TOTALLY WEAK SUPPLEMENTED MODULES

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Serpil TOP

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We approve the thesis of **Serpil TOP**

Date of Signature

. . . . **12 July 2007**

Prof. Dr. Rafail ALİZADE
Supervisor
Department of Mathematics
İzmir Institute of Technology

. . . . **12 July 2007**

Assoc. Prof. Dilek PUSAT-YILMAZ
Department of Mathematics
İzmir Institute of Technology

. . . . **12 July 2007**

Assist. Prof. Engin MERMUT
Department of Mathematics
Dokuz Eylul University

. . . . **12 July 2007**

Prof. Dr. Oğuz YILMAZ
Head of Department
Department of Mathematics
İzmir Institute of Technology

. . . .

Prof. Dr. M. Barış ÖZERDEM
Head of the Graduate School

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ABSTRACT

TOTALLY WEAK SUPPLEMENTED MODULES

The main purpose of this thesis is to give a survey about some classes of modules including supplemented, weakly supplemented, totally supplemented and totally weak supplemented modules over commutative Noetherian rings, in particular over Dedekind domains based on results of H. Zöschinger, P. Rudlof and P. F. Smith. A module is weakly supplemented if and only if the factor of that module by a finite direct sum of its hollow submodules is weakly supplemented. A module is weakly supplemented (totally weak supplemented) if and only if the factor of it by a linearly compact submodule is weakly supplemented (totally weak supplemented).

ÖZET

TÜMDEN ZAYIF TÜMLENMİŞ MODÜLLER

Bu tezde temel olarak H. Zöschinger, P. Rudlof ve P. F. Smith'in sonuçlarına dayanan değişmeli Noether halkaları, özel olarak Dedekind tamlık bölgeleri üzerinde tümlenmiş, zayıf tümlenmiş, tümden tümlenmiş ve tümden zayıf tümlenmiş modülleri içeren bazı modül sınıf ları üzerine inceleme yapılması amaçlanmıştır. Bir modülün zayıf tümlenmiş olması için gerek ve yeter koşul o modülün oyuk altmodüllerinin sonlu dik toplamına göre bölüm modülünün zayıf tümlenmiş olmasıdır. Bir modülün zayıf tümlenmiş (tümden zayıf tümlenmiş) olması için gerek ve yeter koşul o modülün lineer kompakt bir altmodülüne göre bölüm modülünün zayıf tümlenmiş (tümden zayıf tümlenmiş) olmasıdır.

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CHAPTER 1

INTRODUCTION

Supplement submodules and some generalizations were intensively investigated in 1970's mainly by H. Zöschinger. During the past ten years there has been an extensive research in this topic. The main results on this topic are published in monografs (Wisbauer 1991, Clark et al. 2006).

In this thesis, we study the classes of supplemented modules, weakly supplemented modules, totally supplemented modules and totally weak supplemented modules. We consider these modules over commutative Noetherian rings, in particular over Dedekind domains.

In Chapter 2, we introduce our basic terminology for rings and modules, as well as the fundamental results to be used in this thesis. Moreover we investigate some well known results about supplements and supplemented modules. If every submodule U of a module M has a *supplement* V in M, i.e. V is minimal with respect to $M = U + V$ then M is said to be *supplemented*. A submodule of a supplemented module need not be supplemented (Zöschinger 1974a). However, a supplemented module M over a commutative Noetherian ring is characterized in terms of a supplemented submodule U and supplemented factor module M/U such that M/U is reduced (Rudlof 1991). At the end of this chapter, we give a general view about the relations between supplemented, weakly supplemented, totally supplemented and totally weak supplemented modules by a diagram.

In Chapter 3, we deal with weakly supplemented modules. If every submodule U of M has a weak supplement, i.e. $M = U + V$ and $U \cap V \ll M$ for some submodule V of M, then M is called weakly supplemented. The main result of Chapter 3 characterizes the weakly supplemented modules in terms of a finite direct sum of hollow submodules and factor modules by them over arbitrary rings. Consequently, weakly supplemented modules have a characterization in terms of a finite direct sum of local submodules and factor modules by them over arbitrary rings. Particularly, a module M is weakly supplemented if and only if $M/\operatorname{Soc} M$ is weakly supplemented for finitely generated $\operatorname{Soc} M$. Finally, in this chapter we study characterizations of weakly supplemented modules over Dedekind domains.

In Chapter 4, we study the main results about totally supplemented modules. In general, the finite sum of totally supplemented modules need not be totally supplemented but a module M is totally supplemented if and only if M is the direct sum of a semisimple module and a totally supplemented module (Smith 2000). We have improved the characterization about supplemented modules (which is given in Chapter 2) for totally supplemented modules and investigated its consequences in this chapter. A supplemented module M over a commutative Noetherian ring is characterized in terms of a supplemented submodule U and supplemented factor module M/U such that U is semi-Artinian (Rudlof 1991). As a result of that characterization, M is totally supplemented if and only if $M/\operatorname{Soc} M$ is totally supplemented. We show that a module M over a Discrete Valuation Ring (DVR) is totally supplemented if and only if $\operatorname{Rad}(M)$ is totally supplemented. In addition, a module M over a DVR is totally supplemented if and only if $T(M)$ and $M/T(M)$ are totally supplemented.

In Chapter 5, we deal with totally weak supplemented modules. For an R-module M, M is supplemented (totally supplemented) if and only if M/K is supplemented (totally supplemented) for a linearly compact submodule K of M (Smith 2000). We have improved this characterization for weakly supplemented (totally weak supplemented) modules in this chapter. A module M is weakly supplemented (totally weak supplemented) if and only if M/K is weakly supplemented (totally weak supplemented) for a linearly compact submodule K of M. Similarly, a module M is weakly supplemented (totally weak supplemented) if and only if M/U is weakly supplemented (totally weak supplemented) for a uniserial submodule U of M. We also show that a module M over a semilocal Dedekind domain is totally weak supplemented if and only if $T(M)$ and $M/T(M)$ are totally weak supplemented. Finally, we study the results on the relations between supplemented, weakly supplemented, totally supplemented and totally weak supplemented modules in this chapter.

CHAPTER 2

PRELIMINARIES

2.1. Isomorphism Theorems

Definition 2.1 *Let R be a ring with identity 1, M be an abelian group and $f : R \times M \longrightarrow M$, $(f(r, m) = rm)$ be a function where $r \in R$, $m \in M$. Then M is called a left R-module (or a module in brief) if the following are satisfied:*
(i) $r(m + n) = rm + rn$ for every $r \in R$ and $m, n \in M$.
(ii) $(r + s)m = rm + sm$ for every $r, s \in R$ and $m \in M$.
(iii) $(rs)m = r(sm)$ for every $r, s \in R$ and $m \in M$.
(iv) $1.m = m$ for every $m \in M$.

If the function $f : M \times R \longrightarrow M$, $(f(m, r) = mr)$ with similar conditions are given in Definition 2.1, then M is called a right R-module. If M is a left R-module, right S-module and $(rm)s = r(ms)$ for every $r \in R$, $m \in M$, $s \in S$ then M is said to be an $R - S$-module or bimodule.

Vector space is an example of module.

A subset N of an R-module M is called a submodule if N satisfies the module conditions.

Throughout this thesis all rings are associative and have an identity. Unless it is stated otherwise, the symbol R stands for a ring, and when R is a domain, Q stands for its field of fractions.

Basic information about modules can be found in related references (Kasch 1982, Anderson and Fuller 1992, Alizade and Pancar 1999). Throughout this study we will use the following definitions, theorems, lemmas and propositions.

Definition 2.2 *Let M be an R-module and N be a submodule of M. The set of cosets $M/N = \{x + N \mid x \in M\}$ is a module relative to the addition and scalar multiplication defined by $(x + N) + (y + N) = (x + y) + N$, $r(x + N) = rx + N$. The resulting module M/N is called a factor module of M by N.*

Definition 2.3 *If M and N are two modules, then a function $f : M \longrightarrow N$ is a homomorphism in case for all $r,\ s \in R$ and all $x,\ y \in M$*

$$f(rx + sy) = rf(x) + sf(y).$$

If $N = M$, then the homomorphism f is called endomorphism.

Theorem 2.1 *Let M be a module over a commutative ring R. The set of endomorphisms of M is a ring with a unit element if addition and multiplication of endomorphisms are defined as:*

$$(f_1 + f_2)(m) = f_1(m) + f_2(m)$$
$$(f_1 f_2)(m) = f_1(f_2(m)).$$

Definition 2.4 *The given ring in Theorem 2.1 is called endomorphism ring and denoted by* $\mathrm{End}_R(M)$.

Definition 2.5 *A homomorphism $f : M \longrightarrow N$ is called an epimorphism if it is onto. It is called a monomorphism if it is one-to-one (injective).*

Definition 2.6 *Kernel of f:* $\mathrm{Ker}\, f = \{m \in M \mid f(m) = 0\} \subseteq M$. *Image of f:* $\mathrm{Im}\, f = \{f(m) \mid m \in M\} \subseteq N$.

Hence f is an epimorphism if and only if $\mathrm{Im}\, f = N$, and f is an monomorphism if and only if $\mathrm{Ker}\, f = 0$.

Definition 2.7 *A homomorphism f is called an isomorphism if it is both an epimorphism and a monomorphism (i.e. it is a bijection).*

Definition 2.8 *If K is a submodule of M, then the monomorphism $i_K : K \longrightarrow M$ is called the inclusion map, in other words natural embedding of K in M. Then the mapping $\sigma_K : M \longrightarrow M/K$ from M onto the factor module M/K defined by $\sigma_K(m) = m + K \in M/K,\ m \in M$ is called the natural (canonical) epimorphism of M onto M/K. In this case,* $\mathrm{Ker}\, \sigma_K = K$.

Definition 2.9 *Let $\{M_i\}_{i\in I}$ be a family of left R-modules. The set $M = \prod_{i\in I} M_i$ with operations $(x_i)_I + (y_i)_I = (x_i + y_i)_I$ and $r(x_i)_I = (rx_i)_I$ where $x_i,\ y_i \in M_i,\ r \in R$ is called the cartesian product or direct product of the family $\{M_i\}_{i\in I}$. The subset $\bigoplus_{i\in I} M_i = \{(x_i)_{i\in I} \in \prod_{i\in I} M_i \mid x_i = 0$ for all but finite $i \in I\}$ of $\prod_{i\in I} M_i$ is a submodule and it is called external direct sum of the family $\{M_i\}_{i\in I}$.*

Definition 2.10 *Let M be an R-module and $\{N_i \mid i \in I\}$ be the set of submodules of M. $M = \bigoplus_{i \in I} N_i$ is called internal direct sum (or direct sum) if the following conditions hold:*
1. $M = \sum_{i \in I} N_i$
2. For every $j \in I$, $N_j \cap \sum_{i \neq j} N_i = 0$
Then $M = \bigoplus_{i \in I} N_i$ is also said to be a decomposition of M.

Definition 2.11 *Let M be an R-module.*
1. A submodule A is called direct summand of M if $M = A \oplus B$ for some submodule $B \subseteq M$.
2. M is called indecomposable if $M \neq 0$ and it can not be written as a direct sum of non-zero submodules.

Theorem 2.2 ***Fundamental Homomorphism Theorem***
Let M and N be left R-modules and $f : M \longrightarrow N$ be a homomorphism, then

$$M/\operatorname{Ker} f \cong \operatorname{Im} f.$$

If f is an epimorphism, then $M/\operatorname{Ker} f \cong N$.

Theorem 2.3 ***Second Isomorphism Theorem***
If N and K are submodules of M, then

$$(N + K)/K \cong N/(N \cap K).$$

Theorem 2.4 ***Third Isomorphism Theorem***
If $K \subseteq N \subseteq M$, then

$$(M/K)/(N/K) \cong M/N.$$

Definition 2.12 *A sequence*

$$\mathbb{S} : \ldots \longrightarrow M_{n+1} \xrightarrow{f_{n+1}} M_n \xrightarrow{f_n} M_{n-1} \longrightarrow \ldots$$

of modules $\{M_n\}_{n \in \mathbb{Z}}$ and homomorphisms $\{f_n\}_{n \in \mathbb{Z}}$ is exact if $\operatorname{Im} f_{n+1} = \operatorname{Ker} f_n$ for each $n \in \mathbb{Z}$.

Proposition 2.1 *If the sequence $0 \xrightarrow{f} A \xrightarrow{g} B \xrightarrow{h} C \xrightarrow{e} 0$ is exact, then g is a monomorphism and h is an epimorphism, so $\operatorname{Im} g \cong A$ and $C \cong B/\operatorname{Im} g$. Thus we can say $C \cong B/A$.*

Definition 2.13 *The exact sequence* $0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$ *is called a short exact sequence.*

Theorem 2.5 *For a short exact sequence*

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

the following are equivalent:

1. *There exists a homomorphism* $h : B \longrightarrow A$ *such that* $h \circ f = 1_A$
2. $\operatorname{Im} f$ *is a direct summand of* B*, i.e.* $B = \operatorname{Im} f \oplus T$ *where* $T \cong C$
3. *There exists a homomorphism* $k : C \longrightarrow B$ *such that* $g \circ k = 1_C$*.*

Definition 2.14 *If any of these conditions of Theorem 2.5 is satisfied, then the short exact sequence*

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

is called a splitting short exact sequence.

Definition 2.15 *Let* M*,* N *and* P *be* R*-modules.* P *is projective if for every epimorphism* $M \xrightarrow{f} N \longrightarrow 0$ *and homomorphism* $g : P \longrightarrow N$ *there is a homomorphism* $h : P \longrightarrow M$ *such that* $f \circ h = g$*. This can be shown by the following commutative diagram.*

$$\begin{tikzcd}
& P \arrow[d, "g"] \arrow[dl, dotted, "h"'] & \\
M \arrow[r, "f"'] & N \arrow[r] & 0
\end{tikzcd}$$

Definition 2.16 *Let* I*,* M *and* N *be* R*-modules.* I *is injective if for every monomorphism* $0 \longrightarrow M \xrightarrow{f} N$ *and homomorphism* $g : M \longrightarrow I$ *there is a homomorphism* $h : N \longrightarrow I$ *such that* $h \circ f = g$*. This can be shown by the following commutative diagram.*

$$\begin{tikzcd}
0 \arrow[r] & M \arrow[r, "f"] \arrow[d, "g"'] & N \arrow[dl, dotted, "h"] \\
& I &
\end{tikzcd}$$

Definition 2.17 *Let* M *be an* R*-module. A submodule* N *is called cofinite if* M/N *is finitely generated (Alizade et al. 2001).*

Definition 2.18 *Let* M *be an* R*-module. A submodule* K *of* M *is small (superfluous) in* M *if for all proper submodules* L *of* M*,* $L + K \neq M$ *holds. Small submodule is denoted by* $K \ll M$ *and* M *is called a small cover of* M/K*.*

Definition 2.19 *Let M be an R-module. A submodule K of M is called a minimal (simple), respectively a maximal submodule of M* $:\Leftrightarrow$

$$0 \neq K \wedge \forall N \subseteq M\,[N \subsetneqq K \Rightarrow N = 0]$$
$$resp.\, K \neq M \wedge \forall N \subseteq M\,[K \subsetneqq N \Rightarrow N = M]$$

Definition 2.20 *Let R be a commutative ring. The Jacobson radical of R is the intersection of all the maximal ideals of R and the Jacobson radical is denoted by* $\mathrm{J}(R)$.

