STATIONARY AND 2+1 DIMENSIONAL INTEGRABLE REDUCTIONS OF AKNS HIERARCHY

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Stationary and 2+1 Dimensional Integrable Reductions of AKNS Hierarchy

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ABSTRACT

The main concepts of the soliton theory and infinite dimensional Hamiltonian Systems, including AKNS (Ablowitz, Kaup, Newell, Segur) integrable hierarchy of nonlinear evolution equations are introduced. By integro-differential recursion operator for this hierarchy, several reductions to KDV, MKdV, mixed KdV/MKdV and Reaction-Diffusion system are constructed. The stationary reduction of the fifth order KdV is related to finite-dimensional integrable system of Henon-Heiles type. Different integrable extensions of Henon-Heiles model are found with corresponding separation of variables in Hamilton-Jacobi theory. Using the second and the third members of AKNS hierarchy, new method to solve 2+1 dimensional Kadomtsev-Petviashvili(KP-II) equation is proposed. By the Hirota bilinear method, one and two soliton solutions of KP-II are constructed and the resonance character of their mutual interactions are studied. By our bilinear form we first time created new four virtual soliton resonance solution for KPII. Finally, relations of our two soliton solution with degenerate four soliton solution in canonical Hirota form of KPII are established.

ÖZET

Soliton teorisinin ana kavramları ve lineer olmayan evrim denklemlerinin AKNS (Ablowitz, Kaup, Newel, Segur) integrallenebilir hiyerarşisini içeren sonsuz boyutlu hamiltoniyen sistemlerine bir giriş yapıldı. Bu hiyerarşide çeşitli indirgemeler yapılarak, integro-differensiyel tekrarlama operatörü yardımı ile, KdV, MKdV, ve karışık KdV/MKdV nin ve de reaksiyon-difuzyon denklemleri elde edildi. Beşinci derece KdV nin durağan indirgenmesi, Henon-Heiles tipi sonlu boyutlu integrallenebilir sistemi ile ilişkilidir. Henon-Heiles tipi sonlu boyutlu integrallenebilir uzantilari, Hamilton-Jacobi teorisindeki ilgili değişkenlerin yardımı ile bulundu. AKNS hiyerarşisinin ikinci ve üçüncü üyelerini kullanarak 2+1 boyutlu Kadomtsev-Petviashivili (KPII) denklemini çözmek için yeni bir yontem sunuldu. Hirota bilineer yontemi vasıtasıyla KPII nin bir ve iki soliton çözümleri bulundu ve ayrıca bunların karşılıklı etkileşimlerinin rezonans karakteri çalışıldı. Yeni bulduğumuz bilineer form ile ilk defa KP için dört sanal soliton rezonans çözümünü elde ettik. Son olarak Satsuma ve Hirotanın yozlaşmış dört soliton çözümü ile bizim iki soliton çözümümüz arasındaki ilişki kuruldu.

TABLE OF CONTENTS

LIST OF FIGURES

Chapter 1

INTRODUCTION

In August of 1834, John Scott Russel (1808-1882) was studying the motion of a small boat in a canal and he observed that when the boat suddenly stopped, a lump of water formed at the front of the boat, moved forward with constant speed and shape. He called this phenomenon as the WAVE OF TRANSLATION. In a number of experiments he determined the shape of a solitary wave to be that of $sech^2 x$ function. This was the first recorded observation of a soliton [1]. At that time there was no equation describing such water waves.

But in 1895 Korteweg and de Vries [2] were studying propagation of waves on the surface of shallow water and derived the following nonlinear partial differential equation called the KdV equation,

$$U_t = U_{xxx} - 6UU_x \tag{1.1}$$

where $U_t = \partial U/\partial t$ and $U_{xxx} = \partial^3 U/\partial x^3$. They found the solution of this equation exactly in the same form as derived by J.Scott Russel. Later this is called soliton by Kruskal and Zabusky [3], in their study of the KdV as a continuum limit of the Fermi-Pasta-Ulam chain problem. Two years later the method of solution of the initial-value problem associated with the KdV equation, for solutions, rapidly vanishing as $x \to \pm\infty$, was found in terms of the spectral problem for the one-dimensional Schrödinger operator [4]. As it was shown, solutions of the KdV equation corresponding to a purely continuous spectrum are similar to wave packets, and disperse as time goes on. Instead, solutions with a purely discrete spectrum describe the interaction of N solitons with each other. The method of solving the "inverse problem" , i.e. reconstruction of a potential from the given spectral data had been established by the Russian mathematical-physics community in the early 1950s. This method applied for solving the KdV equation is known as the "inverse-scattering method" [5].

It then was found that for the KdV equation infinitely many constants of the motion exist and are all functionally independent of each other, like in the definition of Liouville intergability of finite-dimensional systems [6]. This point was further clarified by showing that the KdV equation can be written in the Hamiltonian form [7]. Relation between solitons and conservation laws has been realized by P. Lax in his representation for the soliton equation, associating the KdV flow with an isospectral change of the linear Schrödinger operator [8]. Later it was shown that soliton equations include other nonlinear evolution equations [9], [10] with rich and beautiful mathematical structures of Hamiltonian integrable systems[11]. Several methods were developed to study such equations: (a) the Hirota bilinear method [12], [13], allowing to construct the N-soliton solutions as an alternative to the spectral method [14], [15], [16]. (b) Bäcklund and Darboux transformations [17], [18]. (c) The zero-curvature formulation of the linear problem [19], [20], [21], [22]. (d) The algebraic-geometric formulation for the class of periodic, finite-gap solutions [23], [24], [25], [26]. (e) The generalized Wronskian method [15]. (g) The Riemann-Hilbert problem and its generalization to the so-called DBAR problem [27], [28], [29], [30]. During the last decades, it has become more widely recognized in many areas of physics [31], [32], such as hydrodynamics [33], [34], [35], plasma physics [36], solid state and condensed matter physics [37], [38], [39], [40], [41], [42], biology [38], [43], nonlinear optics [44], [45], [46], general relativity [47], [48] and elementary particle physics [49], [50], [51] that solitons can result in qualitatively new phenomena which cannot be constructed by perturbation theory of the linear systems [52], [53], [54], [55], [56]. In the last 20 years, optical solitons have been discovered and investigated in different systems like optical fibers [57], [58]. Trans-oceanic high-rate transmission of information by one soliton in nonlinear herbium-doped fibers, as well as ultrafast pulse generation by a soliton laser, and a number of all-optical switching devices, with potential use as integrated components of optical computers, show the influence of quite exotic idea on modern development in science and technology. Soliton idea has stimulated also development and discoveries in mathematics itself [59], [60], [61], [62]. It becomes interdisciplinary subject attracting researchers from different fields [63], [64], [65], [66], [67], [68], and appeared now in some textbooks [69], [70], [71].

In this thesis we study the problem of separation of variables for several integrable extensions of Henon-Heiles system, relations with AKNS integrable hierarchy and KP equation, corresponding soliton solutions and their resonance dynamics.

In the next section we present a brief discussion of the main idea of soliton theory.

In Chapter 2 we review the current approach to the integrable evolution equations as Hamiltonian dynamical systems. In section 2.1 the Hamilton theory in the symplectic geometry approach and basic ingredients of the Liouville theorem for integrability of finite dimensional models are introduced. In section 2.2 we discuss separation of variables and canonical transformation to the action-angle variables. Definitions and examples of evolution equations and dynamical systems are subject of section 2.3. Then, in section 2.4 we show that an evolution equation can be considered as an infinite dimensional dynamical system. For soliton equations these dynamical systems are Hamiltonian systems. The Hamiltonian structure of the KdV equation, with the Faddeev-Zakharov Poisson bracket and infinite set of integrals of motion in involution are given in section 2.5. The idea of spectral transform is illustrated for the linear equations in section 2.6, while for the nonlinear equations in section 2.7. The Lax representation, isospectrality conditions and the zero-curvature representation of a nonlinear integrable system are exposed in section 2.8.

In the third chapter we introduce the AKNS hierarchy of evolution equations and its reductions. We show that every integral of motion generates the Hamiltonian flow from infinite hierarchy (section 3.1). The AKNS hierarchy, recursion operator and first members of the hierarchy are studied in details in section 3.2. In section 3.3 we obtain two important reductions of AKNS hierarchy to the KdV, MKdV equations, corresponding recursion operators and reduced hierarchies. In section 3.4 we found new reduction to the mixed KdV-MKdV equation, corresponding recursion operator and generated by it an infinite hierarchy.

Chapter 4 is devoted to the Henon-Heiles model which is the subject of recent intensive studies in integrability and chaos. Formulation of the Henon-Heiles model and its integrable cases are given in section 4.1. Then, in the next section, following results of A. Fordy we show how these integrable cases appear from the stationary reductions of the fifth order soliton equations, namely, the KdV, the Sawada-Kotera and the Kaup-Kupershmidt equations. For solving Henon-Heiles model we use the Hamilton-Jacobi theory and separation of variable technique from this theory, which we introduce in section 4.4

Separation of variables in the Henon-Heiles model and its integrable extensions are considered in Chapter 5. First, we are discussing an additional integral of motion, Liouville integrability and separation of variables for the Hamiltonian characteristic function. Then, in section 5.1 we show that the second integral can be considered as an additional, second Hamiltonian of the Henon-Heiles system. This type of systems with two Hamiltonians structure are called the bi-Hamiltonian systems. They are subject of recent studies on algebraic formulation of integrability. Separation of variables in the extended Henon-Heiles model we present in section 5.2. Extension with the constant C term and corresponding bi-Hamiltonian formulation, in the subsection 5.2.1, extension with the inverse cubic term of strength D, in subsection 5.2.2, harmonic term extension in subsection 5.2.3, and mixed harmonic C and harmonic D extensions in subsections 5.2.4, 5.2.5. Separation of variables and bi-Hamiltonian formulation in the general extension with harmonic terms and C, D terms, is given in subsection 5.2.6.

In the Chapter 6 we deal with exact one and two soliton solutions of the first two nonlinear systems from the AKNS hierarchy, and their resonance dynamics. In section 6.1 we introduce the main ingredients of the Hirota bilinear method to solve soliton equations. Bilinear representation and one and two dissipative soliton solutions for the Reaction-Diffusion equations are given in section 6.2. The resonance character of soliton interactions in this case illustrates section 6.3. Beautiful geometrical interpretation of equations as a constant curvature surface in pseudo-Riemannian space we demonstrate in section 6.4. One soliton metric in this case develops the so called causal singularity, similar to black hole horizons in the General Relativity Theory. New dissipative solitons for the third flow of AKNS we construct in section 6.5; one soliton solution in subsection 6.5.1 and two-soliton solution in subsection 6.5.2. Reductions of bilinear equations and corresponding soliton solutions to the MKdV equation and mixed KdV-MKdV equations are found in subsections 6.5.3 and 6.5.4 respectively.

In Chapter 7 we propose a new method to generate solutions of 2+1 dimensional extension of KdV equation, known as the KP equation. In section 7.1 we show that if one considers a simultaneous solution of the second and the third flows from the AKNS hierarchy, then the product e^+e^- satisfies the KPII equation (Theorem 7.1.1). Using this theorem and results of Chapter 6 we construct new bilinear representation of KPII equation. Then, by our method we construct one soliton solution (section 7.2) and two soliton solution (section 7.3.). In section 7.4 we compare our two soliton solution of KPII with the one of the known before bilinear Hirota representation. As a result we find that our two-soliton solution corresponds to the degenerate four soliton solution in the standard Hirota form. Resonance character of our soliton interactions is studied in section 7.5

In Chapter 8 we discuss main results of this thesis and conclusions. In Appendix we remained basic formulas of the Hirota bilinear method explored in our work.

1.1 Basic idea of Soliton Theory

To characterize the main property of solitons, we will start from KdV equation (1.1), considering two different limits. The first one is given by the dispersive linear equation

$$U_t = U_{xxx}. \tag{1.2}$$

The particular solution of this equation is $U(x,t) = U_0 e^{i(kx-\omega t)}$, with the dispersion relation $\omega = k^3$. Then, the phase velocity of this elementary wave $\frac{\omega}{k} = k^2$ depends on k. Due to linearity of the equation any superposition of these waves $U = \sum_k a_k e^{i(kx-k^3 t)}$ is also a solution of equation(1.2). But since each component of this "wave packet" will travel with different velocity (they are dispersive), the wave will change shape during propagation, and, in general,will spread out.

In another (non-dispersive) limit, nonlinear equation (1.1) has the form

$$U_t + UU_x = 0. \tag{1.3}$$

To understand the behaviour of a solution of this equation, we will consider first the simple linear wave equation

$$U_t + cU_x = 0, \tag{1.4}$$

describing wave, propagating with velocity c, with shape given in terms of an arbitrary smooth function

$$U(x,t) = U(x - ct).$$

Then, a solution of nonlinear equation (1.3) has to appear in unexplicit form, simply replacing c by function U(x,t), as follows

$$U(x,t) = U(x - U(x,t)t).$$

As easy to see, at finite time, the form of U_x becomes infinite and it indicates an appearance of the shock wave.

But in the KdV equation (1.1) both terms (dispersion and nonlinearity) appear simultaneously. Therefore, their effects are opposite in character, exactly compensate one another and provide the stable structure.

In general, solitons are defined to be special solutions of some nonlinear partial differential equations with the following properties

- Solitons are traveling waves
- The energy of the wave is finite. It is continuous, bounded and localized in space.

- They are stable
- They possesses an elastic collision and keep their identities after pairwise collisions.
- An initial wave will asymptotically decompose into one or more solitons depending on amplitude and other properties.

Chapter 2

EVOLUTION EQUATIONS AND HAMILTONIAN SYSTEMS

2.1 Hamiltonian Dynamical Systems

The Hamiltonian theory is an important tool in the classical mechanics [79] and plays crucial role in the soliton theory [11]. Hamilton's canonical equations for a system with n degrees of freedom, defined in $2n$ dimensional phase space,

$$\dot{p}_i = -\frac{\partial H(p_i, q_i)}{\partial q_i}, \qquad \dot{q}_i = \frac{\partial H(p_i, q_i)}{\partial p_i}, \text{ where } \quad i = (1, 2, ..., n) \tag{2.1}$$

represent special case of finite dimensional dynamical systems generated by one Hamiltonian function $H(q_i, p_i)$ where $q_1, ..., q_n$ are generalized coordinates and $p_1, ..., p_n$ are generalized momenta. In this phase space the Poisson bracket of two functions with respect to canonical variables (q, p) is defined as the following skew-symmetric bilinear form

$$\{F, G\} = \sum_{i=1}^{n} \frac{\partial F}{\partial q_i}\frac{\partial G}{\partial p_i} - \frac{\partial F}{\partial p_i}\frac{\partial G}{\partial q_i}, \tag{2.2}$$

that satisfies the following properties:

1. Skew-Symmetry

$$\{F, G\} = -\{G, F\}, \quad \{F, F\} = 0 \tag{2.3}$$

2. Linearity

$$\{\lambda F_1 + \mu F_2, G\} = \lambda\{F_1, G\} + \mu\{F_2, G\} \tag{2.4}$$

where λ and μ are arbitrary constant

3. Leibnitz Rule

$$\{F_1 F_2, G\} = F_1\{F_2, G\} + F_2\{F_1, G\} \tag{2.5}$$

4. The Jacobi Identity

$$\{F_1, \{F_2, F_3\}\} + \{F_2, \{F_3, F_1\}\} + \{F_3, \{F_1, F_2\}\} = 0 \tag{2.6}$$

The Poisson brackets can be written in a compact form as follows

$$\{F, G\} = (\nabla F)^T J(\nabla G) = \sum_{i,k=1}^{n} \frac{\partial F}{\partial X_i} J_{ik} \frac{\partial G}{\partial X_k}, \tag{2.7}$$

where generalized gradients and symplectic metric are defined as

$$\nabla F = \begin{pmatrix} \frac{\partial F}{\partial X_1} \\ . \\ . \\ . \\ \frac{\partial F}{\partial X_{2n}} \end{pmatrix}, \quad \nabla G = \begin{pmatrix} \frac{\partial G}{\partial X_1} \\ . \\ . \\ . \\ \frac{\partial G}{\partial X_{2n}} \end{pmatrix}, \quad J_n = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}_{2n \times 2n} \tag{2.8}$$

where

$$X_i = (q_1, q_2, ..., q_n, p_1, p_2, ..., p_n). \tag{2.9}$$

Then, the Hamilton equations(2.1) take the form of 2n dimensional dynamical system (the gradient system) as

$$\dot{X}_i = J_{ik} \frac{\partial H}{\partial X_k}. \tag{2.10}$$

Definition 2.1.0.1 *A function $F(q_i, p_i)$ is a first integral of Hamiltonian system with Hamiltonian function $H(q_i, p_i)$ if and only if the Poisson bracket $\{H, F\} = 0$.*

Definition 2.1.0.2 *Two functions $F_1(q_i, p_i)$ and $F_2(q_i, p_i)$ are in involution if their Poisson bracket is equal zero, $\{F_1, F_2\} = 0$.*

Theorem 2.1.0.3 (Liouville) *If a system with n degrees of freedom (in 2n dimensional phase space)admits n independent first integrals of motion in involution then the system is integrable by quadratures.*

2.2 Separation of Variables

An integrable system admits separation of variables by canonical transformation to the action-angle variables. We can represent this by the following diagram

$H(q_i, p_i)$	$\overrightarrow{\text{Canonical Transformation}}$	$H(I_1, ..., I_n)$
$q_1, ..., q_n$		$\varphi_1, ..., \varphi_n$
$p_1, ..., p_n$		$I_1, ..., I_n$
Canonical variables		*Action − angle variables*

$\dot{q}_i = \frac{\partial H}{\partial p_i}$ $\qquad\qquad$ $\dot{\varphi}_i = \frac{\partial H(I)}{\partial I_i} = \omega_i(I)$

$\dot{p}_i = -\frac{\partial H}{\partial q_i}$ $\qquad\qquad$ $\dot{I}_i = -\frac{\partial H(I)}{\partial \varphi_i} = 0$

One uses canonical transformation from (q, p) to (φ, I) variables, called the action-angle variables [79], such that Hamiltonian H is a function of only action variables I_i. Then, equations of motion in these variables show that action variables $(I_1, ..., I_n)$ are independent integrals of motion, while angle variables $\varphi_1, ..., \varphi_n$ are linear functions of time

$$\varphi_i = \omega_i t + \varphi_i(0).$$

Using these canonical transformations we can solve Hamiltonian's dynamics according to diagram

$$\begin{array}{ccc} q(0), p(0) & \xrightarrow[Direct\ CT]{} & I(0), \varphi(0) \\ \downarrow & & \downarrow \\ q(t), p(t) & \xleftarrow[Inverse\ CT]{} & I(t), \varphi(t) \end{array}$$

2.3 Evolution Equations and Dynamical Systems

Definition 2.3.0.4 *Partial differential equation for function $U(x, t)$ in the form*

$$U_t = F(U, U_x, U_{xx}, ...)$$

is called the evolution equation.

Example: The KdV equation

$$U_t = U_{xxx} - 6UU_x$$

Definition 2.3.0.5 *The first order system of equations in the form*

$$\dot{a}_l = \sum_{n=1}^{N} F_{ln}(a_1, ..., a_N)$$

where $a_l(t)$, $(l = 1, .., N)$ is a vector field in N-dimensional space, is called the dynamical system.

Example:The Henon-Heiles system

$$\dot{q}_1 = p_1, \qquad \dot{p}_1 = -3q_1^2 - \frac{1}{2}q_2^2$$

$$\dot{q}_2 = p_2, \qquad \dot{p}_2 = -q_1 q_2.$$

2.4 Infinite Dimensional Dynamical Systems

An evolution equation can be considered as a dynamical system in infinite dimensional space. For example, if function $U(x,t)$ is given in the interval $x \in [0, 2\pi]$, then we can expand it to the Fourier series

$$U(x,t) = \sum_{n=-\infty}^{\infty} a_n(t) e^{inx}. \tag{2.11}$$

Substituting this expansion to the KdV equation (1.1),then multiplying this equality by e^{-ilx}, taking integral and using definition of the Dirac delta function ($\delta(l-m) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i(l-m)x}dx$) we obtain the following system of equations for the Fourier components of $U(x,t)$

$$\dot{a}_l = (il)^3 a_l + 6 \sum_{n=-\infty}^{\infty} (in) a_n a_{l-n}, \qquad \text{where} \qquad (l = 0, \pm 1, \pm 2, ...). \tag{2.12}$$

It represents an infinite dimensional nonlinear dynamical system. Moreover, this dynamical system is the Hamiltonian dynamical system in the form of Eq. (2.10).

