SUBMODULES THAT HAVE SUPPLEMENTS

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ABSTRACT

SUBMODULES THAT HAVE SUPPLEMENTS

In this thesis we study the κ-elements of extension modules where R is a principal ideal domain. In general κ-elements need not form a submodule in an extension module but if C is divisible and almost all primary components of C are zero, they coincide with torsion elements of extension module. If C is divisible and torsion, not all primary components of C are zero, and A is torsion-free of rank 1, then a nonzero element of extension module is a κ-element if and only if the type of the element in extension module is less than or equal to the type of A. Also we define β-elements which form a submodule of extension module and study their relation with κ-elements.

ÖZET

TÜMLEYENİ OLAN ALTMODÜLLER

Bu tezde, R temel idealler bölgesi olmak üzere genişleme modülünün κ-elemanları incelenmiştir. Genel durumda κ-elemanlar genişleme modülünün bir altmodülünü oluşturmayabilir, fakat C bölünebilir modül ise ve C'nin hemen hemen tüm asal bileşenleri sıfır ise, κ-elemanlar genişleme modülünün burulma elemanlarıyla çakışıyor. C bölünebilir burulma modülü ise, C'nin asal bileşenleri hepsi aynı anda sıfır değilse, ve A rankı 1 olan burulmasız modül ise, genişleme modülünün sıfırdan farklı elemanının κ-eleman olması için gerek ve yeterli koşul elemanın genişleme modülündeki tipinin A'nın tipinden küçük veya eşit olmasıdır. Ayrıca genişleme modülünün bir altmodülünü oluşturan β-elemanları tanımladık ve bunların κ-elemanlarla bağlantılarını inceledik.

TABLE OF CONTENTS

CHAPTER 1

INTRODUCTION

In order to study the structure of the module $\mathrm{Ext}_R(C, A)$, one can try to determine the elements of the standard submodules, e.g. in the case of abelian groups divisible part of $D(\mathrm{Ext}(C, A))$, Ulm's subgroup $\mathrm{Ext}(C, A)^1$, Frattini subgroup $\mathrm{Rad}(\mathrm{Ext}(C, A))$ or torsion subgroup $T(\mathrm{Ext}(C, A))$ in details as in (Fuchs 1970). Interpreting $\mathrm{Ext}(C, A)$ as a module of extensions of A by C, the question is to find the properties of the short exact sequence

$$E = 0 \longrightarrow A \xrightarrow{\alpha} B \xrightarrow{\beta} C \longrightarrow 0$$

of modules so that equivalence class $[E]$ is an element of the prescript submodule of $\mathrm{Ext}_R(C, A)$. For example, in a well-known manner, $[E]$ belongs to $D(\mathrm{Ext}(C, A))$ if and only if $\mathrm{Im}\,\alpha$ is a direct summand in $\mathrm{Im}\,\alpha + T(B)$ (see e.g. (Fuchs 1970)).

Conversely, given any class $\mathscr{C}$ of short exact sequences of modules, the problem of finding the corresponding elements in the module $\mathrm{Ext}_R(C, A)$ arises. Perhaps the best known example is the class $\mathscr{P}$ of pure-exact sequences: if R is a principal ideal domain the elements of $\mathrm{Ext}_R(C, A)$ with the representation from $\mathscr{P}$ form a submodule $\mathrm{Pext}_R(C, A)$ which coincides with Ulm's submodule $\mathrm{Ext}_R(C, A)^1$. Our interest in this thesis is the class of κ-exact sequences, where E is called *κ-exact* if $\mathrm{Im}\,\alpha$ has a supplement in B, i.e. a minimal element in the set $\{V \subset B | V + \mathrm{Im}\,\alpha = B\}$. In this case we say that $[E] \in \mathrm{Ext}_R(C, A)$ is a *κ-element* and the set of all κ-elements will be denoted by $\mathrm{Ext}_R(C, A)^\kappa$. For abelian groups the properties of κ-elements were studied in (Zöschinger 1978). We generalize these results and give the "description" of κ-elements for modules over a principal ideal domain R in the following two cases:

(I) *C is divisible and almost all primary components of C are zero.* In this case the κ-elements coincide with torsion elements of $\mathrm{Ext}_R(C, A)$ for arbitrary A.

(II) *C is divisible and torsion, primary components of C are all nonzero, and A is torsion-free of rank 1.* Although then $\mathrm{Ext}_R(C, A)$ is torsion-free, there are still sufficiently many κ-elements in $\mathrm{Ext}_R(C, A)$, because as it will be shown in Chapter

7, they form a generating system. And from our main result Theorem 7.1 it follows: $0 \neq [\mathrm{E}] \in \mathrm{Ext}_R(C, A)$ is a κ-element if and only if the type of $[E]$ in $\mathrm{Ext}_R(C, A)$ is less than or equal to the type of A.

In Chapter 3, the relation between torsion and κ-elements of $\mathrm{Ext}_R(C, A)$ is investigated where a "functorial" subgroup $\mathrm{Ext}_R(C, A)^\beta$ is introduced as the set of elements $[0 \longrightarrow A \xrightarrow{\alpha} B \longrightarrow C \longrightarrow 0]$ such that $\mathrm{Im}\,\alpha$ has a supplement V in B and such that $V \cap \mathrm{Im}\,\alpha$ is bounded. Although in general $\mathrm{Ext}_R(C, A)^\beta \subsetneqq \mathrm{Ext}_R(C, A)^\kappa$ holds, certain statements about $\mathrm{Ext}_R(C, A)^\beta$ can be used while studying κ-elements. For example: If $\mathrm{Ext}_R(C, A)^\beta \subset \mathrm{Rad}(\mathrm{Ext}_R(C, A))$ holds, then it follows $\mathrm{Ext}_R(C, A)^\beta = 0$ and we will show the same in for κ instead of β Theorem 5.1.

Since the κ-elements are preserved under $\mathrm{Ext}(g, f) : \mathrm{Ext}_R(C, A) \to \mathrm{Ext}_R(C', A')$ with respect to the second variable, but not the first variable (and therefore $\mathrm{Ext}_R(C, A)^\kappa$ need not be a submodule), we study in Chapter 4, the homomorphisms $g : C' \to C$ where we have the decomposition $g = \beta \circ \alpha$: If β is a small epimorphism (i.e. surjective with small kernel), then β is an isomorphism. We call it coneat and show that this is equivalent with $g(\mathrm{Soc}(C')) = \mathrm{Soc}(C)$. And for such g, also $g^* : \mathrm{Ext}_R(C, A) \to \mathrm{Ext}_R(C', A)$ preserves κ-elements. Together with the dual concept of neat homomorphisms, like the concept of Enochs introduced while studying torsion-free coverings, it will be shown in Theorem 4.3 the functors Hom resp. Ext preserves neat (coneat) homomorphisms (and other variations).

In Chapter 5, the question when $\mathrm{Ext}_R(C, A)^\kappa = \mathrm{Ext}_R(C, A)^\beta$ is investigated. The extremal case $\mathrm{Ext}_R(C, A)^\kappa = 0$ is quickly solved by Theorem 3.2: There is no κ-element in $\mathrm{Ext}(C, A)$ if and only if the inequality $T_p(C) \neq 0$ implies the divisibility of $T_p(A)$, and if, all $T_p(C) \neq 0$, A is already divisible. Theorem 5.2 gives the answer to the original question at least in the case when $T(A)$ is a direct summand of A.

Chapter 6 summarizes the results about p-height, the property of divisibility, which we call the p-depth of $x \in G$: $t_p^G(x)$ is defined as the smallest p-power which divides x, but the quotient is no more divisible by prime element p of R. By the introduction of this depth concept we can reduce the condition "C divisible", in the case(II) which is mentioned above and with its help we measure the complement characteristics of $x \in \mathrm{Ext}_R(C, A)$. For $x \in G$ we have $t_p^G(x) = min(h_p^G(x), t_p^G(0))$, and for $\varphi \in \mathrm{Hom}(M, R(p^\infty))$, we obtain the formula

$t_p^{\mathrm{Hom}}(\varphi) = inf\{i \in \mathbb{N} \mid M[p] \nsubseteq p^i(\mathrm{Ker}\,\varphi)\}$. Finally we point to the characterization of $t_p^G(0)$ which is the dual statement of a well-known Theorem of Khabbaz (Khabbaz 1961): If, V is a supplement of $G[p^n]$ in G, then V is a direct summand in G.

In the last two chapters, C is torsion, and A is torsion-free of rank 1. Everything follows from the main result Theorem 7.1: $0 \neq [E]$ is an κ-element in $\mathrm{Ext}_R(C, A)$ if and only if the primary components of C are all nonzero and the depth of the class of $[E]$ in $\mathrm{Ext}_R(C, A)$ less than or equal to the type of A. Simple criteria for the fact is that $\mathrm{Ext}_R(C, A)$ consists only of κ-elements or that $g^* : \mathrm{Ext}_R(C, A) \to \mathrm{Ext}_R(C', A)$ preserves κ-elements. And it is shown in Theorem 7.3 that, difference between depth and height gives really different supplement concepts. Since in the sequence $[E]$, given above in which $\mathrm{Im}\,\alpha$ is small in B, $T(B)$ need not be splitting in B, we give a "Splitting Criterion" which is interesting itself.

In Chapter 8, we consider a triple $\mathrm{Im}\,\alpha \subset X \subset B$, such that $\mathrm{Im}\,\alpha$ has a supplement in X and X has a supplement in B. Also these elements of $\mathrm{Ext}_R(C, A)$ give a depth sequence described in Theorem 8.1, and it is shown that the κ-elements form a proper big subset.

Throughout R is a principal ideal domain. By module we will mean a left R-module. K is the field of fractions of R. We write $R(p^\infty)$ for the p-primary component of K/R. If a module has a composition series $0 = M_0 \subsetneq M_1 \subsetneq \ldots \subsetneq M_n = M$, M is called a module of length n, and is denoted by $L(M)$. For a homomorphism $\alpha : A \to B$, $\mathrm{Coker}\,\alpha = B/\mathrm{Im}\,\alpha$. "$U$ is a direct summand in M" and "U has a supplement in M" will be denoted by $U \subset^{\oplus} M$ and $U \subset^{\kappa} M$ respectively. For undefined terms and simple facts see (Fuchs and Salce 2001) and (Kaplansky 1969).

CHAPTER 2

PRELIMINARIES

This Chapter is a short summary of Chapter IX from (Fuchs 1970) and Chapter 3 from (Mac Lane 1995), so one can find missing proofs in (Fuchs 1970) and (Mac Lane 1995).

2.1. Extensions as Short Exact Sequences

If the extension B of A by C is visualized as an exact sequence

$$0 \longrightarrow A \xrightarrow{\mu} B \xrightarrow{\nu} C \longrightarrow 0\,,$$

then one can try to build up a category in which the objects are just the short exact sequences. An adequate definition of a morphism between two exact sequences is rather clear: it is a triple (α, β, γ) of module homomorphisms such that the diagram

$$\begin{tikzcd} E : 0 \arrow[r] & A \arrow[r, "\mu"] \arrow[d, "\alpha"'] & B \arrow[r, "\nu"] \arrow[d, "\beta"] & C \arrow[r] \arrow[d, "\gamma"] & 0 \\ E' : 0 \arrow[r] & A' \arrow[r, "\mu'"] & B' \arrow[r, "\nu'"] & C' \arrow[r] & 0 \end{tikzcd} \tag{1}$$

has commutative squares. It is straightforward to show that in this way a category $\mathscr{E}$ arises.

In accordance with the definition of equivalent extensions, we say that the extensions E and E' with $A = A'$, $C = C'$ are *equivalent*, in sign: $E \equiv E'$, if there is a morphism $(1_A, \beta, 1_C)$ with $\beta : B \to B'$ is an isomorphism. Actually, the condition β being an isomorphism can be omitted, since this follows from (Fuchs 1970) (2.3).

First we study extensions with A fixed. If $\gamma : C' \to C$ is any homomorphism, then to the extension E in (1), there is, by (Fuchs 1970) (10.1), a pullback square

$$\begin{tikzcd} & & B' \arrow[r, "\nu'"] \arrow[d, "\beta"'] & C' \arrow[d, "\gamma"] & \\ 0 \arrow[r] & A \arrow[r, "\mu"] & B \arrow[r, "\nu"] & C \arrow[r] & 0 \end{tikzcd}$$

with suitable B', β and ν'. From (Fuchs 1970) 10 we know that ν' is epic [since ν is epic], and a glance at (3) in (Fuchs 1970) 10 shows that $\operatorname{Ker}\nu' \cong \operatorname{Ker}\nu \cong A$, hence there is a monomorphism $\mu' : A \to B'$ [namely, $\mu' a = (\mu a, 0) \in B'$ if $B' \leqq B \oplus C'$] such that the diagram

$$\begin{tikzcd} E_\gamma : & 0 \arrow[r] & A \arrow[r, "\mu'"] \arrow[d, equal] & B' \arrow[r, "\nu'"] \arrow[d, "\beta"] & C' \arrow[r] \arrow[d, "\gamma"] & 0 \\ E : & 0 \arrow[r] & A \arrow[r, "\mu"] & B \arrow[r, "\nu"] & C \arrow[r] & 0 \end{tikzcd}$$

with exact rows and pullback right square commutes. The top row is an extension of A by C' which we have denoted by $E\gamma$ to indicate its origin from E and γ. Notice that $\gamma^* = (1_A, \beta, \gamma)$ is a morphism $E\gamma \to E$ in $\mathscr{E}$.

If the diagram

$$\begin{tikzcd} E^\circ : & 0 \arrow[r] & A^\circ \arrow[r, "\mu^\circ"] \arrow[d, equal] & B' \arrow[r, "\nu^\circ"] \arrow[d, "\beta^\circ"] & C' \arrow[r] \arrow[d, "\gamma"] & 0 \\ E : & 0 \arrow[r] & A \arrow[r, "\mu"] & B \arrow[r, "\nu"] & C \arrow[r] & 0 \end{tikzcd}$$

has exact rows and commutes, then by (Fuchs 1970) (10.1) there is a unique $\phi : B^\circ \to B'$ such that $\nu'\phi = \nu^\circ$ and $\beta\phi = \beta^\circ$. Since the maps $\phi\mu^\circ, \mu' : A \to B'$ are such that $\beta(\phi\mu^\circ) = \beta^\circ\mu^\circ = \mu = \beta\mu'$ and $\nu'(\phi\mu^\circ) = \nu^\circ\mu^\circ = 0 = \nu'\mu'$, the uniqueness assertion in (Fuchs 1970) (10.1) implies $\phi\mu^\circ = \mu'$. This shows that $E\gamma$ is unique up equivalence and this yields the equivalences

$$E1_C \equiv E \qquad \text{and} \qquad E(\gamma\gamma') \equiv (E\gamma)\gamma'$$

for $C'' \xrightarrow{\gamma'} C' \xrightarrow{\gamma} C$. Now the contravariance of E on C is evident.

Next we keep C fixed and let A vary. Given $\alpha : A \to A'$, let B' be defined by the pushout square

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, "\mu"] \arrow[d, "\alpha"] & B \arrow[r, "\nu"] \arrow[d, "\beta"] & C \arrow[r] & 0 \\ & A' \arrow[r, "\mu'"] & B' & & \end{tikzcd}$$

Here μ' is a monomorphism. Moreover, if B' is defined as a quotient module of $A' \oplus B$, then $\nu'(a', b) + H = \nu b$ makes the diagram

$$\begin{tikzcd} E : & 0 \arrow[r] & A \arrow[r, "\mu"] \arrow[d, "\alpha"] & B \arrow[r, "\nu"] \arrow[d, "\beta"] & C \arrow[r] \arrow[d, equal] & 0 \\ \alpha E : & 0 \arrow[r] & A' \arrow[r, "\mu'"] & B' \arrow[r, "\nu'"] & C \arrow[r] & 0 \end{tikzcd}$$

with exact rows commutative. The bottom row is an extension of A' by C which we have denoted by αE. Here $\alpha_* = (\alpha, \beta, 1_C)$ is a morphism $E \to \alpha E$ in $\mathscr{E}$.

If

$$\begin{array}{llllllllll} E: & 0 & \longrightarrow & A & \xrightarrow{\mu} & B & \xrightarrow{\nu} & C & \longrightarrow & 0 \\ & & & \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta_\circ} & & \| & & \\ E_\circ: & 0 & \longrightarrow & A' & \xrightarrow{\mu_\circ} & B_\circ & \xrightarrow{\nu_\circ} & C & \longrightarrow & 0 \end{array}$$

is a commutative diagram with exact rows, then in view of (Fuchs 1970) (10.2) there exists a unique $\phi : B' \to B_\circ$ such that $\phi\beta = \beta_\circ$ and $\phi\mu' = \mu_\circ$. From $(\nu_\circ\phi)\beta = \nu_\circ\beta_\circ = \nu\nu'\beta$, $(\nu_\circ\phi)\mu' = 0 = \nu'\mu'$ we infer that $\nu_\circ\phi = \nu'$, thus $(1_{A'}, \phi, 1_C)$ is a morphism $\alpha E \to E_\circ$. Consequently, $\alpha E \equiv E_\circ$, i.e., αE is unique up to equivalence. Hence

$$1_A E \equiv E \qquad \text{and} \qquad (\alpha\alpha')E \equiv \alpha(\alpha' E)$$

for $A \xrightarrow{\alpha} A' \xrightarrow{\alpha'} A''$, establishing the covariant dependence of E on A.

With $\alpha : A \to A'$ and $\gamma : C' \to C$ we have the important associative law

(2) $$\alpha(E\gamma) \equiv (\alpha E)\gamma.$$

Indeed, by making use of the pullback property of $(\alpha E)\gamma$, it is easy to prove the existence of a morphism $(\alpha, \beta', 1) : E\gamma \to (\alpha E)\gamma$ and to show the commutativity of the square

$$\begin{tikzcd} E_\gamma \arrow[r, "{(1,\beta_1,\gamma)}"] \arrow[d, "{(\alpha,\beta',1)}"'] & E \arrow[d, "{(\alpha,\beta_2,1)}"] \\ (\alpha E)\gamma \arrow[r, "{(1,\beta,\gamma)}"] & \alpha E. \end{tikzcd}$$

Assume we are given two extensions E_1 and E_2 of A by C. The extensions of A by C [more correctly their equivalence classes] form a module.

In order to describe the module operation in the language of short exact sequences, we make use of diagonal map $\triangle_G : g \mapsto (g, g)$ and the codiagonal map $\nabla_G : (g_1, g_2) \mapsto g_1 + g_2$ of a module G. If we understand by the *direct sum* of two extensions

$$E_i: \quad 0 \longrightarrow A_i \xrightarrow{\mu_i} B_i \xrightarrow{\nu_i} C_i \longrightarrow 0 \qquad (i = 1, 2)$$

the extension

$$E_1 \oplus E_2 : 0 \longrightarrow A_1 \oplus A_2 \xrightarrow{\mu_1 \oplus \mu_2} B_1 \oplus B_2 \xrightarrow{\nu_1 \oplus \nu_2} C_1 \oplus C_2 \longrightarrow 0\,,$$

then we have :

Proposition 2.1 *The sum of extensions E_1, E_2 of A by C is the extension*

$$E_1 + E_2 = \nabla_A(E_1 \oplus E_2)\triangle_C. \tag{3}$$

Proof What we have to verify is that if $f_i : C \times C \to A$ is a factor set belonging to E_i ($i = 1, 2$), then $f_1 + f_2$ belongs to $\nabla_A(E_1 \oplus E_2)\triangle_C$. Clearly, $(f_1(c_1, c_2), f_2(c_1{}', c_2{}'))$ with $c_i, c_i{}' \in C$ is a factor set belonging to the direct sum $E_1 \oplus E_2$, and $(f_1(c_1, c_2), f_2(c_1, c_2))$ is one corresponding to $(E_1 \oplus E_2)\triangle_C$. An application of ∇_A yields the factor set $f_1(c_1, c_2) + f_2(c_1, c_2)$. □

It is of course possible to avoid any reference to factor sets and to develop extensions solely *qua* short exact sequences. In doing so, (3) would serve as the definition of the sum of extensions and then Proposition 2.1 should be replaced by the assertion that $E_1 + E_2$ is actually an extension of A by C which stays in the same equivalence class if E_1 and E_2 are replaced by equivalent extensions, and moreover, the equivalence classes of extensions form a module under this operation.

From what has been said above about the factor sets belonging to $E\gamma$ and αE it is now evident that for some homomorphisms $\alpha : A \to A'$ and $\gamma : C' \to C$, the following equivalences hold true for extensions E_1, E_2, E of A by C:

$$\alpha(E_1 + E_2) \equiv \alpha E_1 + \alpha E_2, \qquad (E_1 + E_2)\gamma \equiv E_1\gamma + E_2\gamma, \tag{4}$$

$$(\alpha_1 + \alpha_2)E \equiv \alpha_1 E + \alpha_2 E, \qquad E(\gamma_1 + \gamma_2) \equiv E\gamma_1 + E\gamma_2. \tag{5}$$

The equivalences of (4) express the fact that $\alpha_* : E \mapsto \alpha E$ and $\gamma^* E \mapsto E\gamma$ are module homomorphisms

$\alpha_* : \mathrm{Ext}_R(C, A) \to \mathrm{Ext}_R(C, A')$, $\qquad \gamma^* : \mathrm{Ext}_R(C, A) \to \mathrm{Ext}_R(C', A)$,

while (5) asserts that $(\alpha_1 + \alpha_2)_* = (\alpha_1)_* + (\alpha_2)_*$ and $(\gamma_1 + \gamma_2)^* = (\gamma_1)^* + (\gamma_2)^*$, i.e., the correspondence

$$\mathrm{Ext}_R : C \times A \mapsto \mathrm{Ext}_R(C, A), \qquad \gamma \times \alpha \mapsto \gamma^*\alpha_* = \alpha_*\gamma^*$$

is an additive bifunctor on $\mathscr{A} \times \mathscr{A}$ to $\mathscr{A}$ [the last equality is just another form of (2)].

Theorem 2.1 Ext_R *is an additive bifunctor on $\mathscr{A} \times \mathscr{A}$ to $\mathscr{A}$ which is contravariant in the first and covariant in the second variable.* □

In order to be consistent with the functorial notation for homomorphisms, we shall use the notation

$$\mathrm{Ext}_R(\gamma,\alpha) : \mathrm{Ext}_R(C,A) \to \mathrm{Ext}_R(C',A')$$

instead of $\gamma^*\alpha_* = \alpha_*\gamma^*$; that is, $\mathrm{Ext}_R(\gamma,\alpha)$ acts as shown by

$$\mathrm{Ext}_R(\gamma,\alpha) : E \mapsto \alpha E\gamma.$$

Let us keep in the mind that if the extension E is given by (1), then for $\gamma : C' \to C$, $E\gamma$ is represented by $0 \longrightarrow A \xrightarrow{\mu'} B' \xrightarrow{\nu'} C' \longrightarrow 0$ where

(6) $\quad B' = \{(b,c') \mid b \in B, c' \in C', \nu b = \gamma c'\}, \mu' a = (\mu a, 0), \nu'(b,c') = c',$

and for $\alpha : A \to A'$, αE is represented by $0 \longrightarrow A' \xrightarrow{\mu'} B' \xrightarrow{\nu'} C \longrightarrow 0$ where

$$(7)\qquad \begin{aligned} B' &= \{(a'+b)+H \mid a' \in A', b \in B\}, \\ \mu' a' &= (a',0)+H, \qquad \nu'((a',b)+H) = \nu b \end{aligned}$$

with $H = \{(\alpha a - \mu a) \mid a \in A\}$. These formulas for $E\gamma$ and αE are helpful in subsequent computations.