Definition 2.21 *Let M be an R-module. The radical of M is the sum of all small submodules of M, equivalently intersection of all maximal submodules of M. The radical of M is denoted by* $\mathrm{Rad}(M)$.

Definition 2.22 *Let* $(T_\alpha)_{\alpha\in A}$ *be an indexed set of simple (minimal) submodules of M. If M is the direct sum of this set, then*

$$M = \bigoplus_A T_\alpha$$

is a semisimple decomposition of M. A module M is called semisimple in case it has a semisimple decomposition.

Definition 2.23 *Let* $\mathfrak{U}$ *be a class of modules. A module M is (finitely) generated by* $\mathfrak{U}$ *(or* $\mathfrak{U}$ *(finitely) generates M) in case there is a (finite) indexed set* $(U_\alpha)_{\alpha\in A}$ *in* $\mathfrak{U}$ *and an epimorphism*

$$\bigoplus_A U_\alpha \longrightarrow M \longrightarrow 0.$$

Definition 2.24 *Let* $\mathfrak{U}$ *be a class of modules. A module M is (finitely) cogenerated by* $\mathfrak{U}$ *(or* $\mathfrak{U}$ *(finitely) cogenerates M) in case there is a (finite) indexed set* $(U_\alpha)_{\alpha\in A}$ *in* $\mathfrak{U}$ *and a monomorphism*

$$0 \longrightarrow M \longrightarrow \prod_A U_\alpha.$$

Theorem 2.6 *For an R-module M, the following statements are equivalent:*

(a) M is semisimple;

(b) M is generated by simple modules;

(c) M is the sum of some set of simple modules;

(d) M is the sum of its simple submodules;

(e) Every submodule of M is a direct summand;

(f) Every short exact sequence

$$0 \longrightarrow K \longrightarrow M \longrightarrow N \longrightarrow 0$$

of R-modules splits (Anderson and Fuller 1992).

Definition 2.25 *Let M be an R-module. A submodule K of M is said to be large or essential if $K \cap L \neq 0$ for every non-zero submodule $L \subseteq M$ and this denoted by $K \trianglelefteq M$.*

Definition 2.26 *Let M be an R-module. The socle of M, denoted by* $\operatorname{Soc}(M)$, *is the sum of all simple submodules of M, equivalently intersection of all essential submodules of M.*

Remark 2.1 *Note that* $\operatorname{Soc} M$ *is the largest semisimple submodule and M is semisimple if and only if* $M = \operatorname{Soc}(M)$.

Definition 2.27 *Let M be an R-module. A pair* (P, p) *is a projective cover of M in case P is a projective R-module and*

$$P \xrightarrow{p} M \longrightarrow 0$$

is a small epimorphism $(\operatorname{Ker} p \ll P)$.

Definition 2.28 *Let M be an R-module. A pair* (I, ε) *is an injective hull of M in case I is an injective R-module and*

$$0 \longrightarrow M \xrightarrow{\varepsilon} I$$

is an essential monomorphism $(\operatorname{Im} \varepsilon \trianglelefteq I)$.

Lemma 2.1 *Let K, N, L be submodules of M and $N \subseteq K$ then*

$$K \cap (N + L) = N + K \cap L.$$

Proof ($\subseteq$) Let $k \in K \cap (N + L)$. Then k can be represented as $k = n + l$ for some $n \in N$, $l \in L$. Since $N \subseteq K$, $n \in K$ we have $l = k - n \in K + N \subseteq K + K = K$. Hence $l \in K \cap L$ and $k = n + l \in N + K \cap L$.

($\supseteq$) Obvious. □

Lemma 2.2 *Let M be an R-module and $K \subseteq L$ and $L_i\,(1 \leq i \leq n)$ be submodules of M for some positive integer n. Then the following hold:*

1. $L \ll M$ if and only if $K \ll M$ and $L/K \ll M/K$.

2. $L_1 + L_2 + \cdots + L_n \ll M$ if and only if $L_i \ll M\,(1 \leq i \leq n)$.

3. If M' is an R-module and $\varphi : M \longrightarrow M'$ is a homomorphism, then $\varphi(L) \ll M'$ whenever $L \ll M$.

4. If L is a direct summand of M, then $K \ll L$ if and only if $K \ll M$.

Proof 1. ($\Rightarrow$) Let $K + N = M$ for some $N \subseteq M$. Since $K \subseteq L$ we have $L + N = M$. Thus $N = M$ since $L \ll M$. Hence $K \ll M$.

Let $L/K + T/K = M/K$ for some $T \subseteq M$ containing K. Then $L + T = M$. Since $L \ll M$ we have $T = M$ and this implies that $T/K = M/K$. Thus $L/K \ll M/K$.

($\Leftarrow$) Let $L + N = M$ for some submodule N of M. Thus $L/K + (N + K)/K = M/K$. Since $L/K \ll M/K$, $(N + K)/K = M/K \Rightarrow N + K = M$. Since $K \ll M$, $N = M$. Hence $L \ll M$.

2. ($\Rightarrow$) Let $L_i + N = M$ for some submodule N of M. For $i \neq j\,(j = 1, 2, \ldots, n)$, $L_1 + L_2 + \cdots + L_i + \cdots + L_n + N = M$. By hypothesis, $L_1 + L_2 + \cdots + L_n \ll M$, so $N = M$, therefore $L_i \ll M$.

($\Leftarrow$) Let each $L_i \ll M$ and $L_1 + L_2 + \cdots + L_n + N = M$. Since $L_1 \ll M$, $L_2 + \cdots + L_n + N = M$. Then since $L_2 \ll M$, $L_3 + \cdots + L_n + N = M$. Continuing in this way $N = M$, therefore $L_1 + L_2 + \cdots + L_n \ll M$.

3. Let $\varphi(L) + N = M'$ for some submodules $N \subseteq M'$ and $L \subseteq M$. $M = \varphi^{-1}(M') = \varphi^{-1}(\varphi(L) + N) = \varphi^{-1}(\varphi(L)) + \varphi^{-1}(N) = (L + \operatorname{Ker}\varphi) + \varphi^{-1}(N) = L + \varphi^{-1}(N)$. Since $L \ll M$, $\varphi^{-1}(N) = M$. $M' = \varphi(L) + N \subseteq \varphi(M) + N = \varphi(\varphi^{-1}(N)) + N \subseteq N \Rightarrow M' = N$. Hence $\varphi(L) \ll M'$.

4. ($\Rightarrow$) Let $K + T = M$ for some submodule T of M. Then $(K + T) \cap L = L$. By Modular Law, $K + (T \cap L) = L$. Since $K \ll L$, $T \cap L = L \Rightarrow L \subseteq T$. Since $K \subseteq L$, $K \subseteq T$, i.e. $M = K + T = T \Rightarrow M = T \Rightarrow K \ll M$.

($\Leftarrow$) Let $K \ll M$. Suppose L is a direct summand of M. There exists a submodule N of M such that $L + N = M$ and $L \cap N = 0$. Let $K + T = L$ for some submodule T of L. $M = L + N = K + T + N$. Since $K \ll M$, $T + N = M$. Then by Modular Law $L = (T + N) \cap L = T + N \cap L$. Since $N \cap L = 0$, $L = T$, therefore $K \ll L$. $\square$

Definition 2.29 *An R-module M is called Noetherian if every non-empty set of sub-*

modules of M has a maximal element. M is artinian if every non-empty set of submodules has a minimal element.

Theorem 2.7 *Let M be an R-module and A be a submodule of M. The following properties are equivalent:*

1. M is Noetherian.

2. A and M/A are Noetherian.

3. Every ascending chain $A_1 \subset A_2 \subset A_3 \subset \cdots$ of submodules holds ascending chain condition, i.e. every ascending chain of submodules of M is stationary.

4. Every submodule of M is finitely generated.

5. In every set $\{A_i \mid i \in I\} \neq \emptyset$ of submodules $A_i \subset M$ there is a finite subset $\{A_i \mid i \in I_0\}$ (i.e. finite $I_0 \subset I$) with

$$\sum_{i \in I} A_i = \sum_{i \in I_0} A_i.$$

Theorem 2.8 *Let M be an R-module and A be a submodule of M. The following properties are equivalent:*

1. M is artinian.

2. A and M/A are artinian.

3. Every descending chain $A_1 \supset A_2 \supset A_3 \supset \cdots$ of submodules holds descending chain condition, i.e. every descending chain of submodules of M is stationary.

4. Every factor module of M is finitely cogenerated.

5. In every set $\{A_i \mid i \in I\} \neq \emptyset$ of submodules $A_i \subset M$ there is a finite subset $\{A_i \mid i \in I_0\}$ (i.e. finite $I_0 \subset I$) with

$$\bigcap_{i \in I} A_i = \bigcap_{i \in I_0} A_i.$$

Definition 2.30 *A submodule N of M is called radical if* $\mathrm{Rad}(N) = N$.

Definition 2.31 *For an R-module M, let* $P(M) = \sum\{N \subseteq M \mid \mathrm{Rad}(N) = N\}$. *The module M is said to be reduced if* $P(M) = 0$ *equivalently, every non-zero submodule has a maximal submodule.*

Remark 2.2 *Every submodule of a reduced module is reduced.*

Definition 2.32 *A module M is called coatomic if* $\mathrm{Rad}(M)/U \neq M/U$ *for every proper submodule U of M equivalently, every proper submodule of M is contained in a maximal submodule of M.*

Remark 2.3 *Every factor module of a coatomic module is coatomic. Finitely generated modules and semisimple modules are coatomic. Note that* $\operatorname{Rad}(M) \ll M$ *for every coatomic module* M.

Theorem 2.9 *Let* M *be a coatomic module over a commutative Noetherian ring. Then every submodule of* M *is coatomic (Zöschinger 1980).*

2.2. Supplement

Definition 2.33 *Let* U *be a submodule of an* R*-module* M. *If there exists a submodule* V *of* M *minimal with respect to the property* $M = U + V$, *then* V *is called a supplement of* U *in* M.

Lemma 2.3 V *is a supplement of* U *in* M *if and only if* $U + V = M$ *and* $U \cap V \ll V$.

Proof ($\Rightarrow$) Let V be a supplement of U in M such that $M = U + V$. Suppose $(U \cap V) + X = V$ for some $X \subseteq V$, then $M = U + V = U + (U \cap V) + X = U + X$. By minimality of V, $X = V$. Thus $U \cap V \ll V$.
($\Leftarrow$) Let $M = U + V$ and $U \cap V \ll V$. Suppose $M = U + Y$ for some $Y \subseteq V$. $V = M \cap V = (U + Y) \cap V = (U \cap V) + Y$ by Modular Law. Then $Y = V$ since $U \cap V \ll V$. Hence V is a supplement of U in M. □

The following proposition gives some properties of supplement.

Proposition 2.2 *Let* $U, V \subseteq M$ *and* V *be a supplement of* U *in* M.
1. *If* $W + V = M$ *for some* $W \subseteq U$, *then* V *is a supplement of* W.
2. *If* M *is finitely generated, then* V *is also finitely generated.*
3. *If* U *is a maximal submodule of* M, *then* V *is cyclic and* $U \cap V = \operatorname{Rad}(V)$ *is a (the unique) maximal submodule of* V.
4. *If* $K \ll M$ *then,* V *is a supplement of* $U + K$.
5. *If* $K \ll M$, *then* $V \cap K \ll V$ *and* $\operatorname{Rad}(V) = V \cap \operatorname{Rad}(M)$.
6. *If* $\operatorname{Rad}(M) \ll M$, *then* U *is contained in a maximal submodule of* M.
7. *If* $L \subseteq U$, $V + L/L$ *is a supplement of* U/L *in* M/L.
8. *If* $\operatorname{Rad}(M) \ll M$ *or* $\operatorname{Rad}(M) \subseteq U$ *and* $p : M \longrightarrow M/\operatorname{Rad}(M)$ *is canonical epimorphism, then* $M/\operatorname{Rad}(M) = p(U) \oplus p(V)$ *(Wisbauer 1991).*

Remark 2.4 *Zero module is a trivial supplement of every module.*

Lemma 2.4 *Let M be an R-module. If every submodule of M is a supplement in M, then M is semisimple.*

Proof Suppose $\mathrm{Rad}(M) \neq 0$. Then there exists a non-zero element in $\mathrm{Rad}(M)$, say x. By hypothesis Rx is a supplement that is $Rx + K = M$ and $Rx \cap K \ll Rx$ for some $K \subseteq M$. Since $x \in \mathrm{Rad}\, M$, $Rx \ll M$ and $K = M$. Thus $Rx \ll Rx$, contradiction. Therefore $\mathrm{Rad}\, M = 0$. Now let N be a submodule of M. Since N is a supplement $N + N' = M$ and $N \cap N' \ll N$ for some $N' \subseteq M$. It is clear that $N \cap N' \subseteq \mathrm{Rad}(M) = 0 \Rightarrow N \cap N' = 0$. Hence $M = N \oplus N'$ and M is semisimple. $\square$

2.3. Coclosed Submodules

Definition 2.34 *A submodule N of an R-module M is called closed if N has no proper essential extension in M, i.e. if $N \trianglelefteq K$ for some $K \subseteq M$, then $K = N$.*

Definition 2.35 *A submodule N of an R-module M is coclosed in M, if whenever $N/K \ll M/K$ for some $K \subseteq M$ implies that $N = K$.*

The relation between supplements and coclosed submodules is given in the following proposition.

Proposition 2.3 *Let N be a submodule of M. Consider the following statements:*
(i) N is a supplement in M,
(ii) N is coclosed in M,
(iii) For all $K \subseteq N$, $K \ll M$ implies $K \ll N$.

Then $(i) \Rightarrow (ii) \Rightarrow (iii)$ holds and if N is a weak supplement in M, then $(iii) \Rightarrow (i)$ holds (Lomp 1996).

2.4. Hollow and Uniform Modules

Definition 2.36 *Let M be an R-module. If every proper submodule of M is small in M, then M is called a hollow module.*

Definition 2.37 *An R-module M is called uniform if every non-zero submodule of M is essential in M.*

Definition 2.38 *M is said to have finite uniform dimension (or finite Goldie dimension) if there exists a sequence*

$$0 \longrightarrow \bigoplus_{i=1}^{n} U_i \xrightarrow{f} M$$

where all the U_i are uniform and the image of f is essential in M. Then n is called the uniform dimension (Goldie dimension) of M and we write $udim(M) = n$. If such an integer doesn't exist, we write $udim(M) = \infty$.

Definition 2.39 *M is said to have hollow dimension (or finite dual Goldie dimension) if there exists an exact sequence*

$$M \xrightarrow{g} \bigoplus_{i=1}^{n} H_i \longrightarrow 0$$

where all the H_i are hollow and the kernel of g is small in M. Then n is called the hollow dimension (dual Goldie dimension) of M and we write $hdim(M) = n$.