2.5 KdV as a Hamiltonian System

The KdV equation can be written as the Hamiltonian system

$$U_t = -\frac{\partial(3U^2 - U_{xx})}{\partial x} = \frac{\partial}{\partial x}\frac{\delta H}{\delta U(x)}, \tag{2.13}$$

where the Hamiltonian functional H is

$$H[U] = -\int_{-\infty}^{\infty} (\frac{U_x^2}{2} + U^3) dx, \tag{2.14}$$

and symbol $\delta/\delta U$ denotes variational derivative. The Poisson bracket in this case, called the Faddeev-Zakharov [7] bracket, is defined as follows

$$\{S, R\} = \int_{-\infty}^{\infty} \frac{\delta S}{\delta U(x)} \frac{\partial}{\partial x} \frac{\delta R}{\delta U(x)} dx, \tag{2.15}$$

where $\partial/\partial x$ is the skew symmetric operator.

If $\frac{\partial}{\partial x}\frac{\delta H}{\delta U(x)}$ is expressed as

$$\int_{-\infty}^{\infty} \delta(x-y) \frac{\delta H}{\delta U(y)} dy = \{U, H\}, \tag{2.16}$$

then equation (2.13) becomes in Hamiltonian form,

$$\frac{\partial U}{\partial t} = \{U, H\}. \tag{2.17}$$

Thus the KdV can be considered as the Hamiltonian dynamical system. Moreover it admits the second Hamiltonian structure

$$U_t = (\partial_x^3 - 2U\partial_x - 2\partial_x U)\frac{\delta}{\delta U(x)}\int_{-\infty}^{\infty}\frac{1}{2}U^2 dx, \tag{2.18}$$

with Magri bracket [130] and this shows bi-Hamiltonian nature of KdV equation.

Furthermore, it is integrable Hamiltonian system. But to discuss infinite dimensional dynamical integrable systems we need an infinite number of integrals of motion $(I_1, ..., I_n)$ in involution. Then, according to Liouville theorem the system is formally integrable.

For the KdV equation following functionals

$$I_1 = \int_{-\infty}^{\infty} U(x)dx \tag{2.19}$$

$$I_2 = \int_{-\infty}^{\infty} U^2(x)dx \tag{2.20}$$

$$I_3 = \int_{-\infty}^{\infty} (\frac{U_x^2}{2} + U^3)dx \tag{2.21}$$

$$I_4 = \int_{-\infty}^{\infty} (\frac{U_{xx}^2}{4} - UU_x + U^4)dx \tag{2.22}$$

$$...$$

are integrals of motion, which are in involution according to the Faddeev-Zakharov bracket (2.15), $\{I_n, I_m\} = 0$. So the system is integrable.

2.6 Spectral Transform for Linear equations

For separation of variables in this system most powerful method is the Inverse Scattering Method(ISM) [5] or the Nonlinear Fourier transform [19].

To illustrate the idea let us consider Fourier transform for the linear equation (1.2)(Initial Value Problem)

$$U_t = U_{xxx}, \qquad U(x,0) = F(x); \tag{2.23}$$

$$U(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx}\hat{U}(k,t)dk \qquad \text{FourierTransform} \tag{2.24}$$

$$\hat{U}(k,t) = \int_{-\infty}^{\infty} e^{-ikx}U(x,t)dx \qquad \text{InverseFourierTransform} \tag{2.25}$$

Then,the Initial Value Problem is solved by the following diagram

$$\begin{array}{ccccl} U(x,0) & \xrightarrow[Fourier\ Transform]{} & \hat{U}(k,0) & & \\ \downarrow & & \downarrow & & \text{Linear time evolution} \\ U(x,t) & \xleftarrow[Inverse\ Fourier\ Transform]{} & \hat{U}(k,t) & & \end{array}$$

2.7 Nonlinear Fourier Transform

Similarly works the nonlinear Fourier transform for KdV equation [15]

$$
\begin{array}{ccc}
U(x,0) & \xrightarrow[\textit{Direct Spectral Transform}]{} & S(0) \\
\downarrow & & \downarrow \\
U(x,t) & \xleftarrow[\textit{Inverse Spectral Transform}]{} & S(t)
\end{array}
$$

where $S(t)$ is Nonlinear Fourier image.

2.8 Linearization of Nonlinear Problem

To develop the spectral transform to a Nonlinear evolution equation one needs the so called linear representation. This auxiliary linear problem is known as the Lax pair representation [8]. For the KdV equation it can be represented by the following linear system

$$-\phi_{xx} = (\lambda - U)\phi, \tag{2.26}$$

$$-\phi_t = -4\frac{\partial^3 \phi}{\partial x^3} + 3U\phi_x + U_x\phi + \phi_x U, \tag{2.27}$$

where $\phi = \phi(x, \lambda, t)$ and λ is a spectral parameter.

Let L be the Schrödinger operator

$$L = -\frac{\partial^2}{\partial x^2} + U, \tag{2.28}$$

and A be

$$A = -4\frac{\partial^3}{\partial x^3} + 3(U\frac{\partial}{\partial x} + \frac{\partial}{\partial x}U). \tag{2.29}$$

If we write equations (2.26) and (2.27) in terms of these operators we get the system

$$L\phi = \lambda^2\phi, \tag{2.30}$$

$$\phi_t = A\phi, \tag{2.31}$$

which is called the Lax representation. If isospectrality condition satisfies (which means that $\partial\lambda/\partial t = 0$), then KdV equation is equivalent to the operator equation

$$L_t = [A, L]. \tag{2.32}$$

In general many nonlinear partial differential equations which are integrable are related to existence of the Lax pair [16].

Another form of linear representation relates to commutativity (compatibility) condition

$$U_t - V_x + [U, V] = 0, \tag{2.33}$$

for the following linear system of equations

$$\frac{\partial \psi}{\partial x} = U\psi, \tag{2.34}$$

$$\frac{\partial \psi}{\partial t} = V\psi. \tag{2.35}$$

This form of the linear problem is known as the zero-curvature representation [5], [20]. And integrable hierarchy can be naturally developed in this approach [19].

Chapter 3

AKNS INTEGRABLE HIERARCHY

3.1 Hierarchy of Integrable Evolutions

The hierarchy of integrals of motion (2.19- 2.22) for the KdV equation, generates hierarchy of nonlinear evolution equations in the form

$$U_{t_n} = \{U, I_n\}, \tag{3.1}$$

with Faddeev-Zakharov bracket (2.15). I_n can be considered as Hamiltonians of corresponding evolution equations in times $t_1, ..., t_n, ...$The hierarchy of these equations is related to the hierarchy of corresponding linear problems.

3.2 AKNS Hierarchy

In 1974 American mathematicians (Ablowitz, Kaup, Newell, Segur) introduced the AKNS Hierarchy of nonlinear evolution equations [19]. This hierarchy includes several nonlinear evolution equations as the Nonlinear Schrödinger equation, and Modified KdV equation, as special cases of equations of degree two and three respectively.

The AKNS hierarchy arises from a sequence of linear evolutions

$$\frac{\partial \psi}{\partial t_n} = V_n \psi, \ (n = 0, 1, 2, ...) \tag{3.2}$$

and the Zakharov-Shabat spectral problem

$$\frac{\partial \psi}{\partial x} = \begin{pmatrix} \lambda & q \\ r & -\lambda \end{pmatrix} \psi = U\psi. \tag{3.3}$$

The ψ is a vector $(\psi_1, \psi_2)^T$ and q, r are potentials. Differentiating these equations

$$\frac{\partial}{\partial x}(\frac{\partial \psi}{\partial t_n}) = (V_n)_x \psi + V_n \psi_x = (V_n)_x \psi + V_n U \psi, \tag{3.4}$$

$$\frac{\partial}{\partial t_n}(\frac{\partial \psi}{\partial x}) = U_{t_n}\psi + U\psi_{t_n} = U_{t_n}\psi + UV_n\psi, \tag{3.5}$$

we get compatibility condition for the system (3.2),(3.3) in the form

$$\frac{\partial U}{\partial t_n} - \frac{\partial V_n}{\partial x} + UV_n - V_nU = 0. \tag{3.6}$$

Similarly, by the compatibility of t_n and t_m evolutions (3.2)

$$\partial_{t_n}\partial_{t_m}\psi = \partial_{t_m}\partial_{t_n}\psi \tag{3.7}$$

we have

$$\frac{\partial V_m}{\partial t_n} - \frac{\partial V_n}{\partial t_m} + V_nV_m - V_mV_n = 0. \tag{3.8}$$

To construct AKNS hierarchy let us suppose that U,V in equations(2.34),(2.35) have the form

$$V = \begin{pmatrix} a(\lambda) & b(\lambda) \\ c(\lambda) & -a(\lambda), \end{pmatrix}, \quad U = \begin{pmatrix} \lambda & q \\ r & -\lambda \end{pmatrix}, \tag{3.9}$$

Then substituting in equation (2.33)

$$U_t - V_x + [U, V] = 0, \tag{3.10}$$

we can find the following restrictions on functions a, b, c,

$$\begin{aligned} a_x &= qc - rb, \\ b_x - 2\lambda b &= q_t - 2qa, \ \cdot \\ c_x + 2\lambda c &= r_t + 2ra. \end{aligned} \tag{3.11}$$

We assume V_N as a polynomial in spectral parameter λ degree N. It implies the following equations

$$a = \sum_{n=0}^{N} \lambda^n a_n, \ b = \sum_{n=0}^{N} \lambda^n b_n, \ c = \sum_{n=0}^{N} \lambda^n c_n. \tag{3.12}$$

Substituting a, b, c to the equation (3.11) we have the recurrence relations as

$$\begin{aligned} b_{0x} &= q_t - 2\,q\,a_0, \\ c_{0x} &= r_t - 2\,r\,a_0, \\ a_{0x} &= q\,c_0 - r\,b_0; \end{aligned} \tag{3.13}$$

$$
\begin{aligned}
b_{nx} - 2\,b_{n-1} &= -2\,q\,a_n, \\
c_{nx} + 2\,c_{n-1} &= 2\,r\,a_n, \\
a_{nx} = q\,c_n - r\,b_n&.\ \ (n = 1, 2, ..., N)
\end{aligned}
\tag{3.14}
$$

Solving the last equation as $a_n = \int^x (q\,c_n - r\,b_n)$ and substituting it to first couple of equations (3.13) we get

$$
\begin{aligned}
q_t &= b_{0_x} + 2\,q\int^x (q\,c_0 - r\,b_0), \\
r_t &= c_{0_x} - 2\,r\int^x (q\,c_0 - r\,b_0).
\end{aligned}
\tag{3.15}
$$

These equations can be written in the matrix form as

$$
\begin{pmatrix} q \\ r \end{pmatrix}_t = R \begin{pmatrix} b_0 \\ c_0 \end{pmatrix}, \tag{3.16}
$$

where integro-differential matrix operator R is

$$
R = \begin{pmatrix} \partial_x - 2q\int^x r & 2q\int^x q \\ 2r\int^x r & \partial_x - 2r\int^x q \end{pmatrix}. \tag{3.17}
$$

The first couple of equations (3.14) in terms of R is

$$
\begin{pmatrix} b_{n-1} \\ c_{n-1} \end{pmatrix} = \frac{1}{2}\sigma_3 R \begin{pmatrix} b_n \\ c_n \end{pmatrix}, \qquad n = 1, ..., N\ , \tag{3.18}
$$

and recursively we have

$$
\begin{pmatrix} b_0 \\ c_0 \end{pmatrix} = \Re^N \begin{pmatrix} b_N \\ c_N \end{pmatrix}, \tag{3.19}
$$

where $\Re = \frac{1}{2}\sigma_3 R$ is called the recursion operator. This recursion operator generates the hierarchy of evolutions

$$
\frac{1}{2}\sigma_3 \begin{pmatrix} q \\ r \end{pmatrix}_{t_N} = \Re \begin{pmatrix} b_0 \\ c_0 \end{pmatrix} = \Re^{N+1} \begin{pmatrix} b_N \\ c_N \end{pmatrix}. \tag{3.20}
$$

To have equations in a closed form let us fix b_N, c_N in the simplest form

$$
\begin{pmatrix} b_N \\ c_N \end{pmatrix} = \begin{pmatrix} q \\ r \end{pmatrix}.
$$

Then we find the hierarchy of evolution equations

$$\frac{1}{2}\sigma_3 \begin{pmatrix} q \\ r \end{pmatrix}_{t_N} = \Re^{N+1} \begin{pmatrix} q \\ r \end{pmatrix}. \tag{3.21}$$

For particular values of N we have equations;
for N=0

$$\begin{aligned} q_{t_0} &= q_x, \\ r_{t_0} &= r_x; \end{aligned} \tag{3.22}$$

for N=1

$$\begin{aligned} q_{t_1} &= \tfrac{1}{2}q_{xx} - q^2\, r, \\ r_{t_1} &= -\tfrac{1}{2}r_{xx} + r^2\, q; \end{aligned} \tag{3.23}$$

for N=2

$$\begin{aligned} q_{t_2} &= \tfrac{1}{4}q_{xxx} - \tfrac{1}{2}(q^2 r)_x + \tfrac{1}{2}q^2 r_x - \tfrac{1}{2}qrq_x, \\ r_{t_2} &= \tfrac{1}{4}r_{xxx} - \tfrac{1}{2}(r^2 q)_x + \tfrac{1}{2}r^2 q_x - \tfrac{1}{2}rqr_x. \end{aligned} \tag{3.24}$$

By the following identification

$$\begin{aligned} q &= \sqrt{\tfrac{-\lambda}{8}}\, e^+, \\ r &= \sqrt{\tfrac{-\lambda}{8}}\, e^-, \end{aligned} \tag{3.25}$$

the recursion operator $\Re$ (integro differential) becomes

$$\Re = \begin{pmatrix} \partial_x - \frac{\lambda}{4}e^+ \int^x e^- & -\frac{\lambda}{4}e^+ \int^x e^+ \\ -\frac{\lambda}{4}e^- \int^x e^- & \partial_x + \frac{\lambda}{4}e^- \int^x e^+ \end{pmatrix}. \tag{3.26}$$

Then the first three members of AKNS hierarchy (3.22), (3.23), (3.24) in terms of e^+ and e^- appear as

$$\partial_{t_0} e^{\pm} = \partial_x e^{\pm}; \tag{3.27}$$

$$\pm\partial_{t_1} e^{\pm} = \partial_x^2 e^{\pm} + \frac{\lambda}{4} e^+ e^- e^{\pm}; \tag{3.28}$$

$$\partial_{t_2} e^{\pm} = \partial_x^3 e^{\pm} + \frac{3\lambda}{4} e^+ e^- \partial_x e^{\pm}. \tag{3.29}$$

The first couple of equations (3.27) are the linear wave equations. The second system (3.28) is called the Reaction-Diffusion system [20]. It is connected with low dimensional gravity, constant curvature surfaces and quantum theory and has been studied in [76]. The last system with cubic dispersion has reductions to KdV, MKdV and mixed KdV-MKdV equations.

3.3 KdV and MKdV reductions

The third member of hierarchy (3.29) admits following reductions [20].

1) The first reduction $e^+ = U,\ e^- = 1$, leads to the KdV equation

$$\partial_{t_2} U = \partial_x^3 U + \frac{3\lambda}{4} U \partial_x U. \tag{3.30}$$

Under this reduction the reduced hierarchy (3.21) becomes

$$\frac{\partial}{\partial_{t_{2k}}} \begin{pmatrix} U \\ 0 \end{pmatrix} = \Re^k_{KdV} \partial_x \begin{pmatrix} U \\ 0 \end{pmatrix}, \tag{3.31}$$

or in the scalar form, the KdV hierarchy

$$\partial_{t_{2k}} U = R^k_{KdV}(\partial_x U), \tag{3.32}$$

with the recursion operator of the KdV hierarchy given by [130]

$$R_{KdV} = (\partial_1^2 + \frac{\lambda}{2} U + \frac{\lambda}{4} \partial_1 U \int^x). \tag{3.33}$$

2) Under the second reduction, $e^+ = e^- = U$, we obtain MKdV equation

$$\partial_{t_2} U = \partial_x^3 U + \frac{3\lambda}{4} U^2 \partial_x U, \tag{3.34}$$

and the corresponding reduced hierarchy

$$\frac{\partial}{\partial_{t_{2k}}} \begin{pmatrix} U \\ U \end{pmatrix} = \Re^k_{MKdV} \partial_x \begin{pmatrix} U \\ U \end{pmatrix}. \tag{3.35}$$

In the scalar form it gives MKdV hierarchy

$$\partial_{t_{2k}} U = R^k_{MKdV}(\partial_x U), \tag{3.36}$$

with the recursion operator

$$R_{MKdV} = (\partial_1^2 + \frac{\lambda}{2} U^2 + \frac{\lambda}{2} \partial_1 U \int^x U). \tag{3.37}$$

3.4 The mixed KdV-MKdV hierarchy

We will consider here also a new reduction of system (3.29) in the form

$$\begin{aligned} e^+ &= (\alpha + \beta) U \\ e^- &= \alpha U + \beta \end{aligned} \tag{3.38}$$

where α, β are arbitrary real constants.

For this mixed case, the reduced equation is of the form of mixed KdV-MKdV equation

$$\partial_{t_2} U = \partial_x^3 U + \frac{3\lambda}{4}(\alpha + \beta)(\alpha\, U^2 \partial_x U + \beta\, U \partial_x U). \tag{3.39}$$

Then, the corresponding hierarchy (3.21)can be reduced to

$$\frac{\partial}{\partial_{t_{2k}}} \begin{pmatrix} (\alpha+\beta)U \\ -\alpha U \end{pmatrix} = \Re^{2k}_{mix} \partial_x \begin{pmatrix} (\alpha+\beta)U \\ -\alpha U \end{pmatrix}, \tag{3.40}$$

or

$$\begin{pmatrix} (\alpha+\beta) \\ -\alpha \end{pmatrix} \frac{\partial}{\partial_{t_{2k}}} U = \begin{pmatrix} (\alpha+\beta) \\ -\alpha \end{pmatrix} R^k_{mix} \partial_x U \tag{3.41}$$

and in scalar form

$$\partial_{t_{2k}} U = R^k_{mix} \partial_x U \tag{3.42}$$

The recursion operator corresponding to this mixed KdV and MKdV hierarchy is

$$R_{mix} = (\partial_1^2 + \frac{\lambda}{2}(\alpha+\beta)U(\alpha U + \beta) + \frac{\lambda}{4}(\alpha+\beta)\partial_1 U \int^x 2\alpha U + \beta). \tag{3.43}$$

In particular cases it reduces to recursion operators

$$\begin{aligned} &a)\, \alpha = 0,\ \beta = 1 \quad \Rightarrow \quad MKdV \\ &b)\, \alpha = 1,\ \beta = 0 \quad \Rightarrow \quad\ KdV \end{aligned} \tag{3.44}$$

Chapter 4

GENERALIZED HENON-HEILES SYSTEM

4.1 Henon-Heiles system

One of the most popular model to study integrability and chaos is called the Henon-Heiles system. Introduced by Henon and Heiles in 1964 [83], this system describes the motion of a star in the gravitational field of a galaxy. The model describes two one-dimensional harmonic oscillators with a cubic interaction and has been discussed in applications to the cosmic rays [101] , for the oscillations of atoms in a three-atomic molecule [86], for geodesic flows on SO(4) [84] and three-particle Toda lattice theory [85].

The generalized form of the Henon-Heiles system is

$$\begin{cases} \ddot{q_1} + c_1 q_1 = b q_1^2 - a q_2^2, \\ \ddot{q_2} + c_2 q_2 = -2a q_1 q_2, \end{cases} \tag{4.1}$$

and its energy is given as

$$E = (\frac{1}{2}\dot{q_1}^2 + \dot{q_2}^2 + c_1 q_1^2 + c_2 q_2^2) + a q_1 q_2^2 - \frac{1}{3b} q_1^3. \tag{4.2}$$

This model is an example of Hamiltonian system with a mixed phase space structure, i.e., partially ordered and partially chaotic. For generic parameters a, b, c, the system possesses chaotic orbits and the energy (4.2) is the only conserved quantity. By increasing the total energy, a transition from an integrable to an ergodic system is induced. This model, firstly introduced to describe the chaotic motion of stars in a galaxy, it later became an important milestone in the development of the theory of chaos [100], partly because of the conceptual simplicity of the model.

