3, M_1 and M_2 are strongly noncosingular. But M_2 is small module, and so $M_2 = 0$. Therefore, $M_1 = M$ is injective. □

Definition 3.9 *((Lam, 1999), 8.26) A ring R is called right Kasch ring if every simple right R-module S can be embedded in R_R. "Left Kasch ring" is defined similarly. As usual, R is called a Kasch ring if it is both right and left Kasch.*
We note that any commutative Artinian ring is both right and left Kasch.

Lemma 3.7 *Let R be a right Kasch ring. An R-module M is strongly noncosingular if and only if M is semisimple and every simple submodule of M is injective.*

Proof Suppose M is not semisimple, i.e. $Soc(M) \neq M$. Since M is strongly noncosingular, M is coatomic by Theorem 3.2. Then the proper submodule $Soc(M)$ is contained in a maximal submodule of M, say K. Since M is strongly noncosingular, M/K is injective by Theorem 3.2. By the hypothesis, M/K embeds in R. But M/K is injective, and so it is direct summand of R by Proposition 2.5(4). Hence, M/K is projective by Proposition 2.7. Then K is a direct summand of M by Proposition 2.8. So, $M = K \bigoplus S$ for some submodules S of M such that $S \cong M/K$ simple. Then $S \subseteq Soc(M)$, and $S \subseteq Soc(M) \cap S \subseteq K \cap S = 0$, a contradiction. Thus we must have $M = Soc(M)$. Therefore M is semisimple. Since M is semisimple, every simple submodule N of M is isomorphic to a simple homomorphic image of M. So, N is injective by Theorem 3.2. The converse follows from Theorem 3.2, since semisimple modules are coatomic. □

In general, rings do not have any semisimple factor rings. However, rings R for which $R/J(R)$ is semisimple are of considerable interest. A ring R is said to be *semilocal* if $R/J(R)$ is a semisimple ring (see (Lam, 1999), §20). Any right or left Artinian ring, any serial ring, and any semiperfect ring is semilocal.

Proposition 3.11 *((Anderson and Fuller, 1992), Proposition 15.17) For a ring R with radical $J(R)$ the following statements are equivalent:*

(1) $R/J(R)$ is semisimple;

(2) $R/J(R)$ is right Artinian;

(3) Every product of simple right R modules is semisimple;

(4) Every product of semisimple right R modules is semisimple.

Lemma 3.8 *Let R be a semilocal ring. An R-module M is strongly noncosingular and every maximal submodule of M is direct summand if and only if M is semisimple injective.*

Proof Suppose for the contrary that $Soc(M) \neq M$. Since M is stongly noncosingular, M is coatomic, and so the proper submodule $Soc(M)$ is contained in a maximal submodule of M , say K. By the assumption, every maximal submodule of M is a direct summand of M. So, $M = K \bigoplus S$ for some submodule S of M. Since K is maximal in M, M/K is simple.

Thus, $S \cong M/K$ is simple such that $S \subseteq Soc(M)$, so $S \subseteq Soc(M) \cap S \subseteq K \cap S = 0$, a contradiction. Thus we must have $M = Soc(M)$. Therefore, M is semisimple, and $M = \bigoplus_{\lambda\in\Lambda} S_\lambda$ for some index set Λ and simple submodules S_λ of M. Then $M \subseteq N := \prod_{\lambda\in\Lambda} S_\lambda$. Since R is semilocal, by Proposition 3.11, the right side N is also a semisimple R-module. Every simple summand $(S_\lambda, \lambda \in \Lambda)$ of M is injective since M is strongly noncosingular. Thus, $N = \prod_{\lambda\in\Lambda} S_\lambda$ is injective. Then the direct summand M of N is injective. So, M is semisimple injective. The converse is clear. □

Lemma 3.7 and Lemma 3.8 yield the following.

Corollary 3.8 *Let R be a semilocal right Kasch ring. An R-module M is strongly noncosingular if and only if M is semisimple injective.*

CHAPTER 4

STRONGLY NONCOSINGULAR MODULES OVER COMMUTATIVE RINGS

In this chapter we investigate strongly noncosingular R-modules over commutative rings.

For a ring R let $Z(R) = \{r \in R \mid rs = sr,$ for each $s \in R\}$ be the center of R. It is easy to check that $Z(R)$ is a subring of R. Of course, $Z(R)$ is commutative and R is commutative if and only if $R = Z(R)$.

Corollary 4.1 *((Anderson and Fuller, 1992), Corollary 15.18) Let R be a ring with radical $J = J(R)$. Then for every left R module M, $J(R).M \subseteq Rad(M)$. If R is semisimple modulo its radical, then for every left R-module M, $J(R).M = Rad(M)$ and M/JM is semisimple.*

Proposition 4.1 *Let R be a ring and M a strongly noncosingular R-module. Then $Z^*(R) \cap Z(R) \subseteq Ann(M)$.*

Proof Let $r \in Z^*(R) \cap Z(R)$. Since $r \in Z(R)$, the map $f : M \to M$, defined by $f(m) = mr$ for each $m \in M$ is an R-homomorphism. On the other hand, $r \in Z^*(R)$ implies that $Im(f) = Mr \subseteq Rad(E(M))$. Therefore, $f = 0$ and so $Mr = 0$ by the hypothesis. Hence, $r \in Ann(M)$. □

Corollary 4.2 *Let R be a ring and M a strongly noncosingular R-module. Then $J(R) \cap Z(R) \subseteq Ann(M)$.*

Corollary 4.3 *Let R be a commutative ring and M a strongly noncosingular R-module. Then $Z^*(R).M = J(R).M = 0$.*

Corollary 4.4 *Let R be a commutative semilocal ring and M an R-module. Then, M is strongly noncosingular if and only if M is a semisimple injective module.*