2.5. Local and Semilocal Rings

Definition 2.40 *A ring is called local if it has a unique maximal ideal.*

Definition 2.41 *Let R be a ring. If every non-zero element of R is invertible, then R is called division ring.*

Proposition 2.4 *The following are equivalent for a ring R with radical* $\mathrm{J}(R)$*:*

(a) R is local;

(b) R has a unique maximal left ideal;

(c) $\mathrm{J}(R)$ *is a maximal left ideal;*

(d) The set of elements of R without left inverses is closed under addition;

(e) $\mathrm{J}(R) = \{x \in R \mid Rx \neq R\}$*;*

(f) $R/\mathrm{J}(R)$ *is a division ring;*

(g) $\mathrm{J}(R) = \{x \in R \mid x \text{ is not invertible}\}$*;*

(h) If $x \in R$ then either x or $1 - x$ is invertible (Anderson and Fuller 1992).

Definition 2.42 *A ring R is called semilocal if* $R/\mathrm{J}(R)$ *is a semisimple ring.*

Proposition 2.5 *For a ring R, consider the following two conditions:*

(i) R is semilocal,

(ii) R has finitely many maximal ideals.

In general, we have $(ii) \Rightarrow (i)$. The converse holds if $R/\,\mathrm{J}(R)$ is commutative (Lam 2001).

Theorem 2.10 *For any ring R, the following statements are equivalent:*

(a) R is semilocal,

(b) Every left R-module is semilocal,

(c) Every left R-module is the direct sum a semisimple module and a semilocal module with essential radical,

(d) Every left R-module with small radical is weakly supplemented,

(e) Every finitely generated left R-module has finite hollow dimension,

(f) Every product of semisimple left R-modules is semisimple,

(g) There exists an $n \in \mathbb{N}$ *and a map* $d : R \longrightarrow \{0, 1, \ldots, n\}$ *such that for all* $a,\ b \in R$ *the following holds:*

(i) $d(a) = 0 \Rightarrow a$ *is a unit,*

(ii) $d(a(1 - ba)) = d(a) + d(1 - ba)$*,*

(h) There exists a partial ordering $(R, \leq)$ *such that:*

(i) $(R, \leq)$ *is an artinian poset (partially ordered set),*

(ii) For all $a,\ b \in R$ *such that* $1 - ba$ *is not a unit, we have* $a > a(1 - ba)$ *(Lomp 1999).*

2.6. Perfect and Semiperfect Rings

Definition 2.43 *A ring R is called perfect if every R-module has a projective cover.*

Lemma 2.5 *For a ring R, the following are equivalent:*

(i) R is perfect ring;

(ii) Every left R-module is supplemented;

(iii) Every left R-module is amply supplemented;

(iv) The left R-module $R^{(\mathbb{N})}$ *is supplemented (Smith 2000).*

Definition 2.44 *Let M be an R-module. M is called cofinitely supplemented if every cofinite submodule of M has a supplement in M.*

Definition 2.45 *A ring R is called semiperfect if every finitely generated R-module has a projective cover.*

Lemma 2.6 *The following statements are equivalent for a ring R:*

(i) R is semiperfect;

(ii) Every finitely generated left (respectively, right) R-module is supplemented;

(iii) Every finitely generated left (respectively, right) R-module is amply supplemented;

(iv) The left (respectively, right) R-module R is cofinitely supplemented;

(v) For every left (respectively, right) R-module M, every maximal submodule has ample supplements in M (Smith 2000).

Definition 2.46 *Let M be an R-module. M is called semiperfect if every factor module of M has a projective cover (Wisbauer 1991).*

Definition 2.47 *Let M be an R-module. M is called f-semiperfect if for every finitely generated submodule $K \subseteq M$, the factor module M/K has a projective cover (Wisbauer 1991).*

Definition 2.48 *Let M be an R-module. M is called perfect if for every index set Λ, the sum $M^{(\Lambda)}$ is semiperfect (Wisbauer 1991).*

Definition 2.49 *Let R be a ring. An element a of R is called idempotent if $a^2 = a$.*

Definition 2.50 *Let R be a ring and $e, f \in R$ be two idempotents. If $ef = fe = 0$, then e, f are called orthogonal idempotents.*

Definition 2.51 *An ideal (right, left, two sided) I is called right (resp. left) T-nilpotent if for any sequence $a_1, a_2, \ldots, a_n, \ldots$ of elements $a_i \in I$ there exists a positive integer k such that $a_k a_{k-1} \ldots a_1 = 0$ (resp. $a_1 a_2 \ldots a_k = 0$). An ideal I is called T-nilpotent if it is right and left T-nilpotent.*

Definition 2.52 *An element a is called nilpotent if there exists a positive integer n such that $a^n = 0$. An ideal is called a nil-ideal if all its elements are nilpotent.*

Remark 2.5 *Clearly a T-nilpotent ideal is a nil-ideal.*

The following proposition gives the relation between perfect ring and T-nilpotent ideal. ${}_R R^{(\mathbb{N})}$ denotes the direct sum of R-module R by index set $\mathbb{N}$.

Proposition 2.6 *For a ring R, the following assertions are equivalent:*

(a) ${}_RR$ is perfect;

(c) Every left R-module (or only $R^{(\mathbb{N})}$) is semiperfect;

(d) Every left R-module has a projective cover;

(e) Every left R-module is (amply) supplemented;

(f) $R/\operatorname{J}(R)$ is left semisimple and $\operatorname{Rad}(R^{(\mathbb{N})}) \ll_R R^{(\mathbb{N})}$;

(g) The ascending chain condition for cyclic left R-modules holds;

(h) $End_R(R^{(\mathbb{N})})$ is f-semiperfect;

(i) $R/\operatorname{J}(R)$ is left semisimple and $\operatorname{J}(R)$ is right T-nilpotent;

(j) R satisfies the descending chain condition for cyclic right ideals;

(k) R contains no infinite set of orthogonal idempotents and every non-zero right R-module has non-zero socle (Wisbauer 1991).

2.7. Dedekind Domains

Definition 2.53 *Let R be a ring. R is called an integral domain if it is commutative and has no zero divisors.*

Definition 2.54 *Let R be an integral domain and M be an R-module. The submodule $T(M) = \{m \in M \mid rm = 0$ for some $0 \neq r \in R\}$ of M is said to be torsion submodule of M. If $T(M) = M$, then M is called a torsion module, and if $T(M) = 0$, then M is called a torsion-free module.*

Definition 2.55 *Let R be a ring and $\mathfrak{p}$ be a prime ideal of R. The submodule $\{m \in M \mid \mathfrak{p}^n m = 0$ for some $n \geq 1\}$ is called $\mathfrak{p}$-primary part of M. This submodule is indicated by $T_{\mathfrak{p}}(M)$.*

Definition 2.56 *Let M be an R-module. The maximum number of linearly independent elements of M is called rank of M.*

Definition 2.57 *A commutative ring R is a valuation ring if its ideals are totally ordered by inclusion. Additionally, if R is an integral domain, it is called a valuation domain. A Noetherian valuation domain is said to be discrete valuation ring (shortly DVR). If R is a DVR, then all of its non-zero ideals are: $R > Rp > \cdots > Rp^n > \cdots$ where $p \in R$ is the unique prime element (Fuchs and Salce 1985).*

Definition 2.58 *Let R be an integral domain and Q be its field of fractions. An element of Q is said to be integral over R if it is a root of a monic polynomial in* $R[X]$ *(the ring of polynomials in X with coefficients in R). A commutative domain R is integrally closed if the elements of Q which are integral over R are just the elements of R.*

Definition 2.59 *A ring R is said to be Noetherian if it satisfies the following three equivalent conditions:*
(i) Every non-empty set of ideals in R has a maximal element,
(ii) Every ascending chain of ideals in R is stationary,
(iii) Every ideal of R is finitely generated.

Definition 2.60 *An integral domain R is a Dedekind domain if the following hold:*
1. R is a Noetherian ring.
2. R is integrally closed in its field of fractions Q.
3. All non-zero prime ideals of R are maximal.

Remark 2.6 *Local Dedekind domain is a DVR (Zöschinger 1974a).*

Definition 2.61 *Let R be an integral domain. R is called principal ideal domain if every ideal of R is cyclic. A principal ideal domain is a Dedekind domain.*

Theorem 2.11 *For a Noetherian domain R, the following are equivalent:*
(i) Every coclosed module is closed,
(ii) Every closed module is coclosed,
(iii) R is Dedekind (Zöschinger 1974a).

Definition 2.62 *Let R be a ring and*

$$I_1 \supseteq I_2 \supseteq I_3 \cdots$$

be a descending chain of two-sided ideals of R. Such a chain is called filtration of R and can be used to give R the structure of topological space (Sharpe and Vamos 1972).

Definition 2.63 *Let R be a ring with a filtration*

$$I_1 \supseteq I_2 \supseteq I_3 \cdots$$

A subset U of R is said to be open if, whenever $r \in U$*, then there exists n such that* $r + I_n \subseteq U$*. The empty set and R itself are open (Sharpe and Vamos 1972).*

Clearly:

(i) The empty set and R itself are open.

(ii) The union of an arbitrary family of open subsets of R is open.

(iii) The intersection of a finite number of open subsets of R is open.

Thus R is a topological space. If $\mathfrak{p}$ is a prime ideal in a domain R, then the topology given by the filtration $I_n = \mathfrak{p}^n$ is called the $\mathfrak{p}$-adic topology.

Remark 2.7 *Note that the sets $r + I_n$, where $r \in R$, are open subsets of R. In particular, the two sided ideals*

$$I_n\,(n = 1, 2, \ldots)$$

are open in R.

Definition 2.64 *Let R be topological space. A neighborhood of a point x is a set containing an open set which involves x. R is called Hausdorff if for every r, $s \in R$, $r \neq s$ there exists disjoint neighborhoods I and I' such that $r \in I$ and $s \in I'$.*

Remark 2.8 *Let R be a topological space with a filtration*

$$I_1 \supseteq I_2 \supseteq I_3 \cdots$$

Suppose that $\bigcap_{n=1}^{\infty} I_n = 0$ and let r, $s \in R$, $r \neq s$. Then there exists n such that $r - s \notin I_n$. It follows that $r + I_n$ and $s + I_n$ do not meet. Thus disjoint open subsets U and V of R such that $r \in U$ and $s \in V$ are found. This says that R is a Hausdorff space. On the other hand, if $\bigcap_{n=1}^{\infty} I_n \neq 0$, then there exists $r \in \bigcap_{n=1}^{\infty} I_n$, $r \neq 0$, and every open set which contains 0 also contains r. Hence R is Hausdorff space if and only if $\bigcap_{n=1}^{\infty} I_n = 0$.

Definition 2.65 *A sequence $\{r_n\}$ of elements of R is convergent with limit r if, given a positive integer k, there exists an integer N such that $r_n - r \in I_k$ whenever $n \geq N$.*

Definition 2.66 *Let R be topological space with a filtration. A sequence $\{r_n\}$ of elements of R is a Cauchy sequence if, given a positive integer k, there exists an integer N such that*

$$r_m - r_n \in I_k$$

whenever m, $n \geq N$. Every convergent sequence is a Cauchy sequence (Sharpe and Vamos 1972).

Definition 2.67 *If every Cauchy sequence is convergent and if, further, R is Hausdorff, then R is said to be complete in its topology (Sharpe and Vamos 1972).*

Definition 2.68 *Let R be a DVR. R is said to be complete if it is complete in its $\mathfrak{p}$-adic topology where $\mathfrak{p} = Rp$ is the unique maximal ideal in R (Kaplansky 1969). R is called incomplete if it is not complete.*

2.8. Local Modules

Definition 2.69 *A module is called local if it has a largest proper submodule. Equivalently, a module is local if and only if it is cyclic, non-zero, and has a unique maximal proper submodule.*

Proposition 2.7 *An R-module M is local if and only if there is a maximal submodule N such that $N \ll M$.*

Proof ($\Rightarrow$) Let M be an R-module and with $X + N = M$ for some submodule X and a maximal submodule N of M. If $X \neq M$, then $X \subseteq N$. Thus $N = X + N = M \Rightarrow N = M$, contradiction, since N is maximal submodule. Hence $X = M \Rightarrow N \ll M$.
($\Leftarrow$) Let X be a proper submodule of M. Then $N \subseteq N + X \subseteq M$. In this case either $N = N + X$ or $N + X = M$. If $N + X = M$, then $X = M$ since $N \ll M$, contradiction. Therefore $N = N + X \Rightarrow X \subseteq N \Rightarrow N$ is the largest submodule of M. □

Proposition 2.8 *If an R-module M is local, then M is hollow.*

Proof Let M be an R-module. For every proper submodule X of M, $X \subseteq \operatorname{Rad} M \ll M$. Hence $X \ll M$ and M is hollow. □

Proposition 2.9 *Let M be an R-module. The following assertions are equivalent:*
(a) M is hollow and $\operatorname{Rad}(M) \neq M$*,*
(b) M is hollow and cyclic (or finitely generated),
(c) M is local (Wisbauer 1991).

2.9. Semilocal Modules

Definition 2.70 *An R-module M is called semilocal if* $M/\operatorname{Rad}(M)$ *is semisimple.*

Proposition 2.10 *For a proper submodule N of M, the following are equivalent:*

(i) M/N is semisimple;

(ii) For every $L \subseteq M$ there exists a submodule $K \subseteq M$ such that $L+K = M$ and $L \cap K \subseteq N$;

(iii) There exists a decomposition $M = M_1 \oplus M_2$ such that M_1 is semisimple, $N \trianglelefteq M_2$ and M_2/N is semisimple (Lomp 1999).

2.10. Supplemented Modules

Definition 2.71 *Let M be an R-module. If every submodule of M has a supplement, then M is called a supplemented module.*

Note that each module need not be supplemented by the following example.

Example 2.1 *In the $\mathbb{Z}$-module $\mathbb{Z}$ every non-zero proper submodule has no supplements.*

Proof Let $n > 1$ and suppose $m\mathbb{Z}$ is a supplement of $n\mathbb{Z}$: $n\mathbb{Z} + m\mathbb{Z} = \mathbb{Z} \Rightarrow (n, m) = 1$. If $m = \mp 1$ take some $m' \neq 1$ with $(n, m') = 1 \Rightarrow n\mathbb{Z} + m'\mathbb{Z} = \mathbb{Z}$ such that $m'\mathbb{Z} \neq \mathbb{Z}$. If $m \neq \mp 1$, then $(n, m^2) = 1 \Rightarrow n\mathbb{Z} + m^2\mathbb{Z} = \mathbb{Z}$ such that $m^2\mathbb{Z} \subsetneq m\mathbb{Z}$. Therefore there is no proper submodule which has a supplement in $\mathbb{Z}$-module $\mathbb{Z}$. □

Example 2.2 *Artinian and semisimple modules are supplemented modules.*

Proposition 2.11 *If an R-module M is hollow, then M is supplemented.*

Proof Let M be an R-module and K be a submodule of M. Then $K + M = M$. By hypothesis, $K \cap M = K \ll M$. Therefore M is supplemented. □

Corollary 2.1 *If an R-module M is local, then M is supplemented.*

Definition 2.72 *Let M be an R-module. Then an R-module N is called (finitely) M-generated if it is a homomorphic image of a (finite) direct sum of copies of M.*

The following proposition gives some properties of supplemented modules.

Proposition 2.12 *For an R-module M, the following properties hold:*

(i) Let U and V be submodules of M such that U is supplemented and U + V have a supplement in M. Then V has a supplement in M.

(ii) If $M = M_1 + M_2$ with M_1 and M_2 are supplemented modules, then M is also supplemented.

