RESONANCE SOLITONS AND DIRECT METHODS IN SOLITON THEORY

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ABSTRACT

RESONANCE SOLITONS AND DIRECT METHODS IN SOLITON THEORY

The Long-Short Wave interaction equations with adding quantum potential term and the Davey-Stewartson equation with addition of both, the quantum potential and the Hamiltonian terms are studied. These equations are reduced to different cases according to the choice of the quantum potential strength. For over critical case reductions to the non-linear diffusion-antidiffusion systems are derived. By the Hirota Direct Method one dissipaton solution of the system is derived. Two and three dissipaton (soliton) solutions are constructed explicitly. For special choice of the parameters they show the resonance character of interaction by fusion and fission of solitons.

ÖZET

REZONANS SOLİTONLAR VE SOLİTON TEORİSİNDE DİREKT METOD

Tezimde Uzun-Kısa dalga etkileşim denklemleri kuantum potansiyel terim eklenerek ve Davey-Stewartson denklemleri hem kuantum potansiyel hem de Hamilton terim eklenerek calışıldı. Her iki sistem de kuantum potansiyel terimin kuvvet katsayısı seçimine gc̈re farklı denklemlere indirgendi. Kritik üstü durum için lineer olmayan difüzyon-antidifüzyon sistemleri elde edildi. Hirota direkt metod ile bir dissipaton çözümü bulundu. İki ve üç dissipaton (soliton) çözümleri de açık formda oluşturuldu. Özel parametre seçimlerine göre, solitonlar için füzyon (birleşme) ve fisyon (ayrılma) etkileşmelerinin rezonans karakter gösterdiği saptandı.

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CHAPTER 1

INTRODUCTION

”I believe I shall best introduce the phenomenon by describing the circumstances of my own first acquaintance with it. I was observing the motion of a boat which was rapidly drawn along a narrow channel by a pair of horses, when the boat suddenly stopped-not so the mass of water in channel which it had put in motion; it accumulated round the prow of vessel in a state of violent agitation, then suddenly leaving it behind, rolled forward with great velocity, assuming the form of a large solitary elevation, a rounded, smooth and well-defined heap of water, which continued its course along the channel apparently without change of form or diminution of speed. I followed it on horse back, and overtook it still rolling on at a rate of some eight or nine miles an hour, preserving its original figure some thirty feet long and a foot to a foot and half in height. its height gradually diminished, and after a chase of one or two miles I lost it in the windings of channel.”

J.S.Russell

J. Scott Russell explained the motion of a solitary wave by these words in Report on Waves which he reported to British Association according to his observations (Russel 1844). He performed laboratory experiments and empirically found the speed,c, of the solitary wave as

$$c^2 = g(h+a) \tag{1.1}$$

where a is the maximum amplitude above the free surface of liquid, h is the finite depth and g is the acceleration due to gravity.

It tooks sixty years to find any theoretical treatment for Russells observation of the shallow water solitary wave. The subject discussed briefly by many mathematicians who conjectured that such waves can not propagate in a liquid medium without change of form (Stokes 1847, Airy 1845). Russells prediction was confirmed independently and it was shown that the solitary wave profile is given by

$$\eta(x,t) = a\,sech^2[\beta(x - Ut)] \tag{1.2}$$

where a is the amplitude, h is the depth, U is the velocity, g is the acceleration due to gravity and $\beta^{-2} = 4h^2(h+a)/3a$ for any $a > 0$ (Boussinesq 1992, Rayleigh 1876). But they could not find an equation that admits this solution. Boussinesq constructed a nonlinear evolution equation for such long water waves in the form

$$\eta_{tt} = c^2\left[\eta_{xx} + \left(\frac{\eta^2}{h}\right)_{xx} + \frac{1}{3}\eta_{xxxx}\right] \tag{1.3}$$

where $c = \sqrt{gh}$ is the speed of shallow water waves. This equation is known as the Boussinesq (bidirectional) equation, which admits solutions

$$\eta(x,t) = a\,sech^2\left[\left(\frac{3a}{h^3}\right)^{1/2}(x \pm Ut)\right] \tag{1.4}$$

representing solitary waves traveling in both positive and negative x-directions with the velocity U.

The initial theoretical confirmation was made by Kortweg and de Vires. They derived an equation for the propagation of waves on the surface of a shallow canal which became very famous and convenient to understand the phenomenon (Korteweg and de Vires 1895). Its solution is a hump-like wave which moves at a constant velocity without changing shape.

$$KdV Equation : \; u_t + 6u\,u_{xx} + u_{xxx} = 0 \tag{1.5}$$

$$Solution of the KdV Equation : \; u = \frac{v^2}{2}\cosh^{-2}\frac{v}{2}(x - v^2 t) \tag{1.6}$$

Fermi, Pasta and Ulam worked on a problem which at the first sight seemed totally unrelated (Fermi, et al. 1965). FPU problem lead numerical studies of Zabusky and Kruskal considering the KdV equation. Remarkable property of these solitary waves was their collision in which they preserved their original shape with only a small change in their phase. They called these solitary waves ”SOLITONS” to point out particle-like behavior (Zabusky and Kruskal 1965).

Using ideas of quantum scattering theory, a new method for solution of the KdV equation developed (Gardner, et al. 1967). Peter Lax generalized these ideas and Zakharov and Shabat showed that this method works also for the nonlinear Schrödinger equation (Lax 1968, Zakharov and Shabat 1972). These ideas were improved and it was shown that the method works for a wide class of nonlinear evolution equations and this procedure named as the Inverse Scattering Transform (IST) (Ablowitz, et al. 1974).

After this, several methods have been developed to find exact solutions for nonlinear equations which is a very complicated area of study. We may summarize some of these approaches. In general it is not easy to find exact solutions for nonlinear partial differential equations. Moreover, a useful method which is suitable for some equation can not be applied to another one. In general, the special form of solutions as traveling wave solutions can be found almost for any equation by special substitution, reducing PDE into an ODE. In some cases it is also possible to use a direct transformation to linearize the problem, like Burgers equation and the heat equation (Cole 1951, Hopf 1950). It works for a narrow class of nonlinear equations called C-integrable systems (Calogero and Degasperis 1982). Another part of the theory, taking origin from differential geometry, are Bäcklund, Darboux and Miura transformations (Rogers and Schief 2002). The Inverse Scattering Transform, which is used to solve initial value problems is based on auxiliary linear problem for nonlinear equations and uses powerful analytical methods under making strong assumptions (Novikov, et al. 1984). Among these approaches we mention the Hirota Direct Method, that can be applied to a wide class of nonlinear soliton equations. Hirota introduced the direct method to construct soliton solutions for KdV equation (Hirota 1971) and then for mKdV, sine-Gordon (Hirota 1972) and NLS equations (Hirota 1973). This new method became also useful to find appropriate Bäcklund transformations of the equations. Sato's theory followed Hirota's invention and explained the essences of the method in a rigorous mathematical way (Sato 1981). Another important direction given by Jimbo and Miwa is related with properties of τ functions, as a new dependent variable of Hirota's substitution (Jimbo and Miwa 1983).

In 1998 Pashaev has applied the Hirota Direct Method to construct resonance interaction of solitons in NLS equation with addition of the so called quantum potential (Pashaev and Lee 2002,a). Interpretation of soliton resonances in terms of black hole physics was also constructed (Martina,et al. 1997) and (Pashaev and Lee 2002,b). Direct application of soliton resoances in plasma physics has been also discussed (Lee, et al. 2007). In all these systems due to complicated analytical structure the Hirota direct method is the only efficient treatment to construct soliton resonances.

Soliton resonances have great importance to understand the generation of extreme waves like tsunamies, rouge and freak waves (Maruno, et al. 2007). In the case of resonant Y-shape soliton which was found by Miles, the maximum amplitude of a solitary wave can reach four times the amplitude of an incoming solitary wave (Miles 1977) which means that two resonance solitons would be sixteen times of the incoming wave. The observations for soliton resonance have been reported and the relevant physical applications have also been discussed such as convective motion of a plasma (Tajiri and Maesono 1997), Toda lattice (Maruno and Biondini 2004), plasma experiments (Nakamura and Lonngren 1999), water wave interaction in shallow water (Osborne, et al. 1998). Moreover, in certain fluid (Pedlosky 1987) and deep ocean (Ibragimov 2006), the most important mechanism resulting in the transfer of energy from one wave to another is believed to be resonant triad interactions. Recently, it has been pointed out (Soomere and Engelbrecht 2006) that soliton resonance has the applications in maritime security and coastal engineering in areas adjacent to intense high-speed ship traffic. The high and steep waves produced by ship wakes may bring considerable impact on the ship traffic (Soomere 2005).

In the present thesis we apply the Hirota Direct Method to construct soliton resonances for several NLS equations like the Long-Short wave interaction equations and the Davey-Stewartson equation and their generalizations.

In Chapter 2 we summarize some of the elementary principles of both linear and nonlinear evolution equations. Effects of dispersion and diffusion are discussed. Propagation type of the traveling wave solutions are explained. Comparison of each case leads us to the definition and general properties of solitons and to the dissipaton concept.

In Chapter 3 we give the basic idea of the Hirota Direct Method and some useful definitions. We use the KdV equation as a model equation and explain the bilinearization process. Important properties of the D operator, which enables us to construct the solutions are given.

In Chapter 4 we study the Zakharov equation and its exactly integrable limits. Some well known results, such as the Lax pair, bilinear representation and soliton solutions are reviewed. We discuss the Long-Short wave interaction equations, which is related with two integrable limits of the Zakharov equation.

In Chapter 5 we introduce the system of L-S wave equations with addition of the quantum potential term and point out the specific cases . For the undercritical case we encounter the already known integrable systems and give the Lax pair, bilinear representation and soliton solutions.

In Chapter 6 the new system of equations for the overcritical case are derived. The Lax pair, the bilinear representation, one soliton and dissipaton solutions are given to explain the general properties. Two and three dissipaton solutions are constructed . Regularity conditions and the resonance cases are discussed. Exponential decompositions of the functions give raise to a new system admitting shock wave profile.

In Chapter 7 integrals of motion for the system are calculated. The conserved identities are used to construct Y-shaped resonance interactions. Relation with regularity conditions and conserved identities are established.

In Chapter 8 we introduce a modified Davey-Stewartson equation. Reductions of the system help us to classify the system into undercritical and overcritical situations. We give bilinear representation and apply Hirota direct method to obtain one and two soliton solutions with resonance interactions.

CHAPTER 2

NONLINEARITY, DISPERSION AND DIFFUSION

In this chapter we consider some important concepts, such as, nonlinearity, dispersion and diffusion (Whitham 1974). Different combinations of these properties effect the systems in a distinct manner and explaining these situations will be helpful to understand the physical state of the problem (Drazin and Johnson 1996).

2.1. Linear Non-Dispersive Waves

The equation governing electromagnetic waves, acoustics (sound waves in air and liquids) and elasticity (stress waves, earthquakes) is the well known wave equation which is the simplest hyperbolic equation to start the discussion. It does not cover the majority of waves but occurs in many problems. We will consider one space dimensional wave equation and two methods to solve it.

One Space Dimensional Wave Equation:
Let $u(x,t)$ satisfy

$$u_{tt} - c^2 u_{xx} = 0 \quad -\infty < x < \infty,\ t > 0 \tag{2.1}$$

In which c is a real valued constant. It is common to introduce the characteristic coordinates $\xi = x - ct$ and $\eta = x + ct$ to reduce the system into

$$u_{\xi\eta} = 0 \tag{2.2}$$

and the general solution can easily be obtained as

$$u(x,t) = h(x+ct) + l(x-ct). \tag{2.3}$$

where $h(x)$ and $l(x)$ are arbitrary functions (Courant and Hilbert 1989, John 1982).

These functions satisfy the one directional wave equations

$$h_t - c\,h_x = 0 \;\; and \;\; l_t + c\,l_x = 0 \tag{2.4}$$

The Cauchy problem for the wave equation with initial data

$$u(x,0) = f(x), \quad u_t(x,0) = g(x) \tag{2.5}$$

then gives d'Alambert's solution.

$$u(x,t) = \frac{f(x+ct)+f(x-ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)\,ds \tag{2.6}$$

From another side the equation (2.1) can be solved by the Fourier transform (Zauderer 1989).

Setting

$$U(k,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} u(x,t)\,e^{-ikx}\,dx \tag{2.7}$$

$$u(k,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} U(k,t)\,e^{ikx}\,dk \tag{2.8}$$

we obtain the second order ordinary differential equation

$$U_{tt} + c^2k^2U = 0 \tag{2.9}$$

with initial conditions

$$U(k,0) = F(k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)\,e^{-ikx}\,dx \tag{2.10}$$

$$U_t(k,0) = G(k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} g(x)\,e^{-ikx}\,dx \tag{2.11}$$

Solution of the initial value problem for $U(k,t)$ is found to be

$$U(k,t) = \left[\frac{1}{2}F(k) + \frac{1}{2ick}G(k)\right] e^{ickt} + \left[\frac{1}{2}F(k) - \frac{1}{2ick}G(k)\right] e^{-ickt} \tag{2.12}$$

Inverting this transform gives solution for $u(x,t)$ as

$$u(x,t) = \frac{1}{\sqrt{2\pi}} \left(\int_{-\infty}^{\infty} \left[\frac{1}{2}F(k) + \frac{1}{2ick}G(k)\right] e^{ik(x+ct)}\, dk + \int_{-\infty}^{\infty} \left[\frac{1}{2}F(k) - \frac{1}{2ick}G(k)\right] e^{ik(x-ct)}\, dk \right) \tag{2.13}$$

We recall that

$$f(x \pm ct) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(k) e^{ik(x\pm ct)}\, dk \tag{2.14}$$

$$\int^{x\pm ct} g(s)\, ds = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{G(k)}{ik} e^{ik(x\pm ct)}\, dk \tag{2.15}$$

then the expression(2.13) again gives d'Alambert's solution (2.6).

The general solution (2.6) is superposition of that two waves propagating with constant velocity c, in opposite directions. Hence, they do not interact with each other during the propagation process.

The wave equation (2.1) is also invariant under the transformation $x \to -x$ and $t \to -t$. In other words, it is bidirectional and time reversible. Due to this we can use double Fourier Transform

$$u(x,t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} A(k,\omega) e^{i(kx-\omega t)}\, dk\, d\omega \tag{2.16}$$

It means that any arbitrary disturbance can be decomposed into harmonic wave components and harmonic wave propagation may serve as the basis for discussing the various shapes of waves. Hence, an elementary solution called harmonic wave, represents a solution for the wave equation (2.1), can be given as

$$u(x,t) = A \exp[i(kx - \omega t)] \tag{2.17}$$

where A is the amplitude, $k = 2\pi/\lambda$ is the wave number related to the wave length λ and $\omega = 2\pi f$ is the angular frequency related to frequency f (Remoissenet 1992). By direct substitution we derive a relation between ω and k as $\omega = \pm ck$.

The relation which admits us to write ω as a function of k ($k \varepsilon R$), is called the **dispersion relation** and for this case it is linear with respect to k. A wave governed by a linear dispersion relation is called a **non-dispersive** wave. For the wave equation (2.1), the phase velocity is the speed of phase ($\varphi = \omega t - kx$), $v_p = \omega(k)/k = \pm c$ and the group velocity is the speed of the bulk wave $v_g = d\omega/dk = \pm c$.

A future of such wave is that an initial profile taking the form of a pulse does not change its shape (see Figure 2.1). This is because waves with different wave numbers travel with the same speed c and the speed is independent of the amplitude of the wave.

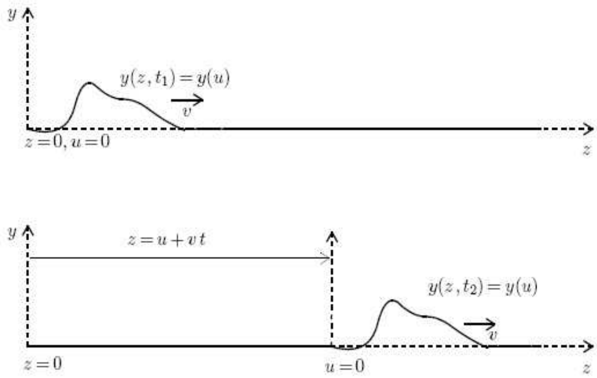


Figure 2.1. Linear Non-Dispersive Wave.

2.2. Nonlinear Non-Dispersive Waves

A large number of problems, including traffic flow, shock waves, flood waves, waves in glaciers and chemical exchange process in chromatography can be represented by the kinematic wave equation (John 1982).

Kinematic Wave Equation:

Let $u(x,t)$ satisfy

$$u_t + v(u)u_x = 0, \quad -\infty < x < \infty, \quad t > 0 \tag{2.18}$$

A traveling wave solution $u = h(x-ct)$ with constanat c which transforms the partial differential equation into an ordinary differential equation if
$v(u) \neq constant$ would only gives a trivial solution where h is a constant. It is possible to obtain a non-trivial solution by introducing the associated characteristic equations

$$\frac{dt}{ds} = 1, \quad \frac{dx}{ds} = v(u), \quad \frac{du}{ds} = 0 \tag{2.19}$$

to clarify that u is a constant on the characteristics which propagate with speed $v(u)$. Hence, this equation has the implicit solution

$$u(x,t) = f(x - v(u)t) \tag{2.20}$$

where $f(x)$ is an arbitrary function. If $v(u)$ is an increasing function in u, this formula tells us that a wave travels faster as its amplitude increases in contrast with linear waves. Setting $v(u) = u$ we can examine the case specifically for nonlinear transport equation by considering the first derivative with respect to x.

$$u_x = \frac{f'}{1 + tf'} \tag{2.21}$$

This relation indicates that at time $t = -1/f'$ (critical/breaking time), the profile of u first develops an infinite slope (the slope blows up) (See Figure 2.2). The dependence of v on u produces the typical nonlinear distortion of the wave as it propagates. Wave steepens and then breaks. This is a common behavior for nonlinear partial differential equations without dispersion (Whitham 1974).

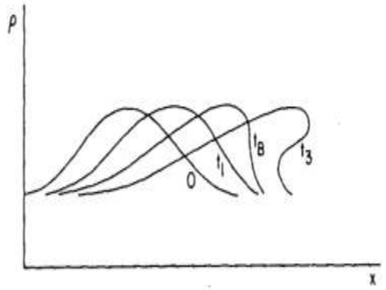


Figure 2.2. Nonlinear Non-Dispersive Wave.

2.3. Linear Dispersive Waves

For linear dispersive waves, we consider the linear Korteweg-de Vires equation which is one of the simplest models for the unidirectional propagation of long waves.

Linear Korteweg-de Vires equation:

Let $u(x,t)$ satisfy

$$u_t + u_{xxx} = 0, \quad -\infty < x < \infty, \quad t > 0 \tag{2.22}$$

Since the equation is linear, we may solve the initial value problem (specifying the initial disturbance) for $u(x,0) = f(x)$ by using Fourier transform. Rewriting u interms of its Fourier transform,

$$U(k,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u(x,t)\, e^{-ikx}\, dx \tag{2.23}$$

$$u(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} U(k,t)\, e^{ikx}\, dk \tag{2.24}$$

we convert the equation into a first order, linear ordinary differential equation with corresponding initial data

$$U_t + (ik)^3 U = 0 \tag{2.25}$$

$$U(k,0) = F(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-ikx}\, dx \tag{2.26}$$

The solution of this initial value problem is $U(k,t) = F(k)e^{ik^3 t}$ and the solution of the equation (2.22) is

$$u(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(k) e^{i(kx+k^3 t)}\, dk \tag{2.27}$$

Specifying the initial disturbance as $u(x,t) = \delta(x)$, the fundamental solution for this dispersive wave equation can be given in terms of the Airy function (Abramowitz and Stegun 1970)

$$u(x,t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i(kx-\omega t)}\, dk = \frac{1}{\pi} \int_{0}^{\infty} \cos(kx + k^3 t)\, dk = \frac{1}{\sqrt[3]{3t}} Ai\left(\frac{x}{\sqrt[3]{3t}}\right) \tag{2.28}$$

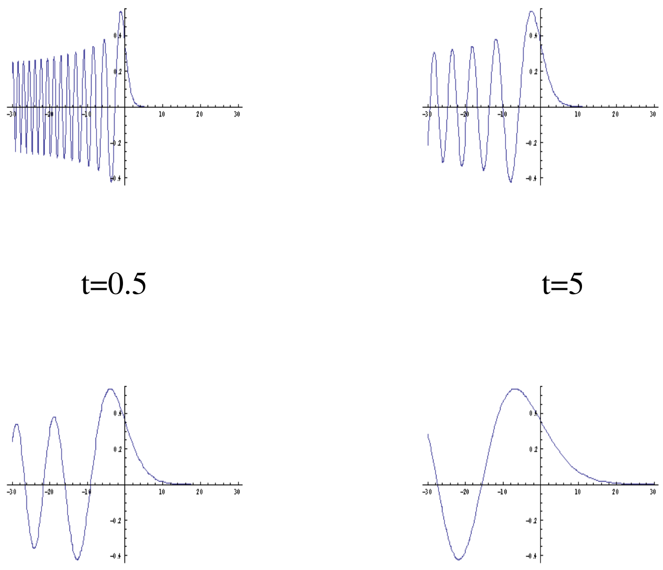
We may generalize this solution by considering the initial delta impulse at $x = \xi$ and the initial data as a superposition of delta functions

$$u(x,0) = f(x) = \int_{-\infty}^{\infty} f(\xi)\, \delta(x-\xi)\, d\xi \tag{2.29}$$

It gives the general solution of the initial value problem:

$$u(x,t) = \frac{1}{\sqrt[3]{3t}} \int_{-\infty}^{\infty} f(\xi) Ai\left(\frac{x-\xi}{\sqrt[3]{3t}}\right) d\xi \tag{2.30}$$

The dispersion relation for elementary harmonics ($\omega = -k^3$) of this equation gives us the opportunity to understand the behavior of dispersive waves. According to the dispersion relation, this wave propagates with a phase velocity $v_p = \omega/k = -k^2$. Also the group velocity of this wave is $v_g = \partial\omega/\partial k = -3k^2$. These equations tell us that waves of different wave number propagate at different velocities, independent of the amplitude. Thus, the resulting superposition of waves spread out as they travel and do not preserve the original shape of superposition (See Figure 2.3).



t=0.5 t=5

t=20 t=100

Figure 2.3. Linear Dispersive Wave (Airy Function).