Proof If R is semilocal, then, by Corollaries 4.1 and 4.3, $Rad(M) = J(R)M = 0$. So, $M/JM = M$ is semisimple by Corollary 4.1. The injectivity of M follows by the proof of Lemma 3.8. The converse is clear. □

Proposition 4.2 *Let R be a commutative ring and M a strongly noncosingular R-module. Then $Ann(M/Rad(M)) = Ann(M)$.*

Proof Let $r \in Ann(M/Rad(M))$. Then $rM \subseteq Rad(M)$, and so from the proof of Proposition 4.1 we get $rM = 0$. Therefore, $r \in Ann(M)$ and $Ann(M/Rad(M)) \subseteq Ann(M)$. On the other hand, we always have $Ann(M) \subseteq Ann(M/Rad(M))$. This completes the proof. □

Lemma 4.1 *((Anderson and Fuller, 1992), Exercises 15.(5)) Let R be a commutative ring and Ω the set of all maximal ideals of R. Then $Rad(M) = \bigcap_{P \in \Omega} PM$ for each R-module M.*

Proposition 4.3 *Let R be a commutative ring and M an R-module with a unique maximal submodule. Then M is strongly noncosingular if and only if M is simple injective.*

Proof We first claim that M is a simple R-module. By the hypothesis $Rad(M)$ is a maximal submodule of M, i.e. $M/Rad(M)$ is simple. Then $M/Rad(M) \cong R/P$ for some maximal ideal P of R by Proposition 2.11.
Since M is strongly noncosingular, $Ann(M) = Ann(M/Rad(M)) = P$ by Proposition 4.2. Then $P.M = 0$, and so $Rad(M) = 0$ by Lemma 4.1. Therefore, M is a simple R-module and, by the hypothesis, M is injective. The converse is clear. □

Lemma 4.2 *((Ware, 1971), Lemma 2.6) Suppose R is a commutative ring and S is a simple R-module. Then S is flat if and only if S is injective.*

Lemma 4.3 *Let R be a commutative noetherian ring and M an R-module. Then, M is strongly noncosingular if and only if M is semisimple injective.*

Proof Suppose for the contrary that $Soc(M) \neq M$. Since M is strongly noncosingular, the proper submodule $Soc(M)$ is contained in a maximal submodule K of M such that M/K is injective. So, M/K is flat by Lemma 4.2. Since M/K is finitely generated, it is finitely presented by Proposition 2.10. Therefore, M/K is projective by Theorem 2.7. Then K is a direct summand of M i.e. $M = K \bigoplus S$ for some simple submodule S of M. But $S \subseteq Soc(M) \cap S \subseteq (K \cap S) = 0$, a contradiction. Hence, M is semisimple. Then $M = \bigoplus_{i \in I} S_i$, where S_i is simple module for each $i \in I$. Since M is strongly noncosingular, every simple summand of M is strongly noncosingular by Proposition 3.3, and so they are injective by Theorem 3.2. Then M is injective since direct sums of injective modules are injective by Proposition 2.6. The converse is clear. □

Lemma 4.3 is not true in noncommutative case, in general. Now, we will give an example of a module over a noncommutative noetherian ring which is strongly noncosingular but not semisimple.

Example 4.1 *Let* $R = \begin{bmatrix} F & F \\ 0 & F \end{bmatrix}$ *be upper triangular matrices over a field F.* R *is a right hereditary Artinian ring , and so it is noetherian (see (Lam, 1999)). The socle of* R_R *is* $Soc(R_R) = \begin{bmatrix} 0 & F \\ 0 & F \end{bmatrix}$*. Let* $A = \begin{bmatrix} F & F \\ 0 & 0 \end{bmatrix}$ *and* $B = \begin{bmatrix} 0 & 0 \\ 0 & F \end{bmatrix}$ *be the right ideals of* R_R *such that* $R_R = A \oplus B$*.* A *is injective by (Goodearl, 1976) Exercise 3B 20-21. Since* R *is right hereditary Artinian ring,* A *is a strongly noncosingular by Corollary 3.5. However* A *is not a semisimple* R *module, otherwise* $Soc(R_R) = R_R$ *, a contradiction .*

Theorem 4.1 *Let* R *be a commutative ring. Then the following statements are equivalent:*

(1) The class of injective R*-modules coincides with the class of strongly noncosingular* R*-modules.*

(2) R *is semisimple.*

Proof $(1) \Rightarrow (2)$ Since strongly noncosingular R-modules closed under direct sums by Proposition 3.3, direct sums of injective R-modules are injective by the assumption. Then R is Noetherian ring by Proposition 2.6. By (1) and by Lemma 4.3, every injective R-module is semisimple, and so every R-module is semisimple. Then R is semisimple by Theorem 3.3. $(2) \Rightarrow (1)$ is obvious. □

CHAPTER 5

CONCLUSIONS

Motivated by the noncosingular modules, in this thesis, we introduced the concept of strongly noncosingular modules. An R-module M is called *strongly noncosingular* if for every nonzero module N and every nonzero homomorphism $f : M \to N$, $Im(f)$ is not a Rad-small submodule of N. The aim of this study is to work on the concept of strongly noncosingular modules and investigate the rings and modules that can be characterized via these modules. We proved that (1) A right R-module M is strongly noncosingular if and only if M is coatomic and noncosingular; (2) a right perfect ring R is Artinian hereditary serial if and only if the class of injective right R-modules coincides with the class of (strongly) noncosingular right R-modules; (3) a right hereditary ring R is Max-ring if and only if absolutely coneat right R-modules are strongly noncosingular; (4) a commutative ring R is semisimple if and only if the class of injective R-modules coincides with the class of strongly noncosingular R-modules.

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