2.2. Exact Sequences for Ext_R

As we have seen in the preceding section, Ext_R is a functor in both of its variables. The main result of this section states that this functor is right exact, moreover, the exact sequences on Hom and Ext_R can be amalgamated into long exact sequences.

Given an extension

$$(1) \qquad E : 0 \longrightarrow A \xrightarrow{\alpha} B \xrightarrow{\beta} C \longrightarrow 0$$

representing an element of $\mathrm{Ext}_R(C,A)$, and a homomorphism $\eta : A \to G$, we know from the preceding section that ηE is an extension of G by C, i.e., ηE represents an element of $\mathrm{Ext}_R(C,G)$. In this way we get a map

$$E^* : \mathrm{Hom}(A,G) \to \mathrm{Ext}_R(C,G)$$

defined as

$$E^* : \eta \mapsto \eta E.$$

Analogously, a homomorphism $\xi : G \to C$ yields from E an extension $E\xi$ of A by G, and

$$E_* : \mathrm{Hom}(G, C) \to \mathrm{Ext}_R(G, A)$$

is a map acting as follows:

$$E_* : \xi \mapsto E\xi.$$

From (5) in the previous section it results at once that E^* and E_* are homomorphisms. They are natural, for if $\phi : G \to H$ is any homomorphism, then because of $(\phi\eta)E \equiv \phi(\eta E)$ and $E(\xi\phi) \equiv (E\xi)\phi$ the diagrams

$$\begin{tikzcd} \mathrm{Hom}(A,G) \arrow[r] \arrow[d] & \mathrm{Ext}_R(C,G) \arrow[d] \\ \mathrm{Hom}(A,H) \arrow[r] & \mathrm{Ext}_R(C,H) \end{tikzcd} \qquad \begin{tikzcd} \mathrm{Hom}(H,C) \arrow[r] \arrow[d] & \mathrm{Ext}_R(H,A) \arrow[d] \\ \mathrm{Hom}(G,C) \arrow[r] & \mathrm{Ext}_R(G,A) \end{tikzcd}$$

with the obvious maps commute. E^* and E_* are called *connecting homomorphisms* for the short exact sequence (1). This terminology is justified in the light of Theorem 2.2.

Before stating this theorem, we prove two technical lemmas.

Lemma 2.1 *Given a diagram*

$$\begin{tikzcd} E: & 0 \arrow[r] & A \arrow[r, "\alpha"] \arrow[d, "\eta"] & B \arrow[r, "\beta"] \arrow[dl, dashed, "\xi"] & C \arrow[r] & 0 \\ & & G & & & \end{tikzcd}$$

with exact row, there exists a $\xi : B \to G$ making the triangle commute if and only if ηE splits.

Proof If there is such a ξ, then the diagram

$$\begin{tikzcd} E: & 0 \arrow[r] & A \arrow[r, "\alpha"] \arrow[d, "\eta"] & B \arrow[r, "\beta"] \arrow[d, "(\xi\oplus\beta)\triangle"] & C \arrow[r] \arrow[d, equal] & 0 \\ & 0 \arrow[r] & G \arrow[r, "(1_G\oplus 0)\triangle"] & G \oplus C \arrow[r, "\nabla(0\oplus 1_C)"] & C \arrow[r] & 0 \end{tikzcd}$$

commutes hence the bottom row is $\equiv \eta E$. Conversely, if $\eta E : 0 \to G \to B' \to C \to 0$ splits, then $B \to B'$ followed by the projection $B' \to G$ yields a map ξ with the desired property. $\square$

The dual of this argument establishes the exact dual of preceding lemma:

Lemma 2.2 *If the diagram*

$$\begin{tikzcd} & & & G \arrow[dl, dashed, "\xi"'] \arrow[d, "\eta"] & \\ E: \quad 0 \arrow[r] & A \arrow[r, "\alpha"] & B \arrow[r, "\beta"] & C \arrow[r] & 0 \end{tikzcd}$$

has exact row, then there is a $\xi : G \to B$ such that $\beta\xi = \eta$ if, and only if, $E\eta$ splits.

With the aid of these lemmas, the following theorem on the exact sequences for Ext_R becomes a straightforward, though mildly intricate calculation.

Theorem 2.2 *If* (1) *is an exact sequence, then the sequences*

$$(2) \qquad \begin{aligned} &0 \longrightarrow \mathrm{Hom}(C,G) \longrightarrow \mathrm{Hom}(B,G) \longrightarrow \mathrm{Hom}(A,G) \longrightarrow \\ &\xrightarrow{E^*} \mathrm{Ext}_R(C,G) \xrightarrow{\beta^*} \mathrm{Ext}_R(B,G) \xrightarrow{\alpha^*} \mathrm{Ext}_R(A,G) \longrightarrow 0, \end{aligned}$$

$$(3) \qquad \begin{aligned} &0 \longrightarrow \mathrm{Hom}(G,A) \longrightarrow \mathrm{Hom}(G,B) \longrightarrow \mathrm{Hom}(G,C) \longrightarrow \\ &\xrightarrow{E_*} \mathrm{Ext}_R(G,A) \xrightarrow{\beta_*} \mathrm{Ext}_R(G,B) \xrightarrow{\alpha_*} \mathrm{Ext}_R(G,C) \longrightarrow 0, \end{aligned}$$

are exact for every module G.

Proof Owing to (Fuchs 1970) (44.4) we may begin the proof of exactness of (2) at $\mathrm{Hom}(A,G)$. We have to show that $\eta : A \to G$ is extendable to $\xi : B \to G$ exactly if $\eta E \in \mathrm{Ext}_R(C,G)$ is splitting; but this is just the statement of Lemma 2.1. The next step is to show the exactness at $\mathrm{Ext}_R(C,G)$. By Lemma 2.2, $E\beta$ splits, thus for $\eta \in \mathrm{Hom}(A,G)$, $\beta^*E^*\eta = \eta E\beta = 0$. Let $E_1 : 0 \longrightarrow G \xrightarrow{\mu} H \xrightarrow{\nu} C \longrightarrow 0$ $\in \mathrm{Ext}_R(C,G)$ be such that $E_1\beta$ splits. By Lemma 2.2, there is a $\xi : B \to H$ such that $\nu\xi = \beta$. Since $\nu\xi\alpha = \beta\alpha = 0$ by (Fuchs 1970) (2.1) there is an $\eta : A \to G$ satisfying $\mu\nu = \xi\alpha$, hence $(\eta, \xi, 1_C)$ maps E upon E_1, i.e., $E_1 = \eta E$. To show exactness at $\mathrm{Ext}_R(B,G)$, notice that obviously $\alpha^*\beta^* = (\beta\alpha)^* = 0^* = 0$. Conversely, to prove that the kernel is contained in image, let $E_2 : 0 \longrightarrow G \xrightarrow{\mu} H \xrightarrow{nu} B \longrightarrow 0$ $\in \mathrm{Ext}_R(B,G)$ satisfy $E_2\alpha = 0$. By Lemma 2.2, there is a $\xi : A \to H$ such that $\nu\xi = \alpha$; ξ is monic. Since $\beta\nu\xi = \beta\alpha = 0$, there is a $\lambda : H/\xi A \to C$ such that $\beta\nu = \lambda\rho$ with $\rho : H \to H/\xi A$ the canonical map. Consequently, we have a commutative diagram

$$
\begin{tikzcd}
 & & & 0 \arrow[d] & 0 \arrow[d] & \\
 & & 0 \arrow[r] \arrow[d] & A \arrow[r, equal] \arrow[d, "\xi"] & A \arrow[r] \arrow[d, "\alpha"] & 0 \\
E_2 : & 0 \arrow[r] & G \arrow[r, "\mu"] \arrow[d, equal] & H \arrow[r, "\nu"] \arrow[d, "\rho"] & B \arrow[r] \arrow[d, "\beta"] & 0 \\
 & 0 \arrow[r] & G \arrow[r, "\rho\mu"] \arrow[d] & H/\xi A \arrow[r, "\lambda"] \arrow[d] & C \arrow[r] \arrow[d] & 0 \\
 & & 0 & 0 & 0 &
\end{tikzcd}
$$

where all three columns and the first two rows are exact. By the 3×3- lemma, the bottom row is exact, hence it represents an element of $\mathrm{Ext}_R(C, G)$ that is mapped by β^* upon E_2. The exactness of (2) at $\mathrm{Ext}_R(A, G)$ express the fact that every extension of G by A can be prolonged to one of G by B: this is true as is shown by (Fuchs 1970) (24.6).

Turning to proof of (3), by (Fuchs 1970) (44.4) and Lemma 2.2 we may begin the proof at $\mathrm{Ext}_R(G, A)$. For $\eta \in \mathrm{Hom}(G, C)$, $\alpha_* E_* \eta = \alpha E \eta = 0$, as αE splits because of Lemma 2.2. Assume $E_1 : 0 \longrightarrow A \xrightarrow{\mu} H \xrightarrow{\nu} G \longrightarrow 0 \in \mathrm{Ext}_R(G, A)$ satisfies $\alpha E_1 = 0$; then by Lemma 2.1 there is a $\xi : H \to B$ such that $\xi\mu = \alpha$. From $\beta\xi\mu = \beta\alpha = 0$ and (Fuchs 1970) (2.2) we infer the existence of an $\eta : G \to C$ such that $\eta\nu = \beta\xi$, and so $(1_A, \xi, \eta)$ maps E upon E_1, i.e., $E_1 = E\eta$. Next we show exactness at $\mathrm{Ext}_R(G, B)$. By $\beta_*\alpha_* = (\beta\alpha)_* = 0_* = 0$, it suffices to show that kernel is contained in image. Assume $E_2 : 0 \longrightarrow B \xrightarrow{\mu} H \xrightarrow{\nu} G \longrightarrow 0 \in \mathrm{Ext}_R(G, B)$ satisfies $\beta E_2 = 0$; then by Lemma 2.1 there is a $\xi : H \to C$ with $\xi\mu = \beta$. Now $\xi\mu\alpha = 0$ implies the existence of a map $\lambda : A \to \mathrm{Ker}\,\xi$ with $\rho\lambda = \mu\alpha$, where

$\rho : \mathrm{Ker}\,\xi \to H$ is the injection. Therefore the diagram

$$
\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
 & 0 \arrow[r] & A \arrow[r, "\lambda"] \arrow[d, "\alpha"] & \mathrm{Ker}\,\xi \arrow[r, "\nu\rho"] \arrow[d, "\rho"] & G \arrow[r] \arrow[d, equal] & 0 \\
E_2 : & 0 \arrow[r] & B \arrow[r, "\mu"] \arrow[d, "\beta"] & H \arrow[r, "\nu"] \arrow[d, "\xi"] & G \arrow[r] \arrow[d] & 0 \\
 & 0 \arrow[r] & C \arrow[r, equal] \arrow[d] & C \arrow[r] \arrow[d] & 0 & \\
 & & 0 & 0 & &
\end{tikzcd}
$$

is commutative, has exact columns and the two bottom rows are exact. By the 3×3-lemma, the top row is exact, thus it is an element of $\mathrm{Ext}_R(G, A)$ which is mapped by α upon E_2. Finally, the epimorphic character of β_* follows again from (Fuchs 1970) (24.6). $\square$

The exact sequences (2) and (3) are of cardinal importance in dealing with Hom and Ext_R. They are extensively made use of in the description of Ext_R, in particular, in the theory of cotorsion modules. They establish a close connection between Hom and Ext_R [exploited to a great extent in homological algebra].

It is worhthwhile pointing out this connection more closely, since yields a method of discussing Ext_R. Given A, C, let $E_0 : 0 \longrightarrow H \xrightarrow{\phi} F \xrightarrow{\psi} C \longrightarrow 0$ be a free resolution of C, i.e., both F and H are free. For an $\eta : H \to A$ we can find a B and a $\chi : F \to B$ such that the diagram

$$
\begin{tikzcd}
E_0 : & 0 \arrow[r] & H \arrow[r, "\phi"] \arrow[d, "\eta"] & F \arrow[r, "\psi"] \arrow[d, "\chi"] & C \arrow[r] \arrow[d, equal] & 0 \\
\eta E_0 : & 0 \arrow[r] & A \arrow[r, "\mu"] & B \arrow[r, "\nu"] & C \arrow[r] & 0
\end{tikzcd}
$$

commutes and the bottom row is exact. Now

$$E_0^* : \qquad \mathrm{Hom}(H, A) \longrightarrow \mathrm{Ext}_R(C, A)$$

is easily seen to be an epimorphism whose kernel consists of all $\eta : H \to A$ that can be extended to an $F \to A$. Notice that if

$$F = \bigoplus_{i \in I} < x_i > \text{ and } H = \bigoplus_{j \in J} < y_j > \text{ with } y_j = \sum_i m_{ji} x_i$$

($m_{ji} \in Z$, almost all m_{ji} with fixed j vanish), then the extension ηE_0 of A by C is the module

$$B = < A, x_i\, (i \in I) : \textstyle\sum_i m_{ji} x_i = \eta y_j\, (j \in J) >.$$

Two homomorphisms $\eta_1, \eta_2 : H \to A$ give rise to equivalent extensions exactly if their difference is extendable to a homomorphism $F \to A$.

2.3. Elementary Properties of Ext_R

Our objective in this section is to record a number of elementary but most useful properties of extensions. We shall make frequent use of the exact sequences stated in Theorem 2.2.

In order not to interrupt our discussion, first we formulate a simple lemma. In accordance with definitions in Section 2.1 if $E : 0 \longrightarrow A \xrightarrow{\mu} B \xrightarrow{\nu} C \longrightarrow 0$ is an extension of A by C, and if $\alpha : A \to A$, $\gamma : C \to C$ are endomorphisms of A and C, respectively, then αE an $E\gamma$ will again be extensions of A by C. The correspondences

$$\alpha_* : E \mapsto \alpha E \qquad \text{and} \qquad \gamma^* : E \mapsto E\gamma$$

are evidently endomorphisms of $\mathrm{Ext}_R(C, A)$; we call them *induced endomorphisms* of Ext_R. The formulas $(\alpha_1 + \alpha_2)_* = (\alpha_1)_* + (\alpha_2)_*$ and $(\gamma_1 + \gamma_2)^* = (\gamma_1)^* + (\gamma_2)^*$ show that the endomorphism ring of A acts on $\mathrm{Ext}_R(C, A)$ and similarly the dual of the endomorphism ring C operates on $\mathrm{Ext}_R(C, A)$. These commute as is shown by $\alpha_*\gamma^* = \gamma^*\alpha_*$; hence $\mathrm{Ext}_R(C, A)$ is a (unital) bimodule over endomorphism rings of A and C, acting from the left and right, respectively. Now our lemma asserts the following remarkable fact.

Lemma 2.3 *Multiplication by an element $n \in R$ on A or C induces multiplication by n on* $\mathrm{Ext}_R(C, A)$.

We begin with two rather trivial observations.

(A) *A module C satisfies* $\mathrm{Ext}_R(C, A) = 0$ *for every A if and only if C is free.*

(B) *A module A satisfies* $\mathrm{Ext}_R(C, A) = 0$ *for every C exactly if A is divisible.*

(C) Let us turn next to the following theorem.

Theorem 2.3 *There exist natural isomorphisms*

(1) $\mathrm{Ext}_R(\bigoplus_{i\in I} C_i, A) \cong \prod_{i\in I} \mathrm{Ext}_R(C_i, A)$,

(2) $\mathrm{Ext}_R(C, \prod_{j\in J} A_j) \cong \prod_{j\in J} \mathrm{Ext}_R(C, A_j)$.

(D) *For every module A and for every $m \in R$,*

$$\mathrm{Ext}_R(R/Rm, A) \cong A/mA.$$

(E) *If $mA = 0$ or $mC = 0$ for some $m \in R$, then $m\,\mathrm{Ext}_R(C, A) = 0$.*

(F) *For $m \in R$,*

$$\mathrm{Ext}_R(C, R/Rm) \cong \mathrm{Ext}_R(C[m], R/Rm).$$

(G) *If $mA = A$ for some $m \in R$, then $m\,\mathrm{Ext}_R(C, A) = \mathrm{Ext}_R(C, A)$*

(H) *An automorphism α of A induces an automorphism α_* of $\mathrm{Ext}_R(C, A)$.*

Furthermore, if A is torsion-free divisible, then $\mathrm{Ext}_R(C, A)$, too, is torsion-free divisible.

(I) $C[m] = 0$ *implies* $m\,\mathrm{Ext}_R(C, A) = \mathrm{Ext}_R(C, A)$. *In particular,* $\mathrm{Ext}_R(C, A)$ *is divisible if C is torsion-free.*

(J) *Let γ be an automorphism of C. Then γ^* is an automorphism of* $\mathrm{Ext}_R(C, A)$. *Thus if $mC = C$ and $C[m] = 0$, then $m\,\mathrm{Ext}_R(C, A) = \mathrm{Ext}_R(C, A)$ and $\mathrm{Ext}_R(C, A)[m] = 0$;* and *if C is torsion-free divisible, then the same holds for $\mathrm{Ext}_R(C, A)$.*

(K) *If A is p-divisible and C is p-module, then $\mathrm{Ext}_R(C, A) = 0$.*

(L) The following theorem provides us with an essential isomorphism.

Theorem 2.4 *If A is torsion-free and C is torsion, then*

$$\mathrm{Ext}_R(C, A) \cong \mathrm{Hom}(C, D/A)$$

where D is divisible hull of A. Hence $\mathrm{Ext}_R(C, A)$ is a reduced algebraically compact module.

The choice $A = R$ leads us the following interesting isomorphism.

Corollary 2.1 *If C is a torsion module, then*

$$\mathrm{Ext}_R(C, R) \cong \mathrm{Char}\, C$$

(M) *If A is a torsion-free module whose p-basic submodule is of rank m, then*

$$\mathrm{Ext}_R(R(p^\infty), A) \cong p\text{-adic completion of } \bigoplus_m J_p.$$

(N) *If A is torsion-free, $\mathrm{Ext}_R(C, A)$ is algebraically compact, whatever C is.*

(O) *If A is algebraically compact, then $\mathrm{Ext}_R(C, A)$ is a reduced algebraically compact module.*

2.4. Cotorsion Modules

A module is called *cotorsion* if $\mathrm{Ext}_R(J, G) = 0$ for every torsion-free module J.

In other words, G is cotorsion if every extension of G by a torsion-free module splits. Since this means that a cotorsion module is a direct summand in every module in which it is contained with torsion-free quotient module, it is evident that algebraically compact modules are cotorsion. We shall see that the converse is not true.

It is convenient to list here following more or less elementary results on cotorsion modules.

(A) *An epimorphic image of a cotorsion module is cotorsion.*

(B) *Let G be reduced and cotorsion. For a submodule H of G to be cotorsion it is necessary and sufficient that G/H is reduced.*

(C) *If G is reduced and cotorsion, then for every endomorphism θ of G, both* $\mathrm{Ker}\,\theta$ *and* $\mathrm{Im}\,\theta$ *are cotorsion.*

(D) *If H is a submodule of G such that both H and G/H are cotorsion, then G is cotorsion.*

(E) *A direct product $\prod_{i\in I} G_i$ is cotorsion if and only if every summand of G_i is cotorsion.*

(F) *The inverse limit of a reduced cotorsion module is a reduced cotorsion module.*

(G) *If G is cotorsion, then* $\mathrm{Hom}(A, G)$ *is cotorsion for any A.*

(H) *For a reduced cotorsion module G, there is a natural isomorphism*

$$\mathrm{Ext}_R(K/R, G) \cong G.$$

(I) *A reduced cotorsion module G may be written uniquely in the form*

$$G = \prod_p G_p$$

where, for each prime element p of R, G_p is a reduced cotorsion module which is a p-adic module.

By (H), we may write

$$G \cong \mathrm{Ext}_R(K/R, G) = \mathrm{Ext}_R(\bigoplus_p R(p^\infty), G) = \prod_p \mathrm{Ext}_R(R(p^\infty), G).$$

Proposition 2.2 *A module is cotorsion if and only if it is an epimorphic image of an algebraically compact module.*

Proposition 2.3 *A reduced cotorsion module is algebraically compact if and only if its Ulm submodule vanishes.*

Theorem 2.5 *The Ulm submodules of cotorsion modules are again cotorsion, and the Ulm factors of cotorsion modules are algebraically compact.*

Corollary 2.2 *A torsion module is cotorsion if and only if it is a direct sum of a divisible module and a bounded module.*

Corollary 2.3 *A necessary and sufficient condition for a torsion-free module to be cotorsion is algebraically compactness.*

Theorem 2.6 $\mathrm{Ext}_R(C, A)$ *is cotorsion for all* A, C.

CHAPTER 3

THE κ-ELEMENTS OF $\mathrm{Ext}_R(C, A)$ AS TORSION ELEMENTS

Definition 3.1 *A short exact sequence E is called κ-*exact *if* $\mathrm{Im}\,\alpha$ *has a supplement in B i.e. a minimal element in the set $\{V \subset B | V + \mathrm{Im}\,\alpha = B\}$. In this case we say that* $[E] \in \mathrm{Ext}_R(C, A)$ *is a κ-*element *and the set of all κ-elements will be denoted by* $\mathrm{Ext}_R(C, A)^\kappa$.

In the following we will permanently use the following result from (Zöschinger 1974b), which can be easily proved:
If $h : H \to C$ is a small cover of C, and at least one primary component of C is zero, then $\mathrm{Ker}\,h$ is torsion; even if almost all primary components in C are equal to zero, then $\mathrm{Ker}\,h$ is bounded. Thus one obtains:

Theorem 3.1 *Let C be a divisible module and almost all primary components of C be zero, then the κ-elements of* $\mathrm{Ext}_R(C, A)$ *are exactly the torsion elements.*

Proof Let $E = 0 \longrightarrow A \xrightarrow{\alpha} B \longrightarrow C \longrightarrow 0$ represent a torsion element in $\mathrm{Ext}_R(C, A)$. It follows from (Fuchs 1970) (53.1) that there exists a nonzero element $n \in R$ such that $\mathrm{Im}\,\alpha/(\mathrm{Im}\,\alpha)[n]$ is a direct summand in $(\mathrm{Im}\,\alpha + nB)/(\mathrm{Im}\,\alpha)[n]$ where $G[n] = \{x \in G \,|\, nx = 0\}$ as usual. Since C is divisible, $\mathrm{Im}\ \alpha/(\mathrm{Im}\ \alpha)[n] \subset^{\oplus} B/(\mathrm{Im}\ \alpha)[n]$, indicates $V + \mathrm{Im}\ \alpha = B$ with $V \cap \mathrm{Im}\,\alpha$ is bounded. By (Zöschinger 1974b), a bounded module has a supplement in every extension, therefore $V \cap \mathrm{Im}\ \alpha \subset^{\kappa} V$, hence $\mathrm{Im}\ \alpha \subset^{\kappa} B$. Conversely, if an arbitrary sequence E is κ-exact, that is there is a supplement V of $\mathrm{Im}\ \alpha$ in B, then by the remark above, $V \cap \mathrm{Im}\ \alpha$ is bounded. Then there exist an element $n \in R$ such that $n(V \cap \mathrm{Im}\,\alpha) = 0$ so $(V \cap \mathrm{Im}\,\alpha) \subset (\mathrm{Im}\,\alpha)[n]$, therefore $B = \mathrm{Im}\,\alpha + V$ and $(V \cap \mathrm{Im}\,\alpha) \subset (\mathrm{Im}\,\alpha)[n]$ implies $\mathrm{Im}\,\alpha/(\mathrm{Im}\,\alpha)[n] \subset^{\oplus} B/(\mathrm{Im}\,\alpha)[n]$ for some $n \in R$. Again by (Fuchs 1970) (53.1), $[E]$ is a torsion element of $\mathrm{Ext}_R(C, A)$ □

Corollary 3.1 *If* $\operatorname{Ext}_R(R(p^\infty), M)$ *is torsion, then* M *has a supplement in each extension* N *of* M*, with* N/M p*-primary.*

Proof Since as is well-known $\operatorname{Ext}_R(R(p^\infty), M)$ is reduced and cotorsion, by Theorem 2.6 it follows from the condition that it is also bounded, since every divisible p-module C is isomorphic to $\bigoplus R(p^\infty)$ then $\operatorname{Ext}_R(C, M) \cong \operatorname{Ext}_R(\bigoplus R(p^\infty), M) \cong \prod(R(p^\infty), M)$ is bounded, thus $\operatorname{Ext}_R(C, M)$ is torsion for every divisible p-module C. Now if N is as given, then there is a module H such that $N \subset H$ with H/M is divisible and N/M is big in H/M. By the Theorem, M has a supplement in H, thus as a result of (Zöschinger 1974b) (Hilfssatz 5.1) also does in N. $\square$

Corollary 3.2 *There is a* p*-module* N *with a pure submodule* M *such that* M *has a supplement in each* K *such that* $M \subset K \subsetneq N$ *but is not a direct summand in* N*.*

Proof We have the short exact sequence

$0 \longrightarrow R \longrightarrow K \longrightarrow K/R \longrightarrow 0$, so we get

$0 \longrightarrow \operatorname{Hom}(K/R, M) \longrightarrow \operatorname{Hom}(K, M) \longrightarrow \operatorname{Hom}(\mathbb{Z}, M) \longrightarrow \operatorname{Ext}_R(K/R, M)$

$\longrightarrow \operatorname{Ext}_R(K, M) \longrightarrow \operatorname{Ext}_R(\mathbb{Z}, M) \longrightarrow 0$.