2.4. Nonlinear Dispersive Waves

A very important equation arising in many physical problems such as water waves, the ion-acoustic waves in plasma, rotating flow in a tube and acoustic waves in anharmonic crystals is the famous Koteveg-de Vires equation (Drazin and Johnson 1996).

Korteweg-de Vires Equation:

Let $u(x,t)$ satisfy

$$u_t - 6uu_x + u_{xxx} = 0, \quad -\infty < x < \infty \tag{2.31}$$

This equation is a combination of two equations considered in Sections 2.2 and 2.3. Since the equation is nonlinear, a harmonic wave is not a solution for this equation and we can not state the dispersion relation easily. This is why we introduce $\xi = x - ct$ and examine the equation for traveling wave solution of the form $f(x-ct)$ (Drazin and Johnson 1996) and (Ablowitz and Clarkson 1991). The equation in terms of this new function is

$$-cf_\xi - 3(f^2)_\xi + f_{\xi\xi\xi} = 0 \tag{2.32}$$

and can be integrated once with respect to ξ, leading to

$$f_{\xi\xi} = A + cf + 3f^2 \tag{2.33}$$

where A is a constant of integration. Multiplying both sides with f_ξ and integrating once more yields

$$\frac{(f_\xi)^2}{2} = \left(f^3 + \frac{c}{2}f^2 + Af + B\right) \tag{2.34}$$

Impossing the boundary conditions $f, f_\xi, f_{\xi\xi} \to 0$ as $\xi \to \pm\infty$, A and B vanish. We get

$$\frac{df}{f\sqrt{2f+c}} = \pm d\xi \tag{2.35}$$

Setting $c = \kappa^2 \geq 0$ we obtain

$$-\frac{2}{\kappa}\tanh^{-1}\sqrt{\frac{2f+\kappa^2}{\kappa}} = \pm\xi + a \tag{2.36}$$

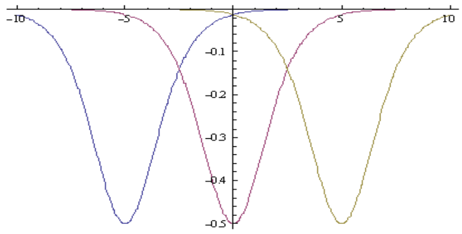
where a is an integration constant. To derive the function f we write

$$\frac{\kappa^2}{2}\left[\left(\tanh^2\frac{\kappa}{2}(\xi+a)\right) - 1\right] = f(\xi) \tag{2.37}$$

Since the solution is an even function, the choice of $\pm$ is redundant. Finally, we have

$$u(x,t) = f(x-\kappa^2 t) = -\frac{\kappa^2}{2}\left(\cosh^{-2}\frac{\kappa}{2}(x-\kappa^2 t + a)\right) \tag{2.38}$$

which is an exact solution of KdV equation in traveling solitary wave form (See Figure 2.4). The amplitude of the wave is proportional to its velocity, so that waves travel faster as their amplitude becomes greater.



t=-5 t=0 t=5

Figure 2.4. Traveling Solitary Wave.

2.4.1. Cnoidal Wave Solution

It is relevant to choose arbitrary constants c, A and B, without restriction for the equation (2.34)

$$\frac{(f_\xi)^2}{2} = \left(f^3 + \frac{c}{2}f^2 + Af + B\right) \equiv g(f) \tag{2.39}$$

However, we should restrict $(f_\xi)^2 \geq 0$, to obtain real and bounded solutions. Thus, it becomes important to examine the roots of $g(f)$ (Drazin and Johnson 1996) and (Ablowitz and Clarkson 1991).

Let $a_0,\ a_1,\ a_2$ represent three real roots of $g(f)$, satisfying $a_0 \geq a_1 \geq a_2$.Thus, we have $c = -2(a_0+a_1+a_2),\ \ A = a_0a_1+a_0a_2+a_1a_2,\ \ B = -a_0a_1a_2$.

- **Case I**

  $a_0 \neq a_1 \neq a_2,\ \ a_0 > a_1 > a_2$

  For this case we should integrate both sides of the following equation

$$\pm\frac{df}{[2(f-a_0)(f-a_1)(f-a_2)]^{1/2}} = d\xi \tag{2.40}$$

  Substituting $f = a_2+(a_1-a_2)\sin^2\theta$, and setting $m = (a_1-a_2)/(a_0-a_2)$ yields

$$\pm\left(\frac{2}{a_0-a_2}\right)^{1/2}\int_0^{\varphi}\frac{d\theta}{\sqrt{1-m\sin^2\theta}} = \xi-\xi_0 \tag{2.41}$$

with integration constant ξ_0 and can be represented in terms of the Jacobi elliptic function of modulus m (cnoidal wave form)

$$cn[(\xi - \xi_0)\sqrt{(a_0 - a_2)/2}|m] = \cos\varphi \tag{2.42}$$

where $\pm$ is suppressed since cn is an even function. The solution is

$$u(x,t) = a_1 - (a_1 - a_2)cn^2[(x - ct - \xi_0)\sqrt{(a_0 - a_2)/2}|m] \tag{2.43}$$

- **Case II**

  $a_0 = a_1 \neq a_2$

  In this case $m = 1$. Since $cn(z|1) = sech(z)$ (2.42) reduces to

$$u(x,t) = a_1 - (a_1 - a_2)sech^2[(x - ct - \xi_0)\sqrt{(a_1 - a_2)/2}] \tag{2.44}$$

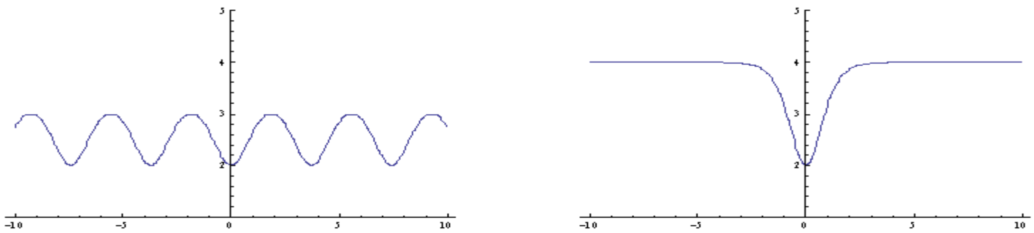


Figure 2.5. Cnoidal Wave and Solitary Wave.

- **Case III**

  $a_0 \neq a_1 = a_2$

  This case will give rise to a constant solution $u(x,t) = a_1$.

- **Case IV**

  $a_0 = a_1 = a_2$

  If the equation has a triple zero at $c = -6a_0$, it will admit the following form

$$\pm\frac{df}{d\xi} = [2(f-a_0)]^{3/2} \tag{2.45}$$

which has the following solution

$$f(\xi) = a_0 - \frac{2}{(\xi-\xi_0)^2} = a_0 - \frac{2}{(x-ct-\xi_0)^2} \tag{2.46}$$

For $\xi \to \xi_0$ this solution is unbounded, but for $|\xi| \to \infty$ it is bounded and the solution is $u(x,t) = a_0$. We observe that this solution has a movable singularity.

- **Case V**
  Let f has a single zero a_0. For this case if $f' > 0$ then $g > 0$ for all $f > a_0$ which meas that the solution is unbounded. However, if $f' < 0$, f will decrease until it reaches $f = a_0$ then f will change its sign and become unbounded again.

2.5. Linear Diffusion

The heat (diffusion) equation describing the flow of a quantity, such as the heat, is the simplest model equation of parabolic type.

Heat Equation:

Let $u(x,t)$ satisfy

$$u_t - u_{xx} = 0, \quad -\infty < x < \infty, \quad t > 0 \tag{2.47}$$

for the initial data $u(x,0) = f(x)$.

To solve this initial value problem, we apply Fourier transform in the x variable (Zauderer 1989). The result is an ordinary differential equation

$$U_t + k^2 U = 0 \tag{2.48}$$

where $U(k,t)$ is the Fourier transform of $u(x,t)$ with the initial condition

$$U(k,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u(x,t)\, e^{-ikx}\, dx \tag{2.49}$$

$$U(k,0) = F(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{-ikx}\, dx \tag{2.50}$$

The solution of this initial value problem is $U(k,t) = F(k)e^{-k^2 t}$. Applying the inverse Fourier transform leads to the solution of the heat equation

$$u(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(k)\, e^{ikx-k^2 t}\, dk \tag{2.51}$$

To find the fundamental solution, we take the initial temperature profile to be a delta function $\delta(x-\xi)$ concatenated at $x = \xi$. Substituting its Fourier transform $F(k) = 1/\sqrt{2\pi}$ into the solution of the heat equation yields

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ik(x-\xi)-k^2 t}\, dk = \frac{1}{2\sqrt{\pi t}} e^{-(x-\xi)^2/4t} \tag{2.52}$$

and it is illustrated in Figure 2.6.

With the fundamental solution in hand we can reconstruct the general solution for arbitrary initial data

$$f(x) = \int_{-\infty}^{\infty} f(\xi)\delta(x-\xi)\, d\xi$$

Hence, when the initial disturbance is $f(x)$, solution of the heat equation is

$$u(x,t) = \frac{1}{2\sqrt{\pi t}} \int_{-\infty}^{\infty} f(\xi)\, e^{-(x-\xi)^2/4t}\, d\xi \tag{2.53}$$

As time increases, the solution shrinks and widens, eventually decaying everywhere to zero. The total energy, which is the area under the graph, remains fixed while gradually spreading out over the entire real line (See Figure 2.6). As we can interpret from the graphics, solution of the heat equation can not propagate in a permanent form.

We may verify this fact by examining the traveling wave solution for the heat equation. Setting $u(x,t) = f(x-ct)$ the equation reduces to $f'' + cf' = 0$. This linear differential equation has the general solution

$$u(x,t) = A + Be^{-c(x-ct)} \tag{2.54}$$

where A and B are arbitrary constants. The only possibility for u to be a constant at both minus and plus infinity is to require B = 0. Thus, traveling wave solution is a constant which is not a usual wave.

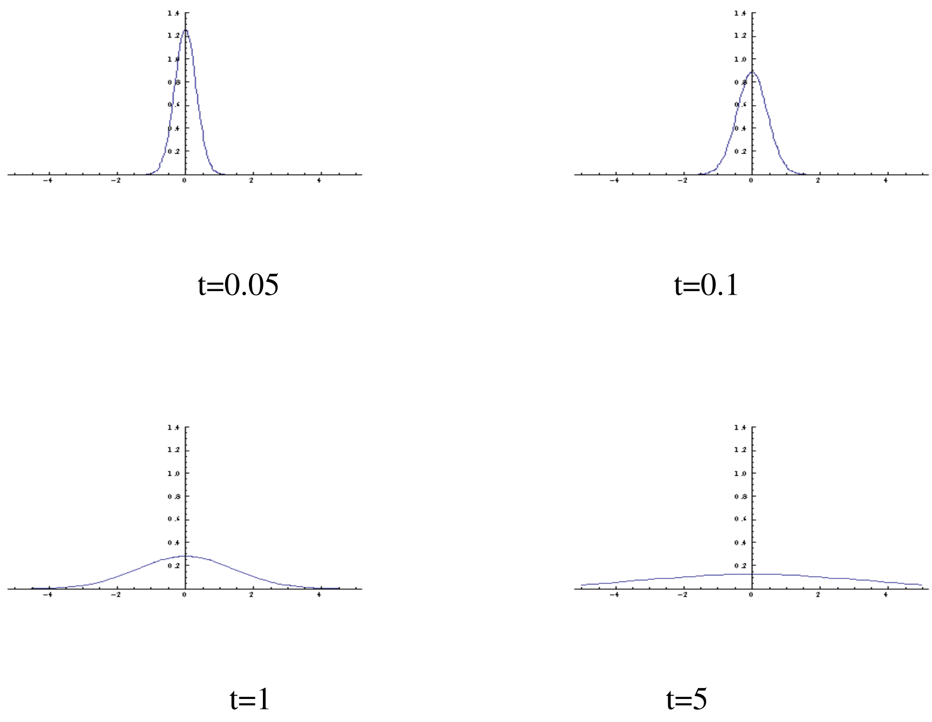


Figure 2.6. Fundamental Solution of the Heat Equation.

2.6. Nonlinear Diffusion

The simplest nonlinear diffusion equation is known as Burgers equation, combining both nonlinearity and diffusion (Drazin and Johnson 1996). Hence the equation represents a very simplified version of the equations of viscous fluid flows (Debnath 2005).

Burgers' Equation:

Let $u(x,t)$ satisfy

$$u_t + uu_x = u_{xx}, \ -\infty < x < \infty, \ t > 0 \tag{2.55}$$

We search for a traveling wave solution to obtain an ordinary differential equation and introduce the function v as $u(x,t) = v(\xi)$ where $\xi = x - ct$ (Whitham 1974). Substituting v into (2.55) gives the nonlinear second order ordinary differential equation

$$-cv_\xi + vv_\xi = v_{\xi\xi} \tag{2.56}$$

and integrating both sides yields the Riccati equation

$$k - cv + \frac{v^2}{2} = v_\xi \tag{2.57}$$

where k is an integration constant. To find a bounded traveling wave solution for $v(\xi)$, the quadratic polynomial on the left hand side must have two real roots. Hence, c and k can be determined in terms of this real roots a and b satisfying the equation (2.57).

$$2\frac{dv}{d\xi} = (v-a)(v-b), \ c = \frac{a+b}{2}, \ k = \frac{ab}{2}, \ for\, k < \frac{c^2}{2} \tag{2.58}$$

Integration gives

$$v(\xi) = u(x,t) = \frac{a + b\,e^{(b-a)(x-ct-x_0)/2}}{1 + e^{(b-a)(x-ct-x_0)/2}} \tag{2.59}$$

where x_0 is an integration constant and $a < v < b$. Also we observe that

$$\lim_{x \to -\infty} u(x,t) = b, \ \lim_{x \to \infty} u(x,t) = a \tag{2.60}$$

We write this solution in Taylor shock profile form (Kink Soliton) (Debnath 2005)

$$a + \frac{b-a}{2}\left[1 - \tanh\left(\frac{b-a}{4}(x - ct - \xi_0)\right)\right] \tag{2.61}$$

moving at constant speed $c = (a+b)/2$ (See Figure 2.7).

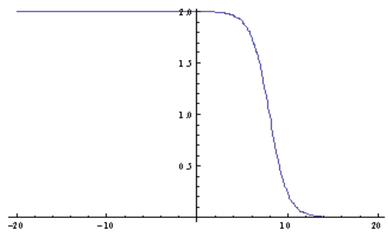


Figure 2.7. The Diffusion Shock Soliton.

If we have two degenerate real roots then we have to solve the ordinary differential equation

$$2\frac{dv}{d\xi} = (v-a)^2 \tag{2.62}$$

and get the solution

$$v(\xi) = a - \frac{2}{\xi - \xi_0} = a - \frac{2}{x - ct - \xi_0} \tag{2.63}$$

which is bounded for $|\xi| \to \infty$. This solution also has a movable singularity at ξ_0 $(t = 0)$.

2.6.1. Lineatization of the Burgers Equation and Cole-Hopf Transformation

In fact, Burgers equation can be converted into the linear heat equation. In general, linearization is extremely challenging but the Cole- Hopf transformation admits us to solve Burgers equation explicitly (Cole 1951) and (Hopf 1950). The Cole-Hopf Transformation for $u(x,t)$ is

$$u(x,t) = -2\frac{T_x}{T} = -2(\ln T)_x \tag{2.64}$$

where $u(x,t)$ is a solution of (2.55). After substituting $u(x,t)$ in terms of $T(x,t)$ and integrating once with respect to x, Burgers' equation reduces to

$$T_t - T_{xx} = -Tk(t) \tag{2.65}$$

The term in the right hand side can be easily removed by transforming
$T \to Te^{-\int^t k(\tau)d\tau}$. For simplicity we take arbitrary function $k(t) = 0$. Then, if T is a solution of heat equation with initial data $T(x,0) = f(x)$ (see 2.53), we may obtain a solution for Burgers equation as $u(x,t) = -2(\ln T)_x$ (Whitham 1974), that can be written in the form

$$u(x,t) = \frac{\int_{-\infty}^{\infty} \frac{x-\xi}{t} f(\xi) \exp\left[\frac{-(x-\xi)^2}{4t}\right] d\xi}{\int_{-\infty}^{\infty} f(\xi) \exp\left[\frac{-(x-\xi)^2}{4t}\right] d\xi} \tag{2.66}$$

The initial value problem for the heat equation with

$$T(x,0) = \exp\left(-\frac{1}{2}\int^x F(\eta)\,d\eta\right) \tag{2.67}$$

and F as the step function

$$F(x)\begin{cases} a_1, & x > 0; \\ a_2, & x < 0. \end{cases} \tag{2.68}$$

creates shock soliton profile (Whitham 1974).

$$u(x,t) = a_1 + \frac{a_2 - a_1}{1 + \exp\left[\left(\frac{a_2-a_1}{2}\right)x - \left(\frac{a_1+a_2}{2}t\right)\right]} \tag{2.69}$$

The discussion above shows that any solution of the heat equation may be used to construct a solution for Burgers' equation. We recall the traveling wave solution of the heat equation (2.54), after applying Cole-Hopf transformation for
$T(x,t) = A + B\exp[-c(x-ct)]$, we obtain Taylor shock profile (for $AB > 0$) as

$$u(x,t) = -2\frac{T_x}{T} = c\left[1 - \tanh\frac{c}{2}(x - ct + x_0)\right] \tag{2.70}$$

2.7. Dissipatons and Envelope Solitons

For complex nonlinear equations like the Nonlinear Schrödinger Equation

$$i\psi_t + \psi_{xx} + |\psi|^2\psi = 0 \tag{2.71}$$

the soliton solution has the following form

$$\psi = \frac{\lambda \exp[i(\mu x + (\mu^2 - \lambda^2)t)]}{\cosh[\lambda(x - \mu t)]} \tag{2.72}$$

It has an oscillatory motion and the envelope

$$|\psi|^2 = \frac{\lambda^2}{\cosh^2[\lambda(x - \mu t)]} \tag{2.73}$$

is preserving its shape. Such solution is called the envelope soliton and it can be considered as a complex root of soliton.

From another side if we consider the coupled nonlinear heat, anti-heat equation

$$q_t = q_{xx} + (qr)q \tag{2.74}$$

$$-r_t = r_{xx} + (qr)r \tag{2.75}$$

It admits the following type solution

$$q = k\frac{\exp(kx + k^2 t)}{\cosh k(x - vt)} \tag{2.76}$$

$$r = k\frac{\exp -(kx + k^2 t)}{\cosh k(x - vt)} \tag{2.77}$$

These waves propagate in a distinct manner from the envelope solitons. They do not oscillate but exponentially grow and decay respectively. However, the product $qr = k^2/\cosh^2 k(x - vt)$ has the soliton profile like in equation (2.73). It is natural to consider q and r as the real roots of the soliton. To indicate the dissipative behavior and the soliton character of this solution Pashaev called these special waves **dissipaton** (Pashaev 1997).

It was shown that dissipaton interaction in equations (2.76) and (2.77) has the resonant character (Pashaev and Lee 2002,a,,). Moreover, relation of this system with the problem of NLS soliton in quantum potential was derived (Pashaev and Lee 2002,a). Corresponding equation

$$i\psi_t + \psi_{xx} + |\psi|^2\psi = s\frac{|\psi|_{xx}}{|\psi|}\psi \tag{2.78}$$

was called the Resonant Nonlinear Schr$\ddot{o}$dinger equation (Pashaev and Lee 2002,a). Recently these equations were derived in plasma physics as descriptive of cold collisionless plasma in magnetic field (Lee, et al. 2007).

CHAPTER 3

THE DIRECT METHOD

In this chapter we give the leading ideas of the Direct Method and the basic properties of the D operator. The KdV equation is considered to find the bilinear representation by the help of D operator and one soliton solution is constructed as an application of this method (Hirota 2004).

3.1. Linearization of Nonlinear Differential Equations and Bilinearization of the KdV Equation

Since the superposition principle does not hold for nonlinear equations, it is very difficult to find exact solutions for nonlinear partial differential equations. In the previous chapter we gave examples of traveling wave solutions which is a common way to construct exact solutions in solitary wave form and also used the Cole-Hopf transformation to linearize the Burgers' equation which reduces the problem into a simpler case, admitting us to obtain the solution for nonlinear equation. Although, it is not possible to categorize nonlinear differential equations that can be linearized, we may consider some well known examples to have some insight for bilinearization, which is important to construct soliton solutions by the direct method (Hirota 2004).

- **The Riccati Equation**

  The Riccati equation,

$$\frac{d}{dx}u(x) = a(x) + 2b(x)u(x) + u^2(x) \tag{3.1}$$

  can be linearized by setting $u = g/f$ to obtain

$$f_x + b(x)f + g = \lambda(x)f \tag{3.2}$$

$$g_x - a(x)f - b(x)g = \lambda(x)g \tag{3.3}$$

where $\lambda(x)$ is an arbitrary function.

The Cole-Hopf transformation which has a similar rational function form (T_x/T) for the Burgers' equation also linearizes the problem as discussed in the previous chapter.