On the other hand, the system was found to have a second independent integral of motion and to be integrable only for some fixed values of parameters [94]. The integrable cases of the Henon-Heiles system (4.1)are

1.

$$a/b = -1 \qquad , c_1 = c_2; \tag{4.3}$$

2.

$$a/b = -1/6 \qquad , c_1, c_2 \ \ arbitrary; \tag{4.4}$$

3.

$$a/b = -1/16 \qquad , c_1 = 16c_2. \tag{4.5}$$

At the end of 1970's, very important discovery was made. Bogoyavlenskii and Novikov [91] showed that each of the stationary reductions of the KdV hierarchy constitutes a completely integrable, finite dimensional Hamiltonian system. Then Fordy [98] has observed that the above integrable cases of the Henon-Heiles system are closely related to stationary flows [90], [99] of integrable fifth-order nonlinear evolution equations, including the higher KdV equation.

4.2 Stationary reductions from Soliton Equations

The above integrable cases of Henon-Heiles system can be reduced from soliton equations as 5-th KdV, Sawada-Kotera and Kaup-Kupershmidt equations. To illustrate this relation we consider equations (4.1), where for simplicity we take $c_1 = c_2 = 0$. Differentiating twice the first equation of the Henon-Heiles system (4.1) and eliminating q_2 variable by the use of the second equation of the system, and the energy equation (4.2), we get for q_1 the fourth order equation

$$\ddot{\ddot{q}}_1 = 2(a+b)\dot{q}_1^2 - 4aE + \frac{20}{3}abq_1^3 + (2a-8b)q_1\ddot{q}_1. \tag{4.6}$$

This equation by following identification

$$U = q_1, U_x = \dot{q}_1, U_{xx} = \ddot{q}_1, ... \tag{4.7}$$

has form of the fourth-order ordinary differential equation for U(x):

$$U_{xxxx} - 2(a+b)U_x^2 - \frac{20}{3}abU^3 + (8a-2b)UU_{xx} = -4aE. \tag{4.8}$$

The last one is the stationary reduction of the general evolution equation

$$U_t = (U_{xxxx} - 2(a+b)U_x^2 - \frac{20}{3}abU^3 + (8a-2b)UU_{xx})_x. \tag{4.9}$$

The known integrable reductions of the last equation are:

1. the Sawada-Kotera equation ($a = 1/2, b = -1/2$)

$$U_t = (U_{xxxx} + 5UU_{xx} + \frac{5}{3}U^3)_x \tag{4.10}$$

2. the fifth-order KdV equation ($a = 1/2, b = -3$)

$$U_t = (U_{xxxx} + 10UU_{xx} + 5U_x^2 + 10U^3)_x \tag{4.11}$$

3. the Kaup-Kupershmidt equation ($a = 1/4, b = -4$)

$$U_t = (U_{xxxx} + 10UU_{xx} + \frac{15}{2}U_x^2 + \frac{20}{3}U^3)_x \tag{4.12}$$

The stationary reductions of these three equations coincide with corresponding integrable cases (4.3)-(4.5) of the Henon-Heiles model (5.1).

4.3 The Hamilton-Jacobi theory

In the Hamilton-Jacobi theory canonical transformations may be used to provide a general procedure for solving mechanical problems [79]. If we consider a canonical transformation from coordinate and momenta (q(t),p(t)) at time t to new set of constant quantities (Q(t),P(t)) which may be $2n$ initial values $Q = q_0, P = p_0$ at time t=0, then equations of transformation relating the old and the new canonical variables give desired solution of the mechanical problem:

$$q = q(q_0, p_0, t) \tag{4.13}$$

$$p = p(q_0, p_0, t) \tag{4.14}$$

Since new variables are constant in time, it requires the transformed Hamiltonian $\tilde{H}$ to be identically zero

$$\frac{\partial \tilde{H}}{\partial P_i} = \dot{Q}_i = 0 \tag{4.15}$$

$$\frac{\partial \tilde{H}}{\partial Q_i} = \dot{P}_i = 0 \tag{4.16}$$

If the generating function $S(q, P, t)$ is a function of old coordinates q and new momenta P, canonical transformations are given by formulas

$$p_i = \frac{\partial S}{\partial q_i}, \qquad Q_i = -\frac{\partial S}{\partial P_i}, \tag{4.17}$$

$$\tilde{H} = H + \frac{\partial S}{\partial t}. \tag{4.18}$$

Since $\tilde{H} = 0$, from the last equation, after substituting old momenta according to the first Eq. (4.17), we have the Hamilton-Jacobi equation

$$H(q_1,...,q_n;\frac{\partial S}{\partial q_1},...,\frac{\partial S}{\partial q_n};t)+\frac{\partial S}{\partial t}=0 \tag{4.19}$$

This equation is the first order partial differential equation (PDE) in $(n+1)$ variables $q_1,...,q_n,t$. Solution of this equation, function S, is Hamilton's Principal function. Complete solution of Eq.(4.19) as the first order PDE, depends on $(n+1)$ constants of integration $\alpha_1,...,\alpha_{n+1}$:

$$S=S(q_1,...,q_n,\alpha_1,...,\alpha_{n+1},t). \tag{4.20}$$

But since only derivatives of S (but not S itself)are involved in the Hamilton-Jacobi equation, one constant is irrelevant. Indeed, if S is some solution of the equation, then $S+\alpha$ is also solution, where α is an arbitrary constant. It appears as an additive constant. Therefore complete solution with non-additive constants can be written in the form

$$S=S(q_1,...,q_n,\alpha_1,...,\alpha_n,t). \tag{4.21}$$

We can take n constants of integration to be new (constant)momenta

$$P_i=\alpha_i, \qquad i=1,...,n\ . \tag{4.22}$$

At time t_0

$$p_i=\frac{\partial S(q,\alpha,t)}{\partial q_i},$$

these constitute n-equations relating n α's with initial q and p. Other half of equations of transformation

$$Q_i=\beta_i=\frac{\partial S(q,\alpha,t)}{\partial \alpha_i},$$

provides new constant coordinates β_i at time t_0 which can be obtained from initial conditions with known initial values of q_i. Then, expressions

$$q_j=q_j(\alpha,\beta,t), \tag{4.23}$$

$$p_i=p_i(\alpha,\beta,t), \tag{4.24}$$

solves the problem giving coordinates and momenta as functions of time and initial conditions. Hamilton's Principal function as the generator of a canonical transformation to constant coordinates and momenta has physical meaning of the action.

4.4 Separation of Variables

Separation of variables is an efficient method of solving Hamilton-Jacobi equations [79].

Separation of Time Variable

If Hamiltonian H is not an explicit function of t

$$\frac{\partial H}{\partial t} = 0, \tag{4.25}$$

the time variable can be separated in the Hamilton-Jacobi equation. Assuming solution in the form

$$S(q,t) = W(q) - \alpha_1 t. \tag{4.26}$$

we have reduced equation

$$H(q, \frac{\partial W}{\partial q}) = \alpha_1. \tag{4.27}$$

From this equation, one constant of integration in S, namely α_1, is thus equal to the constant value of H. Here time independent function W is Hamilton's characteristic function. This function generates canonical transformation in which all new coordinates are cyclic

$$\dot{P}_i = -\frac{\partial \tilde{H}}{\partial Q_i} = 0, \quad P_i = \alpha_i \tag{4.28}$$

Because the new Hamiltonian depends on only one of the momenta $P_1 = \alpha_1$, the equations of motion are

$$\dot{Q}_i = \frac{\partial \tilde{H}}{\partial \alpha_i} = \begin{cases} 1, & i = 1 \\ 0, & i \neq 1 \end{cases} \tag{4.29}$$

Then we have solution

$$Q_1 = t + \beta_1 = \frac{\partial W_i}{\partial \alpha_i}, \tag{4.30}$$

$$Q_i = \beta_i = \frac{\partial W_i}{\partial \alpha_i} \quad i \neq 1. \tag{4.31}$$

It shows that the only coordinate that is not simply a constant of motion is Q_1.

Generalizing, instead of α_1 and constants of integration as a new momenta, one can choose new momenta as an independent functions $\gamma_1, ..., \gamma_n$:$P_i = \gamma_i(\alpha_i, ..., \alpha_n)$. Then, characteristic function W can be expressed in terms of q_i and γ_i as independent variables:

$$W = W(q_i, \gamma_i)$$

Thus, equations of motion become

$$\dot{Q}_i = \frac{\partial \tilde{H}}{\partial \gamma_i} = \nu_i(\gamma), \tag{4.32}$$

and in this case all new coordinates are linear functions of time

$$Q_i = \nu_i t + \beta_i. \tag{4.33}$$

Separable Hamilton-Jacobi Equation

If the Hamilton-Jacobi equation admit a separating variable then solution can be reduced to quadratures.

Definition 4.4.0.6 *Coordinate* q_1 *is said to be separable in Hamilton-Jacobi equation, when Hamilton's principal function can be split into two additive parts*

$$S(q_1, ..., q_n, \alpha_1, ..., \alpha_n, t) = S_1(q_1, \alpha_1, ..., \alpha_n, t) + S'(q_2, ..., q_n, \alpha_1, ..., \alpha_n, t). \tag{4.34}$$

In this case the Hamilton-Jacobi equation can be split in two equations, the first for S_1 and the second for S'.

Definition 4.4.0.7 *Hamilton-Jacobi equation is completely separable if all coordinates in problem are separable. Hamilton's Characteristic function for completely separable problem has the form*

$$W(q_1, ..., q_n, \alpha_1, ..., \alpha_n) = \sum_{i=1}^{n} W_i(q_i, \alpha_1, ..., \alpha_n) \tag{4.35}$$

For this solution the Hamilton-Jacobi equation will split into n equations

$$H_i(q_i, \frac{\partial W}{\partial q_i}, \alpha_1, ..., \alpha_n) = \alpha_i \ \ , (i = 1, ..., n) \tag{4.36}$$

where α_i are separation constants. This set of first order ordinary differential equations is always reducible to quadratures. We solve it first for $\partial S/\partial q_i$ and then integrate over q_i.

Chapter 5

SEPARATION OF VARIABLES IN H-H SYSTEM

The Henon-Heiles model (4.1) in the integrable case 2, Eq. (4.4), where $a = 1/2, b = -3, c_1 = c_2 = 0$, is two dimensional Hamiltonian dynamical system

$$\begin{cases} \ddot{q_1} = -3q_1^2 - \frac{1}{2}q_2^2, \\ \ddot{q_2} = -q_1 q_2. \end{cases} \tag{5.1}$$

The Hamiltonian function H

$$H = \frac{p_1^2}{2} + \frac{p_1^2}{2} + \frac{q_1 q_2^2}{2} + q_1^3 \tag{5.2}$$

is the first integral of motion of the system. The system admits in addition the second integral of motion F as

$$F = -2q_2 p_1 p_2 + 2q_1 p_2^2 - \frac{1}{4}q_2^4 - q_1^2 q_2^2. \tag{5.3}$$

According to Liouville theorem for integrability of this system one needs two independent integrals of motion H and F. Then choosing these integrals as a new momenta and performing corresponding canonical transformation one can represent the system in the action-angle variables. But explicit realization of this program can be done only in some particular cases. The first step in this realization is separation of variables in the Hamilton-Jacobi equation [77], [80], [89], [92]. For this separation we need to find proper canonical transformation from original coordinates q_1, q_2 to new coordinates Q_1, Q_2 with generating function $\mathcal{F}(p, Q)$, where p_1, p_2 are old momenta. Then using

$$P_i = \frac{\partial \mathcal{F}}{\partial Q_i}, \quad q_i = \frac{\partial \mathcal{F}}{\partial p_i}, \ (i = 1, 2)$$

one can derive the old momenta p_1, p_2 in terms of the new coordinates Q_1, Q_2 and conjugate momenta P_1, P_2. Rewriting conserved functions $h_1 = H$ and $h_2 = F$ in terms of new coordinates and new momenta, we have to solve the corresponding Hamilton-Jacobi equations

$$h_i = h_i(Q_i, \frac{\partial W_i}{\partial Q_i}, t) = \alpha_i\ , \ (i = 1, 2)$$

for Hamilton's characteristic functions W_i ($i = 1, 2$). Our problem is separable in new coordinates if one can represent W_i as

$$\begin{aligned} W_1 &= W_{11}(Q_1, C_1, C_2) + W_{12}(Q_2, C_1, C_2) \\ W_2 &= W_{21}(Q_1, C_1, C_2) + W_{22}(Q_2, C_1, C_2) \end{aligned} \tag{5.4}$$

where C_1, C_2 are constants, and reduce problem to find W_{ik} as solution of ordinary differential equation.

5.1 Bi-Hamiltonian Formulation

The Hamilton equations (2.1) in the matrix form(2.10) for the Henon-Heiles system (5.1) are

$$\dot{\vec{x}} = \begin{pmatrix} p_1 \\ p_2 \\ q_1 \\ q_2 \end{pmatrix}_t = J_0 \nabla H, \tag{5.5}$$

where J_0 is a constant symplectic matrix

$$J_0 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}. \tag{5.6}$$

For the Henon-Heiles model we have the second integral of motion F given by Eq.(5.3). The similar form satisfies for F

$$\dot{\vec{x}} = \begin{pmatrix} p_1 \\ p_2 \\ q_1 \\ q_2 \end{pmatrix}_t = J_1 \nabla F \tag{5.7}$$

where J_1 is a skew symmetric matrix

$$J_1 = \frac{1}{q_2^2} \begin{pmatrix} 0 & 0 & q_1 & \frac{1}{2}q_2 \\ 0 & 0 & \frac{1}{2}q_2 & 0 \\ -q_1 & -\frac{1}{2}q_2 & 0 & \frac{1}{2}p_2 \\ -\frac{1}{2}q_2 & 0 & -\frac{1}{2}p_2 & 0 \end{pmatrix}, \tag{5.8}$$

so that F is also Hamiltonian and this can be rewritten as

$$J_1 \nabla F = J_0 \nabla H = \dot{\vec{x}}. \tag{5.9}$$

The system having two Hamiltonians and satisfying this equality is called bi-Hamiltonian [80]. This shows that the Henon-Heiles model is bihamiltonian dynamical system [72], [95]. Since F is integral of motion, according to the first Hamiltonian structure it is in involution with Hamiltonian H. Then the Henon-Heiles system is integrable by the Liouville theorem. Bi-Hamiltonian structure means in addition that evolutions generated by two integrals of motions as independent momenta, with corresponding Poisson structures, are the same.

5.2 Integrable Extensions

Now we consider several new integrable extensions of the generalized Henon-Heiles system 5.1 and corresponding separation of variables.

5.2.1 C-Extended Henon-Heiles System

For the case 2 ($a/b = -1/6$) in equation 5.1 let $c_1 = c_2 = 0, b = 3, a = -1/2$. Then the first extended version with an additional constant term C is

$$\begin{cases} \ddot{q}_1 = -3q_1^2 - \frac{1}{2}q_2^2 + C, \\ \ddot{q}_2 = -q_1 q_2. \end{cases} \tag{5.10}$$

The corresponding Hamiltonian is

$$H = \frac{p_1^2}{2} + \frac{p_1^2}{2} + \frac{q_1 q_2^2}{2} + q_1^3 - C q_1 \tag{5.11}$$

while additional integral of motion F is given by

$$F = -2q_2 p_1 p_2 + 2q_1 p_2^2 - \frac{1}{4}q_2^4 - q_1^2 q_2^2 + C q_2^2. \tag{5.12}$$

Hamiltonian form of these equations with Hamiltonian function H is defined in extended five dimensional phase space, where C is considered as an extra dynamical variable, and are given in the form

$$\begin{pmatrix} q_1 \\ q_2 \\ p_1 \\ p_2 \\ C \end{pmatrix}_t = \begin{pmatrix} p_1 \\ p_2 \\ -3q_1^2 - \frac{1}{2}q_2^2 + C \\ -q_1 q_2 \\ 0 \end{pmatrix} =$$

$$= \underbrace{\begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}}_{J_0} \underbrace{\begin{pmatrix} 3q_1^2 + \frac{1}{2}q_2^2 - C \\ q_1 q_2 \\ p_1 \\ p_2 \\ -q_1 \end{pmatrix}}_{\nabla H}, \tag{5.13}$$

while the form with F as a Hamiltonian

$$\begin{pmatrix} q_1 \\ q_2 \\ p_1 \\ p_2 \\ C \end{pmatrix}_t = \frac{1}{q_2^2} \underbrace{\begin{pmatrix} 0 & 0 & q_1 & \frac{1}{2}q_2 & 0 \\ 0 & 0 & \frac{1}{2}q_2 & 0 & 0 \\ -q_1 & -\frac{1}{2}q_2 & 0 & \frac{1}{2}p_2 & 0 \\ -\frac{1}{2}q_2 & 0 & -\frac{1}{2}p_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}}_{J_1}$$

$$\times \underbrace{\begin{pmatrix} -\frac{1}{2}p_2^2 + \frac{1}{2}q_1 q_2^2 \\ \frac{1}{2}p_1 p_2 + \frac{1}{4}q_2^3 + \frac{1}{2}q_1^2 q_2 - \frac{1}{2}q_2 C \\ \frac{1}{2}q_2 p_2 \\ \frac{1}{2}q_2 p_1 - q_1 p_2 \\ -\frac{1}{4}q_2^2 \end{pmatrix}}_{\nabla F}. \tag{5.14}$$

We notice that due to the odd dimension of our extended phase space the skew-symmetric matrices J_0 and J_1 are degenerate. Both systems (5.2.1), (5.14) can be rewritten as

$$J_0 \nabla H = J_1 \nabla F = \dot{\vec{x}}. \tag{5.15}$$

which means the system (5.10) is bi-Hamiltonian system.

