NON-INTEGER ORDER DERIVATIVES

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ABSTRACT

NON-INTEGER ORDER DERIVATIVES

This thesis is devoted to integrals and derivatives of arbitrary order and applications of the described methods in various fields. This study intends to increase the accessibility of fractional calculus by combining an introduction to the mathematics with a review of selected recent applications in physics. It is described general definitions of fractional derivatives. This definitions are compared with their advantages and disadvantages. Fractional calculus concerns the generalization of differentiation and integration to non-integer (fractional) orders. The subject has a long mathematical history being discussed for the first time already in the correspondence of G. W. Leibnitz around 1690. Over the centuries many mathematicians have built up a large body of mathematical knowledge on fractional integrals and derivatives. Although fractional calculus is a natural generalization of calculus, and although its mathematical history is equally long, it has, until recently, played a negligible role in physics. In the first chapter, Grünwald-Letnikov approache to generalization of the notion of the differentation and integration are considered. In the second chapter, the Riemann –Liouville definition is given and it is compared with Grünwald-Letnikov definition. The last chapter, Caputo’s definition is given. In appendices, two applications are given including tomography and solution of Bessel equation.

ÖZET

TAM MERTEBELİ OLMAYAN TÜREVLER

Bu tezde tam olmayan mertebeli türevler incelenmiş ve bunların uygulamaları üzerine birkaç örnek verilmiştir. Üç bölüm halinde tam olmayan mertebeli türevlerin genel tanımları verilmiştir. Bu tanımların birbirlerine göre avantaj ve dezavantajları tartışılmıştır. Bu konunun Leibnizt'e uzanan detaylı bir geçmişi vardır. Birçok ünlü matematikçi bu konu üzerinde çalışmıştır. Bu çalışmaların derlemesinden oluşan kullanılan kaynaklar tezin sonunda verilmiştir. Çalışmanın birinci bölümünde bahsedilen türev tanımlarından ilki olan Grünwald-Letnikov tanımı verilmiştir. İkinci bölümde Riemann-Liouville tanımı üzerinde çalışıldı. Üçüncü bölümde Caputo tanımından bahsedildi.Bu tanımla birlikte bütün verilen tanımlar karşılaştırıldı. Daha sonra türevdeki en önemli özelliklerden biri olan Leibnitz kuralı ispatlandı. Ekler kısmında türev tanımlarında ihtiyaç duyulan Gamma fonksiyonlarının bazı özellikleri verildi. Ayrıca bu eklerde tomografi cihazında deneysel olarak daha iyi sonuç veren rasyonel mertebeli türev uygulaması ve Bessel denkleminin rasyonel mertebeli dönüşümler kullanılarak çözümünü veren bazı çalışmalar da bulunmaktadır.

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CHAPTER 1

GRUNWALD-LETNIKOV FRACTIONAL DERIVATIVE

1.1. Unification of Integer-order Derivatives and Integrals

Let us consider a continuous function y=f(t). We write the well-known definition for the first-order derivative of the function f(t);

$$f'(t) = \frac{df}{dt} = \lim_{h\to 0}\frac{f(t)-f(t-h)}{h}. \qquad (1)$$

the second-order derivative:

$$f''(t) = \frac{d^2 f}{dt^2} = \lim_{h\to 0}\frac{f'(t)-f'(t-h)}{h} = \lim_{h\to 0}\frac{1}{h}\left\{\frac{f(t)-f(t-h)}{h} - \frac{f(t-h)-f(t-2h)}{h}\right\} \qquad (2)$$

$$= \lim_{h\to 0}\frac{f(t)-2f(t-h)+f(t-2h)}{h^2}$$

$$f'''(t) = \frac{d^3 f}{dt^3} = \lim_{h\to 0}\frac{f(t)-3f(t-h)+3f(t-2h)-f(t-3h)}{h^3} \qquad (3)$$

by induction;

$$f^{(n)}(t) = \frac{d^n f}{dt^n} = \lim_{h\to 0}\frac{1}{h^n}\sum_{r=0}^{n}(-1)^r\binom{n}{r}f(t-rh), \quad \binom{n}{r} = \frac{n(n-1)(n-2)...(n-r+1)}{r!} \qquad (4)$$

now let us consider generalizing the fractions in (1)-(4):

$$f_h^{(p)}(t) = \frac{1}{h^p}\sum_{r=0}^{n}(-1)^r\binom{p}{r}f(t-rh) \qquad (6)$$

where p is an arbitrary integer number; n is also integer for $p \le n$ we have

$$\lim_{h \to 0} f_h^{(p)}(t) = f^{(p)}(t) = \frac{d^p f}{dt^p}, \tag{7}$$

all coefficients in the numerator after $\begin{pmatrix} p \\ p \end{pmatrix}$ are equal to 0.

Let us consider negative values of p. To denote the following expression is better:

$$\begin{bmatrix} p \\ r \end{bmatrix} = \frac{p(p+1)(p+2)...(p+r-1)}{r!} \tag{8}$$

then we have

$$\begin{pmatrix} -p \\ r \end{pmatrix} = \frac{-p(-p-1)...(-p-r+1)}{r!} = (-1)^r \begin{bmatrix} p \\ r \end{bmatrix} \tag{9}$$

and replacing p in (6) with –p we can write

$$f_h^{(-p)}(t) = \frac{1}{h^{-p}} \sum_{r=0}^{n} \begin{bmatrix} p \\ r \end{bmatrix} f(t - rh) \tag{10}$$

where p is a positive integer number.

If n is fixed, then $f_h^{(-p)}(t)$ tends to the limit 0 as $h \to 0$. To reach at non- zero limit, we have to suppose that $n \to \infty$ as $h \to 0$. We can take $h = \frac{t-a}{n}$, where is a real constant, and consider the limit value, either finite or infinite, of $f_h^{(-p)}(t)$, which we will denote as

$$\lim_{\substack{h \to 0 \\ nh = t-a}} f_h^{(-p)}(t) = {}_aD_t^{-p} f(t) \tag{11}$$

now let us consider particular cases;

p=1 we have:

$$f_h^{(-1)}(t) = h\sum_{r=0}^{n} f(t-rh) \tag{12}$$

if we take t-nh=a

$$\lim_{h\to 0} f_h^{(-1)}(t) = \int_0^{t-a} f(t-z)dz = \int_a^t f(\tau)d\tau \tag{13}$$

for p=2

$$\begin{bmatrix} 2 \\ r \end{bmatrix} = \frac{2.3.....(2+r-1)}{r!} = r+1$$

$$f_h^{(-2)}(t) = h\sum_{r=0}^{n} (rh) f(t-rh) \tag{14}$$

denoting t+h=y

$$f^{(-2)}(y-h) = h\sum_{r=0}^{n} (rh) f(y-h-rh)$$

$$= h\sum_{r=0}^{n} (rh) f(y-(r+1)h)$$

$$= h\sum_{r=1}^{n+1} (r-1)hf(y-rh)$$

$$= h\sum_{r=1}^{n+1} (rh) f(y-rh) - h\sum_{r=1}^{n+1} hf(y-rh)$$

$h \to 0$

$$f^{(-2)}(y) = h\sum_{r=1}^{n+1} rhf(y-rh)$$

$$f_h^{(-2)}(t) = h\sum_{r=1}^{n+1}(rh)f(t-rh) \quad (15)$$

taking $h \to 0$;

$$\lim_{\substack{h\to 0\\ nh=t-a}} f_h^{(-2)}(t) = {}_aD_t^{-2}f(t) = \int_0^{t-a} zf(t-z)dt = \int_a^t (t-\tau)f(\tau)d\tau \qquad (16)$$

because $y \to t$ as $h \to 0$.

$$\begin{array}{lll} z=0 & \tau = t & t-z=\tau \\ z=t-a & \tau=a & -dz=d\tau \end{array} \quad \int_t^a -(t-\tau)f(\tau)d\tau = \int_a^t (t-\tau)f(\tau)d\tau$$

$$\begin{bmatrix} 2 \\ r \end{bmatrix} = \frac{3.4.....(3+r-1)}{r!} = \frac{(r+1)(r+2)}{1.2}$$

$$f_h^{(-3)}(t) = \frac{h}{1.2}\sum_{r=0}^{n}(r+1)(r+2)h^2 f(t-rh) \qquad (17)$$

denoting as above t+h=y

$$f_h^{(-3)}(t) = \frac{h}{1.2}\sum_{r=1}^{n+1} r(r+1)h^2 f(y-rh) \qquad (18)$$

$$f_h^{(-3)}(t) = \frac{h}{1.2}\sum_{r=1}^{n+1}(rh)^2 f(y-rh) + \frac{h^2}{1.2}\sum_{r=1}^{n+1} rhf(y-rh) \qquad (19)$$

taking $h \to 0$ we obtain;

$${}_aD_t^{-3}f(t) = \frac{1}{2!}\int_0^{t-a} z^2 f(t-z)dz = \int_a^t (t-\tau)^2 f(\tau)d\tau \qquad (20)$$

because $y \to t$ as $h \to 0$

$$\lim_{\substack{h\to 0\\ nh=t-a}} \frac{h^2}{1.2}\sum_{r=1}^{n+1} rhf(y-rh) = \lim_{\substack{h\to 0\\ nh=t-a}} h\int_a^t (t-\tau)f(\tau)d\tau = 0$$

Relationships (13)-(20)

$${}_aD_t^{-p} f(t) = \lim_{\substack{h\to 0\\ nh=t-a}} h^p \sum_{r=0}^{n} \begin{bmatrix} p \\ r \end{bmatrix} f(t-rh) = \frac{1}{(p-1)!}\int_a^t (t-\tau)^{p-1} f(\tau)d\tau \qquad (21)$$

to prove by induction let us write the function g

$$\text{g(t)}=\int_a^t f(\tau)d\tau \quad \text{(g(a)=0)} \qquad (22)$$

$${}_aD_t^{-p-1} f(t) = \lim_{\substack{h\to 0\\ nh=t-a}} h^{p+1} \sum_{r=0}^{n} \begin{bmatrix} p+1 \\ r \end{bmatrix} f(t-rh)$$

$$= \lim_{\substack{h\to 0\\ nh=t-a}} h^p \sum_{r=0}^{n} \begin{bmatrix} p+1 \\ r \end{bmatrix} g(t-rh) - \lim_{\substack{h\to 0\\ nh=t-a}} h^p \sum_{r=0}^{n} \begin{bmatrix} p+1 \\ r \end{bmatrix} g(t-(r+1)h) \qquad (23)$$

also we can see

$$\begin{bmatrix} p+1 \\ r \end{bmatrix} = \begin{bmatrix} p \\ r \end{bmatrix} + \begin{bmatrix} p+1 \\ r-1 \end{bmatrix} \qquad (24)$$

where we must put

$$\begin{bmatrix} p+1 \\ -1 \end{bmatrix} = 0$$

if we apply (24) to (23) and replacing of r by r-1

$$ {}_aD_t^{-p-1}f(t)=\lim_{\substack{h\to0\\nh=t-a}}h^p\sum_{r=0}^{n}\begin{bmatrix}p\\r\end{bmatrix}g(t-rh)+\lim_{\substack{h\to0\\nh=t-a}}h^p\sum_{r=0}^{n}\begin{bmatrix}p+1\\r-1\end{bmatrix}g(t-rh)-\lim_{\substack{h\to0\\nh=t-a}}h^p\sum_{r=1}^{n+1}\begin{bmatrix}p+1\\r-1\end{bmatrix}g(t-rh) $$

$$ {}_aD_t^{-p}g(t)-\lim_{\substack{h\to0\\nh=t-a}}h^p\begin{bmatrix}p+1\\n\end{bmatrix}g(t-(n+1)h)={}_aD_t^{-p}g(t)-(t-a)^p\lim_{n\to\infty}\begin{bmatrix}p+1\\n\end{bmatrix}\frac{1}{n^p}g(a-\frac{t-a}{n}) $$

it follows from (22)

$$ \lim_{n\to\infty}g(a-\frac{t-a}{n})=0 $$

taking into account the limit definition of the gamma function

$$ \lim_{n\to\infty}\begin{bmatrix}p+1\\n\end{bmatrix}\frac{1}{n^p}=\lim_{n\to\infty}\frac{(p+1)(p+2)...(p+n)}{n^p n!}=\frac{1}{\Gamma(p+1)} $$

$$ {}_aD_t^{-p-1}f(t)={}_aD_t^{-p}g(t)=\frac{1}{(p-1)!}\int_a^t(t-\tau)^{p-1}g(\tau)d\tau=-\frac{(t-\tau)^p g(\tau)}{p!}\Bigg|_{r=a}^{r=t}+\frac{1}{p!}\int_a^t(t-\tau)^p f(\tau)d\tau $$

$$ =\frac{1}{p!}\int_a^t(t-\tau)^p f(\tau)d\tau \qquad (25) $$

Let us show that formula (1. 21) is a representation of a p-fold integral.Integrating the reliationship

$$ \frac{d}{dt}({}_aD_t^{-p}f(t))=\frac{1}{(p-2)!}\int_a^t(t-\tau)^{p-2}f(\tau)d\tau={}_aD_t^{-p+1}f(t) \qquad (26) $$

from a to t we obtain:

$$ {}_aD_t^{-p} f(t) = \int_a^t ({}_aD_t^{-p+1} f(t))dt, $$

$$ {}_aD_t^{-p+1} f(t) = \int_a^t ({}_aD_t^{-p+2} f(t))dt, $$

$$ {}_aD_t^{-p} f(t) = \int_a^t dt \int_a^t ({}_aD_t^{-p+2} f(t))dt = \int_a^t dt \int_a^t dt \int_a^t ({}_aD_t^{-p+3} f(t))dt = \int_a^t dt \int_a^t dt ... \int_a^t f(t)dt $$

we see that the derivative of an integer order n and the p-fold integral of the continuous function f(t) are particular cases of the general expression

$$ {}_aD_t^{p} f(t) = \lim_{\substack{h\to 0 \\ nh=t-a}} h^{-p} \sum_{r=0}^{n} (-1)^r \binom{p}{r} f(t-rh) \tag{27} $$

which represents the derivative of order m if p=m and m-fold integral if p=-m. This observation leads to the idea of a generalization of the notions of differentations and integration by allowing p in (27) to be an arbitrary real or even complex number. We will restrict our attention to real values of p.

1.2. Integrals of Arbitrary Order

Let us consider the case of p<0. For convenience let us replace p by –p in the expression (27). Then (27) takes the form

$$ {}_aD_t^{-p} f(t) = \lim_{\substack{h\to 0 \\ nh=t-a}} h^{p} \sum_{r=0}^{n} \begin{bmatrix} p \\ r \end{bmatrix} f(t-rh), \tag{28} $$

where, as above, the values of h and n relateas nh= t- a

To prove the existance of the limit in (28) and to evaluate that limit we need the following theorem (A.V. Letnikov, [66]):

THEOREM 1.1 Let us take a sequence β_k , (k = 1,2, …) and suppose that

$$\lim_{k\to\infty}\beta_k = 1, \tag{29}$$

$$\lim_{n\to\infty}\alpha_{n,k} = 0 \text{ for all k }, \tag{30}$$

$$\lim_{n\to k}\sum_{k=1}^{n}\alpha_{n,k} = A \text{ for all k}, \tag{31}$$

$$\sum_{k}^{n}\left|\alpha_{n,k}\right| < K \text{ for all n.} \tag{32}$$

Then

$$\lim_{n\to\infty}\sum_{k=1}^{n}\alpha_{n,k}\beta_k = A. \tag{33}$$

Proof: The condition (29) allows us to put

$$\beta_k = 1-\sigma_{k,} \text{ where } \lim_{k\to\infty}\sigma_k = 0. \tag{34}$$

It follows from the condition (30) that for every fixed r

$$\lim_{n\to\infty}\sum_{k=1}^{r-1}\alpha_{n,k}\beta_k = 0 \tag{35}$$

$$\lim_{n\to\infty}\sum_{k=1}^{r-1}\alpha_{n,k} = 0. \tag{36}$$

Using subsequently (35), (34), (31) and (36) we obtain

$$\lim_{n\to\infty}\sum_{k=1}^{n}\alpha_{n,k}\beta_k = \lim_{n\to\infty}\sum_{k=r}^{n}\alpha_{n,k,}\beta_k$$

$$= \lim_{n\to\infty}\sum_{k=r}^{n}\alpha_{n,k} - \lim_{n\to\infty}\sum_{k=r}^{n}\alpha_{n,k}\sigma_k$$

$$= \lim_{n\to\infty}\sum_{k=1}^{n}\alpha_{n,k} - \lim_{n\to\infty}\sum_{k=r}^{n}\alpha_{n,k}\sigma_k$$

$$= \mathrm{A} - \lim_{n\to\infty}\sum_{k=r}^{n}\alpha_{n,k}\sigma_{k.}$$

Now, using (36) and (32), we can perform the following estimation:

$$\left|A - \lim_{n\to\infty}\sum_{k=1}^{n}\alpha_{n,k}\beta_k\right| < \lim_{n\to\infty}\sum_{k=r}^{n}\left|\alpha_{n,k}\right|.\left|\sigma_k\right|$$

$$< \sigma^* \lim_{n\to\infty}\sum_{k=r}^{n}\left|\alpha_{n,k}\right| = \sigma^* \lim_{n\to\infty}\sum\left|\alpha_{n,k}\right|$$

$$< \sigma^* K$$

where $\sigma^* = \max\limits_{k\geq r}\left|\sigma_k\right|$.

It follows from (34) that for each arbitrarily small $\in > 0$ there exists r such that $\sigma^* < \in / K$ and, therefore,

$$\left|A - \lim_{n\to\infty}\sum_{k=1}^{n}\alpha_{n,k}\beta_k\right| < \in,$$

and the statement (33) of the theorem holds.

Theorem 1.1 has a simple consequence. Namely, if we take

$$\lim_{n\to\infty}\beta_k = B,$$

then

$$\lim_{n\to\infty}\sum_{k=1}^{n}\alpha_{n,k}\beta_k = AB. \quad (37)$$

Indeed, introducing the sequence

$$\tilde{\beta}_k = \frac{\beta_k}{B}, \quad \lim_{k\to\infty}\tilde{\beta}_k = 1,$$

we can aplly Theorem 1.1 to obtain

$$\lim_{n\to\infty}\sum_{k=1}^{n}\alpha_{n,k}\tilde{\beta} = \lim_{n\to\infty}\sum_{k=1}^{n}\alpha_{n,k}\frac{\beta_k}{B} = A,$$

from which the statement (37) follows.