- **The Liouville Equation**

  The Liouville equation,

$$u_{xt} = e^u \tag{3.4}$$

can be linearized by the dependent variable transformation (Hirota 2004) and (Lamb 1976)

$$e^u = -2(\log f)_{xt} \tag{3.5}$$

These examples shows that, once a nonlinear differential equation has been linearized, it is relatively easier to solve. The essential of the direct method is to transform a nonlinear differential equation into a type of *bilinear* differential equation (Hirota 2004). A bilinear expression is an extension of a linear expression in x_j, such as

$$\sum_{j=1}^{n} a_j x_j \tag{3.6}$$

to a second degree expression in x_i and y_j, such as

$$\sum_{i,j=1}^{n} a_{ij} x_i y_j \tag{3.7}$$

We again consider the KdV equation

$$u_t + 6uu_x + u_{xxx} = 0 \tag{3.8}$$

and use the normal perturbation method to solve. Expanding u as

$$u = \varepsilon u_1 + \varepsilon^2 u_2 \varepsilon^3 u_3 + \ldots \tag{3.9}$$

where ε is a small parameter and collecting terms in the resulting equation at each order ε, we get first few equations as

$$\left(\frac{\partial}{\partial t}+\frac{\partial^3}{\partial x^3}\right)u_1=0, \tag{3.10}$$

$$\left(\frac{\partial}{\partial t}+\frac{\partial^3}{\partial x^3}\right)u_2=-6u_1u_{1x}, \tag{3.11}$$

$$\left(\frac{\partial}{\partial t}+\frac{\partial^3}{\partial x^3}\right)u_3=-6(u_2u_{1x}+u_1u_{2x}),\ldots \tag{3.12}$$

Choosing $u_1=a_1\exp\eta$ for $\eta=(kx-k^3t)$ where a_1 and k are arbitrary constants , the solution may be written as

$$u=\varepsilon a_1\exp\eta+\varepsilon^2a_2\exp 2\eta+\varepsilon^3a_3\exp 3\eta+\ldots \tag{3.13}$$

This solution diverges as $\eta\to\infty$ and one idea to avoid the divergence would be to express u as a ratio of polynomials G/F (Hirota 2004). Since we have the solution of KdV equation in the following rational form

$$u=\frac{2k^2\exp\eta}{(1+\exp\eta)^2}=\frac{k^2}{2}\cosh^{-2}\frac{\eta}{2} \tag{3.14}$$

we may set $F=(1+\exp\eta)^2,\quad G=2k^2\exp\eta$. With this choice in mind, we substitute $u=G/F$ into (3.8) to obtain

$$\begin{aligned}\frac{G_tF-GF_t}{F^2}+6\frac{G}{F}\frac{G_xF-GF_x}{F^2}+\frac{G_{xxx}F-3G_{xx}F_x-3G_xF_{xx}-GF_{xxx}}{F^2}\\+6\frac{FG_xF_x^2+FGF_{xx}F_x-GF_x^3}{F^4}=0\end{aligned} \tag{3.15}$$

and try to separate the above equation into a set of equations. One possible method would be to set the term with denominator F^2 (or F^4) equal to zero. Although, the term with denominator F^2

$$G_tF - GF_t + G_{xxx}F - 3G_{xx}F_x - 3G_xF_{xx} - GF_{xxx} \tag{3.16}$$

is not zero for our choice of $F = (1 + \exp\eta)^2$ and $G = 2k^2 \exp\eta$ but, replacing fifth term ,$-3G_xF_{xx}$, by $3G_xF_{xx}$ gives zero. Therefore, we may write the coupled equation

$$G_tF - GF_t + G_{xxx}F - 3G_{xx}F_x + 3G_xF_{xx} - GF_{xxx} = 0 \tag{3.17}$$

$$FF_{xx} - F_x^2 - GF = 0 \tag{3.18}$$

This is a system of bilinear differential equations with respect to F and G and it is distinctive in the pattern of derivatives. These derivatives are the same as the Leipnitz derivative of a product, except the crucial sign difference (alternating minus sign) (Hirota 2004).

3.2. The D-Operator

In the above discussion, we recognized that the usual Leipnitz rule does not work to bilinearize the KdV equation. This lead Hirota to introduce the D-operator (Hirota 1974) and (Miura 1976).

D-opetator (Hirota derivative), acting on a pair of functions $a(x)$ and $b(x)$ is defined by

$$D_x^n(a,b) \equiv \left(\frac{\partial}{\partial x} - \frac{\partial}{\partial y}\right)^n a(x)b(y)|_{y=x} = \frac{\partial^n}{\partial y^n} a(x+y)b(x-y)|_{y=0} \tag{3.19}$$

$$D_t^m D_x^n(a,b) \equiv \frac{\partial^m}{\partial s^m}\frac{\partial^n}{\partial y^n} a(t+s,x+y)\,b(t-s,x-y)|_{s=0,y=0} \tag{3.20}$$

where $m,n = 0,1,2,3,....$

Using this definition we may write a few of the first derivatives

$$D_x(G\cdot F) = G_xF - GF_x \tag{3.21}$$

$$D_x^2(G\cdot F) = G_{xx}F - 2G_xF_x - GF_{xx} \tag{3.22}$$

$$D_x^3 = G_tF - GF_t + G_{xxx}F - 3G_{xx}F_x + 3G_xF_{xx} - GF_{xxx} \tag{3.23}$$

and generalize these for n-th order Hirota derivative

$$D_x^n(f\cdot g)=\sum_0^n\left(\begin{array}{c}n\\k\end{array}\right)(-1)^{n-k}\partial_x^k\partial_y^{n-k}f(x)g(y)|_{y=x}=\sum_0^n\left(\begin{array}{c}n\\k\end{array}\right)(-1)^k f^{n-k}(x)g^k(y) \tag{3.24}$$

Some basic properties of Hirota derivative can be given as

- $D_x^n(f\cdot 1)=\frac{\partial^n}{\partial x^n}f$,
- $D_x^n(f\cdot g)=(-1)^n D_x^n(g\cdot f)$,
- $D_x^n(f\cdot f)=0$, if n is odd,
- $D_x^n(e^{\eta_1}\cdot e^{\eta_2})=(p_1-p_2)^n e^{\eta_1+\eta_2}$, for $\eta_i=p_i x-\omega_i t$
- $2(\ln f)_{xx}=[D_{xx}^2(f\cdot f)]/f^2$

3.3. Bilinear Form and One Soliton Solution for The KdV Equation

Considering bilinear form of the KdV equation (3.17), (3.18) and the definition of the D operator we write the system

$$(D_t+D_x^3)(g\cdot f)=0 \tag{3.25}$$

$$D_t^2(f\cdot f)=2gf \tag{3.26}$$

The next step is expanding f and g with respect to a small parameter ε as

$$f=\varepsilon^0 f_0+\varepsilon^1 f_1+\varepsilon^2 f_2+\ldots,\quad g=\varepsilon^0 g_0+\varepsilon^1 g_1+\varepsilon^2 g_2+\ldots \tag{3.27}$$

Substituting the expansion formula of f and g into the bilinear equation and arranging it at each order of ε, we have

$$(D_t+D_x^3)(g_0 f_0)=0 \tag{3.28}$$

$$D_t^2(f_0 f_0) = 2g_0 f_0 \tag{3.29}$$

for the zero order terms. Setting $g_0 = 0$ and $f_0 = 1$ we verify the system. Next system is

$$(D_t + D_x^3)(g_1 f_0 + g_0 f_1) = 0 \tag{3.30}$$

$$D_t^2(f_1 f_0 + f_0 f_1) = 2(g_0 f_1 + g_1 f_0) \tag{3.31}$$

and at this step we set $g_1 = e^{\eta}$ for $\eta = kx + \omega t$ to get the dispersion relation as $\omega = -k^3$, then find $f_1 = e^{\eta}/k^2$. Since we are searching for one soliton solution we set $g_2 = 0$ and derive $f_2 = e^{2\eta}/4k^4$ by the help of

$$(D_t + D_x^3)(g_2 f_0 + g_1 f_1 + g_0 f_2) = 0 \tag{3.32}$$

$$D_t^2(f_2 f_0 + f_1 f_1 + f_0 f_2) = 2(g_2 f_0 + g_1 f_1 + g_0 f_2) \tag{3.33}$$

and write the solution

$$g = e^{\eta}, \quad f = 1 + e^{\eta}/k^2 + e^{2\eta}/4k^4, \quad \frac{g}{f} = \frac{e^{\eta}}{\left(1 + \frac{e^{\eta}}{2k^2}\right)^2} \tag{3.34}$$

then convert into regular soliton form for $A = 1/2k^2 = e^{x_0}$

$$\frac{g}{f} = \frac{e^{\eta}}{(1 + Ae^{\eta})^2} = \frac{e^{\eta}}{[A^{1/2}e^{\eta/2}(A^{-1/2}e^{-\eta/2} + A^{1/2}e^{\eta/2})]^2} = \frac{k^2}{2}\cosh^{-2}[\frac{k}{2}(x - k^2 t + x_0)] \tag{3.35}$$

Traveling (solitary) wave solution has the same form of one soliton solution for the KdV equation. The advantage of the direct method is that it gives us the opportunity to calculate n-soliton solutions algebraically. In the following chapters we will use the same procedure to construct exact soliton solutions for the more complicated systems such as the Zakharov Equation, its two integrable limits as examples of the direct method, the new systems called Long-Short Wave Interaction equations with quantum potential and Modified Davey-Stewartson equation leading to dissipaton solutions.

CHAPTER 4

THE LONG-SHORT WAVE INTERACTION EQUATIONS

4.1. Zakharov Equations and Its Limits

The system of equations for the plasma waves was give by Zakharov as

$$i\frac{\partial E}{\partial t}+\frac{1}{2}\frac{\partial^2 E}{\partial x^2}-nE=0 \tag{4.1}$$

$$\frac{\partial^2 n}{\partial t^2}-\frac{\partial^2 n}{\partial x^2}-2\frac{\partial^2 |E|^2}{\partial x^2}=0 \tag{4.2}$$

where $E(x,t)$ is the complex electric field and $n(x,t)$ is the density of plasma perturbations (Zakharov 1972).

We may solve this system by the direct method to derive one soliton solution. Representing $E = g/f$ for a real valued function f and a complex valued function g, equation (4.1) can be written as

$$i\frac{g_t f-g f_t}{f^2}+\frac{1}{2}\left(\frac{g_{xx} f-g f_{xx}}{f^2}-\frac{2 f f_x (g_x f-g f_x)}{f^4}\right)-n\frac{g}{f}=0 \tag{4.3}$$

and rearranging the terms we obtain

$$\frac{1}{f^2}\left(iD_t(g\cdot f)+\frac{1}{2}D_x^2(g\cdot f)\right)-\frac{g}{f}\left(\frac{D_x^2(f\cdot f)}{2f^2}+n\right)=0 \tag{4.4}$$

Instead of two real functions E_1, E_2 for $E = E_1 + iE_2$, we have introduced three real functions such as $g = Re(g)+iIm(g)$ and f . Hence we have freedom to split this equation to the system

$$\left(iD_t+\frac{1}{2}D_x^2\right)(g\cdot f)=0 \tag{4.5}$$

$$n = -(\ln f)_{xx} \tag{4.6}$$

Substituting n into equation (4.2) and integrating two times with respect to x, and letting integration constants to be zero , equation (4.2) transforms into

$$-(\ln f)_{tt} + (\ln f)_{xx} - 2|E|^2 = 0 \tag{4.7}$$

In terms of D operator it can be expressed as

$$\frac{1}{2f^2}\left[(D_x^2 - D_t^2)(f \cdot f) - (4g\bar{g})\right] = 0 \tag{4.8}$$

Finally we may write the system of equations (4.1) and (4.2) in the bilinear form

$$\left(iD_t + \frac{1}{2}D_x^2\right)(g \cdot f) = 0 \tag{4.9}$$

$$(D_x^2 - D_t^2)(f \cdot f) = (4g\bar{g}) \tag{4.10}$$

where $E = g/f$ and $n = -(\ln f)_{xx}$. So that every solution of this system provides a solution of the original model (4.1) and (4.2). To solve the new system we consider unknown functions by expanding in terms of a small parameter ε.We start by rewriting these functions,

$$g = \varepsilon^0 g_0 + \varepsilon^1 g_1 + \varepsilon^2 g_2 + \ldots \tag{4.11}$$

$$f = \varepsilon^0 f_0 + \varepsilon^1 f_1 + \varepsilon^2 f_2 + \ldots \tag{4.12}$$

By collecting terms with respect to the order ε^0 we get

$$\left(iD_t + \frac{1}{2}D_x^2\right)(g_0 \cdot f_0) = 0 \tag{4.13}$$

$$(D_x^2 - D_t^2)(f_0 \cdot f_0) = (4g_0\bar{g}_0) \tag{4.14}$$

Choosing $g_0 = 0$ and $f_0 = 1$ we verify the system. Next we collect the terms with respect to the power ε^1 and get

$$\left(iD_t + \frac{1}{2}D_x^2\right)(g_0 \cdot f_1 + g_1 \cdot f_0) = 0 \tag{4.15}$$

$$\left(D_x^2 - D_t^2\right)(f_0 \cdot f_1 + f_1 \cdot f_0) = 4(g_0\bar{g}_1 + g_1\bar{g}_0) \tag{4.16}$$

Setting $f_1 = 0$ and $g_1 = e^{\eta}$ for $\eta = kx + \omega t + \eta_0$ we derive the dispersion relation for g_1 as $\omega = ik^2/2$. To calculate f_2 and g_2 we write down equations at the second order of ε

$$\left(iD_t + \frac{1}{2}D_x^2\right)(g_0 \cdot f_2 + g_1 \cdot f_1 + g_2 \cdot f_0) = 0 \tag{4.17}$$

$$\left(D_x^2 - D_t^2\right)(f_0 \cdot f_2 + f_1 \cdot f_1 + f_0 \cdot f_2) = 4(g_0\bar{g}_2 + g_1\bar{g}_1 + g_2\bar{g}_0) \tag{4.18}$$

and choose $g_2 = 0$ to fix the solution for one soliton case. Then calculate f_2.

$$f_2 = 2\frac{e^{\eta+\bar{\eta}}}{(k+\bar{k})^2 - (\omega+\bar{\omega})^2} \tag{4.19}$$

where $k = (\lambda + i\mu), (\lambda, \mu\varepsilon R)$ and $\bar{\eta}$ is the complex conjugate of η. Continuing Hirota's perturbation shows that the higher order terms will be zero. Hence, the solution is exact. It is possible to write f as a secant-hyperbolic function which is the perfect soliton shape. First we should identify the restrictions for the regular soliton solution as $1 - \mu^2 > 0$ so that $-1 < \mu < 1$ (See Figure 4.1). Then write

$$f = 1 + \frac{1}{2\lambda^2(1-\mu^2)}e^{\eta+\bar{\eta}} \tag{4.20}$$

and the constant $1/2\lambda^2(1-\mu^2)$ is greater than zero, so it can be included in exponential function. Setting $2\lambda^2(1-\mu^2) = e^{-\phi_0}$ we rewrite $g = e^{\eta}$, $f = 1 + e^{\eta+\bar{\eta}+\phi_0}$. For $E = g/f$, and $n = -(\ln f)_{xx}$ the solution can be given as

$$n = -\frac{\lambda^2}{\cosh^2[\lambda(x - \mu t + x_0)]} \tag{4.21}$$

$$E = \frac{|\lambda|\sqrt{1-\mu^2}\exp i\left(\mu x + \frac{\lambda^2-\mu^2}{2}t + y_0)\right)}{\sqrt{2}\cosh\lambda\,(x - \mu t + x_0))} \tag{4.22}$$

where $\lambda \neq 0$, $-1 < \mu < 1$, $\lambda x_0 = Re(\eta_0) + (\phi_0/2)$, and $y_0 = Im(\eta_0)$. In the region where

$\lambda > 0$ and $0 < \mu < 1$ soliton is propagating to the right (See Figure 4.2), for $\lambda > 0$ and $-1 < \mu < 0$ it is propagating to the left direction.

Despite that it is not known that Zakharov equation is integrable, two limiting cases, the quasi stationary limit and the ultra sound limit, admit reductions to two different integrable systems namely, The Nonlinear Schödinger equation and the Yajima-Oikawa system.

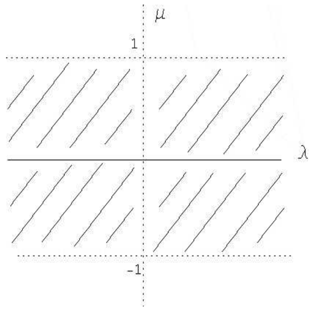
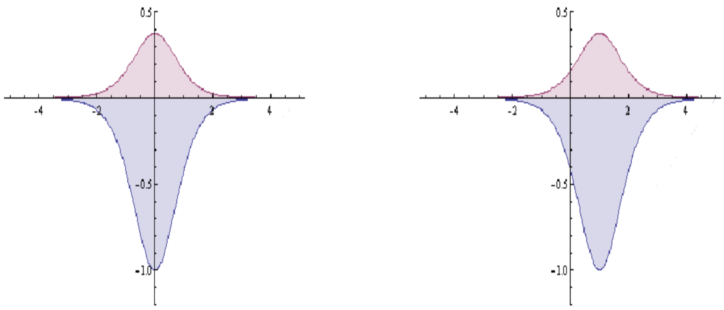


Figure 4.1. Restriction of the parameters (λ, μ) for regular soliton solution.



t=0 t=2

Figure 4.2. n and $|E|^2$ for Zakharov Equation $(\lambda = 1, \mu = 1/2)$

4.2. The Quasi-Stationary Limit of Zakharov Equation

If density of plasma is almost time independent which means that the time variation of n to be slow, then, this is called the quasi-stationary limit of the system. Then the derivative of n with respect to time is very close to zero and we may set

$$\frac{\partial n}{\partial t} \cong 0 \tag{4.23}$$

which means that also $n_{tt} \cong 0$. Substituting into the Zakharov equation (4.2), we get

$$n_{xx} = -2(|E|^2)_{xx} \tag{4.24}$$

Integrating both sides of (4.24) two times with respect to x we get $n = -2|E|^2$, for zero integration constants. Combining this results with equation (4.1) we obtain

$$iE_t + \frac{1}{2}E_{xx} - nE = iE_t + \frac{1}{2}E_{xx} - (-2|E|^2)E = 0 \tag{4.25}$$

This equation is known as the focusing Nonlinear Schrödinger Equation

$$iE_t + \frac{1}{2}E_{xx} + 2|E|^2E = 0 \tag{4.26}$$

The essential difference between this special case and the original equation is that the focusing NLS can be solved by ISM and N-soliton solutions were found (Zakharov and Shabat 1972). Here we give the bilinear representation and the one soliton solution of focusing NLS. Substituting $E = g/f$ we write the equation by the help of D operator

$$\frac{iD_t(g \cdot f)}{f^2} + \frac{1}{2}\frac{D_x^2(g \cdot f)}{f^2} - \frac{1}{2}\frac{g}{f}\frac{D_x^2(f \cdot f)}{f^2} + 2\frac{g}{f}\frac{g\bar{g}}{f^2} = 0 \tag{4.27}$$

after splitting this equation we get bilinear form for NLS.

$$\left(iD_t + \frac{1}{2}D_x^2\right)(g \cdot f) = 0 \tag{4.28}$$

$$D_x^2(f \cdot f) = 4g\bar{g} \tag{4.29}$$

Applying the Hirota method as we did above, this system admits the one soliton solution

$$g = e^{\eta}, \;\; f = 1 + \frac{e^{\eta+\bar{\eta}}}{(k+\bar{k})^2} \tag{4.30}$$

where $\eta = kx + \omega t + \eta_0$, $k = \lambda + i\mu$, $\omega = ik^2/2$ and

$$E = \frac{g}{f} = \frac{|\lambda| \exp i \left(\mu x + \frac{\lambda^2 - \mu^2}{2} t + y_0 \right))}{\sqrt{2} \cosh \lambda (x - \mu t + x_0))} \tag{4.31}$$

for $\lambda \neq 0$, $y_0 = Im\eta_0$, $\lambda x_0 = Re\eta_0$. In contrast with the Zakrarov equation we do not have now any restriction for the parameter k.

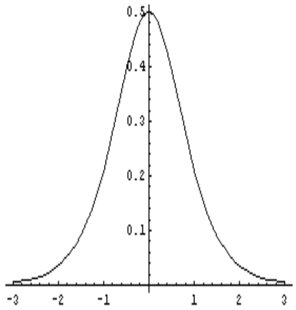


Figure 4.3. $|E|^2$ for NLS equation ($\lambda = 1, \mu = 2$).

4.3. Ultra Sound Limit of Zakharov Equation

If the speed of the wave is very close to the speed of sound ($v \cong c$) which is proportional to 1, this is called the ultra-sound limit. Considering $n = n(x - t)$ as a traveling wave in the positive x direction we may write

$$\frac{\partial n}{\partial t} + \frac{\partial n}{\partial x} \cong 0 \tag{4.32}$$

$$\frac{\partial n}{\partial t} \cong -\frac{\partial n}{\partial x} \tag{4.33}$$

and substitute in to Zakharovs equation (4.2) we get

$$\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial x} \right) + \left(\frac{\partial}{\partial t} - \frac{\partial}{\partial x} \right) n - 2(|E|^2)_{xx} = -2 \frac{\partial}{\partial x} \left(\frac{\partial}{\partial t} + \frac{\partial}{\partial x} \right) n - 2(|E|^2)_{xx} = 0 \tag{4.34}$$

Integrating this equation reduces the system into the following one

$$i\frac{\partial E}{\partial t}+\frac{1}{2}\frac{\partial^2 E}{\partial x^2}-nE=0 \tag{4.35}$$

$$\frac{\partial n}{\partial t}+\frac{\partial n}{\partial x}+\frac{\partial |E|^2}{\partial x}=0 \tag{4.36}$$

where the second equation indicates that the wave motion is unidirectional. This equation is known as Yajima-Oikawa equation and it was solved by using ISM. This system is related with the auxiliary linear problem $A_x = UA$, $A_t = VA$, so that the compatibility condition or zero curvature condition for this equation give us Y-O system (Yajima and Oikawa 1976). For the Lax pair U and V given as

$$U=\frac{i}{2\zeta}\begin{pmatrix} n & -2i\zeta\phi^* & n \\ i\phi & 0 & i\phi \\ -n & 2i\zeta\phi^* & -n \end{pmatrix}+\begin{pmatrix} 3i\zeta & 0 & 0 \\ 0 & i\zeta & 0 \\ 0 & 0 & -i\zeta \end{pmatrix} \tag{4.37}$$

$$V=\begin{pmatrix} i(\frac{2\zeta^2}{3}-2\zeta) & 0 & 0 \\ 0 & -4i\frac{\zeta^2}{3} & 0 \\ 0 & 0 & i(\frac{2\zeta^2}{3}+2\zeta) \end{pmatrix}+H \tag{4.38}$$

$$H=\frac{i}{2\zeta}\begin{pmatrix} n+\frac{|\phi|^2}{2} & -2i\zeta(-\zeta\phi^*+\frac{i}{2}\phi_x^*+\phi^*) & n+\frac{|\phi|^2}{2} \\ -i(\zeta\phi+\frac{i}{2}\phi_x-\phi) & 0 & -i(-\zeta\phi+\frac{i}{2}\phi_x-\phi) \\ -(n+\frac{|\phi|^2}{2}) & 2i\zeta(-\zeta\phi^*+\frac{i}{2}\phi_x^*+\phi^*) & -(n+\frac{|\phi|^2}{2}) \end{pmatrix} \tag{4.39}$$

we write the zero curvature condition $U_t - V_x + UV - VU = 0$ and get

$$i(\phi_t+\phi_x)+\frac{1}{2}\phi_{xx}=u\phi \tag{4.40}$$

$$u_x+u_t+(|\phi|^2)_x=0 \tag{4.41}$$

Yajima and Oikawa reduced this system by setting $\phi(x,t) = e^{i(t/2-x)}E(x,t)$ to obtain (4.35) and (4.36). By this way N-soliton solution for the system was constructed by using the Inverse Scattering Transform (Yajima and Oikawa 1976).