Since $\operatorname{Hom}(K, M) \cong D(M)$ and $\operatorname{Hom}(\mathbb{Z}, M) \cong M$ we get $0 \longrightarrow D(M) \xrightarrow{f} M \xrightarrow{g} \operatorname{Ext}_R(K/R, M) \xrightarrow{h} \operatorname{Ext}_R(K, M) \longrightarrow \cdots$, and also $\operatorname{Ker} h = \operatorname{Im} g \cong M/\operatorname{Ker} g = M/\operatorname{Im} f = M/D(M)$.

So we always have the exact sequence

$0 \longrightarrow M/D(M) \longrightarrow \operatorname{Ext}_R(K/R, M) \longrightarrow \operatorname{Ext}_R(K, M) \longrightarrow 0$, then

$0 \longrightarrow (M/D(M))^1 \longrightarrow (\operatorname{Ext}_R(K/R, M))^1 \longrightarrow 0$. Therefore, for an arbitrary module M we have a monomorphism $M^1/D(M) \longrightarrow \operatorname{Pext}_R(K/R, M)$ where $M^1 = \bigcap_{n \in R} nM$ is as usual with nonzero n. If, one chooses a special p-primary M with $D(M) \subsetneq M^1$, by (Fuchs 1970) (p.150) then $\operatorname{Pext}_R(R(p^\infty), M)$ can not be torsion-free, and for a nonzero torsion element $[0 \longrightarrow M \subset N \longrightarrow R(p^\infty) \longrightarrow 0]$ we have $M \subset^{\kappa} N$ by the theorem; moreover for each $M \subsetneq X \subsetneq N$, there is a cyclic $X_1 \subset X$ with $X_1 + M = X$, and E is splitting by (Fuchs 1970) (28.2) thus $X = M \oplus X_1$ obviously X_1 is a supplement of M in X. $\square$

We still want more details about the short exact sequence

$$0 \longrightarrow A \xrightarrow{\alpha} B \longrightarrow C \longrightarrow 0$$

occurring in the proof of the theorem in which $V + \operatorname{Im} \alpha = B$ for some V where

$V \cap \mathrm{Im}\,\alpha$ is bounded. We call it *β-exact*, and denote $\mathrm{Im}\,\alpha \subset^{\beta} B$. Any β-element of $\mathrm{Ext}_R(C, A)$ is always a κ-element as well as a torsion element. The converse holds in the following special case:

Lemma 3.1 *If C and A are torsion, then*

$$\mathrm{Ext}_R(C, A)^{\beta} = \mathrm{Ext}_R(C, A)^{\kappa} \cap T(\mathrm{Ext}_R(C, A)).$$

Proof With the characterization of the torsion elements of Ext in (Walker 1964) the claim says: If M is a torsion module and $U \subset^{\kappa} M$ with $U/U[n] \subset^{\oplus} (U{+}nM)/U[n]$ for some $0 \neq n \in R$, then $U \subset^{\beta} M$. If we choose a direct supplement $V/U[n]$ of $U/U[n]$, then for all prime elements p of R, with $(p, n) = 1$, we have $T_p(M) = T_p(nM) = T_p(U + nM) = T_p(V) + T_p(U)$. On the other hand, since $U \cap V \subset U[n]$ and $(p, n) = 1$ we have $T_p(V) \cap T_p(U) = 0$, i.e. $T_p(U) \subset^{\oplus} T_p(M)$. Since $U \subset^{\kappa} M$ we can also find a supplement W of U in M with $T_p(W) \cap T_p(U) = 0$ for all prime elements p of R with $(p, n) = 1$ since $T_p(W) = T_p(W) \cap T_p(M) = T_p(U) \oplus K = T_p(M)$. For the rest of the proof, $T_p(W) \cap T_p(U)$ is coatomic after all (i.e. all factors are reduced), thus whole $W \cap U$ is bounded. □

Remark 3.1 *In* $\mathrm{Ext}_R(\mathrm{Soc}(K/R), \mathbb{J}_p)$ *each element is a κ-element as well as a torsion element, but only the zero element is a β-element.*

Lemma 3.2 *If*

$$0 \longrightarrow X \xrightarrow{\iota} Y \xrightarrow{\pi} C \longrightarrow 0$$

is a torsion-free resolution of C and $\delta : \mathrm{Hom}(X, A) \to \mathrm{Ext}_R(C, A)$ *is the relevant connecting homomorphism, then*

$$\mathrm{Ext}_R(C, A)^{\beta} = \delta(T(\mathrm{Hom}(X, A))).$$

Proof If $f \in \mathrm{Hom}(X, A)$, then we obtain $\delta(f) = [E]$ with

$$\begin{tikzcd} & 0 \arrow[r] & X \arrow[r] \arrow[d, "f"'] & Y \arrow[r] \arrow[d, "f'"] & C \arrow[r] \arrow[d, equal] & 0 \\ E = & 0 \arrow[r] & A \arrow[r, "\alpha"'] & B \arrow[r] & C \arrow[r] & 0 \end{tikzcd}$$

and in the diagram, $\mathrm{Im}\, f' + \mathrm{Im}\ \alpha = B$ as well as $\mathrm{Im}\, f' \cap \mathrm{Im}\,\alpha \cong \mathrm{Im}\, f$. So, f is a torsion element of $\mathrm{Hom}(X, A)$, i.e. $\mathrm{Im}\, f$ is bounded, then $[E]$ is a β-element of

$\operatorname{Ext}_R(C, A)$.

Conversely, let now $E = 0 \longrightarrow A \xrightarrow{\alpha} B \xrightarrow{\beta} C \longrightarrow 0$ be β-exact. By (Fuchs 1970) we know if A is torsion-free and B is bounded, then $\operatorname{Ext}_R(A, B) = 0$. So $V + \operatorname{Im}\alpha = B$ with $V \cap \operatorname{Im}\alpha = V \cap \operatorname{Ker}\beta = K$ is bounded. We get the exact sequence $0 \longrightarrow K \longrightarrow V \xrightarrow{\beta|_V} C \longrightarrow 0$. Since $\beta\ |_V$ is surjective and Y is torsion-free we obtain an exact sequence

$0 \longrightarrow \operatorname{Hom}(Y, K) \longrightarrow \operatorname{Hom}(Y, V) \xrightarrow{(\beta|_V)_*} \operatorname{Hom}(Y, C) \longrightarrow \operatorname{Ext}_R(Y, K) = 0$, and for $\pi \in \operatorname{Hom}(Y, C)$ there exist $g \in \operatorname{Hom}(Y, B)$ with $\operatorname{Im} g \subset V$ and $\beta g = \pi$. By the diagram,

$$\begin{tikzcd}
 & 0 \arrow[r] & X \arrow[r, "\iota"] \arrow[d, "f"'] & Y \arrow[r, "\pi"] \arrow[d, "g"] & C \arrow[r] \arrow[d, equal] & 0 \\
E = & 0 \arrow[r] & A \arrow[r, "\alpha"'] & B \arrow[r, "\beta"'] & C \arrow[r] & 0
\end{tikzcd}$$

$\delta(f) = [E]$ as well as $\operatorname{Im} f \cong \operatorname{Im} g \cap \operatorname{Im}\alpha$ is bounded thus $f \in T(\operatorname{Hom}(X, A))$. $\square$

Corollary 3.3 (a) *The β-elements of $\operatorname{Ext}_R(C, A)$ form a submodule.*

(b) *If $f : A \to A'$ is a homomorphism, then $f_* : \operatorname{Ext}_R(C, A) \to \operatorname{Ext}_R(C, A')$ preserves β-elements.*

(c) *If $g : C' \to C$ is a homomorphism, then $g^* : \operatorname{Ext}_R(C, A) \to \operatorname{Ext}_R(C', A)$ preserves β-elements.*

Proof

(a) Clear, since image preserves the β-elements.

(b) For the β-exact sequence E we have the diagram

$$\begin{tikzcd}
 & 0 \arrow[r] & X \arrow[r] \arrow[d, "g"'] & Y \arrow[r] \arrow[d, "i"] & C \arrow[r] \arrow[d, equal] & 0 \\
E: & 0 \arrow[r] & A \arrow[r] \arrow[d, "f"'] & B \arrow[r] \arrow[d, "h"] & C \arrow[r] \arrow[d, equal] & 0 \\
E': & 0 \arrow[r] & A' \arrow[r] & B' \arrow[r] & C \arrow[r] & 0
\end{tikzcd}$$

$g \in T(\operatorname{Hom}(X, A))$ then $fg \in T(\operatorname{Hom}(X, A'))$ hence E' is β-exact.

(c) Similar proof with (b).

$\square$

Remark 3.2 *If C is divisible, then one can also choose a divisible Y in the torsion-free resolution. Then* Coker δ *is torsion-free and* Ker δ *is divisible so that δ is surjective on the torsion elements, i.e.* $T(\mathrm{Ext}_R(C,A)) = \mathrm{Ext}_R(C,A)^{\beta}$.

Proof We know if Y is torsion-free divisible, then $\mathrm{Ext}_R(Y,A)$ is torsion-free and $\mathrm{Hom}(Y,A)$ is divisible for all A. For the exact sequence $0 \longrightarrow \mathrm{Hom}(C,A) \longrightarrow \mathrm{Hom}(Y,A) \xrightarrow{g} \mathrm{Hom}(X,A) \xrightarrow{\delta} \mathrm{Ext}_R(C,A) \xrightarrow{f} \mathrm{Ext}_R(Y,A) \longrightarrow \mathrm{Ext}_R(X,A) \longrightarrow 0$. Then $\mathrm{Coker}\,\delta = \mathrm{Ext}_R(C,A)/\operatorname{Im}\delta = \mathrm{Ext}_R(C,A)/\operatorname{Ker} f = \operatorname{Im} f \subset \mathrm{Ext}_R(Y,A)$ which is torsion-free, and $\operatorname{Ker}\delta = \operatorname{Im} g = \mathrm{Hom}(Y,A)/\operatorname{Ker} g$ is divisible. □

Remark 3.3 *From a result of (Baer 1958) (Proposition 3.2) one can immediately deduce that if $T(\mathrm{Ext}_R(C,R^{(I)})) = 0$ for each I, then $C/D(C)$ is free. Therefore $T(\mathrm{Ext}_R(C,A)) = \mathrm{Ext}_R(C,A)^{\beta}$ for all A if and only if $C/D(C)$ is free.*

Proof We know that if $\mathrm{Ext}_R(A,X) = 0$ for all X, then A is free. Moreover if $\mathrm{Ext}_R(C,A) = 0$ for all torsion X, then A is free by [15]. Hence if $\mathrm{Ext}_R(A,R) = 0$, then A is free. □

Definition 3.2 $\mathrm{Ext}_R(C,A)$ *is called* κ-full *if every element of* $\mathrm{Ext}_R(C,A)$ *is κ-element.*

Example 3.1 *If almost all primary components of C are zero, then the β-elements of* $\mathrm{Ext}_R(C,A)$ *coincide with the κ-elements. If particularly, $C = R/(m)$ where $m \neq 0,1$, then we have the projective resolution $0 \longrightarrow R \xrightarrow{m} R \longrightarrow R/(m) \longrightarrow 0$, and the connecting homomorphism δ yields $\mathrm{Ext}_R(R/(m),A) \cong A/mA$ where the κ-elements correspond exactly the submodule $(T(A)+mA)/mA$. In particular,*

$$\mathrm{Ext}_R(R/(m),A) \textit{ is } \kappa\textit{-full} \Leftrightarrow A/T(A) \textit{ is } m\textit{-divisible.}$$

Proof From the exact sequence $0 \longrightarrow \mathrm{Hom}(R/(m),A) \longrightarrow \mathrm{Hom}(R,A) \xrightarrow{m.n} \mathrm{Hom}(R,A) \longrightarrow \mathrm{Ext}_R(R/(m),A) \longrightarrow \mathrm{Ext}_R(R,A) = 0$, since $\mathrm{Hom}(\mathbb{Z},A) = A$ and $\mathrm{Ext}_R(R,A) \cong A/\operatorname{Ker}\delta = A/mA$, we get $0 \longrightarrow \mathrm{Hom}(R/(m),A) \longrightarrow A \xrightarrow{m} A \xrightarrow{\delta} A/mA \longrightarrow 0$. Then $\mathrm{Ext}_R^{\kappa}(R/mA) = T(\mathrm{Ext}_R(R,A)) \cong T(A/mA) = (T(A)+mA)/mA$.

For the κ-full part, we have $\mathrm{Ext}_R(C,A) = \mathrm{Ext}_R^{\kappa}(R,A)$. Then $(T(A)+mA)/mA = A/mA \Leftrightarrow T(A)+mA = A \Leftrightarrow m(A/T(A)) = A/T(A)$. □

Theorem 3.2 *For any pair* (A, C) *the following are equivalent:*

(i) $\mathrm{Ext}_R(C,A)^\beta \subset \mathrm{Rad}(\mathrm{Ext}_R(C,A))$.

(ii) $\mathrm{Ext}_R(C,A)^\beta = 0$.

(iii) $\mathrm{Rad}(\mathrm{Ext}_R(C,A))$ *is divisible.*

Proof (i ⇒ iii) Since divisibility of $\mathrm{Rad}(\mathrm{Ext}_R(C,A))$ is equivalent to the statement that if $T_p(C) \neq 0$, then $T(A)$ is p-divisible, we must show that if $T_p(C) \neq 0$ then $T(A)$ is p-divisible.

<ins>Case I.</ins> If $T_p(C)$ is reduced, then there is a cyclic direct summand X in $T_p(C)$ where $X \cong R/(Rp^n)$ for nonzero n. We have homomorphisms $\alpha : R/(Rp^n) \to C$ and $p : C \to R/(Rp^n)$ with $\alpha p = 1$. Then we get the maps α^* and p^* between $\mathrm{Ext}_R(R/(Rp^n), A)$ and $\mathrm{Ext}_R(C,A)$ where $\alpha^* p^* = 1$, so we can write $\mathrm{Ext}_R(R/(Rp^n), A)^\beta = \alpha^* p^* (\mathrm{Ext}_R(R/(Rp^n), A))^\beta \subset \alpha^*(\mathrm{Ext}_R(R/(Rp^n), A))^\beta \subset \alpha^*(\mathrm{Rad}(\mathrm{Ext}_R(R/(Rp^n), A))) \subset \mathrm{Rad}(\mathrm{Ext}_R(R/(Rp^n), A)) = p(\mathrm{Ext}_R(R/(Rp^n), A))$. (1.4) indicates that $(T(A) + p^n A)/p^n A \subset p(A/p^n A = pA/p^n A$, thus $T(A)$ is p-divisible.

<ins>Case II.</ins> If $T_p(C)$ is not reduced, then $\mathrm{Ext}_R(R(p^\infty), A)^\beta = \alpha^* p^* (\mathrm{Ext}_R(R(p^\infty), A))^\beta \subset \mathrm{Rad}(\mathrm{Ext}_R(R(p^\infty), A))$, by similar proof as in Case I, the torsion submodule of $\mathrm{Ext}_R(R(p^\infty), A)$ is divisible, so it is zero. Since A can not have a direct summand of the form $R/(p^n)$ for nonzero n the statement follows.

(iii ⇒ ii) <ins>Case I.</ins> $T_p(C) \neq 0$ for all prime elements p of R. Then $T(A)$ is divisible, so $A/D(A)$ is torsion-free. By (Fuchs 1970) $\mathrm{Hom}(C, A/D(A))$ is torsion-free again i.e. $T(\mathrm{Hom}(C, A/D(A))) = 0$. Since $\mathrm{Ext}_R(C, A/D(A))^\beta = \delta(T(\mathrm{Hom}(C, A/D(A))))$ by Lemma 1.3. $\mathrm{Ext}_R(C, A/D(A))^\beta = 0$. The statement follows by the isomorphism $v_* : \mathrm{Ext}_R(C,A) \to \mathrm{Ext}_R(C, A/D(A))$.

▹ $0 \longrightarrow D(A) \longrightarrow A \longrightarrow A/D(A) \longrightarrow 0$ is splitting i.e. $A = D(A) \oplus A/D(A)$. Then we get $\mathrm{Ext}_R(C,A) \cong \mathrm{Ext}_R(C, D(A)) \oplus \mathrm{Ext}_R(C, A/D(A)) = \mathrm{Ext}_R(C, A/D(A))$ ◃

<ins>Case II.</ins> At least one of the primary components of C is zero. We claim that $\mathrm{Ext}_R(C,A)$ has no nonzero κ-element. Furthermore if we use the result from Lemma 4.1 that v_* preserves κ-elements, we can assume that A is reduced, and now our assumption states: $T_p(C) \neq 0 \Longrightarrow T(A)$ is p-divisible so $T_p(A)$ is p-divisible but since A is reduced, $T_p(A) = 0$. Now let $0 \longrightarrow A \xrightarrow{\alpha} B \longrightarrow C \longrightarrow 0$ be κ-

exact, and V be a supplement of $\operatorname{Im}\alpha$ in B. By (Zöschinger 1974a) (Hilfssatz 5.2), $V \cap \operatorname{Im}\alpha$ is torsion and the injective hull of V is isomorphic to the injective hull of C. For some prime element $p \in R$ that $T_p(C) = 0$, we have $T_p(V) = 0$. For some prime element $p \in R$ such that $T_p \neq 0$, since $A \cong \operatorname{Im}\alpha$ we get $T_p(\operatorname{Im}\alpha) = 0$. Then we get for all prime elements p of R, $T_p(V \cap \operatorname{Im}\alpha) = 0$, thus $V \oplus \operatorname{Im}\alpha = B$ i.e. $\operatorname{Ext}_R(C, A) = 0$. □

Remark 3.4 *Of course* $\operatorname{Ext}_R(C, A)^\beta \cap \operatorname{Rad}(\operatorname{Ext}_R(C, A))$ *does not need to be zero, as it is shown in the example (second corollary to Theorem 3.1).*

Proof By second corollary, for $C = R(p^\infty)$ and for some p-primary module M, there exists a pure exact sequence $0 \neq E\colon\ M \longrightarrow N \longrightarrow R(p^\infty) \longrightarrow 0$ which is a κ-element. By Theorem 3.1 $\operatorname{Ext}_R(R(p^\infty), M)^\kappa = T(\operatorname{Ext}_R(R(p^\infty), M))$ and by Lemma 3.1 $\operatorname{Ext}_R(R(p^\infty), M)^\beta = \operatorname{Ext}_R(R(p^\infty), M)^\kappa$. Then $E \in \operatorname{Rad}(\operatorname{Ext}_R(R(p^\infty), M)) \cap \operatorname{Ext}_R(R(p^\infty), M)^\beta$. □

Remark 3.5 *It follows by the proof that if* $A/T(A)$ *is divisible and* $\operatorname{Ext}_R(C, A)^\beta = 0$, *then there is no nonzero* κ*-element in* $\operatorname{Ext}_R(C, A)$.

Finally, with the help of β-concept we will give a necessary criterion that $T(B)$ splits off in B.

Lemma 3.3 *Suppose that every torsion submodule and every torsion factor module of* A *is bounded. Then for* $A \subset B$ *the following are equivalent:*

(i) $T(B) \subset^{\oplus} B$.

(ii) $(T(B) + A)/A \subset^{\beta} B/A$.

Proof (i $\Rightarrow$ ii) From the equality $V \oplus T(B) = B$ we have $(V + A)/A + (T(B) + A)/A = B/A$, and we claim that the intersection is bounded. $[(V + A) \cap (T(B) + A)/A = [A + (V + A) \cap T(B)]/A = (A + T(V + A))/A \cong (T(V + A))/(A \cap T(V + A)) \cong (T(V + A))/T(A)$. On the other hand, $V + A = (V + A) \cap B = (V + A) \cap (V + T(B)) = (V + A) \cap T(B) + V = T(V + A) \oplus V$. So intersection is isomorphic to $(V + A)/(V + T(A)) \cong A/[(V + A) \cap (V + T(A))] \cong A/[(V + T(A)) \cap A] = A/[T(A) + V \cap A] = A/(T(A) \cap V)$, so is a torsion factor of A as desired.

(ii $\Rightarrow$ i) By the equality $X/A + (T(B) + A)/A = B/A$ with bounded $[X \cap (T(B) + A)]/A =$

$(A + T(X))/A \cong T(X)/T(A)$. Since $T(A)$ is bounded by assumption, $T(X)$ is also bounded. By the property (b) (Fuchs 1970) (p.114) $T(X)$ is a pure submodule of X. Then by Theorem 27.5 (Fuchs 1970), $T(X) \subset^{\oplus} X$ i.e. $X = V \oplus T(X)$ for some $V \subset X$. Then $B = X + T(B) + A = X + T(B)$ and $V \cap T(B) = V \cap (X \cap T(B)) = V \cap T(X) = 0$. Thus $V \oplus T(B) = B$. □

As one can easily see, the stated condition on A is equivalent to the condition that A is of the form $A = A_1 \oplus A_2$ where A_1 is finitely generated and free, and A_2 is bounded. Since from $T(C) \subset^{\kappa} C$, we always have $T(C) \subset^{\oplus} C$ we obtain in two special cases $A_1 = 0$ and $A_2 = 0$ respectively:

Corollary 3.4 (Corollary 1 (Papp 1975)) *If $0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$ is an exact sequence and A is bounded, then $T(B) \subset^{\oplus} B$ is equivalent to the statement that $T(C) \subset^{\oplus} C$.*

Corollary 3.5 (Corollary 2 (Stratton 1975)) *If the sequence $0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$ is pure-exact and A is finitely generated and free, then $T(B) \subset^{\oplus} B$ is equivalent to the statement that $T(C) \subset^{\oplus} C$.*

CHAPTER 4

NEAT- AND CONEAT- HOMOMORPHISMS

The main problem with the investigation of the κ-elements in $\mathrm{Ext}_R(C, A)$ is that they need not to form a submodule. The reason for it is the fact that, in general, for a homomorphism $g : C' \to C$ the induced map $g^* : \mathrm{Ext}_R(C, A) \to \mathrm{Ext}_R(C', A)$ need not to preserve κ-elements. For particular homomorphisms which we call coneat, this can not happen, and they are studied in this chapter.