To apply Theorem 1.1 for the evaluation of the limit (28), we write

$${}_aD_t^{-p}f(t) = \lim_{\substack{h\to 0\\ nh=t-a}} h^p\sum_{r=0}^{n}\begin{bmatrix} p\\ r\end{bmatrix} f(t-rh)$$

$$= \lim_{\substack{h\to 0\\ nh=t-a}}\sum_{r=0}^{n}\frac{1}{r^{p-1}}\begin{bmatrix} p\\ r\end{bmatrix} h(rh)^{p-1}f(t-rh)$$

$$= \frac{1}{\Gamma(p)}\lim_{\substack{h\to 0\\ nh=t-a}}\sum_{r=0}^{n}\frac{\Gamma(p)}{r^{p-1}}\begin{bmatrix} p\\ r\end{bmatrix} h(rh)^{p-1}f(t-rh)$$

$$= \frac{1}{\Gamma(p)}\lim_{n\to\infty}\sum_{r=o}^{n}\frac{\Gamma(p)}{r^{p-1}}\begin{bmatrix} p\\ r\end{bmatrix}\frac{t-a}{n}\left(r\frac{t-a}{n}\right)^{p-1} f\left(t-r\frac{t-a}{n}\right)$$

and take

$$\beta_r = \frac{\Gamma(p)}{r^{p-1}} \begin{bmatrix} p \\ r \end{bmatrix},$$

$$\alpha_{n,r} = \frac{t-a}{n}(r\frac{t-a}{n})^{p-1} f(t - r\frac{t-a}{n}).$$

Using (A1.7) we have

$$\lim_{r\to\infty} \beta_r = \lim_{r\to\infty} \frac{\Gamma(p)}{r^{p-1}} \begin{bmatrix} p \\ r \end{bmatrix} = 1. \qquad (38)$$

If f is continuous in the closed interval [a,b], then

$$\lim_{n\to\infty} \sum_{r=0}^{n} \alpha_{n,r} = \lim_{n\to\infty} \sum_{r=0}^{n} \frac{t-a}{n}(r\frac{t-a}{n})^{p-1} f(t - r\frac{t-a}{n})$$

$$= \lim_{h\to 0} \sum_{r=0}^{n} h(rh)^{p-1} f(t-rh)$$

$$= \int_a^t (t-\tau)^{p-1} f(\tau) d\tau \qquad (39)$$

Taking into account (38) and (39) and applying Theorem 1. 1 we have

$$_a D_t^{-p} f(t) = \lim_{\substack{h\to 0 \\ nh=t-a}} h^p \sum_{r=0}^{n} \begin{bmatrix} p \\ r \end{bmatrix} f(t-rh) = \frac{1}{\Gamma(p)} \int_a^t (t-\tau)^{p-1} f(\tau) d\tau \qquad (40)$$

f' is continuous in [a,b]

$$_a D_t^{-p} f(t) = \frac{f(a)(t-a)^p}{\Gamma(p+1)} + \frac{1}{\Gamma(p+1)} \int_a^t (t-\tau)^p f'(\tau) d\tau \qquad (41)$$

and f has m+1 continuous derivatives then

$$ {}_aD_t^{-p} f(t) = \sum_{k=0}^{m} \frac{f^{(k)}(a)(t-a)^{p+k}}{\Gamma(p+k+1)} + \frac{1}{\Gamma(p+k+1)} \int_a^t (t-\tau)^{p+m} f^{(m+1)}(\tau) d\tau \quad (42) $$

1.3. Derivatives of Arbitrary Order

Let us evaluate the limit

$$ {}_aD_t^{p} f(t) = \lim_{\substack{h\to 0 \\ nh=t-a}} h^{-p} \sum_{r=0}^{n} (-1)^r \binom{p}{r} f(t-rh) = \lim_{\substack{h\to 0 \\ nh=t-a}} f_h^{(p)}(t) \quad (43) $$

where

$$ f_h^{(p)}(t) = h^{-p} \sum_{r=0}^{n} (-1)^r \binom{p}{r} f(t-rh) \quad (44) $$

property of the binomial coefficients

$$ \binom{p}{r} = \binom{p-1}{r} + \binom{p-1}{r-1} \quad (45) $$

$$ f_h^{(p)}(t) = h^{-p} \sum_{r=0}^{n} (-1)^r \binom{p-1}{r} f(t-rh) + h^{-p} \sum_{r=0}^{n} (-1)^r \binom{p-1}{r-1} f(t-rh) $$

$$ h^{-p} \sum_{r=0}^{n} (-1)^r \binom{p-1}{r} f(t-rh) + h^{-p} \sum_{r=0}^{n-1} (-1)^{r+1} \binom{p-1}{r} f(t-(r+1)h) = (-1)^n \binom{p-1}{n} h^{-p} f(a) $$

$$ + h^{-p} \sum_{r=0}^{n-1} (-1)^r \binom{p-1}{r} \Delta f(t-rh) \quad (46) $$

where we denote

$$\Delta f(t-rh) = f(t-rh) - f(t-(r+1)h)$$

$\Delta f(t-rh)$ is the first-order backward difference of the function $f(\tau)$ at the point

$$\tau = t - rh .$$

Applying (45) of the binomial coefficient m times, we obtain starting from (46)

$$f_h^{(h)}(t) = (-1)^n \binom{p-1}{n} h^{-p} f(a) + (-1)^{n-1} \binom{p-2}{n-1} h^{-p} \Delta f(a+h) + h^{-p} \sum_{r=0}^{n-2} (-1)^r \binom{p-2}{r} \Delta^2 f(t-rh)$$

$$= (-1)^n \binom{p-1}{n} h^{-p} f(a) + (-1)^{n-1} \binom{p-2}{n-1} h^{-p} \Delta f(a+h) + (-1)^{n-2} \binom{p-3}{n-3} h^{-p} \Delta^2 f(a+2h)$$

$$+ h^{-p} \sum_{r=0}^{n-3} (-1)^r \binom{p-3}{r} \Delta^3 f(t-rh) \qquad (47)$$

$$= \ldots$$

$$= \sum_{k=0}^{m} (-1)^{n-k} \binom{p-k-1}{n-k} h^{-p} \Delta^k f(a+kh) + h^{-p} \sum_{r=0}^{n-m-1} (-1)^r \binom{p-m-1}{r} \Delta^{m+1} f(t-rh) \quad (48)$$

let us evaluate the limit of the k-th term in the first sum in (48):

$$\lim_{\substack{h\to 0\\ nh=t-a}} (-1)^{n-k} \binom{p-k-1}{n-k} h^{-p} \Delta^k f(a+kh) = \lim_{\substack{h\to 0\\ nh=t-a}} (-1)^{n-k} \binom{p-k-1}{n-k} (n-k)^{p-k} \left(\frac{n}{n-k}\right)^{p-k} (nh)^{-p+k} \frac{\Delta^k f(a+kh)}{h^k}$$

$$= (t-a)^{-p+k} \lim_{n\to\infty} (-1)^{n-k} \binom{p-k-1}{n-k} (n-k)^{p-k} . \lim_{n\to\infty} \left(\frac{n}{n-k}\right)^{p-k} . \lim_{h\to 0} \frac{\Delta^k f(a+kh)}{h^k}$$

$$= \frac{f^{(k)}(a)(t-a)^{-p+k}}{\Gamma(-p+k+1)} \tag{49}$$

$$\lim_{n\to\infty}(-1)^{n-k}\binom{p-k-1}{n-k}(n-k)^{p-k} = \lim_{n\to\infty}\frac{(-p+k+1)(-p+k+2)...(-p+n)}{(n-k)^{-p+k}(n-k)!} = \frac{1}{\Gamma(-p+k+1)}$$

$$\lim_{n\to\infty}\left(\frac{n}{n-k}\right)^{p-k} = 1$$

$$\lim_{h\to 0}\frac{\Delta^k f(a+kh)}{h^k} = f^{(k)}(a)$$

We can write easily the limit of the first sum in (48).

To evaluate the limit of the second sum in (48) let us write it in the form

$$\frac{1}{\Gamma(-p+m+1)}\sum_{r=0}^{n-m-1}(-1)^r\Gamma(-p+m+1)\binom{p-m-1}{r}r^{-m+p}.h(rh)^{m-p}\frac{\Delta^{m+1}f(t-rh)}{h^{m+1}} \tag{50}$$

to apply Theorem 1. 1 we take

$$\beta_r = (-1)^r\Gamma(-p+m+1)\binom{p-m-1}{r}r^{-m+p}$$

$$\alpha_{n,r} = h(rh)^{m-p}\frac{\Delta^{m+1}f(t-rh)}{h^{m+1}} \quad h = \frac{t-a}{n}$$

Using (A1. 7) we verify that

$$\lim_{r\to\infty}\beta_r = \lim_{r\to\infty}(-1)^r\Gamma(-p+m+1)\binom{p-m-1}{r}r^{-m+p} = 1 \tag{51}$$

if m-p>-1 then

$$\lim_{n\to\infty}\sum_{r=0}^{n-m-1}\alpha_{n,r}=\lim_{\substack{h\to 0\\ nh=t-a}}\sum_{r=0}^{n-m-1}h(rh)^{m-p}\frac{\Delta^{m+1}f(t-rh)}{h^{m+1}}=\int_a^t(t-\tau)^{m-p}f^{(m+1)}(\tau)d\tau. \qquad (52)$$

Taking into account (51) and (52) and applying Theorem 2. 1 we have that

$$\lim_{\substack{h\to 0\\ nh=t-a}}h^{-p}\sum_{r=0}^{n-m-1}(-1)^r\binom{p-m-1}{r}\Delta^{m+1}f(t-rh)$$

$$=\frac{1}{\Gamma(-p+m+1)}\int_a^t(t-\tau)^{m-p}f^{(m+1)}(\tau)d\tau \qquad (53)$$

Using (49) and (53) we obtain the limit (43):

$${}_aD_t^p f(t)=\lim_{\substack{h\to\infty\\ nh=t-a}}f_h^{(p)}(t)=\sum_{k=0}^{m}\frac{f^{(k)}(a)(t-a)^{-p+k}}{\Gamma(-p+k+1)}+\frac{1}{\Gamma(-p+m+1)}\int_a^t(t-\tau)^{m-p}f^{(m+1)}(\tau)d\tau \quad (54)$$

The formula (54) has been obtained under the assumption that the derivatives $f^{(k)}(t)$, (k=1, 2, 3, … , m+1) are continuous in the closed interval [a, t] and that m is an integer number satisfying the condition m>p-1. The smallest possible value for m is determined by the inequality

$$m<p<m+1.$$

1.4. Fractional Derivative of $(t-a)^{\nu}$

Let us evaluate the Grünwald-Letnikov fractional derivative ${}_aD_t^p f(t)$ of the power function

$$f(t)=(t-a)^{\nu}$$

Where ν is a real number.

Let us start by considering negative values of p, which means that we will start with the evaluation of the fractional integral of order –p. Let us use the formula (40):

$$ {}_a D_t^p (t-a)^\nu = \frac{1}{\Gamma(-p)} \int_a^t (t-\tau)^{-p-1} (\tau - a)^\nu \, d\tau \qquad (55) $$

And suppose $\nu > -1$ for the convergence of the integral. Performing in (55) the substitution $\tau = a + \xi(t-a)$ and then using the definition of beta function, we obtain:

$$ {}_a D_t^p (t-a)^\nu = \frac{1}{\Gamma(-p)} (t-a)^{\nu - p} \int_0^1 \xi^\nu (1-\xi)^{-p-1} d\xi = \frac{1}{\Gamma(-p)} B(-p, \nu+1)(t-a)^{\nu-p} $$

$$ = \frac{\Gamma(\nu+1)}{\Gamma(\nu - p + 1)} (t-a)^{\nu-p}, \ \left(p < 0, \nu > -1\right). \qquad (56) $$

Let us take $0 \le m \le p \le m+1$. To apply the formula (54), we must require $\nu > m$ for the convergence of the integral in (54). Then

$$ {}_a D_t^p (t-a)^\nu = \frac{1}{\Gamma(-p+m+1)} \int_a^t (t-\tau)^{m-p} \frac{d^{m+1}(\tau-a)^\nu}{d\tau^{m+1}} d\tau, \qquad (57) $$

Because all non-integral addends are equal to 0.

Taking into account that

$$ \frac{d^{m+1}(\tau-a)^\nu}{d\tau^{m+1}} = \nu(\nu+1)...(\nu-m)(\tau-a)^{\nu-m-1} = \frac{\Gamma(\nu+1)}{\nu - m} (\tau-a)^{\nu-m-1} $$

and performing the substitution $\tau = a + \xi(t-a)$ we obtain:

$$ {}_a D_t^p (t-a)^\nu = \frac{\Gamma(\nu+1)}{\Gamma(\nu-m)\Gamma(-p+m+1)} \int_a^t (t-\tau)^{m-p} (\tau-a)^{\nu-m-1} d\tau = \frac{\Gamma(\nu+1) B(-p+m+1, \nu-m)}{\Gamma(\nu-m)\Gamma(-p+m+1)} (t-a)^{\nu-p} $$

$$= \frac{\Gamma(\nu+1)}{\Gamma(-p+\nu+1)}(t-a)^{\nu-p}. \tag{58}$$

Noting that the expression (58) is formally identical to the expression (56) we can conclude that the Grünwald-Letnikov fractional derivative of the power function $f(t) = (t-a)^{\nu}$ is given by the Formula

$$_{a}D_{t}^{p}(t-a)^{\nu} = \frac{\Gamma(\nu+1)}{\Gamma(-p+\nu+1)}(t-a)^{\nu-p} \tag{59}$$

$$\left(p < 0, \nu > -1\right) \text{ or } \left(0 \le m \le p \le m+1, \nu > m\right).$$

We will return to Formula (59) for the Grünwald-Letnikov fractional derivative of the power function later, when we consider other approaches to fractional differentiation.

1.5. Composition with Integral Order Derivatives

Noting that we have only one restriction for m in the formula (54), namely the condition m>p-1, let us write s instead of m and rewrite (54)

$$_{a}D_{t}^{p} f(t) = \sum_{k=0}^{s} \frac{f^{(k)}(a)(t-a)^{-p+k}}{\Gamma(-p+k+1)} + \frac{1}{\Gamma(-p+s+1)} \int_{a}^{t} (t-\tau)^{s-p} f^{(s+1)}(\tau) d\tau \tag{60}$$

In what follows we assume that m<p<m+1.

Let us evaluate the derivative of integer order n of the fractional derivative of fractional order p in the form (60), where we take s$\geq$m+n-1. The result is:

$$\frac{d^n}{dt^n}({}_{a}D_{t}^{p} f(t)) = \sum_{k=0}^{s} \frac{f^{(k)}(a)(t-a)^{-p-n+k}}{\Gamma(-p-n+k+1)} + \frac{1}{\Gamma(-p-n+s+1)} \int_{a}^{t} (t-\tau)^{s-p-n} f^{(s+1)}(\tau) d\tau = {}_{a}D_{t}^{p+n} f(t). \tag{61}$$

$$\frac{d^n}{dt^n}({}_aD_t^p f(t)) = {}_aD_t^{p+n} f(t) \tag{62}$$

Since $s \geq m+n-1$ is arbitrary, let us take s=m+n-1. This gives:

$$\frac{d^n}{dt^n}({}_aD_t^p f(t)) = {}_aD_t^{p+n} f(t) = \sum_{k=0}^{m+n-1} \frac{f^{(k)}(a)(t-a)^{-p-n+k}}{\Gamma(-p-n+k+1)} + \frac{1}{\Gamma(m-p)} \int_a^t (t-\tau)^{m-p-1} f^{(m+n)}(\tau)d\tau \tag{63}$$

Let us consider the reverse order of operations and evaluate the fractional derivative of order p of an integer-order derivative $\frac{d^n f(t)}{dt^n}$. Using the formula (60) we obtain:

$${}_aD_t^p(\frac{d^n f(t)}{dt^n}) = \sum_{k=0}^{s} \frac{f^{(n+k)}(a)(t-a)^{-p+k}}{\Gamma(-p+k+1)} + \frac{1}{\Gamma(-p+s+1)} \int_a^t (t-\tau)^{s-p} f^{(n+s+1)}(\tau)d\tau \tag{64}$$

putting here s=m-1 we obtain:

$${}_aD_t^p(\frac{d^n f(t)}{dt^n}) = \sum_{k=0}^{m-1} \frac{f^{(n+k)}(a)(t-a)^{-p+k}}{\Gamma(-p+k+1)} + \frac{1}{\Gamma(m-p)} \int_a^t (t-\tau)^{m-p-1} f^{(m+n)}(\tau)d\tau \tag{65}$$

and comparing (63) and (65) we arrive at the conclusion that

$$\frac{d^n}{dt^n}({}_aD_t^p f(t)) = {}_aD_t^p(\frac{d^n f(t)}{dt^n}) + \sum_{k=0}^{n-1} \frac{f^{(k)}(a)(t-a)^{-p-n+k}}{\Gamma(-p-n+k+1)}. \tag{66}$$

The relationship (66) says that the operations $\frac{d^n}{dt^n}$ and ${}_aD_t^p$ are commutative, that

$$\frac{d^n}{dt^n}({}_aD_t^p f(t)) = {}_aD_t^p\left(\frac{d^n f(t)}{dt^n}\right) = {}_aD_t^{p+n} f(t), \tag{67}$$

only if at the lower terminal t=a of the fractional differentiation we have

$$f^{(k)}(a) = 0, \text{ (k=0, 1, 2, …, n-1)} \tag{68}$$

1.6. Composition with Fractional Derivatives

Let us consider the fractional derivative of order q of a fractional derivative of order p:

$$_aD_t^q(_aD_t^p)$$

Two cases will be considered separately: p<0 and p>0. The first cases means that depending on the sign of q-differetiation of order q>0 or integration of order –q>0 is applied to the fractional integral of order –p>0. In the second case, the object of the outer operation is the fractional derivative of order p>0.

In both cases we will obtain an analogue of the well-known property of integer-order differetiation:

$$\frac{d^n}{dt^n}\left(\frac{d^m f(t)}{dt^m}\right) = \frac{d^m}{dt^m}\left(\frac{d^n f(t)}{dt^n}\right) = \frac{d^{m+n} f(t)}{dt^{m+n}}$$

Case p<0

Let us take q<0. Then we have:

$$_aD_t^q(_aD_t^p f(t)) = \frac{1}{\Gamma(-q)}\int_a^t (t-\tau)^{-q-1}(_aD_\tau^p f(\tau))d\tau$$

$$= \frac{1}{\Gamma(-q)\Gamma(-p)}\int_a^t (t-\tau)^{-q-1}d\tau\int_a^\tau (\tau-\xi)^{-p-1} f(\xi)d\xi$$

$$=\frac{1}{\Gamma(-q)\Gamma(-p)}\int_a^t f(\xi)d\xi\int_\xi^t (t-\tau)^{-q-1}(\tau-\xi)^{-p-1}d\tau$$

$$=\frac{1}{\Gamma(-p-q)}\int_a^t (t-\xi)^{-p-q-1}f(\xi)d\xi={}_aD_t^{p+q}f(t) \qquad (69)$$

where the integral

$$\int_\xi^t (t-\tau)^{-q-1}(\tau-\xi)^{-p-1}d\tau=(t-\xi)^{-p-q-1}\int_0^1 (1-z)^{-q-1}z^{-p-1}dz=\frac{\Gamma(-q)\Gamma(-p)}{\Gamma(-p-q)}(t-\xi)^{-p-q-1}$$

is evaluated with the help of the substitution $\tau=\xi+z(t-\xi)$ and the definition of the beta function.