To separate variables in our system we consider Hamilton's characteristic function $\mathcal{F}(p_1, p_2, Q_1, Q_2)$ in the form

$$\mathcal{F} = p_1(Q_1 + Q_2) + 2p_2\sqrt{-Q_1Q_2}. \tag{5.16}$$

The canonical transformations generated by this function can be written as

$$q_1 = Q_1 + Q_2, \quad p_1 = \tfrac{P_1Q_1 - P_2Q_2}{Q_2 - Q_1}, \tag{5.17}$$

$$q_2 = 2\sqrt{-Q_1Q_2}\ , \quad p_2 = \tfrac{\sqrt{-Q_1Q_2}(P_1 - P_2)}{(Q_2 - Q_1)}. \tag{5.18}$$

Then Hamiltonian H and the second integral of motion F in terms of new coordinates and momenta are

$$\tilde{H} = \frac{1}{2(Q_2 - Q_1)}(P_2^2Q_2 - P_1^2Q_1) + (Q_1 + Q_2)(Q_1^2 + Q_2^2) - C(Q_1 + Q_2), \tag{5.19}$$

$$\tilde{F} = -\frac{2}{(Q_2 - Q_1)}(P_1^2 - P_2^2)Q_1Q_2 + 4Q_1Q_2(Q_1^2 + Q_1Q_2 + Q_2^2 - C). \tag{5.20}$$

The Hamilton-Jacobi equation for the first Hamiltonian H is

$$\frac{1}{2(Q_2 - Q_1)}((\frac{\partial W_1}{\partial Q_2})^2Q_2 - (\frac{\partial W_1}{\partial Q_1})^2Q_1) + (Q_1 + Q_2)(Q_1^2 + Q_2^2 - C) = \alpha_1 \tag{5.21}$$

where W function is denoted as W_1. For the second Hamiltonian F the Hamilton-Jacobi equation is

$$-\frac{2}{(Q_2 - Q_1)}((\frac{\partial W_2}{\partial Q_1})^2 - (\frac{\partial W_2}{\partial Q_2})^2)Q_1Q_2 + 4Q_1Q_2(Q_1^2 + Q_1Q_2 + Q_2^2 - C) = \alpha_2 \tag{5.22}$$

In this case W function is denoted as W_2. We try to separate solution of these equations simultaneously by substitution

$$\begin{aligned} W_1 &= W_{11}(Q_1, C_1, C_2) + W_{12}(Q_2, C_1, C_2), \\ W_2 &= W_{21}(Q_1, C_1, C_2) + W_{22}(Q_2, C_1, C_2). \end{aligned} \tag{5.23}$$

Then, it is obvious that new momenta are given by

$$P_i = \frac{\partial W_i}{\partial Q_i},\ (i = 1, 2). \tag{5.24}$$

So we have separated solution in quadratures

$$\frac{\partial W_{1k}}{\partial Q_k} = \sqrt{\frac{1}{Q_k}(2K_1 + 2CQ_k^2 - 2Q_k^4 + 2\alpha_1 Q_k)},\ (k = 1, 2), \tag{5.25}$$

$$\frac{\partial W_{2k}}{\partial Q_k} = \sqrt{\frac{1}{Q_k}(2K_2Q_k + 2CQ_k^2 - 2Q_k^4 - \frac{\alpha_2}{2})},\ (k = 1, 2). \tag{5.26}$$

5.2.2 D-Extended Henon-Heiles System

The second extension of model(5.1),(4.4)($c_1 = 0, c_2 = 0$)is

$$\begin{cases} \ddot{q_1} = -3q_1^2 - \frac{1}{2}q_2^2 \\ \ddot{q_2} = -q_1 q_2 + \frac{D}{q_2^3} \end{cases} \tag{5.27}$$

where D is an arbitrary constant. The corresponding Hamiltonian H and integral F are

$$H = \frac{p_1^2}{2} + \frac{p_1^2}{2} + \frac{q_1 q_2^2}{2} + q_1^3 + \frac{D}{2q_2^2} \tag{5.28}$$

$$F = -2q_2 p_1 p_2 + 2q_1 p_2^2 - \frac{1}{4}q_2^4 - q_1^2 q_2^2 + \frac{2Dq_1}{q_2^2}. \tag{5.29}$$

Bi-Hamiltonian representation of system (5.27) in extended phase space is

$$\begin{pmatrix} q_1 \\ q_2 \\ p_1 \\ p_2 \\ D \end{pmatrix}_t = \begin{pmatrix} p_1 \\ p_2 \\ -3q_1^2 - \frac{1}{2}q_2^2 \\ -q_1 q_2 + \frac{D}{q_2^3} \\ 0 \end{pmatrix} =$$

$$= \underbrace{\begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}}_{J_0} \underbrace{\begin{pmatrix} 3q_1^2 + \frac{1}{2}q_2^2 \\ q_1 q_2 - \frac{D}{q_2^3} \\ p_1 \\ p_2 \\ \frac{1}{2q_2^2} \end{pmatrix}}_{\nabla H}, \tag{5.30}$$

and

$$\begin{pmatrix} q_1 \\ q_2 \\ p_1 \\ p_2 \\ D \end{pmatrix}_t = -\frac{1}{2q_2^2} \underbrace{\begin{pmatrix} 0 & 0 & 0 & e & -f \\ 0 & 0 & e & -g & -h \\ 0 & -e & 0 & i & -k \\ -e & g & -i & 0 & l \\ f & h & k & -l & 0 \end{pmatrix}}_{J_1}$$

$$\times \underbrace{\begin{pmatrix} -\frac{1}{2}p_2^2 + \frac{1}{2}q_1q_2^2 \\ \frac{1}{2}p_1p_2 + \frac{1}{4}q_2^3 + \frac{1}{2}q_1^2q_2 - \frac{1}{2}q_2C \\ \frac{1}{2}q_2p_2 \\ \frac{1}{2}q_2p_1 - q_1p_2 \\ -\frac{1}{4}q_2^2 \end{pmatrix}}_{\nabla F} \tag{5.31}$$

where

$$e = \frac{1}{q_2}, \qquad f = 2q_2p_2, \qquad g = \frac{2q_1}{q_2^2}, \qquad h = (2q_2p_1 - 4q_1p_2), \tag{5.32}$$

$$i = \frac{p_2}{q_2^2}, \qquad k = (-2q_1q_2^2 + 2p_2^2 + \frac{2D}{q_2^2}), \qquad l = 2p_1p_2 + q_2^3 + 2q_1^2q_2 + \frac{4Dq_1}{q_2^3} \tag{5.33}$$

and

$$J_0\nabla H = J_1\nabla F = \dot{\vec{x}}. \tag{5.34}$$

Now we will do the canonical transformation that is given in the form (5.16), (5.17), (5.18) as for the first extension. After transformation, the Hamiltonians H and F become as

$$\tilde{H} = \frac{1}{2(Q_2 - Q_1)}(P_2^2Q_2 - P_1^2Q_1) + (Q_1 + Q_2)(Q_1^2 + Q_2^2) - \frac{D}{4Q_1Q_2}, \tag{5.35}$$

$$\tilde{F} = -\frac{2}{(Q_2 - Q_1)}((P_1^2 - P_2^2)Q_1Q_2) + 4Q_1Q_2(Q_1^2 + Q_1Q_2 + Q_2^2) - \frac{D}{2Q_1} - \frac{D}{2Q_2}. \tag{5.36}$$

The Hamilton-Jacobi equation defined by these two Hamiltonians are separable in the form (5.23). We find the separated equations as

$$\frac{\partial W_{1k}}{\partial Q_k} = \sqrt{\frac{1}{Q_k}(2K_1 - \frac{D}{2Q_k} + 2Q_k^4 + 2\alpha_1 Q_k)}, \ (k = 1, 2), \tag{5.37}$$

$$\frac{\partial W_{2k}}{\partial Q_k} = \sqrt{\frac{1}{Q_k}(2K_2Q_k - 2Q_k^4 - \frac{\alpha_2}{4} - \frac{D}{4Q_k})}, \ (k = 1, 2). \tag{5.38}$$

5.2.3 The Harmonic Extension

The third extension of the system (5.1),(4.4) with arbitrary $c_1 \neq 0$ and $c_2 \neq 0$ constants, which we denote as $c_1 = \tilde{a}$ and $c_2 = \tilde{b}$ is

$$\begin{cases} \ddot{q}_1 + \tilde{a}q_1 = -3q_1^2 - \frac{1}{2}q_2^2, \\ \ddot{q}_2 + \tilde{b}q_2 = -q_1q_2. \end{cases} \tag{5.39}$$

The corresponding Hamiltonians

$$H = \frac{p_1^2}{2} + \frac{p_1^2}{2} + \frac{\tilde{a}q_1^2}{2} + \frac{\tilde{b}q_2^2}{2} + \frac{q_1q_2^2}{2} + q_1^3; \tag{5.40}$$

and

$$F = -2q_2p_1p_2 + 2q_1p_2^2 - \frac{1}{4}q_2^4 - q_1^2q_2^2 - 2\tilde{b}q_2^2q_1 - (4\tilde{b} - \tilde{a})(p_2^2 + \tilde{b}q_2^2), \tag{5.41}$$

determine bi-Hamiltonian systems respectively

$$\begin{pmatrix} q_1 \\ q_2 \\ p_1 \\ p_2 \end{pmatrix}_t = \begin{pmatrix} p_1 \\ p_2 \\ -3q_1^2 - \frac{1}{2}q_2^2 - \tilde{a}q_1 \\ -q_1q_2 - \tilde{b}q_2 \\ 0 \end{pmatrix} =$$

$$= \underbrace{\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}}_{J_0} \underbrace{\begin{pmatrix} 3q_1^2 + \frac{1}{2}q_2^2 + \tilde{a}q_1 \\ q_1q_2\tilde{b}q_2 \\ p_1 \\ p_2 \end{pmatrix}}_{\nabla H}, \tag{5.42}$$

$$\begin{pmatrix} q_1 \\ q_2 \\ p_1 \\ p_2 \end{pmatrix}_t = \frac{1}{2q_2^2} \underbrace{\begin{pmatrix} 0 & 0 & k & l \\ 0 & 0 & l & 0 \\ -k & -l & 0 & m \\ -l & 0 & -m & 0 \end{pmatrix}}_{J_1}$$

$$\times \underbrace{\begin{pmatrix} 2p_2^2 - 2q_1q_2^2 - 2\tilde{b}q_2^2 \\ -2p_1p_2 - q_2^3 - 2q_1^2q_2 - 4\tilde{b}q_2q_1 - 2\tilde{b}(4\tilde{b} - \tilde{a})q_2 \\ -2q_2p_2 \\ -2q_2p_1 + 4q_1p_2 - 2(4\tilde{b} - \tilde{a})p_2 \end{pmatrix}}_{\nabla F}, \tag{5.43}$$

where

$$k = -2q_1 - (4\tilde{b} - \tilde{a}), \qquad l = -q_2, \qquad m = -p_2. \tag{5.44}$$

Canonical transformation and the characteristic function in this case includes additional terms depending on $\tilde{a}$ and $\tilde{b}$.

$$\mathcal{F} = p_1(Q_1 + Q_2 + 4\tilde{b} - \tilde{a}) + 2p_2\sqrt{-Q_1Q_2},, \tag{5.45}$$

$$q_1 = Q_1 + Q_2 + (4\tilde{b} - \tilde{a}), \quad p_1 = \tfrac{P_1Q_1 - P_2Q_2}{Q_2 - Q_1}\ , \tag{5.46}$$

$$q_2 = 2\sqrt{-Q_1Q_2}, \quad p_2 = \tfrac{\sqrt{-Q_1Q_2}(P_1 - P_2)}{(Q_2 - Q_1)}. \tag{5.47}$$

Hamiltonians in terms of new variables are

$$\tilde{H} = \frac{1}{2(Q_2 - Q_1)}(P_1^2Q_1 - P_2^2Q_2) + [(4\tilde{b} - \tilde{a})(12\tilde{b} - \tilde{a})]\frac{(Q_1 + Q_2)}{4} \tag{5.48}$$

$$+(6\tilde{b} - \tilde{a})(Q_1^2 + Q_2^2 + Q_1Q_2) + (Q_1 + Q_2)(Q_1^2 + Q_2^2) + \frac{\tilde{a}}{2}(\frac{4\tilde{b} - \tilde{a}}{2})^2,$$

$$\tilde{F} = \frac{2}{(Q_2 - Q_1)}(P_1^2 - P_2^2)Q_1Q_2 + 4Q_1Q_2(Q_1^2 + Q_1Q_2 + Q_2^2) \tag{5.49}$$

$$(4\tilde{b} - \tilde{a})(12\tilde{b} - \tilde{a})Q_1Q_2 + 4(6\tilde{b} - \tilde{a})(Q_1 + Q_2)Q_1Q_2.$$

Then, the Hamilton-Jacobi equations determined by these Hamiltonians are separated in the next form

$$\frac{\partial W_{1k}}{\partial Q_k} = \sqrt{\frac{1}{2Q_k}}\Big(4K_1 - (4\tilde{b} - \tilde{a})(12\tilde{b} - \tilde{a})Q_k^2 - 4(6\tilde{b} - \tilde{a})Q_k^3 - 4Q_k^4 \tag{5.50}$$

$$+2(2\alpha_1 - b(4\tilde{b} - 2\tilde{a})^2)Q_k\Big)^{\frac{1}{2}},$$

$$\frac{\partial W_{2k}}{\partial Q_k} = \sqrt{\frac{4}{2Q_k}[K_2Q_k - Q_k^4 - (6\tilde{b} - \tilde{a})Q_k^3 - \frac{(4\tilde{b} - \tilde{a})}{4}(12\tilde{b} - \tilde{a})Q_k^2 - \frac{\alpha_2}{2}]}. \tag{5.51}$$

where $(k = 1, 2)$

5.2.4 The Harmonic C-Extension

The fourth generalization is an arbitrary mixture of the first and the third extensions (5.10),(5.39) with arbitrary constants $\tilde{a}, \tilde{b}, C$

$$\begin{cases} \ddot{q}_1 + \tilde{a}q_1 = -3q_1^2 - \frac{1}{2}q_2^2 + C, \\ \ddot{q}_2 + \tilde{b}q_2 = -q_1q_2. \end{cases} \tag{5.52}$$

Hamiltoninans for this case are

$$H = \frac{p_1^2}{2} + \frac{p_1^2}{2} + \frac{q_1 q_2^2}{2} + q_1^3 + \frac{\tilde{a} q_1^2}{2} + \frac{\tilde{b} q_2^2}{2} - C q_1, \tag{5.53}$$

$$F = -2q_2 p_1 p_2 + 2q_1 p_2^2 - \frac{1}{4} q_2^4 - q_1^2 q_2^2 - 2\tilde{b} q_2^2 q_1 - (4\tilde{b} - \tilde{a})(p_2^2 + \tilde{b} q_2^2) + C q_2^2. \tag{5.54}$$

Bi-Hamiltonian form determined by the first Hamiltonian is given by the following system

$$\begin{pmatrix} q_1 \\ q_2 \\ p_1 \\ p_2 \\ C \end{pmatrix}_t = \begin{pmatrix} p_1 \\ p_2 \\ -3q_1^2 - \frac{1}{2} q_2^2 - \tilde{a} q_1 + C \\ -q_1 q_2 - \tilde{b} q_2 \\ 0 \end{pmatrix} =$$

$$= \underbrace{\begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}}_{J_0} \underbrace{\begin{pmatrix} 3q_1^2 + \frac{1}{2} q_2^2 + \tilde{a} q_1 - C \\ q_1 q_2 + \tilde{b} q_2 \\ p_1 \\ p_2 \\ -q_1 \end{pmatrix}}_{\nabla H}, \tag{5.55}$$

while the form determined by the second Hamiltonian is

$$\begin{pmatrix} q_1 \\ q_2 \\ p_1 \\ p_2 \\ C \end{pmatrix}_t = \frac{1}{2q_2^2} \underbrace{\begin{pmatrix} 0 & 0 & k & l & 0 \\ 0 & 0 & l & 0 & 0 \\ -k & -l & 0 & m & 0 \\ -l & 0 & -m & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}}_{J_1}$$

$$
\times \underbrace{\begin{pmatrix} 2p_2^2 - 2q_1q_2^2 - 2\tilde{b}q_2^2 \\ -2p_1p_2 - q_2^3 - 2q_1^2q_2 - 4\tilde{b}q_2q_1 + 2[C - \tilde{b}(4\tilde{b} - \tilde{a})]q_2 \\ -2q_2p_2 \\ -2q_2p_1 + 4q_1p_2 - 2(4\tilde{b} - \tilde{a})p_2 \\ q_2^2 \end{pmatrix}}_{\nabla F}, \tag{5.56}
$$

where

$$
k = -2q_1 - (4\tilde{b} - \tilde{a}), \qquad l = -q_2, \qquad m = -p_2. \tag{5.57}
$$

Canonical transformation and characteristic function of the form (5.45),(5.46),(5.47) gives for Hamiltonians the next expressions

$$
\tilde{H} = \frac{1}{2(Q_1 - Q_2)}(P_1^2Q_1 - P_2^2Q_2) + [(4\tilde{b} - \tilde{a})(12\tilde{b} - \tilde{a}) - 4C]\frac{(Q_1 + Q_2)}{4} \tag{5.58}
$$

$$
+(6\tilde{b} - \tilde{a})(Q_1^2 + Q_2^2 + Q_1Q_2) + (Q_1 + Q_2)(Q_1^2 + Q_2^2) + (\frac{4\tilde{b} - \tilde{a}}{2})[(4\tilde{b} - \tilde{a})b - C],
$$

and

$$
\tilde{F} = \frac{2}{(Q_2 - Q_1)}(P_1^2 - P_2^2)Q_1Q_2 + 4Q_1Q_2(Q_1^2 + Q_1Q_2 + Q_2^2) \tag{5.59}
$$

$$
[(4\tilde{b} - \tilde{a})(12\tilde{b} - -\tilde{a}) - 4C]Q_1Q_2 + 4(6\tilde{b} - \tilde{a})(Q_1 + Q_2)Q_1Q_2.
$$

Then, separation of Hamilton-Jacobi equations appear as following ODEs

$$
\frac{\partial W_{1k}}{\partial Q_k} = \sqrt{\frac{1}{2Q_k}}\Big(4K_1 - [(4\tilde{b} - \tilde{a})(12\tilde{b} - \tilde{a}) - 4C]Q_k^2 - 4(6\tilde{b} - \tilde{a})Q_k^3 \tag{5.60}
$$

$$
-4Q_k^4 + (4\alpha_1 - E)Q_k\Big)^{\frac{1}{2}}
$$

,

$$
\frac{\partial W_{2k}}{\partial Q_k} = \sqrt{\frac{1}{2Q_k}}\Big(4K_2Q_k - 4Q_k^4 - 4(6\tilde{b} - \tilde{a})Q_k^3 \tag{5.61}
$$

$$
-[(4\tilde{b} - \tilde{a})(12\tilde{b} - \tilde{a}) - C]Q_k^2 - \alpha_2\Big)^{\frac{1}{2}}
$$

where $(k = 1, 2)$,

$$
E = 2(4\tilde{b} - \tilde{a})[(4\tilde{b} - \tilde{a})\tilde{b} - C].
$$

5.2.5 The Harmonic D-Extension

The fifth generalization is a mixture of the second and the third one

$$\begin{cases} \ddot{q}_1 + \tilde{a} q_1 = -3q_1^2 - \frac{1}{2} q_2^2, \\ \ddot{q}_2 + \tilde{b} q_2 = -q_1 q_2 + \frac{D}{q_2^3}. \end{cases} \tag{5.62}$$