Lemma 4.1 (I) *If $f : A \to A'$, then $f_* : \mathrm{Ext}_R(C, A) \to \mathrm{Ext}_R(C, A')$ preserves κ-elements.* (II) *Let $g : C' \to C$ and C' be torsion. If either a primary component of C is zero or A is torsion, then $g^* : \mathrm{Ext}_R(C, A) \to \mathrm{Ext}_R(C', A)$ preserves κ-elements.*

Proof (I) Let the following diagram be commutative with exact lines:

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, "\alpha"] \arrow[d, "f"'] & B \arrow[r] \arrow[d, "f'"] & C \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & A' \arrow[r, "\alpha'"'] & B' \arrow[r] & C \arrow[r] & 0. \end{tikzcd} \tag{I}$$

If V is a supplement of $\mathrm{Im}\,\alpha$ in B, then $f'(V)$ is a supplement of $\mathrm{Im}\,\alpha'$ in B'. Clearly, $f'(V) + \mathrm{Im}\,\alpha' = B'$.
▹ Let $h : B \to C$ and $h' : B' \to C'$ be homomorphisms with $h'(b') = h(b)$ for some $b \in B$ and $b' \in B'$, then $h'(b') = h'(f'(b))$. Thus $b' - f'(b) \in \mathrm{Ker}\,h' = \mathrm{Im}\,\alpha'$, then $b' = f'(b) + \alpha'(a')$ for some $a' \in A'$.◃
Then it is only to be shown that $f'(V) \cap \mathrm{Im}\,\alpha' = f'(V \cap \mathrm{Im}\,\alpha)$ which is small in $f'(V)$. Let $f'(v) = \alpha'(a')$ for some $v \in V$ and $a' \in A'$. Then we have $0 = h'(\alpha'(a')) = h'(f'(v)) = h(v)$ thus $v \in \mathrm{Ker}\,h = \mathrm{Im}\,\alpha$ hence $f'(v) \in f'(V \cap \mathrm{Im}\,\alpha)$.
(II) Let the following diagram be commutative with exact lines:

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r] \arrow[d, equal] & B' \arrow[r, "\beta'"] \arrow[d, "g'"'] & C' \arrow[r] \arrow[d, "g"] & 0 \\ 0 \arrow[r] & A \arrow[r] & B \arrow[r, "\beta"'] & C \arrow[r] & 0. \end{tikzcd} \tag{II}$$

If V is a supplement of $\operatorname{Ker}\beta$ in B, then $g'^{-1}(V)+\operatorname{Ker}\beta' = B'$, and since $g'(g'^{-1}(V)\cap \operatorname{Ker}\beta') = V \cap \operatorname{Ker}\beta$ we can say $g'\,|_{\operatorname{Ker}\beta}= 1_A$, so g' is monic. Since $D' = g'(V) \cap \operatorname{Ker}\beta'$ and $D = V \cap \operatorname{Ker}\beta$ are isomorphic to each other, D and D' are torsion and coatomic. Since each primary component of D' is bounded, it has a supplement in $g'^{-1}(V)$ which is torsion hence $\operatorname{Ker}\beta' \subset^{\kappa} B'$. □

Corollary 4.1 *Every multiple of a κ-element of $\operatorname{Ext}_R(C, A)$ is again a κ-element.*

Proof For each $r \in R$ and $[E] \in \operatorname{Ext}_R(C, A)$, as it is well-known, $r[E] = f_*([E])$ where f is the multiplication of A with r. □

Corollary 4.2 *If C has a torsion-free cover and A is a cotorsion module, then $\operatorname{Ext}_R(C, A)$ is κ-full.*

Proof By any torsion-free cover of C we mean a small epimorphism $h : H \to C$ with torsion-free H (see all modules that have a torsion-free cover below). Thus induced connecting homomorphism $\delta : \operatorname{Hom}(\operatorname{Ker} h, A) \to \operatorname{Ext}_R(C, A)$ has the following property: $\operatorname{Im}\delta$ consists only of κ-elements since each $\varphi \in \operatorname{Hom}(\operatorname{Ker} h, A)$ originates $\delta(\varphi)$ through pushout determined by a κ-exact sequence. $\cdots \longrightarrow \operatorname{Hom}(\operatorname{Ker} h, A) \xrightarrow{\delta} \operatorname{Ext}_R(C, A) \longrightarrow \operatorname{Ext}_R(H, A) = 0$ since H is torsion-free and A is cotorsion. Then we get δ is an epimorphism so we have $\operatorname{Ext}_R(C, A) = \operatorname{Im}\delta = \operatorname{Ext}^{\kappa}_R(C, A)$. □

Corollary 4.3 *If C is torsion, and either a primary component of C is zero or A is torsion, then the κ-elements of $\operatorname{Ext}_R(C, A)$ form a submodule.*

Proof Since the map $\Delta^* : \operatorname{Ext}_R(C \times C, A) \to \operatorname{Ext}_R(C, A)$ preserves κ-elements, for κ-elements $[E_1]$ and $[E_2]$ in $\operatorname{Ext}_R(C, A)$ we have $[E_1] + [E_2] = \Delta^*([\nabla(E_1 \times E_2)])$ which is also a κ-element. □

In connection with the problem of when g^* preserves κ-elements we first give three examples:

(1) The splitting monomorphism $\iota : R(p^\infty) \to K/R$ induces an isomorphism $\iota^* : \operatorname{Ext}_R(K/R, \mathbb{J}_p) \to \operatorname{Ext}_R(R(p^\infty), \mathbb{J}_p)$ and the first module is κ-full by the second corollary while the second has no nonzero κ-element.

(2) If A is a torsion module, for the canonical epimorphism $\nu : K \to K/R$, then $\nu^* : \operatorname{Ext}_R(K/R, A) \to \operatorname{Ext}_R(K, A)$ preserves the κ-elements if and only if A is divisible

by almost all prime elements p of R.
(3) If C is a torsion module with $T_p(C) \neq 0$ for all prime elements p of R, then, as it will be shown in chapter 7 that the κ-elements of $\mathrm{Ext}_R(C, R)$ do not form a submodule. Thus, $\Delta^* : \mathrm{Ext}_R(C \times C, R) \to \mathrm{Ext}_R(C, R)$ can not preserve the κ-elements.

It seems difficult to give necessary conditions for the fact that $g^* : \mathrm{Ext}_R(C, A) \to \mathrm{Ext}_R(C', A)$ preserves κ-elements (see Theorem 7.2). But it is easy to see that the equality $g(\mathrm{Soc}(C')) = \mathrm{Soc}(C)$ is sufficient. Now we want to study on such homomorphisms, and give a relation (in the dual case) with the neat-homomorphisms introduced by Enochs (Enochs 1971). $f : A \to A'$ is called *neat* if for every decomposition $f = \beta\alpha$ where α is essential monomorphism, α is an isomorphism (This is not the original definition, but one of the equivalent condition given by Bowe in (Bowe 1972), (Theorem 1.2)). The dualization is:

Definition 4.1 *A homomorphism $g : C' \to C$ is called* coneat, *if β is an isomorphism for every decomposition $g = \beta\alpha$ where β is small epimorphism.*

For the characterization of the coneat homomorphisms, first we need the following:

Lemma 4.2 (a) *An epimorphism $g : C' \to C$ is coneat if and only if* $\mathrm{Ker}\, g$ *is coclosed in C'.*

(b) *A splitting monomorphism $g : C' \to C$ is coneat if and only if* $\mathrm{Coker}\, g$ *has no small cover.*

(c) *If $g = g_2 g_1$ is coneat, then g_2 is also coneat. In addition, if g_2 is injective, then g_1 is coneat, too.*

Proof A submodule U of M is called coclosed if U/X is not small in M/X for every proper submodule X of U. By (Zöschinger 1974a) (Lemma 3.3) it is equivalent to the fact that $pU = U \cap pM$ for all prime element p of R, i.e. U is a neat submodule of M in the sense of (Fuchs and Salce 2001).

(a) If g is surjective and coneat, and $X \subsetneq \mathrm{Ker}\, g$ such that $\mathrm{Ker}\, g/X$ is small in C'/X, then $g = C' \xrightarrow{\nu} C'/X \xrightarrow{\tilde{g}} C$ and $\tilde{g}$ is small epimorphism, thus $\tilde{g}$ is an isomorphism by assumption, i.e. $X = \mathrm{Ker}\, g$. Conversely, from $g = \beta\alpha$

where β is a small epimorphism, g and α are factorized over $C'/\operatorname{Ker}\alpha$, say g_1 and α_1.

$$\begin{tikzcd}
C' \arrow[r, "g"] \arrow[d, dashed, "\gamma"'] \arrow[dr, "\alpha"] & C \\
C'/\operatorname{Ker}\alpha \arrow[ur, dashed, "g_1"] \arrow[r, dashed, "\alpha_1"'] & A \arrow[u, "\beta"']
\end{tikzcd}$$

Now since g is surjective, α is also surjective thus α_1 is bijective, thus $\operatorname{Ker} g_1 = \operatorname{Ker} g/\operatorname{Ker}\alpha$ is small in $C'/\operatorname{Ker}\alpha$. By assumption g_1 is an isomorphism, thus β is also an isomorphism.

(b) If the splitting monomorphism $g : C' \to C$ is coneat and $h : H \to \operatorname{Coker} g$ is a small cover, then the map $\omega =< g, s >: C' \times \operatorname{Coker} g \to C$ is an isomorphism where s is a right inverse of the canonical map $C \to \operatorname{Coker} g$. Since the following diagram is commutative

$$\begin{tikzcd}
 & & C' \arrow[lld, "{(1,0)}"'] \arrow[d, "g"] \\
C' \times H \arrow[r, "1\times h"'] & C' \times \operatorname{Coker} g \arrow[r, "\omega"'] & C
\end{tikzcd}$$

we have that the lower row is an isomorphism by assumption, so h is also an isomorphism. Conversely, it follows from $g = \beta\alpha$ where β is small a epimorphism that the induced map $\operatorname{Coker}\alpha \to \operatorname{Coker}\beta$ is a small epimorphism, thus by assumption it is an isomorphism, i.e. $\operatorname{Ker}\beta \subset \operatorname{Im}\alpha$; since g is injective $\operatorname{Ker}\beta = 0$.

(c) Only the statement about g_1 to be proved: If $g_1 = \beta_1\alpha_1$ where β_1 is small epimorphism, then, since R is hereditary and g_2 is injective, we have the following commutative diagram with exact rows:

$$\begin{tikzcd}
0 \arrow[r] & \cdot \arrow[r] \arrow[d, equal] & \cdot \arrow[r, "\beta_1"] \arrow[d] & \cdot \arrow[r] \arrow[d, "g_2"] & 0 \\
0 \arrow[r] & \cdot \arrow[r] & \cdot \arrow[r, "\beta"'] & C \arrow[r] & 0.
\end{tikzcd}$$

Then it is clear that, β is also a small epimorphism, thus, since g is an isomorphism, β_1 is bijective, too.

□

Theorem 4.1 *For a homomorphism $g : C' \to C$ the following are equivalent:*

(i) *g is coneat.*

(ii) *$\operatorname{Ker} g$ is coclosed in C', and $\operatorname{Im} g \supset \operatorname{Soc}(C)$.*

(iii) *$g(C'[p]) = C[p]$ for all prime elements p of R.*

(iv) *If the diagram below is a pullback diagram and β is a small epimorphism, then β' is also a small epimorphism.*

$$\begin{tikzcd} B' \arrow[r, "\beta'"] \arrow[d, "g'"'] & C' \arrow[d, "g"] \\ B \arrow[r, "\beta"'] & C \end{tikzcd}$$

Proof (i $\Rightarrow$ ii) The restriction of g on $\operatorname{Im} g$ is again coneat by the lemma, thus $\operatorname{Ker} g$ is coclosed in C'. Naturally, the inclusion $\operatorname{Im} g \subset C$ is also coneat, and for a intermediate module X with $\operatorname{Im} g \subset^{\oplus} X$, X is essential in C gives by (b) that $X/\operatorname{Im} g$ has no small cover, thus it is torsion-free, hence $\operatorname{Soc}(C) = \operatorname{Soc}(X) \subset \operatorname{Im} g$.
(ii $\Rightarrow$ iii) From $c \in C[p]$ we have $c = g(z)$ for some $z \in C'$, $pz \in \operatorname{Ker} g \cap pC'$, $pz = pz_1$ for some $z_1 \in \operatorname{Ker} g$, $g(z - z_1) = c$ with $z - z_1 \in C'[p]$.
(iii $\Rightarrow$ iv) Let a pullback diagram be given as in theorem and β be a small epimorphism. Since β' is surjective and $\operatorname{Ker} \beta'$ is coatomic, it is only to be shown that $\operatorname{Ker} \beta' \subset pB'$ for all prime elements p of R: From $y \in \operatorname{Ker} \beta'$ we have $g'(y) \in \operatorname{Ker} \beta$, $g'(y) = pb$ for some $b \in B$, $\beta(b) \in C[p]$, $\beta(b) = g(z)$ for some $z \in C'[p]$, $z = \beta'(y_1)$ and $b = g'(y_1)$ for some $y_1 \in B'$, $y - py_1 \in \operatorname{Ker} g' \cap \operatorname{Ker} \beta' = 0$, $y = py_1$.
(iv $\Rightarrow$ i) Clear □

Corollary 4.4 *If $g : C' \to C$ is coneat, then $g^* : \operatorname{Ext}_R(C, A) \to \operatorname{Ext}_R(C', A)$ preserves κ-elements.*

Proof Consider the diagram (II) in Lemma 4.1 where V is again a supplement of $\operatorname{Ker} \beta$ in B. Then $V' = g'^{-1}(V)$ is a supplement of $\operatorname{Ker} \beta'$ in B', then it is only to be shown that $V' \cap \operatorname{Ker} \beta'$ is small in V', and since g is coneat this follows from the pullback diagram

$$\begin{tikzcd} V' \arrow[r, "\beta'|_{V'}"] \arrow[d] & C' \arrow[d, "g"] \\ V \arrow[r, "\beta|_V"'] & C. \end{tikzcd}$$

□

Corollary 4.5 *g^* : $\mathrm{Ext}_R(C, A) \to \mathrm{Ext}_R(C', A)$ preserves κ-elements if g satisfies the following two conditions:*

(a) $\mathrm{Im}\, g \supset \mathrm{Soc}(C)$.

(b) $\mathrm{Ker}\, g$ *is supplemented and has a supplement in every extension.*

Proof Since the inclusion $\mathrm{Im}\, g \subset C$ is coneat, so has the desired property, one can assume g is surjective, thus the corresponding pullback diagram is in the particular form

$$\begin{tikzcd} B' \arrow[r] \arrow[d] & B'/Y \arrow[d, "g"] \\ B'/X \arrow[r, "\beta"'] & B'/(X+Y), \end{tikzcd}$$

besides, there is a supplement V/X of $(X+Y)/X$ in B'/X. By the second assumption on $\mathrm{Ker}\, g \cong X$, we now have $X \subset^{\kappa} V$, thus $X + Y \subset^{\kappa} B'$ and from the fact X is supplemented (Zöschinger 1974a) (Lemma 1.3), finally we have $Y \subset B'$. □

Corollary 4.6 *A module M has a torsion-free cover if and only if there is $n \geq 0$ with* $\dim(M[p]) = n$ *for all prime elements p of R.*

Proof *Step 1.* M has a torsion-free cover if and only if $\mathrm{Soc}(M)$ has a torsion-free cover. Namely, if $h : H \to M$ is a torsion-free cover, then, since the inclusion $\mathrm{Soc}(M) \subset M$ is coneat, $h^{-1}(\mathrm{Soc}(M)) \to \mathrm{Soc}(M)$ is also a torsion-free cover. Conversely, if one has a torsion-free cover $h : H \to \mathrm{Soc}(M)$, then, since R is hereditary and h is surjective, there is a commutative diagram with exact rows

$$\begin{tikzcd} 0 \arrow[r] & H \arrow[r] \arrow[d, "h"'] & H_1 \arrow[r] \arrow[d, "h_1"] & \cdot \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & \mathrm{Soc}(M) \arrow[r, phantom, "\subset"] & M \arrow[r] & \cdot \arrow[r] & 0, \end{tikzcd}$$

and it is clear that h_1 is again a small epimorphism, and H_1 is torsion-free.

Step 2. If all p-Socles of M have the same dimension n, then one can define $R \subset S \subset K$ by $S/R = \mathrm{Soc}(K/R)$, and then $S^n \longrightarrow \mathrm{Soc}(K/R)^n \xrightarrow{\cong} \mathrm{Soc}(M)$ is a torsion-free cover. Conversely, if M has a torsion-free cover, then the injective hull of $T(M)$ also has a torsion-free cover i.e. we can assume M is divisible and torsion. For a torsion-free cover $h : H \to M$, $\mathrm{Ker}\, h$ is coatomic and essential in H, thus $\mathrm{Rank}(H) = \mathrm{Rank}(\mathrm{Ker}\, h)$ is finite, therefore $H \cong K^n$ for some $n \geq 0$. If one chooses $F \subset \mathrm{Ker}\, h$ where $F \cong R^n$, then $\mathrm{Ker}\, h/F$ is torsion and direct sum of cyclics, thus $M \cong H/\mathrm{Ker}\, h \cong (H/F)/(\mathrm{Ker}\, h/F) \cong H/F \cong (K/R)^n$. □

We only want to formulate, but not to give the proofs of the corresponding characterization of neat homomorphisms which is simplified by the existence of an injective hull.

Theorem 4.2 *For a homomorphism $f : A \to A'$ the following are equivalent:*

(i) *f is neat.*

(ii) $\operatorname{Im} f$ *is closed in A', and* $\operatorname{Ker} f \subset \operatorname{Rad}(A)$.

(iii) *$f^{-1}(pA') = pA$ for all prime elements p of R.*

(iv) *If the following diagram is a pushout diagram and α is a small monomorphism, then α' is also a small monomorphism.*

$$\begin{tikzcd} A \arrow[r, "\alpha"] \arrow[d, "f"'] & B \arrow[d, "f'"] \\ A' \arrow[r, "\alpha'"'] & B' \end{tikzcd}$$

Corollary 4.7 *$f : A \to A'$ is neat and coneat if and only if* $\operatorname{Ker} f$ *is divisible and* $\operatorname{Coker} f$ *is torsion-free.*

A close connection between neat- and coneat-homomorphisms gives, if one examines what makes pushouts and pullbacks the Hom and Ext functors resp.. We need the assertions in the next chapters always for a single prime element p of R:

Definition 4.2 *$f : A \to A'$ is called p-*neat *if $f^{-1}(pA') = pA$;*
*$g : C' \to C$ is called p-*coneat *if $g(C'[p]) = C[p]$.*

Accordingly, it is favourable to use the functor $Ap = A/pA$ besides the p-Socle $C[p]$ for the following proofs. A homomorphism α is p-neat if and only if αp is a (splitting) monomorphism; and it is p-coneat if and only if $\alpha[p]$ is a (splitting) epimorphism.

Lemma 4.3 *Let the following first diagram be a pushout diagram and the second be a pullback diagram:*

$$\begin{tikzcd} A \arrow[r, "\alpha"] \arrow[d, "f"'] & B \arrow[d, "f'"] \\ A' \arrow[r, "\alpha'"'] & B' \end{tikzcd} \qquad \begin{tikzcd} B' \arrow[r, "\beta'"] \arrow[d, "g'"'] & C' \arrow[d, "g"] \\ B \arrow[r, "\beta"'] & C. \end{tikzcd}$$

Then we have

(I) *If f is p-neat, then so is f'. In addition, if α is p-coneat, then so is α'.*

(II) *If g is p-coneat, then so is g'. In addition, if β is p-neat, then so is β'.*

Proof (I) The functor $\{p\}$ makes again a pushout from the first diagram so that both $f\{p\}$ and $f'\{p\}$ are monomorphisms. From the assumption, f is p-neat, further we have $B'[p] = \alpha'(A'[p]) + f'(B[p])$, so that both α and α' are coneat. $\square$

Theorem 4.3 (1) *For given X and $\varphi : Y \to Y'$ we have $\varphi_* : \mathrm{Hom}(X, Y) \to \mathrm{Hom}(X, Y')$ is p-coneat if and only if φ is p-coneat or X is p-divisible.*

(2) *For given C and $f : A \to A'$ we have $f_* : \mathrm{Ext}_R(C, A) \to \mathrm{Ext}_R(C, A')$ is p-neat if and only if f is p-neat or $T_p(C) = 0$.*

(3) *For given Y and $\gamma : X' \to X$ we have $\gamma^* : \mathrm{Hom}(X, Y) \to \mathrm{Hom}(X', Y)$ is p-coneat if and only if γ is p-neat or $T_p(Y) = 0$.*

(4) *For given A and $g : C' \to C$ we have $g^* : \mathrm{Ext}_R(C, A) \to \mathrm{Ext}_R(C', A)$ is p-neat if and only if g is p-coneat or A is p-divisible.*

Proof As is well known, one has the natural isomorphisms

$$\mathrm{Hom}(X\{p\}, Y[p]) \xrightarrow{\cong} \mathrm{Hom}(X, Y)[p]$$

resp.

$$\mathrm{Ext}_R(C, A)[p] \xrightarrow{\cong} \mathrm{Ext}_R(C[p], A\{p\}).$$

With their help, one obtains the following commutative squares from the given four homomorphisms

$$\begin{tikzcd}
\mathrm{Hom}(X\{p\}, Y[p]) \arrow[r, "\varphi[p]."] \arrow[d, "\cong"'] \arrow[dr, phantom, "(1)"] & \mathrm{Hom}(X\{p\}, Y'[p]) \arrow[d, "\cong"] \\
\mathrm{Hom}(X, Y)[p] \arrow[r, "\varphi_*[p]"] & \mathrm{Hom}(X, Y')[p]
\end{tikzcd}$$

$$
\begin{tikzcd}
\mathrm{Ext}_R(C,A)\{p\} \arrow[rr, "f_*\{p\}"] \arrow[d, "\cong"'] \arrow[drr, phantom, "(2)"] & & \mathrm{Ext}_R(C,A')\{p\} \arrow[d, "\cong"] \\
\mathrm{Ext}_R(C[p],A\{p\}) \arrow[rr, "f\{p\}_{\cdot}"] & & \mathrm{Ext}_R(C[p],A'\{p\})
\end{tikzcd}
$$

$$
\begin{tikzcd}
\mathrm{Hom}(X\{p\},Y[p]) \arrow[rr, "\gamma\{p\}^{\cdot}"] \arrow[d, "\cong"'] \arrow[drr, phantom, "(3)"] & & \mathrm{Hom}(X'\{p\},Y[p]) \arrow[d, "\cong"] \\
\mathrm{Hom}(X,Y)[p] \arrow[rr, "\gamma^*[p]"] & & \mathrm{Hom}(X',Y)[p]
\end{tikzcd}
$$

$$
\begin{tikzcd}
\mathrm{Ext}_R(C,A)\{p\} \arrow[rr, "g^*\{p\}"] \arrow[d, "\cong"'] \arrow[drr, phantom, "(4)"] & & \mathrm{Ext}_R(C',A)\{p\} \arrow[d, "\cong"] \\
\mathrm{Ext}_R(C[p],A\{p\}) \arrow[rr, "g[p]^{\cdot}"] & & \mathrm{Ext}_R(C'[p],A\{p\}).
\end{tikzcd}
$$

So the direction "$\Leftarrow$" is clear in all four cases, i.e. on Hom p-coneat homomorphisms and on Ext_R p-neat homomorphisms are induced. In the cases (1) and (2) resp. also the converse is clear since $\mathrm{Hom}(R/(p),Y[p]) \cong Y[p]$, $\mathrm{Ext}_R(R/(p),A\{p\} \cong A\{p\}$ resp.. In the case (3) and (4) one has, if $Y[p] \neq 0$, $A\{p\} \neq 0$ resp. by the assumption

surjective $(\gamma\{p\})^{\cdot} : \mathrm{Hom}(X\{p\},R/(p)) \to \mathrm{Hom}(X'\{p\},R/(p))$

injective $(g[p])^{\cdot} : \mathrm{Ext}_R(C[p],R/(p)) \to \mathrm{Ext}_R(C'[p],R/(p))$ resp..