Let us suppose that 0<n<q<n+1. Noting that q=(n+1)+(q-n-1), where q-n-1<0 and using the formulas (62) and (69) we obtain:

$$_aD_t^q({}_aD_t^p f(t))=\frac{d^{n+1}}{dt^{n+1}}\left\{{}_aD_t^{q-n-1}({}_aD_t^p f(t))\right\}=\frac{d^{n+1}}{dt^{n+1}}\left\{{}_aD_t^{p+q-n-1}f(t)\right\}={}_aD_t^{p+q}f(t) \qquad (70)$$

Combining (69) and (70) we conclude that if p<0, then for any real q

$$_aD_t^q({}_aD_t^p f(t))={}_aD_t^{p+q}f(t).$$

Case p>0

Let us assume that $0\le m<p<m+1$. Then, according to formula (1. 54), we have

$$_aD_t^p f(t)=\lim_{\substack{h\to\infty\\nh=t-a}} f_h^{(p)}(t)=\sum_{k=0}^m\frac{f^{(k)}(a)(t-a)^{-p+k}}{\Gamma(-p+k+1)}+\frac{1}{\Gamma(-p+m+1)}\int_a^t (t-\tau)^{m-p}f^{(m+1)}(\tau)d\tau \qquad (71)$$

Let us take q<0 and evaluate

$${}_aD_t^q\left({}_aD_t^p f(t)\right).$$

Examining the right-hand side of (71) we see that the functions $(t-a)^{-p+k}$ have non-integrable singularities for k=0, 1, 2, …, m-1. Therefore , the derivative of real order p of ${}_aD_t^p f(t)$ exists only if

$$f^{(k)}(a)=0 \text{ (k=0, 1, …, m-1)} \tag{72}$$

The integral in the right-hand side of (71) is equal to ${}_aD_t^{p-m-1}f(t)$ (the fractional integral of order –p+m+1 of the function f(t)). Therefore, under the conditions (72) the representation (71) of the p-th derivative of f(t) takes the following form:

$${}_aD_t^p f(t)=\frac{f^{(m)}(a)(t-a)^{-p+m}}{\Gamma(-p+m+1)}+{}_aD_t^{p-m-1}f^{(m+1)}(t). \tag{73}$$

we can find the derivative of order q<0 (the integral of order –q>0) of the derivative of order pgiven by (73):

$${}_aD_t^q\left({}_aD_t^p f(t)\right)=\frac{f^{(m)}(a)(t-a)^{-p-q+m}}{\Gamma(-p-q+m+1)}+\frac{1}{\Gamma(-p-q+m+1)}\int_a^t\frac{f^{(m+1)}(\tau)d\tau}{(t-\tau)^{p+q-m}} \tag{74}$$

because

$${}_aD_t^q\left({}_aD_t^{p-m-1}f^{(m+1)}(t)\right)={}_aD_t^{p+q-m-1}f^{(m+1)}(t)=\frac{1}{\Gamma(-p-q+m+1)}\int_a^t\frac{f^{(m+1)}(\tau)d\tau}{(t-\tau)^{p+q-m}}$$

Taking into account the conditions (72) and the formula (71) we obtain

$${}_aD_t^q\left({}_aD_t^p f(t)\right)={}_aD_t^{p+q}f(t) \tag{75}$$

Let us take $0 \le n < q < n+1$. Assuming that f(t) satisfies the conditions (72) and taking into account that q-n-1<0 and, therefore, the formula (75) can be used. We obtain:

$$ {}_aD_t^q({}_aD_t^p f(t)) = \frac{d^{n+1}}{dt^{n+1}}\left\{ {}_aD_t^{q-n-1}({}_aD_t^p f(t))\right\} = \frac{d^{n+1}}{dt^{n+1}}\left\{ {}_aD_t^{p+1-n-1} f(t)\right\} = {}_aD_t^{p+q} f(t) \quad (76)$$

which is the same as (75). Therefore, we can conclude that if p<0, then the relationship (75) holds for arbitrary real q; if $0 \le m < p < m+1$ then the relitionship (75) holds also for arbitrary real q, if the function f(t) satisfies the conditions (72).

Moreover, if $0 \le m < p < m+1$ and $0 \le n < q < n+1$ and the function f(t) satisfies the conditions

$$ f^{(k)}(a) = 0 \text{ (k=0, 1, 2, …, r-1)} \quad (77)$$

where r=max(n, m), then the operators of fractional differentiation ${}_aD_t^p$ and ${}_aD_t^q$ commute:

$$ {}_aD_t^q({}_aD_t^p f(t)) = {}_aD_t^p({}_aD_t^q f(t)) = {}_aD_t^{p+q} f(t) \quad (78)$$

CHAPTER 2

RIEMANN-LIOUVILLE FRACTIONAL DERIVATIVE DEFINITION

Manipulation with the Grünwald-Letnikov fractional derivatives defined a limit of a fractional-order backward difference is not convenient. (54) looks better because of the presense of the integral in it, what about the non-integral terms. To consider (54) as a particular case of the integro-differential expression

$$ {}_aD_t^p f(t) = (\frac{d}{dt})^{m+1} \int_a^t (t-\tau)^{m-p} f(\tau)d\tau, \ \ (m \le p < m+1) . \tag{79}$$

The expression (79) it is the most known definition of the fractional derivative; it is called the Riemann-Liouville definition.

The expression (54), which has been obtained for the Grünwald-Letnikov fractional derivative under the assumption that the function f(t) must be m+1 times continuously differentiable, can be obtained from (79) under the same assumption by performing repeatedly integration by parts and differentiation. This gives

$$ {}_aD_t^p f(t) = (\frac{d}{dt})^{m+1} \int_a^t (t-\tau)^{m-p} f(\tau)d\tau = \sum_{k=0}^{m} \frac{f^{(k)}(a)(t-a)^{-p+k}}{\Gamma(-p+k+1)} + \frac{1}{\Gamma(-p+m+1)} \int_a^t (t-\tau)^{m-p} f^{(m+1)}(\tau)d\tau $$

$$ = {}_aD_t^p f(t), \ \ (m \le p < m+1) . \tag{80}$$

If we consider a class of functions f(t) having m+1 continuous derivatives for $t \ge 0$, then the Grünwald-Letnikov definition (43) is equivalent to the Riemann-Liouville dfinition (73).

From the pure mathematical point of view such a class of functions is norrow; however, this class of functions is very important for applications, because the character of the majority of dynamical processes is smooth enough and does not allow discontinuities. Understanding this fact is important for the proper use of the methods of the fractional calculus in applications, especially because of the fact that the Riemann-Liouville definition (73) provides an opportunity to weaken the conditions on the function f(t). Namely, it is enough to to require the integrability of f(t); then the integral (73) exists for t>a and can be differentiated m+1 times. The weak conditions on the function f(t) in (73) are necessary, for example. For obtaining the solution of the Abel integral equation. Let us look at how the Riemann-Liouville definition (73) appears as the result of the unification of the notions of integral-order integration and differentiation.

2.1. Unification of Integer-order Derivatives and Integrals

Let us suppose that the function f(τ) is continuous and integrable in every finite interval (a, t); the function f(t) may have an integrable singularity of order r<1 at the point $\tau = a$;

$$\lim_{\tau \to a} (\tau - a)^r f(t) = const.(\neq 0) .$$

Then

$$f^{(-1)}(t) = \int_a^t f(\tau) d\tau \quad (81)$$

exists and has a finite value, namely equal to 0, as t$\to$a. Performing the substitution $\tau = a + y(t - a)$ and then denoting $\varepsilon = t - a$, we obtain

$$\lim_{t \to a} f^{(-1)}(t) = \lim_{t \to a} \int_a^t f(\tau) d\tau = \lim_{t \to a} (t - a) \int_0^1 f(a + y(t - a)) dy = \lim_{\varepsilon \to 0} \varepsilon^{1-r} \int_0^1 (\varepsilon y)^r f(a + y\varepsilon) y^{-r} dy = 0 \qquad (82)$$

because r<1. Therefore we can consider the two-fold integral

$$f^{(-2)}(t) = \int_a^t d\tau_1 \int_a^{\tau_1} f(\tau) d\tau = \int_a^t f(\tau) d\tau \int_\tau^t d\tau_1 = \int_a^t (t - \tau) f(\tau) d\tau. \qquad (83)$$

Integration of (77) gives the three-fold integral of f(τ):

$$f^{(-3)}(t)=\int_a^t d\tau_1\int_a^{\tau_1} d\tau_2\int_a^{\tau_2} f(\tau_3)=\int_a^t d\tau_1\int_a^{\tau_1}(\tau_1-\tau)f(\tau)d\tau=\frac{1}{2}\int_a^t (t-\tau)^2 f(\tau)d\tau \qquad (84)$$

by induction in the general case we have the Cauchy formula

$$f^{(-n)}(t)=\frac{1}{\Gamma(n)}\int_a^t (t-\tau)^{n-1} f(\tau)d\tau. \qquad (85)$$

Let us suppose that $n\geq 1$ is fixed and take integer $\mathrm{k}\geq 0$. We will obtain

$$f^{(-k-n)}(t)=\frac{1}{\Gamma(n)}D^{-k}\int_a^t (t-\tau)^{n-1} f(\tau) \qquad (86)$$

where the symbol $D^{-k}\,(k\geq 0)$ denotes k iterated integrations. On the other hand, for a fixed $n\geq 1$ and integer $\mathrm{k}\geq \mathrm{n}$ the (k-n)-th derivative of the function f(t) can be written as

$$f^{(k-n)}(t)=\frac{1}{\Gamma(n)}D^{k}\int_a^t (t-\tau)^{n-1} f(\tau)d\tau. \qquad (87)$$

where the symbol D^k ($k\geq 0$) denotes k iterated differentiations.

The formula (86) and (87) can be considered as particular cases of one of them namely (87), in which n ($n\geq 1$) is fixed and the symbol D^k means k integrations if $k\leq 0$ and k differentiations if k>0. If k=n-1, n-2, …, then the formula (87) gives iterated integrals of f(t); for k=n it gives the function f(t); for k=n+1, n+2, … it gives derivatives of order k-n=1, 2, 3, … of the function f(t).

2.2. Integral of Arbitrary Order

Let us start with the Cauchy formula (85) and replace the integer n in it by a real p>0 to extend the notion of n-fold integration on-integer values of n:

$$ {}_aD_t^{-p} f(t) = \frac{1}{\Gamma(p)} \int_a^t (t-\tau)^{p-1} f(\tau) d\tau. \tag{88} $$

In (85) the integer n must satisfy the condition $n \geq 1$; the corresponding condition for p is weaker: for the existence of the integral (88) we must have p>0.

Also under certain reasonable assumptions

$$ \lim_{p \to 0} {}_aD_t^{-p} f(t) = f(t) \tag{89} $$

so we can put

$$ {}_aD_t^{0} f(t) = f(t) \tag{90} $$

The proof of the relationship (89) is simple if f(t) has continuous derivatives for $t \geq 0$. In such case, integration by parts and the use of (A1. 3) gives

$$ {}_aD_t^{-p} f(t) = \frac{(t-a)^p}{\Gamma(p+1)} + \frac{1}{\Gamma(p+1)} \int_a^t (t-\tau)^p f'(\tau) d\tau $$

and

$$ \lim_{p \to 0} {}_aD_t^{-p} f(t) = f(a) + \int_a^t f'(\tau) d\tau = f(a) + (f(t) - f(a)) = f(t) $$

If f(t) is only continuous for $t \geq$ a, then the proof of (89) is somewhat longer. In such case let us write ${}_aD_t^{-p} f(t)$ in the form:

$$ {}_aD_t^{-p} f(t) = \frac{1}{\Gamma(p)} \int_a^t (t-\tau)^{p-1} (f(\tau) - f(t)) d\tau + \frac{f(t)}{\Gamma(p)} \int_a^t (t-\tau)^{p-1} d\tau $$

$$ = \frac{1}{\Gamma(p)} \int_a^{t-\delta} (t-\tau)^{p-1} (f(\tau) - f(t)) d\tau \tag{91} $$

$$ + \frac{1}{\Gamma(p)} \int_{t-\delta}^{t} (t-\tau)^{p-1} (f(\tau) - f(t)) d\tau \tag{92} $$

$$ + \frac{f(t)(t-a)^p}{\Gamma(p+1)}. \tag{93} $$

Let us consider the integral (92). Since f(t) is continuous, for every δ>0 there exists ε>0 such that

$$ \left| f(\tau) - f(t) \right| < \varepsilon . $$

Then we have the following estimate of the integral (92):

$$ \left| I_2 \right| < \frac{\varepsilon}{\Gamma(p)} \int_{t-\delta}^{t} (t-\tau)^{p-1} d\tau < \frac{\varepsilon \delta^p}{\Gamma(p+1)} \tag{94} $$

and taking into account that $\varepsilon \to 0$ as $\delta \to 0$ we obtain that for all $p \geq 0$

$$ \lim_{\delta \to 0} \left| I_2 \right| = 0 \tag{95} $$

Let us take an arbitrary ε >0 and choose δ such that

$$ \left| I_2 \right| < \varepsilon \tag{96} $$

for all $p \geq 0$. For this fixed δ we obtain the following estimate of the integral (91):

$$|I_2| \leq \frac{M}{\Gamma(p)} \int_a^{t-\delta} (t-\tau)^{p-1} d\tau \leq \frac{M}{\Gamma(p+1)} (\delta^p - (t-a)^p) \qquad (97)$$

from which it follows that for fixed $\delta > 0$

$$\lim_{p \to 0} |I_1| = 0 \qquad (98)$$

Considering

$\left| \ {}_aD_t^{-p} f(t) - f(t) \ \right| \leq |I_1| + |I_2| + |f(t)| \cdot \left| \frac{(t-a)^p}{\Gamma(p+1)} - 1 \right|$ and taking into account the limits (95) and (98) and the estimate (96) we obtain

$$\lim_{p \to 0} \sup \left| {}_aD_t^{-p} f(t) - f(t) \right| \leq \varepsilon$$

where ε can be chosen as small as we possible. So,

$$\lim_{p \to 0} \sup \left| {}_aD_t^{-p} f(t) - f(t) \right| = 0$$

and (89) holds if f(t) is continuous for $t \geq a$.

If f(t) is continuous for t$\geq a$, then integration of arbitrary real order defined by (88) has the following property:

$${}_aD_t^{-p} ({}_aD_t^{-q} f(t)) = {}_aD_t^{-p-q} f(t). \qquad (99)$$

we have

$${}_aD_t^{-p} ({}_aD_t^{-q} f(t)) = \frac{1}{\Gamma(q)} \int_a^t (t-\tau)^{q-1} {}_aD_t^{-p} f(t) d\tau = \frac{1}{\Gamma(p)\Gamma(q)} \int_a^t (t-\tau)^{q-1} d\tau \int_a^\tau (\tau-\xi)^{p-1} f(\xi) d\xi$$

$$=\frac{1}{\Gamma(p)\Gamma(q)}\int_a^t f(\xi)d\xi\int_\xi^t (t-\tau)^{q-1}(\tau-\xi)^{p-1}d\tau=\frac{1}{\Gamma(p+q)}\int_a^t (t-\xi)^{p+q-1}f(\xi)d\xi={}_aD_t^{-p-q}f(t)$$

Obviously, we can interchange p and q, so we have

$$ {}_aD_t^{-p}({}_aD_t^{-q}f(t))={}_aD_t^{-q}({}_aD_t^{-p}f(t))={}_aD_t^{-p-q}f(t) \tag{100}$$

One may note that the rule (100) is similar to well-known property of integer-order derivatives:

$$\frac{d^m}{dt^m}\left(\frac{d^n f(t)}{dt^n}\right)=\frac{d^n}{dt^n}\left(\frac{d^m f(t)}{dt^m}\right)=\frac{d^{m+n}f(t)}{dt^{m+n}}. \tag{101}$$

2.3. Derivatives of Arbitrary Order

The representation (87) for the derivative of an integer order k-n provides an opportunity for extending the notion of differentiation on-integer order We can leave integer k and replace integer n with a real α so that k-α>0. This gives

$$ {}_aD_t^{k-\alpha}f(t)=\frac{1}{\Gamma(\alpha)}\frac{d^k}{dt^k}\int_a^t (t-\tau)^{\alpha-1}f(\tau)d\tau,\ (0<\alpha\le 1) \tag{102}$$

where the only substantial restriction for α is $\alpha>0$, which is neccessary for the convergence of the integral in (102). This restriction , however, can be (without loss of generality) replaced with the narrower condition $0<\alpha\le 1$; this can be shown with the help of the property (100) of the integrals of arbitrary real order and the definition (102). Denoting p=k-α we can write (102)

$$ {}_aD_t^{p}f(t)=\frac{1}{\Gamma(k-p)}\frac{d^k}{dt^k}\int_a^t (t-\tau)^{k-p-1}f(\tau)d\tau,\ (\text{k-1}\le p<k) \tag{103}$$

$$\text{or } {}_aD_t^{p}f(t)=\frac{d^k}{dt^k}({}_aD_t^{-(k-p)}f(t)),\ (k-1\le p<k) \tag{104}$$

If p=k-1, then we obtain a conventional integer-order derivative of order k-1:

$$ {}_aD_t^{k-1}f(t)=\frac{d^k}{dt^k}({}_aD_t^{-(k-(k-1))}f(t))=\frac{d^k}{dt^k}({}_aD_t^{-1}f(t))=f^{(k-1)}(t). $$

Morover, using (90) we see that for p=k≥ 1 and t>a

$$ {}_aD_t^{p}f(t)=\frac{d^k}{dt^k}({}_aD_t^{0}f(t))=\frac{d^k f(t)}{dt^k}=f^{(k)}(t) \quad (105) $$

which means that for t>a the Riemann-Liouville fractional derivative (103) of order p=k>1 coincides with the conventional derivative of order k.

Let us consider some properties of the Riemann-Liouville fractional derivatives. The first property of the Riemann-Liouville fractional derivative is that for p>0 and t>a

$$ {}_aD_t^{p}({}_aD_t^{-p}f(t))=f(t) \tag{106} $$

which means that the Riemann-Liouville fractional differentiation operator is a left inverse to the Riemann-Liouville fractional integration operator of the same order p. To prove the property (106), let us consider the case of integer p=n≥ 1:

$$ {}_aD_t^{n}({}_aD_t^{-n}f(t))=\frac{d^n}{dt^n}\int_a^t (t-\tau)^{n-1}f(\tau)d\tau=\frac{d}{dt}\int_a^t f(\tau)d\tau=f(t) $$

Taking $k-1\leq p<k$ and using the composition rule (94) for the Riemann-Liouville fractional integrals we can write

$$ {}_aD_t^{-k}f(t)={}_aD_t^{-(k-p)}({}_aD_t^{-p}f(t)), \tag{107} $$

and therefore

$$ {}_aD_t^{p}({}_aD_t^{-p}f(t))=\frac{d^k}{dt^k}\left\{{}_aD_t^{-(k-p)}({}_aD_t^{-p}f(t))\right\}=\frac{d^k}{dt^k}({}_aD_t^{-p}f(t))=f(t), $$

and that's proved.