(iii) If M is supplemented, then

(a) Every finitely M-generated module is supplemented.

(b) $M/\operatorname{Rad}(M)$ is semisimple (Wisbauer 1991).

Proof (i) Let X be a supplement of $U + V$ in M, i.e.

$$M = (U + V) + X \text{ and } (U + V) \cap X \ll X$$

and let Y be a supplement of $(V + X) \cap U$ in U, i.e.

$$U = (V + X) \cap U + Y \text{ and } ((V + X) \cap U) \cap Y \ll Y.$$

Since $Y \subseteq U$,

$$Y + V \subseteq U + V \Rightarrow (Y + V) \cap X \subseteq (U + V) \cap X.$$

Thus $(Y + V) \cap X \ll X$ since $(U + V) \cap X \ll X$. Now

$$M = U + V + X = (V + X) \cap U + Y + V + X = Y + V + X$$

and

$$Y \cap (V + X) = Y \cap U \cap (V + X) \ll Y,$$

i.e. Y is a supplement of $V + X$ in M. Hence we obtain

$$(X + Y) \cap V \subseteq (X \cap (Y + V)) + (Y \cap (V + X)) \ll X + Y.$$

Thus $X + Y$ is a supplement of V in M.

(ii) Let U be a submodule of M. Then $M = M_1 + M_2 + U$ and since 0 (zero) module is a supplement for $M_1 + M_2 + U$ and M_1 is supplemented, $M_2 + U$ has a supplement. Hence U has a supplement in M since M_2 is supplemented. Therefore M is supplemented.

(iii)a Let N be a finitely M-generated module. Then there exists an epimorphism $\bigoplus_F M \xrightarrow{f} N \longrightarrow 0$ such that F is finite. Since M is supplemented a finite sum of M is also supplemented. By first isomorphism theorem $\bigoplus_F M/\operatorname{Ker} f \cong N$. Since every factor module of a supplemented module is supplemented, $\bigoplus_F M/\operatorname{Ker} f$ is supplemented. Hence N is supplemented.

(iii)b Let $N/\operatorname{Rad}(M)$ be a submodule of $M/\operatorname{Rad}(M)$. Since M is supplemented there exists a supplement K of N in M, i.e. $N + K = M$ and $N \cap K \ll K$. Then we obtain

$$M/\operatorname{Rad}(M) = (N + K)/\operatorname{Rad}(M) = (N/\operatorname{Rad}(M)) + ((K + \operatorname{Rad}(M))/\operatorname{Rad}(M))$$

$$(N/\operatorname{Rad}(M)) \cap ((K + \operatorname{Rad}(M))/\operatorname{Rad}(M)) = (N \cap K + \operatorname{Rad}(M))/\operatorname{Rad}(M)$$

Since $N \cap K \ll K$, $N \cap K \ll M$. Hence

$$(N \cap K) \subseteq \operatorname{Rad}(M) \Rightarrow (N \cap K) + \operatorname{Rad}(M) = \operatorname{Rad}(M).$$

Thus

$$(N/\operatorname{Rad}(M)) \cap ((K + \operatorname{Rad}(M))/\operatorname{Rad}(M)) = 0$$

so $N/\operatorname{Rad}(M)$ is a direct summand of $M/\operatorname{Rad}(M)$. Thus $M/\operatorname{Rad}(M)$ is semisimple. □

Corollary 2.2 *Let R be a ring and M be an R-module. If M is supplemented and* $\operatorname{Rad}(M) = 0$, *then M is semisimple.*

Proof Clear by Proposition 2.12 ((iii)b). □

Remark 2.9 *If*

$$A \xrightarrow{\alpha} B \xrightarrow{\beta} C$$

are two epimorphisms, then $\beta\alpha$ is small if and only if α and β are small, i.e. $\ker \beta\alpha \ll A$ *if and only if* $\ker \alpha \ll A$ *and* $\ker \beta \ll B$ *(Wisbauer 1991).*

Proposition 2.13 *Let R be a ring and M be an R module with $N \subseteq M$. If in the exact sequence*

$$0 \longrightarrow N \longrightarrow M \longrightarrow M/N \longrightarrow 0$$

N, M/N are supplemented and N has a supplement in every H with $N \subseteq H \subseteq M$, then M is supplemented.

Proof Let U be a submodule of M, V/N be a supplement of $(U+N)/N$ in M/N and W be a supplement of N in V. Then $W \cap N \ll W$ and $(V/N) \cap ((U+N)/N) \ll V/N$. Hence the following epimorphism

$$W \longrightarrow V/N \longrightarrow M/(U+N)$$

is small by Remark 2.9, i.e. $W \cap (U+N) \ll W$ so W is a supplement of $(U+N)$ in M. Then by Proposition 2.12 (i), U has a supplement. Therefore M is supplemented. □

Definition 2.73 *Let M and M_i, $(i \in I)$ be R modules. If $M = \sum_I M_i$ and for every $j \in I$, $M \neq \sum_{i \neq j} M_i$, then the sum $\sum_I M_i$ is called irredundant.*

The following proposition gives a characterization of supplemented modules.

Proposition 2.14 *Let M be an R-module.*

1. For a finitely generated module M, the following are equivalent:

(a) M is supplemented.

(b) Every maximal submodule of M has a supplement in M.

(c) M is a sum of hollow submodules.

(d) M is an irredundant sum of local submodules.

2. If M is supplemented and $\mathrm{Rad}(M) \ll M$, then M is an irredundant sum of local modules (Wisbauer 1991).

Theorem 2.12 *Let M be a module over a commutative Noetherian ring. For a submodule U of M such that M/U is reduced, the following are equivalent:*

(i) M is supplemented;

(ii) U and M/U are supplemented (Rudlof 1991).

Corollary 2.3 *Let R be a commutative Noetherian ring and M be a coatomic R-module. M is supplemented if and only if U and M/U are supplemented for every submodule U of M (Büyükaşık 2005).*

Proof Clearly M/U is coatomic. Then by Theorem 2.9, every submodule of M/U is coatomic. Hence every submodule of M/U contains a maximal submodule, i.e. $P(M/U) = 0$. This means that M/U is reduced. Therefore the proof is completed by Theorem 2.12. □

Theorem 2.13 *If M is a supplemented module, then every submodule X of M with $P(M) \subseteq X \subseteq M$ is supplemented (Rudlof 1991).*

Definition 2.74 *Let M be an R-module. M is called uniserial if its submodules are linearly ordered by inclusion.*

The following proposition gives some characterizations of uniserial module.

Proposition 2.15 *Let M be an R-module. The following are equivalent:*
(a) M is uniserial;
(b) Every factor module of M is uniform;
(c) Every factor module of M has zero or simple socle;
(d) Every submodule of M is hollow;
(e) Every finitely generated submodule of M is local;
(f) Every submodule of M has at most one maximal submodule (Clark et al. 2006).

Definition 2.75 *Let M be an R-module. A family $\{M_i\}_\triangle$ of submodules is called inverse if the intersection of two of its modules contains again a module in $\{M_i\}_\triangle$.*

Definition 2.76 *Let M be an R-module. M is called linearly compact if for every family of cosets $\{x_i + M_i\}_\triangle$, $x_i \in M$, and submodules $M_i \subset M$ (with M/M_i finitely cogenerated) such that the intersection of any finitely many of these cosets is not empty, the intersection is also not empty.*

The following lemma gives some properties of linearly compact modules.

Lemma 2.7 *Let N be a submodule of the R-module M.*
1. Assume N to be linearly compact and $\{M_i\}_\triangle$ to be an inverse family of submodules of M. Then

$$N + \bigcap_{\triangle} M_i = \bigcap_{\triangle} (N + M_i).$$

2. M is linearly compact if and only if N and M/N are linearly compact.
3. Assume M to be linearly compact. Then
(i) there is no non-trivial decomposition of M as an infinite direct sum;
(ii) $M/\operatorname{Rad}(M)$ is semisimple and finitely generated (Wisbauer 1991).

Definition 2.77 *If for every $V \subset M$ with $U + V = M$ there is a supplement V' of U such that $V' \subseteq V$, then it is said that U has ample supplements in M.*

Definition 2.78 *If every submodule of M has ample supplements in M, then M is called amply supplemented. Every amply supplemented module is supplemented.*

Lemma 2.8 *A module M is amply supplemented if and only if every submodule of M is a sum of a supplemented submodule and a small submodule of M (Smith 2000).*

Lemma 2.9 *Let U be a linearly compact submodule of an R-module M. Then U has ample supplements in M.*

Proof Let U, $V \subseteq M$ such that U is linearly compact and $M = U + V$. Define $\Gamma = \{V' \subseteq V \mid U + V' = M\}$. $\Gamma \neq 0$ since $V \in \Gamma$. Take a chain $\{V_\lambda\}$ in Γ. It is an inverse family of submodules V_λ since $\{V_\lambda\}$ is a chain. $\bigcap V_\lambda$ is a lower bound for $\{V_\lambda\}$. $U + (\bigcap V_\lambda) = \bigcap(U + V_\lambda) = M$ by the property of linearly compact module. Thus $\bigcap V_\lambda \in \Gamma$. By Zorn's Lemma there is a minimal element K in Γ such that $M = U + K$ so K is a supplement of U and $K \subseteq V$. Hence U has ample supplements in M. □

Definition 2.79 *An R-module F with a linearly independent spanning set $\{x_\alpha\}_{\alpha \in A}$ is called a free R-module with free basis $(x_\alpha)_{\alpha \in A}$.*

Definition 2.80 *Let R be a commutative domain and M be an R-module. M is said to be divisible if $M = rM$ for all non-zero $r \in R$.*

Definition 2.81 *Let R be a commutative domain and M be an R-module. M is said to be bounded if $rM = 0$ for some $r \in R$.*

The following lemma gives a characterization of supplemented modules over a DVR.

Lemma 2.10 *Let R be a DVR. For an R-module M, the following are equivalent:*
(i) M has a small radical;
(ii) M is coatomic;
(iii) M is a direct sum of a finitely generated free submodule and a bounded submodule;
(iv) M is reduced and supplemented (Zöschinger 1974a).

The following theorem gives the structure of a supplemented module over DVR.

Theorem 2.14 *Let R be a DVR. An R-module M is supplemented if and only if $M = M_1 \oplus M_2 \oplus M_3 \oplus M_4$ where $M_1 \cong R^{n_1}$, $M_2 \cong Q^{n_2}$, $M_3 \cong (Q/R)^{n_3}$ and $p^{n_4}M_4 = 0$ for some integer $n_i \geq 0$ (Zöschinger 1974a).*

Theorem 2.15 *Let R be a non-local Dedekind domain. An R-module M is supplemented if and only if it is torsion and every primary part is supplemented (Zöschinger 1974a).*

The following diagram gives a general view about relations between supplemented, weakly supplemented, totally supplemented and totally weak supplemented modules. The number beside arrow shows the number of example.

$$\begin{tikzcd}
\textit{Totally Supplemented M.} \arrow[r, Rightarrow, shift left] \arrow[d, Rightarrow, shift left] \arrow[dr, Rightarrow, shift left] & \textit{Supplemented M.} \arrow[l, Rightarrow, shift left, "/" marking, "Ex\,4.2"] \arrow[d, Rightarrow, shift left] \arrow[ul, phantom] \\
\textit{Totally Weak Supplemented M.} \arrow[r, Rightarrow, shift left] \arrow[u, Rightarrow, shift left, "/" marking, "Ex\,5.3"] & \textit{Weakly Supplemented M.} \arrow[l, Rightarrow, shift left, "/" marking, "Ex\,5.2"] \arrow[u, Rightarrow, shift left, "/" marking, "Ex\,3.4"] \arrow[ul, Rightarrow, shift left, "/" marking, "Ex\,3.4"]
\end{tikzcd}$$

Figure 2.1. The relations between Supplemented, Weakly Supplemented, Totally Supplemented and Totally Weak Supplemented Modules

CHAPTER 3

WEAKLY SUPPLEMENTED MODULES

Definition 3.1 *Let M be an R-module and $U, V \subseteq M$. V is called weak supplement of U if $U + V = M$ and $U \cap V \ll M$.*

Definition 3.2 *Let M be an R-module. If every submodule of M has a weak supplement in M, then M is called weakly supplemented module.*

Example 3.1 *Supplemented, artinian, semisimple, linearly compact, uniserial and hollow modules are weakly supplemented modules.*

Proposition 3.1 *Every factor module of a weakly supplemented module is weakly supplemented (Lomp 1999).*

Proof Let M/K be a factor module of a weakly supplemented module M and $L/K \subseteq M/K$. Since M is weakly supplemented there exists a submodule N of M such that $L + N = M$ and $L \cap N \ll M$. Then $M/K = (L + N)/K = L/K + (N + K)/K$ and $(L/K) \cap ((N + K)/K) \ll M/K$ since $L \cap K \ll M$. □

Proposition 3.2 *A small cover of a weakly supplemented module is a weakly supplemented module (Clark et al. 2006).*

Proof Let M be a small cover of a weakly supplemented module N. Then $N \cong M/K$ for some $K \ll M$. Take a submodule L of M and a weak supplement X/K of $(L + K)/K$ in M/K. Since $K \ll M$, by Lemma 2.2(1) we get $(X \cap L) + K = X \cap (L + K) \ll M$ and X is a weak supplement of L in M. Thus M is weakly supplemented. □

In the following proposition there are some properties of weakly supplemented modules.

Proposition 3.3 *Let M be an R-module. If M is weakly supplemented, then the following properties hold:*
(i) M is semilocal;

(ii) $M = M_1 \oplus M_2$ *with* M_1 *semisimple and* $\mathrm{Rad}(M) \trianglelefteq M_2$*;*

(iii) Every supplement in M and every direct summand of M is weakly supplemented (Lomp 1999).

Proof (i) and (ii) follow from Proposition 2.10 since for every $L \subseteq M$ there exists a weak supplement $K \subseteq M$ such that $L + K = M$ and $L \cap K \subseteq \mathrm{Rad}(M)$.