In this case we have the Hamiltonians

$$H = \frac{p_1^2}{2} + \frac{p_1^2}{2} + \frac{q_1 q_2^2}{2} + q_1^3 + \frac{\tilde{a} q_1^2}{2} + \frac{\tilde{b} q_2^2}{2} + \frac{D}{2q_2^2}, \tag{5.63}$$

and

$$\begin{aligned} F = & -2q_2 p_1 p_2 + 2q_1 p_2^2 - \frac{1}{4} q_2^4 - q_1^2 q_2^2 - 2\tilde{b} q_2^2 q_1 - (4\tilde{b} - \tilde{a})(p_2^2 + \tilde{b} q_2^2) \\ & + \frac{2D}{q_2^2} q_1 - \frac{D}{q_2^2}(4\tilde{b} - \tilde{a}), \end{aligned} \tag{5.64}$$

the first Hamiltonian form

$$\begin{pmatrix} q_1 \\ q_2 \\ p_1 \\ p_2 \\ D \end{pmatrix}_t = \begin{pmatrix} p_1 \\ p_2 \\ -3q_1^2 - \frac{1}{2} q_2^2 - \tilde{a} q_1 \\ -q_1 q_2 - \tilde{b} q_2 + \frac{D}{q_2^3} \\ 0 \end{pmatrix} =$$

$$= \underbrace{\begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}}_{J_0} \underbrace{\begin{pmatrix} 3q_1^2 + \frac{1}{2} q_2^2 + \tilde{a} q_1 \\ q_1 q_2 + \tilde{b} q_2 - \frac{D}{q_2^3} \\ p_1 \\ p_2 \\ \frac{1}{2q_2^2} \end{pmatrix}}_{\nabla H}, \tag{5.65}$$

the second Hamiltonian form

$$\begin{pmatrix} q_1 \\ q_2 \\ p_1 \\ p_2 \\ D \end{pmatrix}_t = \frac{1}{2q_2^2} \underbrace{\begin{pmatrix} 0 & 0 & 0 & k & l \\ 0 & 0 & k & m & n \\ 0 & -k & 0 & r & s \\ -k & -m & -r & 0 & t \\ -l & -n & -s & -t & 0 \end{pmatrix}}_{J_1}$$

$$\times \underbrace{\begin{pmatrix} 2p_2^2 - 2q_1q_2^2 - 2\tilde{b}q_2^2 + \frac{2D}{q_2^2} \\ -2p_1p_2 - q_2^3 - 2q_1^2q_2 - 4\tilde{b}q_2q_1 - 2\tilde{b}Aq_2 - 4\frac{Dq_1}{q_2^3} + \frac{DA}{q_2^3} \\ -2q_2p_2 \\ -2q_2p_1 + 4q_1p_2 - 2Ap_2 \\ \frac{2q_1}{q_2^2} - \frac{A}{q_2^2} \end{pmatrix}}_{\nabla F}, \tag{5.66}$$

where

$$k = -q_2^2,\ l = 2q_2^3p_2^2,\ m = 2q_1 - A, \tag{5.67}$$

$$n = 2q_2^3p_1 - 4q_1q_2^2p_2 + 2Aq_2^2p_2,\ r = -p_2, \tag{5.68}$$

$$s = -2q_1q_2^4 + 2p_2^2q_2^2 + 2D - 2\tilde{b}q_2^4, \tag{5.69}$$

$$t = -2p_1p_2q_2^2 - q_2^5 - 2q_1^2q_2^3 - \frac{4Dq_1}{q_2} + \frac{2DA}{q_2} - 4\tilde{b}q_1q_2^3 + 2\tilde{b}Aq_2^3, \tag{5.70}$$

with

$$A = (4\tilde{b} - \tilde{a}).$$

Canonically transformed (see Eqs.(5.45),(5.46),(5.47)) first Hamiltonian

$$\tilde{H} = \frac{1}{2(Q_1 - Q_2)}(P_1^2Q_1 - P_2^2Q_2) + [(4\tilde{b} - \tilde{a})(12\tilde{b} - \tilde{a})]\frac{(Q_1 + Q_2)}{4} \tag{5.71}$$

$$+(6\tilde{b} - \tilde{a})(Q_1^2 + Q_2^2 + Q_1Q_2) + (Q_1 + Q_2)(Q_1^2 + Q_2^2) - \frac{D}{8Q_1Q_2} + 2\tilde{b}(\frac{4\tilde{b} - \tilde{a}}{2})^2,$$

and the second Hamiltonian

$$\tilde{F} = \frac{2}{(Q_2 - Q_1)}(P_2^2 - P_1^2)Q_1Q_2 + 4Q_1Q_2(Q_1^2 + Q_1Q_2 + Q_2^2) \tag{5.72}$$

$$[(4\tilde{b} - \tilde{a})(12\tilde{b} - \tilde{a})]Q_1Q_2 + 4(6\tilde{b} - \tilde{a})(Q_1 + Q_2)Q_1Q_2 - \frac{D(Q_1 + Q_2)}{2Q_1Q_2},$$

determine the first couple of separated equations

$$\frac{\partial W_{1k}}{\partial Q_k} = \sqrt{\frac{1}{2Q_k}}\Big(4K_1 - (4\tilde{b} - \tilde{a})(12\tilde{b} - \tilde{a})Q_k^2 - 4(6\tilde{b} - \tilde{a})Q_k^3 - 4Q_k^4 \tag{5.73}$$

$$+(4\alpha_1 - E)Q_k - \frac{D}{2Q_k}]\Big)^{\frac{1}{2}},$$

and the second couple of separated equations

$$\frac{\partial W_{2k}}{\partial Q_k} = \sqrt{\frac{1}{2Q_k}} \Big(4K_2 Q_k - 4Q_k^4 - 4(6\tilde{b} - \tilde{a})Q_k^3 \tag{5.74}$$

$$-(4\tilde{b} - \tilde{a})(12\tilde{b} - \tilde{a})Q_k^2 - \alpha_2 - \frac{D}{2Q_k}] \Big)^{\frac{1}{2}},$$

correspondingly, where

$$E = 2\tilde{b}(4\tilde{b} - \tilde{a})^2, \ (k = 1, 2).$$

5.2.6 The Harmonic C-D Extension

As a sixth generalization we have a mixture of the first, the second and the third generalizations considered above

$$\begin{cases} \ddot{q}_1 + \tilde{a}q_1 = -3q_1^2 - \frac{1}{2}q_2^2 + C, \\ \ddot{q}_2 + \tilde{b}q_2 = -q_1 q_2 + \frac{D}{q_2^3}; \end{cases} \tag{5.75}$$

Two Hamiltonians

$$H = \frac{p_1^2}{2} + \frac{p_1^2}{2} + \frac{q_1 q_2^2}{2} + q_1^3 + \frac{\tilde{a}q_1^2}{2} + \frac{\tilde{b}q_2^2}{2} + \frac{D}{2q_2^2} - Cq_1; \tag{5.76}$$

and

$$F = -2q_2 p_1 p_2 + 2q_1 p_2^2 - \frac{1}{4}q_2^4 - q_1^2 q_2^2 - 2\tilde{b}q_2^2 q_1 - (4\tilde{b} - \tilde{a})(p_2^2 + \tilde{b}q_2^2) \tag{5.77}$$

$$+\frac{2D}{q_2^2}q_1 - \frac{D}{q_2^2}(4\tilde{b} - \tilde{a}) + Cq_2^2;$$

generate following equations; for the first Hamiltonian

$$\begin{pmatrix} q_1 \\ q_2 \\ p_1 \\ p_2 \\ C \\ D \end{pmatrix}_t = \begin{pmatrix} p_1 \\ p_2 \\ -3q_1^2 - \frac{1}{2}q_2^2 - \tilde{a}q_1 + C \\ -q_1 q_2 - \tilde{b}q_2 + \frac{D}{q_2^3} \\ 0 \\ 0 \end{pmatrix} =$$

$$= \underbrace{\begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}}_{J_0} \underbrace{\begin{pmatrix} 3q_1^2 + \frac{1}{2}q_2^2 + \tilde{a}q_1 - C \\ q_1 q_2 + \tilde{b}q_2 - \frac{D}{q_2^3} \\ p_1 \\ p_2 \\ -q_1 \\ \frac{1}{2q_2^2} \end{pmatrix}}_{\nabla H}, \tag{5.78}$$

and for the second one

$$\begin{pmatrix} q_1 \\ q_2 \\ p_1 \\ p_2 \\ C \\ D \end{pmatrix}_t = \frac{1}{2q_2^2} \underbrace{\begin{pmatrix} 0 & 0 & 0 & k & 0 & l \\ 0 & 0 & k & m & 0 & n \\ 0 & -k & 0 & r & 0 & s \\ -k & -m & -r & 0 & 0 & t \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -l & -n & -s & -t & 0 & 0 \end{pmatrix}}_{J_1}$$

$$\times \underbrace{\begin{pmatrix} 2p_2^2 - 2q_1q_2^2 - 2\tilde{b}q_2^2 + \frac{2D}{q_2^2} \\ -2p_1p_2 - q_2^3 - 2q_1^2q_2 - 4\tilde{b}q_2q_1 - 2\tilde{b}Aq_2 - 4\frac{Dq_1}{q_2^3} + \frac{DA}{q_2^3} + 2Cq_2 \\ -2q_2p_2 \\ -2q_2p_1 + 4q_1p_2 - 2Ap_2 \\ q_2^2 \\ \frac{2q_1}{q_2^2} - \frac{A}{q_2^2} \end{pmatrix}}_{\nabla F}, \tag{5.79}$$

where

$$k = -q_2^2,\ l = 2q_2^3p_2^2,\ m = 2q_1 - A, \tag{5.80}$$

$$n = 2q_2^3p_1 - 4q_1q_2^2p_2 + 2Aq_2^2p_2,\ r = -p_2, \tag{5.81}$$

$$s = -2q_1q_2^4 + 2p_2^2q_2^2 + 2D - 2\tilde{b}q_2^4, \tag{5.82}$$

$$t = -2p_1p_2q_2^2 - q_2^5 - 2q_1^2q_2^3 - \frac{4Dq_1}{q_2} + \frac{2DA}{q_2} - 4\tilde{b}q_1q_2^3 + 2\tilde{b}Aq_2^3 + 2Cq_2, \tag{5.83}$$

and

$$A = (4\tilde{b} - \tilde{a}).$$

By canonical transformation ((5.45),(5.46),(5.47)) we find expressions for Hamiltonian H,

$$\tilde{H} = \frac{1}{2(Q_1 - Q_2)}(P_1^2 Q_1 - P_2^2 Q_2) + [(4\tilde{b} - \tilde{a})(12\tilde{b} - \tilde{a}) - 4C]\frac{(Q_1 + Q_2)}{4} \quad (5.84)$$

$$+(6\tilde{b}-\tilde{a})(Q_1^2+Q_2^2+Q_1Q_2)+(Q_1+Q_2)(Q_1^2+Q_2^2)-\frac{D}{8Q_1Q_2}+[(\frac{4\tilde{b}-\tilde{a}}{2})[(4\tilde{b}-\tilde{a})b-C]$$

and F,

$$\tilde{F} = \frac{2}{(Q_2 - Q_1)}(P_2^2 - P_1^2)Q_1Q_2 + 4Q_1Q_2(Q_1^2 + Q_1Q_2 + Q_2^2) \quad (5.85)$$

$$+[(4\tilde{b} - \tilde{a})(12\tilde{b} - \tilde{a}) - 4C]Q_1Q_2 + 4(6\tilde{b} - \tilde{a})(Q_1 + Q_2)Q_1Q_2 - \frac{D(Q_1 + Q_2)}{2Q_1Q_2}.$$

Corresponding Hamilton-Jacobi equations are separated in the form

$$\frac{\partial W_{1k}}{\partial Q_k} = \sqrt{\frac{1}{2Q_k}}\Big(4K_1 - [(4\tilde{b} - \tilde{a})(12\tilde{b} - \tilde{a}) - 4C]Q_k^2 \quad (5.86)$$

$$-4(6\tilde{b} - \tilde{a})Q_k^3 - 4Q_k^4 + (4\alpha_1 - E)Q_k - \frac{D}{2Q_k}\Big)^{\frac{1}{2}},$$

$$\frac{\partial W_{2k}}{\partial Q_k} = \sqrt{\frac{1}{2Q_k}}\Big(4K_2Q_k - 4Q_k^4 - 4(6\tilde{b} - \tilde{a})Q_k^3 \quad (5.87)$$

$$-[(4\tilde{b} - \tilde{a})(12\tilde{b} - \tilde{a}) - C]Q_k^2 - \alpha_2 - \frac{D}{2Q_k}\Big)^{\frac{1}{2}},$$

where

$$E = 2\tilde{b}(4\tilde{b} - \tilde{a})^2, \ (k = 1, 2).$$

It is worth to note that in all above extensions, except Extension 3, the arbitrary constants C,(and/or)D determining additional terms, appears in the Hamiltonian formulation as independent canonical variables.

Chapter 6

RESONANCE SOLITONS IN AKNS HIERARCHY

In this section we construct one and two soliton solutions of the second and third members of AKNS hierarchy with resonance soliton dynamics.

6.1 Hirota Bilinear Method in Soliton Theory

In 1971 Hirota introduced a new direct method for constructing soliton solutions to integrable nonlinear evolution equations [13]. The idea is to make transformation to new variables, so that in these variables a nonlinear evolution equation become represented in the bilinear form , and multisoliton solutions appear in particularly simple form. Multisoliton solutions can, of course derived by many other methods, by the inverse scattering transform , dressing method, Bäcklund and Darboux transformations, and so on. Particularly, the Inverse Scattering Method(ISM) is very powerful, but at the same time it is most complicated and needs information about analytic behaviour of scattering data. Comparing with this, the advantage of Hirota's method is its algebraic rather than analytic structure [12], [74], [114]. It allows one to construct soliton solution in a simple algebraic form avoiding analytic difficulties of ISM [73], [111], [112], [113].

6.2 Bilinear Representation of Reaction-Diffusion System

The second member of the AKNS (6.1)is called the Reaction-Diffusion Equation

$$\begin{aligned} \partial_y e^+ &= \partial_1^2 e^+ + \tfrac{\lambda}{4} e^+ e^- e^+, \\ -\partial_y e^- &= \partial_1^2 e^- + \tfrac{\lambda}{4} e^+ e^- e^-. \end{aligned} \tag{6.1}$$

For Hirota representation of this equation [20] we substitute

$$e^{\pm} = \sqrt{\frac{-8}{\lambda}} \frac{G^{\pm}(x,t)}{F(x,t)}, \tag{6.2}$$

Then, partial derivatives of the ratio of two functions in the Hirota method [12] can be represented in terms of so called Hirota derivatives (see Appendix I) defined as

$$D_x^n(f \cdot g) = (\partial_{x_1} - \partial_{x_2})^n f(x_1)g(x_2)|_{x_2=x_1=x}. \tag{6.3}$$

In explicit form we have

$$\begin{aligned} &D_x(f \cdot g) = f'g - g'f, \\ &D_x^2(f \cdot g) = f''g - 2f'g' + g''f, \\ &D_x^3(f \cdot g) = f'''g - 3f''g' + 3g''f' - g'''f, \\ &\ldots \end{aligned} \tag{6.4}$$

Then we have

$$\frac{d}{dx}(\frac{G^\pm}{F}) = \frac{D_x(G^\pm \cdot F)}{F^2}, \tag{6.5}$$

$$\frac{d^2}{dx^2}(\frac{G^\pm}{F}) = [\frac{D_x^2(G^\pm \cdot F)}{F^2} - \frac{G^\pm}{F}\frac{D_x^2(F \cdot F)}{F^2}]. \tag{6.6}$$

For bilinear representation of the system (6.1) we need the next derivatives

$$e_y^\pm = \sqrt{\frac{-8}{\lambda}}\frac{D_y(G^\pm \cdot F)}{F^2}, \tag{6.7}$$

$$e_x^\pm = \sqrt{\frac{-8}{\lambda}}\frac{D_x(G^\pm \cdot F)}{F^2}, \tag{6.8}$$

$$e_{xx}^\pm = \sqrt{\frac{-8}{\lambda}}[\frac{D_x^2(G^\pm \cdot F)}{F^2} - \frac{G^\pm}{F}\frac{D_x^2(F \cdot F)}{F^2}]. \tag{6.9}$$

Substituting to the Reaction-Diffusion system (6.1) we can split the last one to the couple of bilinear equations

$$\begin{aligned} &(\pm D_y - D_x^2)(G^\pm \cdot F) = 0, \\ &D_x^2(F \cdot F) = -2G^+G^-. \end{aligned} \tag{6.10}$$

Then any solution of this system determines a solution of the Reaction-Diffusion system (6.1). Simplest solution of bilinear system (6.10) has been derived in the form [76]

$$G^\pm = \pm e^{\eta_1^\pm}, \quad F = 1 + \frac{e^{(\eta_1^+ + \eta_1^-)}}{(k_1^+ + k_1^-)^2}, \tag{6.11}$$

where $\eta_1^\pm = k_1^\pm x \pm (k_1^\pm)^2 y + \eta_1^{\pm(0)}$. This solution determines soliton-like solution of the Reaction-Diffusion system called the dissipaton [20], with exponentially growing and decaying amplitudes. But for the product e^+e^- one have perfect one-soliton shape

$$e^+e^- = \frac{8k^2}{\lambda \cosh^2[k(x - vy - x_0)]}, \quad (6.12)$$

of the amplitude $k = (k_1^+ + k_1^-)/2$, propagating with velocity $v = -(k_1^+ - k_1^-)$, where the initial position $x_0 = -\ln(k_1^+ + k_1^-)^2 + \eta_1^{+(0)} + \eta_1^{-(0)}$. For two-soliton(dissipaton) solution we have

$$G^\pm = \pm(e^{\eta_1^\pm} + e^{\eta_2^\pm} + \alpha_1^\pm e^{\eta_1^+ + \eta_1^- + \eta_2^\pm} + \alpha_2^\pm e^{\eta_2^+ + \eta_2^- + \eta_1^\pm}), \quad (6.13)$$

$$F = 1 + \frac{e^{\eta_1^+ + \eta_1^-}}{(k_{11}^{+-})^2} + \frac{e^{\eta_1^+ + \eta_2^-}}{(k_{12}^{+-})^2} + \frac{e^{\eta_2^+ + \eta_1^-}}{(k_{21}^{+-})^2} + \frac{e^{\eta_2^+ + \eta_2^-}}{(k_{22}^{+-})^2} + \beta e^{\eta_1^+ + \eta_1^- + \eta_2^+ + \eta_2^-}, \quad (6.14)$$

where

$$\eta_i^\pm = k_i^\pm x \pm (k_i^\pm)^2 y + \eta_i^\pm(0)$$

$$k_{ij}^{ab} = k_i^a + k_j^b, (i,j = 1,2), (a,b = +-)$$

$$\alpha_1^\pm = \frac{(k_1^\pm - k_2^\pm)^2}{(k_{11}^{+-} k_{21}^{\pm\mp})^2}, \qquad \alpha_2^\pm = \frac{(k_1^\pm - k_2^\pm)^2}{(k_{22}^{+-} k_{12}^{\pm\mp})^2}$$

$$\beta = \frac{(k_1^+ - k_2^+)^2 (k_1^- - k_2^-)^2}{(k_{11}^{+-} k_{12}^{+-} k_{21}^{+-} k_{22}^{+-})^2}.$$

6.3 Resonance Dynamics of Dissipatons

The degenerate case of this solution, when $k_1^+ = k_1^- \equiv p_1$, $k_2^+ = k_2^- \equiv p_2$, can be simplified in the form

$$e^\pm = \pm\sqrt{\frac{-8}{\lambda}} p_+ p_- \frac{p_1 \cosh\theta_2 e^{\pm p_1^2 t} + p_2 \cosh\theta_1 e^{\pm p_2^2 t}}{p_-^2 \cosh\theta_+ + p_+^2 \cosh\theta_- + 4 p_1 p_2 \cosh(p_+ p_- t)}, \quad (6.15)$$

where $p_\pm \equiv p_1 \pm p_2$,$\theta_\pm \equiv \theta_1 \pm \theta_2$, $\theta_i \equiv p_i(x - x_{0i}), (i = 1,2)$. In this solution we have substituted t instead of y variable. It allows us to interpret it dynamically in terms of time evolution $t = y$. Then, reduced solution (6.15) describes a collision of two dissipatons with identical amplitudes $p_+/2$, moving in opposite directions with equal velocities $\|v\| = \|p_-\|$, and creating the resonance bound state [76]. The lifetime of this state, $\triangle T \approx 2p_2 d/p_+ p_-$, linearly depends on the relative distance d, where $x_{01} = 0, x_{02} = d$.

In a more general case, tractable analytically, when $k_i^\pm > 0, (i = 1,2)$, and $k_1^+ - k_1^- > 0$, $k_2^+ - k_2^- > 0$, $k_1^+ - k_2^- > 0$, $k_2^+ - k_1^- < 0$, solution (6.13), (6.14) describes collision of solitons with velocities $v_{12} = -(k_1^+ - k_2^-)$ and $v_{21} = -(k_2^+ - k_1^-)$, correspondingly. Depending on the relative position's shift, also in this general case the resonance states can be created [76].

As a simplest example we consider conditions for decay of a dissipaton at rest ($v = 0$) on two dissipatons with parameters (k_1, v_1) and (k_2, v_2). From the conservation laws one obtains the following relations $v_1^2 = 4k_2^2$, $v_2^2 = 4k_1^2$ leading to two possibilities :

(a)$\|v_1\| = \|v_2\|$. In this case $\|k_1\| = \|k_2\|$, and both dissipatons have equal masses $M_1 = M_2 = M/2$ and velocities, satisfying the critical values $v_i^2 = 4k_i^2, (i = 1, 2)$

(b)$\|v_1\| > \|v_2\|$(without lose of generality). In this case $v_1^2 > 4k_1^2$ and $v_2^2 > 4k_2^2$ so that the initial dissipaton decays on a couple non-equal mass dissipatons.

The process of creation of resonant dissipaton is illustrated in Fig. 1.

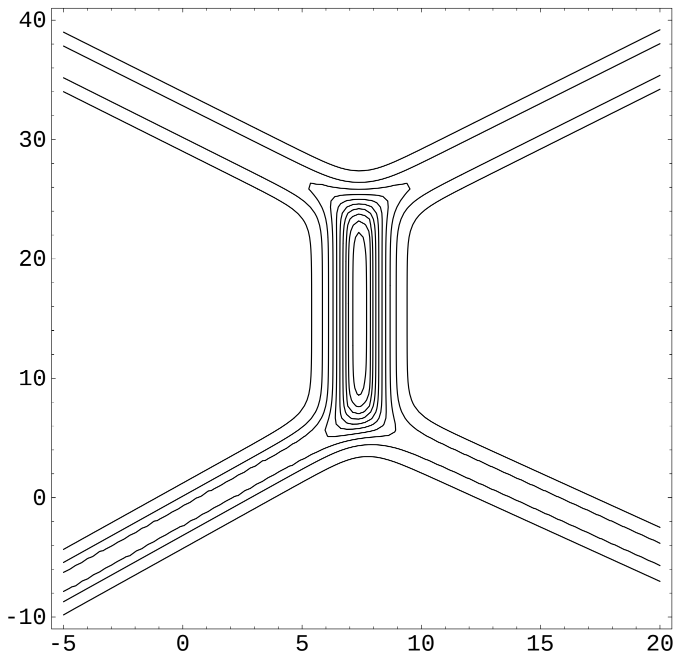

Figure 6.1: Resonant dissipaton creation

Figure 2 shows interaction of two dissipatons by exchange of a third dissipaton.