As with conventional intefer-order differentiation and integration fractional differentiation and integration do not commute. If the fractional derivative ${}_aD_t^p f(t),(k-1 \le p < k)$, of a function f(t)is integrable , then

$$ {}_aD_t^{-p}\left({}_aD_t^p f(t)\right) = f(t) - \sum_{j=1}^{k}\left[{}_aD_t^{p-j} f(t)\right]_{t=a} \frac{(t-a)^{p-j}}{\Gamma(p-j+1)} \tag{108} $$

in fact, on the one hand we have

$$ {}_aD_t^{-p}\left({}_aD_t^p f(t)\right) = \frac{1}{\Gamma(p)}\int_a^t (t-\tau)^{p-1}\, {}_aD_\tau^p f(\tau)d\tau = \frac{d}{dt}\left[\frac{1}{\Gamma(p+1)}\int_a^t (t-\tau)^p\, {}_aD_\tau^p f(\tau)d\tau\right] \tag{109} $$

On the other hand, using (100) we obtain

$$ \frac{1}{\Gamma(p+1)}\int_a^t (t-\tau)^p\, {}_aD_\tau^p f(\tau)d\tau = \frac{1}{\Gamma(p+1)}\int_a^t (t-\tau)^p \frac{d^k}{d\tau^k}\left[{}_aD_\tau^{-(k-p)} f(\tau)\right]d\tau $$

$$ = \frac{1}{\Gamma(p-k+1)}\int_a^t (t-\tau)^{p-k}\left[{}_aD_\tau^{-(k-p)} f(\tau)\right]d\tau - \sum_{j=1}^{k}\left[\frac{d^{k-j}}{dt^{k-j}}\left({}_aD_t^{-(k-p)} f(t)\right)\right]_{t=a} \frac{(t-a)^{p-j+1}}{\Gamma(2+p-j)} $$

$$ = \frac{1}{\Gamma(p-k+1)}\int_a^t (t-\tau)^{p-k}\left[{}_aD_\tau^{-(k-p)} f(\tau)\right]d\tau - \sum_{j=1}^{k}\left[{}_aD_t^{p-j} f(t)\right]_{t=a} \frac{(t-a)^{p-j+1}}{\Gamma(2+p-j)} \tag{110} $$

$$ = {}_aD_t^{-(p-k+1)}\left({}_aD_t^{-(k-p)} f(t)\right) - \sum_{j=1}^{k}\left[{}_aD_t^{p-j} f(t)\right]_{t=a} \frac{(t-a)^{p-j+1}}{\Gamma(2+p-j)} \tag{111} $$

$$ = {}_aD_t^{-1} f(t) - \sum_{j=1}^{k}\left[{}_aD_t^{p-j} f(t)\right]_{t=a} \frac{(t-a)^{p-j+1}}{\Gamma(2+p-j)} \tag{112} $$

The existence of all terms in (110) follows from the integrability of ${}_aD_t^p f(t)$ because of this condition the fractional derivatives ${}_aD_t^{p-j} f(t)$ (j=1, 2, …, k) are all bounded at t=a. Combining (109) and (110) ends the proof the relitionship (108).

An important particular case, if 0<p<1, then

$$ {}_aD_t^{-p}({}_aD_t^p f(t)) = f(t) - \left[{}_aD_t^{p-1} f(t)\right]_{t=a} \frac{(t-a)^{p-1}}{\Gamma(p)} \tag{113}$$

The property (106) is a particular case of a more general property

$$ {}_aD_t^p({}_aD_t^{-q}) = {}_aD_t^{p-q} f(t) \tag{114}$$

where we assume that f(t) is continuous and, if $p \geq q \geq 0$, that the derivative ${}_aD_t^{p-q} f(t)$ exists.

Two cases must be considered: $q \geq p \geq 0$ or $p > q \geq 0$. If $q \geq p \geq 0$, then using the properties (100) and (106) we obtain

$$ {}_aD_t^p({}_aD_t^{-q} f(t)) = {}_aD_t^p({}_aD_t^{-p}\,{}_aD_t^{-(q-p)} f(t)) = {}_aD_t^{-(q-p)} f(t) = {}_aD_t^{p-q} f(t).$$

Now let us consider the case p>q≥ 0. Let us denote by m and n integers such that $0 \leq m-1 \leq p < m$ and $0 \leq n \leq p-q < n.(n \leq m)$. Then using the definition (103) and the property (100) we obtain

$$ {}_aD_t^p({}_aD_t^{-q} f(t)) = \frac{d^m}{dt^m}\left[{}_aD_t^{-(m-p)}({}_aD_t^{-q} f(t))\right] = \frac{d^m}{dt^m}\left[{}_aD_t^{p-q-m} f(t)\right]$$
$$ = \frac{d^n}{dt^n}\left[{}_aD_t^{p-q-n} f(t)\right] = {}_aD_t^{p-q} f(t).$$

The above property (102) is a particular case of the more general property

$$ {}_aD_t^{-p}({}_aD_t^q f(t)) = {}_aD_t^{q-p} f(t) - \sum_{j=1}^{k}\left[{}_aD_t^{q-j} f(t)\right]_{t=a} \frac{(t-a)^{p-j}}{\Gamma(1+p-j)} \tag{115}$$

$$(0 \leq k-1 \leq q < k)$$

To prove the formula (115) we use property (100) ($q \le p$) or property (114) ($q \ge p$) and then property (108). This gives

$$
{}_aD_t^{-p}\left({}_aD_t^{q} f(t)\right) = {}_aD_t^{q-p}\left[{}_aD_t^{-q}\left({}_aD_t^{q} f(t)\right)\right] = {}_aD_t^{q-p}\left[f(t) - \sum_{j=1}^{k}\left[{}_aD_t^{q-j} f(t)\right]_{t=a} \frac{(t-a)^{q-j}}{\Gamma(p-j+1)}\right]
$$

$$
= {}_aD_t^{q-p} f(t) - \sum_{j=1}^{k}\left[{}_aD_t^{q-j} f(t)\right]_{t=a} \frac{(t-a)^{p-j}}{\Gamma(1+p-j)}
$$

$$
{}_aD_t^{q-p}\left[\frac{(t-a)^{q-j}}{\Gamma(1+q-j)}\right] = \frac{(t-a)^{p-j}}{\Gamma(1+p-j)}
$$

2.4. Fractional Derivative of $(t-a)^{\nu}$

Let us evaluate the Riemann-Liouville fractional derivative ${}_aD_t^{p} f(t)$ of the power function

$$
f(t) = (t-a)^{\nu} \quad (\nu \text{ is real})
$$

For this purpose let us assume that $n-1 \le p < n$ and recall that by the definition of the Riemann-Liouville derivative

$$
{}_aD_t^{p} f(t) = \frac{d^n}{dt^n}\left({}_aD_t^{-(n-p)} f(t)\right) \quad (n-1 \le p < n) \tag{116}
$$

Substituting into the formula (116) the fractional integral order $\alpha = n - p$ of this function, which we have evaluated in (56)

$$
{}_aD_t^{-\alpha}\left((t-a)^{\nu}\right) = \frac{\Gamma(1+\nu)}{\Gamma(1+\nu+\alpha)}(t-a)^{\nu+\alpha}
$$

and we have

$$ {}_aD_t^p((t-a)^\nu) = \frac{\Gamma(1+\nu)}{\Gamma(1+\nu-p)}(t-a)^{\nu-p} \tag{117}$$

and the only restriction for f(t)= $(t-a)^\nu$ is its integrability, namely $\nu > -1$.

2.5. Composition with Integer-order Derivatives

Let us consider the n-th derivative of the Riemann-Liouville fractional derivative of real order p. Using the definition (102) of the Riemann-Liouville derivative we have:

$$ \frac{d^n}{dt^n}({}_aD_t^{k-\alpha} f(t)) = \frac{1}{\Gamma(\alpha)}\frac{d^{n+k}}{dt^{n+k}}\int_a^t (t-\tau)^{\alpha-1} f(\tau)d\tau = {}_aD_t^{n+k-\alpha} f(t) \quad (0<\alpha\le 1) \tag{118}$$

and denoting p=k- α we have

$$ \frac{d^n}{dt^n}({}_aD_t^p f(t)) = {}_aD_t^{n+p} f(t)\,. \tag{119}$$

To consider the reversed order of operations, we must take into account that

$$ {}_aD_t^{-n} f^{(n)}(t) = \frac{1}{(n-1)!}\int_a^t (t-\tau)^{n-1} f^{(n)}(\tau)d\tau = f(t) - \sum_{j=0}^{n-1}\frac{f^{(j)}(a)(t-a)^j}{\Gamma(j+1)} \tag{120}$$

and that

$$ {}_aD_t^p g(t) = {}_aD_t^{p+n}({}_aD_t^{-n} g(t)). \tag{121}$$

Using (120), (121) and (117) we have:

$$ {}_aD_t^p(\frac{d^n f(t)}{dt^n}) = {}_aD_t^{p+n}({}_aD_t^{-n} f^{(n)}(t)) = {}_aD_t^{p+n}(f(t) - \sum_{j=0}^{n-1}\frac{f^{(j)}(a)(t-a)^j}{\Gamma(j+1)})$$

$$= {}_aD_t^{p+n} f(t) - \sum_{j=0}^{n-1} \frac{f^{(j)}(a)(t-a)^{j-p-n}}{\Gamma(1+j-p-n)} \tag{122}$$

which is the same as the relitionship (66).

Therefore, as in the case of the Grünwald-Letnikov derivatives, we see that the Riemann-Liouville fractional derivative operator ${}_aD_t^p$ commute with $\frac{d^n}{dt^n}$ that

$$\frac{d^n}{dt^n}({}_aD_t^p f(t)) = {}_aD_t^p (\frac{d^n f(t)}{dt^n}) = {}_aD_t^{p+n} f(t) \tag{123}$$

only if at the lower terminal t=a of the fractional differentiation the function f(t) satisfies the coditions

$$f^{(k)}(a) = 0, \text{ (k=0, 1, 2, …, k-1)} \tag{124}$$

2.6. Composition with Fractional Derivatives

Let us turn to two fractional Riemann-Liouville derivative operators: ${}_aD_t^p, (m-1 \le p < m)$ and ${}_aD_t^q, (n-1 \le q < n)$.

Using the definition of the Riemann-Liouville fractional derivative (98), the formula (102) and the composition with integer-order derivatives (113) we have:

$${}_aD_t^p ({}_aD_t^q f(t)) = \frac{d^m}{dt^m}\left\{ {}_aD_t^{-(m-p)} ({}_aD_t^q f(t)) \right\} = \frac{d^m}{dt^m}\left\{ {}_aD_t^{p+q-m} f(t) - \sum_{j=1}^{n} \left[{}_aD_t^{q-j} f(t) \right]_{t=a} \frac{(t-a)^{m-p-j}}{\Gamma(1+m-p-j)} \right\}$$

$$= {}_aD_t^{p+q} f(t) - \sum_{j=1}^{n} \left[{}_aD_t^{q-j} f(t) \right]_{t=a} \frac{(t-a)^{-p-j}}{\Gamma(1-p-j)} \tag{125}$$

Interchanging p and q, we can write:

$${}_aD_t^q ({}_aD_t^p f(t)) = {}_aD_t^{p+q} f(t) - \sum_{j=1}^{m} \left[{}_aD_t^{p-j} f(t) \right]_{t=a} \frac{(t-a)^{-q-j}}{\Gamma(1-q-j)}. \tag{126}$$

The comparison of the relationship (125) and (126) says that in the general case the Riemann-Liouville fractional derivative operators ${}_aD_t^p$ and ${}_aD_t^q$ dont commute, with only one exception: for $p \neq q$ we have

$$ {}_aD_t^p\,({}_aD_t^q f(t)) = {}_aD_t^q\,({}_aD_t^p f(t)) = {}_aD_t^{p+q} f(t), \tag{127} $$

only if both sums in the right-hand sides of (125) and (126) vanish.We have to require the simultaneous fulfillment of the conditions

$$ [{}_aD_t^{p-j} f(t)]_{t=a} = 0, (j = 1,2,...,m) \tag{128} $$

and the conditions

$$ [{}_aD_t^{q-j} f(t)]_{t=a} = 0, (j = 1,2,...,n) \tag{129} $$

As will be shown below in section 2. 7, if f(t) has a sufficent number of continous derivatives, then the conditions (128) are equivalent to

$$ f^{(j)}(a) = 0, (j = 0,1,2,...,m-1) \tag{130} $$

and the conditions (129) are equivalent to

$$ f^{(j)}(a) = 0, (j = 0,1,2,...,n-1) \tag{131} $$

and the relitionship (127) holds if

$$ f^{(j)}(a) = 0, (j = 0,1,2,...,r-1), \tag{132} $$

where r=max(n, m).

2.7. Link to Grünwald-Letnikov Approach

There exists a link between the Riemann-Liouville and the Grünwald-Letnikov approaches to differentiation of arbitrary real order. The exact conditions of the equivalance of these two approaches are the following. Let us suppose that the function f(t) is (n-1)-times continuously differentiable in the interval [a, T] and that $f^{(n)}(t)$ is integrable in [a, T]. Then for every p (0<p<n) the Riemann-Liouville derivative ${}_aD_t^p f(t)$ exists and coincides with the Grünwald-Letnikov derivative ${}_aD_t^p f(t),$ and if $0 \leq m-1 \leq p < m \leq n,$ then for a<t<T the following holds:

$$ {}_aD_t^p f(t) = \sum_{j=1}^{m-1} \frac{f^{(j)}(a)(t-a)^{j-p}}{\Gamma(1+j-p)} + \frac{1}{\Gamma(m-p)} \int_a^t \frac{f^{(m)}(\tau)d\tau}{(t-\tau)^{p-m+1}}. \tag{133} $$

On the one hand the right-hand side of formula (133) is equal to the Grünwald-Letnikov derivative ${}_aD_t^p f(t).$ On the other hand, it can be written as

$$ \frac{d^m}{dt^m} \left\{ \sum_{j=0}^{m-1} \frac{f^{(j)}(a)(t-a)^{m+j-p}}{\Gamma(1+m+j-p)} + \frac{1}{\Gamma(2m-p)} \int_a^t (t-\tau)^{2m-p-1} f^{(m)}(\tau)d\tau \right\} $$

which after m integrations by parts takes the form of the Riemann-Liouville derivative ${}_aD_t^p f(t)$

$$ \frac{d^m}{dt^m} \left\{ \frac{1}{\Gamma(m-p)} \int_a^t (t-\tau)^{m-p-1} f(\tau)d\tau \right\} = \frac{d^m}{dt^m} \left\{ {}_aD_t^{-(m-p)} f(t) \right\} = {}_aD_t^p f(t). $$

The following particular case of the relationship (133) is important from the viewpoint of numerous applied problems.

If f(t) is continuous and $f'(t)$ is integrable in the interval [a, T], then for every p (0<p<1) both Riemann-Liouville and Grünwald-Letnikov derivatives exists and can be written in the form

$$ {}_aD_t^p f(t) = \frac{f(a)(t-a)^{-p}}{\Gamma(1-p)} + \frac{1}{\Gamma(1-p)} \int_a^t (t-\tau)^{-p} f'(\tau) d\tau. \qquad (134) $$

the derivative given by the expression (134) is integrable. Another important property following from (133) is that the existence of the derivative of order p>0 implies the existence of the derivative of order q for all q such that 0<q<p.

If for a given continuous function f(t) having integrable derivative the Riemann-Liouville (Grünwald-Letnikow) derivative ${}_aD_t^p f(t)$ exists and is integrable, then for every q such that (0<q<p) the derivative ${}_aD_t^q f(t)$ also exists and integrable.

If we denote g(t)=${}_aD_t^{-(1-p)} f(t),$ then we can write

$$ {}_aD_t^p f(t) = \frac{d}{dt}({}_aD_t^{-(1-p)} f(t)) = g'(t). $$

Noting that $g'(t)$ is integrable and taking into account the formula (128) and the inequality 0<1+q-p<1 we conclude that the derivative ${}_aD_t^{1+q-p} g(t)$ exists and integrable. Then, using the property (114), we have:

$$ {}_aD_t^{1+q-p} g(t) = {}_aD_t^{1+q-p} ({}_aD_t^{-(1-p)} f(t)) = {}_aD_t^q f(t). $$

The relitionship (127) between the Grünwald-Letnikov and the Riemann-Liouville definitions also has another consequence which is important for the formulation of applied problems, manipulation with fractional derivatives and the formulation of physically meaningfull initial-value problems for fractional-order differential equations.

Under the same assumptions on the function f(t) (f(t) is (m-1)-times continuously differentiable and its m-th derivative is integrable in [a- T]) and on p (m-1$\leq$p<m) the condition

$$ [{}_aD_t^p f(t)]_{t=a} = 0 \qquad (135) $$

is equivalent to the conditions

$$f^{(j)}(a) = 0, (j = 0,1,2,\ldots,m-1). \tag{136}$$

If the conditions (135) are fulfilled, then putting $t \to a$ in (133) we obtain (135).

On the other hand, if the conditions (135) is fulfilled, the multiplying both sides of (127) subsequently by $(t-a)^{p-j}$ (j=m-1, m-2, …, 2, 1, 0) and taking the limits as $t \to a$ we obtain $f^{(m-1)}(a) = 0, f^{(m-2)}(a) = 0, \ldots, f'(a) = 0, f(a) = 0$ the conditions (136).

Therefore, (135) holds iff (136) holds.

From the equivalance of the conditions (135) and (136) it follows that if for some p>0 the p-th derivative of f(t) is equal to zero at the terminal t=a, then all derivatives of order q (0<q<p) are also equal to zero at t=a:

$$[{}_aD_t^q f(t)]_{t=a} = 0.$$

CHAPTER 3

CAPUTO'S FRACTIONAL DERIVATIVES

3.1. Caputo's Fractional Derivative

Applied problems require definitions of fractional derivatives allowing the utilization of physically interpretable initial conditions, which contain $f(a), f'(a), etc.$

The Riemann-Liouville approachleads to initial conditions containing the limit values of the Riemann-Liouville fractional derivatives at the lower terminal t=a,

$$\begin{aligned} &\lim_{t\to a} {}_aD_t^{\alpha-1} f(t) = b_1, \\ &\lim_{t\to a} {}_aD_t^{\alpha-2} f(t) = b_2, \\ &\ldots \\ &\lim_{t\to a} {}_aD_t^{\alpha-n} f(t) = b_n, \end{aligned} \tag{137}$$

where b_k (k=1, 2, …, n) are given constants.