(iii) Let $N \subseteq M$ be a supplement of M. Then $N + K = M$ and $N \cap K \ll N$ for some $K \subseteq M$. By Proposition 3.1, $M/K \cong N/N \cap K$ is weakly supplemented and by Proposition 3.2, N is weakly supplemented. Direct summands are supplements and so they are weakly supplemented. □

Lemma 3.1 *Let M be an R-module with submodules K and* M_1*. Assume* M_1 *is weakly supplemented and* $M_1 + K$ *has a weak supplement in M. Then K has a weak supplement in M (Clark et al. 2006).*

Proof Let X be a weak supplement of $M_1 + K$ in M, i.e.

$$M = M_1 + K + X \text{ and } (M_1 + K) \cap X \ll M$$

and let Y be a weak supplement of $(K + X) \cap M_1$ in M_1, i.e.

$$M_1 = (K + X) \cap M_1 + Y \text{ and } ((K + X) \cap M_1) \cap Y \ll M_1.$$

Since $Y \subseteq M_1$,

$$Y + K \subseteq M_1 + K \Rightarrow (Y + K) \cap X \subseteq (M_1 + K) \cap X.$$

Thus $(Y + K) \cap X \ll M$ since $(M_1 + K) \cap X \ll M$. Now

$$M = M_1 + K + X = ((K + X) \cap M_1) + Y + K + X = Y + K + X$$

and

$$Y \cap (K + X) = Y \cap M_1 \cap (K + X) \ll M_1 \subseteq M.$$

Hence Y is a weak supplement of $K + X$ in M. Then we obtain

$$(X + Y) \cap K \subseteq (X \cap (Y + K)) + (Y \cap (K + X)) \ll M \Rightarrow (X + Y) \cap K \ll M.$$

Therefore $X + Y$ is a weak supplement for K in M. □

Proposition 3.4 *Let $M = M_1 + M_2$, where M_1 and M_2 are weakly supplemented, then M is weakly supplemented (Clark et al. 2006).*

Proof Let U be a submodule of M. Then $M = U + M_1 + M_2$. Since 0 (zero) submodule is a weak supplement of $U+M_1+M_2$ and M_1 is weakly supplemented, $U + M_2$ has a weak supplement by Lemma 3.1. Hence U has a weak supplement since M_2 is weakly supplemented again by Lemma 3.1. □

Corollary 3.1 *Every finite sum of weakly supplemented modules is weakly supplemented.*

Proposition 3.5 *Let M be an R-module. If M is weakly supplemented, then every finitely M-generated module is weakly supplemented.*

Proof Let N be a finitely M-generated module. Then there exists an epimorphism $\bigoplus_F M \xrightarrow{f} N \longrightarrow 0$ such that F is finite. Since M is weakly supplemented, a finite sum of M is also weakly supplemented. By first isomorphism theorem $\bigoplus_F M/\operatorname{Ker} f \cong N$. Since every factor module of a weakly supplemented module is weakly supplemented, $\bigoplus_F M/\operatorname{Ker} f$ is weakly supplemented. Hence N is weakly supplemented. □

Example 3.2 *$\mathbb{Q}/\mathbb{Z}$ is a weakly supplemented $\mathbb{Z}$-module.*

Proof Firstly write $M := \mathbb{Q}/\mathbb{Z} = \bigoplus_p M_p$ as the direct sum of its primary p-components $M_p := \mathbb{Z}_{p^\infty}$. Every submodule N of M is of the form $N = \bigoplus N_p$ where $N_p = N \cap M_p \subseteq M_p$ are the p-components of N. Since M_p is hollow, either $N_p = M_p$ or $N_p \ll M_p$. Therefore $N \ll M$ if and only if $N_p \neq M_p$ for all p. If N is not small in M, set $\Lambda = \{p \mid N_p \neq M_p\}$ and $L := \bigoplus_{p\in\Lambda} M_p$. Then $N+L = M$ and $N \cap L = \bigoplus_{p\in\Lambda} N_p \ll M$. Hence L is a weak supplement of N in M. □

Example 3.3 *$\mathbb{Q}$ is a weakly supplemented $\mathbb{Z}$-module.*

Proof Since $\mathbb{Q}$ is a small cover of the weakly supplemented module $\mathbb{Q}/\mathbb{Z}$, $\mathbb{Q}$ is also weakly supplemented module. □

Supplemented modules are weakly supplemented but converse does not hold in general by the following example.

Example 3.4 *The $\mathbb{Z}$-module $\mathbb{Q}$ is weakly supplemented but not supplemented (Clark et al. 2006).*

Proof Let p be a prime number and $\mathbb{Z}_{(p)} = \{a/b \in \mathbb{Q} \mid p$ does not divide $b\}$. Assume that $\mathbb{Z}_{(p)}$ has a supplement K in $\mathbb{Q}$. Then $(K + \mathbb{Z})/\mathbb{Z}$ is a supplement of $\mathbb{Z}_{(p)}/\mathbb{Z}$ in $\mathbb{Q}/\mathbb{Z}$. Note that $\mathbb{Z}_{(p)}/\mathbb{Z}$ is the sum of all the q-component of $\mathbb{Q}/\mathbb{Z}$ where q runs through all primes different from p. Since $(K + \mathbb{Z})/\mathbb{Z} + \mathbb{Z}_{(p)}/\mathbb{Z} = \mathbb{Q}/\mathbb{Z}$ and since the p-component of $\mathbb{Z}_{(p)}/\mathbb{Z}$ is zero, the p-component of $\mathbb{Q}/\mathbb{Z}$ must equal the p-component of $(K + \mathbb{Z})/\mathbb{Z}$. On the other hand, if q is a prime different from p, then the q-component of $(K+\mathbb{Z})/\mathbb{Z}$ is a submodule of the q-component of $\mathbb{Z}_{(p)}/\mathbb{Z}$, that is

$$[(K + \mathbb{Z})/\mathbb{Z}]_q \subseteq ((K + \mathbb{Z})/\mathbb{Z}) \cap (\mathbb{Z}_{(p)}/\mathbb{Z}) \ll (K + \mathbb{Z})/\mathbb{Z}.$$

However, the q-component of a torsion $\mathbb{Z}$-module is a direct summand. Hence the q-components of $(K + \mathbb{Z})/\mathbb{Z}$ must be all zero, that is $(K + \mathbb{Z})/\mathbb{Z}$ is equal to its p-component, namely the p-component of $\mathbb{Q}/\mathbb{Z}$. Since $(\mathbb{Q}/\mathbb{Z})_p = \mathbb{Z}[1/p]/\mathbb{Z}$, where $\mathbb{Z}[1/p] := \{a/p^k \in \mathbb{Q} \mid a \in \mathbb{Z}, k \geq 0\}$, we have that $(K + \mathbb{Z})/\mathbb{Z} = \mathbb{Z}[1/p]/\mathbb{Z}$ and so $K + \mathbb{Z} = \mathbb{Z}[1/p]$. Now $K \cap \mathbb{Z} \subset K \cap \mathbb{Z}_{(p)} \ll K$ and so $n\mathbb{Z} = K \cap \mathbb{Z} \ll \mathbb{Z}[1/p]$ for some non-zero number n. On the other hand, if q is a prime that does not divide n nor p, then $\mathbb{Z}[1/p] = n\mathbb{Z} + q\mathbb{Z}[1/p]$, because by the Euclidean algorithm, for any $k \geq 0$, there are integers r and s such that $1 = rq + snp^k$. Thus $1/p^k = q(r/p^k) + sn \in q\mathbb{Z}[1/p] + n\mathbb{Z}$. Since $q \neq p$, $q\mathbb{Z}[1/p] \neq \mathbb{Z}[1/p]$, that is, $n\mathbb{Z}$ is not small in $\mathbb{Z}[1/p]$, contradiction. Thus $\mathbb{Z}_{(p)}$ cannot have a supplement in $\mathbb{Q}$, and so, in particular, $\mathbb{Q}$ is not supplemented. □

Theorem 3.1 *Let $0 \longrightarrow L \longrightarrow M \longrightarrow N \longrightarrow 0$ be a short exact sequence for R-modules L, M, N. If L and N are weakly supplemented and L has a weak supplement in M, then M is weakly supplemented.*

If L is coclosed, then the converse holds; that is if M is weakly supplemented, then L and N are weakly supplemented (Büyükaşık 2005).

Proof Without loss of generality we will assume $L \subseteq M$. Let S be a weak supplement of L in M, i.e. $L + S = M$ and $L \cap S \ll M$. Then we have,

$$M/L \cap S \cong L/L \cap S \oplus S/L \cap S.$$

$L/L \cap S$ is weakly supplemented as a factor module of L which is weakly supplemented. On the other hand

$$S/L \cap S \cong M/L \cong N$$

is weakly supplemented. Then $M/L \cap S$ is weakly supplemented module as a sum of weakly supplemented modules. Therefore M is weakly supplemented by Proposition 3.2.

Conversely, if L is coclosed, for $L \cap S \subseteq L$, $L \cap S \ll M$ implies $L \cap S \ll L$ (see Proposition 2.3), i.e. L is a supplement of S in M. Then by Proposition 3.3 (iii), L is weakly supplemented and by Proposition 3.1, N is weakly supplemented. □

Proposition 3.6 *Let M be an R-module. M is weakly supplemented if and only if $M/\left(\bigoplus_{i=1}^{n} L_i\right)$ is weakly supplemented where each L_i is a hollow submodule of M.*

Proof (⇒) Clear.

(⇐) Suppose $n = 1$ and M/L is weakly supplemented. Consider the following exact sequence:

$$0 \longrightarrow L \longrightarrow M \longrightarrow M/L \longrightarrow 0.$$

Case 1: If $L \ll M$, then M is weakly supplemented since it is small cover of M/L.

Case 2: If $L \not\ll M$, then $M = L + T$ for a proper submodule T of M. Since L is hollow $L \cap T \ll L \subseteq M$. Hence T is a weak supplement of L in M. Since M/L and L are weakly supplemented, by Theorem 3.1 M is weakly supplemented.

Now suppose it holds when $i < n$. Let $M/\left(\bigoplus_{i=1}^{n} L_i\right)$ be weakly supplemented. We get the following exact sequence:

$$0 \longrightarrow \left(\bigoplus_{i=1}^{n} L_i\right)/\left(\bigoplus_{i=1}^{n-1} L_i\right) \longrightarrow M/\left(\bigoplus_{i=1}^{n-1} L_i\right) \longrightarrow M/\left(\bigoplus_{i=1}^{n} L_i\right) \longrightarrow 0.$$

Since $\left(\bigoplus_{i=1}^{n} L_i\right)/\left(\bigoplus_{i=1}^{n-1} L_i\right) \cong L_n,$ is a hollow submodule of $M/\left(\bigoplus_{i=1}^{n-1} L_i\right)$ and $M/\left(\bigoplus_{i=1}^{n} L_i\right)$ is weakly supplemented, $M/\left(\bigoplus_{i=1}^{n-1} L_i\right)$ is weakly supplemented. Therefore M is weakly supplemented by induction. □

Corollary 3.2 *Let M be an R-module. M is weakly supplemented if and only if $M/\left(\bigoplus_{i=1}^{n} L_i\right)$ is weakly supplemented where each L_i is a local submodule of M.*

Proof Since local modules are hollow the proof is clear by Proposition 3.6. □

Corollary 3.3 *Let M be an R-module. If $\operatorname{Soc}(M)$ is finitely generated, then $M/\operatorname{Soc}(M)$ is weakly supplemented if and only if M is weakly supplemented.*

Proof Since simple modules are local, the proof is clear by Corollary 3.2. □

Corollary 3.4 *Let M be an R-module. M is weakly supplemented if and only if M/S is weakly supplemented for a finitely generated supplemented submodule S of M.*

Proof Since finitely generated supplemented modules are the irredundant sum of local submodules, the proof is clear by Corollary 3.2. □

Lemma 3.2 *Let M be a finitely generated module with zero radical and let N be a non-finitely generated submodule of M. Then N does not have any weak supplement in M.*

Proof Suppose that L is a weak supplement of N in M, i.e. $M = N + L$ and $N \cap L \ll M$. Now $N \cap L \subseteq \operatorname{Rad}(M) = 0$. Hence $M = N \oplus L$ and N is finitely generated, a contradiction. □

Definition 3.3 *Let R be a ring. An element $a \in R$ is said to be von Neumann regular if $a \in aRa$. If every $a \in R$ is von Neumann regular, then R is called a von Neumann regular ring.*

Example 3.5 *Let F be a field and S be the direct product $\prod_{n\in\mathbb{N}} F_n$, where $F_n = F\,(n \geq 1)$. Then the element of S are the sequences $\{a_n\}$, where $a_n \in F\,(n \in \mathbb{N})$. Let R be the subring of S consisting of all sequences $\{a_n\}$ such that there exist $a \in F,\ k \in \mathbb{N}$ with $a_n = a$ for all $n \geq k$. Then R is a von Neumann regular ring so that the R-module R has zero radical. The $\operatorname{Soc}(R)$ of the R-module R consists of all sequences $\{a_n\}$ in R such that $a_n = 0$ for all $n \geq k$ for some $k \in \mathbb{N}$. Hence $\operatorname{Soc} R$ is not finitely generated and $\operatorname{Soc} R$ does not have any weak supplement.*

Proof Let $(a_1, \ldots, a_{k-1}, a, a, \ldots)$ be an element of R such that $a_i\,(1 \leq i \leq k-1)$ and $a_n = a$ for $n \geq k$. If each $a_i\,(1 \leq i \leq k-1)$ and a are non-zero elements of F, they have inverse. Hence $(a_1, \ldots, a_{k-1}, a, a, \ldots)(a_1^{-1}, \ldots, a_{k-1}^{-1}, a^{-1}, a^{-1}, \ldots)(a_1, \ldots, a_{k-1}, a, a, \ldots) = (a_1, \ldots, a_{k-1}, a, a, \ldots)$. Thus R is a von Neumann regular ring. Clearly $\operatorname{Rad}(R) = 0$. Now let T be a simple submodule of R and $0 \neq \overline{a} = (a_1, \ldots, a_{k-1}, a, a, \ldots) \in T \subseteq R$. For an element $r = (a_1^{-1}, 0, 0, \ldots, 0, \ldots) \in R$, $r\overline{a} = (1, 0, 0, \ldots, 0, \ldots) \in T$. Then

$A_1 =< (1, 0, 0, \ldots, 0, \ldots) >$ is an submodule of R which is generated by $r\bar{a} \in T$. Hence $A_1 \subseteq T$. Since T is simple and A_1 is different from zero, $A_1 = T$. Hence $\mathrm{Soc}(R)$ is not finitely generated and by Lemma 3.2, $\mathrm{Soc}(R)$ does not have any weak supplement. □

Corollary 3.3 does not hold when $\mathrm{Soc}(M)$ is not finitely generated by the following example.

Example 3.6 *Let R be a ring and M be R-module R as in the Example 3.5. Then $M/\operatorname{Soc}(M)$ is simple but M is not weakly supplemented.*

Proof Let $\mathrm{Soc}(M) \subsetneqq A \subseteq M$. Since $\mathrm{Soc}(M) \subsetneqq A$ there exists an element x of $A \setminus \mathrm{Soc}(M)$ such that $x = (a_1, \ldots, a_n, a, a, \ldots)$. For an element $r = (0, \ldots, 0, a^{-1}, a^{-1}, \ldots)$ of R, $rx = (0, \ldots, 0, 1, 1, \ldots) \in A$. Now let $\overline{m} = (m_1, \ldots, m_k, m, m, \ldots) \in M$. Then $\overline{m}$ can be represented in the following way:

$$\overline{m} = (m_1, \ldots, m_k, 0, 0, \ldots) + (0, \ldots, 0, m, m, \ldots)$$

where $(m_1, \ldots, m_k, 0, 0, \ldots) \in \mathrm{Soc}(M) \subsetneqq A$ and $(0, \ldots, 0, m, m, \ldots) \in A$ so $\overline{m} \in A$ implies $M = A$. Hence $\mathrm{Soc}(M)$ is a maximal submodule of M and $M/\operatorname{Soc}(M)$ is simple. Thus $M/\operatorname{Soc}(M)$ is weakly supplemented but M is not weakly supplemented by Example 3.5. □

Since simple modules are local and Corollary 3.3 does not hold when $\mathrm{Soc}(M)$ is not finitely generated, Corollary 3.2 does not hold when the direct sum of local submodules is not finite. Hence Proposition 3.6 does not hold when the direct sum of hollow submodules is not finite because local modules are hollow.

