HIGHER CURVATURE GRAVITY IN LARGE EXTRA DIMENSIONS: PHENOMENOLOGICAL IMPLICATIONS

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ABSTRACT

HIGHER CURVATURE GRAVITY IN LARGE EXTRA DIMENSIONS: PHENOMENOLOGICAL IMPLICATIONS

This thesis is devoted to a detailed study of the higher curvature gravity and its phenomenological implications in large extra dimensions. This work is intended as a discussion of effective interactions among brane matter induced by modifications of higher dimensional Einstein gravity via the replacement of Einstein-Hilbert term with a generic function $f(\mathcal{R})$ of the curvature scalar $\mathcal{R}$.

In this work, following the introductory chapters on extra dimensions and higher curvature gravity in large extra dimensions, we derive the graviton propagator and then we analyze impact of virtual graviton exchange on interactions among brane matter. We find that $f(\mathcal{R})$ gravity effects are best probed by high-energy processes involving massive gauge bosons, heavy fermions or the Higgs boson. We perform a comparative analysis of the predictions of $f(\mathcal{R})$ gravity and of Arkani–Hamed, Dimopoulos and Dvali (ADD) scenario, and find that the former competes with the latter when $f''(0)$ is positive and comparable to the fundamental scale of gravity in higher dimensions (Demir and Tanyıldızı 2006). In addition, we briefly discuss graviton emission from the brane as well as its decays into brane-localized matter and we find that they hardly compete with the ADD expectations.

Consequently, we discussed that possible existence of higher-curvature gravitational interactions in large extra spatial dimensions opens up various signatures to be confronted with existing and future collider experiments.

ÖZET

UZUN EK BOYUTLARDA YÜKSEK EĞRİLİKLİ ÇEKİM: FENOMENOLOJİK UYGULAMALAR

Bu tez, uzun ek boyutlarda yüksek eğrilikli çekimin ve onun fenomenolojik uygulamalarının ayrıntılı bir çalışmasına adanmıştır. Bu tez, Einstein-Hilbert terimindeki eğrilik skaleri $\mathcal{R}$ yerine $\mathcal{R}$'nin bir genel fonksiyonu olan $f(\mathcal{R})$'nin konulması yoluyla yüksek boyutlu Einstein kütleçekiminin değiştirilmesiyle oluşan, zar maddeleri arasındaki etkin etkileşimlerin bir tartışması olarak tasarlanmıştır.

Bu çalışmada, ek boyutlarla ve uzun ek boyutlarda yüksek eğrilikli çekimle ilgili bölümleri takiben, graviton yayıcıyı türettik ve daha sonra parçacık etki- leşimlerindeki sanal graviton alışverişinin gücünü çözümledik. $f(\mathcal{R})$ kütleçekim etkilerinin, kütleli ayar bosonlarını, ağır fermionları ya da Higgs bosonunu içeren yüksek-enerji yöntemleriyle en iyi şekilde çözümlendiği sonucuna vardık. $f(\mathcal{R})$ kütleçekim öngörülerinin ve Arkani-Hamed-Dvali-Dimopoulos (ADD) senaryosu tahminlerinin karşılaştırmalı bir çüzümlemesini canlandırdık ve $f''(0)$ artı değerli ve yüksek boyutlardaki kütleçekiminin temel ölçeğiyle karşılaştırılabilir olduğunda, birincisinin sonrakiyle çekiştiğini bulduk (Demir ve Tanyıldızı 2006). Ayrıca, gravitonun zarda lokalize maddeye bozunumunun yanısıra zardan graviton yayılmasını da kısaca tartıştık ve bunların ADD beklentileriyle hemen hemen hiç çekişemediğini bulduk.

Son olarak uzun ek uzaysal boyutlarda yüksek eğrilikli kütleçekim etkileşimlerinin olası varlığının günümüzdeki ve gelecekteki çarpıştırıcı deneylerinde karşılaşabileceğimiz çeşitli işaretler sunduğunu tartıştık.

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CHAPTER 1

INTRODUCTION

Problem of extra dimensions has been of great importance in high energy physics during the last decade since this framework has given a new solution to the Higgs mass problem without any contradiction with the expectations of physics below 1 TeV.

In this work, first, we step to explain what the extra dimensions are, then we clarify the Kaluza–Klein approach to the unification of electromagnetism and gravitation, and finally we emphasize the way that physicists prompt to think about, using the large extra dimensions, to solve the Higgs mass problem.

The relative feebleness of gravity with respect to the weak force and its stability under quantum fluctuations, the gauge hierarchy problem, has been pivotal for introducing a number of 'new physics' models to complete the standard electroweak theory (SM) above Fermi energies. The idea (Witten 1996, Horava and Witten 1996a, Horava and Witten 1996b, Antoniadis 1990) that the scale of quantum gravity can be much lower than the Planck scale, possibly as low as the electroweak scale itself (Lykken 1996, Arkani–Hamed et al. 1998, Arkani–Hamed et al. 1999) (see also the recent standard-like models found in intersecting D-brane models (Cremades et al. 2002, Kokorelis 2004)) since this extreme is not excluded by the present experimental bounds (Long et al. 1999, Long and Price 2003), has opened up novel lines of thought and a number of phenomena which possess observable signatures in laboratory, astrophysical and cosmological environments.

The basic setup of the Arkani–Hamed–Dimopoulos–Dvali (ADD) scenario (Arkani–Hamed et al. 1998, Arkani–Hamed et al. 1999) is that (1+3)–dimensional universe we live in is a field-theoretic brane (Rubakov and Shaposhnikov 1983) which traps all flavors of matter except the SM singlets $e.g.$ the graviton and right-handed neutrinos. As long as the surface tension of the brane does not exceed the fundamental scale $\overline{M}_D$ of D–dimensional gravity, at distances $\gg 1/\overline{M}_D$ the spacetime metric g_{AB} remains essentially flat. In other words, for singlet emissions (from brane) with transverse (to brane) momenta $|\vec{p}_T| \ll \overline{M}_D$ the background spacetime is basically Minkowski. Therefore, it is admissible to expand D–dimensional metric about a flat background

$$g_{AB} = \eta_{AB} + 2\overline{M}_D^{1-D/2} h_{AB} \tag{1.1}$$

where $\eta_{AB} = \text{diag}\,(1,-1,-1,\cdots,-1)$ and h_{AB} are perturbations (see Chapter 4 and Sec. 5.2). The gravitational sector is described by Einstein gravity

$$S_{ADD} = \int d^D x \sqrt{-g} \left\{ -\frac{1}{2} \overline{M}_D^{D-2} \mathcal{R} + \mathcal{L}_{matter}\left(g_{AB}, \psi\right) \right\} \tag{1.2}$$

where ψ collectively denotes the matter fields localized on the brane. There are various ways (Arkani–Hamed et al. 1998, Arkani–Hamed et al. 1999) to see that the Planck scale seen on the brane is related to the fundamental scale of gravity in higher dimensions via

$$\overline{M}_{Pl} = \sqrt{V_\delta}\, \overline{M}_D^{1+\delta/2} \tag{1.3}$$

which equals $(2\pi R)^{1/2} \overline{M}_D^{1+\delta/2}$ when $\delta \equiv D-4$ extra spatial dimensions are compactified over a torus of radius R (see Chapter 4). Obviously, larger the R closer the $\overline{M}_D$ to the electroweak scale (Arkani–Hamed et al. 1998, Arkani–Hamed et al. 1999). Experimentally, size of the extra dimensions, R, can be as large as a small fraction of millimeter (Long et al. 1999, Long and Price 2003), and thus, quantum gravitational effects can already show up at experimentally accessible energy domains provided that the strength of gravitational interactions on the brane drives from higher dimensional gravity as in (1.3). Upon compactification, the higher dimensional graviton gives rise to a tower of massive S, P and D states on the brane, and they participate in various scattering processes involving radiative corrections to SM parameters, missing energy signals as well as graviton exchange processes. These processes and their collider signatures have been discussed in detail in seminal papers (Giudice et al. 1999, Han et al. 1999).

The ADD mechanism is based on higher dimensional Einstein gravity with metric (1.1). Given the very fact that general covariance does not forbid the action density in (1.2) to be generalized to a generic function $f\big(\mathcal{R},\, \Box\mathcal{R},\, \nabla_A\mathcal{R}\nabla^A\mathcal{R},\, \mathcal{R}_{AB}\mathcal{R}^{AB},\, \mathcal{R}_{ABCD}\mathcal{R}^{ABCD}, \dots\big)$ of curvature invariants, in this work we will derive and analyze effective interactions among brane matter induced by such modifications of higher dimensional Einstein gravity, and compare them in strength and structure with those predicted by the ADD mechanism. The simplest generalization of (1.2) would be to consider, as we will do in what follows, a generic function $f(\mathcal{R})$ of the curvature scalar. Such modified gravity theories are known to be equivalent to Einstein gravity (with the same fundamental scale) plus a scalar field theory with the scalar field

$$\phi = \overline{M}_D^{(D-2)/2} \sqrt{\frac{D-1}{D-2}} \log \left| \frac{\partial f}{\partial R} \right| \tag{1.4}$$

in a frame accessible by the conformal transformation $g_{AB} \to (\partial f/\partial\mathcal{R})g_{AB}$ (Barrow and Cotsakis 1988, Kalara et al. 1990, Maeda 1989, Magnano and Sokolowski 1994). Therefore, generalized action densities of the form $f(\mathcal{R})$ are equivalent to scalar-tensor theories of gravity, and thus, matter species are expected to experience an additional interaction due to the exchange of the scalar field ϕ (Brans and Dicke 1961). This is the fundamental signature of $f(\mathcal{R})$ gravity compared to Einstein gravity for which simply $f(\mathcal{R}) = \mathcal{R}$. Though remains outside the scope of this work, see the discussions of Lovelock higher-curvature terms in (Rizzo 2005a, Rizzo 2005b).

In this work we study how $f(\mathcal{R})$ gravity influences interactions among brane matter and certain collider processes to observe them. In Sec. 5.2 below we derive graviton propagator and describe how it interacts with brane matter. Here we put special emphasis on virtual graviton exchange. In Sec. 6.1 we study a number of higher dimensional operators which are sensitive to $f(\mathcal{R})$ gravity effects. In Chapter 6 we briefly discuss some further signatures of $f(\mathcal{R})$ gravity concerning graviton production and decay as well as certain loop observables on the brane. Consequently, in Chapter 7 we conclude.

CHAPTER 2

EXTRA DIMENSIONS

There is no compelling reason to claim that spacetime is four and only four dimensional. It is actually a matter of system's energetics to decide on dimensionality of spacetime. A butterfly walking on a surface with jelly feels only a two–dimensional space plus time. If it is energetic enough, however, it can escape from the surface and fly away. In this case it starts feeling a 3–dimensional space plus time. It is in this sense that extra dimensions can indeed exist but we may not feel them due to insufficient energy budget we have. Therefore, extra dimensions can exits yet they can be too small to sense by low-energy phenomena.

The surface with jelly mentioned above is an imperfect analogy of the concept of ”brane”. Brane localizes sources of all the force fields we know of . For example, electric charges are localized on the brane and hence the electromagnetic field with its well-known $1/r^2$ behaviour.

In Nature, not all fields and forces are localized on a brane, however. In general “neutral” (not necessarily in electromagnetic sense, i.e. neutral particle means the particle transmits neutral current for its own symmetry law) particles cannot be localized on a brane. Right-handed neutrino, being a complete singlet under Stantdard Model gauge group, is free to wander in entire space. There is nothing special about brane for right-handed neutrino. The other example is graviton which is a singlet under all gauge groups and it is free to propagate in entire space. (Giudice et al. 1999 and Arkani–Hamed et al. 1998)

Consequently, we can drive entire matter as those that one localized on the brane and those that are not. It is via singlets that we can probe the extra dimensions since they are the fields which can sense the extra space off the brane. In particular, graviton modulates gravitational interactions in 4– and higher–dimensional spacetimes.

Experimentally, we are certain that the gravity obeys Newtonian inverse–square law down to distances 10^{-4}cm (Long et al. 1999, Long and Price 2003). However, Newtonian behavior can fail below these distances due to extra dimensional effects. This constitutes motivation as well as phenomenological relevance of extra dimensions. Besides, there have been various theoretical motivations for considering of extra dimensions. These may be summarize as follows:

- Unification of gravity and gauge interactions of elementary particles was the first scientific exploration which has been proposed by Kaluza (Kaluza 1921) and Klein (Klein 1926) (see Chapter 3). The theory of Kaluza–Klein is based on such an idea that gravitational and electromagnetic interactions may be descendants of a common origin. Therefore, unification of gravity and gauge interactions of elementary particles is the first reason of why extra dimensions are studied.

- String theory, believed to be the correct quantum theory of gravity, cannot be formulated consistently without introduction of extra dimensions. In M theory, for instance, one formulates an 11–dimensional supergravity as the quantum theory of gravity. Hence, the second reason to study extra dimensions stems from quantization of gravity.

- In standard model of electroweak interactions (SM), the Higgs boson mass diverges quadratically with the ultraviolate (UV) scale. Arkani–Hamed, Dimopoulos and Dvali in 1998 (Arkani–Hamed et al. 1998, Arkani–Hamed et al. 1999, Antoniadis et al. 1998), addressed this problem in the framework of large extra dimensions – as large as experiments permit. Another solution was proposed by Randall and Sundrum in 1999 (Randall and Sundrum 1999, Dvali et al. 2000). Thus the third reason for considering extra dimensions stems from solving the Higgs mass problem.

In the next chapter we will mainly dwell on unification of gravity and electromagnetism i.e. Kaluza–Klein approach. In other chapters to follow, we will discuss large extra dimensions motivated by quadratic divergence of the Higgs boson mass. Quantum theory of gravity and string theory will not be discussed in this thesis work.

CHAPTER 3

KALUZA–KLEIN APPROACH

We consider the higher–dimensional unification from the general relativity perspective rather than the particle physics side. Ideas of Nordström (Nordström 1914) in 1914 and independently Kaluza (Kaluza 1921), which were inspired by the close ties between Minkowski's 4D–spacetime and Maxwell's unification of electricity and magnetism, were the first attempts to unify gravity with electromagnetism in an extra dimensional theory. According to their approach, the universe is a four–dimensional hypersurface in a five–dimensional spacetime.

Kaluza demonstrated that general relativity, which is interpreted as a five–dimensional theory in vacuum (i.e. ${}^5G_{AB} = 0$), contains four–dimensional general relativity in the presence of an electromagnetic field (i.e. $G_{\mu\nu} = \mathcal{T}^{EM}_{\mu\nu}$, where $\mu, \nu = 0, 1, 2, 3$). That is to say, all derivatives with respect to x^4 in five–dimensional spacetime vanish due to an unknown physical reason (the infamous cylinder condition). These assumptions provide very useful tools for obtaining the field equations of both electromagnetism and gravity from the five–dimensional spacetime successfully. Kaluza's realization shows that five–dimensional general relativity contains both Einstein's four–dimensional theory of gravity and Maxwell's theory of electromagnetism. The cylinder condition, which he imposed as an artificial restriction on the coordinates, barred the direct appearance of the fifth dimension in the laws of physics.

On the other hand, Klein's idea (Klein 1926a, Klein 1926b) modifies Kaluza's five–dimensional scheme by introducing 'compactified extra dimension' in that x^4–independence of physical phenomena in 4D is attributed to smallness and periodicity of x^4. In other words, his contribution was to make Kaluza's restriction less artificial through the compactification of the fifth dimension by suggesting a plausible physical basis for the theory. In this sense, extra spatial dimensions are thought to be curled-up, or compactified. Going back to the example with the butterfly in Chapter 2, let us roll up the jelly surface. The butterfly then starts crawling in the direction of the curvature. It will eventually come back to the same point it started from. According to compactified extra dimensional theories, we live in a universe where our three familiar spatial dimensions are nearly flat, but there are additional dimensions which are curled-up very tightly so that they have an extremely small radius.