In spite of the fact that initial value problems with such initial conditions can be solved mathematically, their solutions are practcally unless, because there is no known physical interpretation for such types of initial conditions. Here we observe a conflict between the well-established and polished mathematical theory and practical needs. A certain solution to this conflict was proposed by M. Caputo in his paper and in his book and recently by El-Sayed. Caputo's definition can be written as

$${}_a^C D_t^\alpha f(t) = \frac{1}{\Gamma(\alpha-n)} \int_a^t \frac{f^{(n)}(\tau)d\tau}{(t-\tau)^{\alpha+1-n}}, (n-1<\alpha<n) \tag{138}$$

Under conditions on the function f(t), for $\alpha \to n$ the Caputo derivative becomes a conventional n-th derivative of the function f(t). Let us assume that $0 \le n-1 < \alpha < n$ and that the function f(t) has n+1 continuous bounded derivatives in [a, T] for every T>a. Then

$$\lim_{\alpha \to n} {}_a^C D_t^\alpha f(t) = \lim_{\alpha \to n} \left(\frac{f^{(n)}(a)(t-a)^{n-\alpha}}{\Gamma(n-\alpha+1)} + \frac{1}{\Gamma(n-\alpha+1)} \int_a^t (t-\tau)^{n-\alpha} f^{(n+1)}(\tau) d\tau \right)$$

$$= f^n(a) + \int_a^t f^{(n+1)}(\tau) d\tau = f^{(n)}(t), n = 1,2,...$$

This says that, similarly to the Grünwald-Letnikov and Riemann-Liouville approaches, the Caputo approach also provides an interpolation between integer-order derivatives. The main advantage of Caputo's approach ism that the initial conditions for fractional differetial equations with Caputo derivatives take on the same form as for integer-order differential equations, contain the limit values of integer-order derivatives of unknown functions at the lower terminal t=a. The formula for the Laplace transform of the Riemann-Liouville fractional derivative is

$$\int_0^\infty e^{-pt} \left\{ {}_0D_t^\alpha f(t) \right\} dt = p^\alpha F(p) - \sum_{k=0}^{n-1} p^k \, {}_0D_t^{\alpha-k-1} f(t) \Big|_{t=0}, (n-1 \le \alpha < n) \qquad (139)$$

whereas Caputo's formula, for the Laplace transform of the Caputo derivative is

$$\int_0^\infty e^{-pt} \left\{ {}_0D_t^\alpha f(t) \right\} dt = p^\alpha F(p) - \sum_{k=0}^{n-1} p^{\alpha-k-1} f^{(k)}(0), (n-1 < \alpha \le n). \qquad (140)$$

We see that the Laplace transform of the Riemann-Liouville fractional derivative allows utilization of initial conditions of the type (137) which can cause problems with their physical interpretation. On the contrary, the Laplace transform of the Caputo derivative allows utilitization of initial values of classical integer-order derivatives with known physical interpretation. The Laplace transform method is frequently used for solving applied problems. To choose the suitable Laplace transform formula, it is important to understand which type definition of fractional derivative must be used. Another difference between the Riemann-Liouville definition (103) and Caputo definition (138) is that the Caputo derivative of a constant is 0, whereas in the cases of a finite value of the lower terminal a the Riemann-Liouville fractional derivative of a constant C is not equal to 0, but

$$
{}_0D_t^{\alpha}C = \frac{Ct^{-\alpha}}{\Gamma(1-\alpha)} \tag{141}
$$

This fact led Ochmann and Makarov [67] to using the Riemann-Liouville definition with $a = -\infty$, because, on the one hand, from the physical point of view they need the fractional derivative of a constant equal to zero and on the other hand formula (141) gives 0 if $a \to -\infty$. The pyhsical meaning of this step is that the starting time of the physical process is set to $-\infty$. In such a case transient effects can not be studied. However, taking a=$-\infty$ is the necessarry abstraction for the consideration of the steady-state processes, for example for studying the response of the fractional-order dynamic system to the periodic input signal, wave propagation in viscoelastic materials. Putting $a = -\infty$ in both definitions and requiring reasonable behaviour of f(t) and its derivatives for $t \to -\infty$, we arrive at the same formula

$$
{}_{-\infty}D_t^{\alpha} f(t) = {}_{-\infty}^{\;C}D_t^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_{-\infty}^{t} \frac{f^{(n)}(\tau)d\tau}{(t-\tau)^{\alpha+1-n}}, \tag{142}
$$

(n-1<$\alpha < n$) which shows that for the study of steady-state dynamical process the Riemann-Liouville definition and Caputo definition must give the same results. There is also another diifference between the Riemann-Liouville and the Caputo approaches, which we would like to mention here and which seems to be important for applications. Namely, for the Caputo derivative we have

$$
{}_a^CD_t^{\alpha}({}_a^CD_t^{m} f(t)) = {}_a^CD_t^{\alpha+m} f(t), \text{ (m=0, 1, 2, … ; n-1<}\alpha\text{<n)} \tag{143}
$$

while for the Riemann-Liouville derivative

$$
{}_aD_t^{m}({}_aD_t^{\alpha} f(t)) = {}_aD_t^{\alpha+m} f(t), \text{ (m=0, 1, 2, … ; n-1<}\alpha < n\text{)} \tag{144}
$$

The interchange of the differentiation operators in formulas (143) and (144) is allowed under different conditions:

$$ {}_{a}^{C}D_t^{\alpha}({}_{a}^{C}D_t^{m} f(t)) = {}_{a}^{C}D_t^{m}({}_{a}^{C}D_t^{\alpha} f(t)) = {}_{a}^{C}D_t^{\alpha+m} f(t), \tag{145} $$

$f^{(s)}(0) = 0$, s=n, n+1, … , m (m=0, 1, 2, … ; n-1< $\alpha < n$)

$$ {}_{a}D_t^{m}({}_{a}D_t^{\alpha} f(t)) = {}_{a}D_t^{\alpha}({}_{a}D_t^{m} f(t)) = {}_{a}D_t^{\alpha+m} f(t), \tag{146} $$

$f^{(s)}(0) = 0$, s=0, 1, 2, …, m (m=0, 1, 2, … ; n-1< $\alpha < n$).

We see that contrary to the Riemann-Liouville approach, in the case of the Caputo derivative there are no restrictions on the values $f^{(s)}(0), (s = 0,1,...,n-1)$.

3.2. The Leibnitz Rule For Fractional Derivatives

Let us take two functions, $\varphi(t)$ and $f(t)$, and start with the known Leibnitz rule for evaluating the n-th derivative of the product $\varphi(t) f(t)$:

$$ \frac{d^n}{dt^n}(\varphi(t) f(t)) = \sum_{k=0}^{n} \binom{n}{k} \varphi^{(k)}(t) f^{(n-k)}(t). \tag{147} $$

Let us now take the right-hand side the formula (147) and replace the integer parameter n with the real-valued parameter p. This means that the integer-order derivative $f^{(n-k)}(t)$ will be replaced with the Grünwald-Letnikov fractional-order derivative ${}_{a}D_t^{p-k} f(t)$. Denoting

$$ \Omega_n^p(t) = \sum_{k=0}^{n} \binom{p}{k} \varphi^{(k)}(t)\, {}_{a}D_t^{p-k} f(t) \tag{148} $$

let us evaluate the sum (148).

First, let us suppose that p=q<0. Then we have also p-k=q-k<0 for all k, and according to (40)

$$ {}_aD_t^{p-k} f(t) = \frac{1}{\Gamma(-q+k)} \int_a^t (t-\tau)^{-q+k-1} f(\tau) d(\tau), \tag{149} $$

which leads to $\Omega_n^q(t) = \sum_{k=0}^{n} \binom{q}{k} \frac{1}{\Gamma(-q+k)} \int_a^t (t-\tau)^{-q+k-1} \varphi^{(k)}(t) f(\tau) d\tau$ (150)

$$ = \int_a^t \left\{ \sum_{k=0}^{n} \binom{q}{k} \frac{1}{\Gamma(-q+k)} \varphi^{(k)}(t)(t-\tau)^k \right\} \frac{f(\tau)}{(t-\tau)^{q+1}} d\tau. \tag{151} $$

Taking into account the reflection formula (A1. 16) for the gamma function, we have

$$ \binom{q}{k} \frac{1}{\Gamma(-q+k)} = \frac{\Gamma(q+1)}{k!\Gamma(q-k+1)} \cdot \frac{1}{\Gamma(-q+k)} \tag{152} $$

$$ = \frac{\Gamma(q+1)}{k!} \cdot \frac{\sin(k-q)\pi}{\pi} \tag{153} $$

$$ = (-1)^{k+1} \frac{\Gamma(q+1)}{k!} \frac{\sin(q\pi)}{\pi}, \tag{154} $$

and, therefore, the expression (151) takes form:

$$ \Omega_n^q(t) = -\frac{\sin(q\pi)}{\pi} \Gamma(q+1) \int_a^t \left\{ \sum_{k=0}^{n} \frac{(-1)^k}{k!} \varphi^{(k)}(t)(t-\tau)^k \right\} \frac{f(\tau)}{(t-\tau)^{q+1}} f(\tau) d\tau. \tag{155} $$

Using the Taylor theorem we can write

$$ \sum_{k=0}^{n} \frac{(-1)^k}{k!} \varphi^{(k)}(t)(t-\tau)^k = \varphi(t) + \varphi'(t)(t-\tau) + \ldots + \frac{\varphi^{(n)}(t)}{n!}(t-\tau)^n = \varphi(\tau) + \frac{1}{n!} \int_a^t \varphi^{(n+1)}(\xi)(\tau-\xi)^n d\xi $$

and therefore we obtain

$$\Omega_n^q(t) = -\frac{\sin(q\pi)\Gamma(q+1)}{\pi}\int_a^t (t-\tau)^{-q-1}\varphi(\tau)f(\tau)d\tau$$

$$-\frac{\sin(q\pi)\Gamma(q+1)}{\pi n!}\int_a^t (t-\tau)^{-q-1} f(\tau)d\tau\int_a^t \varphi^{(n+1)}(\xi)(\tau-\xi)^n d\xi$$

$$=\frac{1}{\Gamma(-q)}\int_a^t (t-\tau)^{-q-1}\varphi(\tau)f(\tau)d\tau + \frac{1}{n!\Gamma(-q)}\int_a^t (t-\tau)^{-q-1} f(\tau)d\tau\int_\tau^t \varphi^{(n+1)}(\xi)(\tau-\xi)^n d\xi$$

$$= {}_aD_t^q(\varphi(t)f(t)) + R_n^q(t). \qquad (156)$$

where

$$R_n^q(t) = \frac{1}{n!\Gamma(-q)}\int_a^t (t-\tau)^{-q-1} f(\tau)d\tau\int_\tau^t \varphi^{(n+1)}(\xi)(\tau-\xi)^n d\xi. \qquad (157)$$

Let us consider the case of p>0. Our first step is to show that the evaluation of $\Omega_n^p(t)$ can be reduced to the evaluation of Ω_n^q for a certain negative q.

Taking into account that $\Gamma(0)=\infty$ we have to put

$$\binom{p-1}{-1} = 0$$

and using the known property of the binomial coefficients

$$\binom{p}{k} = \binom{p-1}{k} + \binom{p-1}{k-1}$$

we can write

$$\Omega_n^p(t) = \sum_{k=0}^{n}\binom{p-1}{k}\varphi^{(k)}(t)\,{}_aD_t^{p-k}f(t) + \sum_{k=1}^{n}\binom{p-1}{k-1}\varphi^{(k)}(t)\,{}_aD_t^{p-k}f(t). \qquad (158)$$

Replacing k with k+1 in the second sum gives

$$\Omega_n^p(t) = \sum_{k=0}^{n} \binom{p-1}{k} \varphi^{(k)}(t) \frac{d}{dt}({}_aD_t^{p-k-1} f(t)) + \sum_{k=0}^{n-1} \binom{p-1}{k} \frac{d\varphi^{(k)}(t)}{dt} \cdot {}_aD_t^{p-k-1} f(t), \quad (159)$$

which can be written as

$$\Omega_n^p(t) = \binom{p-1}{n} \varphi^{(n)}(t)\, {}_aD_t^{p-n} f(t) + \frac{d}{dt} \sum_{k=0}^{n-1} \binom{p-1}{k} \varphi^{(k)}(t)\, {}_aD_t^{p-k-1} f(t). \qquad (160)$$

Adding and subtracting the expression

$$\frac{d}{dt}\left\{ \binom{p-1}{n} \varphi^{(n)}(t)\, {}_aD_t^{p-n-1} f(t) \right\}$$

we obtain

$$\Omega_n^p(t) = \frac{d}{dt} \sum_{k=0}^{n} \binom{p-1}{k} \varphi^{(k)}(t)\, {}_aD_t^{p-k-1} f(t) \qquad (161)$$

$$- \binom{p-1}{n} \varphi^{(n+1)}\, {}_aD_t^{p-n-1} f(t) \qquad (162)$$

Or

$$\Omega_n^p(t) = \frac{d}{dt} \Omega_n^{p-1}(t) - \binom{p-1}{n} \varphi^{(n+1)}(t)\, {}_aD_t^{p-k-1} f(t). \qquad (163)$$

The relationship (163) says that the evaluation of $\Omega_n^p(t)$ can be reduced to the evaluation of $\Omega_n^{p-1}(t)$. Repeating this procedure we can reduce the evaluation of $\Omega_n^p(t)(p > 0)$ to evaluation of $\Omega_n^q(t)(q < 0)$.

Let us suppose that 0<p<1. Then p-1<0, and according to (156) we have

$$\Omega_n^{p-1}(t) = {}_aD_t^{p-1}(\varphi(t)f(t)) + R_n^{p-1}(t). \tag{164}$$

To combine (164) and (165), we have to differentiate (164) with respect to t. Taking into account that

$$\frac{d}{dt}R_n^{p-1}(t) = \frac{-p}{n!\Gamma(-p+1)}\int_a^t (t-\tau)^{-p-1} f(\tau)d\tau \int_\tau^t \varphi^{n+1}(\xi)(\tau-\xi)^n d\xi + \frac{(-1)^n \varphi^{n+1}(t)}{n!\Gamma(-p+1)}\int_a^t (t-\tau)^{-p+n} f(\tau)d\tau \tag{165}$$

and that

$$\int_a^t (t-\tau)^{-p+n} f(\tau)d\tau = \Gamma(-p+n+1)\,{}_aD_t^{p-n-1} f(t) \tag{166}$$

(n-p>0), we obtain

$$\frac{d}{dt}\Omega_n^{p-1}(t) = {}_aD_t^p(\varphi(t)f(t)) + \frac{(-1)^n\Gamma(-p+n+1)\varphi^{(n+1)}(t)}{n!\Gamma(-p+1)} \cdot {}_aD_t^{p-n-1} f(t) + R_n^p(t)$$

$$= {}_aD_t^p(\varphi(t)f(t)) + \binom{p-1}{n}\varphi^{(n+1)}(t)\,{}_aD_t^{p-n-1} f(t) + R_n^p(t), \tag{167}$$

and the substitution of this expression into (163) gives

$$\Omega_n^p(t) = {}_aD_t^p(\varphi(t)f(t)) + R_n^p(t), \tag{168}$$

which has the same form as (156).

Using mathematical induction we can prove that the relationship (168) holds for all p such that p+1<n.

The relationship (168) gives, in fact, the rule for the fractional differentiation of the product of two functions. This rule is a generalization of the Leibnitz rule for integer-order differentiation, so it is convenient to preserve Leibnitz's name also in the case of fractional differentiation.

The Leibnitz rule for fractional differentiation is the following. If $f(\tau)$ is continuous in
[a, t] and $\varphi(\tau)$ has n+1 continuous derivatives in [a, t], then the fractional derivative of the product $\varphi(t)f(t)$ is given by

$$ {}_aD_t^p(\varphi(t)f(t)) = \sum_{k=0}^{n} \binom{p}{k} \varphi^{(k)}(t)\, {}_aD_t^{p-k} f(t) - R_n^p(t) \qquad (169)$$

where $n \geq p+1$ and

$$R_n^p(t) = \frac{1}{n!\Gamma(-p)} \int_a^t (t-\tau)^{-p-1} f(\tau) d\tau \int_\tau^t \varphi^{(n+1)}(\xi)(\tau-\xi)^n d\xi. \qquad (170)$$

The sum in (169) can be considered as a partial sum of an infinite series and $R_n^p(t)$ as a remainder of that series.

Performing two subsequent changes of integration variables, first $\xi = \tau + \varsigma(t-\tau)$ and then $\tau = a + \eta(t-a)$ we obtain the following expression for $R_n^p(t)$:

$$R_n^p(t) = \frac{(-1)^n}{n!\Gamma(-p)} \int_a^t (t-\tau)^{n-p} f(\tau) d\tau \int_0^1 \varphi^{(n+1)}(\tau + \varsigma(t-\tau))\varsigma^n d\varsigma$$

$$= \frac{(-1)^n (t-a)^{n-p+1}}{n!\Gamma(-p)} \int_0^1 \int_0^1 F_a(t,\varsigma,\eta) d\eta d\varsigma, \qquad (171)$$

$$F_a(t,\varsigma,\eta) = f(a+\eta(t-a))\varphi^{(n+1)}(a+(t-a)(\varsigma+\eta-\varsigma\eta)),$$

from which it follows that

$$\lim_{n\to\infty} R_n^p(t) = 0$$

If $f(\tau)$ and $\varphi(\tau)$ along with all its derivatives are continuous in [a, t]. Under this condition the Leibnitz rule for fractional differentiation takes the form:

$$ {}_aD_t^p(\varphi(t)f(t)) = \sum_{k=0}^{\infty} \binom{p}{k} \varphi^{(k)}(t)\, {}_aD_t^{p-k} f(t). \tag{172}$$

The Leibnitz rule (172) is useful for the evaluation of fractional derivatives of a function which is a product of a polynomial and a function with known fractional derivative.

To justify the above operations on $R_n^p(t)$ we have to show that $R_n^p(t)$ has a finite value for p>0. The function

$$\frac{f(\tau)\int_\tau^t \varphi^{(n+1)}(\xi)(\tau-\xi)^n d\xi}{(t-\tau)^{p+1}} \tag{173}$$

gives an infinite expression $\frac{0}{0}$ for $\tau = t.$ To find the limit we can use the L'Hospital rule. Differentiating the numerator and the denominator with respect to τ we obtain

$$\frac{f'(\tau)\int_\tau^t \varphi^{(n+1)}(\xi)(\tau-\xi)^n d\xi + nf(\tau)\int_\tau^t \varphi^{(n+1)}(\xi)(\tau-\xi)^{n-1} d\xi}{-(p+1)(t-\tau)^p}, \tag{174}$$

which again gives an indefinite expression $\frac{0}{0}$ for $\tau = t.$ However, if $m < p \le m+1,$ then applying the L'Hospital rule m+2 times we will obtain $(t-\tau)^{p-m-1}$ in the denominator (giving infinity for $\tau = t$), while the numerator will consist of the terms containing the multipliers of the form

$$\int_{\tau}^{t} \varphi^{(n+1)}(\xi)(\tau-\xi)^{n-k}\, d\xi \tag{175}$$

which vanish as $\tau \to t$ if n>k. k cannot be greater than m+2, so we can take $n \geq m+2$ and the function (173) will tend to 0 for $\tau \to t$. This means that the integral in (170) exists in the classical sense even for p>-1.