Definition 3.4 *An R-module is called decomposable if it is a direct sum of cyclic modules and finitely generated torsion-free modules of rank one. If R is a principal ideal domain, then a decomposable module is exactly a direct sum of cyclic modules (Kaplansky 1952).*

Definition 3.5 *Let M be an R-module. A submodule N is called pure if $rN = N \cap rM$ for every $r \in R$.*

Theorem 3.2 *Let R be a Dedekind domain, M be an R-module and S be a pure submodule such that M/S is decomposable. Then S is a direct summand of M (Kaplansky 1952).*

As a result of this theorem, the following corollary can be given.

Corollary 3.5 *Let R be a Dedekind domain, L, M, N be R-modules and*

$$0 \longrightarrow L \longrightarrow M \longrightarrow N \longrightarrow 0$$

be an exact sequence with L pure in M and N decomposable. L and N are weakly supplemented if and only if M is weakly supplemented.

Proof (⇒) By Theorem 3.2, the sequence is splitting so M is weakly supplemented since L and N are weakly supplemented.
(⇐) Since direct summands of weakly supplemented modules are weakly supplemented, L and N are weakly supplemented. □

Theorem 3.3 *Let R be a Dedekind domain, M be an R-module and S be a pure submodule of bounded order (that is, $rS = 0$ for some non-zero r in R). Then S is a direct summand of M (Kaplansky 1952).*

Corollary 3.6 *Let R be a Dedekind domain, L, M, N be R-modules and*

$$0 \longrightarrow L \longrightarrow M \longrightarrow N \longrightarrow 0$$

be an exact sequence with L pure submodule of bounded order. L and N are weakly supplemented if and only if M is weakly supplemented.

Proof (⇒) Clear since the sequence is splitting by Theorem 3.3.
(⇐) By Proposition 3.3 (iii). □

By Ω we denote the set of all maximal ideals of R.
A characterization of weakly supplemented modules over Dedekind domain is given in the following theorem.

Theorem 3.4 *Let R be a Dedekind domain and M an R-module. Then M is weakly supplemented module if and only if*
(i) $M/\operatorname{Rad}(M)$ is semisimple,
(ii) $M/T(M)$ has a finite Goldie dimension (finite rank),
(iii) $T_{\mathfrak{p}}(M)$ is a direct sum of an Artinian and a bounded submodule for every $\mathfrak{p} \in \Omega$ (Zöschinger 1986).

Lemma 3.3 *Let R be a domain and M be an R-module. Then the torsion submodule $T(M)$ of M is closed in M (Büyükaşık 2005).*

Proof Suppose $T(M) \trianglelefteq K$ for some $K \subseteq M$. Let $k \in K$, then since $T(M)$ is essential in K, we have $0 \neq rk \in T(M)$ for some $r \in R$. Then $srk = 0$ for some $0 \neq s \in R$. Since R is a domain, $sr \neq 0$. Therefore $k \in T(M)$, i.e. $K = T(M)$. Hence $T(M)$ is closed in M. □

Theorem 3.5 *Let R be a Dedekind domain and M be an R-module. Then M is weakly supplemented if and only if*
(i) $M/\operatorname{Rad}(M)$ is semisimple;
(ii) $T(M)$ and $M/T(M)$ are weakly supplemented (Büyükaşık 2005).

Proof $(\Rightarrow)$(i) By Theorem 3.4, $M/\operatorname{Rad}(M)$ is semisimple.
(ii) $M/T(M)$ is weakly supplemented as a factor module of a weakly supplemented module. By Lemma 3.3, $T(M)$ is a closed submodule of M. Then by Theorem 2.11, $T(M)$ is a coclosed submodule of M. By Proposition 2.3, $T(M)$ is a supplement in M. Every supplement in M is weakly supplemented by Proposition 3.3 (iii). Hence $T(M)$ is weakly supplemented.
$(\Leftarrow)$ $M/T(M)$ is weakly supplemented so it has finite rank by Theorem 3.4. Since $T(M)$ is weakly supplemented, $T_{\mathfrak{p}}(M)$ is a direct sum of artinian and a bounded submodule for every $\mathfrak{p} \in \Omega$. Now by Theorem 3.4, M is weakly supplemented. □

Proposition 3.7 *Let R be a Dedekind domain and M be an R-module. If $T(M)$ has a weak supplement in M, then M is weakly supplemented if and only if $T(M)$ and $M/T(M)$ are weakly supplemented.*

Proof $(\Leftarrow)$ By Theorem 3.1.
$(\Rightarrow)$ By Theorem 3.5. □

The following theorem gives the structure of weakly supplemented modules over a DVR.

Theorem 3.6 *Let R be a DVR and M be an R-module. Then M is weakly supplemented if and only if $M = Q^{n_1} \oplus (Q/R)^{n_2} \oplus B \oplus N$, where B is bounded, N is reduced torsion-free with finite rank and $n_i \geq 0$ (Büyükaşık 2005).*

CHAPTER 4

TOTALLY SUPPLEMENTED MODULES

Definition 4.1 *Let M be an R-module. If every submodule of M is supplemented, then M is called a totally supplemented module.*

Example 4.1 *Artinian and semisimple modules are totally supplemented.*

Lemma 4.1 *Every factor module of totally supplemented module is totally supplemented (Smith 2000).*

Proof Let M be a totally supplemented module and N/K be a submodule of M/K for some submodule N containing K. Since M is totally supplemented N is supplemented. Hence N/K is supplemented as a factor module of supplemented module. Thus M/K is totally supplemented. □

Corollary 4.1 *Every homomorphic image of a totally supplemented module is totally supplemented module.*

Remark 4.1 *Let M be an R-module. If M is totally supplemented, then M is amply supplemented by Lemma 2.8 because for every submodule N of M, $N = N + 0$, i.e. N is the sum of a supplemented and a small module.*

A totally supplemented module is supplemented but converse does not hold in general. The following example shows that a supplemented module need not be totally supplemented.

Example 4.2 *Let S be any commutative local domain which is not a field and let X be any free S-module of infinite rank. Let R be the commutative ring $[S,X]$ which denotes the commutative ring of matrices*

$$\begin{pmatrix} s & x \\ 0 & s \end{pmatrix}$$

where $s \in S$, $x \in X$ with the usual matrix addition and multiplication. Then the R-module R is local, i.e. supplemented but not totally supplemented.

Proof Since S is local it has the unique maximal ideal, say $\mathfrak{m}$. Then $[\mathfrak{m}, X]$ is the unique maximal ideal of R, because for an element $r \notin \mathfrak{m}$, r^{-1} exists so that

$$\begin{pmatrix} r & x \\ 0 & r \end{pmatrix}\begin{pmatrix} r^{-1} & x \\ 0 & r^{-1} \end{pmatrix} = \begin{pmatrix} 1 & (rx + r^{-1}x) \\ 0 & 1 \end{pmatrix}$$

and

$$\begin{pmatrix} 1 & (rx + r^{-1}x) \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & -(rx + r^{-1}x) \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Hence $\begin{pmatrix} r & x \\ 0 & r \end{pmatrix} \notin [\mathfrak{m}, X]$ since $r \notin \mathfrak{m}$ but there exists inverse of $\begin{pmatrix} r & x \\ 0 & r \end{pmatrix}$ in R. Thus R is a local ring. Furthermore R-module R is supplemented. Let a denote the ideal $[0, X]$ of R. It is clear that if Y is an S-submodule of X, then $[0, Y]$ is an R-submodule of a and the mapping $Y \longrightarrow [0, Y]$ is an isomorphism from the lattice of S-submodules of X to the lattice of R-submodules of a. But the Jacobson radical $\mathrm{J}(S)$ is not T-nilpotent because if $\mathrm{J}(S)$ were a T-nilpotent ideal, it would be a nil-ideal. However, not every nil-ideal is T-nilpotent. Hence nil-ideals situated between T-nilpotent ideals and the ring itself. Since $\mathrm{J}(S)$ is the unique maximal ideal, it is nil-ideal, i.e. for every non-zero element of $\mathrm{J}(S)$, say x there exists $n_x > 0$ such that $x^{n_x} = 0$, implies $x = 0$ so $\mathrm{J}(S) = 0$. If $\mathrm{J}(S) = 0$, then S is field but this contradicts with assumption. Therefore $\mathrm{J}(S)$ is not T-nilpotent so S is not perfect ring by Proposition 2.6. Thus X is not a supplemented S-module by Lemma 2.5. Moreover the R-module a is not supplemented, because $Y \longrightarrow [0, Y]$ is an isomorphism. It follows that the R-module R is not totally supplemented (Smith 2000). □

In general a supplemented module is not totally supplemented but under some conditions they are equivalent. The following theorem clarifies this claim.

Theorem 4.1 *Let R be a non-local Dedekind domain. Then the following statements are equivalent for an R-module M:*

(i) M is supplemented.

(ii) M is amply supplemented.

(iii) M is totally supplemented.

(iv) M is a torsion module such that $T_{\mathfrak{p}}(M)$ is a direct sum of an Artinian submodule and a bounded submodule for each maximal ideal $\mathfrak{p}$ of R (Smith 2000).

If R is a local Dedekind domain (i.e. a DVR), then there are two cases to consider, namely when R is complete and when R is incomplete.

The following theorem gives a characterization of totally supplemented modules over complete DVR.

Theorem 4.2 *Let R be a complete DVR with field of fractions Q. Then the following are equivalent for an R-module M:*
(i) M is supplemented.
(ii) M is amply supplemented.
(iii) M is totally supplemented.
(iv) There exist non-negative integers a, b, c such that $M = M_1 \oplus M_2 \oplus M_3 \oplus M_4$ *is a direct sum of a submodule* $M_1 \cong R^a$*, a submodule* $M_2 \cong Q^b$*, a submodule* $M_3 \cong (Q/R)^c$ *and a bounded submodule* M_4 *(Smith 2000).*

Remark 4.2 *In case R is an incomplete DVR with field of fractions Q, an R-module M is supplemented if and only if M satisfies (iv) of Theorem 4.2. However the following statements are equivalent for an R-module M:*
(a) M is amply supplemented,
(b) M is totally supplemented,
(c) M satisfies (iv) of Theorem 4.2 with $b \leq 1$ *(Zöschinger 1974b). In particular, if R is an incomplete DVR with field of fractions Q, then the R-module* $M = Q \oplus Q$ *is supplemented but not amply supplemented so it is not totally supplemented. Hence a finite direct sum of totally supplemented module is not totally supplemented (Smith 2000).*

It is known that a finite direct sum of supplemented modules is supplemented but this is not true for amply (or totally) supplemented modules by Remark 4.2. The following theorem shows that direct sum of a totally supplemented module and a semisimple module is totally supplemented.

Theorem 4.3 *Let a module* $M = M_1 \oplus M_2$ *be a direct sum of submodules* M_1*,* M_2 *such that* M_2 *is semisimple. Then M is totally supplemented if and only if* M_1 *is totally supplemented (Smith 2000).*

Proof ($\Rightarrow$) Clear by Lemma 4.1.
($\Leftarrow$) Suppose that M_1 is totally supplemented. Let N be a submodule of M. Then

$M_2 = (N \cap M_2) \oplus L$ for some submodules L of M_2 since M_2 is semisimple. It follows that

$$M = M_1 \oplus M_2 = M_1 \oplus (N \cap M_2) \oplus L$$

and hence

$$N = N \cap M = N \cap (M_1 \oplus (N \cap M_2) \oplus L) = (N \cap M_2) \oplus (N \cap (M_1 \oplus L))$$

by Modular Law.

Consider the submodule $H = N \cap (M_1 \oplus L)$ of $M_1 \oplus L$. Note that

$$H \cap L = N \cap (M_1 \oplus L) \cap L = N \cap L = 0 \Rightarrow H \cap L = 0.$$

Thus H embeds in M_1 because

$$H \cap M_2 = N \cap (M_1 \oplus L) \cap M_2 = N \cap [M_1 \cap M_2 + L] = 0$$

and for the projection $\pi_1 : M \longrightarrow M_1$; we have $\operatorname{Ker} \pi_1 = M_2$, therefore $\pi_1 |_H : H \longrightarrow M_1$ is a monomorphism. By hypothesis, H is supplemented. But, being semisimple, $N \cap M_2$ is supplemented. Therefore

$$N = (N \cap M_2) \oplus (M_1 \oplus L) \cap N = (N \cap M_2) \oplus H$$

is supplemented as a sum of supplemented modules. Hence N is supplemented and M is totally supplemented. □

Definition 4.2 *Let M be an R-module. The annihilator of M is* $\operatorname{ann}(M) = \{r \in R \mid rm = 0 \text{ for all } m \in M\}$.

Lemma 4.2 *Let a module $M = M_1 \oplus \cdots \oplus M_n$ be a finite direct sum of submodules M_i $(1 \leq i \leq n)$, for some $n \geq 2$, such that $R = \operatorname{ann}(M_i) + \operatorname{ann}(M_j)$ for all $1 \leq i < j \leq n$. Then*

$$N = (N \cap M_1) \oplus \cdots \oplus (N \cap M_n)$$

for every submodule N of M (Smith 2000).

The following lemma shows that a finite direct sum of amply (or totally) supplemented modules is amply (or totally) supplemented by using Lemma 4.2.

Lemma 4.3 *Let a module $M = M_1 \oplus \cdots \oplus M_n$ be a finite direct sum of amply (totally) supplemented submodules M_i $(1 \leq i \leq n)$, for some positive integer $n \geq 2$, such that $R = \mathrm{ann}(M_i) + \mathrm{ann}(M_j)$ for all $1 \leq i < j \leq n$. Then M is amply (totally) supplemented (Smith 2000).*

Proof First suppose that M_i is amply supplemented for all $1 \leq i \leq n$. Let N, K be submodules of M such that $M = N + K$. By Lemma 4.2,

$$N = (N \cap M_1) \oplus \cdots \oplus (N \cap M_n) \text{ and } K = (K \cap M_1) \oplus \cdots \oplus (K \cap M_n).$$

For each $1 \leq i \leq n$, $M_i = (N \cap M_i) + (K \cap M_i)$ and there exists a submodule L_i of $K \cap M_i$ such that $M_i = (N \cap M_i) + L_i$ and $N \cap L_i \ll L_i$. Let $L = L_1 \oplus \cdots \oplus L_n$. Then L is a submodule of K, $M = N + L$ and $N \cap L = (N \cap L_1) \oplus \cdots \oplus (N \cap L_n) \ll L$ by Lemma 2.2 (2). Hence L is a supplement of N in M. It follows that M is amply supplemented.

Now suppose that M_i is totally supplemented for all $1 \leq i \leq n$. Let $V \subseteq U$ be submodules of M. By Lemma 4.2,

$$U = (U \cap M_1) \oplus \cdots \oplus (U \cap M_n) \text{ and } V = (V \cap M_1) \oplus \cdots \oplus (V \cap M_n).$$

For each $1 \leq i \leq n$, $V \cap M_i$ is a submodule of $U \cap M_i$ and, by hypothesis, there exists a supplement W_i of $V \cap M_i$ in $U \cap M_i$. Let $W = W_1 \oplus \cdots \oplus W_n$. By argument used in the first part of this proof, W is a supplement V in U. Thus U is supplemented. It follows that M is totally supplemented. □

Lemma 4.4 *Let R be a commutative ring and an R-module $M = M_1 \oplus \cdots \oplus M_n$ be a finite direct sum of local submodules M_i $(1 \leq i \leq n)$, for some positive integer n. Then M is amply supplemented. If, in addition, M is Noetherian, then M is totally supplemented (Smith 2000).*

For any module M, $\mathrm{Loc}(M)$ will denote the sum of all local submodules of M and $\mathrm{Cof}(M)$ will denote the sum of all cofinitely supplemented submodules of M.