4.4. Long-Short Wave Interaction

NLS and the Yajima-Oikawa system are particular realizations of the Long-Short Wave interaction equations. In 1977 D.J. Benney presented a general theory for nonlinear partial differential equations which admit both short and long wave equations (Benney 1977). These equations arise in the study of surface waves with both gravity and capillary modes. Benney's first equation governing the interaction of $L(x,y)$ (long wave profile) and $S(x,y)$ (short wave envelope) is

$$i(S_t + c_g S_x) = \beta S_{xx} + \gamma |S|^2 S + \delta L S \tag{4.42}$$

$$L_t + c_l L_x = \alpha(|S|^2)_x \tag{4.43}$$

where c_g, c_l, α, β, γ and δ are real constants. The quantity c_l is the long wave speed and c_g is the group velocity of the short waves. If $\delta = 0$ equation (4.42) can be solved directly, since it uncouples from (4.43) and represents nonlinear Schrödinger equation (NLS). From the analysis of long time behavior, Benney concluded that with $\delta \neq 0$ existence of solitary wave solutions restricts the parameters by an inequality.

$$\beta\left(\gamma + \frac{\alpha\delta}{c_l - c_g}\right) > 0 \tag{4.44}$$

A special case $c_l = c_g = c$ corresponds to the resonance condition and the Galilean transformation $x' = x - ct, t' = t$ eliminates the first x derivative terms. Thus, we have

$$iS_{t'} = \beta S_{x'x'} + \gamma |S|^2 S + \delta L S \tag{4.45}$$

$$L_{t'} = \alpha(|S|^2)_{x'} \tag{4.46}$$

The Yajima-Oikawa system (4.35), (4.36) is a particular case of the long-short wave interaction equations (4.42),(4.43) with constants $c_g = c_l = 1$, $\beta = -1/2$, $\gamma = 0, \delta = 1$ and $\alpha = -1$.

Below we will solve both the long-short wave and the Yajima-Oikawa equations and their modifications by adding quantum potential. For this reason we will use different notation and give the equation in a more general form such as

$$i\psi_t = \psi_{xx} + 2N\psi + 2\beta|\psi|^2\psi \tag{4.47}$$

$$\alpha N_x + N_t = (|\psi|^2)_x - \beta(|\psi|^2)_t \tag{4.48}$$

By shifting N as $N = n - \beta|\psi|^2$ we obtain the reduced system

$$i\psi_t = \psi_{xx} + 2n\psi \tag{4.49}$$

$$\alpha N_x + N_t = (1+\alpha\beta)|\psi|^2)_x \tag{4.50}$$

representing the L-S resonance interaction for $\alpha = 0$ and Y-O for $\alpha = 1$. In the next chapter we are going to discuss the modification of this system adding the quantum potential.

The procedure to derive resonant soliton equations in 1+1 dimensions for the NLS type equations was introduced in a series of papers (Pashaev and Lee 2002,a),(Pashaev and Lee 2002,b),(Lee, et al. 2007),(Pashaev 1997). It was shown that by adding the quantum potential term $s|\psi|_{xx}/\psi$ with strength $s < 1$ and $s > 1$ the corresponding NLS type equation is reducible to the system without the quantum potential term or nonlinear diffusion-anti diffusion equations respectively. The last one admits dissipaton solutions, providing resonant interaction of original envelope solitons. In the next chapters we apply this idea to the Long-Short wave system and to the Davey-Stewartson system.

CHAPTER 5

THE L-S WAVE EQUATION WITH QUANTUM POTENTIAL

In this chapter we introduce the Long-Short (L-S) Wave interaction equations with quantum potential. It was shown that the Nonlinear Schrödinger equation with quantum potential, which is a third version of the NLS between the defocusing and focusing cases, separates two distinct regions of behavior due to the choice of strength of the quantum potential (Pashaev and Lee 2007). We apply this idea to examine its effect on the L-S wave equation. The L-S wave interaction equations with quantum potential can be considered as a particular realization of the L-S soliton propagating in the so called "quantum potential". This potential, responsible for producing the quantum behaviour, was introduced (de Broglie 1926) and was subsequently used (Bohm 1952)to develop a hidden variable theory in quantum mechanics (Nelson 1966).

We consider the system of equations which defines interaction of the long-short wave with adding the quantum potential term ,$\frac{|\psi|_{xx}}{|\psi|}\psi$, with strength s, as follows (Pashaev and Duruk 2009)

$$i\psi_t = \psi_{xx} + 2n\psi - s\frac{|\psi|_{xx}}{|\psi|}\psi \tag{5.1}$$

$$\alpha n_x + n_t = (1+\alpha\beta)(|\psi|^2)x \tag{5.2}$$

For $s = 0$ the system is reduced to (4.49), (4.50). If we set $\psi = e^{R+iS}$ for real $S(x,t)$ and $R(x,t)$, then the system yields

$$-S_t = (1-s)(R_{xx} + R_x^2) - S_x^2 + 2n$$

$$R_t = S_{xx} + 2R_x Sx \tag{5.3}$$

$$\alpha n_x + n_t = 2(1+\alpha\beta)R_x e^{2R}$$

5.1. Undercritical Case

If $s < 1$, then on rescaling time and phase of the wave function according to

$$S = \sqrt{1-s}\tilde{S}, \quad t = \frac{\tilde{t}}{\sqrt{1-s}}, \quad R = \tilde{R}, \quad x = \tilde{x}, \quad n = \tilde{n}, \quad \alpha = \tilde{\alpha}, \quad \beta = \tilde{\beta} \tag{5.4}$$

we obtain

$$i\tilde{\psi}_{\tilde{t}} = \tilde{\psi}_{\tilde{x}\tilde{x}} + \frac{2\tilde{n}}{1-s}\tilde{\psi} \tag{5.5}$$

$$\tilde{\alpha}\tilde{n}_{\tilde{x}} + \sqrt{1-s}\tilde{n}_{\tilde{t}} = (1+\tilde{\alpha}\tilde{\beta})(|\ \tilde{\psi}\ |^2)_{\tilde{x}} \tag{5.6}$$

in $\tilde{\psi} = e^{\tilde{R}+i\tilde{S}}$. Rescaling $\tilde{n}$, $\tilde{\psi}$, $\tilde{\alpha}$ and $\tilde{\beta}$

$$\hat{\beta} = \sqrt{1-s}\tilde{\beta},\ \hat{\alpha} = \frac{\tilde{\alpha}}{\sqrt{1-s}},\ \hat{n} = \frac{\tilde{n}}{1-s},\ \hat{\psi} = \frac{\tilde{\psi}}{(1-s)^{3/4}},\ \hat{x} = \tilde{x},\ \hat{t} = \tilde{t} \tag{5.7}$$

reduces the system to

$$i\hat{\psi}_{\hat{t}} = \hat{\psi}_{\hat{x}\hat{x}} + 2\hat{n}\hat{\psi} \tag{5.8}$$

$$\hat{\alpha}\hat{n}_{\hat{x}} + \hat{n}_{\hat{t}} = (1+\hat{\alpha}\hat{\beta})(|\ \hat{\psi}\ |^2)_{\hat{x}} \tag{5.9}$$

It shows that our modification of L-S wave equation with additional quantum potential term in undercritical case is reduced to the L-S wave equation without quantum potential term by modification of parameters (5.7).

5.1.1. Bilinear Representation and Soliton Solutions

- **CASE I**

  If we set $\hat{\alpha} = 0$ and $\hat{\beta}$ as an arbitrary real number, the system (5.8),(5.9) reduces to

$$i\hat{\psi}_{\hat{t}} = \hat{\psi}_{\hat{x}\hat{x}} + 2\hat{n}\hat{\psi} \tag{5.10}$$

$$\hat{n}_{\hat{t}} = (|\ \hat{\psi}\ |^2)_{\hat{x}} \tag{5.11}$$

  It describes the resonance interaction between long wave and short wave and was formulated by Djordjevic and Redekopp, when they studied two dimensional packets of capillary-gravity waves. The physical significance of this system is that the dispersion of the short wave is balanced by nonlinear interaction of the long wave with the short wave, while the evolution of the long wave is driven by the self interaction of the short wave (Djordevic and Redekop 1977).

  1. **Lax Pair**

     The Lax pair of the system (5.10),(5.11) was given in the form (Ma 1978)

$$U = \begin{pmatrix} 3i\xi & A & iB \\ 0 & 2i\xi & A^* \\ -i & 0 & i\xi \end{pmatrix} \tag{5.12}$$

$$V = \begin{pmatrix} i\xi^2 & 2(A\xi - iA_x) & -2i|A|^2 \\ -2A^* & -i\xi^2 & 2(iA_x^* - \xi A^*) \\ 0 & -2A & i\xi^2 \end{pmatrix} \tag{5.13}$$

     according to zero curvature condition for U and V, $U_t - V_x + UV - VU = 0$ we find the equations

$$iA_t = A_{xx} - 2AB \tag{5.14}$$

$$B_t = -4(|A|^2)_x \tag{5.15}$$

     and by rescaling

$$A = 2^{-3/4}\hat{\psi},\ \ B = -\hat{n},\ \ x = \sqrt{2}\hat{x},\ \ t = \hat{t} \tag{5.16}$$

we recover the system (5.10),(5.11).

2. **Bilinear Representation**

   To construct the bilinear form we introduce real valued function $f(\hat{x},\hat{t})$ and a complex valued function $g(\hat{x},\hat{t})$ as $\hat{\psi} = g/f$. Hence, we get

$$(iD_{\hat{t}} - D_{\hat{x}}^2)(g \cdot f) = 0 \tag{5.17}$$

$$D_{\hat{x}\hat{t}}(f \cdot f) = 2g\bar{g} \tag{5.18}$$

   where $\hat{n} = (\ln f)_{\hat{x}\hat{x}}$has a regular (non-singular) soliton solution

$$g = e^{\eta}, \quad f = 1 + \frac{e^{\eta+\hat{\eta}}}{8\lambda^2\mu} \tag{5.19}$$

   for real valued constants λ, μ satisfying $\lambda \neq 0$, $\mu > 0$ and $\eta = k\hat{x} - ik^2\hat{t} + \eta_0$ where $k = \lambda + i\mu$ and η_0 is a constant.

3. **One Soliton Solution**

   It is possible to write $\hat{n}$ and $\hat{\psi}$ as

$$\hat{n} = \frac{\lambda^2}{\cosh^2[\lambda(\hat{x} + \mu\hat{t} + x_0)]} \tag{5.20}$$

$$\hat{\psi} = \frac{|\lambda|\,\sqrt{2\mu}\exp[i(\mu\hat{x} + (\mu^2 - \lambda^2)\hat{t} + t_0)]}{\cosh[\lambda(\hat{x} + \mu\hat{t} + x_0)]} \tag{5.21}$$

   The amplitude of soliton $\hat{n}$ is dependent on the real part of k, $(Re(k) = \lambda)$, while velocity $v = -\mu$. Since $\mu > 0$ soliton is propagating only in the left direction (chiral soliton)(See Figure 5.1).

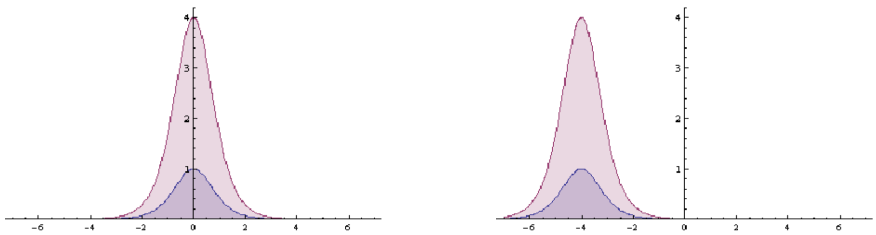


Figure 5.1. $\hat{n}$ (amplitude 1) and $|\hat{\psi}|^2$(amplitude 4) for L-S Wave Equation.

- **CASE II**

  For $\hat{\alpha} \neq 0$, by one more rescaling on (5.8)(5.9)

$$\hat{x} = \frac{\check{x}}{\hat{\alpha}}, \quad \hat{t} = \frac{\check{t}}{\hat{\alpha}^2}, \quad \hat{n} = \hat{\alpha}^2 u, \quad \hat{\psi} = \hat{\alpha}^{3/2}\phi \tag{5.22}$$

we have

$$i\phi_{\check{t}} = \phi_{\check{x}\check{x}} + 2u\phi \tag{5.23}$$

$$u_{\check{x}} + u_{\check{t}} = (1+\hat{\alpha}\hat{\beta})(|\phi|^2)_{\check{x}} \tag{5.24}$$

which is the Yajima-Oikawa equation that we encountered in the previous chapter.

1. **Lax Pair**

   The Lax pair of the Yajima-Oikawa system (Yajima and Oikawa 1976), which we have discussed in previous chapter, is

$$U = \frac{i}{2\zeta}\begin{pmatrix} n & -2i\zeta\vartheta^* & n \\ i\vartheta & 0 & i\vartheta \\ -n & 2i\zeta\vartheta^* & -n \end{pmatrix} + \begin{pmatrix} 3i\zeta & 0 & 0 \\ 0 & i\zeta & 0 \\ 0 & 0 & -i\zeta \end{pmatrix} \tag{5.25}$$

$$V = \begin{pmatrix} i(\frac{2\zeta^2}{3} - 2\zeta) & 0 & 0 \\ 0 & -4i\frac{\zeta^2}{3} & 0 \\ 0 & 0 & i(\frac{2\zeta^2}{3} + 2\zeta) \end{pmatrix} + H \tag{5.26}$$

$$H = \frac{i}{2\zeta}\begin{pmatrix} n+\frac{|\vartheta|^2}{2} & -2i\zeta(-\zeta\vartheta^* + \frac{i}{2}\vartheta_x^* + \vartheta^*) & n+\frac{|\vartheta|^2}{2} \\ -i(\zeta\vartheta + \frac{i}{2}\vartheta_x - \vartheta) & 0 & -i(-\zeta\vartheta + \frac{i}{2}\vartheta_x - \vartheta) \\ -(n+\frac{|\vartheta|^2}{2}) & 2i\zeta(-\zeta\vartheta^* + \frac{i}{2}\vartheta_x^* + \vartheta^*) & -(n+\frac{|\vartheta|^2}{2}) \end{pmatrix} \tag{5.27}$$

and for $\vartheta = Ee^{i(t/2-x)}$ can be written as

$$iE_t = \frac{1}{2}E_{xx} + nE \tag{5.28}$$

$$n_t + n_x + (|E|^2)_x = 0 \tag{5.29}$$

We may recover the system (5.23),(5.24) by rescaling

$$E = 2\sqrt{1+\hat{\alpha}\hat{\beta}}\phi, \quad n = -4u, \quad t = -\check{t}/2, \quad x = -\check{x}/2 \tag{5.30}$$

2. **Bilinear Representation**

To construct bilinear form of (5.23) and (5.24), we introduce a real valued function $f(\check{x},\check{t})$ and a complex valued function $g(\check{x},\check{t})$ as $\phi = g/f$. Hence, we get

$$(iD_{\check{t}} - D_{\check{x}}^2)(g\cdot f) = 0 \tag{5.31}$$

$$(D_{\check{x}}^2 + D_{\check{x}\check{t}})(f\cdot f) = 2(1+\hat{\alpha}\hat{\beta})g\bar{g} \tag{5.32}$$

where $u = (\ln f)_{\check{x}\check{x}}$ has a regular (non-singular) one soliton solution

$$g = e^{\eta}, \;\; f = 1 + \frac{(1+\hat{\alpha}\hat{\beta})e^{\eta+\hat{\eta}}}{4\lambda^2(1+2\mu)} \tag{5.33}$$

for real valued constants λ, μ satisfying $\lambda \neq 0$, $(\hat{\alpha}\hat{\beta}) > -1$, $\mu > -1/2$ and $\eta = k\check{x} - ik^2\check{t} + \eta_0$ where $k = \lambda + i\mu$.

3. **One Soliton Solution**

We may write the solution in soliton form as follows

$$u = \frac{\lambda^2}{\cosh^2[\lambda(\check{x} + 2\mu\check{t} + \check{x}_0)]} \tag{5.34}$$

$$\phi = \frac{|\lambda|\sqrt{1+2\mu}\exp[i(\mu\check{x} + (\mu^2 - \lambda^2)\check{t} + \check{y}_0)]}{\sqrt{1+\hat{\alpha}\hat{\beta}}\cosh[\lambda(\check{x} + 2\mu\check{t} + \check{x}_0)]} \tag{5.35}$$

where $\check{y}_0 = Im\eta_0$, $\check{x}_0 = Re\eta_0 + \phi_0/2$ and $\exp\phi_0 = (1+\hat{\alpha}\hat{\beta})/4\lambda^2(1+2\mu)$. For $-0.5 < \mu < 0$ soliton is moving in the right direction (See Figure 5.2) and for $\mu > 0$ in the left direction (See Figure 5.3), at time $t = 0$ and $t = 2$.

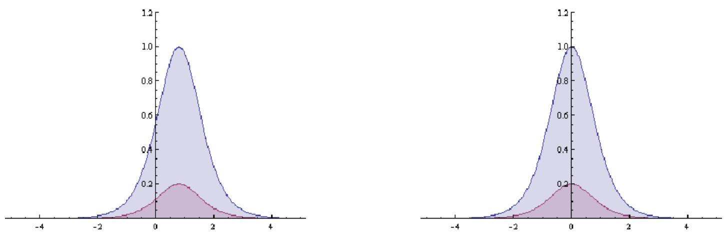


Figure 5.2. u (amplitude 1) and $|\phi|^2$(amplitude 0.2) for Y-O Equation .

$(\lambda = 1, \mu = -0.2, \hat{\alpha} = 1, \hat{\beta} = 2)$

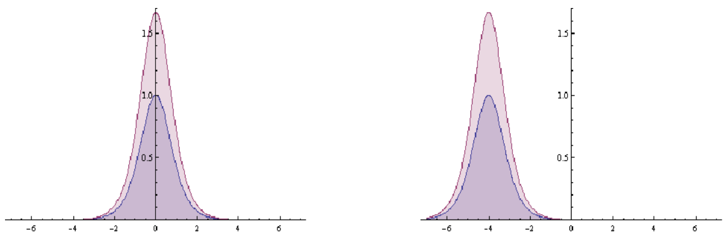


Figure 5.3. u (amplitude 1) and $|\phi|^2$(amplitude 5/3) for Y-O Equation .

$(\lambda = 1, \mu = 2, \hat{\alpha} = 1, \hat{\beta} = 2)$

CHAPTER 6

OVERCRITICAL CASE OF L-S WAVE EQUATION

In case $s > 1$ it is impossible to reduce the original system (5.1),(5.2) to the standard L-S wave equation. However, by rescaling and introducing two real valued functions we may obtain a reaction-diffusion system. By rescaling

$$S = \sqrt{s-1}\tilde{S}, \quad t = \frac{\tilde{t}}{\sqrt{s-1}}, \quad R = \tilde{R}, \quad x = \tilde{x}, \quad n = \tilde{n}, \quad \alpha = \tilde{\alpha}, \quad \beta = \tilde{\beta} \tag{6.1}$$

and introduction of two real valued functions $q = e^{\tilde{R}+\tilde{S}}$ and $r = e^{\tilde{R}-\tilde{S}}$ from (5.3)we get the following system

$$q_{\tilde{t}} = q_{\tilde{x}\tilde{x}} - \frac{2\tilde{n}}{s-1}q \tag{6.2}$$

$$-r_{\tilde{t}} = r_{\tilde{x}\tilde{x}} - \frac{2\tilde{n}}{s-1}r \tag{6.3}$$

$$\tilde{\alpha}\tilde{n}_{\tilde{x}} + \sqrt{s-1}\tilde{n}_{\tilde{t}} = (1+\tilde{\alpha}\tilde{\beta})(qr)_{\tilde{x}} \tag{6.4}$$

By rescaling $\tilde{n}$, $\tilde{\alpha}$, $\tilde{\beta}$, q and r,

$$\hat{n} = \frac{-\tilde{n}}{s-1}, \quad \hat{\alpha} = \frac{\tilde{\alpha}}{\sqrt{s-1}}, \quad \hat{\beta} = \sqrt{s-1}\tilde{\beta}, \quad \hat{q} = \frac{q}{(s-1)^{3/4}}, \quad \hat{r} = \frac{-r}{(s-1)^{3/4}}, \quad \tilde{x} = \hat{x}, \quad \tilde{t} = \hat{t} \tag{6.5}$$

this system reduces to

$$\hat{q}_{\hat{t}} = \hat{q}_{\hat{x}\hat{x}} + 2\hat{n}\hat{q} \tag{6.6}$$

$$-\hat{r}_{\hat{t}} = \hat{r}_{\hat{x}\hat{x}} + 2\hat{n}\hat{r} \tag{6.7}$$

$$\hat{\alpha}\hat{n}_{\hat{x}} + \hat{n}_{\hat{t}} = (1+\hat{\alpha}\hat{\beta})(\hat{q}\hat{r})_{\hat{x}} \tag{6.8}$$

6.1. Bilinear Representation and Soliton Solutions for Overcritical Case

- **CASE I**

  If we set $\hat{\alpha} = 0$ and $\hat{\beta}$ as an arbitrary real number, the system (6.6)-(6.8) reduces to the one

$$\hat{q}_{\hat{t}} = \hat{q}_{\hat{x}\hat{x}} + 2\hat{n}\hat{q} \tag{6.9}$$

$$-\hat{r}_{\hat{t}} = \hat{r}_{\hat{x}\hat{x}} + 2\hat{n}\hat{r} \tag{6.10}$$

$$\hat{n}_{\hat{t}} = (\hat{q}\hat{r})_{\hat{x}} \tag{6.11}$$

representing the real version (diffusion-antidiffusion) of Long-Short wave resonance interaction equations.