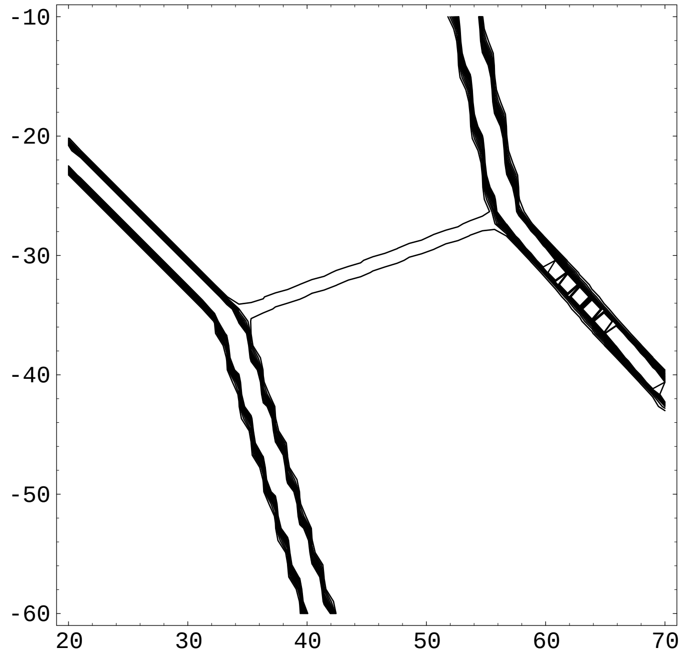

Figure 6.2: Exchange interactions of dissipatons

6.4 Geometrical Interpretation

Reaction-Diffusion system has a geometrical interpretation in a language of constant curvature surfaces [76]. We define two-dimensional metric tensor in terms of $e^+$and e^-,

$$g_{\mu\nu} = e^a_\mu e^b_\nu \eta_{ab} = \frac{1}{2}(e^+_\mu e^-_\nu + e^+_\nu e^-_\mu), \tag{6.16}$$

where $e^\pm_\mu = e^0_\mu \pm e^1_\mu = (e^\pm_0, e^\pm_1)$, $\eta_{ab} = diag(-1, 1)$,

$$e^\pm_0 = \pm\frac{\partial}{\partial x}e^\pm, \;\; e^\pm_1 \equiv e^\pm, \tag{6.17}$$

such that

$$g_{00} = -\frac{\partial e^+}{\partial x}\frac{\partial e^-}{\partial x}, \;\; g_{11} = e^+e^-, \;\; g_{01} = \frac{1}{2}(\frac{\partial e^+}{\partial x}e^- - e^+\frac{\partial e^-}{\partial x}), \tag{6.18}$$

implying identification $x_0 \equiv t(y)$, $x_1 \equiv x$. It follows that when $e^\pm$ satisfy the Reaction-Diffusion equations (6.1), this metric describes two-dimensional pseudo-Riemannian space-time with constant curvature Λ:

$$R = g^{\mu\nu}R_{\mu\nu} = \Lambda. \tag{6.19}$$

If we calculate the metric (6.16) for one soliton solution

$$ds^2 = \frac{-8k^2}{\lambda \cosh^2 k(x - vt - x_0)} \tag{6.20}$$
$$\times[(k^2 \tanh^2 k(x - vt) - \frac{1}{4}v^2)(dt)^2 - (dx)^2 - vdxdt]$$

then for $\|v\| < 2\|k\| \equiv \|v_{max}\|$, it shows a singularity (sign changing) at

$$\tanh k(x - vt) = \pm\frac{v}{2k}. \tag{6.21}$$

It was shown [76], that this singularity (called the casual singularity)has physical interpretation in terms of black hole physics [106], [107] and relates with resonance properties of solitons.

6.5 Dissipative solitons for the Third Flow

For the third flow of AKNS hierarchy we have the qubic dispersion system

$$\begin{cases} \partial_t e^+ = \partial_1^3 e^+ + \frac{3\lambda}{4} e^+ e^- \partial_1 e^+, \\ \partial_t e^- = \partial_1^3 e^- + \frac{3\lambda}{4} e^+ e^- \partial_1 e^-, \end{cases} \tag{6.22}$$

For the bilinear representation of this system first of all we represent functions $e^{(\pm)}(x, t)$ satisfying Eqs.(6.22) in terms of three real functions $G^\pm, F$. Hirota's derivatives are defined as before (6.3)(6.4). But for the third derivative term we need also following expressions

$$\frac{\partial^3}{\partial x^3}(\frac{g}{f}) = \frac{D_x^3(g \cdot f)}{f^2} - 3[\frac{D_x^2(g \cdot f)}{f^2}\frac{D_x^2(f \cdot f)}{f^2}],$$

and

$$e^\pm_{xxx} = \sqrt{\frac{-8}{\lambda}}[\frac{D_x^3(G^\pm \cdot F)}{F^2} - 3[\frac{D_x^2(G^\pm \cdot F)}{F^2}\frac{D_x^2(F \cdot F)}{F^2}]. \tag{6.23}$$

Then, we have the bilinear form of Eqs.(6.22)

$$\begin{aligned} (D_t + D_x^3)(G^\pm \cdot F) = 0, \\ D_x^2(F \cdot F) = -2G^+G^-. \end{aligned} \tag{6.24}$$

From the last equation we have for the product

$$U = e^{(+)}e^{(-)} = \frac{8}{-\lambda}\frac{G^+G^-}{F^2} = \frac{4}{\lambda}\frac{D_x^2(F \cdot F)}{F^2} = \frac{8}{\lambda}\frac{\partial^2}{\partial x^2}\ln F. \tag{6.25}$$

As in the canonical Hirota approach, we search solution of this bilinear system in the form

$$G^{\pm} = \varepsilon G_1^{\pm} + \varepsilon^3 G_3^{\pm} + ... \qquad , \; F = F_0 + \varepsilon^2 F_2 + \varepsilon^4 F_4 + ... \; , \tag{6.26}$$

where ε is a parameter which is not small. If we substitute $G^{\pm}$ and F to the system 6.24 we get a sequence of equations in ε,

$$\begin{aligned}(D_t + D_x^3)(\varepsilon G_1^{\pm} + \varepsilon^3 G_3^{\pm} + ...) \cdot (F_0 + \varepsilon^2 F_2 + \varepsilon^4 F_4 + ...) = 0, \\ D_x^2(F_0 + \varepsilon^2 F_2 + \varepsilon^4 F_4 + ...) \cdot (F_0 + \varepsilon^2 F_2 + \varepsilon^4 F_4 + ...) \\ = -2(\varepsilon G_1^+ + \varepsilon^3 G_3^+ + ...)(\varepsilon G_1^- + \varepsilon^3 G_3^- + ...).\end{aligned} \tag{6.27}$$

In the zero order approximation ε^0 we have equation $D_x^2(F_0 \cdot F_0) = 0$, with an arbitrary constant solution $F_0 =$ constant. Without loss of generality we can put this constant to the one. Indeed, the Hirota substitution is invariant under multiplication of functions G and F with arbitrary function h(x,t):

$$e^{\pm} = \frac{G^{\pm}}{F} = \frac{hG^{\pm}}{hF} = \frac{h(\varepsilon G_1^{\pm} + \varepsilon^3 G_3^{\pm} + ...)}{h(F_0 + \varepsilon^2 F_2 + \varepsilon^4 F_4 + ...)}, \tag{6.28}$$

so that we can always choose $h = 1/F_0$.

1. For ε^1 we have the system

$$(D_t + D_x^3)(G_1^{\pm} \cdot F_0) = 0, \tag{6.29}$$

with solution

$$G_1^{\pm} = \pm e^{\eta_1^{\pm}}, \tag{6.30}$$

where

$$\eta_1^{\pm} = k_1^{\pm} x - (k_1^{\pm})^3 t + \eta_1^{\pm(0)}. \tag{6.31}$$

2. Then ε^2 equation

$$2D_x^2(1 \cdot F_2) = -2G_1^+ G_1^-, \tag{6.32}$$

provides solution

$$F_2 = \frac{e^{(\eta_1^+ + \eta_1^-)}}{(k_1^+ + k_1^-)^2}. \tag{6.33}$$

3. For ε^3 the system is

$$(D_t + D_x^3)(G_1^{\pm} \cdot F_2 + G_3^{\pm} \cdot 1) = 0 \tag{6.34}$$

6.5.1 One Dissipative Soliton Solution

Simplest solution of this system $G_3 = 0$ implies to take all higher order terms $G_n = 0$, $n = 3, 5, 7, \ldots$ and $F_n = 0$, $n = 4, 6, 8, \ldots$. It is easy to check that this truncated solution is an exact solution of our system,

$$G^{\pm} = \pm e^{\eta_1^{\pm}}, F = 1 + \frac{e^{(\eta_1^{+}+\eta_1^{-})}}{k_1^{+}+k_1^{-}}, \tag{6.35}$$

where $\eta_1^{\pm} = k_1^{\pm}x - (k_1^{\pm})^3 t + \eta_1^{\pm(0)}$. It defines one dissipative soliton solution of the system (6.22) in the form

$$e^{\pm} = \pm\sqrt{\frac{8}{-\Lambda}}\frac{|k_{11}^{\pm}|}{2}\frac{e^{\pm\frac{1}{2}(\eta_1^{+}-\eta_1^{-})}}{cosh\frac{\eta_1^{+}+\eta_1^{-}+\phi_{11}}{2}}, \tag{6.36}$$

$$\eta_1^{+} + \eta_1^{-} = (k_1^{+}+k_1^{-})[x - v\tau - x_0], \tag{6.37}$$

where

$$v = ({k_1^{+}}^2 - k_1^{+}k_1^{-} + {k_1^{-}}^2),\ \ x_0 = \frac{{\eta_1^{+}}^{(0)} + {\eta_1^{+}}^{(0)}}{k_1^{+}k_1^{-}},\ \ \frac{\phi_{11}}{2} = ln\frac{1}{k_{11}^{+-}}.$$

6.5.2 Two Dissipative Soliton Solution

Another, nontrivial choice for G_3 we find as

$$G_3^{\pm} = \pm e^{\eta_2^{\pm}}, \tag{6.38}$$

where

$$\eta_2^{\pm} = k_2^{\pm}x - (k_2^{\pm})^3 t + \eta_2^{\pm}(0). \tag{6.39}$$

4. In this case for ε^4 we have the equation

$$2D_x^2(1\cdot F_4) + D_x^2(F_2\cdot F_2) = -2(G_1^{+}G_3^{-} + G_3^{+}G_1^{-}), \tag{6.40}$$

with following solution

$$F_4 = \frac{e^{(\eta_1^{+}+\eta_2^{-})}}{(k_1^{+}+k_1^{-})^2} + \frac{e^{(\eta_2^{+}+\eta_1^{-})}}{(k_2^{+}+k_1^{-})^2}. \tag{6.41}$$

5. For ε^5 for the system

$$(D_t + D_x^3)(G_5^{\pm}\cdot 1 + G_3^{\pm}\cdot F_2 + G_1^{\pm}\cdot F_4) = 0 \tag{6.42}$$

and we find solution

$$G_5^{\pm} = \pm\alpha_1{}^{\pm} e^{\eta_1^+ + \eta_1^- + \eta_2^{\pm}}, \tag{6.43}$$

where coefficients

$$\alpha_1^{\pm} = \frac{(k_1^{\pm} - k_2^{\pm})^2}{(k_{11}^{+-} k_{21}^{\pm\mp})^2}. \tag{6.44}$$

6. At the next level ε^6 equation

$$2D_x^2(1 \cdot F_6) + 2D_x^2(F_2 \cdot F_4) = -2(G_1^+ G_5^- + G_3^+ G_3^- + G_5^+ G_1^-), \tag{6.45}$$

admits solution

$$F_6 = \frac{e^{(\eta_2^+ + \eta_2^-)}}{(k_2^+ + k_2^-)^2}. \tag{6.46}$$

7. For ε^7 it is

$$(D_t + D_x^3)(G_7^{\pm} \cdot 1 + G_5^{\pm} \cdot F_2 + G_3^{\pm} \cdot F_4 + G_1^{\pm} \cdot F_6) = 0, \tag{6.47}$$

and we find

$$G_7^{\pm} = \pm\alpha_2{}^{\pm} e^{(\eta_2^+ + \eta_2^- + \eta_1^{\pm})}, \tag{6.48}$$

where

$$\alpha_2^{\pm} = \frac{(k_1^{\pm} - k_2^{\pm})^2}{(k_{22}^{+-} k_{12}^{\pm\mp})^2}. \tag{6.49}$$

8. Next level ε^8 gives equation

$$2D_x^2(1 \cdot F_8) + 2D_x^2(F_2 \cdot F_6) + D_x^2(F_4 \cdot F_4) \tag{6.50}$$

$$= -2(G_1^+ G_7^- + G_3^+ G_5^- + G_5^+ G_3^- + G_7^+ G_1^-)$$

with solution

$$F_8 = \beta e^{(\eta_1^+ + \eta_1^- + \eta_2^+ + \eta_2^-)}, \tag{6.51}$$

where

$$\beta = \frac{(k_1^+ - k_2^+)^2 (k_1^- - k_2^-)^2}{(k_{11}^{+-} k_{12}^{+-} k_{21}^{+-} k_{22}^{+-})^2}. \tag{6.52}$$

9. For higher order terms in ε we can choose $G_n = 0$, with index $n = 9, 11, 13, ...$ and $F_n = 0$, with index $n = 10, 12, 14, ...$ Then, by direct substitution we checked that with this choice, bilinear equations are satisfied in all orders of ε. Therefore we have an exact solution.

This solution is the two soliton solution in the form

$$G^{\pm} = \pm(e^{\eta_1^{\pm}} + e^{\eta_2^{\pm}} + \alpha_1^{\pm} e^{\eta_1^{+}+\eta_1^{-}+\eta_2^{\pm}} + \alpha_2^{\pm} e^{\eta_2^{+}+\eta_2^{-}+\eta_1^{\pm}}), \tag{6.53}$$

$$F = 1 + \frac{e^{\eta_1^{+}+\eta_1^{-}}}{(k_{11}^{+-})^2} + \frac{e^{\eta_1^{+}+\eta_2^{-}}}{(k_{12}^{+-})^2} + \frac{e^{\eta_2^{+}+\eta_1^{-}}}{(k_{21}^{+-})^2} + \frac{e^{\eta_2^{+}+\eta_2^{-}}}{(k_{22}^{+-})^2} + \beta e^{\eta_1^{+}+\eta_1^{-}+\eta_2^{+}+\eta_2^{-}}, \tag{6.54}$$

where

$$\eta_i^{\pm} = k_i^{\pm} x - (k_i^{\pm})^3 t + \eta_i^{\pm(0)},$$

$$k_{ij}^{ab} = k_i^a + k_j^b, (i, j = 1, 2), (a, b = +-),$$

$$\alpha_1^{\pm} = \frac{(k_1^{\pm} - k_2^{\pm})^2}{(k_{11}^{+-} k_{21}^{\pm\mp})^2}, \qquad \alpha_2^{\pm} = \frac{(k_1^{\pm} - k_2^{\pm})^2}{(k_{22}^{+-} k_{12}^{\pm\mp})^2}, \tag{6.55}$$

$$\beta = \frac{(k_1^{+} - k_2^{+})^2 (k_1^{-} - k_2^{-})^2}{(k_{11}^{+-} k_{12}^{+-} k_{21}^{+-} k_{22}^{+-})^2}. \tag{6.56}$$

6.5.3 The MKdV Reduction

From Section 5.3,we know that the third order of AKNS flow under special reduction

$$e^{+} = e^{-} = U \tag{6.57}$$

reduces to MKdV equation (3.34). For the bilinear equation (6.24), it means the following reduction

$$G^{+} \equiv G^{-} \equiv G, \tag{6.58}$$

under this reduction

$$\begin{cases} (D_t + D_x^3)(G \cdot F) = 0 \\ D_x^2(F \cdot F) = -2G^2 \end{cases} \tag{6.59}$$

we have

$$e^{\eta_1^{+}} = e^{\eta_1^{-}} \equiv e^{\eta}, \tag{6.60}$$

$$k_1^{+} = k_1^{-} \equiv k, \;\; {\eta_1}^{+(0)} = {\eta_1}^{-(0)}, \tag{6.61}$$

and for one-soliton solution of MKdV

$$e^{+} = e^{-} = U = \sqrt{\frac{8}{-\Lambda}} |k| \frac{1}{\cosh(\eta + \frac{\phi_{11}}{2})}, \tag{6.62}$$

or

$$U(x, t) = \sqrt{\frac{8}{-\Lambda}} \frac{|k|}{\cosh k(x - k^2\tau - x_0)}, \tag{6.63}$$

where

$$x_0 = -\frac{\eta^{(0)} + \frac{\phi_{11}}{2}}{k}, \quad \frac{\phi_{11}}{2} = ln\frac{1}{k_{11}^{+-}}.$$

For two soliton solution we find the reduction

$$\begin{aligned} \eta_1^{+(0)} &= \eta_1^{-(0)} = \eta_1^{(0)}, \\ \eta_2^{+(0)} &= \eta_2^{-(0)} = \eta_2^{(0)}, \\ k_1^+ &= k_1^- \equiv k_1, \\ k_2^+ &= k_2^- \equiv k_2, \end{aligned} \tag{6.64}$$

and

$$G = (e^{\eta_1} + e^{\eta_2} + \alpha_1 e^{2\eta_1+\eta_2} + \alpha_2 e^{2\eta_2+\eta_1}), \tag{6.65}$$

$$F = 1 + \frac{e^{\eta_1}}{4{k_1}^2} + 2\frac{e^{\eta_1+\eta_2}}{(k_1+k_2)^2} + \frac{e^{2\eta_2}}{4{k_1}^2} + \beta e^{2\eta_1+2\eta_2}, \tag{6.66}$$

where

$$\alpha_1 = \frac{(k_1-k_2)^2}{4{k_1}^2(k_1+k_2)^2}, \qquad \alpha_2 = \frac{(k_1-k_2)^2}{4{k_2}^2(k_1+k_2)^2},$$

$$\beta = \frac{(k_1-k_2)^4}{16{k_1}^2{k_2}^2(k_1+k_2)^4}.$$

It gives 2-soliton solution of MKdV equation in the form

$$U = \sqrt{\frac{8}{-\Lambda}}\frac{G}{F} = \sqrt{\frac{8}{-\Lambda}}\,\frac{(e^{\eta_1} + e^{\eta_2} + \alpha_1 e^{2\eta_1+\eta_2} + \alpha_2 e^{2\eta_2+\eta_1})}{1 + \frac{e^{\eta_1}}{4{k_1}^2} + 2\frac{e^{\eta_1+\eta_2}}{(k_1+k_2)^2} + \frac{e^{2\eta_2}}{4{k_1}^2} + \beta e^{2\eta_1+2\eta_2}} \tag{6.67}$$

or

$$U = \sqrt{\frac{8}{-\Lambda}}\frac{2k_1k_2|k_1^2-k_2^2|[k_2\cosh\tilde{\eta_1} + k_1\cosh\tilde{\eta_2}]}{(k_1-k_2)^2\cosh(\tilde{\eta_1}+\tilde{\eta_2}) + (k_1+k_2)^2\cosh(\tilde{\eta_1}-\tilde{\eta_2}) + 4k_1k_2} \tag{6.68}$$

where

$$\begin{aligned} \tilde{\eta_1} &= \eta_1 + \psi + \phi_1 = k_1(x - k_1^2\tau - X_0^1), \qquad X_0^1 = \tfrac{1}{k_1}{\eta_1}^{(0)} + \ln\tfrac{|k_1-k_2|}{|k_1+k_2|} - \tfrac{1}{2}\ln 4k_1^2, \\ \tilde{\eta_2} &= \eta_2 + \psi + \phi_2 = k_2(x - k_2^2\tau - X_0^2), \qquad X_0^2 = \tfrac{1}{k_2}{\eta_2}^{(0)} + \ln\tfrac{|k_1-k_2|}{|k_1+k_2|} - \tfrac{1}{2}\ln 4k_2^2. \end{aligned}$$