With the equation $\gamma\{p\} = \gamma\{p\} \circ \sigma \circ \gamma\{p\}$, $g[p] = g[p] \circ \rho \circ g[p]$ resp. one has $(1-\sigma \circ \gamma\{p\})^{\cdot} = 0$, $(1-g[p] \circ \rho)^{\cdot} = 0$ resp., both times thus a semisimple p-module G with an endomorphism u, for $Fu = 0$ holds where $F = \mathrm{Hom}(-,R/(p))$, $F = \mathrm{Ext}_R(-,R/(p))$ resp.. But however, if $u = 0$, then the decomposition $u = G \xrightarrow{\pi} \mathrm{Im}\, u \xrightarrow{\iota} G$ yields $0 = FG \xrightarrow{F\iota} F\,\mathrm{Im}\, u \xrightarrow{F\pi} FG$ where $F\pi$ is a splitting monomorphism, $F\iota$ is a splitting epimorphism, thus $F\,\mathrm{Im}\, u = 0$, $\mathrm{Im}\, u = 0$. This implies however $\sigma \circ \gamma\{p\} = 1$, $g[p] \circ \rho = 1$ resp. as desired. □

Corollary 4.8 *If C is a module where $T_p(C) \neq 0$ for all prime elements p of R and* $\mathrm{Ext}_R(C,A)^{\kappa} \subset q\,\mathrm{Ext}_R(C,A)$, *then every element of* $\mathrm{Ext}_R(C,A)$ *is divisible by q.*

Proof We must show that A is q-divisible. Since all primary components of C are nonzero, there is a coneat-homomorphism $g : C \to K/R$ such that $g^* : \operatorname{Ext}_R(K/R, A) \to \operatorname{Ext}_R(C, A)$ is neat and also preserves κ-elements, and then $\operatorname{Ext}_R(K/R, A)^\kappa \subset q \operatorname{Ext}_R(K/R, A)$. For the connecting homomorphism $\delta : A \to \operatorname{Ext}_R(K/R, A)$ we have $\operatorname{Im}\delta \subset \operatorname{Ext}_R(K/R, A)^\kappa$ such that $\operatorname{Im}\delta$, as pure submodule of Ext_R is itself q-divisible. Since also $\operatorname{Ker}\delta = D(A)$, the assertion follows. □

Under additional conditions one can also obtain that Hom preserves p-neat homomorphisms and Ext preserves p-coneat homomorphisms:

Lemma 4.4 *Let the four homomorphisms be given as in Theorem 4.3 :*

(1) *If X is torsion-free and Y is cotorsion, then we have $\varphi_* : \operatorname{Hom}(X, Y) \to \operatorname{Hom}(X, Y')$ is p-neat if and only if φ is p-neat or X is p-divisible.*

(2) *If C is divisible and A' is reduced, then we have $f_* : \operatorname{Ext}_R(C, A) \to \operatorname{Ext}_R(C, A')$ is p-coneat if and only if f is p-coneat or $T_p(C) = 0$.*

(3) *If Y is divisible, then we have $\gamma^* : \operatorname{Hom}(X; Y) \to \operatorname{Hom}(X', Y)$ is p-neat if and only if γ is p-coneat or $T_p(Y) = 0$.*

(4) *If C' is torsion and A is torsion-free, then we have $g^* : \operatorname{Ext}_R(C, A) \to \operatorname{Ext}_R(C', A)$ is p-coneat if and only if g is p-neat or A is p-divisible.*

Proof We will prove only the first one. By (1) one has a short exact sequence $0 \longrightarrow X \longrightarrow Q \longrightarrow C \longrightarrow 0$ where Q is torsion-free divisible, and hence a commutative diagram with exact rows:

$$\begin{tikzcd}
\operatorname{Hom}(Q, Y) \arrow[r] & \operatorname{Hom}(X, Y) \arrow[r, "\delta"] \arrow[d, "\varphi_*"'] & \operatorname{Ext}_R(C, Y) \arrow[r] \arrow[d, "\varphi_."] & 0 \\
\operatorname{Hom}(Q, Y') \arrow[r] & \operatorname{Hom}(X, Y') \arrow[r, "\delta'"] & \operatorname{Ext}_R(C, Y') \arrow[r] & \operatorname{Ext}_R(Q, Y').
\end{tikzcd}$$

It is obvious that δ and δ' are both neat and coneat. Now if φ is p-neat, then by Theorem (4.3, 2) $\varphi_.$ is also p-neat. Therefore φ_* is p-neat; but if X is divisible, then $\operatorname{Hom}(X, Y)$ is also divisible, such that φ_* is trivially p-neat. Conversely, let φ_* be p-neat and X is not p-divisible: Then $\varphi_.$ is p-neat, due to the surjectivity of δ. Furthermore $T_p(C) \neq 0$, and again by Theorem (4.3, 2) φ is p-neat. □

CHAPTER 5

FOR THE PROBLEM $\mathrm{Ext}_R(C,A)^\kappa = \mathrm{Ext}_R(C,A)^\beta$

The submodule of the β-elements of $\mathrm{Ext}_R(C,A)$ can be described according to Lemma 3.2 with the help of projective resolution of C. If almost all primary components in C are zero, it coincides with the set of κ-elements of $\mathrm{Ext}_R(C,A)$. We want to set a question by this similarity, and give an answer in case $T(A) \subset^{\oplus} A$.

In the extreme case, when $\mathrm{Ext}_R(C,A)$ has no κ-elements at all, it was already done by the proof of Theorem 3.2 and by the Corollary to Theorem 4.3:

Theorem 5.1 *For a pair (A,C) the following are equivalent:*

(i) $\mathrm{Ext}_R(C,A)^\kappa \subset \mathrm{Rad}(\mathrm{Ext}_R(C,A))$.

(ii) $\mathrm{Ext}_R(C,A)^\kappa = 0$.

(iii) $\mathrm{Ext}_R(C,T(A))$ *is divisible, and if* $T_p(C) \neq 0$ *for all prime elements p of R, then A is divisible.*

Lemma 5.1 *If $g : C' \to C$ is a monomorphism, then* $g^* : \mathrm{Ext}_R(C,A) \to \mathrm{Ext}_R(C',A)$ *is also surjective on the κ-elements.*

Proof *Step 1.* Assume the sequence
$E' = 0 \longrightarrow A \longrightarrow B' \xrightarrow{\beta'} C' \longrightarrow 0$ is not only κ-exact, but also $\mathrm{Ker}\,\beta'$ is small in B'. Since g is injective, there is a commutative diagram with exact rows

$$\begin{tikzcd}
E' = 0 \arrow[r] & A \arrow[r] \arrow[d, equal] & B' \arrow[r, "\beta'"] \arrow[d] & C' \arrow[r] \arrow[d, "g"] & 0 \\
E = 0 \arrow[r] & A \arrow[r] & B \arrow[r, "\beta"'] & C \arrow[r] & 0,
\end{tikzcd}$$

and necessarily $\mathrm{Ker}\,\beta$ is small in B, thus certainly $[E] \in \mathrm{Ext}_R(C,A)^\kappa$ with $g^*([E]) = [E']$.

Step 2. Now let E' in be κ-exact, V be a supplement of $\mathrm{Ker}\,\beta'$ in B'. We obtain the following two diagrams where $A_1 = V \cap \mathrm{Ker}\,\beta'$

$$\begin{tikzcd}
E_1 = 0 \arrow[r] & A_1 \arrow[d, "f"'] \arrow[r, phantom, "\subset"] & V \arrow[r, "\beta'|_V"] \arrow[d, phantom, "\cap"] & C' \arrow[r] \arrow[d, equal] & 0 \\
E' = 0 \arrow[r] & A \arrow[r] & B' \arrow[r] & C' \arrow[r] & 0
\end{tikzcd}$$

and

$$\begin{tikzcd}
\mathrm{Ext}_R(C,A_1) \arrow[r, "g^{\cdot}"] \arrow[d, "f_*"'] & \mathrm{Ext}_R(C',A_1) \arrow[d, "f_\cdot"] \\
\mathrm{Ext}_R(C,A) \arrow[r, "g^*"'] & \mathrm{Ext}_R(C',A).
\end{tikzcd}$$

However there is just one $x \in \mathrm{Ext}_R(C,A_1)$ for E_1 where $g^{\cdot}(x) = [E_1]$ such that $f_*(x) \in \mathrm{Ext}_R(C,A)^\kappa$ with $g^*(f*(x)) = f_\cdot([E_1]) = [E']$ as desired. (Similarly, one can show by the second step that g^* is also surjective on β-elements.) □

Lemma 5.2 (a) *Let $(C_i \mid i \in I)$ be a nonempty family of modules,*

$$\omega : \mathrm{Ext}_R(\coprod C_i, A) \to \prod \mathrm{Ext}_R(C_i, A)$$

be the canonical isomorphism and $x \in \mathrm{Ext}_R(\coprod C_i, A)$. Then; if all projections of $w(x)$ are κ-elements, and almost every projection of $w(x)$ is equal to zero, then x is also a κ-element.

(b) *Let C be a torsion module, $\omega : \mathrm{Ext}_R(C,A) \to \prod \mathrm{Ext}_R(T_p(C),A)$ be the canonical isomorphism and $x \in \mathrm{Ext}_R(C,A)$. Then; if all projections of $w(x)$ are κ-elements, then x is also a κ-element.*

Proof

(a) Let $E = 0 \longrightarrow A \xrightarrow{\alpha} B \xrightarrow{\beta} \coprod C_i \longrightarrow 0$ be an exact sequence where $[E] = x$. With the inclusion $\epsilon_j : C_j \to \coprod C_i$ and with $B_j = \beta^{-1}(\mathrm{Im}\,\epsilon_j)$ we obtain

$$\begin{tikzcd}
E\epsilon_j \equiv 0 \arrow[r] & A \arrow[r] \arrow[d, equal] & B_j \arrow[r] \arrow[d, phantom, "\cap"] & C_j \arrow[r] \arrow[d, "\epsilon_j"] & 0 \\
E = 0 \arrow[r] & A \arrow[r, "\alpha"'] & B \arrow[r, "\beta"'] & \coprod C_i \arrow[r] & 0,
\end{tikzcd}$$

as well as $B/\,\mathrm{Im}\,\alpha = \bigoplus_{i\in I} B_i/\,\mathrm{Im}\,\alpha$. By assumption we have a supplement V_j of $\mathrm{Im}\,\alpha$ in B_j, for each $j \in I$, and the additional condition $V_j \cap \mathrm{Im}\,\alpha = 0$ for almost all j. Then $(\sum V_j) + \mathrm{Im}\,\alpha = B$ and $(\sum V_j) \cap \mathrm{Im}\,\alpha = \sum(V_j \cap \mathrm{Im}\,\alpha)$, the sum is finite due to the additional condition, thus it is small in $\sum V_i$.

(b) With the corresponding descriptions each V_p is a small cover of $B_p/\,\mathrm{Im}\,\alpha \cong T_p(C)$ and p-primary, thus without any extra condition $\sum(V_p \cap \mathrm{Im}\,\alpha)$ is small in $\sum V_p$.

□

Remark 5.1 *$x \in \mathrm{Ext}_R(\coprod C_i, A)$ can be a κ-element, without any projection of $\omega(x)$. As an example one can choose $\omega : \mathrm{Ext}_R(K/R, R) \xrightarrow{\cong} \prod \mathrm{Ext}_R(R(p^\infty), R$ and $x = [0 \to R \subset K \to K/R \to 0]$.*

Remark 5.2 *By (a) it follows directly that if each $\mathrm{Ext}_R(C_1, A), \ldots, \mathrm{Ext}_R(C_n, A)$ is κ-full, then $\mathrm{Ext}_R(\bigoplus_{i=1}^n C_i, A)$ is also κ-full. It is not true for infinitely many summands. For example, let $C = R/(p)$ and A be a reduced unbounded p-module. Indeed $\mathrm{Ext}_R(C, A)$ is κ-full, however $\mathrm{Ext}_R(C^{(\mathbb{N})}, A)$ is not, since there is an epimorphism of A on $M = \coprod_{n=1}^\infty R/(Rp^n)$, and $\mathrm{Ext}_R(M/pM, pM)$ is not κ-full by (Zöschinger 1974b) (Satz 5.3).*

Lemma 5.3 (a) *If $\mathrm{Ext}_R(C, A)^\kappa \subset^\kappa T(\mathrm{Ext}_R(C, A))$, then $\mathrm{Ext}_R(C, T(A))$ is divisible by almost all prime elements p of R.*

(b) *If $\mathrm{Ext}_R(C, T(A))$ is divisible by almost all prime elements p of R, and either a primary component of C is equal to zero or A is torsion, then $\mathrm{Ext}_R(C, A)^\kappa = \mathrm{Ext}_R(C, A)^\beta$.*

Proof

(a) By Lemma 5.1 each κ-element in $\mathrm{Ext}_R(T(C), A)$ is also a torsion element, and since the claimed divisibility condition depends only on $T(C)$, we can assume that C is torsion. Choose $x \in \mathrm{Ext}_R(C, A)$ such that by the isomorphism $\omega : \mathrm{Ext}_R(C, A) \to \prod \mathrm{Ext}_R(T_p(C), A)$ all projections of $\omega(x)$ are κ-elements, and only the p-th projection is zero when $\mathrm{Ext}_R(T_p(C), A)^\kappa = 0$. By Lemma 5.2b we have $x \in \mathrm{Ext}_R(C, A)^\kappa$, so the assumption gives $\omega(x) \in T(\prod \mathrm{Ext}_R(T_p(C), A)) = \coprod T(Ext_R(T_p(C), A))$. By choice of $\omega(x)$ therefore $\mathrm{Ext}_R(T_p(C), A)^\kappa = 0$ for almost all prime elements p of R, and this indicates just the p-divisibility of $\mathrm{Ext}_R(C, T(A))$ for almost all prime elements p of R by Theorem 5.1.

(b) The assumptions also hold for $A' = A/D(A)$, and since the canonical isomorphism $v_* : \mathrm{Ext}_R(C, A) \to \mathrm{Ext}_R(C, A')$ preserves κ-elements and also reflects β-elements, we will assume that A is reduced. Now let the sequence $0 \longrightarrow A \xrightarrow{\alpha} B \longrightarrow C \longrightarrow 0$ be κ-exact, and V be a supplement of $\mathrm{Im}\,\alpha$ in B.

Case 1. At least one primary component of C is zero. Then $V \cap \mathrm{Im}\,\alpha$ is torsion and coatomic, in addition $T_p(V \cap \mathrm{Im}\,\alpha) = 0$ for almost all prime elements p of R, thus $V \cap \mathrm{Im}\,\alpha$ is bounded.

Case 2. A is torsion. For a moment we can assume that $T_p(C) \neq 0$ for all prime elements p of R (otherwise we have done) such that currently according to assumption $T_p = 0$ for almost all prime elements p of R, thus again $V \cap \operatorname{Im} \alpha$ is bounded.

□

Corollary 5.1 *If* $\operatorname{Ext}_R(C, \operatorname{Soc}(K/R))^\kappa \subset T(\operatorname{Ext}_R(C, \operatorname{Soc}(K/R)))$, *then almost all primary components of C are zero, and then* $\operatorname{Ext}_R(C, A)^\kappa = \operatorname{Ext}_R(C, A)^\beta$ *for all prime elements p of R.*

Lemma 5.4 *For a module A the following are equivalent:*

(i) $\operatorname{Ext}_R(\operatorname{Soc}(K/R), A)^\kappa = \operatorname{Ext}_R(\operatorname{Soc}(K/R), A)^\beta$.

(ii) *A is divisible by all prime elements p of R, and $A/T(A)$ is divisible.*

(iii) $\operatorname{Ext}_R(C, A)$ *is β-full for each torsion module C where all of the primary components are finitely generated.*

Proof (i ⇒ ii) Simply $T(A)$ is divisible by almost all prime elements p of R. Now let $S/R = \operatorname{Soc}(K/R)$ and $\delta : A \to \operatorname{Ext}_R(S/R, A)$ be the connecting homomorphism belonging to the κ-exact sequence $0 \longrightarrow R \subset S \longrightarrow S/R \longrightarrow 0$. Obviously, each element of $\operatorname{Im} \delta$ is a κ-element, so by assumption it is a β-element, and this indicates $T(A) + \operatorname{Ker} \delta = A$ by Lemma 3.2. But since $\operatorname{Ker} \delta = \operatorname{Rad}(A)$, the divisibility of $A/T(A)$ holds.
(ii ⇒ iii) From the assumption on C, $\operatorname{Ext}_R(C, T(A))$ is κ-full, hence also β-full by Lemma 5.3b. Since $\iota_* : \operatorname{Ext}_R(C, T(A)) \to \operatorname{Ext}_R(C, A)$ is an isomorphism, the claim follows. □

Theorem 5.2 *Let C be arbitrary,* $T(A) \subset^\oplus A$. *Then the following are equivalent:*

(i) $\operatorname{Ext}_R(C, A)^\kappa = \operatorname{Ext}_R(C, A)^\beta$.

(ii) $\operatorname{Ext}_R(C, T(A))$ *is divisible by almost all prime elements p of R, and if* $T_p(C) \neq 0$ *for all prime elements p of R, then* $A/T(A)$ *is divisible.*

Proof With the statements on hand there is almost nothing more to be shown.
(i $\Rightarrow$ ii) By Lemma 5.3a we can assume $T_p(C) \neq 0$ for all prime elements p of R. Then there is a monomorphism $g : \mathrm{Soc}(K/R) \to C$, and since $g^* : \mathrm{Ext}_R(C, A) \to \mathrm{Ext}_R(\mathrm{Soc}(K/R), A)$ is surjective on the κ-elements, $A/T(A)$ is divisible by Lemma 5.4.
(ii $\Rightarrow$ i) Again due to Lemma 5.3b we can assume $T_p(C) \neq 0$ for all prime elements p of R. Since A splits and $A/T(A)$ is divisible, the isomorphism $\iota_* : \mathrm{Ext}_R(C, T(A)) \to \mathrm{Ext}_R(C, A)$ is also surjective on κ-elements, and by Lemma 5.3b the κ-elements coincide with the β-elements in $\mathrm{Ext}_R(C, T(A))$, and also in $\mathrm{Ext}_R(C, A)$. □

Remark 5.3 *The proof shows that inclusion (i $\Rightarrow$ ii) also holds without the important assumption $T(A) \subset^{\oplus} A$. It is not the case for (ii $\Rightarrow$ i): Let M be a reduced unbounded p-module and $0 \longrightarrow M \longrightarrow A \longrightarrow K \longrightarrow 0$ be representative of non zero elements of* $\mathrm{Ext}_R(K, M)$*. Then* $\mathrm{Ext}_R(K/R, T(A))$ *is divisible by all prime elements p of R except p and $A/T(A) \cong K$. However since $T(A)$ does not split, the image of connecting homomorphism $A \to \mathrm{Ext}_R(K/R, A)$ does not consist of only torsion elements, and hence* $\mathrm{Ext}_R(K/R, A)^{\beta} \subsetneq \mathrm{Ext}_R(K/R, A)^{\kappa}$.

CHAPTER 6

THE DEPTH SEQUENCE OF MODULE ELEMENTS

In the next chapter, the κ-elements of $\mathrm{Ext}_R(C, A)$ will be characterized by divisibility property. In Theorem 7.3 it is found that why the usual measure – the p-height of $x \in G$ – is too strong. While the p-height of x is the *greatest* p-power which can be withdrawn from x, we are interested in the *smallest* p-power which must be withdrawn from x, therefore "the rest" (of course not uniquely determined) is no longer divisible by p.

Definition 6.1 *For $x \in G$, $t_p^G(x) = \inf\{i \in \mathbb{N} \mid$ there is $y \in G \setminus pG$ where $x = p^i y\}$ is called the p-*depth *of x in G and*

$$t^G(x) = (t_p(x)) \in \textstyle\prod_{p \in P} \mathbb{N}^p$$

is called the depth sequence *of x in G where p_i are all prime elements of R.*

Thus $t_p^G(x)$ is an element of $\mathbb{N} \cup \{\infty\}$, and it coincides with $h_p^G(x)$ for a torsion-free G. However in general the p-depth is smaller than the p-height, e.g. for $n \geqslant 1$ the p-depth of zero elements of $R/(Rp^n)$ are just n. First we want to derive something over $t_p^G(0)$.

Lemma 6.1 *If V is a supplement of $G[p^n]$ in G, then V is a direct summand in G.*

Proof It remains show that the canonical epimorphism $G[p^n] \to G/V$ splits, thus its kernel $V[p^n]$ is p-pure in $G[p^n]$, and it is obviously equivalent with $\{x \in G \mid p^n x = 0, p^i x \in V\} \subset G[p^i] + V$ for all $0 \leqslant i \leqslant n$. We show this by induction on i. For $i = 0$ there is nothing to prove. For $i + 1$ we choose $x \in G$ where $p^n x = 0$, $p^{i+1} x \in V$. By inductive hypothesis, $px = x_1 + v$ where $p^i x_1 = 0$. Since $v = px - x_1 \in V \cap G[p^n] \subset pV$, we have $v = pv_1$, thus $x = (x - v_1) + v_1$ where $p^{i+1}(x - v_1) = p^i(x_1 + v - v) = 0$. □

Remark 6.1 *The dual case is a well-known theorem of Khabbaz (Khabbaz 1961): Every intersection complement of $p^n[G]$ is a direct summand. Our statement is true in general, indeed $G[p^n]$ (more generally every bounded submodule) has a supplement in G.*

Lemma 6.2 *For an R-module G, the following are equivalent:*

(i) $t_p^G(0) \leqslant n$.

(ii) $p^n G$ *is not essential in* G.

(iii) $G[p^n]$ *is not small in* G.

(iv) $G = G_1 \oplus G_2$ *with* $G_1 \cong R/(p^e)$ *where* $1 \leqslant e \leqslant n$.

Proof (i ⇒ iii) There is $y \in G \setminus pG$ where $y \in G[p^n]$, i.e. $G[p^n] \not\subset pG$. But then $G[p^n]$ is not small in G.
(iii ⇒ iv) For a supplement V of $G[p^n]$ in G we get $V \oplus X = G$ by what we have done above. By assumption $X \neq 0$, and of course bounded by p^n, so that the claim holds.
(iv ⇒ ii) $G_1[p] \not\subset p^n G_1$, therefore $p^n G$ is not essential in G.
(ii ⇒ i) There is an element $y \notin G[p]$ such that $y \notin p^n G$, thus there is also an element $y_1 \in G \setminus pG$ such that $y = p^e y_1,\ e < n$. From $0 = p^{e+1} y_1$ the claim holds. □

Remark 6.2 *Obviously,* $t_p^G = \infty$ *if and only if* $T_p(G)$ *is divisible. If* $t_p^G(0)$ *is finite, then* $\sigma = t_p^G(0) - 1$ *where* σ *is the smallest such number where* σ*-th Ulm invariant of* $T_p(G)$ *is nonzero.*

Theorem 6.1 *For* $x \in G$ *we have* $t_p^G(x) = \min(h_p^G(x), t_p^G(0))$.