Taking into account the link between the Grünwald-Letnikov fractional derivatives and the Riemann-Liouville ones we see that under the above conditions on f(t) and $\varphi(t)$ the Leibnitz rule (172) holds also for Riemann-Liouville derivatives.

3.3. Examples

Example 1

Let's apply Leibnitz rule in Grünwald Letnikov definition

$$\frac{d^{-p}(t.f(t))}{dt^{-p}} = \lim_{h\to 0} h^{p} \sum_{r=0}^{n} \begin{bmatrix} p \\ r \end{bmatrix} (t-rh) f(t-rh)$$

Let's take $p = \frac{1}{2}$

$$\frac{d^{-\frac{1}{2}}(t.f(t))}{dt^{-\frac{1}{2}}} = \lim_{h\to 0} h^{\frac{1}{2}} \sum_{r=0}^{n} \begin{bmatrix} \frac{1}{2} \\ r \end{bmatrix} (t-rh) f(t-rh) = \lim_{h\to 0} h^{\frac{1}{2}} \sum_{r=0}^{n} \begin{bmatrix} \frac{1}{2} \\ r \end{bmatrix} tf(t-rh)$$

$$\frac{d^{-\frac{1}{2}}t}{dt^{-\frac{1}{2}}} = \lim_{h\to 0} h^{\frac{1}{2}} \sum_{r=0}^{n} \begin{bmatrix} \frac{1}{2} \\ r \end{bmatrix} (t-rh)$$

$$\frac{d^{-\frac{1}{2}}f(t)}{dt^{-\frac{1}{2}}} = \lim_{h\to 0} h^{\frac{1}{2}} \begin{bmatrix} \frac{1}{2} \\ r \end{bmatrix} f(t-rh)\ \mathrm{f(t)}.$$

$$\lim_{h\to 0} h^{\frac{1}{2}} \sum_{r=0}^{n} \begin{bmatrix} \frac{1}{2} \\ r \end{bmatrix} (t-rh) + t.\lim_{h\to 0} h^{\frac{1}{2}} \sum_{r=0}^{n} \begin{bmatrix} \frac{1}{2} \\ r \end{bmatrix} f(t-rh) = \lim_{h\to 0} h^{\frac{1}{2}} \sum_{r=0}^{n} \begin{bmatrix} \frac{1}{2} \\ r \end{bmatrix} \left[f(t)(t-rh) + t.f(t-rh) \right]$$

$$= \lim_{h\to 0} h^{\frac{1}{2}} \sum_{r=0}^{n} \begin{bmatrix} \frac{1}{2} \\ r \end{bmatrix} tf(t) + \lim_{h\to 0} h^{\frac{1}{2}} \sum_{r=0}^{n} \begin{bmatrix} \frac{1}{2} \\ r \end{bmatrix} \left[(-rh)f(t) + tf(t-rh) \right] = \lim_{h\to 0} h^{\frac{1}{2}} \sum_{r=0}^{n} \begin{bmatrix} \frac{1}{2} \\ r \end{bmatrix} tf(t-rh).$$

Example 2

Let us take derive Dirac Delta function in order of ½. We will use Caputo's Definition.

$$\frac{d^{\frac{1}{2}}\delta(t)}{dt^{\frac{1}{2}}} = \frac{1}{\Gamma(\frac{1}{2}-1)} \int_a^t \frac{\delta'(\tau)d\tau}{(t-\tau)^{\frac{1}{2}+1-1}} = \frac{1}{\Gamma(-\frac{1}{2})} \int_a^t \delta'(\tau)(t-\tau)^{-\frac{1}{2}} d\tau$$

Integrating by part

$$= \frac{1}{\sqrt{\pi}} [(t-\tau)^{-\frac{1}{2}} \delta(\tau) \Big|_a^t + \frac{1}{2} \int_a^t (t-\tau)^{-\frac{3}{2}} \delta(\tau) d\tau] = \frac{1}{2\sqrt{\pi}} \int_a^t (t-\tau)^{-\frac{3}{2}} \delta(\tau) d\tau = \frac{1}{2\sqrt{\pi}} t^{-\frac{3}{2}}$$

CONCLUSION

The goal of this study is to analyze the basic concept of fractional calculus. In the first chapter Grünwald-Letnikov Definition has given. In the second chapter and the third chapter The Riemann-Liouville Definition and Caputo's Definition has given. After every definition this definitions are dissued. In the appendices some application has given about fractional derivatives. Interest in fractional calculus for many years was purely mathematic and it is not hard to see why. Only the very basic concepts regarding the fractional order calculus were addressed here, and yet it is evident that the study fractional calculus opens the mind to entirely new branches of thought. It fills in the gaps of traditional calculus in ways that as of yet, no one completely understands. But the goal of this study is not only to expose the reader to the basic concepts of fractional calculus, but also to whet his/her appetite with some appendices.

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APPENDIX A

THE GAMMA FUNCTIONS

One of the basic functions of the fractional calculus is Euler's gamma function Γ(z), which generalizes the factorial n! and allows n to take also non-integer and even complex values.

The gamma function Γ(z) is defined by the integral

$$\Gamma(z)=\int_0^\infty e^{-t}t^{z-1}dt, \qquad \text{(A1. 1)}$$

Which converges in the right half of the complex plane Re(z)>0. Indeed, we have

$$\Gamma(x+iy)=\int_0^\infty e^{-t}t^{x-1+iy}dt=\int_0^\infty e^{-t}t^{x-1}\ e^{iy\log(t)}dt$$

$$=\int_0^\infty e^{-t}t^{x-1}\left[\cos\left(y\log(t)\right)+i\sin\left(y\log(t)\right)\right]dt \qquad \text{(A1. 2)}$$

The expression in the square bracket in (A1. 2) is bounded for all t; convergence at infinity is provided by e^{-t}, and for the convergence at t=0 we must have x=Re(z)>1.

One of the basic properities of the gamma function is that it satisfies the following functional equation:

$$\Gamma(z+1)=z\ \Gamma(z), \qquad \text{(A1. 3)}$$

This can be shown easily by integrating by parts:

$$\Gamma(z+1)=\int_0^\infty e^{-t}t^{z}dt=\left[-e^{-t}t^{z}\right]_{t=0}^{t=\infty}+z\int_0^\infty e^{-t}t^{z-1}dt=z\Gamma(z).$$

Clearly, $\Gamma(1)=1$ and,

$$\Gamma(2)=1.\Gamma(1)=1=1!$$
$$\Gamma(3)=2.\Gamma(2)=2.1!=2!$$
$$\ldots\ \ldots\ \ldots.$$
$$\Gamma(n+1)=n.\Gamma(n)=n.(n-1)!=n!.$$

Also the gamma function has simple poles at points z=-n, (n=0, 1, 2, …).

To show this, let us write the definition (A1. 1) in the form:

$$\Gamma(z)=\int_0^1 e^{-t}t^{z-1}dt+\int_1^\infty e^{-t}t^{z-1}dt. \tag{A1. 4}$$

The first integral in (A1. 4) can be evaluated by using the series expansion for the exponential function . If Re(z)=x>0 (z is in the right half-plane), then Re(z+k)=x+n>0 and $t^{x+k}=0$ (t=0). Therefore,

$$\int_0^1 e^{-t}t^{z-1}dt=\int_0^1\sum_{k=0}^\infty\frac{(-t)^k}{k!}t^{z-1}dt=\sum_{k=0}^\infty\frac{(-1)^k}{k!}\int_0^1 t^{k+z-1}dt=\sum_{k=0}^\infty\frac{(-1)^k}{k!(k+z)}.$$

The second integral defines an entire function of the complex variable z. Let us write,

$$\varphi(z)=\int_1^\infty e^{-t}t^{z-1}dt=\int_1^\infty e^{(z-1)\log(t)-t}dt. \tag{A1. 5}$$

The function $e^{(z-1)\log(t)-t}$ is a continuous function of z and t for arbitrary z and $t\geq 1$.

If $t\geq 1$ ($\log(t)\geq 0$), then it is an entire function of z . let us consider an arbitrary bounded closed domain D in the complex plane and denote $x_0=\max\limits_{x\in D}\mathrm{Re}(z)$. Then we have,

$$\left|e^{-t}t^{z-1}\right|=\left|e^{(z-1)\log(t)-t}\right|=\left|e^{(x-1)\log(t)-t}\right|.\left|e^{iy\log(t)}\right|=\left|e^{(x-1)\log(t)-t}\right|\leq e^{(x_0-1)\log(t)-t}=e^{-t}t^{x_0-1}.$$

So, the integral (A1. 5) converges uniformly in D and the function φ(z) is regular in D and differentiation under the integral in (A1. 5) is allowed. Because the domain D has been chosen arbitrarily, we concullude that the the function φ(z) has the above properities in the whole complex plane. Therefore, φ(z) is an entire function allowing differentiation under the integral.

$$\Gamma(z)=\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\frac{1}{k+z}+\int_1^{\infty}e^{-t}t^{z-1}dt=\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\frac{1}{k+z}+\text{entire function} \qquad \text{(A1. 6)}$$

and $\Gamma(z)$ has only simple poles at the points z=-n, n=0, 1, 2, …

The gamma function can be represented by the limit

$$\Gamma(z)=\lim_{n\to\infty}\frac{n!n^z}{z(z+1)...(z+n)},\ z\neq 0,\ -1,\ -2,\ \ldots \qquad \text{(A1. 7)}$$

To prove (A1. 7) let us introduce the following function

$$f_n(z)=\int_0^n\left(1-\frac{t}{n}\right)^n t^{z-1}dt.$$

$$\frac{t}{n}=u \quad \frac{dt}{n}=du$$

$$f_n(z)=\int_0^1(1-u)^n(nu)^{z-1}ndu=n^z\int_0^1(1-u)^n u^{z-1}du \qquad \text{(A1. 8)}$$

we obtain

$$f_n(z)=n^z\frac{n.(n-1).(n-2)...1}{z.(z+1).(z+2)...(z+n-1)}\int_0^1 u^{z+n-1}du=\frac{1.2.3...n}{z.(z+1).(z+2)...(z+n)}n^z$$

$$\lim_{n\to\infty} f_n(z) = \Gamma(z)$$

Also another representations of the gamma function;

$$\Gamma(z) = 2\int_0^\infty e^{-t^2} t^{2z-1} dt \text{ , (1. 7) } \Gamma(z) = \int_0^1 \left[In\left(\frac{1}{t}\right)\right]^{z-1} dt \text{ .} \qquad \text{(A1. 9)}$$

Integrating by parts we showed:

$$\Gamma(z) = (\frac{1}{z})\int_0^\infty e^{-t} t^z dt = \left(\frac{1}{z}\right)\Gamma(1+z),$$

$$\Gamma(1+z) = z\Gamma(z), \qquad \text{(A1. 10)}$$

if n is positive integer,

$$\Gamma(z+n) = z(z+1)(z+2)...(z+n-1)\Gamma(z), \text{ and this follows;} \qquad \text{(A1. 11)}$$

$$\frac{\Gamma(z)}{\Gamma(z-n)} = (z-1)(z-2)...(z-n) = (-1)^n \frac{\Gamma(-z+n+1)}{\Gamma(-z+1)}, \qquad \text{(A1. 12)}$$

$$\frac{\Gamma(-z+n)}{\Gamma(-z)} = (-1)^n z(z-1)...(z-n+1) = (-1)^n \frac{\Gamma(z+1)}{\Gamma(z-n+1)}, \qquad \text{(A1. 13)}$$

$$\frac{1}{\Gamma(z)} = ze^{\gamma z} \prod_{n=1}^{\infty} \left[(1+z/n)e^{-\frac{z}{n}} \right], \text{ where} \qquad \text{(A1. 14)}$$

$$\gamma = \lim_{n\to\infty}(\sum_{n=1}^{m} 1/n - \log m) = 0.5772156649... \text{ (Euler's or Mascheroni's constant)}$$

from (A1. 14),

$$\Gamma(z)\Gamma(-z) = -z^{-2}\prod_{n=1}^{\infty}(1-\frac{z^2}{n^2})^{-1} \text{ and since,} \tag{A1. 15}$$

$$\sin(\pi z)=\pi z\prod_{n=1}^{\infty}\left(1-\frac{z^2}{n^2}\right) \text{ and we have;}$$

$$\Gamma(z)\Gamma(-z) = -\pi \;\; z^{-1}\csc(\pi z) \tag{A1. 16}$$

so that

$$\Gamma(z)\Gamma(1-z) = \pi \;\; \csc(\pi z), \text{ or} \tag{A1. 17}$$

$$\Gamma(\frac{1}{2}+z)\Gamma(\frac{1}{2}-z) = \pi\sec(\pi z), \tag{A1. 18}$$

$$\frac{\Gamma(n+z)\Gamma(n-z)}{\left[(n-1)!\right]^2} = \frac{\pi z}{\sin(\pi z)}\prod_{m=1}^{n-1}\left(1-\frac{z^2}{m^2}\right) \text{ n=1, 2, 3, …} \tag{A1. 19}$$

$$\frac{\Gamma(n+\frac{1}{2}+z)\Gamma(n+\frac{1}{2}-z)}{\left[\Gamma\left(n+\frac{1}{2}\right)\right]^2} = \frac{1}{\cos(\pi z)}\prod_{m=1}^{n}\left[1-\frac{4z^2}{(2m-1)^2}\right] \text{ n=1, 2, 3,…}$$

z=1/2

$$\Gamma(1/2) = 2\int_0^{\infty} e^{-t^2}dt = \sqrt{\pi} \tag{A1. 20}$$

The integration variable t in the definition of the gamma function (A1. 1) is real.If t is complex, the function $e^{(z-1)\log(t)-t}dt$ has a branch point t=0. Cutting the complex plane (t) along the real semi-axis from t=0 to t=∞ makes this function single-valued. Therefore, according to Cauchy's Theorem, the integral

$$\int_C e^{-t} t^{z-1} dt = \int_C e^{(z-1)\log(t)-t} dt$$

has the same value for any contour C running around the point t=0 with both ends at $+\infty$.

Let us consider the contour C consisting of the part of the upper edge $(+\infty, \varepsilon)$ of the cut, the circle C_ε of radius ε with the centre at t=0 and the part of the lower cut edge $(\varepsilon, +\infty)$.

Taking log(t) to be real on the upper cut edge,

$$e^{(z-1)\log(t)-t} = e^{(z-1)\log|t|-|t|}$$

$$t = |t|e^{i\Phi} \quad \Phi = 0 \quad t = |t|$$

$$e^{(z-1)\log(t)-t} = t^{z-1}e^{-t}$$

On the lower cut edge we must replace log(t) by log(t)+2πi:

$$e^{(z-1)\log(t)-t} = e^{(z-1)\log|t|e^{2i\pi}-|t|e^{2i\pi}} = e^{(z-1)[2i\pi+\ln|t|]-|t|} = e^{2(z-1)\pi i}e^{(z-1)\ln(t)-t} \quad (\Phi = 2\pi)$$

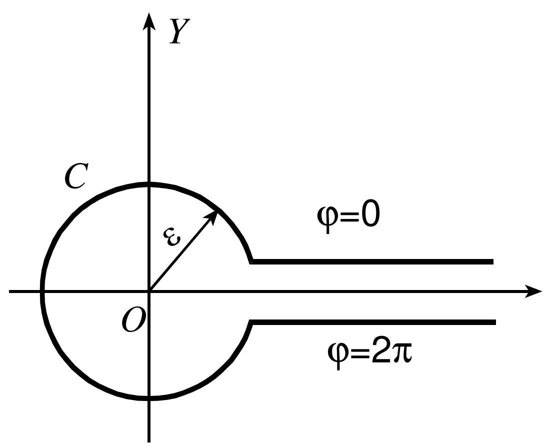

Figure 1.1. Contour C

Therefore,

$$\int_C e^{-t}t^{z-1}dt = \int_{+\infty}^{\varepsilon} e^{-t}t^{z-1}dt + \int_{C_\varepsilon} e^{-t}t^{z-1}dt + e^{2(z-1)\pi i}\int_{\varepsilon}^{+\infty} e^{-t}t^{z-1}dt$$

The integral along C_ε tends to zero as $\varepsilon \to 0$. Taking into account that $|t| = \varepsilon$ on C_ε and denoting

$$\text{M}=\max_{t\in C\varepsilon}\left|e^{-y\arg(t)-t}\right| \text{ (y=Im(z)),}$$

Where M is independent of t, we obtain;

$$\left|\int_{C_\varepsilon} e^{-t}t^{z-1}dt\right| \le \int_{C_\varepsilon}\left|e^{-t}t^{z-1}\right|dt = \int_{C_\varepsilon}\left|t^{z-1}\right|.\left|e^{-y\arg(t)-t}\right|dt \le M\varepsilon^{x-1}\int_{C_\varepsilon} dt = M\varepsilon^{x-1}.2\pi\varepsilon = 2\pi M\varepsilon^{x}$$

$$\lim_{\varepsilon\to 0}\int_{C_\varepsilon} e^{-t}t^{z-1}dt = 0$$

$$\int_C e^{-t}t^{z-1}dt = \int_{+\infty}^{0} e^{-t}t^{z-1}dt + e^{2(z-1)\pi i}\int_{0}^{+\infty} e^{-t}t^{z-1}dt.$$

Using (A1.1)

$$\int_C e^{-t}t^{z-1}dt = \int_{+\infty}^{0} e^{-t}t^{z-1}dt + e^{2(z-1)\pi i}\int_{0}^{+\infty} e^{-t}t^{z-1}dt = -\int_{0}^{+\infty} e^{-t}t^{z-1}dt + e^{2(z-1)\pi i}\int_{0}^{+\infty} e^{-t}t^{z-1}dt$$

$$\int_0^{+\infty} e^{-t}t^{z}dt = \frac{1}{e^{2z\pi i}-1}\int_C e^{-t}t^{z}dt$$

$$\Gamma(z) = \frac{1}{e^{2\pi iz}-1}\int_C e^{-t}t^{z-1}dt. \qquad \text{(A1. 21)}$$

The function $e^{2\pi i z}-1$ has its zeros at the points z=0, $\pm 1, \pm 2, ...$

The points z=1, 2, … are not the poles of $\Gamma(z)$, because in this case the function $e^{-t}t^{z-1}$ is single-valued and regular in the comlex plane (t) and according to Cauchy's theorem

$$\int_C e^{-t}t^{z-1}dt = 0 .$$

If z=0, -1, -2, … then the function $e^{-t}t^{z-1}$ is not an entire function of t and the integral of it along the contour C is not equal to zero. Therefore, the points z=0, -1, -2, … are the poles of $\Gamma(z)$. According to the principle of analytic continuation, the integral representation (A1. 21) holds not only for Re(z)>0, as assumed at the beginning, but in the whole complex plane (z).