1. **Lax Pair**

   The Lax pair, of this system can be given as

$$V = \begin{pmatrix} 2\xi & \hat{r} & -2\hat{n} \\ 0 & 2\xi & \hat{q} \\ 1 & 0 & 2\xi \end{pmatrix}, \quad U = \begin{pmatrix} \xi^2 & -\hat{r}_{\hat{x}} & -\hat{q}\hat{r} \\ -\hat{q} & \xi^2 & \hat{q}_{\hat{x}} \\ 0 & -\hat{r} & \xi^2 \end{pmatrix} \tag{6.12}$$

   and the zero curvature condition $U_{\hat{x}} - V_{\hat{t}} + UV - VU = 0$ gives the system of equations (6.9)-(6.11).

2. **Bilinear Representation**

   Bilinear representation or this system is

$$(\pm D_{\hat{x}}^2 + D_{\hat{t}})(G^{\mp} \cdot F) = 0 \tag{6.13}$$

$$D_{\hat{x}\hat{t}}(F \cdot F) = 2G^{+}G^{-} \tag{6.14}$$

   where $\hat{q} = G^{+}/F$, $\hat{r} = G^{-}/F$ and $\hat{n} = (\ln F)_{\hat{x}\hat{x}}$, for three real valued functions $G^{\pm}(\hat{x},\hat{t})$ and $F(\hat{x},\hat{t})$.

3. **One Dissipaton Solution**

Simplest solution of this system is

$$G^{\pm} = e^{\eta^{\pm}}, \quad F = 1 + \frac{e^{(\eta^{+}+\eta^{-})}}{(k^{+}+k^{-})^{2}(k^{+}-k^{-})} \tag{6.15}$$

where $\eta^{\pm} = k^{\pm}\hat{x} \pm (k^{\pm})^{2}\hat{t} + \eta_{0}^{\pm}$ is regular if $(k^{+} > k^{-})$. I t defines one-soliton solution of long-short wave equation with quantum potential in (5.1),(5.2) or the one dissipaton solution (Pashaev 1997) of (6.9)-(6.11).

$$\hat{n} = \frac{(k^{+}+k^{-})^{2}}{4\cosh^{2}\left[\frac{(k^{+}+k^{-})}{2}(\hat{x}+(k^{+}-k^{-})\hat{t}-x_{0})\right]} \tag{6.16}$$

$$\hat{q} = \frac{|k^{+}+k^{-}|\sqrt{k^{+}-k^{-}}\exp\left(\frac{k^{2}+v^{2}}{4}\hat{t}-\frac{1}{2}v\hat{x}+x_{0}\right)}{2\cosh\left(k(\hat{x}-v\hat{t}+c_{0})\right)} \tag{6.17}$$

$$\hat{r} = \frac{|k^{+}+k^{-}|\sqrt{k^{+}-k^{-}}\exp-\left(\frac{k^{2}+v^{2}}{4}\hat{t}-\frac{1}{2}v\hat{x}+x_{0}\right)}{2\cosh\left(k(\hat{x}-v\hat{t}+c_{0})\right)} \tag{6.18}$$

where $k = (k^{+}+k^{-})$ and $v = (k^{+}-k^{-})$. The above solution does not preserve the form for $\hat{q}$ and $\hat{r}$ separately. During the time evolution and the space alteration one of the fields is growing exponentially, while the other one is decaying. Meanwhile, the product has the perfect soliton form (see Figure 6.1).

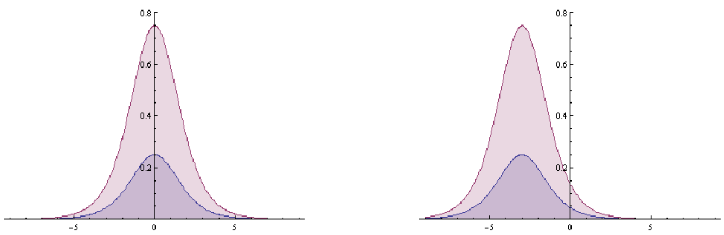


Figure 6.1. $\hat{n}$ and $\hat{q}\hat{r}$ for Real Version of L-S Wave Equation ($k^{+} = 2$, $k^{-} = -1$).

4. **Two Dissipaton Solution**

Continuing Hirotas perturbation we find two dissipaton solution in the form

$$G^{+} = (e^{\eta_1^+} + e^{\eta_2^+} + \alpha_1^+ e^{\eta_1^+ + \eta_1^- + \eta_2^+} + \alpha_2^+ e^{\eta_1^+ + \eta_2^+ + \eta_2^-}) \tag{6.19}$$

$$G^{-} = (e^{\eta_1^-} + e^{\eta_2^-} + \alpha_1^- e^{\eta_1^+ + \eta_1^- + \eta_2^-} + \alpha_2^- e^{\eta_1^- + \eta_2^+ + \eta_2^-}) \tag{6.20}$$

$$F = 1 + \sum_{i,j=1}^{2} \frac{\exp(\eta_{ij}^{+-})}{\Omega_{ij}^{+-}} + \frac{\bar{\Omega}_{12}^{++}\bar{\Omega}_{12}^{--}\exp(\eta_1^+ + \eta_1^- + \eta_2^+ + \eta_2^-)}{\Omega_{11}^{+-}\Omega_{12}^{+-}\Omega_{21}^{+-}\Omega_{22}^{+-}} \tag{6.21}$$

$$\alpha_1^+ = \frac{(\breve{k}_{12}^{++})^2(k_{12}^{++})}{(k_{11}^{+-}k_{21}^{+-})^2\breve{k}_{11}^{+-}\breve{k}_{21}^{+-}} \quad , \quad \alpha_2^+ = \frac{(\breve{k}_{12}^{++})^2(k_{12}^{++})}{(k_{12}^{+-}k_{22}^{+-})^2\breve{k}_{12}^{+-}\breve{k}_{22}^{+-}} \tag{6.22}$$

$$\alpha_1^- = \frac{(\breve{k}_{12}^{--})^2(k_{12}^{--})}{(k_{11}^{+-}k_{12}^{+-})^2\breve{k}_{11}^{+-}\breve{k}_{21}^{-+}} \quad , \quad \alpha_2^- = \frac{(\breve{k}_{12}^{--})^2(k_{12}^{--})}{(k_{21}^{+-}k_{22}^{+-})^2\breve{k}_{12}^{-+}\breve{k}_{22}^{-+}} \tag{6.23}$$

where

$$\begin{aligned} &\eta_{ij}^{+-} = \eta_i^+ + \eta_j^-, \ \ \Omega_{ij}^{+-} = (k_{ij}^{+-})^2(\breve{k}_{ij}^{+-}), \ \ \bar{\Omega}_{ij}^{\pm\pm} = (\breve{k}_{ij}^{\pm\pm})^2(k_{ij}^{\pm\pm}) \\ &k_{ij}^{ab} = k_i^a + k_j^b, \quad \breve{k}_{ij}^{ab} = k_i^a + k_j^b \quad \eta_i^{\pm} = k_i^{\pm}\hat{x} \pm (k_i^{\pm})^2\hat{t} + \eta_i^{\pm(0)} \end{aligned} \tag{6.24}$$

Regularity condition for this solution requires restriction of the parameters $\alpha_i^{\pm}$ to be positive. Thus we set $k_i^+ > 0$ and $k_i^- < 0$, to avoid singularities. We visualize the counterplots of two soliton solutions of the $(\hat{q}\hat{r})$ which is the dissipaton density, for appropriate parameters in Figure 6.2 and 6.3.

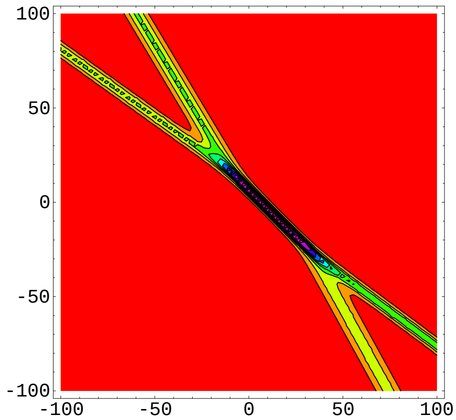


Figure 6.2. Zero Loop Interaction for Real Version of L-S Wave Equation

$$k_1^+ = 0.9, k_1^- = -0.5, k_2^+ = 0.1, k_2^- = -0.45,$$
$$\eta_1^{+(0)} = \eta_1^{-(0)} = -2,\ \eta_2^{+(0)} = \eta_2^{-(0)} = 2$$

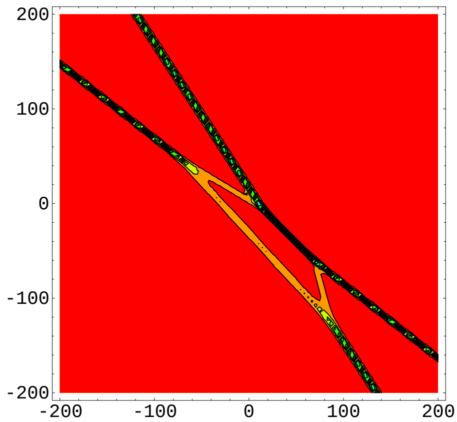


Figure 6.3. One Loop Interaction for Real Version of L-S Wave Equation

$$k_1^+ = 0.65,\ k_1^- = -1, k_2^+ = 10^{-5}, k_2^- = -0.27,$$
$$\eta_1^{+(0)} = \eta_1^{-(0)} = 0,\ \eta_2^{+(0)} = \eta_2^{-(0)} = 5$$

So if non-resonant solitons behave like particles, the resonance solitons can be considered as quantum type particles. In these pictures , we recognize quantum diagrams with zero and one loop interactions.

5. **Three Dissipaton Solution**

For three soliton we are getting

$$
\begin{aligned}
G^{+} = (&e^{\eta_1^+} + e^{\eta_2^+} + e^{\eta_3^+} + \gamma_1^+ e^{\eta_1^+ + \eta_1^- + \eta_2^+} + \gamma_2^+ e^{\eta_1^+ + \eta_1^- + \eta_3^+} + \gamma_3^+ e^{\eta_1^+ + \eta_2^- + \eta_2^+} \\
&+ \gamma_4^+ e^{\eta_2^+ + \eta_1^- + \eta_3^+} + \gamma_5^+ e^{\eta_1^+ + \eta_2^- + \eta_3^+} + \gamma_6^+ e^{\eta_1^+ + \eta_3^- + \eta_2^+} + \gamma_7^+ e^{\eta_2^+ + \eta_2^- + \eta_3^+} \\
&+ \gamma_8^+ e^{\eta_1^+ + \eta_3^- + \eta_3^+} + \gamma_9^+ e^{\eta_2^+ + \eta_3^- + \eta_3^+} + \gamma_{10}^+ e^{\eta_1^+ + \eta_1^- + \eta_2^+ + \eta_2^- + \eta_3^+} \\
&+ \gamma_{11}^+ e^{\eta_1^+ + \eta_1^- + \eta_2^+ + \eta_3^+ + \eta_3^-})
\end{aligned}
\tag{6.25}
$$

$$
\begin{aligned}
G^{-} = (&e^{\eta_1^-} + e^{\eta_2^-} + e^{\eta_3^-} + \gamma_1^- e^{\eta_1^+ + \eta_1^- + \eta_2^-} + \gamma_2^- e^{\eta_1^+ + \eta_1^- + \eta_3^-} + \gamma_3^- e^{\eta_2^+ + \eta_2^- + \eta_1^-} \\
&+ \gamma_4^- e^{\eta_2^+ + \eta_1^- + \eta_3^-} + \gamma_5^- e^{\eta_1^+ + \eta_2^- + \eta_3^-} + \gamma_6^- e^{\eta_3^+ + \eta_1^- + \eta_2^-} + \gamma_7^- e^{\eta_3^+ + \eta_1^- + \eta_3^-} \\
&+ \gamma_8^- e^{\eta_2^+ + \eta_2^- + \eta_3^-} + \gamma_9^- e^{\eta_3^+ + \eta_2^- + \eta_3^-} + \gamma_{10}^- e^{\eta_1^+ + \eta_1^- + \eta_2^+ + \eta_2^- + \eta_3^-} \\
&+ \gamma_{11}^- e^{\eta_1^+ + \eta_1^- + \eta_2^- + \eta_3^+ + \eta_3^-})
\end{aligned}
\tag{6.26}
$$

$$
\gamma_1^+ = \frac{(\breve{k}_{12}^{++})^2 k_{12}^{++}}{(k_{11}^{+-} k_{21}^{+-})^2 \breve{k}_{11}^{+-} \breve{k}_{21}^{+-}} \quad , \quad \gamma_2^+ = \frac{(\breve{k}_{13}^{++})^2 k_{13}^{++}}{(k_{11}^{+-} k_{31}^{+-})^2 \breve{k}_{11}^{+-} \breve{k}_{31}^{+-}} \tag{6.27}
$$

$$
\gamma_3^+ = \frac{(\breve{k}_{12}^{++})^2 k_{12}^{++}}{(k_{12}^{+-} k_{22}^{+-})^2 \breve{k}_{12}^{+-} \breve{k}_{22}^{+-}} \quad , \quad \gamma_4^+ = \frac{(\breve{k}_{23}^{++})^2 k_{23}^{++}}{(k_{21}^{+-} k_{31}^{+-})^2 \breve{k}_{21}^{+-} \breve{k}_{31}^{+-}} \tag{6.28}
$$

$$
\gamma_5^+ = \frac{(\breve{k}_{13}^{++})^2 k_{13}^{++}}{(k_{12}^{+-} k_{32}^{+-})^2 \breve{k}_{12}^{+-} \breve{k}_{32}^{+-}} \quad , \quad \gamma_6^+ = \frac{(\breve{k}_{12}^{++})^2 k_{12}^{++}}{(k_{13}^{+-} k_{23}^{+-})^2 \breve{k}_{13}^{+-} \breve{k}_{23}^{+-}} \tag{6.29}
$$

$$
\gamma_7^+ = \frac{(\breve{k}_{23}^{++})^2 k_{23}^{++}}{(k_{32}^{+-} k_{22}^{+-})^2 \breve{k}_{32}^{+-} \breve{k}_{22}^{+-}} \quad , \quad \gamma_8^+ = \frac{(\breve{k}_{13}^{++})^2 k_{13}^{++}}{(k_{13}^{+-} k_{33}^{+-})^2 \breve{k}_{13}^{+-} \breve{k}_{33}^{+-}} \tag{6.30}
$$

$$
\gamma_9^+ = \frac{(\breve{k}_{23}^{++})^2 k_{23}^{++}}{(k_{23}^{+-} k_{33}^{+-})^2 \breve{k}_{23}^{+-} \breve{k}_{33}^{+-}} \tag{6.31}
$$

$$
\gamma_{10}^+ = \frac{(\breve{k}_{12}^{++} \breve{k}_{13}^{++} \breve{k}_{23}^{++} \breve{k}_{12}^{--})^2 k_{12}^{++} k_{13}^{++} k_{23}^{++} k_{12}^{--}}{(k_{11}^{+-} k_{21}^{+-} k_{12}^{+-} k_{22}^{+-} k_{31}^{+-} k_{32}^{+-})^2 \breve{k}_{11}^{+-} \breve{k}_{21}^{+-} \breve{k}_{12}^{+-} \breve{k}_{22}^{+-} \breve{k}_{31}^{+-} \breve{k}_{23}^{-+}} \tag{6.32}
$$

$$
\gamma_{11}^+ = \frac{(\breve{k}_{12}^{++} \breve{k}_{13}^{++} \breve{k}_{23}^{++} \breve{k}_{13}^{--})^2 k_{12}^{++} k_{13}^{++} k_{23}^{++} k_{13}^{--}}{(k_{11}^{+-} k_{13}^{+-} k_{21}^{+-} k_{23}^{+-} k_{33}^{+-} k_{31}^{+-})^2 \breve{k}_{11}^{+-} \breve{k}_{13}^{+-} \breve{k}_{21}^{+-} \breve{k}_{23}^{+-} \breve{k}_{33}^{+-} \breve{k}_{13}^{-+}} \tag{6.33}
$$

$$
\gamma_1^- = -\frac{(\breve{k}_{12}^{--})^2 k_{12}^{--}}{(k_{11}^{+-} k_{12}^{+-})^2 \breve{k}_{11}^{+-} \breve{k}_{12}^{+-}} \quad , \quad \gamma_2^- = -\frac{(\breve{k}_{13}^{--})^2 k_{13}^{--}}{(k_{11}^{+-} k_{13}^{+-})^2 \breve{k}_{11}^{+-} \breve{k}_{13}^{+-}} \tag{6.34}
$$

$$\gamma_3^- = -\frac{(\breve{k}_{12}^{--})^2 k_{12}^{--}}{(k_{22}^{+-}k_{21}^{+-})^2\breve{k}_{22}^{+-}\breve{k}_{21}^{+-}} \quad , \quad \gamma_4^- = -\frac{(\breve{k}_{13}^{--})^2 k_{12}^{--}}{(k_{21}^{+-}k_{23}^{+-})^2\breve{k}_{21}^{+-}\breve{k}_{23}^{+-}} \tag{6.35}$$

$$\gamma_5^- = -\frac{(\breve{k}_{23}^{--})^2 k_{23}^{--}}{(k_{13}^{+-}k_{12}^{+-})^2\breve{k}_{13}^{+-}\breve{k}_{12}^{+-}} \quad , \quad \gamma_6^- = -\frac{(\breve{k}_{12}^{--})^2 k_{12}^{--}}{(k_{31}^{+-}k_{32}^{+-})^2\breve{k}_{31}^{+-}\breve{k}_{32}^{+-}} \tag{6.36}$$

$$\gamma_7^- = -\frac{(\breve{k}_{13}^{--})^2 k_{13}^{--}}{(k_{31}^{+-}k_{33}^{+-})^2\breve{k}_{31}^{+-}\breve{k}_{33}^{+-}} \quad , \quad \gamma_8^- = -\frac{(\breve{k}_{23}^{--})^2 k_{23}^{--}}{(k_{22}^{+-})^2(k_{23}^{+-})^2\breve{k}_{22}^{+-}\breve{k}_{23}^{+-}} \tag{6.37}$$

$$\gamma_9^- = -\frac{(\breve{k}_{23}^{--})^2 k_{23}^{--}}{(k_{32}^{+-})^2(k_{33}^{+-})^2\breve{k}_{32}^{+-}\breve{k}_{33}^{+-}} \tag{6.38}$$

$$\gamma_{10}^- = -\frac{(\breve{k}_{12}^{++}\breve{k}_{12}^{--}\breve{k}_{13}^{--}k_{23}^{--})^2 k_{13}^{++}k_{12}^{--}k_{13}^{--}k_{23}^{--}}{(k_{11}^{+-}k_{21}^{+-}k_{12}^{+-}k_{22}^{+-}k_{13}^{+-}k_{23}^{+-})^2\breve{k}_{11}^{+-}\breve{k}_{21}^{+-}\breve{k}_{12}^{+-}\breve{k}_{22}^{+-}\breve{k}_{13}^{+-}\breve{k}_{23}^{-+}} \tag{6.39}$$

$$\gamma_{11}^- = -\frac{(\breve{k}_{13}^{++}\breve{k}_{12}^{--}\breve{k}_{13}^{--}k_{23}^{--})^2 k_{13}^{++}k_{12}^{--}k_{13}^{--}k_{23}^{--}}{(k_{11}^{+-}k_{12}^{+-}k_{31}^{+-}k_{32}^{+-}k_{13}^{+-}k_{33}^{+-})^2\breve{k}_{11}^{+-}\breve{k}_{12}^{+-}\breve{k}_{31}^{+-}\breve{k}_{32}^{+-}\breve{k}_{13}^{+-}\breve{k}_{33}^{-+}} \tag{6.40}$$

$$f = 1 + \sum_{i,j=1,2,3} \frac{\exp(\eta_{ij}^{+-})}{\Omega_{ij}^{+-}} + \sum_{i,j,m,l=1,2,3} \frac{\bar{\Omega}_{im}^{++}\bar{\Omega}_{jl}^{--}\exp(\eta_i^+ + \eta_j^- + \eta_k^+ + \eta_l^-)}{\Omega_{ij}^{+-}\Omega_{il}^{+-}\Omega_{mj}^{+-}\Omega_{lm}^{-+}}$$
$$+A\exp(\eta_1^+ + \eta_1^- + \eta_2^+ + \eta_2^- + \eta_3^+ + \eta_3^-) \tag{6.41}$$

$$A = \frac{\bar{\Omega}_{12}^{++}\bar{\Omega}_{13}^{++}\bar{\Omega}_{23}^{++}\bar{\Omega}_{12}^{--}\bar{\Omega}_{13}^{--}\bar{\Omega}_{23}^{--}}{\Omega_{11}^{+-}\Omega_{12}^{+-}\Omega_{13}^{+-}\Omega_{22}^{+-}\Omega_{21}^{+-}\Omega_{23}^{+-}\Omega_{31}^{+-}\Omega_{32}^{+-}\Omega_{33}^{-+}} \tag{6.42}$$

where $i \neq m$, $j \neq l$ and $\Omega_{ij}^{ab} = (k_{ab}^{ij})^2\breve{k}_{ab}^{ij}$, $\bar{\Omega}_{ij}^{ab} = (\breve{k}_{ab}^{ij})^2 k_{ab}^{ij}$.

We visualize the counterplots of soliton resonances for special values of parameters in Figures 6.4, 6.5 and 6.6.