Proof (1) We always have $t_p^G(x) \leqslant h_p^G(x)$, because for $h_p^G(x)$ there is nothing to prove, and it follows from $h_p^G(x) = n$ that $x = p^n y$ where $y \in G \setminus pG$, so $t_p^G(x) \leqslant n$.
(2) We always have $t_p^G(x) \leqslant t_p^G(0)$, because for $t_p^G(0) = \infty$ there is nothing to prove, and from $t_p^G(0) = n$ we get $0 = p^n z$ where $z \in G \setminus pG$. For all $y \in G$, now we have $z + py \notin pG$, thus $t_p^G(p^{n+1} y_1) \leqslant n$; however if $x \notin p^{n+1} G$, then $t_p^G(x) \leqslant n$.
(3) From $t_p^G(x) < h_p^G(x)$ where $t_p^G(x) = n$, we get $y_1 \in G$ and $y_2 \in G \setminus pG$ with $x = p^{n+1} y_1 = p^n y_2$, so that $0 = p^n(py_1 - y_2)$ gives $t_p^G(0) \leqslant n$, thus $t_p^G(x) = t_p^G(0)$ □

Corollary 6.1 $t_p^G(x) \leqslant n$ *if and only if there is an element* $y \in G \setminus pG$ *where* $p^n y \in Rx$.

Lemma 6.3 *For* $x \in G$, $0 \neq r \in R$ *we have* $t_p^G(rx) = \min(t_p^G(x) + e, t_p^G(0))$, *where e is the highest p-power in r (and if necessary* $\infty + e = \infty$*).*

Proof (1) We have $t_p^G(rx) \leqslant t_p^G(x) + e$, because for $t_p^G(x) = \infty$ there is nothing to prove, and from $t_p^G(x) = n$ we get $x = p^n y$ with $y \in G \setminus pG$, so that where $r = r'p^e$, further we get $rx = p^{n+e}(r'y)$, $r'y \notin pG$, thus $t_p^G(rx) \leqslant n + e$.
(2) Clearly, by the theorem $t_p^G(rx) \leqslant t_p^G(0)$.
(3) Again by the theorem $t_p^G(rx) = \min(h_p^G(rx), t_p^G(0)) \geqslant \min(h_p^G(x) + e, t_p^G(0)) \geqslant \min(t_p^G(x) + e, t_p^G(0))$. □

Remark 6.3 *The formula means in particular that if $t_p^G(x) = t_p^G(y)$, then $t_p^G(rx) = t_p^G(ry)$ for all $r \in R$. The corresponding statement is not true for the p-height.*

Corollary 6.2 (a) *We always have $t_p^G(x) \leqslant t_p^G(px) \leqslant t_p^G(x) + 1$, and $t_p^G(x) = t_p^G(px)$ if and only if $t_p^G(x) = t_p^G(0)$.*

(b) *If $t_p^G(p^e x \leqslant e)$, then $t_p^G(x) = 0$ or $e \geqslant t_p^G(0)$.*

(c) *If $rx = sy$ where both r and s are nonzero, then $t_p^G(x) \sim t_p^G(y)$, where $\sim$ is the usual equivalence of sequences, see (Fuchs 1973) (p. 109). Particularly the depth sequence of torsion elements is equivalent to the depth sequence of zero elements.*

Lemma 6.4 *Let $f : A \to A'$ be a homomorphism. Then*

(I) *$t_p^{A'}(fa) \leqslant t_p^A(a)$ for all $a \in A$ if and only if f is p-neat.*

(II) *$t_p^{A'}(fa) \geqslant t_p^A(a)$ for all $a \in A$ if and only if $t_p^{A'}(0) \geqslant t_p^A(0)$. If it is satisfied, f is p-coneat.*

Proof

(I) If the inequality holds, f must be p-neat; and conversely, since f is p-neat, $t_p^A(a) = n$, we get $a = p^n a_1$ where $a_1 \in A \setminus pA$, thus $fa = p^n(fa_1)$ where $fa_1 \in A' \setminus pA'$, therefore $t_p^{A'}(fa) \leqslant n$.

(II) The equivalence is clear by Theorem 6.1. Now let f be p-coneat. From $t_p^{A'}(0) = n$, there exists $a' \in A'[p]$ where $a' \notin p^n A'$, additionally there is also $a \in A[p]$ where $fa = a'$, so that $a \notin p^n A$ must hold, and so $t_p^A(0) \leqslant n$.

□

Corollary 6.3 *If $x \in G$ and $f : G \to G$ is an endomorphism, then $t_p^G(fx) \geqslant t_p^G(x)$.*

Lemma 6.5 (a) *If $(A_i \mid i \in I)$ is a nonempty family of modules, and $x \in G = \prod_{i\in} A_i$, then $t_p^G(x) = \min\{t_p^{A_i}(x_i) \mid i \in I\}$; the same formula holds if $G = \coprod A_i$.*

(b) *If $U \subset \bigcap_{i\in I}^{\infty} p^i G$, then $t_p^{G/U}(\overline{x}) = t_p^G(x)$ for all $x \in G$.*

(c) *If $x \in U \subset G$ and U is p-pure in G, then $t_p^G(x) = \min(t_p^U(x), t_p^{G/U}(0))$.*

(d) *If X and Y are torsion modules and $\varphi \in \mathrm{Hom}(X, Y)$, then the p-depth of φ in $\mathrm{Hom}(X, Y)$ is equal to the p-depth of φ_p in $\mathrm{Hom}(T_p(X), T_p(Y))$.*

Proof

(a) For all $i \in I$, $t_p^G(x) \leqslant t_p^{A_i}(x_i)$, since the projections $G \to A_i$ are coneat; however also if $t_p^G(x) \geqslant \min$, then for $t_p^G(x) = \infty$ there is nothing to prove, and from $t_p^G(x) = n$ we get $x = p^n y$ where $y \in G \setminus pG$, so that at least one y_j is not divisible by p, thus $t_p^{A_j}(x_j) \leqslant n$. The proof is similar for the sum.

(b) Since the canonical map $G \to G/U$ is p-neat, $\leqslant$ holds; however $t_p^{G/U}(\overline{x}) = n$, hence from $\overline{x} = p^n\overline{y}$, we get $\overline{y}$ is not divisible by p, that $x - p^n y \in U$, $x - p^n y = p^{n+1}y_1$, thus $x = p^n(py_1 + y)$ where $py_1 + y \notin pG$, therefore $t_p^G(x) \leqslant n$.

(c) *Step 1.* $x = 0$. Since $U \subset G$ is p-neat, we get $t_p^G(0) \leqslant t_p^U(0)$, and since $G \to G/U$ p-coneat we get $t_p^G(0) \leqslant t_p^{G/U}(0)$, hence together $t_p^G(0) \leqslant \min$; however $t_p^G(0) < \min$ thus $U[p] \subset p^n U$ as well as $(G/U)[p] \subset p^n(G/U)$ for $n = t_p^G(0)$, so it follows from the p-purity that $G[p] \subset p^n G$ which is not possible.
Step 2. Let $x \in U$ be arbitrary. Then we have
$t_p^G(x) = \min(h_p^G(x), t_p^G(0)) = \min(h_p^U(x), t_p^U(0), t_p^{G/U}(0)) = \min(t_p^U(x), t_p^{G/U}(0))$.

(d) The canonical map $\mathrm{Hom}(X, Y) \to \mathrm{Hom}(T_p(X), T_p(Y))$ is an epimorphism with p-divisible kernel so that the claim yields by (b).

□

In view of Chapter 7, we are interested up until now the p-depth of elements of $\mathrm{Hom}(X, Y)$ and $\mathrm{Ext}_R(C, A)$, resp.. After Theorem 6.1 we proceed about the p-depth of zero elements, and we first want to obtain a lemma for the special case $X = R/(p^n)$ and $C = R/(p^n)$ respectively. Let $G\{p^n\} = G/p^nG$ as before.

Lemma 6.6 *For a module G;*

(I) *If $G[p^n] \neq 0$, then $t_p^{G[p^n]}(0) = \min(t_p^G(0), n)$.*

(II) *If $G\{p^n\} \neq 0$, then $t_p^{G\{p^n\}}(0) = \min(t_p^G(0), n)$ as before.*

Proof

(I) Since $n \geqslant 1$, the map $G[p^n] \subset G$ is p-coneat, thus $t_p^{G[p^n]}(0) \leqslant t_p^G(0)$, and since $G[p^n] \neq 0$, we have $t_p^{G[p^n]}(0) \leqslant n$. But if $e = t_p^{G[p^n]}(0)$ is properly smaller than $t_p^G(0)$, then it follows from $0 = p^e y$ where $y \in G[p^n]$, $y \notin p(G[p^n])$, that $y = py_1$ for some $y_1 \in G$, thus $e + 1 > n$, $e = n$.

(II) Similarly, for $e = t_p^{G\{p^n\}}(0)$, we have $e \leqslant \min(t_p^G(0))$. But if $e < n$, it follows from $0 = p^e\overline{y}$ where $\overline{y} \notin p(G\{p^n\})$ that $p^e y = p^{e+1}y_1$ for some $y_1 \in G$, thus $0 = p^e(py_1 - y)$ where $py_1 - y \notin pG$, so $e = t_p^G(0)$.

□

Theorem 6.2 *For a pair of modules (X, Y) and (A, C) resp. we have:*

(I) *If X is not p-divisible and $T_p(Y) \neq 0$, then*

$$t_p^{\mathrm{Hom}(X,Y)}(0) = \min(t_p^X(0), t_p^Y(0)).$$

(II) *If A is not p-divisible and $T_p(C) \neq 0$, then*

$$t_p^{\mathrm{Ext}_R(C,A)}(0) = \min(t_p^A(0), t_p^C(0)).$$

Proof The conditions on X, Y and A, C resp. indicate no proper limitation, because without the p-depth of zero elements of Hom and Ext, resp. they are infinite. Only for the proof we will write shortly $t(G)$, instead of $t_p^G(0)$.

(I) In particular, if $X = \bigoplus_{i \in I} X_i$, where $I \neq \emptyset$, $0 \neq X_i$ is cyclic and p-primary for all $i \in I$, then the formula is true, because by the lemma $t(\mathrm{Hom}(X_i, Y)) =$

$\min(t(X_i), t(Y))$, by Lemma 6.5a, thus $t(\mathrm{Hom}(X,Y)) = \min\{\min(t(X_i), t(Y)) \mid i \in I\} = \min(\min\{t(X_i) \mid i \in I\}, t(Y)) = \min(t(X), t(Y))$. Therefore for each non p-divisible X and each $n \geqslant 1$ we have:

$$\begin{aligned}\min(t(\mathrm{Hom}(X,Y)), n) = \quad & t(\mathrm{Hom}(X,Y)[p^n]) = \quad t(\mathrm{Hom}(X\{p^n\}, Y)) \\ = \quad & \min(t(X), t(Y), n). \qquad\qquad (*)\end{aligned}$$

Now if $t(X)$ or $t(Y)$ is finite, $\mathrm{Hom}(X,Y)$ contains a direct summand which is nonzero and bounded by a power of p, so that $t(\mathrm{Hom}(X,Y))$ is also finite. But if $t(X) = t(Y) = \infty$, then (*) implies that $t(\mathrm{Hom}(X,Y)) \geqslant n$ for all $n \geqslant 1$, thus also $t(\mathrm{Hom}(X,Y)) = \infty$.

(II) In particular, if $C = \bigoplus_{i\in I} C_i$, where $I \neq \emptyset$, each C_i is cyclic, p-primary and nonzero, then one can see that the formula is correct again by Lemma 6.5a and by the lemma. But if C is arbitrary where $T_p(C) \neq 0$, then for each $n \geqslant 1$ we have

$$\begin{aligned}\min(t(\mathrm{Ext}_R(C,A)), n) = \quad & t(\mathrm{Ext}_R(C,A)\{p^n\}) = \quad t(\mathrm{Ext}_R(C[p^n], A)) \\ = \quad & \min(t(A), t(C), n), \qquad\qquad (**)\end{aligned}$$

and the same conclusions as in (I) prove the claim. □

Corollary 6.4 *If X is arbitrary, Y is divisible with $T_p(Y) \neq 0$ and $\varphi \in \mathrm{Hom}(X,Y)$, then*

$$t_p^{\mathrm{Hom}(X,Y)}(\varphi) = \inf\{i \in \mathbb{N} \mid X[p] \not\subset p^i(\mathrm{Ker}\,\varphi)\}.$$

Proof We will first show that $h_p(\varphi) = \sup\{i \in \mathbb{N} \mid X[p^i] \subset \mathrm{Ker}\,\varphi\}$. If $n \in \mathbb{N}$ where $h_p(\varphi) \geqslant n$, then from $\varphi = p^n\psi$ we get immediately $X[p^n] \subset \mathrm{Ker}\,\varphi$. Conversely, it follows from $X[p^n] \subset \mathrm{Ker}\,\varphi$ that φ is factorized through $X \to p^nX$, say φ_0, and due to the divisibility of Y this ψ_0 is induced by $\psi \in \mathrm{Hom}(X,Y)$. One can obtain $\varphi = p^n\psi$, so $h_p(\varphi) \geqslant n$. Therefore the height formula is clear.
If $n \in \mathbb{N}$ where $t_p(\varphi) \leqslant n$, then we have $\varphi = p^e\psi$ where $e \leqslant n$, ψ not divisible by p. It just implies that $X[p] \not\subset \mathrm{Ker}\,\psi$, such that from $p^n(\mathrm{Ker}\,\varphi) \subset \mathrm{Ker}\,\psi$, we also have $X[p] \not\subset p^n(\mathrm{Ker}\,\varphi)$. Conversely, it follows from $X[p] \not\subset p^n(\mathrm{Ker}\,\varphi)$ also $t_p(\varphi) \leqslant n$, because if $h_p(\varphi) \leqslant n$ there is nothing to prove, and if $h_p(\varphi) \nleqslant n$, thus $\varphi = p^{n+1}\psi$, it follows from $p^n(\mathrm{Ker}\,\varphi) = \mathrm{Ker}(p\psi) \cap p^nX$ that $X[p] \not\subset p^nX$, thus by the theorem $t_p^{\mathrm{Hom}(X,Y)} \leqslant n$. □

Corollary 6.5 *If M is p-primary and $\varphi \in \mathrm{Hom}(M, R(p^\infty))$, then*

(a) *φ is coneat if and only if $t_p(\varphi) = 0$.*

(b) *If M is indecomposable, then $t_p(\varphi) = L(\mathrm{Ker}\,\varphi)$.*

(c) *If $m \leqslant t_p^M(0)$, then $m - L(\varphi(M[p^m])) = \min(t_p(\varphi), m)$.*

Proof The first statement which is just shown gives the second, because for indecomposable M, the condition $M[p] \not\subset p^i(\mathrm{Ker}\,\varphi)$ is equivalent with the statement $\mathrm{Ker}\,\varphi \subset M[p^i]$. By (c), we can assume $m > 0$ and $t_p(\varphi) < \infty$. With $n = m - 1$ now the assumption becomes $t_p^{\mathrm{Hom}(X,Y)} \nleqslant n$, so that for each $e \geqslant 0$ the series are equivalent to the following: $L(\varphi(M[p^m])) > e$, $p^e\varphi(M[p^m]) \neq 0$, $h_p(p^e\varphi) \leqslant n$, $t_p(p^e\varphi) = \min(t_p(\varphi) + e, t_p(0)) \leqslant n$, $m - t_p(\varphi) > e$. Hence it follows directly that $L(\varphi(M[p^m])) = \max(0, m - t_p(\varphi))$ as claimed. (It is easy to give an example that, for $m > t_p^M(0)$ the investigated length does not only depend on $t_p(\varphi)$. □

CHAPTER 7

THE κ-ELEMENTS OF $\mathrm{Ext}_R(C, A)$ FOR C TORSION, A TORSION-FREE OF RANK 1

Throughout this chapter the pair (A, C) is as in the title, and we will assume that $R \subset A \subset K$. For each $[E] \in \mathrm{Ext}_R(C, A)$ there is a commutative diagram with exact rows

$$(\square) \qquad \begin{tikzcd} E = 0 \arrow[r] & A \arrow[r, "\alpha"] \arrow[d, equal] & B \arrow[r, "\beta"] \arrow[d, "f'"] & C \arrow[r] \arrow[d, "f"] & 0 \\ 0 \arrow[r] & A \arrow[r, phantom, "\subset"] & K \arrow[r, "\nu"'] & K/A \arrow[r] & 0, \end{tikzcd}$$

that we cite by $(\square)$ in the following. By means of the connecting isomorphism $\vartheta : \mathrm{Hom}(C, K/A) \to \mathrm{Ext}_R(C, A)$, we write $\vartheta(f) = [E]$, and we want to study the supplement property of E by describing f.

Lemma 7.1 *Let* $(\square)$ *be given,* $0 \neq a_0 \in A$*. Then the following are equivalent:*

(i) $\mathrm{Im}\,\alpha$ *has a supplement* V *in* B *where* $\alpha(a_0) \in V$.

(ii) *There are homomorphisms* λ, μ, φ *where* $\mu\varphi = f$, $\mu\lambda = \nu$, φ *is coneat,* λ *is small epimorphism and* $a_0 \in \mathrm{Ker}\,\lambda$*:*

$$\begin{tikzcd} & & C \arrow[dl, "\varphi"'] \arrow[d, "f"] \\ K \arrow[r, "\lambda"'] & \cdot \arrow[r, "\mu"'] & K/A. \end{tikzcd}$$

(iii) *For each prime element* p *of* R*,* $T_p(C) \neq 0$ *and* $t_p^{\mathrm{Hom}}(f) \leqslant h_p^A(a_0)$*.*

Proof For the equivalence of (i) with (ii), we consider the following two diagrams:

$$
\begin{tikzcd}
B \arrow[rrr] \arrow[ddd] & & & C \arrow[ddd] \arrow[ddl, dashed, "\omega'"] \\
& V \arrow[ul, hook', "\supseteq"'] \arrow[urr, "\gamma"] \arrow[ddl, "\omega"'] \arrow[dr, phantom, "\mathscr{PO}"] & & \\
& & \bullet \arrow[dr, dashed, "\mu"] & \\
K \arrow[rrr] \arrow[urr, dashed, "\gamma'"'] & & & K/A
\end{tikzcd}
\qquad
\begin{tikzcd}
B \arrow[rrr] \arrow[ddd] & & & C \arrow[ddd] \arrow[ddl, "\varphi"] \\
& \bullet \arrow[ul, dashed, "\alpha"] \arrow[urr, dashed, "\lambda'"] \arrow[ddl, dashed, "\varphi'"'] \arrow[dr, phantom, "\mathscr{PB}"] & & \\
& & \bullet \arrow[dr, "\mu"] & \\
K \arrow[rrr] \arrow[urr, "\lambda"'] & & & K/A
\end{tikzcd}
$$

(i $\Rightarrow$ ii) We have a supplement V of $\operatorname{Ker}\beta$ in B where $\alpha(a_0) \in V$ so one forms the pushout diagram from $\omega = f' \mid_V$ and $\gamma = \beta \mid_V$; since $\nu\omega = f\gamma$ exist and also by μ where $\mu\omega' = f$ and $\mu\gamma' = v$, and it remains to show that ω' and γ' have the desired property: With γ naturally small epimorphism, γ' is also a small epimorphism, and by Lemma 4.3 ω and ω' are coneat; finally we get $\gamma'(a_0) = \gamma'\omega(\alpha a_0) = \omega'\gamma(\alpha a_0) = 0$, thus $a_0 \in \operatorname{Ker}\gamma'$.

(ii $\Rightarrow$ i) We have λ, μ, φ as stated, so we construct a pullback diagram from λ and φ, since $f\lambda' = \nu\varphi'$ exist ϵ where $\beta\epsilon = \lambda'$ and $f'\epsilon = \varphi'$. Since φ is coneat, λ and λ' are small epimorphisms, in particular $\epsilon(\operatorname{Ker}\lambda') = \operatorname{Im}\epsilon \cap \operatorname{Ker}\beta$ is small in $\operatorname{Im}\epsilon$ such that $\operatorname{Im}\epsilon$ is a supplement of $\operatorname{Ker}\beta$ in B, finally since there is an x with $\varphi'(x) = a_0$ and $\lambda'(x) = 0$, the element $\alpha(a_0) = \epsilon(x)$ is in $\operatorname{Im}\epsilon$.

(ii $\Rightarrow$ iii) Obviously, we can assume $f = C \xrightarrow{\varphi} K/U \xrightarrow{\mu} K/A$ where $a_0 \in U \subset A$, U is coatomic and μ is canonical. Since φ is coneat and $K/U \cong K/R$, no primary component can be zero in C. If $h_p^A(a_0) = \infty$, the claimed inequality is certainly true. Let $h_p^A(a_0) < \infty$. Then $\operatorname{Ker}\mu_p = T_p(A/U)$ has the finite length $e_p = h_p^A(a_0) - h_p^U(a_0)$, so that $\mu_p = p^{e_p}\omega_p$ holds for any isomorphism ω_p, and since $\omega_p\varphi_p$ is coneat with $f_p = p^{e_p}(\omega_p\varphi_p)$, we have $t_p(f) = t_p(f_p) \leqslant e_p \leqslant h_p^A(a_0)$ as claimed.

(iii $\Rightarrow$ ii) Let $P = \{p \mid p \text{ is prime element of } R \text{ and } A \text{ is not } p\text{-divisible}\}$. For each $p \in P$ one has, $f_p = p^{e_p} g_p$ where g_p is coneat $e_p \leqslant h_p^A(a_0) < \infty$, by assumption. In addition, there is precisely one intermediate module $a_0 \in U \subset K$ where $h_p^U(a_0) = h_p^A(a_0) - e_p$ if $p \in P$, $h_p^U(a_0) = 0$ if $p \notin P$. It follows that U is a coatomic submodule of A, and that for the canonical map $\mu : K/U \to K/A$ there is an isomorphism ω_p ($p \in P$) where

$\mu_p = p^{e_p}\omega_p$. Now one can define $\varphi : C \to K/U$: For $p \notin P$, φ is coneat (which is possible since $T_p \neq 0$), for $p \in P$ $\varphi_p = \omega_p^{-1} g_p$, and so that follows $\mu\varphi = f$ as well as φ is coneat. □

Remark 7.1 *For the equivalence ($i \Leftrightarrow ii$) one can also accept that $a_0 = 0$. Then the concept "supplement" seems as the reduction of "direct summand," because the latter asserts naturally in diagram (□) that f can be completely factorized through ν.*

Theorem 7.1 *Let (□) be given. Then the following are equivalent:*

(i) $\operatorname{Im}\alpha \subset^{\kappa} B$.