Let us write representation of **1/**$\Gamma(z)$**.** We will replace z by 1-z in the formula (A1. 21)

$$\int_C e^{-t}t^{z}dt = (e^{-2z\pi i}-1)\Gamma(1-z) \qquad \text{(A1. 22)}$$

and then perform the substitution t = $\tau e^{\pi i}$ = - τ . The transformation t=$\tau e^{i\pi}$ corresponds to the anticloclwise rotation of the complex plane by which the upper cut edge in t-plane goes over into the lower cut edge in τ - plane (extending from 0 to -∞). The contour C will be transformed to Hankel's contour Ha shown in Fig.1.2

$$\int_C e^{-t}t^{-z}dt = -\int_{Ha} e^{\tau}(e^{\pi i}\tau)^{-z}d\tau = -e^{-z\pi i}\int_{Ha} e^{\tau}\tau^{-z}d\tau . \qquad \text{(A1. 23)}$$

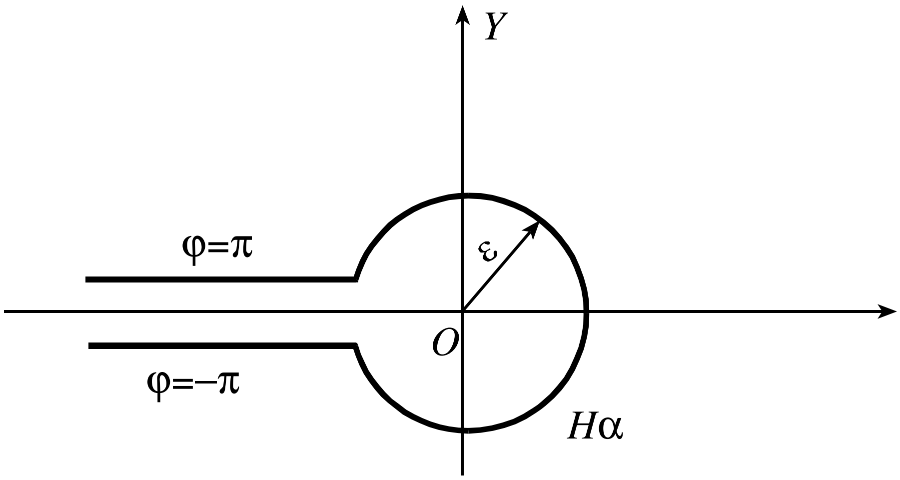

Figure 1.2. The Hankel contour *Hα*

Taking into account the relationships (A1. 9) and … we obtain

$$\int_{Ha} e^{\tau}\tau^{-z}d\tau = (e^{z\pi i} - e^{-z\pi i})\Gamma(1-z) = 2i\sin(\pi z)\Gamma(1-z) = \frac{2\pi i}{\Gamma(z)}. \qquad \text{(A1. 24)}$$

Therefore, we have the following integral representation for reciprocal gamma function:

$$\frac{1}{\Gamma(z)} = \frac{1}{2\pi i}\int_{Ha} e^{\tau}\tau^{-z}d\tau \qquad \text{(A1. 25)}$$

Let us denote by $\gamma(\varepsilon,\varphi)$ $(\varepsilon > 0, 0 < \varphi \le \pi)$ the contour consisting of the following three parts;

$$\arg\ \tau = -\varphi, |\tau| \ge \varepsilon;$$

$$-\varphi \le \arg\tau \le \varphi, |\tau| = \varepsilon;$$

$$\arg\tau = \varphi, |\tau| \ge \varepsilon$$

The contour is traced so that $\arg\tau$ is non-decreasing. It is shown in Fig1. 2

The contour $\gamma(\varepsilon,\varphi)$ divides the complex plane τ into two domains, which we denote by $G^{-}(\varepsilon,\varphi)$ and $G^{+}(\varepsilon,\varphi)$, lying on the left and on the right side of the contour $\gamma(\varepsilon,\varphi)$

If $0<\varphi<\pi$, then both $G^{-}(\varepsilon,\varphi)$ and $G^{+}(\varepsilon,\varphi)$ are infinite domains. If $\varphi=\pi$, then $G^{-}(\varepsilon,\varphi)$ becomes a circle $|\tau|<\varepsilon$ and $G^{+}(\varepsilon,\varphi)$ becomes a complex plane excluding the circle $|\tau|<\varepsilon$ and the line $|\arg\varphi|=\pi$.

Let us show that instead of integrating along Hankel's contour Ha in (A1. 12) we can integrate along the contour $\gamma(\varepsilon,\varphi)$,where $\frac{\pi}{2}<\varphi<\pi$

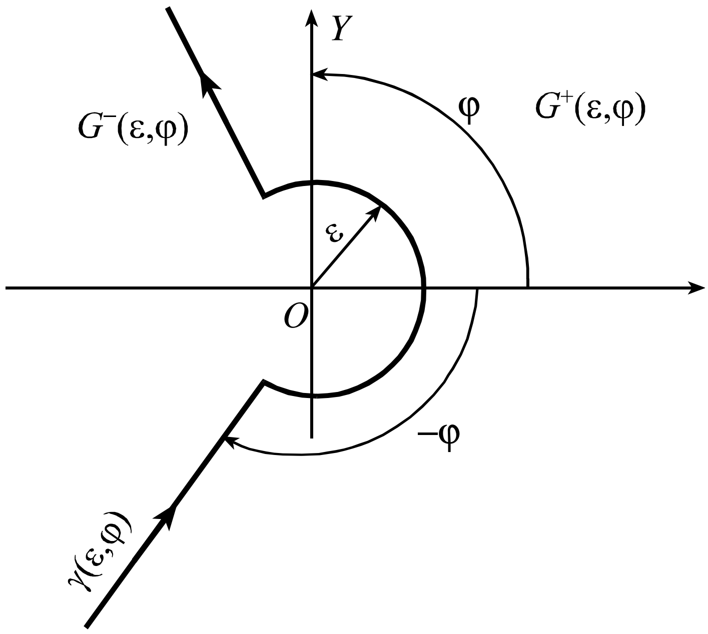

Figure 1.3. Contour *γ(ε,φ)*

$$\frac{1}{\Gamma(z)}=\frac{1}{2\pi i}\int_{\gamma(\varepsilon,\varphi)}e^{\tau}\tau^{-z}d\tau,\ (\varepsilon>0,\frac{\pi}{2}<\varphi\leq\pi) \tag{A1. 26}$$

Let us consider the contour ($A^{+}B^{+}C^{+}D^{+}$) shown in Fig 1. 3 Using the Cauchy theorem for contour gives:

$$\int_{(A^+B^+C^+D^+)} e^{\tau}\tau^{-s}d\tau = \int_{A^+}^{B^+} + \int_{B^+}^{C^+} + \int_{C^+}^{D^+} + \int_{D^+}^{A^+} \tag{A1. 27}$$

On the arc (A^+B^+) we have $|\tau| = R$ and

$$\mathrm{e}^{\tau}\tau^{-z} = e^{\tau}e^{-z\log\tau} \;\; (\tau = |\tau|e^{i\arg\tau} = \mathrm{Re}^{i\arg\tau})$$

$$e^{Re^{i\,arg\,\tau} - z\,log(\,R\,)e^{i\,arg\,\tau}} = e^{R(\cos\varphi + i\sin\varphi) - z(\log R + i\varphi)} = e^{R(\cos\varphi + i\sin\varphi) - (x+iy)(\log R + i\varphi)}$$

$$e^{R\cos\varphi - x\log R + y\varphi}$$

$$\left|e^{\tau}\tau^{-z}\right| = e^{R\cos(\arg\tau) - x\log R + y\arg\tau} \le e^{-R\cos(\pi-\varphi) - x\log R + 2\pi y} \text{ and}$$

$$\lim_{R\to\infty}\int_{A^+}^{B^+} = 0 \tag{A1. 28}$$

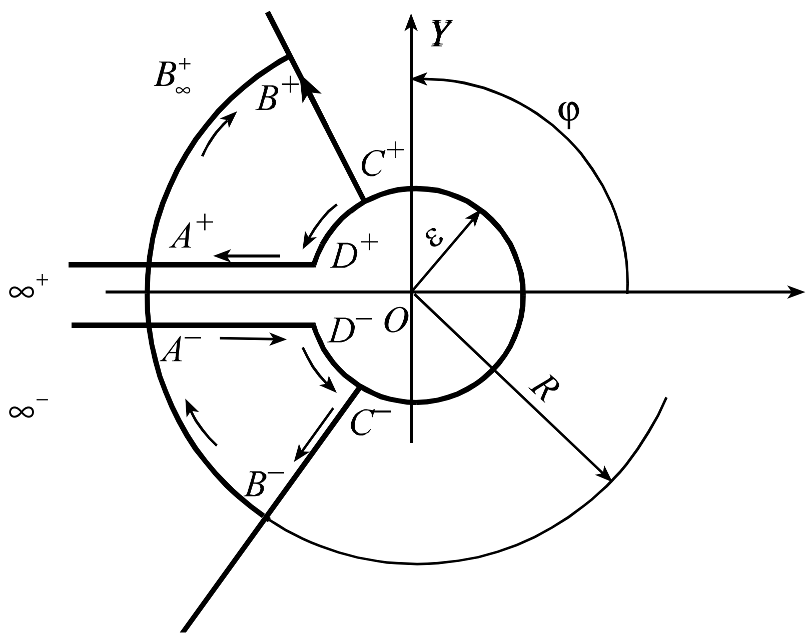

Figure 1.4. Transformation of the contour Hα to the contour *y(ε,φ)*

Taking $R \to \infty$ in (A1. 14) and using (A1. 15) we obtain:

$$\int_{C^+}^{D^+} + \int_{D^+}^{\infty^+} + \int_{B_\infty^+}^{C^+} = 0$$

$$\int_{C^+}^{D^+} + \int_{D^+}^{\infty^+} = \int_{C^+}^{B_\infty^+} \quad \text{similarly,} \tag{A1. 29}$$

$$\int_{\infty^-}^{D^-} + \int_{D^-}^{C^-} = \int_{B_\infty^-}^{C^-} \tag{A1. 30}$$

Using (A1. 16) and (A1. 17)

$$\int_{Ha} e^{\tau} t^{-z} d\tau = \left(\int_{B_\infty^-}^{C^-} + \int_{C^-}^{C^+} + \int_{C^+}^{B_\infty^+}\right) e^{\tau} t^{-z} d\tau = \int_{\gamma(\varepsilon,\varphi)} e^{\tau} t^{-z} d\tau$$

APPENDIX B

FRACTIONAL DERIVATIVES, SPLINES AND TOMOGRAPHY

Splines are made up of polynomials and are esentially as easy to manipulate. The term "spline" is used to refer to a wide class of functions that are used in applications requiring data interpolation and/or smoothing. Splines may be used for interpolation and/or smoothing of either one-dimensional or multi-dimensional data. Spline functions for interpolation are normally determined as the minimizers of suitable measures of roughness (for example integral squared curvature) subject to the interpolation constraints. Smoothing splines may be viewed as generalizations of interpolation splines where the functions are determined to minimize a weighted combination of the average squared approximation error over observed data and the roughness measure. For a number of meaningful definitions of the roughness measure, the spline functions are found to be finite dimensional in nature, which is the primary reason for their utility in computations and representation. For the rest of this section, we focus entirely on one-dimensional, polynomial splines and use the term "spline" in this restricted sense.

A (univariate, polynomial) spline is a piecewise polynomial function. In its most general form a polynomial spline $S:[a,b]\to R$ consists of polynomial pieces $P_i:[t_i,t_{i+1}]\to R$, where

$$a=t_0<t_1<...<t_{k-2}<t_{k-1}=b\,.$$

That is,

$$S(t)=P_0(t),t_0\le t<t_1$$

$$S(t)=P_0(t),t_1\le t<t_2$$

$$...$$

$$S(t) = P_{k-2}(t), t_{k-2} \le t \le t_{k-1}.$$

The given k values ti are called knots. The vector $t = (t_0, ..., t_{k-1})$ is called a knot vector for the spline. If the knots are equidistantly distributed in the interval [a,b] we say the spline is uniform, otherwise we say it is non-uniform.

If the polynomial pieces on the subintervals

$$[t_i, t_{i+1}], i = 0, ..., k-2$$

all have degree at most n, then the spline is said to be of degree $\le n$ (or of order n+1).

If $S \in C^{r_i}$ in a neighborhood of ti, then the spline is said to be of smoothness (at least) C^{r_i} at ti. That is, the two pieces Pi − 1 and Pi share common derivative values from the derivative of order 0 (the function value) up through the derivative of order ri. Or stated differently, the two adjacent polynomial pieces connect with loss of smoothness of (at most) ji, defined by ri = n − ji. (Expressing the connectivity as a "loss of smoothness" is reasonable, since if S were a simple polynomial throughout a neighborhood of ti, it would have smoothness Cn at ti, and you would expect to lose smoothness in order to break a polynomial apart into pieces.) A vector $r = (r_0, ..., r_{k-2})$ such that the spline has smoothness C^{r_i} at ti for 0 < i < k − 1 is called a smoothness vector for the spline.

Given a knot vector t, a degree n, and a smoothness vector r for t, one can consider the set of all splines of degree $\le n$ having knot vector t and smoothness vector r. Equipped with the operation of adding two functions (pointwise addition) and taking real multiples of functions, this set becomes a real vector space. This spline space is commonly denoted by $S_n^r(t)$.

In the mathematical study of polynomial splines the question of what happens when two knots, say ti and ti+1, are moved together has an easy answer. The polynomial piece Pi(t) disappears, and the pieces Pi−1(t) and Pi+1(t) join with the sum of the continuity losses for ti and ti+1. That is,

$$S(t) \in C^{n-j_i-j_{i+1}}[t_i = t_{i+1}]$$

This leads to a more general understanding of a knot vector. The continuity loss at any point can be considered to be the result of multiple knots located at that point, and a spline type can be completely characterized by its degree n and its extended knot vector

$$a = t_0 < t_1 = ...t_1 < ... < t_{k-2} = ... = t_{k-2} < t_{k-1} = b$$

where ti is repeated ji times for $i = 1,...,k-2$.

A parametric curve on the interval [a,b]

$$G(t) =< X(t), Y(t) >, t \in [a,b]$$

is a spline curve if both X and Y are splines of the same degree with the same extended knot vectors on that interval.

Examples

Suppose the interval [a,b] is [0,3] and the subintervals are [0,1), [1,2), and [2,3]. Suppose the polynomial pieces are to be of degree 2, and the pieces on [0,1) and [1,2) must join in value and first derivative (at t=1) while the pieces on [1,2) and [2,3] join simply in value (at t=2). This would define a type of spline S(t) for which

$$S(t) = P_0(t) = -1 + 4t - t^2, 0 \le t < 1$$

$$S(t) = P_1(t) = 2t, 1 \le t < 2$$

$$S(t) = P_2(t) = 2 - t + t^2, 2 \le t < 3$$

would be a member of that type, and also

$$S(t) = P_0(t) = -2 - 2t^2, 0 \le t < 1$$

$$S(t) = P_1(t) = 1 - 6t + t^2, 1 \le t < 2$$

$$S(t) = P_2(t) = -1 + t - 2t^2, 2 \le t < 3$$

would be a member of that type. (Note: the polynomial piece 2t is quadratic, since it can be written 2t + 0t2. Any polynomial of one degree is trivially a polynomial of higher degree simply by this trick of adding appropriate powers with zero coefficients.) The extended knot vector for this type of spline would be 0,1,2,3-.

The simplest spline has degree 0. It is also called a step function. The next most simple spline has degree 1. It is also called a linear spline. The corresponding parametric curve having linear spline components X(t) and Y(t) just a polygon.

A common spline is the natural cubic spline of degree 3 with continuity C2. The word "natural" means that the second derivatives of the spline polynomials are set equal to zero at the endpoints of the interval of interpolation

$$. \ S''(a) = S''(b) = 0$$

This forces the spline to be a straight line outside of the interval, while not disrupting its smoothness.

One operation that is especially simple to implement is differentiation. It has the same effect on splines as it has on polynomials: it reduces the degree by one. The derivative of a B-spline of degree n is given by

$$D\beta^n(x) = \Delta\beta^{n-1}(x) = \beta^{n-1}(x + \frac{1}{2}) - \beta^{n-1}(x - \frac{1}{2})$$

Where Δ is denotes the central finite difference operator. The implication of this differentiation Formula is that one can calculate spline derivatives simply by applying finite differences to the B-spline coefficients of the representation. Thus, with splines, one has an exact equivalence between finite diffrences and differentation and not just an approximate one as is usually the case in numerical analysis. This is a property that can be exploited advatageously for implementing differential signal processing operators[6]. The main difficulty with fractional derivatives is that the derivatives of polynomials (or splines) are no-longer polynomial when the order of differetiation in non-integer. This forces us to consider the enlarged family of fractional splines [7]; these are reviewed in

Section 2. In Section 3, it is presented that the differentiation rules fort he fractional splines and shown that this family is closed under fractional differentiation: specifically, the γ th derivative of a fractional spline of degree α is a fractional spline of degree α - γ, where α and γ are not necessarily integer. Finally, in Section 4, it is indicated how these results are useful for improving the implementation of the filtered backprojection (FBP) algorithm for tomographic reconstruction [4, 5].

In this section, it is defined the fractional splines and summarized the main properties of their basic constituents: the fractional B-splines. For more details, refer to [7]. The purest examples of fractional splines of degree α are the one-sided and rectified power functions, x_+^α and $|x_*|^\alpha$, which both exhibit one singularity of order α (Hölder exponent) at the origin. The one-sided power function is defined by:

$$x_+^\alpha = \begin{Bmatrix} x^\alpha, x \geq 0 \\ 0, otherwise \end{Bmatrix} \text{ (A2. 1)}$$

For $\alpha \notin N$, its Fourier transform is $\Gamma(\alpha+1)/(i\omega)^{\alpha+1}$.