According to the gauge-invariant point of view, Kaluza–Klein's compactification process effectively means that the isometries of the extra space, used in constructions where one space is embedded in another space, such as rotational invariance, appears as the continuous Kaluza–Klein invariance in 4–dimensions (see figure 3.1). Hence, an electromagnetic field appears as a vector gauge field in 4–dimensions.

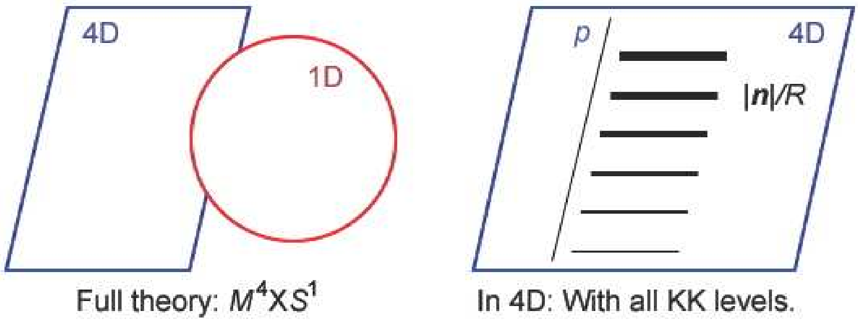

Figure 3.1. Full theory: $M^4 \times S^1$ with rotational invariance on S^1. In 4D: A tower of Kaluza-Klein levels with $U(1)$ invariance corresponding to S^1 invariance along extra dimension. Electromagnetic gauge symmetry is explained as a geometric symmetry of five–dimensional spacetime.

3.1. Kaluza's Approach to Higher–Dimensional Unification

Kaluza unifies gravity with Maxwell's unification of electricity and magnetism, applying Einstein's general theory of relativity. Kaluza's unification is an application of Einstein's general theory of relativity to a five–dimensional spacetime manifold. The five–dimensional Einstein equations are derived, by varying the five–dimensional Einstein action:

$$S = -\frac{1}{16\pi\, {}^5G} \int \sqrt{-\, {}^5g}\; {}^5\mathcal{R}\, d^4x\, dy \tag{3.1}$$

where ${}^5G$ is the five–dimensional gravitational constant.

Then the Einstein equations in 5D–spacetime are defined by:

$${}^5G_{AB} \equiv {}^5\mathcal{R}_{AB} - \frac{1}{2}\, {}^5g_{AB}\, {}^5\mathcal{R} = 0 \tag{3.2}$$

where ${}^5G_{AB}$ is the five–dimensional Einstein tensor, ${}^5\mathcal{R}_{AB}$ is the five–dimensional Ricci tensor, ${}^5\mathcal{R}$ is the five–dimensional Ricci scalar and ${}^5g_{AB}$ is the five–dimensional metric tensor.

The absence of an energy-momentum tensor implies the absence of the matter, which in turn becomes the first assumption of Kaluza in 5D–spacetime. The assumption is that the space in higher dimensions is empty.

The five–dimensional connection coefficients and Ricci tensor in terms of the five–dimensional metric are defined as:

$$ {}^5\Gamma^C_{AB} = \frac{1}{2}\,{}^5g^{CD}\left(\partial_A\,{}^5g_{BD} + \partial_B\,{}^5g_{AD} - \partial_D\,{}^5g_{AB}\right) \tag{3.3} $$

and

$$ {}^5\mathcal{R}_{AB} = \partial_C\,{}^5\Gamma^C_{AB} - \partial_B\,{}^5\Gamma^C_{AC} + \,{}^5\Gamma^C_{AB}\,{}^5\Gamma^D_{CD} - \,{}^5\Gamma^C_{AD}\,{}^5\Gamma^D_{BC} \tag{3.4} $$

Following Kaluza let's consider five–dimensional Riemanian metric in a special form:

$$ {}^5g_{AB} = \begin{pmatrix} g_{\mu\nu} + \kappa^2\phi^2 A_\mu A_\nu & \kappa\phi^2 A_\mu \\ \kappa\phi^2 A_\nu & \phi^2 \end{pmatrix} \tag{3.5} $$

where κ is a constant multiplicative factor in the action, A_α is the electromagnetic potential and ϕ is a scalar field.

After that, by applying the cylinder condition as the third assumption of Kaluza's unification approach, all derivatives with respect to the fourth spatial dimension are dropped. Then, using the field equations (3.2), Ricci tensor (3.4) and gravitational field (3.5) in five-dimensional representation, the corresponding field equations (Lessner 1982, Thiry 1948) are found:

$$ \begin{aligned} G_{\mu\nu} &= \frac{\kappa^2\phi^2}{2}\mathcal{T}^{EM}_{\mu\nu} - \frac{1}{\phi}\left[\nabla_\mu\left(\partial_\nu\phi\right) - g_{\mu\nu}\Box\phi\right] \\ \nabla^\mu F_{[\mu 4]} &= -3\frac{\partial^\mu\phi}{\phi}F_{[\mu 4]} \quad , \quad \Box\phi = \frac{\kappa^2\phi^3}{4}F_{44}F^{44} \end{aligned} \tag{3.6} $$

where

$$ G_{\mu\nu} \equiv \mathcal{R}_{\mu\nu} - \frac{1}{2}g_{\mu\nu}\mathcal{R} \tag{3.7} $$

is the four–dimensional Einstein tensor,

$$ \mathcal{T}^{EM}_{\mu\nu} = \frac{1}{4}g_{\mu\nu}F_{\lambda\rho}F^{\lambda\rho} - F^{\lambda}_{\mu}F_{\nu\lambda} \tag{3.8} $$

is the electromagnetic energy-momentum tensor,

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \tag{3.9}$$

is the field strength tensor of the theory, and the scaling factor κ is identified as $4\sqrt{\pi G}$ in terms of the four–dimensional gravitational constant. In addition, there are fifteen independent Einstein equations because of the fact that the chosen five–dimensional metric is a symmetric tensor. Under these conditions, if the scalar field ϕ is chosen as a constant over all spacetime manifold, the Einstein tensor and the unified Maxwell equations become, respectively,

$$G_{\mu\nu} = 8\pi G\, \phi^2 \mathcal{T}^{EM}_{\mu\nu} \quad , \qquad \nabla^\mu F_{\mu\nu} = 0 \tag{3.10}$$

in four–dimensions. Indeed, the same thing has been done by some authors in subsequent work employing special coordinate systems (Kaluza 1921). The equations (3.10) are originally obtained by Kaluza–Klein, where $\phi = 1$. However, as $F_{\mu\nu}F^{\mu\nu} = 0$, the condition of constant ϕ is only consistent with the equation

$$\Box\phi = \frac{\kappa^2\phi^3}{4} F_{44}F^{44} \tag{3.11}$$

in the equation (3.6), as it was first pointed out by Jordan (Bergmann 1948, Jordan 1947) and Thiry (Thiry 1948).

The same results can be obtained by variation of the action which includes only four–dimensional Lagrange density. The Lagrange density can be obtained considering the cylinder condition:

$$S = \int d^4x\sqrt{-g}\, \phi \left(-\frac{\mathcal{R}}{16\pi G} - \frac{1}{4}\phi^2 F_{\mu\nu}F^{\mu\nu} - \frac{2}{3}\frac{1}{16\pi G}\frac{\partial^\mu\phi\partial_\mu\phi}{\phi^2} \right) \tag{3.12}$$

where

$${}^5G = G\int dy \tag{3.13}$$

is five–dimensional gravitational constant in terms of Newton's gravitational constant. Under the restriction of ϕ being a constant, the first two terms of the action (3.12) are the Einstein-Maxwell action, which is scaled by ϕ, for gravity and electromagnetic radiation. The third term is the action for the Klein-Gordon field equation with a massless scalar field.

The fact that action (3.1) leads to (3.12), or equivalently, the field equations (3.2) without a source lead to (3.6) with a source, shows that the four–dimensional matter or the electromagnetic radiation is generated from the geometry.

To complete the meaning of Kaluza–Klein's ideas, we should emphasize the situation for vanishing $A_\mu = 0$. If ϕ is a variable (rather than a spacetime constant) then Kaluza's five–dimensional theory contains an additional scalar field besides electromagnetism. This would be no more than a choice of coordinates and would not entail any loss of algebraic generality. Conversely, with the cylinder condition, by working in a special set of coordinates, the theory is no longer invariant with respect to five–dimensional coordinate transformations. As a consequence, $A_\mu = 0$ becomes a physical restriction, and it restricts us to the gravi-scalar sector of the Kaluza–Klein's unification.

This is acceptable in the context which entails homogeneous and isotropic situations. For instance, this case could occur when off-diagonal metric coefficients pick up preferred directions.

3.2. Klein's Compactification Mechanism

Klein maneuvered the idea that extra dimensions do exist as new spatial coordinates but they do not play a role in 4D physics. He did this with two assumptions: First assumption is that the extra spatial dimension has a circular topology (S^1) and second assumption is that it is small, that is, one needs considerable amount of energy to detect or feel them. Under the first assumption, any quantity $f(x^\mu, y)$, where $x^\mu = (x^0, x^1, x^2, x^3)$ and $y = x^4$, turns out to be a periodic function of extra spatial dimension. Therefore, $f(x^\mu, y) \to f(x^\mu, y + 2\pi R)$ where R is the radius of the extra space S^1. Here, from the point of view of Kaluza–Klein, f is a generic field in five–dimensional spacetime. Due to periodicity condition f admits a Fourier series expansion

$$f(x^\mu, y) = \sum_{n=-\infty}^{+\infty} f^{(n)}\,(x^\mu) \exp\left[i\frac{n\,y}{R}\right] \tag{3.14}$$

from which we infer that:

- The lowest mode $n = 0$ (the so–called zero–mode) is independent of y. This inference stems directly from the compact nature of the extra space i.e. S^1 on which the extra dimension y extends.

- Higher modes $n \neq 0$ (the so–called higher harmonics) depend explicitly on y with a wavelength (or equivalently inverse–mass, in natural units) $\lambda^{(n)} = R/n$ for n–th harmonic.

- The x^μ–dependence of $f(x^\mu, y)$ does not exhibit any periodicty at all because these macroscopic dimensions are not compact; they extend to infinity in both directions. Thus, the energy spectrum of $f(x^\mu, y)$ in four–dimensional spacetime can be extracted via Fourier intergral rather than Fourier series, considering the case in this item the fact depends on boundary conditions for field $f(x^\mu, y)$.

It is clear that first item above, the one about the zero–mode, comprises Kaluza's cylinder condition. The second item, the other about the non–zero–modes, tells us that higher harmonics can be hidden from present–day experiments as they may not have reached yet the energies $\sim (n/R)$ which is the main reason behind assuming radius R to be small.

Therefore, components of metric can be Fourier-expanded as

$$g_{\mu\nu}(x, y) = \sum_{n=-\infty}^{+\infty} g_{\mu\nu}^{(n)}(x) \exp\left[i \frac{n\, y}{R}\right] \quad , \qquad \phi(x, y) = \sum_{n=-\infty}^{+\infty} \phi^{(n)} \exp\left[i \frac{n\, y}{R}\right]$$

$$A_\mu(x, y) = \sum_{n=-\infty}^{+\infty} A_\mu^{(n)}(x) \exp\left[i \frac{n\, y}{R}\right] \tag{3.15}$$

where the (n) is the nth Fourier mode. Obviously, the zero–modes of $g_{\mu\nu}(x, y)$, $A_\mu(x, y)$, $\phi(x, y)$ don't carry a momentum into the extra space. However, every other mode in the Kaluza–Klein theory carries a momentum of the order $|\vec{n}|/R$, through the extra dimension. If the radius of the extra dimension is small enough, the x^4 component of the momenta becomes sufficiently large even for $n = 1$.

The zero–modes of $g_{\mu\nu}(x, y)$ and $A_\mu(x, y)$ are nothing but the fields which have already been established by experiments, that is, they are the fields which are strictly bound to live in M^4. On the other hand, their higher harmonics do have a sinusoidal extension into the extra space with a wavelength decreasing with increasing Kaluza–Klein index, n. For instance, to be able to disentangle effects of $g^{(9)}(x)$ on a physical process it is necessary to have a collider with a characteristic energy $\sim 9/R$ apart from additional effects that might come from strength of its coupling to colliding matter species.

The radius R is taken usually equal to the Planck length $\ell_{Pl} \sim 10^{-35}$ m because it is both a natural value and small enough to guarantee that the mass of any $n \neq 0$ Fourier modes lies beyond the Planck mass $M_{Pl} \sim 10^{19}$ GeV.

In the higher–dimensional unification, in the sense of metric decomposition (3.5) and action (3.12) there are three key features (Overduin 1997, Puliçe 2006):

- The electromagnetic and gravitational fields are contained in the higher dimensional Einstein tensor ${}^{(4+\delta)}G_{AB}$, that is, in the metric and its derivatives. Therefore, there is no need to have an explicit source of energy and momentum ${}^{(4+\delta)}\mathcal{T}_{AB}$. In this sense, matter species in four–dimensions, follow from pure geometry.

- The simple S^1 compactification illustrated above has extension to more extra dimensions assuming extra dimensions are curled up to form a compact manifold.

Having motivated by use and possible need to extra dimensions via Kaluza-Klein approach to unifying gravity and electromagnetism, we now turn to another possible application of extra dimensions, that is, their role in restricting existing gauge theories at the electroweak scale.

CHAPTER 4

EINSTEIN GRAVITY IN LARGE EXTRA DIMENSIONS

In this chapter our goal is to provide an overview of Einstein's theory of gravitation in spacetimes with number of extra dimensions δ. The physical interest to this stems from (Antoniadis et al. 1998, Arkani-Hamed et al. 1998) which have shown that large extra dimensions (as large as experimental sensitivity to possible failure of Newtonian law of attraction permits) can provide a novel solution for taming the quadratic UV-sensitivity of the Higgs boson mass. The other famous solution to this problem, not mentioned in length in this thesis work, refers to supersymmetry which protects Higgs boson mass against quadratic divergences just like chiral symmetry does fermion masses. Therefore, large extra dimensions, as proposed originally by Arkani-Hamed, Dvali and Dimopoulos, provide a novel way of taming the Higgs mass, and predicts a number of phenomena not expected in supersymmetric theories.

In the presence of extra dimensions, law of gravitational attraction between massive bodies is expected to deviate from 4-dimensional structure at distances comparable to characteristic size of the extra dimensions. In fact, it turns out that we know very well how gravity works at large distances. However, no one has tested how well known laws of gravitational attraction works at distances less than about 1 mm. It is complicated to study gravitational interactions at small distances. Objects positioned so close to each other must be very small and very light, so their gravitational interactions are also small and hard to detect (due to several noise sources such as electromagnetic effects or van der Waals forces). While a new generation of gravitational experiments that should be capable of probing Newton's law at short distances (up to 1 micron) is under way, our current knowledge about gravity stops at distances of the order of 1 mm. We currently cannot say whether there are, or are not, possible extra dimensions smaller than 1 mm.