(ii) *If $[E] \neq 0$, then $T_p(C) \neq 0$ for all p and $c\ell\, t^{\mathrm{Ext}_R}([E]) \leqslant \tau(A)$.*

(iii) *If $f \neq 0$, then*

   (a) *$T_p(C) \neq 0$ for all p,*

   (b) *A is not p-divisible and C is p-divisible $\Rightarrow f_p \neq 0$,*

   (c) *for almost all p, $t_p^{\mathrm{Hom}}(f) \leqslant h_p^A(1)$.*

Proof (i $\Rightarrow$ ii) Let $c\ell$ be the class formation, as the usual height sequences is, and $\tau(A)$ be so called type of A, i.e. $c\ell\, h^A(1)$. Let V be a supplement of $\operatorname{Im}\alpha$ in B. Since $[E] \neq 0$, we have $V \cap \operatorname{Im}\alpha \neq 0$, thus $\alpha(a_0) \in V$ for some $0 \neq a_0 \in A$, and by the Lemma it follows that $T_p(C) \neq 0$ for all prime elements p of R, as well as $c\ell\, t^{\mathrm{Ext}_R}([E]) \leqslant c\ell\, h^A(a_0) = \tau(A)$.
(ii $\Rightarrow$ iii) If A is not p-divisible and C is p-divisible, then the p-depth of zero elements of $\mathrm{Hom}(T_p(C), T_p(K/A))$ are infinite.
(iii $\Rightarrow$ i) For $f = 0$ there is nothing to prove. Let $f \neq 0$, $P = \{p \mid p$ is prime element of R and $t_p(f) \nleqslant h_p^A(1)\}$. If $P = \emptyset$, we are done by the Lemma, otherwise by the assumption P is at least finite, say $P = \{p_1, \ldots, p_k\}$. For these p, we also have $t_p(f) < \infty$ by (b), and we similarly see that there are g and $e_i \geqslant 0$ where $f = p_1^{e_1} \ldots p_k^{e_k} g$, $t_{p_i}(g) = 0$ for all i. By Lemma 6.3 we have $t_p(g) \leqslant h_p^A(1)$ for all prime elements p of R such that g is a κ-element by the Lemma, therefore it is also a multiple of f.
It remains to show that: If G is a module, $x \in G$, and $p_1, \ldots, p_k$ are pairwise distinct prime elements p of R where $t_{p_i}^G(x) = e_i < \infty$ for all i, then there is an element $y \in G$

such that $t^G_{p_i}(y) = 0$ for all i and $x = p_1^{e_1} \dots p_k^{e_k} y$. For $k = 1$, it follows by the definition of the depth, and for $k \geqslant 2$ by induction with the help of Lemma 6.3. $\square$

Remark 7.2 *Whether $\vartheta(f)$ is a κ-element of* $\mathrm{Ext}_R(C, A)$ *or not can be decided only by* Ker f *by the first Corollary to Theorem 6.2. In the special case $C = K/R$ one can also read the condition (iii) of Theorem: If $f \neq 0$, there is an epimorphism from A onto* Ker f.

Corollary 7.1 *If* $\mathrm{Ext}_R(C, A)^\kappa$ *contains a nonzero torsion element, then* $\mathrm{Ext}_R(C, A)$ *is κ-full.*

Proof Let $0 \neq x \in \mathrm{Ext}_R(C, A)$ be both κ-element and torsion element. Then it follows that $T_p(C) \neq 0$ for all p, as well as $c\ell\, t^{\mathrm{Ext}_R}(0) = c\ell\, t\mathrm{Ext}_R(x) \leqslant \tau(A)$, and hence $c\ell\, t^{\mathrm{Ext}_R}(y) \leqslant \vartheta(A)$ for all $y \in \mathrm{Ext}_R(C, A)$. $\square$

Corollary 7.2 *If $T_p(C) \neq 0$ for all prime elements p of R, then every element of* $\mathrm{Ext}_R(C, A)$ *is a sum of two κ-elements.*

Proof Let $f \in \mathrm{Hom}(C, K/A)$. We look for a decomposition of f_p for each prime element p of R:

Case 1. $T_p(C)$ has a nontrivial decomposition, say $M_1 \oplus M_2$. Then there are coneat homomorphisms $\alpha_i : M_i \to T_p(K/A)$, such that both $g_p{}' = < \alpha_1, f_p\,|_{M_2} - \alpha_2 >$, and $g_p{}'' = < f_p\,|_{M_1} - \alpha_1, \alpha_2 >$ are coneat where $g_p{}' + g_p{}'' = f_p$.

Case 2. $T_p(C)$ is indecomposable. For any prime element p of R we have again $f_p = g_p{}' + g_p{}''$ where $g_p{}'$ and $g_p{}''$ are coneat, then in the endomorphism ring $\mathrm{End}_R(T_p(C))$ every element is a sum of two units. $\square$

As an application of the theorem we want to prove that when $\mathrm{Ext}_R(C, A)$ is κ-full, and in addition, examine more generally for any homomorphism $g : C' \to C$ when the kernel resp. image consists only of κ-elements for the induced map $g^* : \mathrm{Ext}_R(C, A) \to \mathrm{Ext}_R(C', A)$.

Lemma 7.2 *Let $g : C' \to C$ be given where C' is torsion, g^* :* $\mathrm{Ext}_R(C, A) \to \mathrm{Ext}_R(C', A)$. *Then*

(I) *If $\Omega = \{p$ is prime element of $R, \mid A$ is not p-divisible and $g_p \neq 0\}$, then the following are equivalent:*

   (i) Im g^* *contains only κ-elements.*

(ii) *If* $\operatorname{Im} g^* \neq 0$, *then*

(a) $T_p(C') \neq 0$ *for all* p,

(b) *A is not p-divisible and C′ is p-divisible* $\Rightarrow \Omega = \{p\}$,

(c) *for almost all prime elements p of R,* $t_p^{C'}(0) \leqslant h_p^A(1)$.

(II) *If* $\Psi = \{p$ *is prime element of* $R \mid A$ *is not p-divisible and* g_p *is not surjective*$\}$, *then the following are equivalent:*

(i) $\operatorname{Ker} g^*$ *contains only* κ*-elements.*

(ii) *If* $\operatorname{Ker} g^* \neq 0$, *then*

(a) $T_p(C) \neq 0$ *for all* p,

(b) *A is not p-divisible and C′ is p-divisible* $\Rightarrow \Psi = \{p\}$,

(c) *for almost all prime elements p of R,* $t_p^{C}(0) \leqslant h_p^A(1)$.

Proof (I) (i $\Rightarrow$ ii) Obviously $\Omega = \emptyset$ is equivalent to the statement that $g^* = 0$. Therefore, let $\operatorname{Im} g^* \neq 0$, $q \in \Omega$. Choose $f : C \to K/A$ such that $f_q g_q \neq 0$ and $f_p = 0$ for all $p \neq q$. Then $0 \neq fg = g^*(f)$ is a κ-element by assumption, thus (a) and (c) are satisfied. For (b) suppose A is not p-divisible, C' is p-divisible: Since $(fg)_p \neq 0$, it follows that $p \in \Omega$; but then it would give another $q' \in \Omega$ where $q' \neq p$, such that $f'g$ (with one f' similar to f) would be κ-element by assumption, in particular $(f'g)_p \neq 0$, which is not possible.
(ii $\Rightarrow$ i) Let $f : C \to K/A$ be given where $fg \neq 0$ (otherwise the proof is done). To show that fg is a κ-element, it remains only to prove the case when A is not p-divisible, C is p-divisible: Then by assumption $(fg)_r = 0$ for all prime elements of R $r \neq p$, thus $(fg)_p \neq 0$.

(II) From the exact sequence $C' \xrightarrow{g} C \xrightarrow{\nu} \operatorname{Coker} g \longrightarrow 0$, since C' and C are torsion and A is torsion-free, one obtains the exact sequence $0 \longrightarrow \operatorname{Ext}_R(\operatorname{Coker} g, A) \xrightarrow{\nu^*} \operatorname{Ext}_R(C, A) \xrightarrow{g^*} \operatorname{Ext}_R(C', A)$, as well as $\Psi = \{p$ *is prime element of* $R \mid A$ is not p-divisible and $\nu_p \neq 0\}$. If one applies part (I) on ν^*, the claim follows. □

Corollary 7.3 *If* $T_p(C) \neq 0$, *the following are equivalent:*

(i) $\operatorname{Ext}_R(C, A)$ *is* κ*-full.*

(ii) *If A is not divisible by at least two prime elements p of R, then* $c\ell\, t^{\mathrm{Ext}_R}(0) \leqslant \tau(A)$*.*

(iii) *If A is not divisible by at least two prime elements p of R, then:*

(a) *A is not p-divisible $\Rightarrow$ C is not p-divisible,*

(b) *for almost all prime elements p of R, $t_p^C(0) \leqslant h_p^A(1)$.*

Proof (i $\Rightarrow$ iii) Let A be as given, $C' = C$ and $g = 0$. In part (II), if $|\Psi| > 1$, then the case (b) in (ii) can not happen, and this is the claim.
(iii $\Rightarrow$ ii) Clear.
(ii $\Rightarrow$ i) It only remains to show that $\mathrm{Ext}_R(C, A)$ is κ-full when A is not divisible by only one prime element q of R: For $f \neq 0$ the parts (a) and (c) in Theorem are trivially satisfied, then (b) is also satisfied, because $f_q \neq 0$. □

Corollary 7.4 *If $(C_i \mid i \in I)$ is a nonempty family of torsion modules, where* $\mathrm{Ext}_R(C_i, A)$ *is κ-full for each i, then* $\mathrm{Ext}_R(\coprod C_i, A)$ *is also κ-full.*

Proof If at least one primary component of each C_i is zero, then all $\mathrm{Ext}_R(C_i, A)$'s are zero, which implies that $\mathrm{Ext}_R(\coprod C_i, A)$ is zero; however there is a $j \in I$ where $T_p(C_j) \neq 0$ for all prime elements p of R, so the coneat homomorphism $\epsilon^* :$ $\mathrm{Ext}_R(\coprod C_i, A) \to \mathrm{Ext}_R(C_j, A)$ yields the claim together with Lemma 6.4 and just proven assertion of first Corollary. (Second Corollary is not true for arbitrary A as it was shown at the remark to Lemma 5.2.) □

By our particular choice of A and C, C' one can answer the question of Chapter 4, when $g^* : \mathrm{Ext}_R(C, A) \to \mathrm{Ext}_R(C', A)$ preserves the κ-elements.

Theorem 7.2 *Let $g : C' \to C$ be given where C' is torsion, $T_p(C) \neq 0$ for all prime elements p of R. Where $\Omega = \{p$ is prime element of $R \mid A$ is not p-divisible and $g_p \neq 0\}$, then*

(I) *If $|\Omega| = 1$, then $g^* : \mathrm{Ext}_R(C, A) \to \mathrm{Ext}_R(C', A)$ preserves κ-elements if and only if*

(a) *$T_p(C') \neq 0$ for all prime elements p of R,*

(b) *A is not p-divisible and C' is p-divisible $\Rightarrow$ $\Omega = \{p\}$,*

(c) *for almost all prime elements p of R, $t_p^{C'}(0) \leqslant h_p^A(1)$.*

(II) *If $|\Omega| \geqslant 2$, then $g^* : \mathrm{Ext}_R(C, A) \to \mathrm{Ext}_R(C', A)$ preserves κ-elements if and only if*

(a) $T_p(C') \neq 0$ *for all prime elements* p *of* R,

(b) A *is not* p*-divisible and* C' *is* p*-divisible* $\Rightarrow g_p$ *is surjective,*

(c) *for almost all prime elements* p *of* R, $t_p^{C'}(0) \leqslant h_p^A(1)$ *or* g *is* p*-coneat.*

Proof The case $\Omega = \emptyset$ indicates that $g^* = 0$ which is not interesting; however if $\Omega \neq \emptyset$ and g^* preserves κ-elements, then a nonzero κ-element lies in $\operatorname{Im} g^*$ by second Corollary to Theorem 7.1, so that $T_p(C) \neq 0$ for all prime elements p of R. Further, we need the following remark:
If $q \in \Omega$, g^* preserves the κ-elements, and if p is prime element of R where A is not p-divisible, C' is divisible and $p \neq q$, then g_p must be surjective.
Namely, the condition $\operatorname{Coker} g_p \neq$ gives $f_p : T_p(C) \to T_p(K/A)$ where $f_p \neq 0$; in addition one can choose $f_q : T_q(C) \to T_q(K/A)$ where $f_q g_q \neq 0$, and if one specifies f in such a way that it is coneat in all different primary components, then f is a κ-element, thus also fg is a κ-element, in particular $(fg)_p \neq 0$ which is not true.

Case I. If the three conditions are all satisfied, then one knows by Lemma 7.2 that $\operatorname{Im} g^*$ consists only of κ-elements. Conversely, if g^* preserves κ-elements, then (a) and (b) are clear by the above remark. For (c) one can choose a function $f : C \to K/A$ such that $f_q g_q \neq 0$ where q is the only element of Ω, and f_p is coneat for all $p \neq q$. Then since f is a κ-element, so $fg \neq 0$ is, in particular, $t_p(f_p g_p) \leqslant h_p^A(1)$ for almost all prime elements p of R. But now $f_p g_p = 0$ for all $f \neq g$, so that (c) follows.

Case II. Suppose three conditions are all satisfied. Let $f : C \to K/A$ be a κ-element where $fg \neq 0$ (otherwise the proof is done): In view of Theorem 7.1, if A is not p-divisible and C' is p-divisible, then by assumption g_p is surjective, C is p-divisible, f_p is surjective, $fg \neq 0$. Besides, there is $n \geqslant 1$ where $p \nmid n$ and A is not p-divisible $\Rightarrow t_p(f) \leqslant h_p^A(1)$ and $[t_p^{C'}(0) \leqslant h_p^A(1)$ or g is p-coneat $]$, thus $t_p(fg) \leqslant h_p^A(1)$.

Conversely, g^* preserves κ-elements. Again by the remark above (a) is clear, similarly (b) is clear, since Ω possesses at least two elements. For (c) choose $q \in \Omega$ and $f_q : T_q(C) \to T_q(K/A)$ where $f_q g_q \neq 0$. If $p \neq q$ and A is not p-divisible, by Lemma 4.4 there is a homomorphism $\varphi_p : T_p(C) \to T_p(K/A)$, for the $\varphi_p g_p$ is coneat only when g_p is coneat. So one can complete f_q to any f in which for $p \neq q$ and A is not p-divisible, set $f_p = p^h \varphi_p$ where $h = h_p^A(1)$. It follows with Theorem 7.1 that f is a κ-element by assumption, we also have $fg \neq 0$, so that in particular, there exists $n \in R$ where $p \nmid n$ and A is not p-divisible $\Rightarrow p \neq q$ and $t_p(fg) \leqslant h_p^A(1)$, thus $t_p(p^h \varphi_p g_p) \leqslant h$, by Lemma 6.3 therefore $t_p^{C'}(0) \leqslant h$ or $\varphi_p g_p$ is coneat; in the second case it follows that g_p is coneat by the choice of φ_p. □

Since p-depth of the zero elements are always greater than or equal to units, in the particular case $A = R$ we obtain several simple descriptions, namely:

Corollary 7.5 *Let $g : C' \to C$ where C' is torsion, and $T_p(C) \neq 0$ for all prime elements p of R. Then the following are equivalent:*

(i) $g^* : \mathrm{Ext}_R(C, R) \to \mathrm{Ext}_R(C', R)$ *preserves κ-elements.*

(ii) *If $g \neq 0$, then*

  (a) *C' is p-divisible $\Rightarrow g_p$ is surjective,*

  (b) *g_p is p-coneat for almost all prime elements p of R.*

The following theorem shows why the investigation of the κ-elements in Ext the height concept was brought into focus. We want to put the arguments in an earlier lemma together which are necessary in each primary component of C.

Lemma 7.3 *Let M be a p-module, $\varphi \in \mathrm{Hom}(M, R(p^\infty))$ and $n \geqslant 0$. Then the following are equivalent:*

(i) *$t_p(\varphi\,|_V) \leqslant n$ for all $V \subset M$ that have a direct supplement under* $\mathrm{Ker}\,\varphi$.

(ii) *$h_p(\varphi) \leqslant n$.*

(iii) *$h_p(\varphi\,|_V) \leqslant n$ for all $V \subset M$ with the following property:*
$V \subset X \subset M$ and X/V is cyclic $\Rightarrow V \oplus U = X$ where $U \subset$ $\mathrm{Ker}\,\varphi$. (*)

Proof We constantly use the characterization of height and depth from Theorem 6.2.

(i $\Rightarrow$ ii) If $h_p(\varphi) \nleqslant n$, then $M[p^{n+1}] \subset \mathrm{Ker}\,\varphi$, hence one can choose a supplement V of $M[p^{n+1}]$ in M that satisfies the assumption (i) by Lemma 6.1. From $t_p(\varphi\,|_V) \leqslant n$, since $V[p^{n+1}]$ is small in V and hence $t_p^V(0) \geqslant n+2$, also $h_p(\varphi\,|_V) \leqslant n$, in fact $h_p(\varphi) \leqslant n$ which is not possible.

(ii $\Rightarrow$ iii) Let V possess the property (*). Assume that $h_p(\varphi\,|_V) \nleq n$, then $V[p^{n+1}] \subset \mathrm{Ker}\,\varphi$, thus (*) contradicts with $M[p^{n+1}] \subset \mathrm{Ker}\,\varphi$, because for $w \in M[p^{n+1}]$ there is a decomposition $V \oplus U = V + Rw$ where $U \subset \operatorname{Ker}\varphi$, hence $w \in V[p^{n+1}] \oplus U[p^{n+1}] \subset \operatorname{Ker}\varphi$.

(iii ⇒ i) If V has a direct supplement under $\operatorname{Ker}\varphi$, then it certainly satisfies the condition (*). □

Theorem 7.3 *Let (□) be given. Then the following are equivalent:*

(i) $\operatorname{Im}\alpha$ *has a direct supplement in every intermediate module.*

(ii) $\operatorname{Im}\alpha$ *has a pure supplement in every intermediate module.*

(iii) *If $f \neq 0$, then*

(a) *$f_p \neq 0$ for all prime elements p of R,*

(b) *$h_p^{\mathrm{Hom}}(f) \leqslant h_p^A(1)$ for almost all prime elements p of R.*

Proof By a direct (resp. pure) intermediate module we understand an X where $\operatorname{Im}\alpha \subset X \subset B$ and $X \subset^{\oplus} B$ (resp. X is pure in B). One can easily describe C: For $L \subset C$, $\beta^{-1}(L) \subset^{\oplus} B$ is equivalent to the fact that L has a direct supplement in $\operatorname{Ker} f$. Clearly, it follows from $\beta^{-1}(L) \oplus V = B$ that $L \oplus \beta(V) = C$, and $V \subset T(B)$, $\beta(V) \subset \operatorname{Ker} f$. Conversely, if one has $L \oplus K = C$ where $K \subset \operatorname{Ker} f$, then by the first we get $\beta^{-1}(L)/\operatorname{Im}\alpha \oplus \beta^{-1}(K)/\operatorname{Im}\alpha = B/\operatorname{Im}\alpha$, by the second, $\beta^{-1}(K) \subset \operatorname{Im}\alpha + T(B)$, and $\operatorname{Im}\alpha \subset^{\oplus} \beta^{-1}(K)$, thus together $\beta^{-1}(L) \subset^{\oplus} B$. Correspondingly, it can be shown that $\beta^{-1}(L)$ is pure in B if and only if there is a decomposition $L \oplus K = N$ with $K \subset \operatorname{Ker} f$ for all $L \subset N \subset C$ where N/L is cyclic.

(i ⇒ iii) Let $f \neq 0$. (a) If there is a prime element q of R with $f_q = 0$, then $L = \bigoplus_{p \neq q} T_p(C)$ would have a direct supplement under $\operatorname{Ker} f$, and it follows that $\beta^{-1}(L) \subset^{\oplus} B$, $\operatorname{Im}\alpha \subset^{\kappa} \beta^{-1}(L)$; since the q-components are missing in $\beta^{-1}(L)/\operatorname{Im}\alpha$, it implies that $\operatorname{Im}\alpha \subset^{\oplus} \beta^{-1}(L)$, $f = 0$ contrary to the assumption. To prove (b), let $P = \{p \text{ is prime element of } R \mid h_p(f) \nleqslant h_p^A(1)\}$. If $P = \emptyset$ it is done, otherwise by the lemma for each $p \in P$ there is a submodule $L_p \subset T_p(C)$ which has a direct supplement under $\operatorname{Ker} f_p$ and we get $t_p(f_p \mid_{L_p}) \nleqslant h_p^A(1)$. Now for $L \subset C$ we define $T_p(L) = L_p$ if $p \in P$, and $T_p(L) = T_p(C)$ if $p \notin P$, then L has a direct supplement under $\operatorname{Ker} f$, and it follows that $\beta^{-1}(L) \subset^{\oplus} B$, $\operatorname{Im}\alpha \subset^{\kappa} \beta^{-1}(L)$, however the latter does not split. By Theorem 7.1 it follows that $t_p(f \mid_L \leqslant h_p^A(1)$ for almost all prime elements p of R, so that P must be finite.

(iii $\Rightarrow$ ii) Assume $\operatorname{Im}\alpha$ is not a direct summand in B, and X is a pure intermediate module. Then for all prime elements p of R, the condition (*) satisfies the inclusion $T_p(\beta X) \subset T_p(C)$ the lemma, so that $h_p(f\mid_{\beta X})$ is finite for all prime elements p of R, and for almost all p smaller than or equal to $h_p^A(1)$, thus $f\mid_{\beta X}$ is a κ-element, i.e. $\operatorname{Im}\alpha \subset^\kappa X$. $\square$

Corollary 7.6 *If $T_p(C) \neq 0$ for all prime elements p of R and $A \subsetneq K$, then the following are equivalent:*

(i) *If* $0 \longrightarrow A \xrightarrow{\alpha} B \xrightarrow{\beta} C \longrightarrow 0$ *is κ exact, then* $\operatorname{Im}\alpha$ *has a supplement in every direct intermediate module.*

(ii) *A is coatomic and C is divisible.*

Proof (i $\Rightarrow$ ii) Since $\operatorname{Ext}_R(C, A)$ is not zero, it has a nonzero κ-element by second corollary to Theorem 7.1 such that $T_p(K/A) \neq 0$ for all prime elements p of R, thus A is coatomic. Let q be a prime element of R where C is not q-divisible: Choose $f : C \to K/A$ such that $f_q \neq 0$ and f_p is coneat for all $p \neq q$. Then $0 \neq f$ is a κ-element, thus by assumption $f_p \neq 0$ for all p contrary to our choice of f.
(ii $\Rightarrow$ i) If V is a supplement of $\operatorname{Im}\alpha$ in B, then $V = C(B)$ from our assumption. But then every direct intermediate module X, $D(X)$ is a supplement of $\operatorname{Im}\alpha$ in X. $\square$

Remark 7.3 *If* $\operatorname{Im}\alpha$ *also has a supplement in each coclosed intermediate module, then* $\operatorname{Im}\alpha$ *is a direct summand or small in B.*

Remark 7.4 *If* $\operatorname{Im}\alpha$ *itself is small in B, then $T(B) \subset^\oplus B$ does not need to hold: For each prime element p of R one can choose a homomorphism $f_p : R/(p^3) \times R/(p) \to R(p^\infty)$ such that* $\operatorname{Ker} f_p$ *is certainly coclosed, but not a direct summand. The direct summand on p yields $f : C \to K/R$ where C is supplemented and reduced, f is coneat. If one forms the corresponding diagram ($\square$) with $A = R$, then v and β are small epimorphisms; however $T(B) \subset^\oplus B$, so* $\operatorname{Ker} f \subset^\beta C$ *must hold by Lemma 3.3 thus* $\operatorname{Ker} f_p \subset^\oplus T_p(C)$ *for almost all p, contrary to our choice of f.*

The criterion Lemma 3.3 for the decomposition of B allows to alter our situation ($\square$); it yields a remarkable connection with the supplement concept:

Theorem 7.4 *Let* ($\square$) *be given.* $T(B) \subset^{\oplus} B$ *if and only if there is a supplement* L *of* $\operatorname{Ker} f$ *in* C *where* $L(T_p(L \cap \operatorname{Ker} f)) \leqslant h_p^A(1)$.