Theorem 4.4 *Let R be any ring. The following statements are equivalent for an R-module M:*

(i) M is cofinitely supplemented;

(ii) Every maximal submodule of M has a supplement in M;

(iii) The module $M/\operatorname{Loc}(M)$ does not contain a maximal submodule;

(iv) The module $M/\operatorname{Cof}(M)$ does not contain a maximal submodule (Alizade et al. 2001).

A supplemented module need not be totally supplemented in general but the following theorem shows that under some conditions a supplemented module can be totally supplemented.

Lemma 4.5 *For any commutative ring R, every Noetherian supplemented R-module is totally supplemented (Smith 2000).*

Proof Let M be a Noetherian supplemented R-module. By Theorem 4.4, $M = \operatorname{Loc} M$ and hence $M = L_1 + \cdots + L_n$ for some positive integer n and local submodules L_i $(1 \leq i \leq n)$. Let L denote the module $L_1 \oplus \cdots \oplus L_n$. Since a finite sum of Noetherian modules is Noetherian, L is Noetherian. Thus L is totally supplemented by Lemma 4.4. M is totally supplemented by Corollary 4.1. □

Theorem 4.5 *Let K be a linearly compact submodule of a module M. Then M is (totally) supplemented if and only if M/K is (totally) supplemented (Smith 2000).*

Proof In each case the necessity is clear. Conversely, suppose that M/K is supplemented. Note that K is supplemented. Moreover for every submodule H of M with $K \subseteq H$, Lemma 2.9 shows that K has a supplement in H. By Proposition 2.13, M is supplemented.

Now suppose M/K is totally supplemented. Let $N \subseteq M$. Then $N \cap K$ is a linearly compact submodule of N and $N/N \cap K \cong (N + K)/K$. $N/N \cap K$ is supplemented since M/K is totally supplemented. By the above argument N is supplemented. Thus M is totally supplemented. □

Corollary 4.2 *Let a module $M = M_1 \oplus M_2 \oplus M_3$ be a direct sum of submodules M_1, M_2, M_3 such that M_2 is linearly compact and M_3 is semisimple. Then M is totally supplemented if and only if M_1 is totally supplemented (Smith 2000).*

Proof ($\Rightarrow$) Clear by Lemma 4.1.

($\Leftarrow$) Suppose M_1 is totally supplemented.

$$M/M_2 = (M_1 \oplus M_2 \oplus M_3)/M_2 \cong M_1 \oplus M_3$$

Since M_1 is totally supplemented and M_3 is semisimple then $M_1 \oplus M_3$ is totally supplemented by Theorem 4.3. Hence M/M_2 is totally supplemented, i.e. M is totally supplemented by Theorem 4.5. □

Proposition 4.1 *Let R be a commutative Noetherian ring and U be a submodule of an R-module M such that M/U is reduced. The following are equivalent:*
(i) M is totally supplemented.
(ii) U and M/U are totally supplemented.

Proof $(i) \Rightarrow (ii)$ Clear.
$(ii) \Rightarrow (i)$ Take the following exact sequence:

$$0 \longrightarrow U \longrightarrow M \longrightarrow M/U \longrightarrow 0.$$

Let N be a submodule of M. If $N \subseteq U$, then N is supplemented. If $N \nsubseteq U$, then

$$(N+U)/U \cong N/N \cap U.$$

Since M/U is reduced $N/N \cap U$ is reduced. Then we have

$$0 \longrightarrow N \cap U \longrightarrow N \longrightarrow N/N \cap U \longrightarrow 0$$

such that $N \cap U$ and $N/N \cap U$ are supplemented. Hence N is supplemented by Theorem 2.12 so M is totally supplemented. □

Corollary 4.3 *Let R be a commutative Noetherian ring and M be an R-module. If* $\operatorname{Rad} M$ *is totally supplemented and* $M/\operatorname{Rad}(M)$ *is supplemented, then M is totally supplemented.*

Conversely, if M is totally supplemented, then $\operatorname{Rad}(M)$ *and* $M/\operatorname{Rad}(M)$ *are totally supplemented.*

Proof The claim can be explained on the exact sequence below.

$$0 \longrightarrow \operatorname{Rad}(M) \longrightarrow M \longrightarrow M/\operatorname{Rad}(M) \longrightarrow 0.$$

Let N be a submodule of M. If $N \subseteq \operatorname{Rad}(M)$, then N is supplemented since $\operatorname{Rad}(M)$ is totally supplemented. If $N \nsubseteq \operatorname{Rad}(M)$, then

$$(N+\operatorname{Rad}(M))/\operatorname{Rad}(M) \cong N/N \cap \operatorname{Rad}(M).$$

Since $M/\operatorname{Rad}(M)$ is supplemented, by Corollary 2.2 $M/\operatorname{Rad}(M)$ is totally supplemented so $N/(N \cap \operatorname{Rad}(M))$ is supplemented. Since $M/\operatorname{Rad}(M)$ is reduced

$N/(N \cap \mathrm{Rad}(M))$ is reduced. Hence N is supplemented by Theorem 2.12. Furthermore M is totally supplemented.

Converse is clear. □

Corollary 4.4 *Let M be a module over a commutative Noetherian ring and K be a cofinite submodule. K and M/K are totally supplemented if and only if M is totally supplemented.*

Proof Since K is cofinite, M/K is coatomic. A coatomic module is reduced over commutative Noetherian ring. Then the proof is completed by Proposition 4.1. □

Corollary 4.5 *Let M be a module over a commutative Noetherian ring. If M has a maximal submodule which is totally supplemented, then M is totally supplemented.*

Proof Since maximal modules are cofinite, the proof is clear by Corollary 4.4. □

Theorem 4.6 *Let R be a DVR. Then the following hold for an R-module M:*
(i) $\mathrm{Rad}(M)$ *is supplemented if and only if M is supplemented.*
(ii) M is supplemented if and only if $T(M)$ and $M/T(M)$ are supplemented (Zöschinger 1974a).

The immediate consequences of Theorem 4.6 can be given.

Corollary 4.6 *Let R be a DVR and M be an R-module.* $\mathrm{Rad}\, M$ *is totally supplemented if and only if M is totally supplemented.*

Proof ($\Leftarrow$) Clear.
($\Rightarrow$) Let U be a submodule of M. Since $\mathrm{Rad}(U) \subseteq \mathrm{Rad}(M)$ and $\mathrm{Rad}(M)$ is totally supplemented, $\mathrm{Rad}(U)$ is supplemented submodule of M. By Theorem 4.6, U is supplemented. Hence M is totally supplemented. □

Corollary 4.7 *Let R be a DVR and M be an R-module. M is totally supplemented if and only if $T(M)$ and $M/T(M)$ are totally supplemented.*

Proof ($\Leftarrow$) Let N be a submodule of M. If $N \subseteq T(M)$, then N is supplemented. If $N \nsubseteq T(M)$, then

$$(N + T(M))/T(M) \cong N/N \cap T(M) = N/T(N).$$

$T(N)$ is supplemented since $T(N) \subseteq T(M)$ and $N/T(N)$ is supplemented since it is isomorphic to a submodule of $M/T(M)$. Now the proof is clear by Theorem 4.6.
($\Rightarrow$) Clear. □

Definition 4.3 *Let M be an R-module. M is called semi-Artinian if all of its non-zero factor modules have minimal submodules, equivalently* $\mathrm{Soc}(M/N) \neq 0$ *for every proper submodule N of M.*

Theorem 4.7 *Let M be a module over a commutative Noetherian ring. For a semi-Artinian submodule U of M, the following are equivalent:*
(i) M is supplemented.
(ii) U and M/U are supplemented (Rudlof 1991).

Corollary 4.8 *Let R be a commutative Noetherian ring. For an R-module M, $M/\mathrm{Soc}(M)$ is totally supplemented if and only if M is totally supplemented.*

Proof ($\Leftarrow$) Clear.
($\Rightarrow$) The claim can be explained on the following exact sequence:

$$0 \longrightarrow \mathrm{Soc}(M) \longrightarrow M \longrightarrow M/\mathrm{Soc}(M) \longrightarrow 0.$$

Let N be a submodule of M. If $N \subseteq \mathrm{Soc}(M)$, then N is supplemented. If $N \nsubseteq \mathrm{Soc}(M)$, then

$$(N + \mathrm{Soc}(M))/\mathrm{Soc}(M) \cong N/N \cap \mathrm{Soc}(M) = N/\mathrm{Soc}(N).$$

$\mathrm{Soc}(N)$ is supplemented since $\mathrm{Soc}(N) \subseteq \mathrm{Soc}(M)$ and $N/\mathrm{Soc}(N)$ is supplemented since it is isomorphic to a submodule of $M/\mathrm{Soc}(M)$. Furthermore $\mathrm{Soc}(N)$ is semi-Artinian because $\mathrm{Soc}\left((\mathrm{Soc}(N))/K\right) \neq 0$ for every proper submodule K of $\mathrm{Soc}(N)$. Hence N is supplemented by Theorem 4.7. Moreover M is totally supplemented. $\square$

CHAPTER 5

TOTALLY WEAK SUPPLEMENTED MODULES

Definition 5.1 *Let M be an R-module. If every submodule of M is weakly supplemented, then M is said to be totally weak supplemented module.*

Example 5.1 *Artinian, semisimple, linearly compact and uniserial modules are totally weak supplemented modules.*

Lemma 5.1 *Every factor module of a totally weak supplemented module is totally weak supplemented.*

Proof Let M be a totally weak supplemented module and N/K be a submodule of M/K for some submodule N which contains K. Since M is totally weak supplemented N is weakly supplemented. Hence N/K is weakly supplemented as a factor module of weakly supplemented module. Therefore M/K is totally weak supplemented module. □

Corollary 5.1 *Every homomorphic image of a totally weak supplemented module is totally weak supplemented module.*

Proposition 5.1 *Let M be an R-module. M is weakly supplemented if and only if M/K is weakly supplemented for a linearly compact submodule K of M.*

Proof (⇐) Clear.
(⇒) Consider the following exact sequence:

$$0 \longrightarrow K \longrightarrow M \longrightarrow M/K \longrightarrow 0.$$

Since K is linearly compact it is weakly supplemented. By Lemma 2.9, K has an ample supplement in M, therefore K has a weak supplement in M. Hence M is weakly supplemented by Theorem 3.1. □

Proposition 5.2 *Let M be an R-module. M is totally weak supplemented if and only if M/K is totally weak supplemented for a linearly compact submodule K of M.*

Proof ($\Rightarrow$) Clear.

($\Leftarrow$) Consider the following exact sequence:

$$0 \longrightarrow K \longrightarrow M \longrightarrow M/K \longrightarrow 0.$$

Take a submodule N of M. If $N \subseteq K$, then N is weakly supplemented since it is a submodule of a linearly compact module. If $N \nsubseteq K$, then

$$(N + K)/K \cong N/N \cap K.$$

Hence we have the following exact sequence:

$$0 \longrightarrow N \cap K \longrightarrow N \longrightarrow N/N \cap K \longrightarrow 0.$$

Since $N \cap K$ is a submodule of a linearly compact module, $N \cap K$ is linearly compact so it is weakly supplemented. $N/N \cap K$ is isomorphic to a submodule of M/K so $N/N \cap K$ is weakly supplemented. Hence N is weakly supplemented by Proposition 5.1. □

Proposition 5.3 *Let M be an R-module. M is weakly supplemented if and only if M/U is weakly supplemented for a uniserial submodule U of M.*

Proof ($\Rightarrow$) Clear.

($\Leftarrow$) Consider the following exact sequence:

$$0 \longrightarrow U \longrightarrow M \longrightarrow M/U \longrightarrow 0.$$

Since U is uniserial, it is hollow by Proposition 2.15 so weakly supplemented.

Case 1: If $U \ll M$, then M is weakly supplemented by Proposition 3.2.

Case 2: If $U \not\ll M$, then $U + N = M$ for a proper submodule N of M. Since $U \cap N \subseteq U$ and U is hollow, every proper submodule is small in U, i.e. $U \cap N \ll U$ so $U \cap N \ll M$. Thus U has a weak supplement. Hence M is weakly supplemented by Theorem 3.1. □

Proposition 5.4 *Let M be an R-module. M is totally weak supplemented if and only if M/U is totally weak supplemented for a uniserial submodule U of M.*

Proof ($\Rightarrow$) Clear.

($\Leftarrow$) Consider the following exact sequence:

$$0 \longrightarrow U \longrightarrow M \longrightarrow M/U \longrightarrow 0.$$

Take a submodule N of M. If $N \subseteq U$, then N is weakly supplemented because submodules of uniserial modules are uniserial and uniserial modules are weakly supplemented. If $N \nsubseteq U$, then

$$(N + U)/U \cong N/(N \cap U).$$

Hence we have the following exact sequence:

$$0 \longrightarrow (N \cap U) \longrightarrow N \longrightarrow N/(N \cap U) \longrightarrow 0.$$

Since $N \cap U$ is uniserial, it is weakly supplemented. $N/(N \cap U)$ is isomorphic to a submodule of M/K so $N/(N \cap U)$ is weakly supplemented. Therefore N is weakly supplemented by Proposition 5.3. □

Lemma 5.2 *Let R be an integral domain and $\mathfrak{p}$ be a maximal ideal of R. Then for every $\mathfrak{p}$-primary R-module M, $M/\operatorname{Rad}(M)$ is semisimple (Büyükaşık 2005).*

Proof Since R is commutative we have

$$\operatorname{Rad}(M) = \bigcap_{\mathfrak{q} \in \Omega} \mathfrak{q}M,$$

where Ω is the set of all maximal ideals of R. Let $\mathfrak{q}$ be a maximal ideal of R and suppose $\mathfrak{q} \neq \mathfrak{p}$. Let $x \in M$. Since M is $\mathfrak{p}$-primary, $\mathfrak{p}^n x = 0$ for some $n \in \mathbb{N}$. Since $\mathfrak{p}^n \nsubseteq \mathfrak{q}$ and $\mathfrak{q}$ is a maximal ideal, $\mathfrak{p}^n + \mathfrak{q} = R$, i.e. $1 = p + q$ for some $p \in \mathfrak{p}^n$ and $q \in \mathfrak{q}$. So $x = px + qx = qx \in \mathfrak{q}M$, hence $M = \mathfrak{q}M$. Therefore

$$\operatorname{Rad}(M) = \bigcap_{\mathfrak{q} \in \Omega} \mathfrak{q}M = \mathfrak{q}M.$$

Then since $R/\mathfrak{p}$ is a field, $M/\operatorname{Rad}(M) = M/\mathfrak{p}M$ is a semisimple $R/\mathfrak{p}$-module, and so it is semisimple as an R-module. □

Corollary 5.2 *Let R be a Dedekind domain and M be a torsion R-module, then $M/\operatorname{Rad}(M)$ is semisimple (Büyükaşık 2005).*