The fundamental novelty in ADD approach is to identify the electroweak scale ($\sim 1\ \mathrm{TeV}$) with Newton's constant of gravity (or fundamental scale of gravity) in higher dimensions so that well-known Newton's constant of gravitation (or Planck scale) in 4D is just a derived concept, as detailed below. The most important issue is to relate Newton's constants of 4– and $(4+\delta)$–dimensional gravity theories. Following, the ADD paper (Arkani-Hamed et al. 1999) we now provide two alternative ways to establish this relation:

- The Einstein field equations follow from extremization of Eintein-Hilbert action

$$S_{4D} = \int d^4x\sqrt{-g}\left\{-\frac{1}{2}M_{Pl}^2\mathcal{R}\left(x\right)+\mathcal{L}_{matter}\right\} \tag{4.1}$$

  with respect to metric field $g_{\mu\nu}(x)$. This action describes relation between matter distribution (whose lagrangian not shown here) and resulting gravitational field. It remains stationary against fluctuations in metric field if Einstein field equations are satisfied.

  The analog of (4.1) in a higher dimensional spacetime reads as

$$S = \int d^4x d^\delta y\sqrt{-g}\left\{-\frac{1}{2}M_D^{D-2}\mathcal{R}\left(x,y\right)\right\} \tag{4.2}$$

  where we separated coordinates pertaining to 4D ($x^\mu, \mu = 0,1,2,3$) from those doing to extra dimensions ($y^i, i = 1,\ldots,\delta$). Obviously, the curvature scalar is a function of x and y. The difference from (4.1) is that every single field (metric and suppressed matter fields) in the lagrangian density depends on both x^μ and y^i.

  The details of integration over $\vec{y}$ in (4.2) depend on in what space $\vec{y}$ are taking values. In other words, geometry and topology of extra space are of prime importance. Here, we follow ADD approach and consider the torus depicted in Fig. 4.1. The

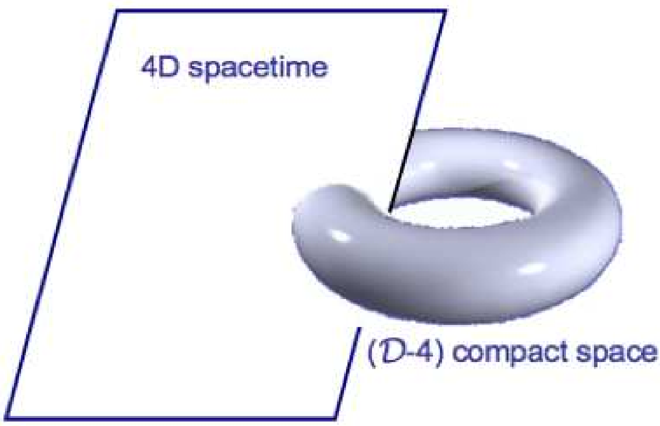

Figure 4.1. Spacetime structure employed in ADD scenario.

compactness of the extra space guarantees that there is a zero-mode for metric and hence all geometrical fields. This enables one to make the reduction

$$\begin{aligned} S &= \int d^4x \int_0^{2\pi R} dy_1 \int_0^{2\pi R} dy_2 ... \int_0^{2\pi R} dy_\delta \sqrt{-g} \left\{ -\frac{1}{2} M_D^{D-2} \mathcal{R}(x) \right\} \\ &= \int d^4x \sqrt{-g} \left\{ -\frac{1}{2} M_D^{D-2} (2\pi R)^\delta \mathcal{R}_D(x) \right\} \end{aligned} \quad (4.3)$$

by considering only the zero-mode. At this point, all one has to do is to compare this very action with that of gravity in 4D, which gives

$$\frac{1}{2} M_{Pl}^2 = \frac{1}{2} M_D^{\delta+2} (2\pi R)^\delta \quad (4.4)$$

and hence

$$M_{Pl}^2 = M_D^{\delta+2} (2\pi R)^\delta \quad . \quad (4.5)$$

This result tells us that gravity in 4D proceeds with the well-known fundamental constant M_{Pl}, and this very quantity is actually a derived one in that it is related to volume of the extra space $V_\delta = (2\pi R)^\delta$ and fundamental scale of gravity in $4+\delta$ dimensions, M_D.

The basic claim of ADD scenario is that when $M_D \sim 1$ TeV one can find M_{Pl} correctly for $R \sim 1$ mm for relatively low values of δ.

- Another way to relate fundamental scales of gravity in 3D space and higher dimensions proceeds via comparison of respective Newtonian laws of attraction. Indeed, two bodies of masses m_1 and m_2 separated by a distance R attract each other via

$$\begin{aligned} F_{(3+\delta)} &= G_{N(3+\delta)} \frac{m_1 m_2}{R^{\delta+2}} \\ F_{(3)} &= G_{N(3)} \frac{m_1 m_2}{R^2} \end{aligned} \quad (4.6)$$

in $(3+\delta)$ and 3 dimensions, respectively.

This method is one of the easiest derivations, and is a trivial application of Gauss' law. Let us compactify the new δ-dimensions y_δ by making the periodic identification $y_\delta \sim y_\delta + L$. Then assume that a point mass m is placed at the origin. This situation can be reproduced in the uncompactified theory by placing "mirror" masses periodically in all the new dimensions. Of course, for a test mass at distances $r \ll L$ from m, the "mirror" masses make a negligibly small contribution to the force and we have the $(3+\delta)$ dimensional force law. However, for $r \gg L$, the discrete distance between the mirror masses cannot be discerned and they look like an infinite δ spatial dimensional "line" with uniform mass density. The problem is analogous to finding the gravitational field of an infinite line of mass with uniform mass per unit length, where Klein's compactification and Gauss' law give the answer. Following exactly the same procedure, we consider a "cylinder" C centered around the δ dimensional line of mass, with side length l and end caps being three dimensional spheres of radius r. We now apply the $(3+\delta)$ dimensional Gauss law which reads

$$\int_{\text{surface } C} \vec{F} \cdot d\vec{S} = S_{(3+\delta)} G_{N(4+\delta)} \times \text{Mass in } C \tag{4.7}$$

where $D = 3 + \delta$

$$S_D = \frac{2\pi^{D/2}}{\Gamma\left(\frac{D}{2}\right)} \tag{4.8}$$

is the surface area of the unit sphere in D spatial dimensions. Equating the two sides, we find the correct $1/R^2$ force law and can identify

$$G_{N(4)} = \frac{S_{(3+\delta)}}{(4\pi)^\delta} \frac{G_{N(4+\delta)}}{V_\delta} \tag{4.9}$$

where $V_\delta = L^\delta$ is the volume of compactified dimensions. This relation is identical to (4.5) derived before.

The main lesson to be inferred from these two alternative methods of relating fundamental scales of gravity in four and higher dimensions is that Planck's constant M_{Pl} is a derived quantity, and this depends exclusively on the compact nature of extra dimensions.

The purpose of this thesis work is not to review them. Rather, in what follows, an analysis of gravitational theories with higher order curvature terms will be given.

CHAPTER 5

HIGHER CURVATURE GRAVITY IN LARGE EXTRA DIMENSIONS

The Einstein-Hilbert action from which Einstein field equations follow is based on a lagrangian density linear in curvature scalar $\mathcal{R}$. However, given that there is no symmetry principle other than general covariance that governs structure of terms that can contribute to action density one automatically realizes that Einstein-Hilbert term is not necessarily unique. In other words, as far as general covariance is respected, any combination of curvature tensors can and must be included in action density for a general analysis of gravitational system. Therefore, a general action density involves not only $\mathcal{R}$ but also a generic function $f\big(\mathcal{R}, \Box\mathcal{R}, \nabla_A\mathcal{R}\nabla^A\mathcal{R}, \mathcal{R}_{AB}\mathcal{R}^{AB}, \mathcal{R}_{ABCD}\mathcal{R}^{ABCD}, \dots\big)$ of curvature invariants. In this chapter we will discuss gravitational theories based on action density $f\left(\mathcal{R}\right)$ rather than $\mathcal{R}$ in large extra dimensions (an analysis of such theories in 4D can be found in (Nunez and Solganik 2004)). More general structures involving invariants constructed out of Riemann and Ricci tensors have been discussed in (Aslan and Demir 2006).

In analyzing $f\left(\mathcal{R}\right)$ theories of gravity, it is necessary first determine gravi–particle content of the theory i.e. it is necessary first to identify propagating physical degrees of freedom in the theory. Then, on the basis of these findings, one can establish a firm analysis of particle spectrum and various scattering processes similar to ADD model.

5.1. Conformal Transformations

In this section we determine physical particle spectrum of $f\left(\mathcal{R}\right)$ gravity by a conformal transformation to reduce theory to Einstein gravity. The action of higher curvature gravity theory is

$$S = \int d^D x\,\sqrt{-g}\left\{-\frac{1}{2}\overline{M}_D^{D-2} f\left(\mathcal{R}\right) + \mathcal{L}_{matter}\left(g_{AB}, \psi\right)\right\} \tag{5.1}$$

where $\eta_{AB} = \text{diag}\left(1, -1, -1, \cdots, -1\right)$.

A conformal transformation implies a local change of length scale. It means also that the transformation is on the same event. The conformal transformations are realized by multiplying metric by a local positive-definite function $\left(e^{\phi(x)}\right)^2$ of spacetime:

$$\widetilde{g}_{AB}(x) \;\; = \;\; e^{2\phi(x)} g_{AB}(x) \qquad (5.2)$$

with inverse transformation

$$\widetilde{g}^{AB}(x) \;\; = \;\; e^{-2\phi(x)} g^{AB}(x)\;. \qquad (5.3)$$

The main objective of this section, as will be seen below, is to find an appropriate scalar (and dimensionless) function $\phi(x)$ such that geometrical part of (5.1) reduces to Einstein gravity.

Before looking at how conformal transformations change the geometrical quantities in spacetime, let us first show that a conformal transformation leaves light cone invariant. We can do this by showing that null–curves remain invariant under conformal transformations. Then if $x^A(\lambda)$ is a null curve with respect to g_{AB}, it will also remain as a null curve with respect to $\widetilde{g}_{AB}$. Reminding that a curve $x^A(\lambda)$ is null iff its tangent vector $d\,x^A(\lambda)/d\,\lambda$ is null

$$g_{AB}(x)\frac{dx^A(\lambda)}{d\lambda}\frac{dx^B(\lambda)}{d\lambda} = 0 \qquad (5.4)$$

we find that

$$\widetilde{g}_{AB}(x)\frac{dx^A(\lambda)}{d\lambda}\frac{dx^B(\lambda)}{d\lambda} = e^{2\phi(x)} g_{AB}(x)\frac{dx^A(\lambda)}{d\lambda}\frac{dx^B(\lambda)}{d\lambda} = 0 \qquad (5.5)$$

which guarantees that a null vector remains null under (5.2).

Under a conformal transformation connection coefficients change as

$$\begin{aligned}
\widetilde{\Gamma}^C_{BD} &= \frac{1}{2}\widetilde{g}^{CF}\left(\partial_B \widetilde{g}_{DF} + \partial_D \widetilde{g}_{FB} - \partial_F \widetilde{g}_{BD}\right) \\
&= \frac{1}{2}\omega^{-2} g^{CF}\left[2\omega(\partial_B\omega^2)g_{DF} + 2\omega(\partial_D\omega^2)g_{FB} - 2\omega(\partial_F\omega^2)g_{BD}\right] \\
&+ \frac{1}{2}\omega^{-2} g^{CF}\left[\omega^2\partial_B g_{DF} + \omega^2\partial_D g_{FB} - \omega^2\partial_F g_{BD}\right] \\
&= \omega^{-1}\left(\delta^C_D\partial_B\omega + \delta^C_B\partial_D\omega - g^{CF}g_{BD}\partial_F\omega\right) + \Gamma^C_{BD} \qquad (5.6)
\end{aligned}$$

where use has been made of $\widetilde{\nabla}_A\omega = \nabla_A\omega = \partial_A\omega$ and $\omega(x) = e^{\phi(x)}$. Thus

$$\widetilde{\Gamma}^C_{BD} \;\; = \;\; \omega^{-1}\left(\delta^C_D\nabla_B\omega + \delta^C_B\nabla_D\omega - g^{CF}g_{BD}\nabla_F\omega\right) + \Gamma^C_{BD} \qquad (5.7)$$

which has the form

$$\widetilde{\Gamma}^{C}_{AB} = \Gamma^{C}_{AB} + \Delta^{C}_{AB} \tag{5.8}$$

where $\widetilde{\Gamma}^{C}_{AB}$ and Γ^{C}_{AB} are both non-tensors so that Δ^{C}_{AB} is necessarily a tensor field. From (5.7) we identify

$$\Delta^{C}_{AB} = \omega^{-1}\left(\delta^{C}_{B}\nabla_{A}\omega + \delta^{C}_{A}\nabla_{B}\omega - g^{CD}g_{AB}\nabla_{D}\omega\right) \tag{5.9}$$

so that Riemann tensor transforms as

$$\begin{aligned} \widetilde{\mathcal{R}}^{C}_{DAB} &= \partial_{A}\widetilde{\Gamma}^{C}_{BD} - \partial_{B}\widetilde{\Gamma}^{C}_{AD} + \widetilde{\Gamma}^{C}_{AE}\widetilde{\Gamma}^{E}_{BD} - \widetilde{\Gamma}^{C}_{BE}\widetilde{\Gamma}^{E}_{AD} \\ &= \mathcal{R}^{C}_{DAB} + \nabla_{A}\Delta^{C}_{BD} - \nabla_{B}\Delta^{C}_{AD} + \Delta^{C}_{AE}\Delta^{E}_{BD} - \Delta^{C}_{BE}\Delta^{E}_{AD} \end{aligned} \tag{5.10}$$

or equivalently

$$\begin{aligned} \widetilde{\mathcal{R}}^{C}_{DAB} &= \mathcal{R}^{C}_{DAB} - 2\Big(\delta^{C}_{A}\delta^{E}_{B}\delta^{F}_{D} - \delta^{C}_{B}\delta^{E}_{A}\delta^{F}_{D} + g_{DB}\delta^{E}_{A}g^{CF} - g_{DA}\delta^{E}_{B}g^{CF}\Big)\omega^{-1}\left(\nabla_{E}\nabla_{F}\omega\right) \\ &+ 4\Big(\delta^{C}_{A}\delta^{E}_{B}\delta^{F}_{D} - \delta^{C}_{B}\delta^{E}_{A}\delta^{F}_{D} + g_{DB}\delta^{E}_{A}g^{CF} - g_{DA}\delta^{E}_{B}g^{CF} \\ &+ \frac{1}{2}g_{DA}\delta^{C}_{B}g^{EF} - \frac{1}{2}g_{DB}\delta^{C}_{A}g^{EF}\Big) \times \omega^{-2}\left(\nabla_{E}\omega\right)\left(\nabla_{F}\omega\right) \end{aligned} \tag{5.11}$$

where we introduced $\omega(x) = e^{\phi(x)}$, for compactness of notation. As usual, contracting first and third indices of $\widetilde{\mathcal{R}}^{C}_{DAB}$ one finds the Ricci tensor

$$\begin{aligned} \widetilde{\mathcal{R}}_{DB} &= \mathcal{R}_{DB} - \left[(D-2)\delta^{E}_{D}\delta^{F}_{B} + g_{DB}g^{EF}\right]\omega^{-1}\left(\nabla_{E}\nabla_{F}\omega\right) \\ &+ \left[2(D-2)\delta^{E}_{D}\delta^{F}_{B} - (D-3)g_{DB}g^{EF}\right]\omega^{-2}\left(\nabla_{E}\omega\right)\left(\nabla_{F}\omega\right) \end{aligned} \tag{5.12}$$

where $\delta^{A}_{A} = D$ gives number of spacetime dimensions.

Further contraction of $\widetilde{\mathcal{R}}_{DB}$ with $\widetilde{g}^{DB}$ yields the Ricci scalar

$$\begin{aligned} \widetilde{\mathcal{R}} &= \omega^{-2}\Big\{\mathcal{R} - 2(D-1)g^{AB}\omega^{-1}\left(\nabla_{A}\nabla_{B}\omega\right) \\ &- (D-1)(D-4)g^{AB}\omega^{-2}\left(\nabla_{A}\omega\right)\left(\nabla_{B}\omega\right)\Big\}. \end{aligned} \tag{5.13}$$

where ω^{-2} in front follows from use of inverse metric $\widetilde{g}^{DB}$.