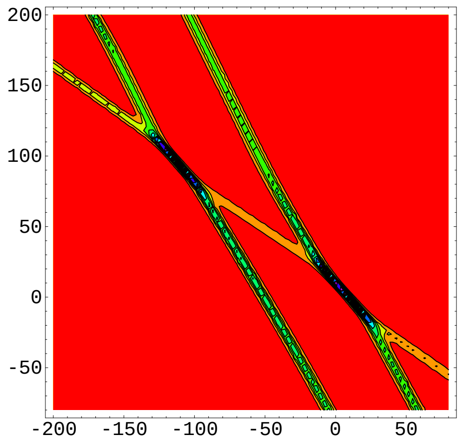


Figure 6.4. Zero Loop Interaction for Real Version of L-S Wave Equation

$$k_1^+ = 0.9, k_1^- = -0.49, k_2^+ = 10^{-7}, k_2^- = -0.54, k_3^+ = 10^{-4},\ k_3^- = -0.6$$
$$\eta_1^{+(0)} = \eta_1^{-(0)} = -5,\ \eta_2^{+(0)} = \eta_2^{-(0)} = 3,\ \eta_3^{+(0)} = \eta_3^{-(0)} = -1$$

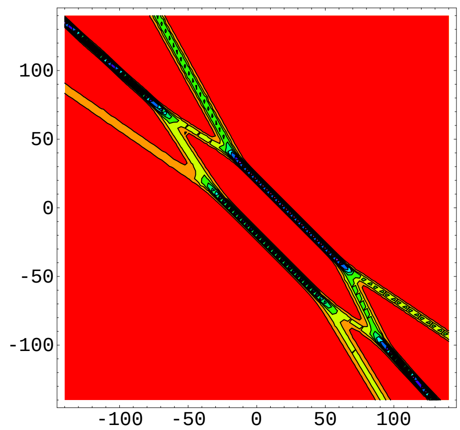


Figure 6.5. One Loop Interaction for Real Version of L-S Wave Equation

$$k_1^+ = 0.55,\ k_1^- = -1, k_2^+ = 10^{-5}, k_2^- = -0.5, k_3^+ = 0.9, k_3^- = -0.1\eta_1^{+(0)} = 0,$$
$$\eta_1^{-(0)} = 10,\ \eta_2^{+(0)} = 0,\ \eta_2^{-(0)} = 10,\ \eta_3^{+(0)} = 0,\ \eta_3^{-(0)} = 5$$

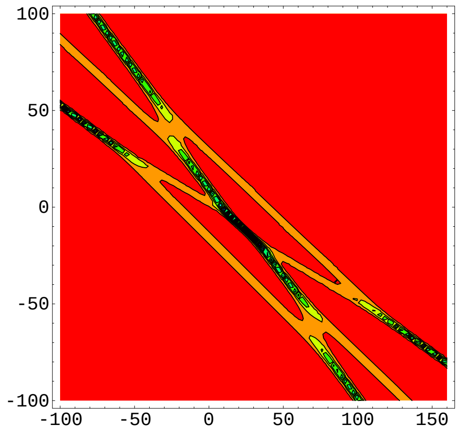


Figure 6.6. Two Loop Interaction for Real Version of L-S Wave Equation

$$k_1^+ = 0.9,\ k_1^- = -0.5,\ k_2^+ = 0.1, k_2^- = -0.45,\ k_3^+ = 1.3,\ k_3^- = -0.85,$$
$$\eta_1^{+(0)} = \eta_1^{-(0)} = -5,\ \eta_2^{+(0)} = \eta_2^{-(0)} = 5,\ \eta_3^{+(0)} = \eta_3^{-(0)} = -1$$

- **CASE II**

If $\hat{\alpha} \neq 0$, then one more scaling on (6.6)-(6.8)

$$\hat{x} = \frac{1}{\hat{\alpha}}\check{x},\ \ \hat{t} = \frac{1}{\hat{\alpha}^2}\check{t},\ \ \hat{x} = \hat{\alpha}^2 u,\ \ \hat{q} = \hat{\alpha}^{3/2} Q,\ \ \hat{r} = \hat{\alpha}^{3/2} R \tag{6.43}$$

admits the following system, real version (diffusion-antidiffusion) of Yajima-Oikawa equation.

$$Q_{\check{t}} = Q_{\check{x}\check{x}} + 2uQ \tag{6.44}$$

$$-R_{\check{t}} = R_{\check{x}\check{x}} + 2uR \tag{6.45}$$

$$u_{\check{x}} + u_{\check{t}} = (1 + \hat{\alpha}\hat{\beta})(QR)_{\check{x}} \tag{6.46}$$

1. **Lax Pair**

The Lax Pair of this system is

$$Y = -\frac{1}{2\lambda}\begin{pmatrix} U & 2\lambda\check{q} & U \\ -\check{r} & 0 & -\check{r} \\ -U & -2\lambda\check{q} & -U \end{pmatrix} + \begin{pmatrix} 3\lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & -\lambda \end{pmatrix} \tag{6.47}$$

$$V=\frac{1}{2\lambda}\begin{pmatrix} U-\frac{\breve{q}\breve{r}}{2} & 2\lambda(-\lambda\breve{q}+\breve{q}-\frac{\breve{q}_x}{2}) & U-\frac{\breve{q}\breve{r}}{2} \\ (\lambda\breve{r}-\breve{r}-\frac{\breve{r}_x}{2}) & 0 & (-\lambda\breve{r}-\breve{r}-\frac{\breve{r}_x}{2}) \\ -U+\frac{\breve{q}\breve{r}}{2} & -2\lambda(\lambda\breve{q}+\breve{q}-\frac{\breve{q}_x}{2}) & -U+\frac{\breve{q}\breve{r}}{2} \end{pmatrix}+K \quad (6.48)$$

$$K=\begin{pmatrix} \frac{2\lambda^2}{3}-2\lambda & 0 & 0 \\ 0 & -\frac{4\lambda^2}{3} & 0 \\ 0 & 0 & \frac{2\lambda^2}{3}+2\lambda \end{pmatrix} \quad (6.49)$$

Zero curvature condition $Y_x - V_t + VY - YV = 0$ leads to

$$U_t + U_x = (\breve{q}\breve{r})_x \quad (6.50)$$

$$\breve{r}U + \breve{r}_t + \breve{r}_x + \frac{\breve{r}_{xx}}{2} = 0 \quad (6.51)$$

$$\breve{q}U - \breve{q}_t - \breve{q}_x + \frac{\breve{q}_{xx}}{2} = 0 \quad (6.52)$$

and by Galilean transformation

$$\breve{r} = \breve{R}e^{(-x+t/2)}, \quad \breve{q} = \breve{Q}e^{(x-t/2)} \quad (6.53)$$

we get

$$U_t + U_x = (\breve{Q}\breve{R})_x \quad (6.54)$$

$$\breve{Q}_t = \breve{Q}U + \frac{\breve{Q}_{xx}}{2} \quad (6.55)$$

$$-\breve{R}_t = \breve{R}U + \frac{\breve{R}_{xx}}{2} \quad (6.56)$$

By resacaling

$$U = 4u, \quad \breve{Q} = 2(1+\hat{\alpha}\hat{\beta})Q, \quad \breve{R} = 2R, \quad x = \frac{\breve{x}}{2}, \; t = \frac{\breve{t}}{2} \quad (6.57)$$

the system can be written in the same form as (6.44)-(6.46).

2. **Bilinear Representation**

   Bilinear representation of the system is

$$(\pm D_{\hat{x}}^2 + D_{\hat{t}})(\hat{G}^{\mp} \cdot \hat{F}) = 0 \tag{6.58}$$

$$(D_{\hat{x}}^2 + D_{\hat{x}\hat{t}})(\hat{F} \cdot \hat{F}) = 2(1 + \hat{\alpha}\hat{\beta})\hat{G}^+\hat{G}^- \tag{6.59}$$

   where $Q = \hat{G}^+/\hat{F}$, $R = \hat{G}^-/\hat{F}$, $u = (\ln \hat{F})_{\hat{x}\hat{x}}$. It gives regular one soliton solution for (5.1),(5.2) or dissipaton solution of (6.44),(6.46).

$$\hat{G}^{\pm} = e^{\eta^{\pm}}, \quad \hat{F} = 1 + \frac{(1 + \hat{\alpha}\hat{\beta})e^{\eta^+ + \eta^-}}{(K^+ + K^-)^2(K^+ - K^- + 1)} \tag{6.60}$$

   where $\eta^{\pm} = K^{\pm}\hat{x} \pm (K^{\pm})^2\hat{t} + \eta^{\pm(0)}$, $\hat{\alpha}\hat{\beta} > -1$ and $K^+ - K^- + 1 > 0$.

3. **One Dissipaton Solution**

$$u = \frac{(K^+ + K^-)^2}{4\cosh^2 \frac{(K^+ + K^-)}{2}[\hat{x} + (K^+ - K^-)\hat{t} - x_0]} \tag{6.61}$$

$$Q = \frac{|K^+ + K^-| \sqrt{K^+ - K^- + 1} \exp\left(\frac{k^2 + v^2}{4}\hat{t} - \frac{1}{2}v\hat{x} + x_0\right)}{2\cosh\left(k(\hat{x} - v\hat{t} + c_0)\right)} \tag{6.62}$$

$$R = \frac{|K^+ + K^-| \sqrt{K^+ - K^- + 1} \exp - \left(\frac{k^2 + v^2}{4}\hat{t} - \frac{1}{2}v\hat{x} + x_0\right)}{2\cosh\left(k(\hat{x} - v\hat{t} + c_0)\right)} \tag{6.63}$$

4. **Two Dissipaton Solution**

$$\hat{G}^+ = (e^{\hat{\eta}_1^+} + e^{\hat{\eta}_2^+} + \hat{\alpha}_1^+ e^{\hat{\eta}_1^+ + \hat{\eta}_1^- + \hat{\eta}_2^+} + \hat{\alpha}_2^+ e^{\hat{\eta}_1^+ + \hat{\eta}_2^+ + \hat{\eta}_2^-}) \tag{6.64}$$

$$\hat{G}^- = (e^{\hat{\eta}_1^-} + e^{\hat{\eta}_2^-} + \hat{\alpha}_1^- e^{\hat{\eta}_1^+ + \hat{\eta}_1^- + \hat{\eta}_2^-} + \hat{\alpha}_2^- e^{\hat{\eta}_1^- + \hat{\eta}_2^+ + \hat{\eta}_2^-}) \tag{6.65}$$

$$\hat{F} = 1 + \sum_{i,j=1}^{2} \frac{\exp(\hat{\eta}_{ij}^{+-})}{\Omega_{ij}^{+-}} + \frac{\bar{\Omega}_{12}^{++}\bar{\Omega}_{12}^{--} \exp(\hat{\eta}_1^+ + \hat{\eta}_1^- + \hat{\eta}_2^+ + \hat{\eta}_2^-)}{\Omega_{11}^{+-}\Omega_{12}^{+-}\Omega_{21}^{+-}\Omega_{22}^{+-}} \tag{6.66}$$

$$\alpha_1^+ = \frac{(\breve{K}_{12}^{++})^2(K_{12}^{++})}{(K_{11}^{+-}K_{21}^{+-})^2\breve{K}_{11}^{+-}\breve{K}_{21}^{+-}} \quad , \quad \alpha_2^+ = \frac{(\breve{K}_{12}^{++})^2(K_{12}^{++})}{(K_{12}^{+-}K_{22}^{+-})^2\breve{K}_{12}^{+-}\breve{K}_{22}^{+-}} \tag{6.67}$$

$$\hat{\alpha}_1^- = \frac{(\breve{K}_{12}^{--})^2(K_{12}^{--})}{(K_{11}^{+-}K_{12}^{+-})^2\breve{K}_{11}^{+-}\breve{K}_{21}^{-+}} \quad , \quad \hat{\alpha}_2^- = \frac{(\breve{K}_{12}^{--})^2(K_{12}^{--})}{(K_{21}^{+-}K_{22}^{+-})^2\breve{K}_{12}^{-+}\breve{K}_{22}^{-+}} \tag{6.68}$$

where

$$\hat{\eta}_i^{\pm} = K_i^{\pm}\hat{x} \pm (K_i^{\pm})^2\hat{t} + \hat{\eta}_i^{\pm(0)}, \quad \hat{\eta}_{ij}^{+-} = \hat{\eta}_i^+ + \hat{\eta}_j^-$$

$$K_{ij}^{ab} = K_i^a + K_j^b, \ \breve{K}_{ij}^{ab} = K_i^a - K_j^b + 1 \tag{6.69}$$

$$\Omega_{ij}^{+-} = (K_{ij}^{+-})^2(\breve{K}_{ij}^{+-}), \ \bar{\Omega}_{ij}^{\pm\pm} = (\breve{K}_{ij}^{\pm\pm})^2(K_i^{\pm} + K_{ij}^{\pm\pm})$$

Regularity conditions for this case are $\breve{K}_{ab}^{ij} > 0$ and we visualize the counter-plots of resonance interaction (See Figure 6.7 and 6.8) for special values of parameters.

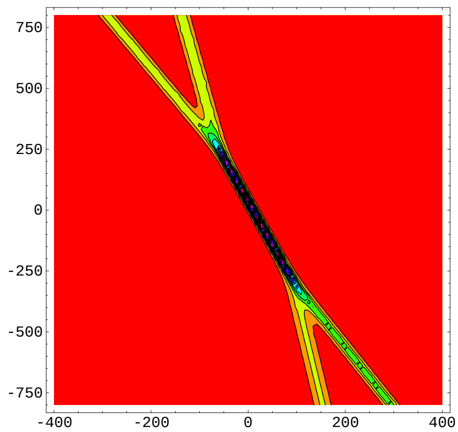


Figure 6.7. Zero Loop Interaction for Real Version of Y-O Wave Equation

$$K_1^+ = 0.14, \ K_1^- = -0.25, \ K_2^+ = 10^{-5}, \ K_2^- = -0.12,$$
$$\hat{\eta}_1^{+(0)} = \hat{\eta}_1^{-(0)} = \hat{\eta}_2^{+(0)} = \hat{\eta}_2^{-(0)} = 0$$

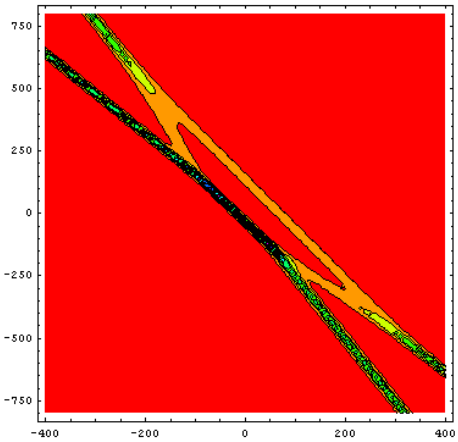


Figure 6.8. One Loop Interaction for Real Version of Y-O Wave Equation

$$K_1^+ = 0.42,\ K_1^- = -0.3,\ K_2^+ = 0.2,\ K_2^- = -0.1,$$
$$\hat{\eta}_1^{+(0)} = 4,\ \hat{\eta}_1^{-(0)} = -3,\ \hat{\eta_2}^{+(0)} = 3,\ \hat{\eta}_2^{-(0)} = -3$$

6.2. Shock Wave Profile Form for Resonance Solitons

- **$\hat{\alpha} = 0$ case**

  We now turn back to the system of equations (6.9)-(6.11). Representing functions Φ and Ψ such that

$$\hat{q} = \exp(\Phi), \quad \hat{r} = \exp(\Psi) \tag{6.70}$$

we get the system of equations

$$\Phi_{\hat{t}} = \Phi_{\hat{x}\hat{x}} + \Phi_{\hat{x}}^2 + 2\hat{n} \tag{6.71}$$

$$-\Psi_{\hat{t}} = \Psi_{\hat{x}\hat{x}} + \Psi_{\hat{x}}^2 + 2\hat{n} \tag{6.72}$$

$$\hat{n}_{\hat{t}} = [\exp(\Phi + \Psi)]_{\hat{x}} \tag{6.73}$$

Integrating the first two equations (6.71) and (6.72) with respect to $\hat{x}$ and taking the logarithm of the third one (6.73) we have the system of equations in terms of $u = -\Phi_{\hat{x}}$ and $v = -\Psi_{\hat{x}}$ as

$$u_{\hat{t}} + 2uu_{\hat{x}} = u_{\hat{x}\hat{x}} - 2\hat{n}_{\hat{x}} \tag{6.74}$$

$$-v_{\hat{t}} + 2vv_{\hat{x}} = v_{\hat{x}\hat{x}} - 2\hat{n}_{\hat{x}} \tag{6.75}$$

$$(u+v)\hat{n}_{\hat{x}\hat{t}} = (u_{\hat{x}} + v_{\hat{x}} - (u+v)^2)\hat{n}_{\hat{t}} \tag{6.76}$$

Thus, (6.73) represents the inhomogeneous Burgers' equation and (6.74)represents the inhomogeneous anti-Burgers' equation. With one dissipaton solution $\hat{q} = e^{\eta^+}/F$ and $\hat{r} = e^{\eta^-}/F$ in hand (6.16)-(6.18) we derive u and v in Taylor shock profile form (See Figure 6.9).

$$u(\hat{x},\hat{t}) = \frac{k^- - k^+}{2} + \left(\frac{k^+ + k^-}{2}\right)\tanh\left[\left(\frac{k^+ + k^-}{2}\right)(\hat{x} + (k^- - k^+)\hat{t} + x_0)\right] \tag{6.77}$$

$$v(\hat{x},\hat{t}) = \frac{k^+ - k^-}{2} + \left(\frac{k^+ + k^-}{2}\right)\tanh\left[\left(\frac{k^+ + k^-}{2}\right)(\hat{x} + (k^- - k^+)\hat{t} + x_0)\right] \tag{6.78}$$

where

$$\hat{n} = \frac{(k^+ + k^-)^2}{4\cosh^2\left[\frac{(k^+ + k^-)}{2}(\hat{x} + (k^- - k^+)\hat{t} + x_0)\right]} \tag{6.79}$$

- **$\hat{\alpha} \neq 0$ case**

Considering equations (6.44)-(6.46) we set

$$Q = \exp(\Phi), \;\; R = \exp\Psi \tag{6.80}$$

and applying the same procedure as we did for the previous case the system can be written as

$$U_{\check{t}} + 2UU_{\check{x}} = U_{\check{x}\check{x}} - 2u_{\check{x}} \tag{6.81}$$

$$-V_{\check{t}} + 2VV_{\check{x}} = V_{\check{x}\check{x}} - 2u_{\check{x}} \tag{6.82}$$

$$(U+V)(u_{\check{x}\check{x}} + u_{\check{x}\check{t}}) = (U_{\check{x}} + V_{\check{x}} - (U+V)^2)(u_{\check{x}} + u_{\check{t}}) \tag{6.83}$$

where $U = -(\Phi)_{\check{x}}$ and $V = -(\Psi)_{\check{x}}$. Then we have shock soliton (See Figure 6.10) of (6.81)-(6.83) as

$$U = K^- - \left(\frac{(K^+ + K^-)^2 \sqrt{K^+ - K^- + 1}\,(1 - \tanh \varpi)}{2\sqrt{1 + \hat{\alpha}\hat{\beta}}} \right) \tag{6.84}$$

$$V = K^+ - \left(\frac{(K^+ + K^-)^2 \sqrt{K^+ - K^- + 1}\,(1 - \tanh \varpi)}{2\sqrt{1 + \hat{\alpha}\hat{\beta}}} \right) \tag{6.85}$$

$$u = \frac{(K^+ + K^-)^2}{4\cosh^2 \frac{K^+ + K^-}{2}[\check{x} + (K^- - K^+)\check{t} + x_0]} \tag{6.86}$$

where

$$\varpi = \left[\frac{K^+ + K^-}{2} (\check{x} + (K^- - K^+)\check{t} + x_0 \right] \tag{6.87}$$

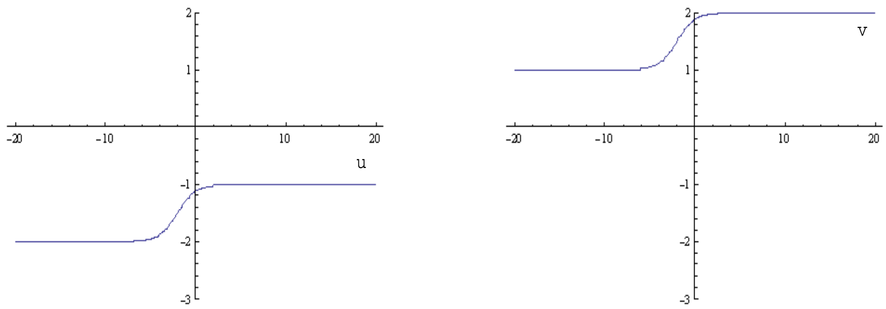


Figure 6.9. u and v (Shock Wave Profile for Real Version of L-S Wave)

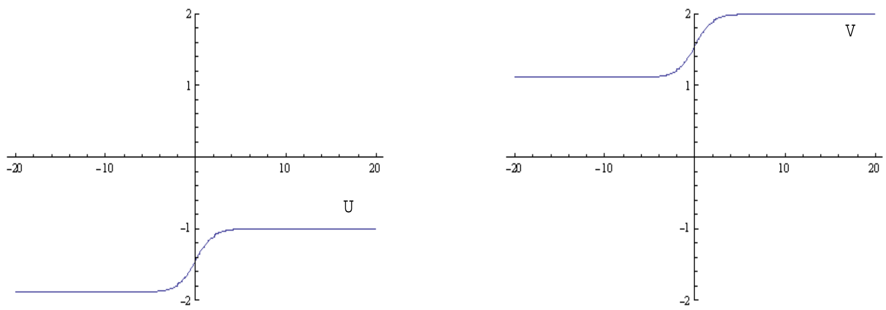


Figure 6.10. U and V (Shock Wave Profile for Real Version of Y-O Equation)

CHAPTER 7

CONSERVATION LAWS AND RESONANCE PHENOMENA

Integrable systems have some common properties. The Lax pair, N-soliton solution and infinite number of conserved identities are among these. In this chapter we will briefly give some definitions and examples apply to the systems which are at the focus of this thesis. We will find the resonance conditions on the integrals of motion.

7.1. Conservation Law

A conservation law in one space dimension is an equation of the form

$$\frac{\partial T}{\partial t}+\frac{\partial X}{\partial x}=0 \tag{7.1}$$

The function T is known as the conserved density and X is the associated flux. In the simplest form $T(x,t,u)$ and $X(x,t,u)$ depend on time t, the position x, and the solution $u(x,t)$ to the physical system. Although, higher order conservation laws, which also depend on derivatives of u are important for integrable partial differential equations. Integrated form of the conservation law on a closed interval $(a<x<b)$ is an immediate consequence of the Fundamental Theorem.

$$\frac{\partial}{\partial t}\int_a^b T\,dx=\int_a^b \frac{\partial T}{\partial t}\,dx=-\int_a^b \frac{\partial X}{\partial x}\,dx=-X|_a^b \tag{7.2}$$

In particular, if there is no net flux into or out of the interval, then the integrated density is conserved, meaning that it remains constant over time.