The second symmetric type, $|x|_*^\alpha$, is defined as the function whose fourier transform is $\Gamma(\alpha+1)/|\omega|^{\alpha+1}$.For α non-even, it is a (rectified) power function; otherwise, it has an additional logarithmic factor:

$$x_*^\alpha = \begin{Bmatrix} \dfrac{|x|^\alpha}{-2\sin(\frac{\pi}{2}\alpha)}, \alpha = 2n-1 \\ \dfrac{x^{2n}\log x}{(-1)^{1+n}\pi}, \alpha = 2n \end{Bmatrix} \qquad \text{(A2. 2)}$$

By analogy with the classical B-splines, one consructs the fractional casual B-splines by taking the (α +1)-fractional difference of the one-sided power function

$$\beta_+^\alpha(x) = \frac{\Delta_+^{\alpha+1} x_+^\alpha}{\Gamma(\alpha+1)} = \frac{1}{\Gamma(\alpha+1)} \sum_{k=0}^{+\infty} (-1)^k \binom{\alpha+1}{k} (x-k)_+^\alpha \qquad \text{(A2. 3)}$$

Where $\Gamma(u+1)=\int x^{u}e^{-x}dx$ is Euler's gamma function. $\Delta_{+}^{\alpha+1}$ is the (α+1)-fractional difference operator; it is a convolution operator whose transfer function is

$$\hat{\Delta}_{+}^{\alpha+1}(\omega)=(1-e^{-i\omega})^{\alpha+1}=\sum_{k=0}^{+\infty}(-1)^{k}\binom{\alpha+1}{k}e^{-i\omega k}. \tag{A2. 4}$$

The fractional B-splines are in L_2 for $\alpha>-\frac{1}{2}$. They are compactly supported for α integer; otherwise, they decay like $|x|^{-(\alpha+2)}$ (cf. [7], Theorem3.1). The Fourier domain equivalent of (A2. 3) is

$$\hat{\beta}_{+}^{\alpha}(\omega)=\left(\frac{1-e^{-i\omega}}{i\omega}\right)^{\alpha+1} \tag{A2. 5}$$

It is constructed the symmetric B-splines by taking (α+1)-symmetric fractional differences of the rectified power function:

$$\beta_{*}^{\alpha}(\omega)=\frac{\Delta_{*}^{\alpha+1}x_{*}^{\alpha}}{\Gamma(\alpha+1)}=\frac{1}{\Gamma(\alpha+1)}\sum_{k=0}^{+\infty}(-1)^{k}\begin{vmatrix}\alpha+1\\ k\end{vmatrix}|x-k|_{*}^{\alpha} \tag{A2. 6}$$

Where $\Delta_{*}^{\alpha}\overset{fourier}{\leftrightarrow}|1-e^{-i\omega}|^{\alpha}$ is the symmetric fractional difference operator. Similar to their casual counterparts, these functions are not compactly supported either unless n is odd, in which case they coincide with the traditional polynomial B-splines. When α is not odd. They decay like $|x|^{-(\alpha+2)}$ and their asymptotic form is available [7]. The Fourier counterpart of (A2. 6) is simply

$$\hat{\beta}_{*}^{\alpha}(\omega)=\left|\frac{\sin(\omega/2)}{\omega/2}\right|^{\alpha+1}. \tag{A2. 7}$$

Note that the expansion coefficients on the right hand side of (A2. 3) and (A2. 6) are generalized versions of the binomials. They are both compatible with the following extended definition:

$$\binom{u}{v} = \frac{\Gamma(u+1)}{\Gamma(v+1)\Gamma(u-v+1)} \tag{A2. 8}$$

Where the gamma function replaces the factorials encountered in the standart Formula when u and v are both integer. The coefficients in (A2. 6) are re-centered version given by

$$\begin{vmatrix} r \\ k \end{vmatrix} = \begin{pmatrix} r \\ k + \frac{r}{2} \end{pmatrix} \tag{A2. 9}$$

In most general terms, fractional splines maybe defined as linear combinations of shifted fractional power functions or fractional B-splines. As in the polynomial case, it is usually more advantageous to use the second type of representation. The fractional B-splines have all the good properties of the conventional B-splines, except that they lack compact support when α is not an integer. In particular, they form a Riesz basis which ensures that B-spline representation is stable numerically. Thus, if we consider the basic integer grid, we may represent a fractional spline signal by its B-spline expansion

$$s(x) = \sum_{k \in z} c(k)\beta^{\alpha}(x-k) \tag{A2. 10}$$

Where it is used the generic notation $\beta^{\alpha}(x)$ to specify any one of the fractional B-splines $(\beta_{+}^{\alpha}(x), or \beta_{*}^{\alpha}(x))$. What this means that a fractional spline signal s(x) with knots at the integers is unambiguously characterized through its B-spline coefficients c(k), $k \in z$ (discrete/continuous representiation). The representation is one-to-one there is exactly one coefficient c(k) by sample value s(k). Note that this spline representation is compatible which the traditional model used in signal processing for it can be shown that the signal (A2. 10) converges to a bandlimited function as the order of the spline increases [1]. It is considered two versions of fractional derivatives which can be defined in the Fourier domain. The first type, which is compatible with Liouville's definition [2], is given by

$$D^{\gamma} f(x) \overset{fourier}{\longleftrightarrow} (i\omega)^{\gamma} \hat{f}(\omega) \qquad \text{(A2. 11)}$$

Where $\hat{f}(\omega) = \int f(x) e^{-i\omega x} dx$ denotes the Fourier transform of f(x) and where $z^{\gamma} = |z|^{\gamma} e^{i\gamma \arg(z)}$ with i=$\sqrt{-1}$ and arg(z)$\in [-\pi, \pi]$.

The second type of derivative, which is a symmetrized version of first, is defined by

$$D_{*}^{\gamma} f(x) \overset{fourier}{\longleftrightarrow} |\omega|^{\gamma} \hat{f}(\omega) \qquad \text{(A2. 12)}$$

Note that the first type agrees with the usual definition of the derivative when α is integer, while the second one only does when α is even.

The general B-spline differentiation rules are

$$D^{\gamma} \beta_{+}^{\alpha}(x) = \Delta_{+}^{\gamma} \beta_{+}^{\alpha-\gamma}(x) \qquad \text{(A2. 13)}$$

$$D_{*}^{\gamma} \beta_{*}^{\alpha}(x) = \Delta_{*}^{\gamma} \beta_{*}^{\alpha-\gamma}(x) \qquad \text{(A2. 14)}$$

Where D^{γ} and D_{*}^{γ} are defined by (A2. 11) and (A2. 12). This is established in the fourier domain. For instance, to obtain (A2. 14) , (A2. 7) is substituted in (A2. 12) and rewritten the fourier transform of $D_{*}^{\gamma} \beta_{*}^{\alpha}(x)$ as

$$|\omega|^{\gamma} \left| \frac{\sin(\omega/2)}{\omega/2} \right|^{\alpha+1} . = \left| \frac{\sin(\omega/2)}{2} \right|^{\gamma} . \left| \frac{\sin(\omega/2)}{\omega/2} \right|^{\alpha+1-\gamma}$$

$$\left|\frac{\sin(\omega/2)}{2}\right|^{\gamma} = \left|1-e^{j\omega}\right|^{\gamma}, \left|\frac{\sin(\omega/2)}{\omega/2}\right|^{\alpha+1-\gamma} = \hat{\beta}_{*}^{\alpha-\gamma}(\omega)$$

Let us now indicate how these rules can be applied to obtain the fractional derivative of the spline signal in (A2. 10). Taking the fractional derivative and interchanging the order of summation,

$$D^{\alpha}s(x) = \sum_{k\in z} c(k)\Delta^{\gamma}\beta^{\alpha-\gamma}(x-k) = \sum_{k\in z}(\Delta^{\gamma} * c)(k)\beta^{\alpha-\gamma}(x-k),\ (\Delta^{\gamma} * c)(k) = d(k) \quad \text{(A2. 15)}$$

Where it has been moved the fractional difference operator into the discrete domain. Thus, the B-spline coefficient d(k) of $D^{\alpha}s(x)$ are obtained by convolving the c(k)'s with the digital filter Δ^{γ} whose frequency response is $(1-e^{-j\omega})^{\gamma}$ or $\left|1-e^{-j\omega}\right|^{\gamma}$, depending on the type of derivative.

The mathematical basis fort he standart filtered backprojection tomographic reconstruction algorithm is the following identity $\forall f \in L_2(R^2)(cf.[3])$

$$f(x,y) = R^{*}KRf(x,y) = R^{*}K\{p_{\theta}(t)\} \quad \text{(A2. 16)}$$

With t=(x,y).$\vec{\theta}$ where $\vec{\theta} = (\cos\theta, \sin\theta) \in S$ is the unit that specifies the direction of the projection:

$p_{\theta}(t) = \iint_{R^2} f(\vec{x})\delta(\vec{x}.\vec{\theta}-t)d\vec{x}$ is Radon transform of f and R^{*} is the so-called backprojection operator; is the adjoint of the Radon or projection operator R. The right hand side of (A2. 16) provides the filtered backprojection solution fort he recovery of the function $f(x,y)$ from its projection data $p_{\theta}(t)$.

The algorithm proceeds in two steps. First, each projection $p_{\theta}(t)$ is filtered continuously with the ramp or Ram-Lak fitler [4]; the crucial observation here is that the filtering operator K is proportional to our fractional derivative $D_{*} \leftrightarrow |\omega|$; i.e., $K = (2\pi)^{-1}D_{*}$. Second, the filtered projections are projected back onto the image and averaged according to the Formula

$$R^{*}K\{p_{\theta}(t)\}=\frac{1}{2\pi}\int_{0}^{\pi}D_{*}p_{\theta}(t)d\theta \cong \frac{1}{2N}\sum_{i=1}^{N}D_{*}p_{\theta}(t) \tag{A2. 17}$$

With $t=(x,y).\vec{\theta}$. The reconstruction Formula (A2. 16) is exact provided that one treats the projection data $p_{\theta}(t)$ as a continum both in terms of t and θ. In practice, however, one has only Access to a finite number of projection at the angles θ_i, and the continuous average in (A2. 17) is usually replaced by the discrete one on the right. The error can be assumed to be negligible provided that the number of projections N is sufficient.

In this method, it is assumed that the projection data at angle θ is a fractional spline of degree α:

$$p_{\theta}(t)=R_{\theta}f(t)=\sum_{k\in z}c(k)\beta_{*}^{\alpha}(t-k) \tag{A2. 18}$$

After symmetric differentiation (ramp filter), it is found that

$$D_{*}p_{\theta}(t)=\sum_{k\in z}d(k)\beta_{*}^{\alpha-1}(t-k) \tag{A2. 19}$$

Where the d(k) are obtained by applying the symmetric finite difference to the c(k) (cf.(15)). Thus, we have an explicit continuous representation of the filtered projection which can then be directly plugged into (A2. 17).

In practice. We are given the sampled values of the projection $p_{\theta}(k)$ and the first step is to determined the B-spline coefficients c(k) such that the spline model interpolates these values exactly. This can be done by digital filtering. Combining both filters together (interpolation and ramp-filter), getting

$$\mathrm{d(k)}=(h_{*} * p_{\theta})(k)) \tag{A2. 20}$$

where h_* is the digital fitler whose transfer function is

$$h_*(k) \overset{fourier}{\longleftrightarrow} \frac{\left|1-e^{j\omega}\right|}{B_*^{\alpha}(e^{j\omega})} = \frac{\sin(\omega/2)/2}{\sum_{n\in Z}\left|\sin c(\omega/2\pi - n)\right|^{\alpha+1}} \tag{A2. 21}$$

İn the implementation, we select α even (typ. $\alpha = 2$ or 4) such that the basis functions in (A2. 19) are polynomial B-splines that are compactly supported. This allows us to use the spline model (A2. 19) to our full advantage in the backprojection part of the algorithm (A2. 17). The digital filtering part of the algorithm (A2. 20) is implemented in the fourier domain since the filter h_* has infinite support. The interesting aspect of the algorithm is that, once we have selected the spline model (A2. 18), all other aspects of the computation are exact. İn particular, the discretization of the ramp filter is achieved implicitly through (A2. 21).

The fractional splines offer the same conceptual case for dealing with fractional derivatives as the polynomial splines do with derivatives. İn the B-spline domain, fractional differentiation gets translated into simple fractional finite differences. This spline calculus provides a general tool fort he discretization and implementation of fractional derivative operators. The Ram-Lak filter, which plays a crucial role in tomography, corresponds to our symmetric differential operator $D_* \leftrightarrow |\omega|$. It is an non-local operator that can be implemented exactly provided that one has a spline representation of the projection data. İt is proposed a modification of the standart FBP algorithm that takes advantage of this property. İt is found that working with splines is also beneficial fort he back-projection part of the reconstruction process.

APPENDIX C

SOLUTION OF BESSEL EQUATION

The modified Bessel equation, which differs only in the sign of the third term, and which arises in a number of diffusion problems, is equally amenable to the approach considered here.

The equation

$$x^2 \frac{d^2\omega}{dx^2} + x\frac{d\omega}{dx} + [\,x - \frac{\nu^2}{4}\,]\,\omega = 0 \qquad \text{(A3. 1)}$$

is a form of Bessel's equation. As is the rule for second-order differential equations, its general solution is a combination of two linearly independent solutions $w1$ and w_2 of x, each of which depends on the parameter v. The usual method of solving (1) is via an infinite series approach, but we shall demonstrate how differentiation procedures lead to a ready solution in terms of elementary functions. We start by making either of the substitutions

$$\omega = x^{+-\frac{1}{2}} \qquad \text{(A3. 2)}$$

where v denotes the nonnegative square root of v^2, so that equation (1) is transformed to

$$x\frac{d^2u}{dx^2} + [1 \pm \nu]\frac{du}{dx} + u = 0 \qquad \text{(A3. 3)}$$

We next assume that for every function u that satisfies (3) there exists a differintegrable function f, related to u by the equation

$$u = \frac{d^{\frac{1}{2}\pm\nu} f}{dx^{\frac{1}{2}\pm\nu}} \qquad \text{(A3. 4)}$$

Moreover, use of equation

$$\frac{d^q f}{dx^q} = \frac{d^n}{dx^n}[\frac{d^{q-n} f}{dx^{q-n}}] \qquad \text{(A3. 5)}$$

where $\frac{d^n}{dx^n}$ effects ordinary n-fold differentiation and n is an integer chosen so large that q- $n < 0$, permits the combination of the equations (3) and (4) to give

$$x\frac{d^{\frac{5}{2}\pm\nu} f}{dx^{\frac{5}{2}\pm\nu}} + [1 \pm \nu]\frac{d^{\frac{3}{2}\pm\nu} f}{dx^{\frac{3}{2}\pm\nu}} + \frac{d^{\frac{1}{2}\pm\nu} f}{dx^{\frac{1}{2}\pm\nu}} = 0 \qquad \text{(A3. 6)}$$

Application of the Leibnitz rule allows the rewriting of equation (6) as

$$\frac{d^{\frac{5}{2}\pm\nu}\{xf\}}{dx^{\frac{5}{2}\pm\nu}} - \frac{3}{2}\frac{d^{\frac{3}{2}\pm\nu} f}{dx^{\frac{3}{2}\pm\nu}} + \frac{d^{\frac{1}{2}\pm\nu} f}{dx^{\frac{1}{2}\pm\nu}} = 0 \qquad \text{(A3. 7)}$$

wherein the parameter v is no longer present as a coefficient. We next plan to decompose the operators, thus

$$\frac{d^{\frac{1}{2}\pm\nu}}{dx^{\frac{1}{2}\pm\nu}}\frac{d^2\{xf\}}{dx^2} - \frac{3}{2}\frac{d^{\frac{1}{2}\pm\nu}}{dx^{\frac{1}{2}\pm\nu}}\frac{df}{dx} + \frac{d^{\frac{1}{2}\pm\nu} f}{dx^{\frac{1}{2}\pm\nu}} = 0 \qquad \text{(A3. 8)}$$

an equation directly convertible to

$$\frac{d^2\{xf\}}{dx^2} - \frac{3}{2}\frac{df}{dx} + f = 0 \qquad \text{(A3. 9)}$$

by the action of the $\frac{d^{-\frac{1}{2}\mp\nu}}{dx^{-\frac{1}{2}\mp\nu}}$ operator. Equations (8) and (7) are equivalent to each other if and only if

$$[xf]_{x=0} = 0 \text{ and } [\frac{d\{xf\}}{dx}]_{x=0} = 0 \tag{A3. 10}$$

$$f(0) = 0 \tag{A3. 11}$$

whereas (9) and (8) are equivalent if

$$\frac{d^{-\frac{1}{2}\mp\nu}}{dx^{-\frac{1}{2}\mp\nu}} \frac{d^{\frac{1}{2}\pm\nu} g}{dx^{\frac{1}{2}\pm\nu}} = g \text{ with g=f, } \frac{df}{dx}, \frac{d^2\{f\}}{dx^2} \tag{A3. 12}$$

Conversion of equation (9) to the canonical form

$$\frac{d^2 f}{[d(2\sqrt{x})]^2} + f = 0 \tag{A3. 13}$$

is straightforward, whereby it follows that the two possible candidate functions *f* are

$$f_1 = \sin(2\sqrt{x}) \text{ and } f_2 = \cos(2\sqrt{x})$$

We must now inquire which, if either, of these candidate functions satisfies the requirements (10), (11), and (12), which we assumed held during our derivation. Because

$$\cos(2\sqrt{x}) = 1 - 2x + \frac{2}{3}x^2 - ...$$

it is evident that f_2 fails to meet requirement (10) or (11) and must be rejected. However, f_1 passes these tests. The requirement

$$\frac{d^{-\frac{1}{2}+\nu}}{dx^{-\frac{1}{2}+\nu}}\frac{d^{\frac{1}{2}-\nu}g}{dx^{\frac{1}{2}-\nu}} = g \text{ with g=f, } \frac{df}{dx}, \frac{d^2\{xf\}}{dx^2}$$

one part of (12), is met by function

$$\sin(2\sqrt{x}) = 2x^{\frac{1}{2}} - \frac{4}{3}x^{\frac{3}{2}} + \frac{4}{15}x^{\frac{5}{2}} - \ldots$$

for all values of ν (recall that we restricted ν to negative values), while the other part,

$$\frac{d^{-\frac{1}{2}-\nu}}{dx^{-\frac{1}{2}-\nu}}\frac{d^{\frac{1}{2}+\nu}g}{dx^{\frac{1}{2}+\nu}} = g \text{ with g=f, } \frac{df}{dx}, \frac{d^2\{xf\}}{dx^2}$$

is met by f_1 for all ν values except the nonnegative integers. Returning to equation (4) then, we conclude that the function

$$u_1 = \frac{d^{\frac{1}{2}-\nu}}{dx^{\frac{1}{2}-\nu}}\sin(2\sqrt{x})$$

is a solution to equation (3) then for all ν values, and that

$$u_2 = \frac{d^{\frac{1}{2}+\nu}}{dx^{\frac{1}{2}+\nu}}\sin(2\sqrt{x})$$

is another solution when ν is not an integer. Our sought solutions to the original Bessel equation are thus

$$\omega_1(\nu,x) = x^{-\frac{1}{2}\nu} u_1 = x^{-\frac{1}{2}\nu} \frac{d^{\frac{1}{2}-\nu} \sin(2\sqrt{x})}{dx^{\frac{1}{2}-\nu}}, \text{ all } \nu \geq 0$$

and

$$\omega_2(\nu,x) = x^{\frac{1}{2}\nu} u_2 = x^{\frac{1}{2}\nu} \frac{d^{\frac{1}{2}+\nu} \sin(2\sqrt{x})}{dx^{\frac{1}{2}+\nu}}, \ 0 \leq \nu \neq 1,2,...$$

The problem is now completely solved, except that a second solution is needed for integer v values. Our technique cannot reveal this second solution. The relationship of w_1 and w_2 to the conventional notation for Bessel functions is Simply

$$\omega_1(\nu,x) = \sqrt{\pi} J_{-\nu}(2\sqrt{x}) \text{ and } \omega_2(\nu,x) = J_\nu(2\sqrt{x}).$$