Consequently, after conformal transformations, Einstein tensor takes the form

$$\begin{aligned} \widetilde{G}_{AB} &= G_{AB} - \omega^{-1}\left(\nabla_{C}\nabla_{D}\omega\right)(D-2)\left(\delta^{C}_{A}\delta^{D}_{B} - g_{AB}g^{CD}\right) \\ &+ \omega^{-2}\left(\nabla_{C}\omega\right)\left(\nabla_{D}\omega\right)(D-2)\left[2\delta^{C}_{A}\delta^{D}_{B} + \frac{1}{2}(D-5)\,g_{AB}g^{CD}\right] \end{aligned} \tag{5.14}$$

which will serve as central object for transforming $f(\mathcal{R})$ gravity to Einstein gravity.

This result must be compared with Einstein equation for $f(\mathcal{R})$ gravity:

$$G_{AB} = f'(\mathcal{R})\,\mathcal{R}_{AB} - \frac{1}{2} g_{AB} f(\mathcal{R}) + (g_{AB}\Box - \nabla_A \nabla_B)\, f'(\mathcal{R}) \tag{5.15}$$

which involves $f(\mathcal{R})$ itself and its derivatives with respect to $\mathcal{R}$ and x^A. Substitution of this expression into (5.14) and enforcement of $\widetilde{G}_{AB}$ into Einstein-Hilbert form yields the correct transformation rule

$$g_{AB} \rightarrow \left(\frac{\partial f(\mathcal{R})}{\partial \mathcal{R}}\right) g_{AB} \tag{5.16}$$

as also mentioned elsewhere (Kalara et al. 1990, Maeda 1989, Magnano and Sokolowski 1994). In fact, the scalar field $\phi(x)$ takes the form (now redefined to be a dimensionful field)

$$\phi = \overline{M}_D^{(D-2)/2} \sqrt{\frac{D-1}{D-2}}\, \ln \left|\frac{\partial f}{\partial \mathcal{R}}\right| \tag{5.17}$$

which appears to be an additional physical degree of freedom in the theory and $\partial f / \partial \mathcal{R} \neq 0$ due to condition in 5.35. To see this in a clearer way, one notes that the original $f(\mathcal{R})$ gravity action (5.1) transforms into

$$S = \int d^D x \sqrt{-\widetilde{g}} \left\{ -\frac{1}{2} \overline{M}_D^{D-2} \widetilde{\mathcal{R}} + g^{AB} \widetilde{\nabla}_A \phi \widetilde{\nabla}_B \phi - \widetilde{V}(\phi) \right\} \tag{5.18}$$

where

$$\widetilde{V}(\phi) = \frac{1}{D}\left[\left(\widetilde{\nabla}\phi\right)^2 \left(1 - \frac{D}{2}\right) - \mathcal{T} \right] \tag{5.19}$$

The action (5.18) describes Einstein gravity (with metric $\widetilde{g}_{AB}$ plus a scalar field theory. Consequently, (5.1) is equivalent to Einstein gravity plus a self-interacting scalar field theory such that matter couplings are modified via the conformal transformation (5.16). Consequently, $f(\mathcal{R})$ gravity is made up of a massless graviton (as in Einstein gravity) and a presumably massive scalar field ϕ with modified couplings between matter and gravity.

5.2. Graviton Propagator and Gravitational Interactions From $f(\mathcal{R})$ Gravity

In this section we will expand (5.1) into small perturbations around flat background and determine graviton propagator. Our results will confirm findings of previous section in that propagator will consist of a massless tensorial exchange atop a massive scalar exchange.

We start with a derivation of equations of motion for $f(\mathcal{R})$ gravity by extremizing (5.1) with respect to variations in metric field $g_{AB}(x)$:

$$\begin{aligned}
\delta S &= \int d^D x \sqrt{-g}\left[\frac{1}{4} g^{AB} \overline{M}_D^{D-2} f(\mathcal{R}) - \frac{1}{2} g^{AB} \mathcal{L}_{matter} + \frac{\delta \mathcal{L}_{matter}}{\delta g^{AB}}\right] \delta g^{AB} \\
&+ \int d^D x \sqrt{-g}\left[-\frac{1}{2} \overline{M}_D^{D-2} \mathcal{R}_{AB} \frac{\partial f(\mathcal{R})}{\partial \mathcal{R}} - \frac{1}{2} \overline{M}_D^{D-2} g^{CD} \frac{\delta R_{CD}}{\delta g^{AB}} \frac{\partial f(\mathcal{R})}{\partial \mathcal{R}}\right] \delta g^{AB} \\
&= \int d^D x \sqrt{-g}\left[\frac{1}{4} g^{AB} \overline{M}_D^{D-2} f(\mathcal{R}) - \frac{1}{2} \overline{M}_D^{D-2} \mathcal{R}_{AB} \frac{\partial f(\mathcal{R})}{\partial \mathcal{R}}\right] \delta g^{AB} \\
&+ \int d^D x \sqrt{-g}\left[-\frac{1}{2} g^{AB} \mathcal{L}_{matter} + \frac{\delta \mathcal{L}_{matter}}{\delta g^{AB}}\right] \delta g^{AB} \\
&+ \int d^D x \sqrt{-g}\left[-\frac{1}{2} \overline{M}_D^{D-2} g^{CD} \frac{\delta \mathcal{R}_{CD}}{\delta g^{AB}} \frac{\partial f(\mathcal{R})}{\partial \mathcal{R}}\right] \delta g^{AB}
\end{aligned} \tag{5.20}$$

where use has been made of

$$\delta \sqrt{-g} = -\frac{1}{2} \sqrt{-g}\, g_{AB} \delta g^{AB} \,. \tag{5.21}$$

To proceed, we need to compute variation of connection coefficients with respect to g_{AB} to derive $\delta \mathcal{R}_{CD}$ in the fifth line of (5.20).

The variation of $\mathcal{R}^C_{DAB}$ with respect to metric tensor can be derived by first varying the connection coefficient with respect to metric, and then substituting into Riemann tensor. However, arbitrary variations, $\delta\Gamma^C_{BD}$, of the connection coefficients are tensors (they are differences between two connection coefficients), and thus, variation of the Riemann tensor can be arranged in terms of the covariant derivatives of $\delta\Gamma^C_{BD}$:

$$\nabla_A(\delta\Gamma^C_{BD}) = \partial_A(\delta\Gamma^C_{BD}) + \Gamma^C_{AE}\delta\Gamma^E_{BD} - \Gamma^E_{AB}\delta\Gamma^C_{ED} - \Gamma^E_{AD}\delta\Gamma^C_{BE} \tag{5.22}$$

so that variations of Riemann and Ricci tensors take the form

$$\begin{aligned}
\delta \mathcal{R}^C_{DAB} &= \nabla_A\left(\delta\Gamma^C_{BD}\right) - \nabla_B\left(\delta\Gamma^C_{AD}\right) \\
\delta \mathcal{R}_{DB} &= \nabla_C\left(\delta\Gamma^C_{BD}\right) - \nabla_B\left(\delta\Gamma^C_{CD}\right)
\end{aligned} \tag{5.23}$$

whose substitution in (5.20) gives for the fifth line

$$\begin{aligned}
\delta S_{fifth\,line} &= \int d^D x \sqrt{-g}\left\{-\frac{1}{2} \overline{M}_D^{D-2} g^{CD} \delta \mathcal{R}_{CD}\right\} \frac{\partial f(\mathcal{R})}{\partial \mathcal{R}} \\
&= \int d^D x \sqrt{-g}\left\{-\frac{1}{2} \overline{M}_D^{D-2} g^{CD}\left[\nabla_E\left(\delta\Gamma^E_{DC}\right) - \nabla_D\left(\delta\Gamma^E_{EC}\right)\right]\right\} \frac{\partial f(\mathcal{R})}{\partial \mathcal{R}} \\
&= \int d^D x \sqrt{-g}\left\{-\frac{1}{2} \overline{M}_D^{D-2} \nabla_A\left[g^{CD}\left(\delta\Gamma^A_{CD}\right) - g^{CA}\left(\delta\Gamma^E_{EC}\right)\right]\right\} \frac{\partial f(\mathcal{R})}{\partial \mathcal{R}} \,.
\end{aligned}$$

At this step, using

$$\delta\Gamma^{A}_{CD} = -\frac{1}{2}\left[g_{EC}\nabla_{D}\left(\delta g^{EA}\right) + g_{ED}\nabla_{C}\left(\delta g^{EA}\right) - g_{CE}g_{DF}\nabla^{A}\left(\delta g^{EF}\right)\right] \tag{5.24}$$

one finds

$$\delta S_{fifth\,line} = \int d^{D}x\sqrt{-g}\left\{-\frac{1}{2}\overline{M}_{D}^{D-2}\left(g_{AB}\Box - \nabla_{A}\nabla_{B}\right)\frac{\partial f(\mathcal{R})}{\partial \mathcal{R}}\right\} \tag{5.25}$$

where $\nabla_{A}\nabla^{A} = \Box$.

Having done with geometrical sector, we now turn to matter lagrangian in (5.1). As usual, its variation with respect to g_{AB} yields the energy-momentum tensor of matter:

$$\begin{aligned} \mathcal{T}_{AB} &= -\frac{2}{\sqrt{-g}}\frac{\delta(\sqrt{-g}\mathcal{L}_{matter})}{\delta g^{AB}} \\ &= -\frac{2}{\sqrt{-g}}\left[\frac{\delta\sqrt{-g}\mathcal{L}_{matter}}{\delta g^{AB}} - \frac{\sqrt{-g}\delta\mathcal{L}_{matter}}{\delta g^{AB}}\right] . \end{aligned} \tag{5.26}$$

Therefore, equation of motion for metric takes the form

$$f'\left(\mathcal{R}\right)\mathcal{R}_{AB} - \frac{1}{2}f\left(\mathcal{R}\right)g_{AB} + \left(g_{AB}\Box - \nabla_{A}\nabla_{B}\right)f'\left(\mathcal{R}\right) = -\frac{\mathcal{T}_{AB}}{\overline{M}_{D}^{D-2}} \tag{5.27}$$

where prime denotes differentiation with respect to $\mathcal{R}$.

The equation of motion (5.27) does not tell much about the set-up we will use to explore extra dimensions. The essence of the dynamics arises when we restrict energy momentum tensor to

$$\mathcal{T}_{AB} = \delta^{\delta}(\vec{y})\delta^{\mu}_{A}\delta^{\nu}_{B}T_{\mu\nu}(z) \tag{5.28}$$

which explicitly expresses the fact that entire matter is localized on a 3-brane situated at the origin of extra space $i.e.$ $\vec{y} = 0$. In this expression, z_{μ} stands for coordinates on the brane. One notes that conservation of energy-momentum in $(4+\delta)$ dimensions, $\nabla^{A}\mathcal{T}_{AB} = 0$, is guaranteed by conservation law on the brane $\nabla^{\mu}T_{\mu\nu} = 0$.

Obviously, the equations of motion (5.27) reduce to Einstein equations when $f(\mathcal{R}) = \mathcal{R}$. In general, for analyzing dynamics of small oscillations about a background geometry, $g_{AB} = g^{0}_{AB}$ with curvature scalar $\mathcal{R}_{0}$, $f(\mathcal{R})$ must be regular at $\mathcal{R} = \mathcal{R}_{0}$. In particular, as suggested by (5.27), $f(\mathcal{R})$ must be regular at the origin and $f(0)$ must vanish ($i.e.$ bulk cosmological constant must vanish) for $f(\mathcal{R})$ to admit a flat background geometry.

For determining how higher curvature gravity influences interactions among the brane matter, it is necessary to determine the propagating modes which couple to the matter stress tensor. This requires expansion of the action density in (5.1) by using (1.1) up to the desired order in small perturbations h_{AB}:

$$g_{AB} = \eta_{AB} + 2\overline{M}_D^{1-D/2} h_{AB} \tag{5.29}$$

which proves to be a useful expansion as long as surface tension of brane does not exceed fundamental scale of gravity in $(4+\delta)$ dimensions:

$$\frac{T_{AB}}{M_*} \ll 1\,. \tag{5.30}$$

In other words, if matter energy-momentum is small enough we can interpret the curved geometry expressed by g_{AB} as a small folding of the flat background. Physically, matter bends the spacetime, and non-flat spacetime influences matter in the form of gravitational force.

In expanding the action (5.1) via (5.29) the zeroth order term vanishes by $f(0) = 0$ $i.e.$ vanishing of the cosmological term. The terms linear in h_{AB} cancel out by equations of motion (5.27). The quadratic terms yield an effective action of the form

$$S_h = \int d^D x \left[\frac{1}{2} h_{AB}(x) \mathcal{O}^{ABCD}(x) h_{CD}(x) - \frac{1}{\overline{M}_D^{(D-2)/2}} h_{AB}(x) \mathcal{T}(x)^{AB} \right] \tag{5.31}$$

such that propagator of $h_{AB}(x)$, defined via the relation

$$\mathcal{O}_{ABCD}(x) \mathcal{D}^{CDEF}(x, x') = \frac{1}{2} \delta^D(x - x') \left(\delta_A^E \delta_B^F + \delta_B^E \delta_A^F \right)\,, \tag{5.32}$$

takes the form (Demir and Tanyıldızı 2006)

$$\begin{aligned}
-i\mathcal{D}^{ABCD}(p^2) \;=\; & - \left(\frac{f'(0) + 2f''(0)p^2}{(D-2)f'(0) + 2(D-1)f''(0)p^2} \right) \frac{1}{f'(0)p^2} \eta^{AB} \eta^{CD} \\
& + \frac{1}{2f'(0)p^2} \left(\eta^{AC} \eta^{BD} + \eta^{AD} \eta^{BC} \right) \\
& + \frac{2f''(0)p^2}{(D-2)f'(0) + 2(D-1)f''(0)p^2} \frac{1}{f'(0)p^4} \left(\eta^{CD} p^A p^B + \eta^{AB} p^C p^D \right) \\
& + \frac{(\xi - 1)}{2f'(0)p^4} \left(\eta^{BD} p^A p^C + \eta^{DA} p^C p^B + \eta^{AC} p^B p^D + \eta^{CB} p^D p^A \right) \\
& + \frac{2(D-2)f''(0)p^2}{(D-2)f'(0) + 2(D-1)f''(0)p^2} \frac{1}{f'(0)p^6} p^A p^B p^C p^D
\end{aligned} \tag{5.33}$$

in momentum space, where

$$\begin{aligned}
\mathcal{O}_{ABCD}(p^2) &= f'(0)\Big\{ \left(-p^2 + \frac{p^2}{3\xi} - \frac{2f''(0)}{f'(0)}p^4\right)\eta_{AB}\eta_{CD} + \frac{p^2}{2}\left(\eta_{AC}\eta_{BD} + \eta_{AD}\eta_{BC}\right) \\
&+ \left(-\frac{1}{\xi} + 1 + \frac{2f''(0)}{f'(0)}p^2\right)\left(\eta_{CD}p_A p_B + \eta_{AB}p_C p_D\right) \\
&+ \left(-\frac{1}{2} + \frac{1}{2\xi}\right)\left(\eta_{BD}p_A p_C + \eta_{DA}p_C p_B + \eta_{AC}p_B p_D + \eta_{CB}p_D p_A\right) \\
&- \frac{2f''(0)}{f'(0)}p_A p_B p_C p_D\Big\} \; . \qquad (5.34)
\end{aligned}$$

Here momentum p_A refers to momentum in $(4+\delta)$ dimensions. The factor of i in the propagator follows from requirement of positive definite transition amplitude (Peskin and Schröeder 1997).