6.5.4 The MKdV-KdV Mixed Reduction

As it was shown in Section 5.3, the third order of AKNS flow under the special reduction (3.38) gives the mixed KdV-MKdV equation (3.39)

$$\partial_{t_2}U = \partial_x^3 U + \frac{3\lambda}{4}(\alpha+\beta)(\alpha\, U^2\partial_x U + \beta\, U\partial_x U). \tag{6.69}$$

To produce bilinear representation for such mixed equation, this reduction can be imposed on the bilinear equations (6.24)

$$\begin{cases} (D_t + D_x^3)(G^{\pm} \cdot F) = 0, \\ D_x^2(F \cdot F) = -2G^+G^-, \end{cases} \tag{6.70}$$

in the form

$$\begin{cases} e^+ = (\alpha+\beta)U = \frac{G^+}{F}, \\ e^- = \alpha U + \beta = \frac{G^-}{F}, \end{cases} \tag{6.71}$$

or

$$\begin{cases} G^+ = (\alpha+\beta)G, \\ G^- = \alpha G + \beta F. \end{cases} \tag{6.72}$$

Under this reduction we get bilinear representation for Eq.(6.69) as follows

$$\begin{cases} (D_t + D_x^3)(G \cdot F) = 0, \\ \frac{1}{2}D_x^2(F \cdot F) = (\alpha+\beta)\alpha G^2 + (\alpha+\beta)\beta GF. \end{cases} \tag{6.73}$$

Solution of this system provides solution of Eq(6.69) according formula

$$\alpha U^2 + \beta U = \frac{1}{2(\alpha+\beta)} \frac{D_x^2(F \cdot F)}{F^2} = \frac{1}{(\alpha+\beta)}(\ln F)_{xx}. \tag{6.74}$$

Then, for one-soliton solution we have

$$U = \frac{G}{F} = \frac{e^{\eta_1}}{1 + \frac{(\alpha+\beta)\beta}{k_1^2}e^{\eta_1} + \frac{1}{4k_1^2}[(\alpha+\beta)\beta + \frac{(\alpha+\beta)^2\beta^2}{k_1^2}]e^{2\eta_1}}, \tag{6.75}$$

or

$$U = \frac{e^{-\phi}}{2\cosh(\eta_1+\phi) + \frac{(\alpha+\beta)\beta}{k_1^2}e^{\phi}}, \tag{6.76}$$

where

$$\phi = \frac{1}{2}\ln\frac{1}{4k_1^2[(\alpha+\beta)\beta + \frac{(\alpha+\beta)^2\beta^2}{k_1^2}]}. \tag{6.77}$$

Chapter 7

THE KADOMTSEV-PETVIASHVILI MODEL

7.1 2+1 Dimensional Reduction of AKNS

AKNS hierarchy allows us to develop also a new method to find solution for (2+1) and higher dimensional integrable systems, namely the Kadomtsev-Petviashvili (KP)equation. Kadomtsev-Petviashvili equation is one of a few soliton equations which describes physical phenomena in two-dimensional space [27]. The equation was presented by Kadomtsev and Petviashvili to discuss the stability of one-dimensional soliton in a nonlinear media with weak dispersion. It has been explored recently in plasma physics, hydrodynamics, string theory and low-dimensional gravity [120], [121], [122], [123], [124]. The hierarchy of KP equations [104], [131] has a reach mathematical structure related with complex analysis and Riemann surfaces, pseudo-differential operators and algebraic geometry [105], [125], [127], [128], [28].

Depending on sign of dispersion, two types of the KP equations are known. The minus sign in the right side of the KP corresponds to the case of negative dispersion and called KPII. Now we describe a new relation of KPII with AKNS hierarchy [117] discussed in Chapter 6 and construct corresponding solutions. Let us consider the pair of functions $e^+(x,y,t)$, $e^-(x,y,t)$ satisfies the following second (7.1) and third (7.2) equations of the hierarchy.

$$\begin{aligned} e^+_y &= e^+_{xx} + \tfrac{\lambda}{4} e^+ e^- e^+, \\ -e^-_y &= e^-_{xx} + \tfrac{\lambda}{4} e^+ e^- e^-, \end{aligned} \tag{7.1}$$

$$\begin{aligned} e^+_t &= e^+_{xxx} + \tfrac{3\lambda}{4} e^+ e^- e^+_x, \\ e^-_t &= e^-_{xxx} + \tfrac{3\lambda}{4} e^+ e^- e^-_x. \end{aligned} \tag{7.2}$$

Differentiating according to t and y, Eqs. (7.1) and (7.2) correspondingly, we can see that they are compatible.

Theorem 7.1.0.1 *Let the functions* $e^{+}(x,y,t)$ *and* $e^{-}(x,y,t)$ *are simultaneously solutions of the equations (7.1) and (7.2). Then the function* $U(x,y,t) = e^{+}e^{-}$ *satisfies the Kadomtsev-Petviashvili (KPII) equation*

$$(4U_t + \frac{3\lambda}{4}(U^2)_x + U_{xxx})_x = -3U_{yy}. \tag{7.3}$$

Proof: We take the derivative of U according to y variable

$$U_y = e_y^+ e^- + e^+ e_y^- \tag{7.4}$$

and substituting e_y^+ and e_y^- from the system (7.1) we have

$$U_y = (e_x^+ e^- - e_x^- e^+)_x, \tag{7.5}$$

$$U_{yy} = (e_{xxx}^+ e^- + e_{xxx}^- e^+ - (e_x^+ e_x^-)_x) + \frac{\lambda}{2} U_x U. \tag{7.6}$$

In a similar way U_t is

$$U_t = e_t^+ e^- + e^+ e_t^- \tag{7.7}$$

and after substitution of e_t^+ and e_t^- we get

$$U_t = -(e_{xxx}^+ e^- + \frac{3\lambda}{4} U e^- e_x^- + e_{xxx}^- e^+ + \frac{3\lambda}{4} U e_x^- e^+), \tag{7.8}$$

and

$$U_{xt} = -(e_{xxx}^+ e^- + e_{xxx}^- e^+ + \frac{3\lambda}{4} U U_x)_x. \tag{7.9}$$

Combining above formulas together

$$4U_{xt} + 3U_{yy} = [-e_{xxx}^+ e^- - e_{xxx}^- e^+ - \frac{3\lambda}{2} U U_x - 3(e_x^+ e_x^-)_x]_x, \tag{7.10}$$

and using

$$U_{xxx} = e_{xxx}^+ e^- + e_{xxx}^- e^+ + 3e_{xx}^+ e_x^- + 3e_x^+ e_{xx}^-, \tag{7.11}$$

we get KPII (7.3)

$$4U_{xt} + 3U_{yy} = -\frac{3\lambda}{4}(U^2)_{xx} - U_{xxxx}. \tag{7.12}$$

Like the KdV equation, KPII is an infinite dimensional Hamiltonian system admitting (2+1) dimensional soliton solution [127], [128], [28]. Each soliton is a planar wave similar to (1+1) KdV type soliton, but traveling in an arbitrary direction in the x-y plane. We can use the above Theorem to generate solutions of KPII in terms of solutions of equations (7.1) and (7.2).

7.2 Bilinear Representation of KPII and AKNS flows

Using bilinear representations for systems (7.1) and (7.2) and Theorem **7.1.0.1** we can find bilinear representation for KPII. The Reaction-Diffusion system (7.1) can be represented in Hirota bilinear form as

$$\begin{cases} (\pm D_y - D_x^2)(G^\pm \cdot F) = 0, \\ D_x^2(F \cdot F) = -2G^+G^-. \end{cases} \tag{7.13}$$

In a similar way the system of the third flow equations (6.22) has the following bilinear form

$$\begin{cases} (D_t + D_x^3)(G^\pm \cdot F) = 0, \\ D_x^2(F \cdot F) = -2G^+G^-. \end{cases} \tag{7.14}$$

Any solution $G^\pm(x, y), F(x, y)$ of bilinear equations (7.13) satisfies the system of equations (7.1) for $e^{(\pm)}(x, y)$, while any solution $G^\pm(x, t), F(x, t)$, of Eqs.(7.14) satisfies the system (6.22) for $e^{(\pm)}(x, t)$. Now we consider $G^\pm$ and F as functions of three variables $G^{(\pm)} = G^{(\pm)}(x, y, t)$, $F = F(x, y, t)$, and require for these functions to be solution of both bilinear systems (7.13), (7.14) simultaneously. Since the second equation in both systems (7.13),(7.14)is the same, it is sufficient to consider the next bilinear system [135]

$$\begin{cases} (\pm D_y - D_x^2)(G^\pm \cdot F) = 0, \\ (D_t + D_x^3)(G^\pm \cdot F) = 0, \\ D_x^2(F \cdot F) = -2G^+G^-. \end{cases} \tag{7.15}$$

Then, according to Theorem **7.1.0.1**, any solution of this system generates solution of KPII.

From the last equation we can derive expression for solution of KPII (7.3), directly in terms of function F only

$$U = e^{(+)}e^{(-)} = \frac{8}{-\lambda}\frac{G^+G^-}{F^2} = \frac{4}{\lambda}\frac{D_x^2(F \cdot F)}{F^2} = \frac{8}{\lambda}\frac{\partial^2}{\partial x^2}\ln F \tag{7.16}$$

As in the canonical Hirota approach we search solution of this bilinear system in the form

$$G^\pm = \varepsilon G_1^\pm + \varepsilon^3 G_3^\pm + \dots\,, \qquad F = F_0 + \varepsilon F_2 + \varepsilon^4 F_4 + \dots\,, \tag{7.17}$$

where ε is a parameter which is not small. If we substitute this expansion to system (7.15) we get a sequence of equations

$$
\begin{aligned}
&(\pm D_y - D_x^2)(\varepsilon G_1^{\pm} + \varepsilon^3 G_3^{\pm} + ...).(F_0 + \varepsilon^2 F_2 + \varepsilon^4 F_4 + ...) = 0, \\
&(D_t + D_x^3)(\varepsilon G_1^{\pm} + \varepsilon^3 G_3^{\pm} + ...).(F_0 + \varepsilon^2 F_2 + \varepsilon^4 F_4 + ...) = 0, \\
&D_x^2(1 + \varepsilon^2 F_2 + \varepsilon^4 F_4 + ...).(F_0 + \varepsilon^2 F_2 + \varepsilon^4 F_4 + ...) \\
&\qquad\qquad = -2(\varepsilon G_1^{+} + \varepsilon^3 G_3^{+} + ...)(\varepsilon G_1^{-} + \varepsilon^3 G_3^{-} + ...).
\end{aligned}
\tag{7.18}
$$

In the zero order level we have equation $D_x^2(F_0.F_0) = 0$ with an arbitrary constant solution $F_0 = \text{const}$. Like before, without lose of generality, we can put this constant to one, since the Hirota substitution is invariant under multiplication of functions G and F with an arbitrary function h(x,t):

$$
U = \frac{G^{\pm}}{F} = \frac{hG^{\pm}}{hF} = \frac{h(\varepsilon G_1^{\pm} + \varepsilon^3 G_3^{\pm} + ...)}{h(F_0 + \varepsilon^2 F_2 + \varepsilon^4 F_4 + ...)} \tag{7.19}
$$

which we choose as $h = 1/F_0$.

1. For ε^1 we have the system

$$
\begin{aligned}
&(\pm D_y - D_x^2)(G_1^{\pm} \cdot F_0) = 0, \\
&(D_t + D_x^3)(G_1^{\pm} \cdot F_0) = 0,
\end{aligned}
\tag{7.20}
$$

   with solution

$$
G_1^{\pm} = \pm e^{\eta_1^{\pm}}, \tag{7.21}
$$

   where

$$
\eta_1^{\pm} = k_1^{\pm} x \pm (k_1^{\pm})^2 y - (k_1^{\pm})^3 t + \eta_1^{\pm(0)} \tag{7.22}
$$

2. For ε^2 equation

$$
2D_x^2(1 \cdot F_2) = G_1^{+} G_1^{-} \tag{7.23}
$$

   after integration

$$
F_2 = \frac{e^{(\eta_1^{+} + \eta_1^{-})}}{(k_1^{+} + k_1^{-})^2}. \tag{7.24}
$$

3. For ε^3 the system is

$$
\begin{aligned}
&(\pm D_y - D_x^2)(G_1^{\pm} \cdot F_2 + G_3^{\pm} \cdot 1) = 0, \\
&(D_t + D_x^3)(G_1^{\pm} \cdot F_2 + G_3^{\pm} \cdot 1) = 0.
\end{aligned}
\tag{7.25}
$$

Simplest solution of this system $G_3 = 0$, implies to take all higher order terms $G_n = 0$, $n = 3, 5, 7, ...$ and $F_n = 0$, $n = 4, 6, 8, ...$ It is easy to check that this truncated solution is an exact solution of our system,

$$G^{\pm} = \pm e^{\eta_1^{\pm}}, \; F = 1 + \frac{e^{(\eta_1^{+} + \eta_1^{-})}}{k_1^{+} + k_1^{-}}, \tag{7.26}$$

where $\eta_1^{\pm} = k_1^{\pm} x \pm (k_1^{\pm})^2 y - (k_1^{\pm})^3 t + \eta_1^{\pm(0)}$, defining one-soliton solution of KPII according to Eq.(7.16)

$$U = \frac{2(k_1^{+} + k_1^{-})^2}{\lambda \cosh^2 \frac{1}{2}[(k_1^{+} + k_1^{-})x + ({k_1^{+}}^2 - {k_1^{-}}^2)y - ({k_1^{+}}^3 + {k_1^{-}}^3)t + \gamma]}, \tag{7.27}$$

where $\gamma = -\ln(k_1^{+} + k_1^{-})^2 + \eta_1^{+(0)} + \eta_1^{-(0)}$

Another, nontrivial choice for G_3 we find as

$$G_3^{\pm} = \pm e^{\eta_2^{\pm}}, \tag{7.28}$$

where

$$\eta_2^{\pm} = k_2^{\pm} x \pm (k_2^{\pm})^2 y - (k_2^{\pm})^3 t + \eta_2^{\pm(0)} \tag{7.29}$$

4. In this case for ε^4 we have the equation

$$2D_x^2(1 \cdot F_4) + D_x^2(F_2 \cdot F_2) = -2(G_1^{+} G_3^{-} + G_3^{+} G_1^{-}), \tag{7.30}$$

and solution

$$F_4 = \frac{e^{(\eta_1^{+} + \eta_2^{-})}}{(k_1^{+} + k_1^{-})^2} + \frac{e^{(\eta_2^{+} + \eta_1^{-})}}{(k_2^{+} + k_1^{-})^2} \tag{7.31}$$

5. For ε^5 it gives

$$\begin{aligned} (\pm D_y - D_x^2)(G_5^{\pm} \cdot 1 + G_3^{\pm} \cdot F_2 + G_1^{\pm} \cdot F_4) = 0, \\ (D_t + D_x^3)(G_5^{\pm} \cdot 1 + G_3^{\pm} \cdot F_2 + G_1^{\pm} \cdot F_4) = 0, \end{aligned} \tag{7.32}$$

and we find

$$G_5^{\pm} = \pm {\alpha_1}^{\pm} e^{\eta_1^{+} + \eta_1^{-} + \eta_2^{\pm}}, \tag{7.33}$$

where

$$\alpha_1^{\pm} = \frac{(k_1^{\pm} - k_2^{\pm})^2}{(k_{11}^{+-} k_{21}^{\pm\mp})^2}. \tag{7.34}$$

6. At the next level ε^6 a single equation is

$$2D_x^2(1 \cdot F_6) + 2D_x^2(F_2 \cdot F_4) = -2(G_1^+ G_5^- + G_3^+ G_3^- + G_5^+ G_1^-) \tag{7.35}$$

and solution

$$F_6 = \frac{e^{(\eta_2^+ + \eta_2^-)}}{(k_2^+ + k_2^-)^2} \tag{7.36}$$

7. For ε^7 it is

$$\begin{aligned} &(\pm D_y - D_x^2)(G_7^\pm \cdot 1 + G_5^\pm \cdot F_2 + G_3^\pm \cdot F_4 + G_1^\pm \cdot F_6) = 0, \\ &(D_t + D_x^3)(G_7^\pm \cdot 1 + G_5^\pm \cdot F_2 + G_3^\pm \cdot F_4 + G_1^\pm \cdot F_6) = 0, \end{aligned} \tag{7.37}$$

and we find

$$G_7^\pm = \pm \alpha_2{}^\pm e^{(\eta_2^+ + \eta_2^- + \eta_1^\pm)}, \tag{7.38}$$

where

$$\alpha_2^\pm = \frac{(k_1^\pm - k_2^\pm)^2}{(k_{22}^{+-} k_{12}^{\pm\mp})^2}. \tag{7.39}$$

8. Next level ε^8 gives

$$2D_x^2(1 \cdot F_8) + 2D_x^2(F_2 \cdot F_6) + D_x^2(F_4 \cdot F_4) \tag{7.40}$$

$$= -2(G_1^+ G_7^- + G_3^+ G_5^- + G_5^+ G_3^- + G_7^+ G_1^-),$$

and

$$F_8 = \beta e^{(\eta_1^+ + \eta_1^- + \eta_2^+ + \eta_2^-)}, \tag{7.41}$$

where

$$\beta = \frac{(k_1^+ - k_2^+)^2 (k_1^- - k_2^-)^2}{(k_{11}^{+-} k_{12}^{+-} k_{21}^{+-} k_{22}^{+-})^2}. \tag{7.42}$$

9. For higher order in ε we can choose $G_n = 0$, with index $n = 9, 11, 13, ...$, and $F_n = 0$, with index $n = 10, 12, 14, ...$. By direct substitution we checked that with this choice bilinear equations are satisfied in all orders of ε. Therefore we have an exact solution.

7.3 Two Soliton Solutions

This solution is two soliton solution in the form [135]

$$G^\pm = \pm(e^{\eta_1^\pm} + e^{\eta_2^\pm} + \alpha_1^\pm e^{\eta_1^+ + \eta_1^- + \eta_2^\pm} + \alpha_2^\pm e^{\eta_2^+ + \eta_2^- + \eta_1^\pm}), \tag{7.43}$$

$$F = 1 + \frac{e^{\eta_1^+ + \eta_1^-}}{(k_{11}^{+-})^2} + \frac{e^{\eta_1^+ + \eta_2^-}}{(k_{12}^{+-})^2} + \frac{e^{\eta_2^+ + \eta_1^-}}{(k_{21}^{+-})^2} + \frac{e^{\eta_2^+ + \eta_2^-}}{(k_{22}^{+-})^2} + \beta e^{\eta_1^+ + \eta_1^- + \eta_2^+ + \eta_2^-}, \qquad (7.44)$$

where

$$\eta_i^{\pm} = k_i^{\pm} x \pm (k_i^{\pm})^2 y - (k_i^{\pm})^3 t + \eta_i^{\pm(0)},$$

$$k_{ij}^{ab} = k_i^a + k_j^b, (i, j = 1, 2), (a, b = +-),$$

$$\alpha_1^{\pm} = \frac{(k_1^{\pm} - k_2^{\pm})^2}{(k_{11}^{+-} k_{21}^{\pm\mp})^2}, \qquad \alpha_2^{\pm} = \frac{(k_1^{\pm} - k_2^{\pm})^2}{(k_{22}^{+-} k_{12}^{\pm\mp})^2},$$

$$\beta = \frac{(k_1^+ - k_2^+)^2 (k_1^- - k_2^-)^2}{(k_{11}^{+-} k_{12}^{+-} k_{21}^{+-} k_{22}^{+-})^2}.$$

According to Eq.(7.16) it provides two-soliton solution of KPII

$$U = e^+ e^- = \frac{-8}{\Lambda} \frac{\partial^2}{\partial x^2} \ln F. \qquad (7.45)$$

7.4 Degenerate Four-Soliton Solution

For KPII another bilinear form, only in terms of one function F, is known [74]

$$(D_x D_t + D_x^4 \pm D_y^2)(F \cdot F) = 0 \qquad (7.46)$$

where

$$U = 2 \frac{\partial^2}{\partial x^2} \ln F. \qquad (7.47)$$

Thus it is natural to compare soliton solutions of our bilinear equations (7.15) with the ones given by above bilinear equation [135]. To solve equation (7.46)we consider

$$F = 1 + \varepsilon F_1 + \varepsilon^2 F_2 + ... \, . \qquad (7.48)$$

1. For ε^1 we have the equation

$$(D_x D_t + D_x^4 + D_y^2)(1 \cdot F_1) = 0, \qquad (7.49)$$

with the solution

$$F_1 = e^{\eta_1}, \qquad (7.50)$$

where

$$\eta_1 = k_1 x + \Omega_1 y - \omega_1 t + \eta_1^0, \qquad (7.51)$$

and dispersion

$$k_1 \omega_1 + k_1^4 + \Omega_1^2 = 0. \qquad (7.52)$$

2. For ε^2 we have equation

$$(D_x D_t + D_x^4 + D_y^2)(1 \cdot 2F_2 + F_1 \cdot F_1) = 0. \tag{7.53}$$

The simplest solution of this equation is the trivial one

$$F_2 = 0. \tag{7.54}$$

Then, truncating the series by putting all higher order terms in ε to zero, $F_n = 0$, (n = 2,3,...), we have one soliton solution of KPII (7.3):

$$U = \frac{k_1}{2\cosh^2 \frac{1}{2}[(k_1 x + ({k_1^+}^2 - {k_1^-}^2)y - ({k_1^+}^3 + {k_1^-}^3)t + \gamma]} \tag{7.55}$$

If we consider one soliton solution of equation (7.46), we realize that it coincides with our one soliton solution (7.27)

$$U = \frac{2(k_1^+ + k_1^-)^2}{\lambda \cosh^2 \frac{1}{2}[(k_1^+ + k_1^-)x + ({k_1^+}^2 - {k_1^-}^2)y - ({k_1^+}^3 + {k_1^-}^3)t + \gamma]} \tag{7.56}$$

where $\gamma = -\ln(k_1^+ + k_1^-)^2 + {\eta_1^+}^{(0)} + {\eta_1^-}^{(0)}$. But two soliton solution of equation (7.46)doesn't correspond to our two-soliton solution (7.43),(7.44). Appearance of four different terms $e^{\eta_i^\pm + \eta_k^\pm}$ in equations (7.43),(7.44), suggest that our two-soliton solution should correspond to some degenerate case of four soliton solution of Eq(7.46)(We thank Prof. J. Hietarinta for this suggestion).