Proof We want to use the criterion of Meggiben (Meggiben 1967) (p.142) and must calculate the height-matrix of $\alpha(1)$ in B:

(I) $p^n\alpha(1) \in p^m(B) \Leftrightarrow m - L(f(C[p^m])) \leqslant n + h_p^A(1)$. For the proof one can assume that A is not p-divisible, thus $h = h_p^A(1)$ can be assumed finite. The statement is true for $n \geqslant m$, so let $n < m$. Obviously, $p^n\alpha \in p^m B$ is equivalent to the statement $v(1/p^{m-n}) \in f(C[p^m])$ and since $Rv(1/p^i)$ is always a cyclic p-module of length $\max(0, i - h)$, here with $i = m - n$, the left side of (I) is equivalent to $\max(0, m - n - h) \leqslant L(f(C[p^m]))$, and that is the claim.

(II) If $T_p(C)$ is indecomposable, $f_p \neq 0$, $e = L(\operatorname{Ker} f_p)$ and $h = H_p^A(1)$, thus we have:

$$
\begin{aligned}
h_p^B(p^n\alpha(1)) &= n + h, & & e > n + h\\
&= n + h + l - e, & if \quad & e \leqslant n + h, T_p(C) \cong R/(p^l)\\
&= \infty, & & e \leqslant n + h, T_p(C) \cong R/(p^\infty).
\end{aligned}
$$

For the proof we will first consider the case $T_p(C) \cong R/(p^\infty)$: Then $f_p = p^e g_p$ where g_p is an isomorphism, thus $L(f(C[p^m])) = L(p^e(R(p^m))) = \max(0, m - e)$, therefore $m - L(f(C[p^m])) = \min(e, m)$, and hence with (I) the claim follows. If however $T_p(C) \cong R(p^l)$, then $e < l$ and $f_p = p^e g_p$, where g_p is a monomorphism, thus $f([p^m]) \cong p^e(C[p^m])$. Since $C[p^m]$ is a cyclic p-module of length $\min(m, l)$, we have $L(f(C[p^m])) = \min(\max(0, m-e), l-e)$, thus $m-L(f(C[p^m])) = \max(\min(e, m), m-l+e)$. With (I) we get $p^n\alpha(1) \in p^m B$ if and only if $\min(e, m) \leqslant n+h$ and $m \leqslant n+h+l-e$, again the claim follows.

(III) If $\mathbb{H}$ is the height-matrix of $\alpha(1)$ in B, $T_p(C)$ is indecomposable and $f_p \neq 0$, then at most p-row of $\mathbb{H}$ has a gap, and it is gap-free if and only if $L(\operatorname{Ker} f_p) \leqslant h_p^A(1)$.

For the notion of height-matrix see (Fuchs 1973) (p.197). Since $T_p(B)$ is either zero or indecomposable, for each $x \in B$ the p-height coincides with the so-called generalized p-height, and our claim follows immediately from (II).

(IV) The theorem is true if in particular each primary component of C is zero or indecomposable.

For the proof we first consider the excluded case in (III): If A is p-divisible, then $h_p^B(p^n\alpha(1)) = \infty$ for all n, but if A is not p-divisible and $f_p = 0$, then it follows from (I) that $h_p^B(p^n\alpha(1)) = n + h_p^A(1)$ for all n. At most for such a prime element p of R where $f_p \neq 0$ the p-row of $\mathbb{H}$ has a gap. By (Meggiben 1967) $T_B \subset^{\oplus} B$ is equivalent with that almost all rows of $\mathbb{H}$ are gap-free, i.e. there is $n \in R$ such that $p \nmid n$ and $f_p \neq 0 \Rightarrow L(\operatorname{Ker} f_p) \leqslant h_p^A(1)$. For the unique supplement L of $\operatorname{Ker} f$ in C this is just the claim of the theorem.

(V) Let C be arbitrary. If L is as given, then all of the primary components of L are zero or indecomposable, and we have $L(\operatorname{Ker} g_p) \leqslant h_p^A(1)$ for almost all p where $g = f\,|_L$. It follows now from (IV) that $V \oplus T(\beta^{-1}(L)) = \beta^{-1}(L)$, together with $\beta^{-1}(L) + T(B) = B$ thus it follows at once that $V \oplus T(B) = B$. Conversely, from any decomposition $V \oplus T(B) = B$ it follows that every primary component is zero or indecomposable in $\beta(V)$. In the induced diagram

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r] \arrow[d, equal] & V + \operatorname{Im}\alpha \arrow[r] \arrow[d] & \beta(V) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & A & \subset \quad K \arrow[r, "\nu"'] & K/A \arrow[r] & 0
\end{tikzcd}$$

again $V \oplus T(V + \operatorname{Im}\alpha) = V + \operatorname{Im}\alpha$, so by (IV) there is a supplement L of $\beta(V) \cap \operatorname{Ker} f$ in $\beta(V)$ where $L(T_p(L \cap \operatorname{Ker} f)) \leqslant h_p^A(1)$ for almost all prime elements p of R. Since $\beta(V) + \operatorname{Ker} f = C$ but L is also a supplement of $\operatorname{Ker} f$ in C, and the claim follows. $\square$

CHAPTER 8

ON THE TRANSITIVITY OF THE RELATION κ; WEAK SUPPLEMENTS

If $U \subset^{\kappa} M$ and Y is an intermediate module which is a direct summand, then $U \subset Y \subset^{\oplus} M$, so U does not need to have any supplement in Y by Theorem 7.3. However, if V is a supplement of U in M, then one can obtain with $V_1 = V \cap Y$ at least $V_1 + U = Y$, and $V_1 \cap U$ is small in Y : We call V_1 a *weak supplement* of U in Y. Further, conversely, it follows that:

Lemma 8.1 *If $X \subset Y$ and X has a weak supplement in Y, then there is a splitting extension $Y \subset^{\oplus} Z$ where $X \subset^{\kappa} Z$.*

Proof Let $V + X = Y$ where $V \cap X$ is small in Y. To the monomorphism $d : Y \ni y \mapsto (y, \overline{y}) \in Y \times (Y/V)$ as we know there is an extension $Y \subset Z$ and an isomorphism $\chi : X \times (Y/V) \to Z$ where $\chi d = Y \subset Z$. It is clear that $Y \subset^{\oplus} Z$ and by means of the canonical isomorphism $\varphi : Y \times (X/V \cap X) \to Y \times (Y/V)$ one can also obtain that $\chi\varphi(Y \times 0)$ is a supplement of X in Z. □

We want to show in the following for particular sequence $0 \longrightarrow A \xrightarrow{\alpha} B \xrightarrow{\beta} C \longrightarrow 0$ that $\operatorname{Im}\alpha$ has a weak supplement in B if and only if there is an intermediate module X where $\operatorname{Im}\alpha \subset^{\kappa} X$ and $X \subset^{\kappa} B$ and hence deduce an example showing that the relation κ is not transitive.

Lemma 8.2 *Let (□) be given as in Chapter 7. Then the following are equivalent:*

(i) $\operatorname{Im}\alpha$ *has a weak supplement in B.*

(ii) *If* $\operatorname{Im}\alpha$ *is not a direct summand in B, then $\alpha(1) \in pB$ for almost all prime elements p of R.*

(iii) *If $f \neq 0$, then $h_p^A(1) > 0$ or f is p-coneat for almost all p.*

Proof (i $\Rightarrow$ ii) If $V + \operatorname{Im}\alpha = B$ where $V \cap \operatorname{Im}\alpha$ is small in B, then $V \cap \operatorname{Im}\alpha \neq 0$ since $\operatorname{Im}\alpha$ does not split, thus $\alpha(a) \in \operatorname{Rad}(B)$ for some element $0 \neq a \in A$, therefore

$\alpha(n) \in \mathrm{Rad}(B)$ for some element $0 \neq n \in \mathbb{N}$, thus $\alpha(1) \in pB$ for all $p \nmid n$.

(ii $\Rightarrow$ i) If $\mathrm{Im}\,\alpha$ splits, there is nothing to show; other cases follow from the assumption that U is small in B for some $0 \neq U \subset \mathrm{Im}\,\alpha$. Since $\mathrm{Im}\,\alpha/U$ is artinian in each primary component, it has a supplement in torsion B/U, we denote it by V/U, and it follows that $V \cap \mathrm{Im}\,\alpha$ is small in B such that V is a weak supplement of $\mathrm{Im}\,\alpha$ in B since $(V \cap \mathrm{Im}\,\alpha)/U$ is small in B/U.

(*ii* $\Leftrightarrow$ *iii*) Obviously, $\alpha(1) \in pB$ is equivalent with that $\nu(1/p) = f(c)$ for some $c \in C[p]$ thus equivalent to $\nu(1/p) \in f(C[p])$. Since $\nu(1/p) \neq 0 \Leftrightarrow h_p^A(1) = 0$ the claim follows. □

Theorem 8.1 *Let* (□) *be given as in Chapter 7. Then the following are equivalent:*

(i) *There is an intermediate module X where* $\mathrm{Im}\,\alpha \subset^\kappa X$ *and* $X \subset^\kappa B$.

(ii) *There is an intermediate module Y where* $\mathrm{Im}\,\alpha \subset^\kappa Y$ *and* $Y + T(B) = B$.

(iii) *If $f \neq 0$, then $T_p(C) \neq 0$ for all prime elements p of R, and $h_p^A(1) = 0$ for almost all prime elements p of R or $[T_p(C)$ is indecomposable and $f_p \neq 0] \Rightarrow t_p^{\mathrm{Hom}}(f) \leqslant h_p^A(1)$.*

Proof (i $\Rightarrow$ ii) More generally we show that if

$$\mathrm{Im}\,\alpha \subset X \subset^\kappa B \text{ and } X + T(B) \subsetneq B,$$

then $\mathrm{Im}\,\alpha \subset^\kappa B$ (thus in this case choose $Y = B$, otherwise $Y = X$). Namely, if W is a supplement of X in B, then it is also a supplement of $\mathrm{Im}\,\alpha$ in $\mathrm{Im}\,\alpha + W$; on the other hand $\mathrm{Im}\,\alpha$ can not be a direct summand in $\mathrm{Im}\,\alpha$, because from $\mathrm{Im}\,\alpha \oplus S = \mathrm{Im}\,\alpha + W$ it would follow that S is torsion, $X + S = B$, $X + T(B) = B$ contrary to our assumption. Thus the sequence

$$0 \longrightarrow A \xrightarrow{\overline{\alpha}} \mathrm{Im}\,\alpha + W \xrightarrow{\overline{\beta}} \beta(W) \longrightarrow 0$$

is κ-exact and does not split, i.e. $\iota^*([E])$ is a nonzero κ-element of $\mathrm{Ext}_R(\beta(W), A)$. While on the contrary $\iota : \beta(W) \subset C$ is a neat-homomorphism since $\beta(W)$ is a supplement of $\beta(X)$ in C, so that ι^* is coneat by Lemma 4.4, 4 and does no make the depth smaller by Lemma 6.4. It follows that $T_p(C) \neq 0$ for all prime elements p of R, and

$$c\ell\, t^{\mathrm{Ext}_R(C,A)}([E]) \leqslant \tau(A)$$

as assumed.

(ii $\Rightarrow$ iii) Let $f \neq 0$. Then $\operatorname{Im}\alpha$ can not be direct summand in Y, since from $\operatorname{Im}\alpha \oplus S = Y$, it would follow that S is torsion, $\operatorname{Im}\alpha \oplus T(B) = B$ which was excluded. In particular, if $T_p(C) \neq 0$ for all prime elements p of R, and for $L = \beta(Y)$ we get $L + \operatorname{Ker} f = C$, $t_p(f\mid_L) \leqslant h_p^A(1)$ for almost all prime elements p of R; but then $t_p \leqslant h_p^A(1)$ must hold in the given particular cases.

(iii $\Rightarrow$ i) To construct X we need the case that $T_p(C)$ is nontrivial decomposable and $h_p^A(1) < \infty$. □

Lemma 8.3 *Let M be a p-module with a nontrivial decomposition $\varphi \in (M, R(p^\infty))$. Then there is a submodule V of M with*

(1) $V + \operatorname{Ker}\varphi = M$,

(2) $V \cap \operatorname{Ker}\varphi \subset^\kappa \operatorname{Ker}\varphi$,

(3) $V[p] \not\subset pV$ *(in particular $t_p(\varphi\mid_V) \leqslant 1$).*

Proof Case 1. $\operatorname{Ker}\varphi \subset^\kappa M$. Then if X is a supplement of $\operatorname{Ker}\varphi$ in M, then it follows from the structure of X and assumption on M that X is not essential in M, thus $X \cap E = 0$ for some simple $E \subset M$. So $V = X + E$ gives the desired result.

Case 2. $\operatorname{Ker}\varphi$ has no supplement in M. Then it follows that $D(M) \subset \operatorname{Ker}\varphi$, since the existence of an $X \subset M$ where $X \cong R(p^\infty)$, $X \notin \operatorname{Ker}\varphi$ would involve $0 \neq (X + \operatorname{Ker}\varphi)/\operatorname{Ker}\varphi \subset M/\operatorname{Ker}\varphi$, such that X would be a supplement of $\operatorname{Ker}\varphi$ in M, which was excluded. Further, it follows that $M/D(M)$ can not be supplemented, thus a decomposition $M = G \oplus H$ exists where H is finitely generated, and not cyclic. Now one can use case 1. on H and $\varphi\mid_H$, thus $H_1 + (H \cap \operatorname{Ker}\varphi) = H$ where $E \subset^\oplus H_1$, and E is simple. For $V = G + H_1$ we have $V + \operatorname{Ker}\varphi = M$ as well as $E \subset^\oplus V$, and M/V is finitely generated as factor of H, it also has a supplement $V \cap \operatorname{Ker}\varphi$ in $\operatorname{Ker}\varphi$.

For proof of (iii $\Rightarrow$ i) in the theorem let $f \neq 0$. Now define $L \subset C$ by,

$$T_p(L) = \begin{cases} \text{simple} & f_p = 0, \\ T_p(C) \quad \text{if} & f_p \neq 0 \text{ and } \begin{bmatrix} h_p^A(1) = 0 \text{ or} \\ T_p(C) \text{ is} \\ \text{indecomposable} \end{bmatrix} \\ \begin{matrix}\text{with the three} \\ \text{properties of } V \\ \text{as in the lemma}\end{matrix} & f_p \neq 0 \text{ and } \begin{bmatrix} hp^A(1) \neq 0 \text{ and} \\ T_p(C) \text{ is nontrivial} \\ \text{decomposable} \end{bmatrix}. \end{cases}$$

Then this L satisfies the following five conditions:

(1) $L + \operatorname{Ker} f = C$;

(2) $L \cap \operatorname{Ker} f \subset^{\kappa} \operatorname{Ker} f$;

(3) $T_p(L) \neq 0$ for all prime elements p of R;

(4) $(f\,|_L)_p = 0 \Rightarrow T_p(L)$ is simple;

(5) $t_p(f\,|_L) \leqslant h_p^A(1)$ for almost all prime elements p of R.

In order to produce the desired $X = \beta^{-1}(L)$, since (3-5) warranted that $\operatorname{Im}\alpha \subset^{\kappa} X$, from (1) it follows that $X + T(B) = B$ such that there is still to show $X \cap T(B) \subset^{\kappa} T(B)$, and it follows from (2) via the induced isomorphism $T(B) \to \operatorname{Ker} f$ of β, $X \cap T(B)$ is precisely mapped on $L \cap \operatorname{Ker} f$. □

Corollary 8.1 *If one has* $\operatorname{Im}\alpha \subset^{\kappa} X_0 \subset^{\kappa} X_1 \subset \cdots \subset^{\kappa} X_n = B$ *where* $n \geqslant 2$, *then there is an intermediate module* X *where* $\operatorname{Im}\alpha \subset^{\kappa} X \subset^{\kappa} B$.

Proof If $X_0 + T(X_1) \subsetneq X_1$ or $X_1 + T(X_2) \subsetneq X_2$, one has $\operatorname{Im}\alpha \subset^{\kappa} X \subset^{\kappa} X_2$ by the step (i $\Rightarrow$ ii), but if two cases are equalities, we have $X_0 + T(X_2) = X_2$, we can find such an X by the step (ii $\Rightarrow$ i). □

Corollary 8.2 *If* $A = R$ *and* $T_p(C) \neq 0$ *for all prime elements* p *of* R, *then* $\operatorname{Im}\alpha$ *has a weak supplement in* B *if and only if there is an intermediate module* X *with* $\operatorname{Im}\alpha \subset^{\kappa} X \subset^{\kappa} B$.

Proof If $f \neq 0$, two assertions are equivalent to the statement that f is p-coneat for almost all prime elements p of R. □

Now one can immediately find an example that the relation κ is not transitive: We define $E = 0 \longrightarrow R \xrightarrow{\alpha} B \xrightarrow{\beta} K/R \longrightarrow 0$ via $f \in \operatorname{Hom}(K/R, K/R)$ with $f_q = 0$, f_p isomorphism for all $p \neq q$ so $\operatorname{Im}\alpha$ has a weak supplement in B, but does not have a supplement in B. On the other hand, one can give simple conditions that for the extensions of A by C this distinction does not exist.

Theorem 8.2 *Let* $(\square)$ *be given as in Chapter 7 and* $T_p(C) \neq 0$ *for all prime elements* p *of* R. *Then the following are equivalent for the pair* (A, C)*:*

(i) *If the sequence* $0 \longrightarrow A \xrightarrow{\alpha} B \xrightarrow{\beta} C \longrightarrow 0$ *is exact and* $\operatorname{Im}\alpha$ *has a weak supplement in* B, *then* $\operatorname{Im}\alpha \subset^{\kappa} B$.

(ii) *If the sequence* $0 \longrightarrow A \xrightarrow{\alpha} B \xrightarrow{\beta} C \longrightarrow 0$ *is exact and there is an intermediate module* X *where* $\operatorname{Im}\alpha \subset^{\kappa} X \subset^{\kappa} B$, *then* $\operatorname{Im}\alpha \subset^{\kappa} B$.

(iii) *If* A *is not divisible by at least two prime elements of* R, *then*

  (a) A *is not* p*-divisible* $\Rightarrow$ C *is not* p*-divisible,*

  (b) $h_p^A(1) = 0$ *or* $t_p^C(0) \leqslant h_p^A(1)$ *for almost all prime elements* p *of* R.

Proof (i $\Rightarrow$ ii) With the help of these X, we must only show that $\operatorname{Im}\alpha$ has a weak supplement in B: Let V be a supplement of $\operatorname{Im}\alpha$ in X and W be a supplement of X in B. If both $V \cap \operatorname{Im}\alpha = 0$ and $W \cap \operatorname{Im}\alpha = 0$, then both V and W are torsion, thus $(V + W) \oplus \operatorname{Im}\alpha = B$; but if both of them are nonzero, one has $0 \neq U \subset \operatorname{Im}\alpha$ where U is small in B, and the claim follows as in Lemma 8.2.
(ii $\Rightarrow$ iii) Let A be as required. (a) Assume that there is a prime element q of R with A is not q-divisible but C is q-divisible: One can choose $f : C \to K/A$ such that $f_q = 0$ and f_p is coneat for all $p \neq q$, so $f \neq 0$, since still there is $q' \neq q$ where $T_{q'}(K/A) \neq 0$; in addition, there is an intermediate module X where $\operatorname{Im}\alpha \subset^{\kappa} X \subset^{\kappa} B$ by Theorem 8.1, however $\operatorname{Im}\alpha$ does not have a supplement in B. This contradicts with the assumption.
(b) One can choose a fixed q with A is not q-divisible, and in addition $f : C \to K/A$ such that $f_p = 0$ if $p \neq q$ and $h_p^A(1) \neq 0$, and that f_p is coneat in all other cases. Again $f \neq 0$, and $\operatorname{Im}\alpha$ possesses an "intermediate supplement". By assumption it follows that $\operatorname{Im}\alpha \subset^{\kappa} B$, in particular, there is $n \in R$ where $t_p^C(0) \leqslant h_p^A(1)$ for all prime elements p of R with $p \nmid n$. For $p \nmid n$, $p \neq q$ and $h_p^A(1) \notin \{0, \infty\}$ is satisfied, f is as we have selected, $t_p^C(0) \leqslant h_p^A(1)$, and this is the claim.
(iii $\Rightarrow$ i) Let $\operatorname{Im}\alpha$ have a weak supplement in B, and f be the homomorphism belonging to the sequence $(\square)$.

Case 1. $f = 0$ or A is not divisible by any prime element of R. In two cases we have $\operatorname{Im}\alpha \subset^{\kappa} B$ (see Lemma 7.2).

Case 2. $f \neq 0$ and A is not divisible by at least two prime elements of R. By Lemma 8.2 and our assumption (b), there is $n \geqslant 1$ such that: $p \nmid n$ and $h_p^A(1) = 0 \Rightarrow t_p(f) = 0$, $p \nmid n$ and $h_p^A(1) \neq 0 \Rightarrow t_p(f) \leqslant h_p^A(1)$. Thus, since $t_p(f) \leqslant h_p^A(1)$ for almost all prime elements p of R, it follows with (a) that f is κ-element. □

However under different additional conditions, the relation κ is transitive, and we want to give two such conditions to finish:

Lemma 8.4 *Let $X \subset^{\kappa} Y \subset^{\kappa} Z$, and V be a supplement of X in Y, W be a supplement of Y in Z. Then*

(a) *If* $\operatorname{Rad}(Y/X) = Y/X \cap \operatorname{Rad}(Z/X)$, *then $V + W$ is a supplement of X in Z.*

(b) *If V is torsion, then $X \subset^{\kappa} Z$.*

Proof (a) From the characterization of the radical, it follows that $(X + (W \cap Y))/X = ((W + X) \cap Y)/X$ is small not only in Z/X, but also in Y/X, thus the canonical map $V \to Y/X \to Z/W + X$ is an essential epimorphism, i.e. V is a supplement of $W + X$ in Z. Of course W is also a supplement of $V + X$ in Z, and both together provide the assertion.

(b) The coatomic module $((W + X) \cap Y)/X$ has a supplement Y'/X in the torsion module Y/X such that $(W + X)/X$ and Y'/X are mutual supplement in Z/X. Since $Y' \subset^{\kappa} Z$ and $\operatorname{Rad}(Y'/X) = Y'/X \cap \operatorname{Rad}(Z/X)$ and since $(V \cap Y') + X = Y'$ it follows the equivalent argument that $X \subset^{\kappa} Y'$, thus $X \subset^{\kappa} Z$. □

CHAPTER 9

CONCLUSION

In this thesis we applied homological methods for description of the submodules of modules over a principal ideal domain that have supplements. The corresponding elements in the module of extensions are called κ-elements. These elements for the case of abelian groups were studied in (Zöschinger 1978). We generalized the results from (Zöschinger 1978) to modules over principal ideal domains. The κ-elements in general need not form a submodule in the extension module $\mathrm{Ext}_R(C, A)$ but if C is divisible and almost all primary components of C are zero, they coincide with torsion elements of $\mathrm{Ext}_R(C, A)$. We have also investigated β-elements which form a submodule of $\mathrm{Ext}_R(C, A)$ and their relation with κ-elements. It is interesting which of these properties hold in more general situation e.g. for modules over discrete valuation domains, Dedekind domains and Prüfer domains.

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