The small perturbations h_{AB} will henceforth be interpreted as gravitational wave though we have not applied and will not apply at all any quantization procedure. The reason for this ignorance is that one cannot directly quantize gravitational interactions as in gauge forces (Peskin and Schröeder 1997); at each order of perturbation theory there appear new types of divergences which cannot be included in a redefinition of tree-level parameters of the theory, that is, theory is not renormalizable (Stelle 1977). However, for analyzing certain processes involving tree-level h_{AB} exchange the formalism at hand suffices and we can call h_{AB} to be graviton.

It is clear that

$$f'(0) > 0 \qquad (5.35)$$

as otherwise all graviton modes become ghost (residues of various poles become negative). Therefore, if one is to prevent ghosty modes participating in physical processes it is necessary to keep $f'(0)$ positive definite. The parameter ξ in (5.33) arises from the gauge fixing term

$$\mathcal{L}_g = \frac{f'(0)}{\xi}\eta^{AC}\left(\partial^B h_{AB} - \frac{1}{2}\partial_A h^B_B\right)\left(\partial^D h_{CD} - \frac{1}{2}\partial_C h^D_D\right) \qquad (5.36)$$

added to the h_{AB} action density in (5.31). The gauge-fixing term, as in gauge theories (Peskin and Schröeder 1997), is needed for lifting degeneracy of system under diffeomorphism invariance.

In the expression above, $f'(0)$ is introduced to match the terms generated by $\mathcal{L}_g$ with the ones in (5.31). The propagator (5.33) depends explicitly on the second derivative of $f(\mathcal{R})$ evaluated at the origin, and it correctly reduces to the graviton propagator in Einstein gravity (Giudice et al. 1999, Han et al. 1999) when $f''(0) = 0$ and $f'(0) = 1$. The specific choice $\xi = 1$ corresponds to de Donder gauge frequently employed in quantum gravity. Obviously, one can probe $f(\mathcal{R})$ with higher and higher precision by computing higher and higher order h_{AB} correlators. Indeed, for probing $f''''(0)$, for instance, it is necessary to expand the action density in (5.1) up to quartic order so as to compute the requisite four-point function. Rather generically, higher the order of correlators higher the dimensions of the operators they induce. The propagator (5.33) induces a dimension-8 operator via graviton exchange between two matter stress tensors (Giudice et al. 1999, Giudice and Strumia 2003). On the other hand, four-point function induces a dimension-16 operator via graviton exchange among four matter stress tensors.

The scattering processes which proceed with graviton exchange do exhibit new features as one switch from ADD setup to $f(\mathcal{R})$ gravity. Indeed, single graviton exchange influences various processes including $2 \rightarrow 2$ scatterings, particle self-energies, box diagrams and as such. The tree level processes are sensitive to virtual states associated with the propagation of graviton in the bulk. On the other hand, loop level processes involve particle virtualities both on the brane and in the bulk. In this sense, tree level processes offer some degree of simplicity and clarity for disentangling the graviton contribution (see (Giudice and Strumia 2003) for a through analysis of the virtual graviton exchange effects) from those of the SM states. Hence, in the following, we will restrict our discussions exclusively to tree level processes.

By imposing compactness of the extra space and taking its shape to be a torus as in the ADD mechanism the energy-momentum tensor (5.28) takes the form

$$\mathcal{T}_{AB}(x) = \sum_{n_1=-\infty}^{+\infty} \cdots \sum_{n_\delta=-\infty}^{+\infty} \int \frac{d^4p}{(2\pi)^4} \frac{1}{\sqrt{V_\delta}} e^{-i\left(k\cdot z - \frac{\vec{n}\cdot\vec{y}}{R}\right)} \delta^\mu_A \delta^\nu_B T_{\mu\nu}(k) \tag{5.37}$$

where $(n_1, \ldots, n_\delta)$ is a δ-tuple of integers. Given this Fourier decomposition of the stress tensor, the amplitude for an on-brane system a to make a transition into another on-brane system b becomes

$$\mathcal{A}(k^2) = \frac{1}{\overline{M}_{Pl}^2} \sum_{\vec{n}} T^{(a)}_{\mu\nu}(k) \mathcal{D}^{\mu\nu\lambda\rho}\left(k^2 \frac{\vec{n}\cdot\vec{n}}{R^2}\right) T^{(b)}_{\lambda\rho}(k) \tag{5.38}$$

where use has been made of (1.3) in obtaining $1/\overline{M}_{Pl}^2$ factor in front. Though we are dealing with a tree-level process the amplitude involves a summation over all Kaluza-Klein levels due to the fact that these states are inherently virtual because of their propagation off the brane.

Conservation of energy and momentum implies that only the first two terms in the propagator (5.33) contributes to (5.38). Therefore, after performing summation the transition amplitude (5.38) takes the form

$$\begin{aligned}\mathcal{A}(k^2) &= \frac{S_{\delta-1}}{(2\pi)^\delta}\frac{1}{\overline{M}_D^4 f'(0)}\left(\frac{\Lambda}{\overline{M}_D}\right)^{\delta-2} R\left(\frac{\Lambda}{\sqrt{k^2}}\right)\left(T^{(a)}_{\mu\nu}T^{(b)\,\mu\nu} - \frac{1}{\delta+2}T^{(a)\,\mu}_{\mu}T^{(b)\,\nu}_{\nu}\right)\\ &+ \frac{(\delta+4)}{2(\delta+2)(\delta+3)}\frac{S_{\delta-1}}{(2\pi)^\delta}\frac{1}{\overline{M}_D^4 f'(0)}\left(\frac{\Lambda}{\overline{M}_D}\right)^{\delta-2} R\left(\frac{\Lambda}{\sqrt{\widetilde{k}^2}}\right)T^{(a)\,\mu}_{\mu}T^{(b)\,\nu}_{\nu} \quad (5.39)\end{aligned}$$

which exhibits a huge enhancement $\mathcal{O}\left(\overline{M}_{Pl}^2/\overline{M}_D^2\right)$ compared to (5.38) due to the contributions of finely-spaced Kaluze-Klein levels (Arkani-Hamed et al. 1998, Arkani-Hamed et al. 1999). Here $S_{\delta-1} = (2\pi^{\delta/2})/\Gamma(\delta/2)$ is the surface area of δ-dimensional unit sphere, $\widetilde{k}^2 = k^2 - m_\phi^2$ (see below eq. (18) for definitions), and Λ (which is expected to be $\mathcal{O}\left(\overline{M}_D\right)$ since above $\overline{M}_D$ underlying quantum theory of gravity completes the classical treatment pursued here) is the ultraviolet cutoff needed to tame divergent summation over Kaluza-Klein levels. In fact, $\mathcal{A}(k^2)$ exhibits a strong dependence on Λ, as suggested by (see also series expressions of $R\left(\Lambda/\sqrt{k^2}\right)$ derived in (Giudice et al. 1999, Han et al. 1999))

$$\begin{aligned}R\left(\frac{\Lambda}{\sqrt{k^2}}\right) &= -i\frac{\pi}{2}\left(\frac{k^2}{\Lambda^2}\right)^{\frac{\delta}{2}-1} + \frac{\pi}{2}\left(\frac{k^2}{\Lambda^2}\right)^{\frac{\delta}{2}-1}\cot\frac{\pi\delta}{2}\\ &- \frac{1}{\delta-2}\,{}_2F_1\left(1, 1-\frac{\delta}{2}, 2-\frac{\delta}{2}, \frac{k^2}{\Lambda^2}\right) \quad (5.40)\end{aligned}$$

for $0 \leq k^2 \leq \Lambda^2$, and

$$R\left(\frac{\Lambda}{\sqrt{k^2}}\right) = \frac{1}{\delta}\frac{\Lambda^2}{k^2}\,{}_2F_1\left(1, \frac{\delta}{2}, 1+\frac{\delta}{2}, \frac{\Lambda^2}{k^2}\right) \quad (5.41)$$

for $k^2 < 0$ or $k^2 > \Lambda^2$, where ${}_2F_1$ are hypergeometric functions. The imaginary part of R, relevant for the timelike propagator (5.40), is generated by exchange of on-shell gravitons $i.e.$ those Kaluza-Klein levels satisfying $k^2 = \vec{n}\cdot\vec{n}/R^2$. On the other hand, its real part follows from exchange of off-shell gravitons. For spacelike propagator, the scattering amplitude (5.41) is real since in this channel Kaluza-Klein levels cannot come on shell.

The first line of $\mathcal{A}(k^2)$ in (5.39), except for the overall $1/f'(0)$ factor in front, is identical to the single graviton exchange amplitude computed within the ADD setup

(Giudice et al. 1999, Han et al. 1999). In fact, operators $T^{(a)}_{\mu\nu} T^{(b)\,\mu\nu}$ and $T^{(a)\,\mu}_{\mu} T^{(b)\,\nu}_{\nu}$ are collectively induced by exchange of $J = 2$ and $J = 0$ modes of gravity waves h_{AB} (Giudice et al. 1999, Han et al. 1999). The second line at right-hand side, on the other hand, is a completely new contribution not found in ADD setup. The structure of the induced interaction, $T^{(a)\,\mu}_{\mu} T^{(b)\,\nu}_{\nu}$, implies that it is induced by exchange of a scalar field, different than the graviscalar which induces the same type operator in the first line of (5.39). The sources of this additional interaction is nothing but the scalar field ϕ defined in (1.4). Therefore, the main novelty in $\mathcal{A}(k^2)$ lies in the second line at right-hand side of (5.39) which is recognized to be generated by the exchange of a scalar field with non-vanishing bare mass-squared

$$m^2_\phi = -\frac{\delta + 2}{2(\delta + 3)} \frac{f'(0)}{f''(0)} \qquad (5.42)$$

so that $\widetilde{k^2} = k^2 - m^2_\phi$ in (5.39). The nature of the scalar field ϕ depends on sign of $f''(0)$: ϕ is a real scalar for $f''(0) < 0$ and a tachyon for $f''(0) > 0$. Moreover, when $f''(0) = 0$ it is clear that $f(\mathcal{R})$ gravity remains Einsteinian up to $\mathcal{O}(\mathcal{R}^3)$ and this reflects itself by decoupling of ϕ from propagator (5.33) and transition amplitude (5.38) since now ϕ is an infinitely massive scalar. On the other hand, when $f''(\mathcal{R})$ is singular at the origin the bare mass of ϕ vanishes and thus $\mathcal{A}(k^2)$ simplifies to the first line of (5.39) such that coefficient of $T^{(a)\,\mu}_{\mu} T^{(b)\,\nu}_{\nu}$ changes from $-1/(\delta+2)$ to $-1/(2(\delta+3))$. A tachyonic scalar, $m^2_\phi < 0$, decouples from the transition amplitude (5.39) as $k^2 - m^2_\phi \to \infty$. This can be seen from the asymptotic behavior of (5.40) by noting that on-shell graviton graviton exchange is shut off for $k^2 - m^2_\phi \geq \Lambda^2$. Similarly, a true scalar, $m^2_\phi > 0$, also decouples from the transition amplitude (5.39) when $k^2 - m^2_\phi \to -\infty$ as suggested by the asymptotic behavior (5.41). In the next section we will study higher dimensional operators induced by $f(\mathcal{R})$ gravity and their collider signatures.

CHAPTER 6

COLLIDER EFFECTS OF HIGHER CURVATURE GRAVITY

In this section we will discuss certain phenomena in which h_{AB} plays a role. These will include certain scattering amplitudes or higher-dimensional operators which influence low-energy processes.

6.1. Higher Dimensional Operators From $f(\mathcal{R})$ Gravity

The impact of $f(\mathcal{R})$ gravity on the transition amplitude (5.39) is restricted to occur via the dimension-8 operator $T_{\mu}^{(a)\,\mu}T_{\nu}^{(b)\,\nu}$. This operator involves traces of the stress tensors of both systems a and b. In general, trace of the energy momentum tensor, at tree level, is directly related to the sources of conformal breaking in the system (Bekenstein and Meisels 1980, Gross and Wess 1970, Polchinski 1988). It may be instructive to determine stress tensors and their traces for fundamental fields. The energy and momentum of a massive vector field A_{μ} is contained in the conserved stress tensor

$$T_{\mu\nu}^{(J=1)} = \eta_{\mu\nu}\left(\frac{1}{4}F^{\lambda\rho}F_{\lambda\rho} - \frac{1}{2}M_A^2 A_{\lambda}A^{\lambda}\right) - \left(F_{\mu}^{\rho}F_{\nu\rho} - M_A^2 A_{\mu}A_{\nu}\right) \tag{6.1}$$

whose trace

$$T_{\mu}^{(J=1)\,\mu} = -M_A^2 A_{\mu}A^{\mu} \tag{6.2}$$

demonstrates that vector boson mass breaks conformal invariance explicitly. On the other hand, conserved energy-momentum tensor for a massive fermion reads as

$$\begin{aligned} T_{\mu\nu}^{(J=1/2)} &= -\eta_{\mu\nu}\left(\overline{\psi}i\partial\psi - m_{\psi}\overline{\psi}\psi\right) + \frac{i}{2}\overline{\psi}\left(\gamma_{\mu}\partial_{\nu} + \gamma_{\nu}\partial_{\mu}\right)\psi \\ &+ \frac{1}{4}\left[2\eta_{\mu\nu}\partial^{\lambda}\left(\overline{\psi}i\gamma_{\lambda}\psi\right) - \partial_{\mu}\left(\overline{\psi}i\gamma_{\nu}\psi\right) - \partial_{\nu}\left(\overline{\psi}i\gamma_{\mu}\psi\right)\right] \end{aligned} \tag{6.3}$$

whose trace

$$T_{\mu}^{(J=1/2)\,\mu} = m_{\psi}\overline{\psi}\psi \tag{6.4}$$

shows that fermion mass breaks conformal invariance explicitly. In contrast to vector fields and spinors, trace of the stress tensor for a scalar field is not directly related to its mass term. In fact, $T_{\mu}^{(J=0)\,\mu}$ is nonzero even for a massless scalar. For a scalar field Φ to have $T_{\mu}^{(J=0)\,\mu}$ to be proportional to its mass term it is necessary to introduce gauging $\Box\Phi \rightarrow (\Box - \zeta_c \mathcal{R})\,\Phi$ with 'gauge coupling' $\zeta_c = (D-2)/(4(D-1))$ (Demir 2004, Iorio et al. 1997). The curvature scalar serves as the gauge field of local scale invariance. This gauging gives rise to additional terms in the stress tensor of Φ, and they do not vanish even in the flat limit. More explicitly, for a massive complex scalar with quartic coupling the stress tensor reads as

$$\begin{aligned} T_{\mu\nu}^{(J=0)} &= -\eta_{\mu\nu}\left[\partial^{\rho}\Phi^{\dagger}\partial_{\rho}\Phi - M_{\Phi}^{2}\Phi^{\dagger}\Phi - \lambda\left(\Phi^{\dagger}\Phi\right)^{2}\right] + \partial_{\mu}\Phi^{\dagger}\partial_{\nu}\Phi + \partial_{\nu}\Phi^{\dagger}\partial_{\mu}\Phi \\ &+ 2\zeta\left(\eta_{\mu\nu}\Box - \partial_{\mu}\partial_{\nu}\right)\Phi^{\dagger}\Phi \end{aligned} \tag{6.5}$$

whose trace

$$T_{\mu}^{(J=0)\,\mu} = -2(1-6\zeta)\left[\partial^{\rho}\Phi^{\dagger}\partial_{\rho}\Phi - \lambda\left(\Phi^{\dagger}\Phi\right)^{2}\right] + 4\left(1-3\zeta\right)M_{\Phi}^{2}\Phi^{\dagger}\Phi \tag{6.6}$$

reduces to $T_{\mu}^{(J=0)\,\mu} = 2M_{\Phi}^{2}\Phi^{\dagger}\Phi$ for $\zeta = \zeta_c \equiv 1/6$, as desired. For $\zeta \neq \zeta_c$, say $\zeta = 0$, $T_{\mu}^{(J=0)\,\mu}$ involves kinetic term, self-interaction potential $\lambda\left(\Phi^{\dagger}\Phi\right)^{2}$ as well as mass term of the scalar field. The terms proportional to ζ in (6.5) might be regarded as either following from coupling of Φ to curvature scalar as discussed above, or as a field-theoretic technicality to improve properties of the dilatation current (Callan et al. 1970).