7.2. Integrals of Motion for Reaction-Diffusion L-S Wave System

Now we will obtain first few integrals of motion for the reaction-diffusion type long-short wave equation.

$$\hat{q}_{\hat{t}} = \hat{q}_{\hat{x}\hat{x}} + 2\hat{n}\hat{q} \tag{7.3}$$

$$-\hat{r}_{\hat{t}} = \hat{r}_{\hat{x}\hat{x}} + 2\hat{n}\hat{r} \tag{7.4}$$

$$\hat{n}_{\hat{t}} = (\hat{q}\hat{r})_{\hat{x}} \tag{7.5}$$

The first conservation law for the system can immediately be written just taking the third equation (7.5) it self.

$$\hat{n}_{\hat{t}} - (\hat{q}\hat{r})_{\hat{x}} = 0 \tag{7.6}$$

The second one is calculated by multiplying (7.3) and (7.4) with $\hat{r}$ and $\hat{q}$ respectively

$$(\hat{q}\hat{r})_{\hat{t}} - (\hat{r}\hat{q}_{\hat{x}} - \hat{q}\hat{r}_{\hat{x}})_{\hat{x}} = 0 \tag{7.7}$$

Now we will calculate the third conservation law as follows

$$\begin{aligned}
(\hat{r}\hat{q}_{\hat{x}} - \hat{q}\hat{r}_{\hat{x}})_{\hat{t}} &= -\hat{q}_{\hat{x}}(\hat{r}_{\hat{x}\hat{x}} + 2\hat{n}\hat{r}) + \hat{r}(\hat{q}_{\hat{x}\hat{x}\hat{x}} + 2(\hat{n}\hat{q})_{\hat{x}}) + \hat{q}(\hat{r}_{\hat{x}\hat{x}\hat{x}} + 2(\hat{n}\hat{r})_{\hat{x}}) - \hat{r}_{\hat{x}}(\hat{q}_{\hat{x}\hat{x}} + 2\hat{n}\hat{q}) \\
&= (\hat{q}\hat{r}_{\hat{x}\hat{x}\hat{x}} + \hat{r}\hat{q}_{\hat{x}\hat{x}\hat{x}} - \hat{q}_{\hat{x}}\hat{r}_{\hat{x}\hat{x}} - \hat{r}_{\hat{x}}\hat{q}_{\hat{x}\hat{x}}) + 4\hat{n}_{\hat{x}}(\hat{q}\hat{r}) \\
&= (\hat{q}\hat{r}_{\hat{x}\hat{x}} + \hat{r}\hat{q}_{\hat{x}\hat{x}} - 2\hat{q}_{\hat{x}}\hat{r}_{\hat{x}} + 4\hat{n}\hat{q}\hat{r})_{\hat{x}} - 4\hat{n}(\hat{q}\hat{r})_{\hat{x}}
\end{aligned} \tag{7.8}$$

Substituting $(\hat{q}\hat{r})_{\hat{x}} = \hat{n}_{\hat{t}}$ we obtain

$$(\hat{r}\hat{q}_{\hat{x}} - \hat{q}\hat{r}_{\hat{x}} + 2\hat{n}^2)_{\hat{t}} - (\hat{q}\hat{r}_{\hat{x}\hat{x}} + \hat{r}\hat{q}_{\hat{x}\hat{x}} - 2\hat{q}_{\hat{x}}\hat{r}_{\hat{x}} + 4\hat{n}\hat{q}\hat{r})_{\hat{x}} = 0 \tag{7.9}$$

One more identity holds by the next equation

$$(\hat{q}_{\hat{x}}\hat{r}_{\hat{x}})_{\hat{t}} = \hat{r}_{\hat{x}}\hat{q}_{\hat{x}\hat{x}\hat{x}} - \hat{q}_{\hat{x}}\hat{r}_{\hat{x}\hat{x}\hat{x}} + 2\hat{n}_{\hat{x}}(\hat{r}_{\hat{x}}\hat{q} - \hat{q}_{\hat{x}}\hat{r})$$

$$= (\hat{r}_{\hat{x}}\hat{q}_{\hat{x}\hat{x}} - \hat{q}_{\hat{x}}\hat{r}_{\hat{x}\hat{x}} + 2\hat{n}(\hat{r}_{\hat{x}}\hat{q} - \hat{q}_{\hat{x}}\hat{r})_{\hat{x}} - 2\hat{n}(\hat{r}_{\hat{x}}\hat{q} - \hat{q}_{\hat{x}}\hat{r})_{\hat{x}} \tag{7.10}$$

and substituting $(\hat{q}\hat{r})_{\hat{t}} = (\hat{q}_{\hat{x}\hat{x}}\hat{r} - \hat{r}_{\hat{x}\hat{x}}\hat{q})$ yields

$$(\hat{q}_{\hat{x}}\hat{r}_{\hat{x}} - 2\hat{n}\hat{q}\hat{r})_{\hat{t}} - (\hat{q}_{\hat{x}\hat{x}}\hat{r}_{\hat{x}} - \hat{r}_{\hat{x}\hat{x}}\hat{q}_{\hat{x}} + 2\hat{n}(\hat{r}_{\hat{x}}\hat{q} - \hat{q}_{\hat{x}}\hat{r}) - (\hat{q}\hat{r})^2)_{\hat{x}} \tag{7.11}$$

The conservation laws (7.6),(7.7), (7.9) give us first three integrals of motion and admits to determine the mass, the momentum and the energy of the system which are conserved quantities.

$$M_{\hat{n}} = \int_{-\infty}^{\infty} \hat{n}\, d\hat{x} = \int_{-\infty}^{\infty} (\ln F)_{\hat{x}\hat{x}}\, d\hat{x} \tag{7.12}$$

$$P = -\int_{-\infty}^{\infty} \hat{q}\hat{r}\, d\hat{x} = -\int_{-\infty}^{\infty} (\ln F)_{\hat{x}\hat{t}}\, d\hat{x} \tag{7.13}$$

$$E = \int_{-\infty}^{\infty} (\hat{r}\hat{q}_{\hat{x}} - \hat{q}\hat{r}_{\hat{x}} + 2\hat{n}^2)\, d\hat{x} \tag{7.14}$$

For one dissipaton solution (6.16)-(6.18) or correspondingly one soliton solution, the masses, momentum and energy conserved quantities can be expressed as

$$M_{\hat{n}} = |k_1^+ + k_1^-| \tag{7.15}$$

$$P = |(k_1^+)^2 - (k_1^-)^2| = M_{\hat{n}} v \tag{7.16}$$

$$E = M_{\hat{n}} v^2 + \frac{M_{\hat{n}}^3}{3} \tag{7.17}$$

where $v = k_1^- - k_1^+$.

Now we will use these identities to show that the resonant system admits Y-shaped soliton resonance. For the process of fusion and fission of two solitons (See Figure 7.1) we have conservation laws.

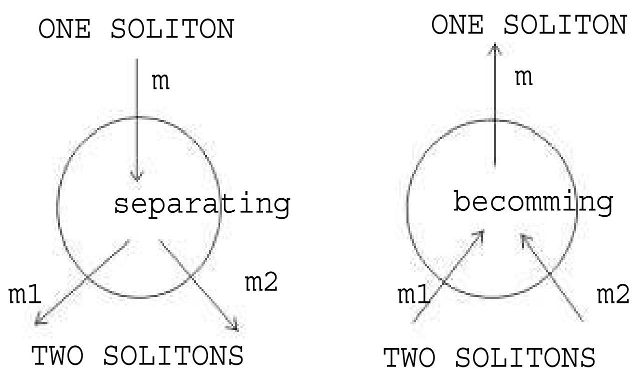


Figure 7.1. Fusion and Fission of Two Solitons

$$M_{\hat{n}} = M_{\hat{n1}} + M_{\hat{n2}} \tag{7.18}$$

$$M_{\hat{n}} v = M_{\hat{n1}} v_1 + M_{\hat{n2}} v_2 \tag{7.19}$$

$$M_{\hat{n}} v^2 + \frac{M_{\hat{n}}^3}{3} = M_{\hat{n1}} v_1^2 + \frac{M_{\hat{n1}}^3}{3} M_{\hat{n2}} v_2^2 + \frac{M_{\hat{n2}}^3}{3} \tag{7.20}$$

By the help of these equalities and regularity conditions we may define the relation between k_1^+,k_2^+,k_1^- and k_2^-. We recall that we stated $k_i^+ > 0$ and $k_i^- < 0$ to obtain regular solitons. We write v as

$$v = \frac{M_{\hat{n1}} v_1 + M_{\hat{n2}} v_2}{M_{\hat{n1}} + M_{\hat{n2}}} \tag{7.21}$$

and substitute in (7.20) to get

$$(M_{\hat{n1}} v_1 + M_{\hat{n2}} v_2)^2 + M_{\hat{n}}^2 M_{\hat{n1}} M_{\hat{n2}} = M_{\hat{n}} (M_{\hat{n1}} v_1^2 + M_{\hat{n2}} v_2^2) \tag{7.22}$$

Finally we have $|v_1 - v_2| = M_{\hat{n1}} + M_{\hat{n2}}$ and in terms of the coefficients $k\pm_i$ $(i = 1,2)$

$$|k_1^+ - k_1^- - k_2^+ + k_2^-| = |k_1^+ + k_1^-| + |k_2^+ - k_2^-| \tag{7.23}$$

The equation (7.23)is the necessary condition to obtain a resonance interaction of soliton solutions for diffusion-antidiffusion type L-S wave equation. Setting $k_1^+ > k_2^+ > k_1^- = k_2^-$ we may visualize the Y-shaped collision (See Figure 7.2).

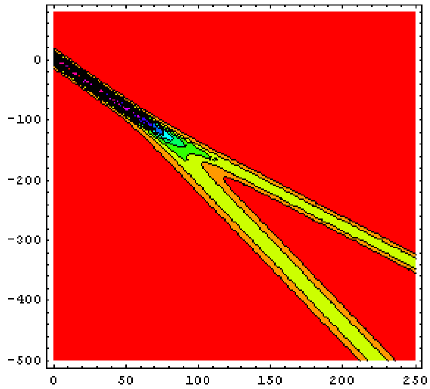


Figure 7.2. Y-shaped collision for Real Version of L-S Wave Equation

$$k_1^+ = 0.5,\ k_2^+ = 0.1,\ k_1^- = k_2^- = -0.3$$

7.3. Integrals of Motion for Reaction-Diffusion Y-O System

We may also construct first few integrals of motion for reaction-diffusion Yajima-Oikawa system.

$$Q_t = Q_{xx} + 2uQ \tag{7.24}$$

$$-R_t = R_{xx} + 2uR \tag{7.25}$$

$$u_x + u_t = (QR)_x \tag{7.26}$$

The first conservation law is

$$u_t - (QR - u)_x = 0 \tag{7.27}$$

The second conserved identity is exactly the same with the second identity of reaction-diffusion L-S wave system.

$$(QR)_t - (RQ_x - QR_x)_x = 0 \tag{7.28}$$

To obtain third law we calculate

$$(RQ_x - QR_x)_t = (RQ_{xxx} + QR_{xxx} - Q_xR_{xx} - R_xQ_{xx}) + 4u(QR)_x$$
$$(RQ_{xx} + QR_{xx} - 2Q_xR_x + 4uQR)_x - 4u(QR)_x \tag{7.29}$$

and substituting $(QR)_x = u_t + u_x$ yields

$$(RQ_x - QR_x + 2u^2)_t - (RQ_{xx} + QR_{xx} - 2Q_xR_x + 4uQR - 2u^2)_x = 0 \tag{7.30}$$

The last conservation law will be derived as follows

$$(Q_xR_x - 2uQR)_t = (R_xQ_{xx} - Q_xR_{xx} + 2u(QR_x - RQ_x) - (QR)^2 + 2uQR)_x - 2u(QR)_x \tag{7.31}$$

Considering $(QR)_x = (u_x + u_t)$ we get

$$(Q_xR_x - 2uQR + u^2)_t - (R_xQ_{xx} - Q_xR_{xx} + 2u(QR_x - RQ_x) - (QR)^2 + 2uQR - u^2)_x \tag{7.32}$$

Conservation laws (7.27),(7.28), (7.30) determine the conserved quantities

$$M_u = \int_{-\infty}^{\infty} u\,dx = \int_{-\infty}^{\infty} (\ln F)_{xx}\,dx \tag{7.33}$$
$$P = \int_{-\infty}^{\infty} QR\,dx = \int_{-\infty}^{\infty} [(\ln F)_{xx} + (\ln F)_{xt}]\,dx \tag{7.34}$$
$$E = \int_{-\infty}^{\infty} (RQ_x - QR_x + 2u^2)\,dx \tag{7.35}$$

For one dissipaton solution (6.61)-(6.63)we have

$$M_u = |k_1^+ + k_1^-| \tag{7.36}$$

$$P = |k_1^+ + k_1^-|(k_1^+ - k_1^- + 1) = (v+1)M_u \tag{7.37}$$

$$E = M_u v^2 + \frac{M_u^3}{3} \tag{7.38}$$

Like in the previous section (7.2) we have resonance conditions

$$M_u = M_{u1} + M_{u2} \tag{7.39}$$

$$M_u v = M_{u1} v_1 + M_{u2} v_2 \tag{7.40}$$

$$M_u v^2 + \frac{M_u^3}{3} = M_{u1} v_1^2 + \frac{M_{u1}^3}{3} M_{u2} v_2^2 + \frac{M_{u2}^3}{3} \tag{7.41}$$

A similar calculation which we applied for the diffusion-antidiffusion L-S wave equation gives the same resonance condition as

$$|K_1^+ - K_1^- - K_2^+ + K_2^-| = |K_1^+ + K_1^-| + |K_2^+ - K_2^-| \tag{7.42}$$

but the regularity condition is $K_i^+ - K_j^- + 1 > 0$. We visualize the Y-shaped collision in Figure 7.3.

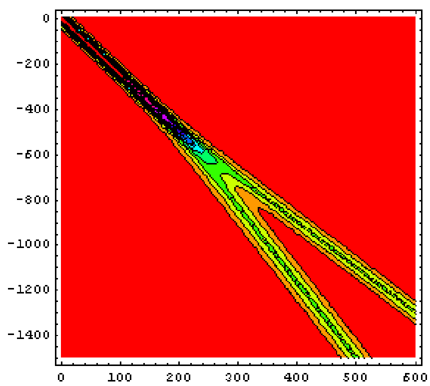


Figure 7.3. Y-shaped collision for Real Version of Y-O Equation

$$k_1^+ = 0.3,\ k_2^+ = 0.1,\ k_1^- = k_2^- = -0.2$$

7.4. Soliton Resonances

The occurrence of resonant interactions between solitons is permitted, for an appropriate set of k_i which satisfy the following equation (which is strictly related with the phase shift parameter)

$$P[\omega(k_1) \pm \omega(k_2), k_1 \pm k_2] = 0 \tag{7.43}$$

where P is the dispersion relation satisfied by ω and k_i (Kako and Yajima 1980). It can be seen that the dispersion relation plays a major role in producing soliton resonance. For example, the KdV equation is not possible to have this phenomenon for any k_i. However, the dispersion relation is a necessary condition but not a sufficient one (Hirota 1983). The relation which is obtained by the conserved identities should correspond to the case in which the phase shift parameter is $-\infty$ or ∞. Equations that have the same dispersion relation would have different type of conserved identities. This is the physical reason why one shows the resonance phenomena and the other not (Hirota 1983).
For two soliton solution of the diffusion-antidiffusion type L-S system

$$F = 1 + \frac{e^{\eta_1^+ + \eta_1^-}}{(k_{11}^{+-})^2(\breve{k}_{11}^{+-})} + \frac{e^{\eta_1^+ + \eta_2^-}}{(k_{12}^{+-})^2(\breve{k}_{12}^{+-})} + \frac{e^{\eta_2^+ + \eta_1^-}}{(k_{21}^{+-})^2(\breve{k}_{21}^{+-})} + \frac{e^{\eta_2^+ + \eta_2^-}}{(k_{22}^{+-})^2(\breve{k}_{22}^{+-})} + \beta e^{\eta_1^+ + \eta_1^- + \eta_2^+ + \eta_2^-} \tag{7.44}$$

the related phase shift parameter $\log \beta$ for

$$\beta = \frac{(\breve{k}_{12}^{++}\breve{k}_{12}^{--})^2(k_{12}^{++}k_{12}^{--})}{(k_{11}^{+-}k_{12}^{+-}k_{21}^{+-}k_{22}^{+-})^2\breve{k}_{11}^{+-}\breve{k}_{12}^{-+}\breve{k}_{12}^{+-}\breve{k}_{22}^{+-}}. \tag{7.45}$$

is $-\infty$ for $k_1^- = k_2^-$. Hence, it is possible to get one soliton solution for the limiting cases $\eta_1^+ + \eta_1^- \to -\infty$ and $\eta_2^+ + \eta_1^- \to -\infty$.

CHAPTER 8

A MODIFIED DAVEY-STEWARTSON EQUATION

The 2+1 dimensional version of the NLS equation preserving integrability property is given by the Davey-Stewartson equation. Davey and Stewartson derived the two coupled nonlinear partial differential equations which describe the evolution of a three dimensional wave packet of wave number k on water of finite depth (Davey and Stewartson 1974). In their work Davey and Stewartson obtained the following system

$$iA_{tt} + \lambda A_{xx} + \mu A_{yy} = \nu |A|^2 A + \nu_1 AQ \tag{8.1}$$

$$\lambda_1 Q_{xx} + \mu_1 Q_{yy} = \kappa_1 |A|^2_{yy} \tag{8.2}$$

where A is a complex field and Q is a real field. The system can be considered as the analogue of (a two-dimensional modification) the one-dimensional NLS equation. Generalization of this equation by adding the quantum potential has been considered in connection with capillary waves theory (Rogers, et al. 2009). From another side, the Hamiltonian form of NLS equation was introduced (Malomed and Stenflo 1991). It was shown that addition of these terms to NLS equation produce soliton resonances for special values of parameters (Pashaev and Rogers 2008). In this chapter we consider the D-S equation by adding a quantum potential term with strength Γ and also a Hamiltonian term with complex coupling constant c, as follows (Pashaev and Duruk 2009)

$$i\psi_t + \sigma^2 \psi_{xx} + \psi_{yy} + 2p \mid \psi \mid^2 \psi - Q\psi = \Gamma \left(\frac{\sigma^2 \mid \psi \mid_{xx} + \mid \psi \mid_{yy}}{\mid \psi \mid} \right) \psi$$

$$+ \left[\bar{c} \frac{\sigma^2 \psi_x^2 + \psi_y^2}{\psi^2} + c \frac{\sigma^2 \bar{\psi}_x^2 + \bar{\psi}_y^2}{\bar{\psi}^2} - 2c \frac{\sigma^2 \bar{\psi}_{xx} + \bar{\psi}_{yy}}{\bar{\psi}} - 2c \frac{\sigma^2 \bar{\psi}_x \psi_x + \bar{\psi}_y \psi_y}{\bar{\psi}\psi} \right] \psi \tag{8.3}$$

$$Q_{xx} - \sigma^2 Q_{yy} = 4p(\mid \psi \mid^2)_{xx} \tag{8.4}$$

Representing $\psi = e^{R+iS}$, then (8.3) and (8.4) yields

$$R_t+(1-2c_1)[\sigma^2(S_{xx}+2R_xS_x)+S_{yy}+2R_yS_y]$$

$$+2c_2(\sigma^2R_{xx}+R_{yy})+4c_2(\sigma^2R_x^2+R_y^2)=0,$$

$$-S_t+(1+2c_1-\Gamma)[\sigma^2(R_{xx}+R_x^2)+R_{yy}+R_y^2] \tag{8.5}$$

$$-(1-2c_1)[\sigma^2S_x^2+S_y^2]+2c_2[\sigma^2S_{xx}+S_{yy}]+2pe^{2R}-Q=0,$$

$$Q_{xx}-\sigma^2Q_{yy}=8p(R_{xx}+2R_x^2)e^{2R}$$

The linear transformation

$$S=\hat{S}+\frac{2c_2}{2c_1-1}\hat{R},\quad \hat{t}=(2c_1-1)t,\quad R=\hat{R} \tag{8.6}$$

allows us to write

$$\begin{gathered}
\hat{S}_{\hat{t}}-\left(\sigma^2\hat{S}_x^2+\hat{S}_y^2\right)+\left[\tfrac{\Gamma}{2c_1-1}-\tfrac{4|c|^2-1}{(2c_1-1)^2}\right]\left(\sigma^2(\hat{R}_{xx}+\hat{R}_x^2)+\hat{R}_{yy}+\hat{R}_y^2\right)\\
-\tfrac{1}{2c_1-1}\left(2pe^{2\hat{R}}-Q\right)=0,\\
-\hat{R}_{\hat{t}}+\sigma^2\left[2\hat{R}_x\hat{S}_x+\hat{S}_{xx}\right]+\left[2\hat{R}_y\hat{S}_y+\hat{S}_{yy}\right]=0,\\
Q_{xx}-\sigma^2Q_{yy}=8p\left(\hat{R}_{xx}+2\hat{R}_x^2\right)e^{2\hat{R}}
\end{gathered} \tag{8.7}$$

Representing $\hat{\psi}=e^{\hat{R}-i\hat{S}}$, the system reduces to

$$\begin{gathered}
i\hat{\psi}_{\hat{t}}+\sigma^2\hat{\psi}_{xx}+\hat{\psi}_{yy}-\hat{\Gamma}\left(\tfrac{\sigma^2|\hat{\psi}|_{xx}+|\hat{\psi}|_{yy}}{|\hat{\psi}|}\right)\hat{\psi}+\gamma\mid\hat{\psi}\mid^2\hat{\psi}-\hat{Q}\hat{\psi}=0,\\
\hat{Q}_{xx}-\sigma^2\hat{Q}_{yy}=2\gamma(\mid\hat{\psi}\mid^2)_{xx}
\end{gathered} \tag{8.8}$$

where $(-2p/2c_1-1)=\gamma$, $(-1/2c_1-1)Q=\hat{Q}$ and $\hat{\Gamma}-1=\frac{4|c|^2-1}{(2c_1-1)^2}-\frac{\Gamma}{2c_1-1}$.