Proof Since R is a Dedekind domain and M is a torsion R-module, we have

$$M = \left(\bigoplus_{\mathfrak{p}\in\Omega} T_{\mathfrak{p}}(M) \right).$$

Then

$$M/\operatorname{Rad}(M) = \left(\bigoplus_{\mathfrak{p}\in\Omega} T_{\mathfrak{p}}(M) \right) / \left(\bigoplus_{\mathfrak{p}\in\Omega} \operatorname{Rad}(T_{\mathfrak{p}}(M)) \right) \cong \bigoplus_{\mathfrak{p}\in\Omega} (T_{\mathfrak{p}}(M)/\operatorname{Rad} T_{\mathfrak{p}}(M))$$

is semisimple by Lemma 5.2. □

Lemma 5.3 *Let R be a Dedekind domain and M be $\mathfrak{p}$-primary for some $\mathfrak{p} \in \Omega$. Then M is divisible if and only if $M = \mathfrak{p}M$ (Büyükaşık 2005).*

Lemma 5.4 *Let R be a Dedekind domain and M be a $\mathfrak{p}$-primary R-module. Suppose M is a direct sum of an artinian submodule and a bounded submodule. Then every submodule of M is a direct sum of an artinian submodule and a bounded submodule (Büyükaşık 2005).*

Proof Suppose

$$M = A \oplus B$$

with A an artinian and B a bounded submodule of M. Let N be a submodule of M and D be the divisible part of N. Then $N = D \oplus S$ where S is a reduced submodule of N. Let

$$\pi : A \oplus B \longrightarrow B$$

be the canonical projection, then $\pi(D)$ is a divisible submodule of B as a homomorphic image of the divisible submodule D. Since B is bounded it has no non-zero divisible submodule, i.e. $\pi(D) = 0$. Therefore $D \subseteq A$, hence D is artinian. Since B is bounded then $\mathfrak{p}^n B = 0$ for some $n \in \mathbb{N}$. Then

$$\mathfrak{p}^n S \subseteq \mathfrak{p}^n M = \mathfrak{p}^n A,$$

so $\mathfrak{p}^n S$ is artinian. Then for the descending chain

$$\mathfrak{p}^n S \supseteq \mathfrak{p}^{n+1} S \supseteq \cdots \supseteq \mathfrak{p}^{n+k} S \supseteq \cdots$$

there exists $t \in \mathbb{N}$ such that $\mathfrak{p}^{n+k} S = \mathfrak{p}^{n+t+1} S$. Then by Lemma 5.3, $\mathfrak{p}^{n+t} S$ is a divisible submodule of S, but S is reduced, so we must have $\mathfrak{p}^{n+k} S = 0$, which shows that S is bounded. □

Now we are able to give the following characterization of totally weak supplemented modules over semilocal Dedekind domains.

Theorem 5.1 *Let R be a semilocal Dedekind domain and M be an R-module. The following are equivalent.*
(i) M is totally weak supplemented,
(ii) M is weakly supplemented,
(iii) $M/T(M)$ has finite Goldie dimension and $T_{\mathfrak{p}}(M)$ is a direct sum of an artinian submodule and a bounded submodule for every $\mathfrak{p} \in \Omega$ (Büyükaşık 2005).

Proof $(i) \Rightarrow (ii)$ Clear.
$(ii) \Rightarrow (iii)$ By Theorem 3.4.
$(iii) \Rightarrow (i)$ Let U be a submodule of M. Since R is semilocal, then $U/\operatorname{Rad}(U)$ is semisimple by Theorem 2.10. $(U + T(M))/T(M)$ has finite Goldie dimension as a submodule of $M/T(M)$, then

$$U/T(U) \cong (U + T(M))/T(M)$$

also has finite Goldie dimension.
By Lemma 5.4, $T_{\mathfrak{p}}(U)$ is a direct sum of an Artinian submodule and a bounded submodule. Therefore by Theorem 3.4, U is weakly supplemented, hence M is a totally weak supplemented module. □

Totally weak supplemented modules are weakly supplemented but converse does not hold in general by the following example.

Example 5.2 *The $\mathbb{Z}$-module $\mathbb{Q}$ is weakly supplemented but it is not totally weak supplemented.*

Proof Take submodule $\mathbb{Z}$ of $\mathbb{Q}$. Let m, $n > 1$ and $(m, n) = 1$. Then $m\mathbb{Z} + n\mathbb{Z} = \mathbb{Z}$ and $m\mathbb{Z} \cap n\mathbb{Z} = mn\mathbb{Z}$. But $mn\mathbb{Z}$ is not small in $\mathbb{Z}$ because for any prime number p which does not divide mn, $mn\mathbb{Z} + p\mathbb{Z} = \mathbb{Z}$. Hence the submodule $\mathbb{Z}$ of $\mathbb{Q}$ is not weakly supplemented. Moreover $\mathbb{Z}$-module $\mathbb{Q}$ is not totally weak supplemented.
□

Totally supplemented modules are totally weak supplemented but converse does not hold in general. The following example shows that totally weak supplemented modules need not be totally supplemented.

Example 5.3 *Let R be incomplete DVR. $M = Q \oplus Q$ is totally weak supplemented but it is not totally supplemented.*

Proof By Remark 4.2, it is known $M = Q \oplus Q$ is not totally supplemented over incomplete DVR. Since M is supplemented, i.e. weakly supplemented it is totally weak supplemented by Theorem 5.1. □

Proposition 5.5 *Let R be a non-semilocal domain and M be an R-module. If M is a totally weak supplemented module, then M is torsion (Büyükaşık 2005).*

Theorem 5.2 *Let R be a non-semilocal Dedekind domain. Then an R-module M is a totally weak supplemented module if and only if M is torsion and $T_{\mathfrak{p}}(M)$ is a direct sum of an artinian and a bounded submodule for every $\mathfrak{p} \in \Omega$ (Büyükaşık 2005).*

Proof ($\Rightarrow$) By Proposition 5.5, M is a torsion module. Since M is weakly supplemented, by Theorem 3.4, $T_{\mathfrak{p}}(M)$ is a direct sum of an Artinian submodule and a bounded for every maximal ideal $\mathfrak{p}$.
($\Leftarrow$) Let U be a submodule of M. Then by Corollary 5.2, $U/\operatorname{Rad}(U)$ is semisimple. Since M is torsion we have $U = T(U)$ and so $U/T(U)$ has finite Goldie dimension. By Lemma 5.4, $T_{\mathfrak{p}}(U)$ is a direct sum of an Artinian submodule and a bounded submodule. Then by Theorem 3.4, U is weakly supplemented. Therefore M is a totally weak supplemented module. □

Corollary 5.3 *Let R be a Dedekind domain and M be a torsion R-module. Then the following are equivalent:*
(i) M is weakly supplemented,
(ii) M is a totally weak supplemented module,
(iii) $T_{\mathfrak{p}}(M)$ is a direct sum of an Artinian submodule and a bounded submodule for every maximal ideal $\mathfrak{p}$ (Büyükaşık 2005).

Proof If R is semilocal, then this follows by Theorem 5.1. If R is non-semilocal, the proof follows from by Theorem 3.4 and Theorem 5.2. □

Proposition 5.6 *Let R be a semilocal Dedekind domain and M be an R-module. M is totally weak supplemented module if and only if $T(M)$ and $M/T(M)$ are totally weak supplemented.*

Proof ($\Rightarrow$) Clear.

($\Leftarrow$) Let $U \subseteq M$. Consider the following exact sequence:

$$0 \longrightarrow T(M) \longrightarrow M \longrightarrow M/T(M) \longrightarrow 0.$$

If $U \subseteq T(M)$, then U is weakly supplemented. If $U \nsubseteq T(M)$, then

$$(U + T(M))/T(M) \cong U/U \cap T(M) = U/T(U).$$

$T(U)$ is weakly supplemented since $T(U) \subseteq T(M)$ and $U/T(U)$ is weakly supplemented since it is isomorphic to a submodule of $M/T(M)$. Since R is semilocal $U/\operatorname{Rad}(U)$ is semisimple by Theorem 2.10. Hence U is weakly supplemented by Theorem 3.5 so M is totally weak supplemented. $\square$

Proposition 5.7 *Let R be a Dedekind domain and M be a torsion module. Suppose M/L is weakly supplemented for some $L \ll M$. Then M is a totally weak supplemented module (Büyükaşık 2005).*

Proof M is weakly supplemented since M is a small cover of M/L with the canonical epimorphism $\sigma : M \longrightarrow M/L$. Hence by Corollary 5.3, M is a totally weak supplemented module. $\square$

Corollary 5.4 *Let R be a Dedekind domain and M be a torsion R-module with* $\operatorname{Rad}(M) \ll M$. *Then M is a totally weak supplemented module (Büyükaşık 2005).*

Proof By Corollary 5.2, $M/\operatorname{Rad}(M)$ is semisimple, so it is weakly supplemented. Then M is weakly supplemented since M is a small cover of $M/\operatorname{Rad}(M)$. Therefore by Proposition 5.7, M is a totally weak supplemented module. $\square$

Corollary 5.5 *Let R be a non-local Dedekind domain. Then an R-module M is supplemented if and only if M is totally supplemented (Büyükaşık 2005).*

Corollary 5.6 *Let R be a non-local Dedekind domain and M be a torsion R-module. Then M is supplemented if and only if M is a totally weak supplemented module (Büyükaşık 2005).*

Corollary 5.7 *Let R be a non-semilocal Dedekind domain and M be an R-module. Then the following are equivalent:*

(i) M is supplemented,

(ii) M is totally supplemented,

(iii) M is a totally weak supplemented module (Büyükaşık 2005).

Proposition 5.8 *Let R be a Dedekind domain and M be an R-module. Suppose either R is semilocal or M is torsion. The following are equivalent:*

(i) $\mathrm{Rad}(M)$ *is weakly supplemented and has a weak supplement in M,*

(ii) M is weakly supplemented,

(iii) M is a totally weak supplemented module (Büyükaşık 2005).

Proof $(i) \Rightarrow (ii)$ In both cases $M/\operatorname{Rad}(M)$ is semisimple and so weakly supplemented. Then by Theorem 3.1, M is weakly supplemented.

$(ii) \Rightarrow (iii)$ By Theorem 5.1 if R is semilocal. By Corollary 5.3 if M is torsion.

$(iii) \Rightarrow (i)$ Clear. □

Lemma 5.5 *Let R be a DVR and M be a torsion R-module. Then the following are equivalent:*

(i) M is supplemented,

(ii) M is totally supplemented,

(iii) M is weakly supplemented,

(iv) M is totally weak supplemented,

(v) The divisible part of M is artinian and the reduced part is bounded (Büyükaşık 2005).

Corollary 5.8 *Let R be a complete DVR and M be an R-module. Then the following are equivalent:*

(i) M is totally supplemented.

(ii) M is supplemented.

(iii) M is weakly supplemented.

(iv) M is totally weak supplemented (Rudlof 1991).

CHAPTER 6

CONCLUSION

The aim of this study is to give a survey to determine the structures, characterizations and properties of supplemented, weakly supplemented, totally supplemented and totally weak supplemented modules. As a result of this survey, we see that they have different properties. Although the finite sum of supplemented (weakly supplemented) modules is supplemented (weakly supplemented), the finite sum of totally supplemented modules is not totally supplemented in general. But we haven't reached any information about whether the finite sum of totally weak supplemented modules is totally weak supplemented or not. In general supplemented, weakly supplemented, totally supplemented and totally weak supplemented modules are not equivalent. But we see that over complete DVR (discrete valuation ring) they are equivalent.

However supplemented, weakly supplemented, totally supplemented and totally weak supplemented modules are all closed under homomorphic images. As a result of this study we see that supplemented, weakly supplemented, totally supplemented and totally weak supplemented modules can be characterized in terms of factor modules of them by linearly compact submodules over arbitrary rings. We have also reached characterizations of supplemented, weakly supplemented, totally supplemented, totally weak supplemented modules on exact sequences under some conditions.

REFERENCES

Alizade, R., Bilhan, G. and Smith, P. F., 2001. "Modules Whose Maximals Have Supplements", *Comm. Algebra*. Vol. 29, No. 6, pp. 2389-2405.

Alizade, R. and Pancar, A., 1999. *Homoloji Cebire Giriş*, (Ondokuz Mayıs Üniversitesi Yayınları, Samsun).

Anderson, F. W. and Fuller K. R., 1992. *Rings and Categories of Modules*, (Springer, NewYork).

Büyükaşık, E., 2005. *Weakly and Cofinitely Weak Supplemented Modules over Dedekind Domains*, (Ph.D.thesis, Dokuz Eylül University, The Graduate School of Natural and Applied Sciences).

Clark, J., Lomp, C., Vanaja, N. and Wisbauer, R., 2006. *Lifting Modules*, (Birkhäuser Verlag, Basel).

Cohn, P.M., 2002. *Basic Algebra*, (Springer-Verlag, London).

Fuchs, L. and Salce, L., 1985. *Modules Over Valuation Domains*, (Marcel Dekker Inc.).

Hazewinkel, M., Gubareni, N. and Kirichenko, V. V., 2004. *Algebras, Rings and Modules*, (Kluwer Academic Publishers, Boston).

Kaplansky, I., 1952. "Modules Over Dedekind Rings and Valuation Rings", *Transaction of the American Mathematical Society*. Vol. 72, No. 2, pp. 327-340.

Kaplansky, I., 1969. *Infinite Abelian Groups*, (University of Michigan Press, Michigan).

Kasch, F., 1982. *Modules and Rings*, (Academic Press Inc., London).

Lam, T. Y., 2001. *A First Course in Noncommutative Rings*, (Sphinger-Verlag, New York).

Lomp, C., 1996. *On Dual Goldie Dimension*, (Diplomarbeit M. Sc. Thesis, Mathematischen Institut der Heinrich-Heine Universität, Düsseldorf. Revised verison (2000)).

Lomp, C., 1999. "On Semilocal Modules and Rings", *Comm. Algebra*. Vol. 27, No. 4, pp. 1921-1935.

Rudlof, P., 1991. "On The Structure of Couniform and Complemented modules", *Journal of Pure and Appl. Algebra*. Vol. 74, No. (1-2), pp. 281-305.

Sharp, R. Y., 2000. *Steps in Commutative Algebra*, (Chambridge University Press, Chambridge).

Sharpe, D. W. and Vamos, P., 1972. *Injective Modules*, (Chambridge University Press).

Smith, P. F., 2000. "Finitely Generated Supplemented Modules are Amply Supplemented", *Arab. J. Sci. Eng. Sect. C Theme Issues*. Vol. 25, No. 2, pp. 69-79.

Wisbauer, R., 1991. *Foundations of Module and Ring Theory*, (Gordon and Breach).

Zöschinger, H., 1974a. "Komplementierte Moduln über Dedekindringen", *J. Algebra*. Vol. 29, pp. 42-56.

Zöschinger, H., 1974b. "Komplemente als direkte Summanden", Vol. 25, pp. 241-253.

Zöschinger, H., 1980. "Koatomare moduln", *Math. Z*. Vol. 170, pp. 221-232.

Zöschinger, H., 1986. "Komplemente als direkte Summanden III", *Arch. Math. (Basel)*. Vol.42, No. 2, pp.125-135.