The stress tensor traces (6.2), (6.4) and (6.6) with $\zeta = 1/6$ show that effects of graviscalar exchange (the operator $T_{\mu}^{(a)\,\mu}T_{\nu}^{(b)\,\nu}$ in the first line of (5.39)) and $f(\mathcal{R})$ gravity (the operator in the second line of (5.39)) can show up only in those scattering processes which involve massive brane matter at their initial and final states. Their phenomenological viability depends on how heavy the brane states compared to $\overline{M}_D$. For instance, high-energy processes initiated by e^+e^- annihilation or $\gamma\gamma$ scattering or $p\overline{p}$ annihilation cannot probe the operator $T_{\mu}^{(a)\,\mu}T_{\nu}^{(b)\,\nu}$ in (5.39). On the other hand, scattering processes which involve heavy fermions ($e.g.$ bottom and top quarks, muon and tau lepton), weak bosons $W^{\pm}$, Z, and Higgs boson h are particularly useful for probing the gravitational effects. Each of these processes provides an arena for probing effects of scalar graviton exchange in general, and $f(\mathcal{R})$ gravity effects in particular. It might be instructive to depict explicitly how $\mathcal{A}(k^2)$ differs from that computed within the ADD setup by a number of specific scattering processes.

Concerning $2 \to 2$ scattering of weak bosons one can consider, for instance, the process $Z_\alpha(p_1)Z_\beta(p_2) \to Z_\gamma(k_1)Z_\lambda(k_2)$ which is described by the amplitude

$$\begin{aligned}
\mathcal{A}_{ZZ\to ZZ}(k^2) &= \mathcal{A}_{SM}(k^2) + \frac{1}{f'(0)}\mathcal{A}_{ADD}(k^2) \\
&+ \frac{(\delta+4)}{2(\delta+2)(\delta+3)}\frac{S_{\delta-1}}{(2\pi)^\delta}\frac{M_Z^4}{\overline{M}_D^4 f'(0)}\left(\frac{\Lambda}{\overline{M}_D}\right)^{\delta-2} \\
&\times \left\{R\left(\frac{\Lambda}{\sqrt{s}}\right)\eta_{\alpha\beta}\eta_{\gamma\lambda} + R\left(\frac{\Lambda}{\sqrt{t}}\right)\eta_{\alpha\gamma}\eta_{\beta\lambda} + R\left(\frac{\Lambda}{\sqrt{u}}\right)\eta_{\alpha\lambda}\eta_{\beta\gamma}\right\} \\
&\times \epsilon_Z^\alpha(p_1)\epsilon_Z^\beta(p_2)\epsilon_Z^{\star\gamma}(k_1)\epsilon_Z^{\star\lambda}(k_2) \qquad (6.7)
\end{aligned}$$

after using (6.1) in (5.39). In this expression, $s = (p_1+p_2)^2 = (k_1+k_2)^2$, $t = (k_1-p_1)^2$ and $u = (k_2-p_1)^2 = 4M_Z^2 - s - t$ are Mandelstam variables, and ϵ_Z^μ stands for the polarization vector of Z boson. The amplitudes $\mathcal{A}_{SM}(k^2)$ and $\mathcal{A}_{ADD}(k^2)$ can be found in (Atwood et al. 2000). Obviously, $f(\mathcal{R})$ gravity effects get pronounced when M_D lies close to M_Z. Clearly, $\sigma\,(ZZ \to ZZ)$ feels $f(\mathcal{R})$ gravity via square of the third term in (6.7) and its interference with SM and ADD contributions.

The fermion scattering $\psi_1(p_1)\psi_1(p_2) \to \psi_2(k_1)\psi_2(k_2)$ is described by

$$\begin{aligned}
\mathcal{A}_{\psi_1\psi_1\to\psi_2\psi_2}(k^2) &= \mathcal{A}_{SM}(k^2) + \frac{1}{f'(0)}\mathcal{A}_{ADD}(k^2) \\
&+ \frac{(\delta+4)}{2(\delta+2)(\delta+3)}\frac{S_{\delta-1}}{(2\pi)^\delta}\frac{m_{\psi_1}m_{\psi_2}}{\overline{M}_D^4 f'(0)}\left(\frac{\Lambda}{\overline{M}_D}\right)^{\delta-2} \\
&\times R\left(\frac{\Lambda}{\sqrt{s}}\right)\overline{\psi}_1(p_1)\psi_1(p_2)\overline{\psi}_2(k_1)\psi_2(k_2) \qquad (6.8)
\end{aligned}$$

after using (6.3) in (5.39). If ψ_1 and ψ_2 are identical fermions then t and u channel contributions must also be included. The SM and ADD pieces in this amplitude can be found in (Giudice et al. 1999, Han et al. 1999). The heavy fermion scatterings ($e.g.$ $tt \to tt$, $bb \to tt$, $\tau\tau \to tt$) are potential processes for highlighting effects of $f(\mathcal{R})$ gravity. The $2 \to 2$ scattering of Higgs bosons provides another interesting channel to probe $f(\mathcal{R})$ gravity effects. Indeed, after expanding (6.5) around the electroweak vacuum $\Phi = (v+h, 0)/\sqrt{2}$ with $v \simeq 246\,\mathrm{GeV}$, the amplitude for $h(p_1)h(p_2) \to h(k_1)h(k_2)$ scattering takes the form

$$\begin{aligned}
\mathcal{A}_{hh\to hh}(k^2) &= \mathcal{A}_{SM}(k^2) + \frac{1}{f'(0)}\mathcal{A}_{ADD}(k^2) \\
&+ \frac{(\delta+4)}{8(\delta+2)(\delta+3)}\frac{S_{\delta-1}}{(2\pi)^\delta}\frac{m_h^4}{\overline{M}_D^4 f'(0)}\left(\frac{\Lambda}{\overline{M}_D}\right)^{\delta-2} \\
&\times \left\{R\left(\frac{\Lambda}{\sqrt{s}}\right) + R\left(\frac{\Lambda}{\sqrt{t}}\right) + R\left(\frac{\Lambda}{\sqrt{u}}\right)\right\} \qquad (6.9)
\end{aligned}$$

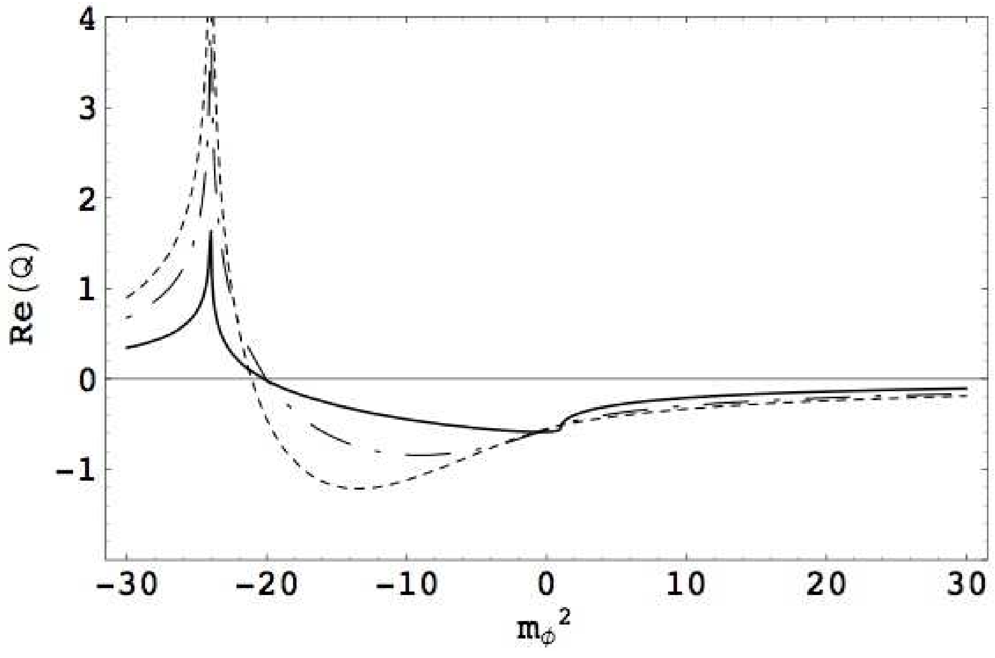

Figure 6.1. The dependence of Re $[Q(k^2)]$ on m_ϕ^2 for $k^2 = (1\,\text{TeV})^2$, $\Lambda = \overline{M}_D = 5\,\text{TeV}$, and $\delta = 3$ (solid curve), $\delta = 5$ (dot-dashed curve) and $\delta = 7$ (short-dashed curve). We vary m_ϕ^2 from $-(30\,\text{TeV})^2$ up to $+(30\,\text{TeV})^2$.

where $m_h^2 = -2M_\Phi^2$ is the Higgs boson mass-squared. It is clear that size of $f(\mathcal{R})$ gravity effects depends crucially on how close M_D is to m_h. Calculations (He 2000) within ADD setup show that graviton exchange can have significant impact on $h(p_1)h(p_2) \to h(k_1)h(k_2)$, and thus, resulting deviation from the SM expectation might be of observable size.

The $2 \to 2$ scattering processes mentioned above illustrate how $f(\mathcal{R})$ gravity influences certain observables to be measured in collider experiments. Beyond these, there are, of course various observables which can sense $f(\mathcal{R})$ gravity. For instance, hZZ coupling, which is crucial for Higgs boson search via Bjorken process, gets also modified by graviton exchange (Choudhury et al. 2003) via $T_{\mu\nu}^{(J=0)} T_{\lambda\rho}^{(J=1)}$ correlator. The discussions above show that, independent of what brane matter species are taking part in a specific process, entire novelty brought about by $f(\mathcal{R})$ gravity is contained in the second line of (5.39), and thus, it proves useful to carry out a comparative analysis of this contribution with the same structure present in the ADD setup, for completeness. In fact,

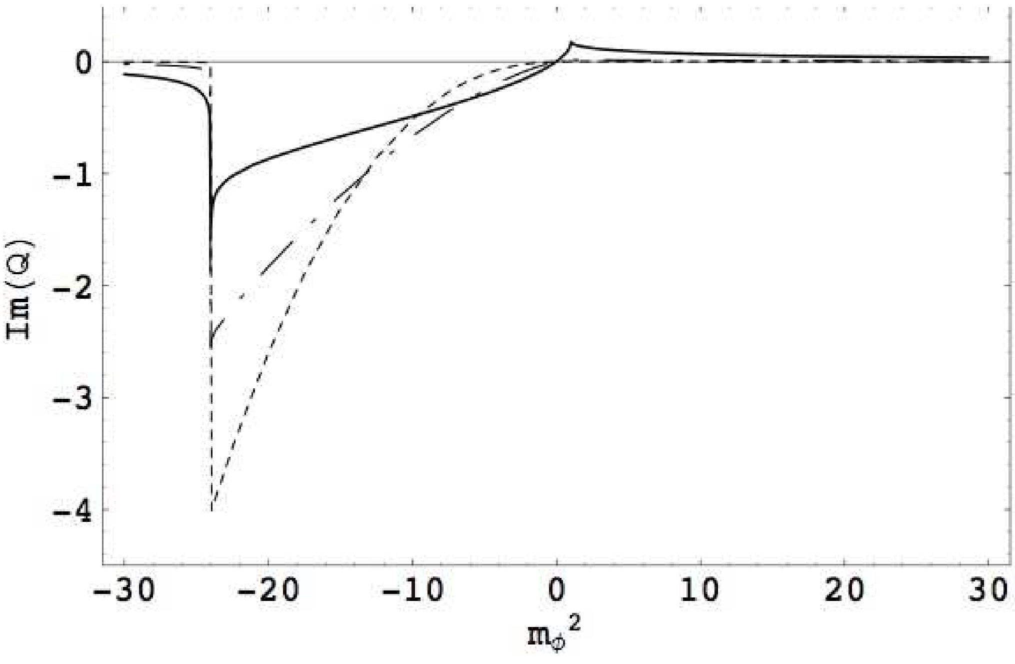

Figure 6.2. The same as in Fig. 6.1 but for Im $[Q(k^2)]$.

ratio of the coefficients of $T_\mu^{(a)\,\mu} T_\nu^{(b)\,\nu}$ in (5.39)

$$Q(k^2) = -\frac{(\delta+4)}{2(\delta+3)} \frac{R\left(\frac{\Lambda}{\sqrt{\widetilde{k}^2}}\right)}{R\left(\frac{\Lambda}{\sqrt{k^2}}\right)} \qquad (6.10)$$

is a useful quantity for such a comparative analysis. For determining how finite $f''(0)$ influences the scattering processes it suffices to determine m_ϕ^2 dependence of $Q(k^2)$ for given values of k^2, δ and $\Lambda \sim \overline{M}_D$. In accord with future collider searches, one can take, for instance, $k^2 = (1\,\mathrm{TeV})^2$ and $\Lambda = \overline{M}_D = 5\,\mathrm{TeV}$, and examine m_ϕ^2 dependencies of Re $[Q(k^2)]$ and Im $[Q(k^2)]$ separately. In fact, depicted in Figs. 6.1 and 6.2 are, respectively, the variations of Re $[Q(k^2)]$ and Im $[Q(k^2)]$ with m_ϕ^2. In the figures 6.1 and 6.2, m_ϕ^2 varies from $-(30\,\mathrm{TeV})^2$ up to $+(30\,\mathrm{TeV})^2$ for each number of extra dimensions considered: $\delta = 3$ (solid), $\delta = 5$ (dot-dashed) and $\delta = 7$ (short-dashed). As suggested by (5.42), positive and negative m_ϕ^2 values in the figures correspond, respectively, to negative and positive values of $f''(0)$ since $f'(0)$ has already been restricted to take positive values to prevent graviton becoming a ghost (see the propagator (5.33). On the other hand, if $\mathcal{A}(k^2)$ in (5.39) exhibits a timelike ($\widetilde{k}^2 > 0$) or spacelike ($\widetilde{k}^2 < 0$) propagation depends on how k^2 compares with m_ϕ^2. With the values of parameters given above, the figures illustrate cases where $k^2 > 0$ yet $\widetilde{k}^2$ varies over a wide range of values comprising spacelike and timelike behaviors as well as a heavy ϕ *i.e.* $\left|m_\phi^2\right| \gg \Lambda^2$.

The overall behaviors of both figures suggest that $f(\mathcal{R})$ gravity effects fade away for large $\left|m_\phi^2\right|$, as expected. Both real and imaginary parts of $Q(k^2)$ exhibit a narrow peak at $m_\phi^2 = -(24\,\mathrm{TeV})^2$ which corresponds to resonating of the transition amplitude by Kaluza-Klein levels with mass-squared $= k^2 - m_\phi^2 = \Lambda^2$. From Fig. 6.1 it is clear that Re $[Q(k^2)]$ becomes significant at large δ and negative m_ϕ^2. This is also seen to hold for Im $[Q(k^2)]$ from Fig. 6.2. Obviously, $f(\mathcal{R})$ gravity predictions differ from ADD ones for moderate (with respect to scale Λ) negative m_ϕ^2 or equivalently for sufficiently small and positive $f''(0)$ (see eq.(5.42) for details). Indeed, for positive values of m_ϕ^2 or equivalently for negative $f''(0)$ the strength of $f(\mathcal{R})$ gravity contribution remains significantly below the ADD one.