This system of equations was introduced as 2+1 generalization of the resonant NLS (Pashaev and Lee 2002,a) and its relation with capillary model system has been derived(Rogers, et al. 2009). We must examine the case to construct restriction on these parameters and the below graphics (See Figure 8.1 and 8.2) is useful to determine specific cases.

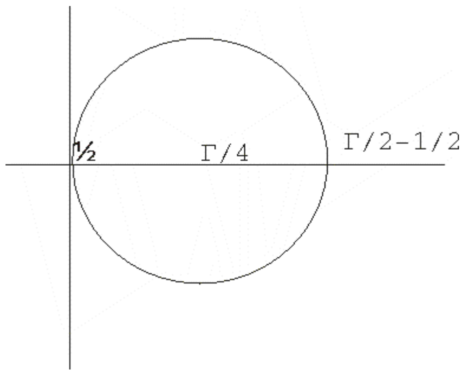


Figure 8.1. The complex c plane for $\Gamma > 2$

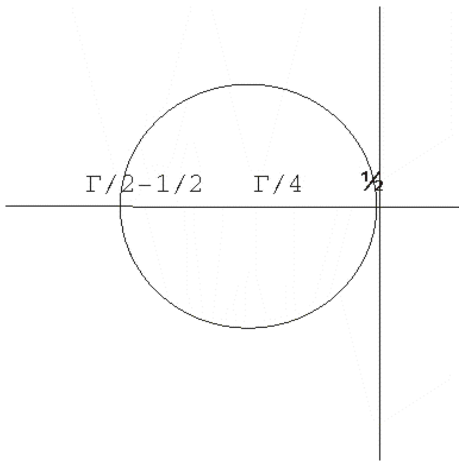


Figure 8.2. The complex c plane for $\Gamma < 2$

In these two graphics, for $\hat{\Gamma} < 1$ inside the circle corresponds to undercritical case, for $\hat{\Gamma} > 1$ outside the circle corresponds to overcritical case.

8.1. Undercritical case

For $\hat{\Gamma} < 1$ by rescaling in (8.7)

$$\hat{t}=\frac{\tilde{t}}{\sqrt{1-\hat{\Gamma}}},\quad \hat{S}=\sqrt{1-\hat{\Gamma}}\tilde{S},\quad \hat{R}=\tilde{R},\quad (-1/2c_1-1)Q=\hat{Q} \tag{8.9}$$

$$(-2p/2c_1-1)=\gamma,\ \ \hat{\Gamma}=1+\frac{4|c|^2-1}{(2c_1-1)^2}-\frac{\Gamma}{2c_1-1} \tag{8.10}$$

For $\tilde{\phi}=e^{\tilde{R}-i\tilde{S}}$we get

$$\begin{gathered} i\tilde{\phi}_{\tilde{t}}+\sigma^2\tilde{\phi}_{xx}+\tilde{\phi}_{yy}+\tfrac{\gamma}{1-\hat{\Gamma}}\mid\tilde{\phi}\mid^2\tilde{\phi}-\tfrac{1}{1-\hat{\Gamma}}\hat{Q}\tilde{\phi}=0,\\ \hat{Q}_{xx}-\sigma^2\hat{Q}_{yy}=2\gamma(\mid\tilde{\phi}\mid^2)_{xx} \end{gathered} \tag{8.11}$$

Then rescaling $\tilde{\phi}$ and $\hat{Q}$ as

$$\tilde{\phi}=(\sqrt{1-\hat{\Gamma}})\varphi,\quad \hat{Q}=(1-\hat{\Gamma})\vartheta \tag{8.12}$$

yields the D-S I(II) system ($\sigma^2=\pm1$ correspondingly)

$$\begin{gathered} i\varphi_{\tilde{t}}+\sigma^2\varphi_{xx}+\varphi_{yy}+\gamma\mid\varphi\mid^2\varphi-\vartheta\varphi=0,\\ \vartheta_{xx}-\sigma^2\vartheta_{yy}=2\gamma(\mid\varphi\mid^2)_{xx} \end{gathered} \tag{8.13}$$

1. **Bilinear Representation**

   D-S I (II) system

$$\begin{gathered} i\varphi_{\tilde{t}}+\sigma^2\varphi_{xx}+\varphi_{yy}+\gamma\mid\varphi\mid^2\varphi-\vartheta\varphi=0,\\ \vartheta_{xx}-\sigma^2\vartheta_{yy}=2\gamma(\mid\varphi\mid^2)_{xx} \end{gathered} \tag{8.14}$$

   can be represented in bilinear form as

$$\begin{gathered} \left(iD_{\tilde{t}}^2+\sigma^2D_x^2+D_y^2\right)(g\cdot f)=0,\\ \left(-\sigma^2D_x^2+D_y^2\right)(f\cdot f)=\gamma g\bar{g} \end{gathered} \tag{8.15}$$

   where $\varphi=g/f$ and $\vartheta=-4\sigma^2(\ln f)_{xx}$, g is a complex valued function and f is a real valued function.

2. **One Soliton Solution**

   One soliton solution of this system can be given as

$$g = e^{\eta}, \quad f = 1 + \gamma \frac{e^{\eta+\bar{\eta}}}{2[(m+\bar{m})^2 - \sigma^2(k+\bar{k})^2]} \tag{8.16}$$

$$\eta = kx + my + i(\sigma^2 k^2 + m^2)t + \eta^{(0)}, \quad k = k_1 + ik_2, \quad m = m_1 + im_2 \tag{8.17}$$

   If $\gamma > 0$ and $\sigma^2 = -1$ the equation admits regular soliton solution, but to obtain a regular soliton solution for $\sigma^2 = 1$ the condition $\mid m_1 \mid > \mid k_1 \mid$ must be satisfied. If $\gamma < 0$ and $\sigma^2 = 1$ the equation admits regular soliton solution, but to obtain a regular soliton solution for $\sigma^2 = -1$ the condition $\mid m_1 \mid < \mid k_1 \mid$ must be satisfied.

8.2. Overcritical case

When $\hat{\Gamma} > 1$ by rescaling in (8.7)

$$\hat{t} = \frac{\tilde{t}}{\sqrt{\hat{\Gamma}-1}}, \quad \hat{S} = \sqrt{\hat{\Gamma}-1}\tilde{S}, \quad \hat{R} = \tilde{R} \tag{8.18}$$

and introducing $q = e^{\tilde{R}+\tilde{S}}$ and $r = e^{\tilde{R}-\tilde{S}}$ we get

$$q_{\tilde{t}} - \sigma^2 q_{xx} - q_{yy} - \gamma(qr)q - uq = 0$$

$$r_{\tilde{t}} + \sigma^2 r_{xx} + r_{yy} + \gamma(qr)r + ur = 0 \tag{8.19}$$

$$u_{xx} - \sigma^2 u_{yy} = -2\gamma(qr)_{xx}$$

where

$$\gamma = \frac{2p}{(2c_1-1)(\hat{\Gamma}-1)}, \quad u = -\frac{Q}{(2c_1-1)(\hat{\Gamma}-1)} \tag{8.20}$$

This system is the system of nonlinear diffusion-antidiffusion equations in 2+1 dimensions.

1. **Bilinear Representation**

   The nonlinear diffusion-antidiffusion system in 2+1 dimensions (8.19) can be represented in bilinear form as

$$\left(D_{\tilde{t}} - \sigma^2 D_{\tilde{x}}^2 - D_{\tilde{y}}^2\right)(g \cdot f) = 0,$$

$$\left(D_{\tilde{t}} + \sigma^2 D_{\tilde{x}}^2 + D_{\tilde{y}}^2\right)(h \cdot f) = 0, \tag{8.21}$$

$$\left(-\sigma^2 D_{\tilde{x}}^2 + D_{\tilde{y}}^2\right)(f \cdot f) = \gamma g h$$

   where $q = g/f$, $r = h/f$ and $u = 4\sigma^2(\ln f)_{xx}$.

2. **One Dissipaton Solution**

   One dissipaton solution for this system is

$$g = e^{\eta_1^+}, \quad h = -e^{\eta_1^-}, \quad f = 1 + \gamma \frac{e^{\eta_1^+ + \eta_1^-}}{2[\sigma^2(k_1^+ + k_1^-)^2 - (m_1^+ + m_1^-)^2]} \tag{8.22}$$

   where $\eta_1^{\pm} = k_1^{\pm}x + m_1^{\pm}y \pm (\sigma^2(k_1^{\pm})^2 + (m_1^{\pm})^2)t + \eta_1^{\pm(0)}$. The regular dissipaton (soliton) for $\sigma^2 = 1$ must be restricted as $|k_1^+ + k_1^-| > |m_1^+ + m_1^-|$ and for $\sigma^2 = -1$ as $|k_1^+ + k_1^-| < |m_1^+ + m_1^-|$.

3. **Two Dissipaton Solution**

$$g = e^{\eta_1^+} + e^{\eta_2^+} + \alpha_1^+ e^{\eta_1^+ + \eta_2^+ + \eta_1^-} + \alpha_2^+ e^{\eta_1^+ + \eta_2^+ + \eta_2^-} \tag{8.23}$$

$$h = -e^{\eta_1^-} - e^{\eta_2^-} + \alpha_1^- e^{\eta_1^+ + \eta_1^- + \eta_2^-} + \alpha_2^- e^{\eta_2^+ + \eta_1^- + \eta_2^-} \tag{8.24}$$

$$f = 1 + \beta_1 e^{\eta_1^+ + \eta_1^-} + \beta_2 e^{\eta_2^+ + \eta_1^-} + \beta_3 e^{\eta_1^+ + \eta_2^-} + \beta_4 e^{\eta_2^+ + \eta_2^-} + \beta_5 e^{\eta_1^+ + \eta_1^- + \eta_2^+ + \eta_2^-} \tag{8.25}$$

   where$\eta_i^{\pm} = k_i^{\pm}x + m_i^{\pm}y \pm (\sigma^2(k_i^{\pm})^2 + (m_i^{\pm})^2)t + \eta_i^{\pm(0)}$ and the constants are

$$\alpha_1^+ = \frac{\gamma}{2} \frac{\sigma^2(\check{k}_{12}^{++})^2 - (\check{m}_{12}^{++})^2}{(k_{11}^{+-})^2 - (m_{11}^{+-})^2)(k_{21}^{+-})^2 - (m_{21}^{+-})^2)} \tag{8.26}$$

$$\alpha_2^+ = \frac{\gamma}{2}\frac{\sigma^2(\breve{k}_{12}^{++})^2 - (\breve{m}_{12}^{++})^2}{(k_{12}^{+-})^2 - (m_{12}^{+-})^2)(k_{22}^{+-})^2 - (m_{22}^{+-})^2)} \tag{8.27}$$

$$\alpha_1^- = -\frac{\gamma}{2}\frac{\sigma^2(\breve{k}_{12}^{--})^2 - (\breve{m}_{12}^{--})^2}{(k_{11}^{+-})^2 - (m_{11}^{+-})^2)(k_{12}^{+-})^2 - (m_{12}^{+-})^2)} \tag{8.28}$$

$$\alpha_2^- = -\frac{\gamma}{2}\frac{\sigma^2(\breve{k}_{12}^{--})^2 - (\breve{m}_{12}^{--})^2}{(k_{21}^{+-})^2 - (m_{21}^{+-})^2)(k_{22}^{+-})^2 - (m_{22}^{+-})^2)} \tag{8.29}$$

$$\beta_1 = \frac{\gamma}{2[\sigma^2(k_{11}^{+-})^2 - (m_{11}^{+-})^2]}, \quad \beta_2 = \frac{\gamma}{2[\sigma^2(k_{21}^{+-})^2 - (m_{21}^{+-})^2]} \tag{8.30}$$

$$\beta_3 = \frac{\gamma}{2[\sigma^2(k_{12}^{+-})^2 - (m_{12}^{+-})^2]}, \quad \beta_4 = \frac{\gamma}{2[\sigma^2(k_{22}^{+-})^2 - (m_{22}^{+-})^2]} \tag{8.31}$$

$$\beta_5 = \frac{4}{\gamma^2}[\sigma^2(\breve{k}_{12}^{++})^2 - (\breve{m}_{12}^{++})^2][\sigma^2(\breve{k}_{12}^{--})^2 - (\breve{m}_{12}^{--})^2]\beta_1\beta_2\beta_3\beta_4 \tag{8.32}$$

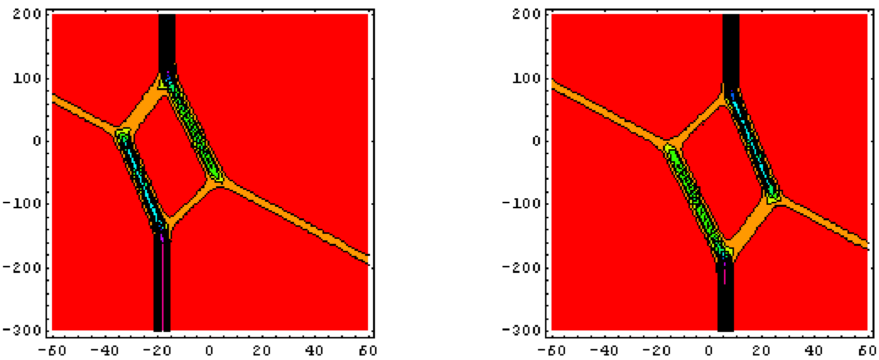


Figure 8.3. One Loop Interaction for Nonlinear Diffusion-Antidiffusion Equation

$$k_1^+ = 0.5,\ k_1^- = 0.9,\ k_2^+ = 0.1,\ k_2^- = 0.35,$$
$$m_1^+ = 0.1,\ m_1^- = -0.1,\ m_2^+ = 0.2,\ m_2^- = 10^{-7}$$
$$\eta_1^{+0} = 10,\ \eta_1^- = 0,\ \eta_2^+ = 10,\ \eta_2^- = 0,\ \gamma = 1,\ \alpha = 1,\ \sigma^2 = 1$$

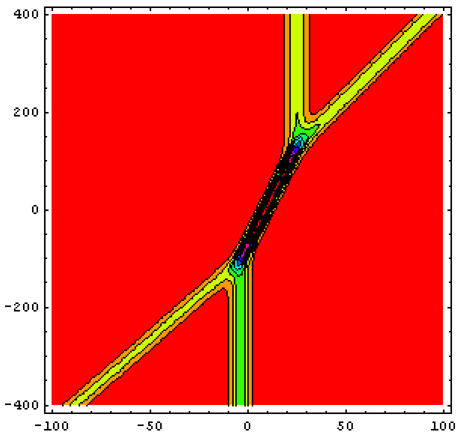


Figure 8.4. Zero Loop Interaction for Nonlinear Diffusion-Antidiffusion Equation

$$k_1^+ = 0.9,\ k_1^- = -0.5,\ k_2^+ = 0.1,\ k_2^- = -0.45,$$
$$m_1^+ = 0.1,\ m_1^- = -0.1,\ m_2^+ = 0.2,\ m_2^- = -0.1$$
$$\eta_1^{+0} = -5,\ \eta_1^- = -5,\ \eta_2^+ = 4,\ \eta_2^- = 4,\ \gamma = 1,\ \alpha = 1,\ \sigma^2 = 1 \text{ at } t = 3$$

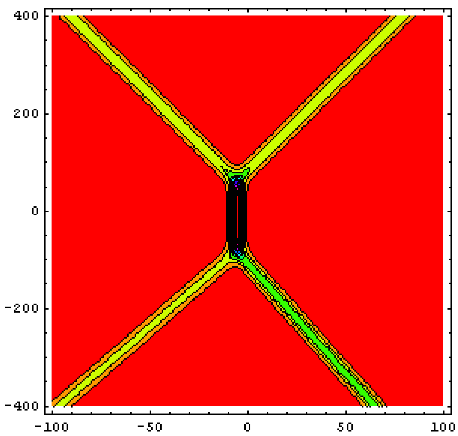


Figure 8.5. Zero Loop Interaction for Nonlinear Diffusion-Antidiffusion Equation

$$k_1^+ = 0.9,\ k_1^- = -0.5,\ k_2^+ = 0.1,\ k_2^- = -0.45,$$
$$m_1^+ = 0.2,\ m_1^- = -0.1,\ m_2^+ = 0.2,\ m_2^- = -0.1$$
$$\eta_1^{+0} = -7,\ \eta_1^- = -7,\ \eta_2^+ = 4,\ \eta_2^- = 4,\ \gamma = 1,\ \alpha = 1,\ \sigma^2 = 1 \text{ at } t = 20$$

In Figures 8.3, 8.4 and 8.5, depending on the special values of parameters, the resonance interaction of solitons are shown as counterplot graphics.

CHAPTER 9

CONCLUSION

In this thesis we have studied the resonance generalization of the well known equations governing wave motions, namely long-short wave interaction and the Davey-Stewartson system, by addition of quantum potential. We showed that the behavior of these systems under quantum potential can be reduced to two different cases and both cases can be decomposed into bilinear representation which admits us to calculate solutions exactly. The existence of the Lax pair for each system gives an opportunity to solve initial value problems by ISM. However, the response of the systems create different structure for these solutions which is called dissipatons for overcritical cases. Resonance type of collisions for dissipatons of these equations are visualized and also showed analytically. Counter plots of each case imitates interaction of quantum particles and structure of solution can include number of loops between zero and $n-1$ for n soliton solutions, which shows the web-like structure. Explicitly we calculated results for $n=2$ and $n=3$.

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APPENDIX A

AIRY FUNCTION

Airy equation is a linear homogeneous second order differential equation.

$$y''(x) - xy(x) = 0 \tag{A.1}$$

Seeking a solution as an integral

$$y(x) = \int_C e^{xz} v(z) dz \tag{A.2}$$

is equivalent to solve first order equation

$$v'(z) - z^2 v(z) = 0 \tag{A.3}$$

Thus we have the solution of (A.1)

$$y(x) = \int_C exp\left(xz + \frac{z^3}{3}\right) dz \tag{A.4}$$

which is called Airy function and represented as $Ai(x)$. This function can also be defined in the following integral forms.

$$Ai(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} expi\left(xz + \frac{z^3}{3}\right) dz \tag{A.5}$$

$$Ai(x) = \frac{1}{\pi} \int_{-\infty}^{\infty} \cos\left(xz + \frac{z^3}{3}\right) dz \tag{A.6}$$

$$Ai(kx) = \frac{1}{k\pi} \int_{-\infty}^{\infty} \cos\left(xz + \frac{z^3}{3k^3}\right) dz \tag{A.7}$$

The asymptotic behavior of the Airy functions as x goes to ∞ is given by (Abramowitz and Stegun 1970)

$$\lim_{x \to \infty} Ai(x) = \frac{\exp(-\frac{2}{3}x^{3/2})}{2\sqrt{\pi}x^{1/4}} \tag{A.8}$$

APPENDIX B

LAX PAIR

In his 1968 seminal paper, Lax developed an formalism for nonlinear equations (Lax 1968). His formulation has the feature of associating certain nonlinear evolution equations with linear equations which are analogs of the Schödinger equation for the KdV equation (Novikov, et al. 1984).

To formulate Lax's Method, we consider two linear operators L and M. The eigenvalue equation related to the operator L corresponds to the Schrödinger equation for the KdV equation. The general form of this eigenvalue equation is

$$L\Psi = \lambda\Psi \tag{B.1}$$

where Ψ is the eigenfunction and λ is the corresponding eigenvalue. The operator M describes the change of eigenvalues with the parameter t, which usually represents time in a nonlinear evolution equation. The general form of this evolution is

$$\Psi_t = M\Psi \tag{B.2}$$

Differentiating (B.1) with respect to t gives

$$L_t\Psi + L\Psi_t = \lambda_t\Psi + \lambda\Psi_t \tag{B.3}$$

We next eliminate Ψ_t and obtain

$$\frac{\partial L}{\partial t}\Psi = \lambda_t\Psi + (ML - LM)\Psi \tag{B.4}$$

Thus, eigenvalues are constant for nonzero eigenfunctions if and only if

$$\frac{\partial L}{\partial t} = -(LM - ML) = -[L, M] \tag{B.5}$$

where $[L, M] = (LM - ML)$ is called the commutator of the operators L and M. The equation (B.5) is the Lax equation and the operators L and M are called the Lax pair. Any evolution equations solvable by IST, can be represented in Lax form. However, the main difficulty is that there is no completely systematic method of finding whether or not a given partial differential equation produces a Lax equation and, if so, how to find the Lax pair L and M (Debnath 2005).

If we set

$$L = -6\frac{d^2}{dx^2} - u, \quad M = -4\frac{d^3}{dx^3} - u\frac{d}{dx} - \frac{1}{2}u_x \tag{B.6}$$

the Lax equation is equivalent to the KdV equation.

$$u_t + uu_x + u_{xxx} = 0$$

To represent the KdV equation in another way, we write

$$\phi_x = U\phi, \quad \phi_t = V\phi \tag{B.7}$$

and the compatibility condition now takes the form of a matrix equation, the zero-curvature condition:

$$U_t - V_x + UV - VU = 0$$

where

$$U = \begin{pmatrix} 0 & 1 \\ -\frac{u+\lambda}{6} & 0 \end{pmatrix}, \quad V = \begin{pmatrix} \frac{1}{6}u_x & \frac{1}{3}(2\lambda - u) \\ -\frac{1}{9}\lambda^2 - \frac{1}{18}\lambda u + \frac{1}{18}u^2 + \frac{1}{6}u_{xx} - \frac{1}{6}u_x & -\frac{1}{6}u_x \end{pmatrix}$$

The name comes from a geometrical interpretation. The equations $(\partial x - U)\phi = 0$ and $(\partial t - V)\phi = 0$ define a connection on a two-dimensional vector bundle over the (x,t)-plane. The first equation describes how to parallel-translate a vectorϕ in the x-direction, and the second equation describes how to parallel-translate a vector ϕ in the t-direction. The matrices U and V are the connection coefficients. A connection is said to have zero curvature if parallel translation of a vector ϕ along a path from a point (x_0,t_0) to another point (x_1,t_1) gives the same result independent of path connecting the points. This is the same thing as asserting the existence of a full two-dimensional basis of simultaneous solutions of the equations $(\partial x - U)\phi = 0$ and $(\partial t - V)\phi = 0$, which is the above zero-curvature condition that must be satisfied by the connection coefficients (Dubrovin, et al. 1992). Therefore, every solution of the KdV equation defines a connection with zero curvature.