6.2. Yet More Signatures of $f(\mathcal{R})$ Gravity

So far we have focussed mainly on higher dimensional operators induced by tree-level virtual graviton exchange. Clearly, effects of higher dimensional gravity on brane matter are not restricted to such processes: graviton can contribute to self-energies, effective vertices or box diagrams of brane matter; graviton can be emitted off the brane matter; and graviton can decay into brane matter. In this section we will discuss such processes briefly for illustrating how $f(\mathcal{R})$ gravity effects (Demir and Tanyıldızı 2006) differ from those found in the ADD setup.

First of all, as suggested by (5.1), couplings of the gravity waves $h_{AB}(x)$ to brane matter are identical in ADD and $f(\mathcal{R})$ gravity setups. Therefore, distinction between the two frameworks rests mainly on the additional scalar field (1.4) imbrued in the $f(\mathcal{R})$ gravity dynamics. Consequently, detection of $f(\mathcal{R})$ gravity effects requires scattering processes on the brane to be sensitive to the new energy threshold m_ϕ not found in the ADD setup.

Let us consider first role of $f(\mathcal{R})$ gravity on brane-localized loops. The simplest of such processes is the self-energy of a brane particle. One may consider, for instance, self-energy of the Z boson (or any of the massive SM fields mentioned in the last section).

At the level of a single graviton exchange one finds

$$
\begin{aligned}
-i\Pi\left(q\right) &= -i\Pi_{SM}\left(q\right) - i\frac{1}{f'(0)}\Pi_{ADD}\left(q\right) - i\Pi_{seagull}\left(q^{2}\right) \\
&- \frac{M_{Z}^{4}}{\overline{M}_{Pl}^{2} f'(0)} \frac{\delta+4}{(\delta+2)(\delta+3)} \\
&\times \sum_{\vec{n}} \int \frac{d^{4}k}{(2\pi)^{4}} \frac{1}{k^{2}-m_{\phi}^{2}-\frac{\vec{n}\cdot\vec{n}}{R^{2}}+i\epsilon} \frac{1}{(q+k)^{2}-M_{Z}^{2}+i\epsilon}
\end{aligned}
\tag{6.11}
$$

where contribution of the four-point vertex that binds gravitons and Z bosons is contained in the seagull contribution. The summation and integration involved in this expression are difficult to evaluate analytically, and therefore, one may eventually need to resort some numerical techniques (Han et al. 1999). However, at least for vanishing external momentum, one can show that $f\left(\mathcal{R}\right)$ gravity contribution in the second line of (6.11) is diminished at large $\left|m_{\phi}\right|^{2}$, and is particularly pronounced when $\left|m_{\phi}\right|^{2} \sim M_{Z}^{2}$ and $m_{\phi}^{2} < 0$. Therefore, when $f''(0) \sim 1/M_{Z}^{2}$ one expects observable enhancements in the Z boson self energy (see (Giudice et al. 2001) and (Han et al. 1999) for analyses of the Higgs boson self energy).

Having discussed effects of $f\left(\mathcal{R}\right)$ gravity on brane-localized loops we now turn to an analysis of production and decays of the graviton. In these processes graviton is a physical particle described by asymptotically free states connected by the S-matrix elements. Therefore, the scalar field ϕ imbrued in $f\left(\mathcal{R}\right)$ dynamics must be endowed with a positive mass-squared for its decays and productions to be observable. Consequently, $f\left(\mathcal{R}\right)$ gravity effects on graviton production and decay exist within $m_{\phi}^{2} > 0$ domain. However, as suggested by Figs. 6.1 and 6.2, $f\left(\mathcal{R}\right)$ gravity contribution, the second line of (5.39), stays significantly below the corresponding contribution in ADD setup. This implies, in particular, that production and decay of ϕ graviton are suppressed relative to those of the $J=2$ and $J=0$ gravitons.

The above observation is confirmed by the fact that when the looping particles come on their mass shells, as dictated by the optical theorem, the Z boson self-energy (6.11) above represents the Drell-Yan production of graviton and Z boson at lepton (via $e^{+}e^{-} \rightarrow Z^{\star} \rightarrow graviton\ + Z$ annihilation) or hadron (via $q\overline{q} \rightarrow Z^{\star} \rightarrow graviton\ + Z$ annihilation) colliders. The main novelty brought about by $f\left(\mathcal{R}\right)$ gravity is the production of ϕ (in addition to $J=0$ and $J=2$ gravitons) when the center-of-mass energy of the collider is sufficiently large $i.e.$ $s \geq m_{\phi}^{2} + M_{Z}^{2}$. This phenomenon reflects itself by a sudden change in the number of events (similar to opening of W^+W^- channel at LEP experiments). The dominant contribution to graviton emission comes from Kaluza–Klein levels in the vicinity of $R^2(M_Z^2 - m_\phi^2)$. The emission of gravitons from the brane is not restricted to such $2 \to 2$ processes, however. Indeed, massive brane localized states can decay into gravitons, including ϕ itself, and this reflects itself as an increase in their invisible widths (see, for instance, (Giudice et al. 2001) for a detailed discussion of the Higgs boson width).

There are, of course, inverse processes to graviton emission. Indeed, gravitons propagating in the bulk can decay into brane matter when they land on the brane. The graviton decay channels can open only if their Kaluza–Klein level is high enough (Han et al. 1999). The only exception to this is the ϕ graviton which can decay into brane matter even at zeroth Kaluza–Klein level provided that its mass, m_ϕ, is larger than those of the daughter particles. Detailed discussions of the production and decays of gravitons (as well as those of the right-handed neutrinos propagating in the bulk (Demir et al. 2002)) in the framework of ADD mechanism can be found in (Arkani–Hamed et al. 1998, Arkani–Hamed et al. 1999).

This section is intended to provide a brief summary of what impact $f(\mathcal{R})$ gravity can have on processes involving brane-loops, missing energy signals in brane matter scatterings, and population of brane via the graviton decays. These processes are of great importance for both collider (Giudice et al. 1999, Han et al. 1999) and cosmological (Arkani–Hamed et al. 1998, Arkani–Hamed et al. 1999, Demir et al. 2002) purposes, and discussions provided in this section is far from being sufficient for a proper description of what effects $f(\mathcal{R})$ gravity can leave on them. From this brief analysis, combined with results of the previous section, one concludes that $f(\mathcal{R})$ gravity effects on decays and emissions of graviton cannot compete with the ADD expectations.

CHAPTER 7

CONCLUSION

In this thesis work we have provided an introduction to notion of extra dimensions together with experimental limits and theoretical motivations for them. We have provided a review of Kaluza-Klein approach to illustrate an important scenario where extra dimensions are needed. Then, we have also reviewed the infamous ADD model where extra dimensions play a crucial role in taming quadratic divergences in Higgs boson mass.

Having summarized existing scenarios involving extra dimensions we have explored certain salient features and phenomenological implications of $f(\mathcal{R})$ gravity in large extra dimensions. In Sec. 5.1 we have determined particle content of this higher-curvature gravity theory by applying conformal transformations. In Sec. 5.2 we have derived graviton propagator about flat Minkowski background, and have determined how it influences interactions among the brane matter. In Sec. 6.1 we have listed down a set of higher dimensional operators which exhibit an enhanced sensitivity to $f(\mathcal{R})$ gravity (compared to those operators involving light fermions or massless gauge fields). Therein we have also performed a comparative study of ADD and $f(\mathcal{R})$ gravity predictions and determined ranges of parameters where the latter dominates over the former. The analysis suggests that $f(\mathcal{R})$ gravity theories with finite and positive $f''(0)$ induce potentially important effects testable at future collider studies. In Sec. 6.2 we have discussed briefly how $f(\mathcal{R})$ gravity influences loop processes on the brane as well as decays and productions of gravitons.

The analysis in this work can be applied to various laboratory, astrophysical and cosmological observables (see (Arkani–Hamed et al. 1998, Arkani–Hamed et al. 1999) for a detailed discussion of major observables) for examining non-Einsteinian forms of general relativity in higher dimensions. The discussions presented here are far from being complete in their coverage and phenomenological investigations. The rule of thumb to be kept in mind is that higher curvature gravity influences scatterings of massive (sufficiently heavy compared to the fundamental scale of gravity) brane matter.

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APPENDIX A

NOTATIONS AND CONVENTIONS

Units are $\hbar = c = 1$ then [length] $=$ [time] $=$ [energy]$^{-1}$ $=$ [mass]$^{-1}$. Indices of 4D–spacetime are

$$\mu, \nu = 0,\ 1,\ 2,\ 3 \tag{A.1}$$

and indices of D–dimensional spacetime are

$$A, B = 0,\ 1,\ 2,\ 3,\ 4,\ \cdots,\ D \tag{A.2}$$

and signature of this work is

$$\eta_{AB} = \text{diag}(+1, -1, -1, \cdots, -1)\ . \tag{A.3}$$

APPENDIX B

SPACETIME AND METRIC

The special relativity is a model that invokes a particular kind of spacetime, with no curvature and hence no gravity, such that the particular spacetime is the Minkowski (flat 4D) spacetime. Minkowski's spacetime is a four–dimensional set established by three spatial dimensions and a temporal dimension as time. An individual point in spacetime is named as event. Then any path is a curve through spacetime, called the worldline and the curve is parameterized as a set of events (B.1).

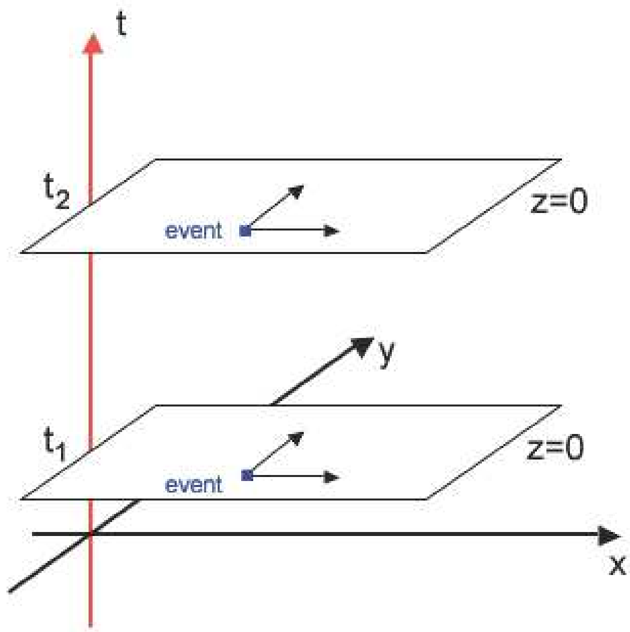

Figure B.1. Spacetime diagram.

According to the description of the time as a dimension, there is an important difference between Newtonian mechanics and special relativity, caused by the notion of simultaneity defined in Newtonian mechanics. In principle the notion of simultaneity includes the fact that there is a basic division of spacetime into well-defined slices of all of space at a fixed moment in time. In other words when two events occur at the same time, simultaneity is naturally defined in Newtonian mechanics. Moreover, there isn't such a thing as a limit on the relative velocity of the particles in Newtonian case. On the other hand, in special relativity there is no well-defined notion of two separate events occurring at the same time. In this case, we might define a new light cone at every event. As another tool, the lightcone is the locus of worldlines through spacetime, as in the B.2 figure.

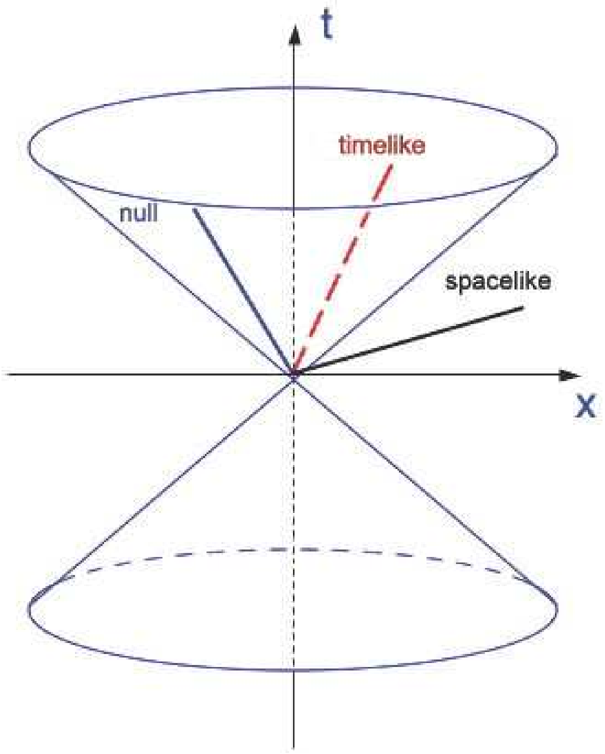

Figure B.2. A lightcone, portrayed on a spacetime diagram.

Obviously, the surface of each cone is the boundary for each subsequent event, then the particles can travel along only paths which always remain inside the cones. In this point, we may understand why the surface of the light cone is the boundary of spacetime. The idea comes from the fact that speed of light is constant and constitutes the maximum relative speed between all objects in Nature. Indeed as a hypothetical way, to define the 4D–spacetime metric is possible by assuming that c is some fixed conversion factor between space and time. Such that, conventionally the velocity c is fixed to 1 in this work. We need a limited relative velocity, because it will provide us the spacetime interval which is invariant under changes of inertial coordinates. In this sense an interval

$$(\Delta s)^2 = (\Delta x^0)^2 - (\Delta x^1)^2 - (\Delta x^2)^2 - (\Delta x^3)^2 \tag{B.1}$$

will remain invariant as in the form

$$(\Delta s)^2 = (\Delta x^{0\prime})^2 - (\Delta x^{1\prime})^2 - (\Delta x^{2\prime})^2 - (\Delta x^{3\prime})^2 \tag{B.2}$$

where $x^0 = c\,t$ is temporal dimension and $x^1,\ x^2$ and x^3 are spatial dimensions. To see that the velocity of light is limited and in consequence to realize that any spacetime interval is to be zero for light, we should setup a transformation which release the interval invariant, in 4D–spacetime. Then the transformation matrix is

$$\eta_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{B.3}$$

It is called as the spacetime metric (in particular, Minkowski metric, here). Consequently, special relativity is a theory of Minkowski spacetime which is defined by metric (B.3).

APPENDIX C

CURVATURE TENSOR

In the general case of a space, the infinitesimal parallel displacement of a vector is defined as displacement in which the components of the vector are not changed along the given infinitesimal closed curve in the coordinate system which is galilean.

If $x^\mu = x^\mu(s)$ is the parametric equation of an infinitesimal (in particular closed) curve where s is the arc length measured from some point and the tangent unit vector to the curve is $A^\mu = dx^\mu/ds$. If the considered curve is a geodesic, along the curve DA^μ equal to zero. In other words, following the vector A^μ is subjected to a parallel displacement from a point x^μ on the geodesic curve to the point $x^\mu + dx^\mu$ on the same geodesic curve, the parallel transport leaves the vector $A^\mu + dA^\mu$ parallel to the tangent line at the point $x^\mu + dx^\mu$ on the geodesic curve. Therefore the tangent is displaced parallel to itself, when the tangent to a geodesic moves along the curve. In other words, during the parallel displacement of a vector along any geodesic curve on the space, the angle between the vector and the tangent to the geodesic doesn't change.

As a result, we may say that the parallel displacement of a vector, which is in a non-flat space, from a given point to another causes appearance of different vectors if it is moved along different paths. In particular, if a vector is displaced parallel to itself along some closed curve (closed contour), it will lost the original value.

In order to illuminate the case, let us consider two dimensional flat space, see figure C.1. In moving along the line (dashed curve) AB, BC and CA, the vector A^μ, always retaining its angle with the curve unchanged, goes over into the vector A^ν which is coincide with A^μ.

Then let us consider two dimensional non-flat space, see figure C.2. In moving along the lines AB, BC and CA consecutively, the vector A^μ, always retaining its angle with the curve unchanged, goes over into the vector A^ν which is not coincide with its original value.

The general formula for the change in the vector may be derived following parallel displacement around the infinitesimal closed curve. This change ΔA_μ may be written in

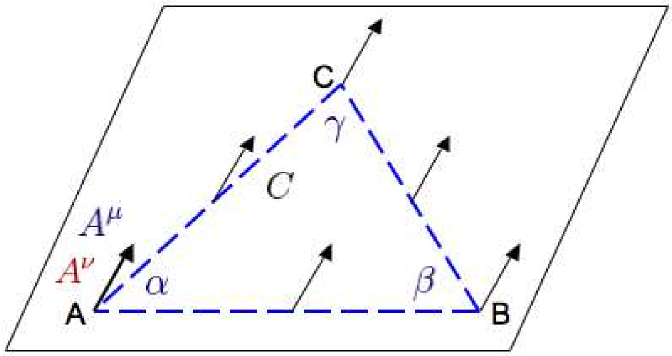

Figure C.1. In flat space: Total change in $A^\mu \propto (\alpha+\beta+\gamma-180^o) = 0$, $\delta A^\mu = 0$.

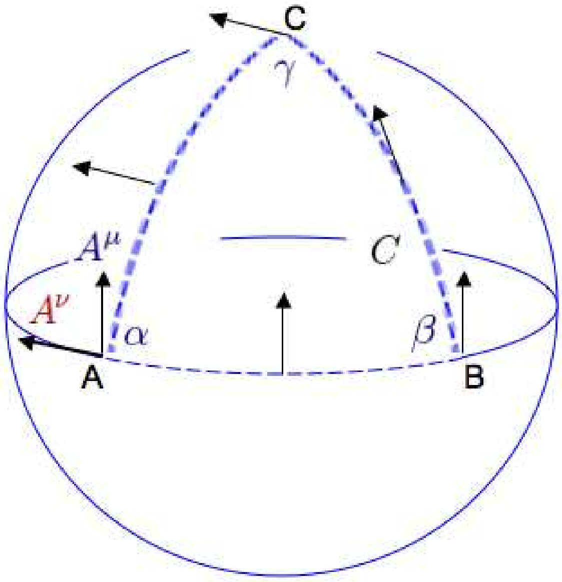

Figure C.2. In curved space: Total change in $A^\mu \propto (\alpha+\beta+\gamma-180^o) > 0$, $\delta A^\mu \neq 0$.

the form:

$$\Delta A_\mu = \oint \delta A_\mu \tag{C.1}$$

where the integral is taken over the infinitesimal closed curve in figure C.2. Substituting in place of

$$\delta A_\mu = \Gamma^\nu_{\mu\lambda} A_\nu dx^\lambda \tag{C.2}$$

the integration, we obtain

$$\Delta A_\mu = \oint \Gamma^\lambda_{\mu\nu} A_\lambda dx^\nu \tag{C.3}$$

where the vector A_λ, which appears here, as the vector A_μ is moved along the closed curve.

The values of the vector A_λ depend on the path along which we approach the particular point. It means that they are not unique at points inside the curve. However the non-uniqueness is related to second-order terms. Therefore, with the first-order accuracy which is sufficient for the transformation, it is regarded that the components of the vector A_λ at points inside the infinitesimal closed curve as being uniquely determined by their values on the contour itself by the formulas (C.2), by the derivatives

$$\frac{\partial A_\lambda}{\partial x^\nu} = \Gamma^\rho_{\lambda\nu} A_\rho \ . \tag{C.4}$$

Applying Stoke's theorem:

$$\oint A_\lambda dx^\lambda = \int df^{\mu\lambda} \frac{\partial A_\lambda}{\partial x^\mu} = \frac{1}{2}\int df^{\lambda\mu}\left(\frac{\partial A_\mu}{\partial x^\lambda} - \frac{\partial A_\lambda}{\partial x^\mu}\right) \tag{C.5}$$

where $dx^\lambda \rightarrow df^{\mu\lambda}(\partial/\partial x^\mu)$, to the integral (C.3) and considering that the area enclosed by the closed curve has the infinitesimal value $\Delta f^{\nu\beta}$, it is obtained that

$$\begin{aligned} \Delta A_\mu &= \frac{1}{2}\left[\frac{\partial\left(\Gamma^\lambda_{\mu\beta}A_\lambda\right)}{\partial x^\nu} - \frac{\partial\left(\Gamma^\lambda_{\mu\nu}A_\lambda\right)}{\partial x^\beta}\right]\Delta f^{\nu\beta} \\ &= \frac{1}{2}\left[\frac{\partial \Gamma^\lambda_{\mu\beta}}{\partial x^\nu}A_\lambda - \frac{\partial\Gamma^{\mu\nu}}{\partial x^\beta}A_\lambda + \Gamma^\lambda_{\mu\beta}\frac{\partial A_\lambda}{\partial x^\nu} - \Gamma^\lambda_{\mu\nu}\frac{\partial A_\lambda}{\partial x^\beta}\right]\Delta f^{\nu\beta} \end{aligned} \tag{C.6}$$

Hence the total change ΔA_μ is obtained in the form:

$$\Delta A_\mu = \frac{1}{2}\mathcal{R}^\lambda_{\mu\nu\beta} A_\lambda \Delta f^{\nu\beta} \tag{C.7}$$

substituting the values of the derivatives (C.4), where $\mathcal{R}^\lambda_{\mu\nu\beta}$ is the Riemann curvature tensor:

$$\mathcal{R}^\lambda_{\mu\nu\beta} = \frac{\partial\Gamma^\lambda_{\mu\beta}}{\partial x^\nu} - \frac{\partial\Gamma^\lambda_{\mu\nu}}{\partial x^\beta} + \Gamma^\lambda_{\rho\nu}\Gamma^\rho_{\mu\beta} - \Gamma^\lambda_{\rho\beta}\Gamma^\rho_{\mu\nu} \tag{C.8}$$

The total change δA_μ between the values of vectors at one and the same point is a vector, in this sence it is clear that $\mathcal{R}^\lambda_{\mu\nu\beta}$ is a tensor.

It is easy to obtain a similar formula for a contravariant vector A^μ using the fact that a scalar doesn't change under parallel displacement, $\Delta(B^\mu A_\mu) = 0$. Considering the equation (C.7):

$$\begin{aligned}\Delta\left(A_\mu B^\mu\right) &= B^\mu \Delta A_\mu + A_\mu \Delta B^\mu \\ &= \frac{1}{2} B^\mu A_\lambda \mathcal{R}^\lambda_{\mu\nu\beta} \Delta f^{\nu\beta} + A_\mu \Delta B^\mu \\ &= A_\mu \left(\frac{1}{2} B^\lambda \mathcal{R}^\mu_{\lambda\nu\beta} \Delta f^{\nu\beta} + \Delta B^\mu\right) \\ &= 0 \end{aligned} \tag{C.9}$$

and an arbitrary choosing of the vector A_μ provides us

$$\Delta B^\mu = -\frac{1}{2} \mathcal{R}^\mu_{\lambda\nu\beta} B^\lambda \Delta f^{\nu\beta} \quad . \tag{C.10}$$

As a vector A_λ is differentiated covariantly with respect to x^μ and x^ν, the result depends on the order of differentiation, on the contrary for ordinary differentiation the result does not depend on the order of differentiation. In this sense, the difference appear as the form:

$$\frac{\partial A_\lambda}{\partial x^\mu \partial x^\nu} - \frac{\partial A_\lambda}{\partial x^\nu \partial x^\mu} = A_\rho \mathcal{R}^\rho_{\lambda\mu\nu} \tag{C.11}$$

considering the direct calculation in the local-geodesic coordinate system. For the contravariant form of a vector, the formula above takes the form:

$$\frac{\partial A^\lambda}{\partial x^\mu \partial x^\nu} - \frac{\partial A^\lambda}{\partial x^\nu \partial x^\mu} = A^\rho \mathcal{R}^\lambda_{\rho\mu\nu} \quad . \tag{C.12}$$

Clearly, in flat space the curvature tensor is zero, the coordinates may be chosen such that over all space all $\Gamma^\lambda_{\mu\nu} = 0$, and therefore also $\mathcal{R}^\lambda_{\mu\nu\beta} = 0$. This situation is related to the fact that parallel displacement is a single-valued operation in a flat space. Hereby the vector doesn't change in moving on a closed curve in flat space. Additionally, $\mathcal{R}^\lambda_{\mu\nu\beta} = 0$ means that the space is flat, as the converse theorem.

Consequently, we can propose that vanishing or nonvanishing of the curvature tensor is a criterion to determine whether a space is flat or non-flat, also the Galilean coordinate system can be presumed over an infinitesimal region in a space.

We may obtain the Ricci tensor and Ricci scalar by using our definitions and Einstein's summation convention.

$$\mathcal{R}^\lambda_{\mu\lambda\beta} = \mathcal{R}_{\mu\beta} \ , \quad g^{\mu\beta} \mathcal{R}_{\mu\beta} = \mathcal{R} \tag{C.13}$$

where, $\mathcal{R}_{\mu\beta}$ is Ricci tensor which is symmetric and $\mathcal{R}$ is Ricci scalar.

To give a physical example in which the Riemann curvature tensor appears, let us consider a free relativistic particle. According to Newton's first law, it obeys

$$\frac{d^2\xi^\alpha(\tau)}{d\tau} = 0 \tag{C.14}$$

which implies a straight line trajectory, as expected. Here, $d\tau^2 = \eta_{\alpha\beta}d\xi^\alpha d\xi^\beta$ is particle's eigentime and $\eta_{\alpha\beta} = \text{diag.}(+1, -1, -1, -1)$ is the flat space metric. Now let us switch to another coordinate system, x^μ, which may be cartesian, curvilinear, accelerated, rotating, whatever is imagined. The two coordinate systems are related via an invertible relation:

$$\xi^\alpha \equiv \xi^\alpha(x) \iff x^\mu = x^\mu(\xi) \ . \tag{C.15}$$

In this coordinate system (C.14) takes also the form which forms an invertible relation called as "geodesic equation":

$$\frac{d^2x^\lambda}{d\tau^2} + \Gamma^\lambda_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} = 0 \tag{C.16}$$

where

$$\Gamma^\lambda_{\mu\nu} = \frac{\partial x^\lambda}{\partial \xi^\alpha}\frac{\partial^2\xi^\alpha}{\partial x^\mu \partial x^\nu} \tag{C.17}$$

are the connection coefficients. Under a general coordinate transformation from x^μ to $(x')^\mu \equiv x^{\mu'}$ the connection coefficients transform as

$$\Gamma^{\lambda'}_{\mu'\nu'} = \underbrace{\frac{\partial x^{\lambda'}}{\partial x^\lambda}\frac{\partial x^\mu}{\partial x^{\mu'}}\frac{\partial x^\nu}{\partial x^{\nu'}}\Gamma^\lambda_{\mu\nu}}_{\textit{tensorial piece}} + \underbrace{\frac{\partial x^{\lambda'}}{\partial x^\lambda}\frac{\partial^2 x^\lambda}{\partial x^{\mu'}\partial x^{\nu'}}}_{\textit{non-tensorial piece}} \tag{C.18}$$

which is clearly a non-tensorial object. Indeed, the connection coefficients must not be tensors for them to represent gravitational force which changes from frame to frame. Therefore, $\Gamma^\lambda_{\mu\nu}$ is not a tensor. The non-tensorial piece can be expanded as, considering the invertibility relation as in (C.15),

$$\frac{\partial^2 x^{\lambda'}}{\partial x^\mu \partial x^\nu} = \frac{\partial x^{\lambda'}}{\partial x^\lambda}\Gamma^\lambda_{\mu\nu} - \frac{\partial x^{\mu'}}{\partial x^\mu}\frac{\partial x^{\nu'}}{\partial x^\nu}\Gamma^{\lambda'}_{\mu'\nu'} \ . \tag{C.19}$$

This is a defining relation between the inhomogeneous term and connection coefficients in $\left\{x^\mu\right\}$ and $\left\{x^{\mu'}\right\}$ frames.

Let us try to eliminate inhomogeneous term. It is simply impossible since both sides are symmetric in $\left\{\mu, \nu\right\}$ and there seems to be no symmetry relation to eliminate it.

Let us, therefore, consider derivatives of this relation with the hope that the left-hand side can be eliminated at some stage. In this sense, following the derivation of (C.19) with respect to x^β, we make $\beta \leftrightarrow \nu$ changing (possible to chose $\beta \leftrightarrow \mu$), then we see the left hand side is symmetric:

$$\frac{\partial x^{\lambda'}}{\partial x^\beta \partial x^\mu \partial x^\nu} - \frac{\partial x^{\lambda'}}{\partial x^\nu \partial x^\mu \partial x^\beta} = 0 \qquad \text{(C.20)}$$

Hence, the right hand side must be

$$\begin{aligned} 0 &= \frac{\partial x^{\lambda'}}{\partial x^\lambda} \left(\frac{\partial \Gamma^\lambda_{\mu\nu}}{\partial x^\beta} - \frac{\partial \Gamma^\lambda_{\mu\beta}}{\partial x^\nu} + \Gamma^\lambda_{\beta\theta}\Gamma^\theta_{\mu\nu} - \Gamma^\lambda_{\nu\theta}\Gamma^\theta_{\mu\beta} \right) \\ &- \frac{\partial x^{\mu'}}{\partial x^\mu} \frac{\partial x^{\nu'}}{\partial x^\nu} \frac{\partial x^{\beta'}}{\partial x^\beta} \left(\frac{\partial \Gamma^{\lambda'}_{\mu'\nu'}}{\partial x^{\beta'}} - \frac{\partial \Gamma^{\lambda'}_{\mu'\beta'}}{\partial x^{\nu'}} + \Gamma^{\lambda'}_{\beta'\theta'}\Gamma^{\theta'}_{\mu'\nu'} - \Gamma^{\lambda'}_{\nu'\theta'}\Gamma^{\theta'}_{\mu'\beta'} \right) \end{aligned} \qquad \text{(C.21)}$$

Consequently, first derivation of (C.19) implies that

$$\mathcal{R}^{\lambda'}_{\mu'\nu'\beta'} = \frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial x^\nu}{\partial x^{\nu'}} \frac{\partial x^\beta}{\partial x^{\beta'}} \frac{\partial x^{\lambda'}}{\partial x^\lambda} \mathcal{R}^\lambda_{\mu\nu\beta} \qquad \text{(C.22)}$$

which is a clear-cut tensor transformation law. Here,

$$\mathcal{R}^\lambda_{\mu\nu\beta} = \frac{\partial \Gamma^\lambda_{\mu\nu}}{\partial x^\beta} - \frac{\partial \Gamma^\lambda_{\mu\beta}}{\partial x^\nu} + \Gamma^\lambda_{\beta\theta}\Gamma^\theta_{\mu\nu} - \Gamma^\lambda_{\nu\theta}\Gamma^\theta_{\mu\beta} \qquad \text{(C.23)}$$

is called the Riemann curvature tensor as we said before. It is a measure of whether a given manifold is curved or not. It, however, does not carry any information on what the source of curvature is in physical manner.