To construct two soliton solution we choose another solution of bilinear equation (7.46)

$$F_2 = e^{\eta_2} \tag{7.57}$$

where

$$\eta_2 = k_2 x + \Omega_2 y - \omega_2 t + \eta_2^0 \tag{7.58}$$

3. For ε^3 we have the equation

$$(D_x D_t + D_x^4 + D_y^2)(1 \cdot 2F_3 + 2F_1 \cdot F_2) = 0, \tag{7.59}$$

with the solution

$$F_3 = \alpha_{12} e^{\eta_1 + \eta_2}, \tag{7.60}$$

where

$$\alpha_{12} = -\frac{(k_1 - k_2)(\omega_1 - \omega_2) + (k_1 - k_2)^4 + (\Omega_1 - \Omega_2)^2}{(k_1 + k_2)(\omega_1 + \omega_2) + (k_1 + k_2)^4 + (\Omega_1 + \Omega_2)^2}. \tag{7.61}$$

4. For ε^4 equation

$$(D_xD_t + D_x^4 + D_y^2)(1 \cdot 2F_4 + F_1 \cdot 2F_3 + F_2 \cdot F_2) = 0 \tag{7.62}$$

has solution

$$F_4 = e^{\eta_3}, \tag{7.63}$$

where

$$\eta_3 = k_3x + \Omega_3y - \omega_3t + \eta_3^0. \tag{7.64}$$

5. For ε^5 equation

$$(D_xD_t + D_x^4 + D_y^2)(1 \cdot 2F_5 + F_1 \cdot 2F_4 + F_2 \cdot 2F_3) = 0 \tag{7.65}$$

admits

$$F_5 = \alpha_{13}e^{\eta_1+\eta_3}, \tag{7.66}$$

where

$$\alpha_{13} = -\frac{(k_1 - k_3)(\omega_1 - \omega_3) + (k_1 - k_3)^4 + (\Omega_1 - \Omega_3)^2}{(k_1 + k_3)(\omega_1 + \omega_3) + (k_1 + k_3)^4 + (\Omega_1 + \Omega_3)^2}. \tag{7.67}$$

6. For ε^6 equation is

$$(D_xD_t + D_x^4 + D_y^2)(1 \cdot 2F_6 + F_1 \cdot 2F_5 + F_2 \cdot 2F_4 + F_3 \cdot F_3) = 0, \tag{7.68}$$

with solution

$$F_6 = \alpha_{23}e^{\eta_1+\eta_3}, \tag{7.69}$$

where

$$\alpha_{23} = -\frac{(k_2 - k_3)(\omega_2 - \omega_3) + (k_2 - k_3)^4 + (\Omega_2 - \Omega_3)^2}{(k_2 + k_3)(\omega_2 + \omega_3) + (k_2 + k_3)^4 + (\Omega_2 + \Omega_3)^2}. \tag{7.70}$$

Now we will do the special parameterizations of our solution

$$\begin{aligned} k_1 &= k_1^+ + k_1^-, \quad \omega_1 = -4(k_1^{+3} + k_1^{-3}), \quad \Omega_1 = \sqrt{3}(k_1^{+2} - k_1^{-2}),\\ k_2 &= k_2^+ + k_2^-, \quad \omega_2 = -4(k_2^{+3} + k_2^{-3}), \quad \Omega_2 = \sqrt{3}(k_2^{+2} - k_2^{-2}),\\ k_3 &= k_1^+ + k_2^-, \quad \omega_3 = -4(k_1^{+3} + k_2^{-3}), \quad \Omega_3 = \sqrt{3}(k_1^{+2} - k_2^{-2})\\ k_4 &= k_2^+ + k_1^-, \quad \omega_4 = -4(k_2^{+3} + k_1^{-3}), \quad \Omega_4 = \sqrt{3}(k_2^{+2} + k_1^{-2}), \end{aligned} \tag{7.71}$$

Then, substituting these parameterizations we find that

$$\alpha_{13} = 0 \;\Rightarrow\; F_5 = 0 \tag{7.72}$$

$$\alpha_{23} = 0 \;\Rightarrow\; F_6 = 0 \tag{7.73}$$

7. For ε^7 we have the equation

$$(D_x D_t + D_x^4 + D_y^2)(1 \cdot 2F_7 + F_1 \cdot 2F_6 + F_2 \cdot 2F_5 + F_3 \cdot 2F_4) = 0, \tag{7.74}$$

with the solution

$$F_7 = e^{\eta_4}, \tag{7.75}$$

where

$$\eta_4 = k_4 x + \Omega_4 y - \omega_4 t + \eta_4^0. \tag{7.76}$$

8. For ε^8 the system

$$(D_x D_t + D_x^4 + D_y^2)(1 \cdot 2F_8 + F_1 \cdot 2F_7 + F_2 \cdot 2F_6 + F_3 \cdot 2F_5 + F_4 \cdot F_4) = 0, \tag{7.77}$$

gives solution

$$F_8 = \alpha_{14} e^{\eta_1 + \eta_4}, \tag{7.78}$$

where

$$\alpha_{14} = -\frac{(k_1 - k_4)(\omega_1 - \omega_4) + (k_1 - k_4)^4 + (\Omega_1 - \Omega_4)^2}{(k_1 + k_4)(\omega_1 + \omega_4) + (k_1 + k_4)^4 + (\Omega_1 + \Omega_4)^2}. \tag{7.79}$$

After the parameterizations given above, we also get

$$\alpha_{14} = 0 \;\Rightarrow\; F_8 = 0 \tag{7.80}$$

9. For ε^9 we have equation

$$(D_x D_t + D_x^4 + D_y^2)(1 \cdot 2F_9 + F_1 \cdot 2F_8 + F_2 \cdot 2F_7 + F_3 \cdot 2F_6 + F_4 \cdot 2F_5) = 0, \tag{7.81}$$

with solution

$$F_8 = \alpha_{24} e^{\eta_2+\eta_4}, \tag{7.82}$$

where

$$\alpha_{24} = -\frac{(k_2-k_4)(\omega_2-\omega_4)+(k_2-k_4)^4+(\Omega_2-\Omega_4)^2}{(k_2+k_4)(\omega_2+\omega_4)+(k_2+k_4)^4+(\Omega_2+\Omega_4)^2}, \tag{7.83}$$

which also leads to

$$\alpha_{24} = 0 \;\Rightarrow\; F_8 = 0. \tag{7.84}$$

10. For ε^{10} equation

$$\begin{gathered}(D_xD_t + D_x^4 + D_y^2)(1\cdot 2F_{10} + F_1\cdot F_9 + F_2\cdot 2F_8 \\ +F_3\cdot 2F_7 + F_4\cdot 2F_6 + F_5\cdot F_5) = 0,\end{gathered} \tag{7.85}$$

is satisfied by solution

$$F_{10} = 0. \tag{7.86}$$

11. For ε^{11} we have the equation

$$\begin{gathered}(D_xD_t + D_x^4 + D_y^2)(1\cdot 2F_{11} + F_1\cdot 2F_{10} + F_2\cdot 2F_9 \\ +F_3\cdot 2F_8 + F_4\cdot 2F_7 + F_4\cdot 2F_7) = 0,\end{gathered} \tag{7.87}$$

and corresponding solution

$$F_{11} = \alpha_{34} e^{\eta_3+\eta_4}, \tag{7.88}$$

$$\alpha_{34} = -\frac{(k_3-k_4)(\omega_3-\omega_4)+(k_3-k_4)^4+(\Omega_3-\Omega_4)^2}{(k_3+k_4)(\omega_3+\omega_4)+(k_3+k_4)^4+(\Omega_3+\Omega_4)^2}. \tag{7.89}$$

When it is checked for higher order terms we find that

$$F_{12} = F_{13} = \ldots = 0. \tag{7.90}$$

Thus, we have degenerate four-soliton solution of equations (7.46)

$$F = 1 + e^{\eta_1} + e^{\eta_2} + e^{\eta_3} + e^{\eta_4} + \alpha_{12}e^{\eta_1+\eta_2} + \alpha_{34}e^{\eta_3+\eta_4}. \tag{7.91}$$

The above consideration shows that our two-soliton solution of KP-II corresponds to the degenerate four soliton solution in the canonical Hirota form (7.46). Moreover, it allows us to find new four virtual soliton resonance for KPII.

7.5 Resonance Interaction of Planar Solitons

In section 7.3 we constructed two soliton solution of the KPII equation. Choosing different values of parameters we find resonance character of our soliton interaction. For the next choice of parameters $k_1^+ = 2, k_1^- = 1, k_2^+ = 1, k_2^- = 0.3$, and vanishing value of the position shift constants, we obtained two soliton solution moving in the plane with constant velocity, with creation of the four, so called virtual solitons (solitons without asymptotic states at infinity [115]).

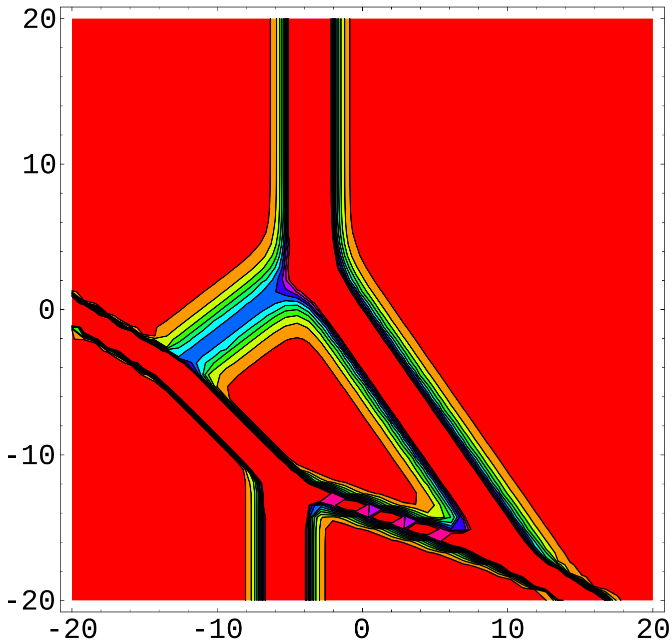

Figure 7.1: Resonance KP soliton dynamics

Fig. 7.1-7.5 illustrates that at the negative time, the four virtual solitons are decreasing in the size and at time zero collapse and disappear completely. Then, when time positively growing, they start to grow in the size.

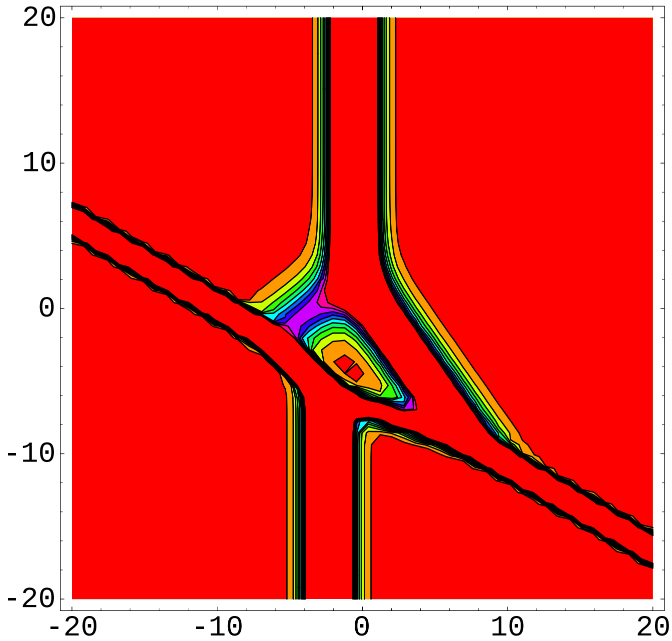

Figure 7.2: Resonance KP soliton dynamics

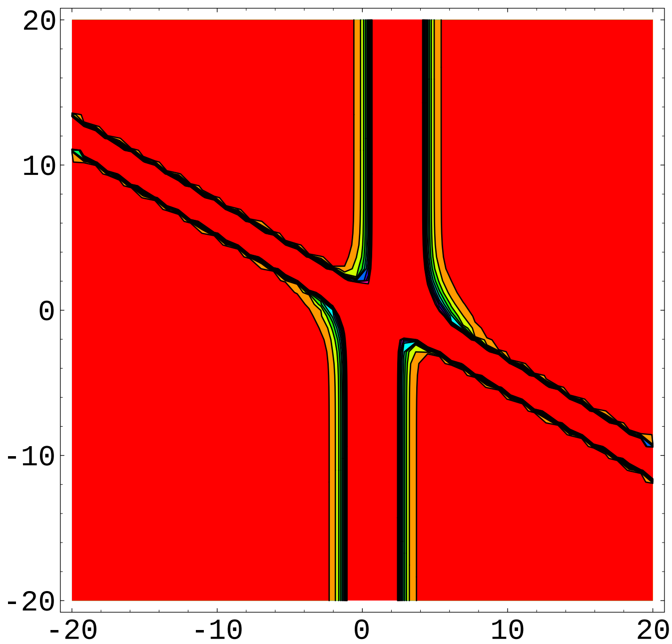

Figure 7.3: Resonance KP soliton dynamics

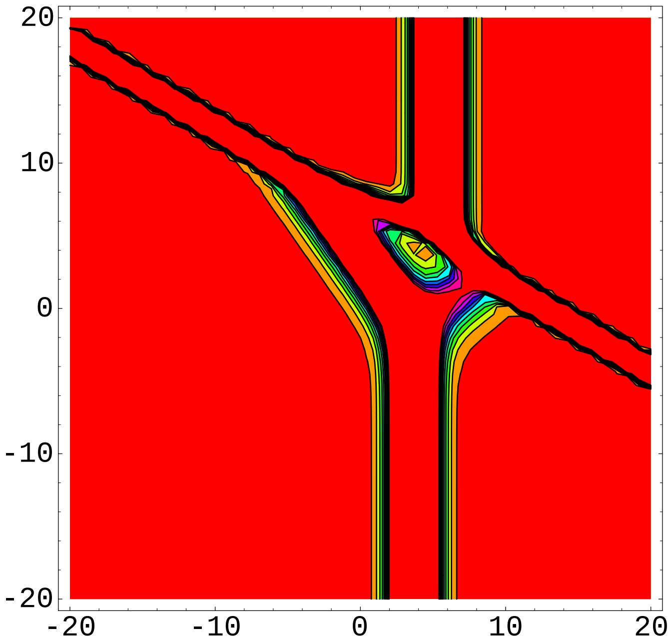

Figure 7.4: Resonance KP soliton dynamics

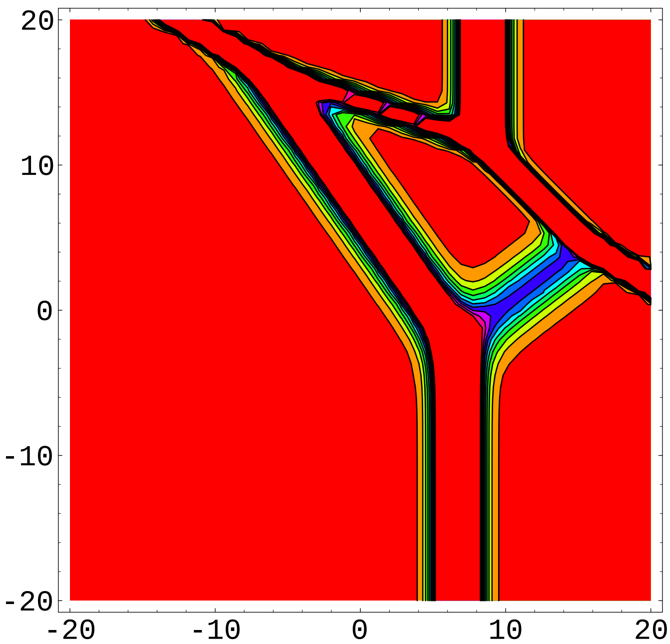

Figure 7.5: Resonance KP soliton dynamics

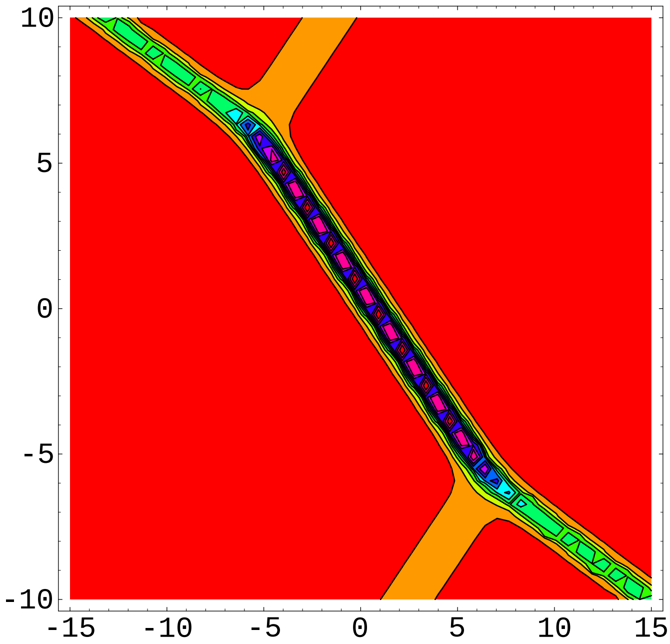

Figure 7.6: One Virtual Soliton Solution

On Fig 7.6 we show two soliton solution with parameters $k_1^+ = 2, k_1^- = 1, k_2^+ = 0.001, k_2^- = 0.001$ which includes only one virtual soliton. This virtual soliton can be considered as created from the pair of real solitons and is decaying into a pair of real solitons.

The resonance character of our planar soliton interactions is related with resonance nature of dissipatons considered in Chapter 6, section 6.3. It has been reported also in several systems like the Sawada-Kotera equation [118] , the Boussinesq [116] type equation, the KP equation [75]. But the four virtual soliton resonance does not seem to have been done for KPII [134] prior our work [135]. Experimentally, the soliton resonance of ion-acoustic solitons [119] has been observed.

In Fig. 7.7 we show two solitons observed in the ocean with similar to the KP structure.

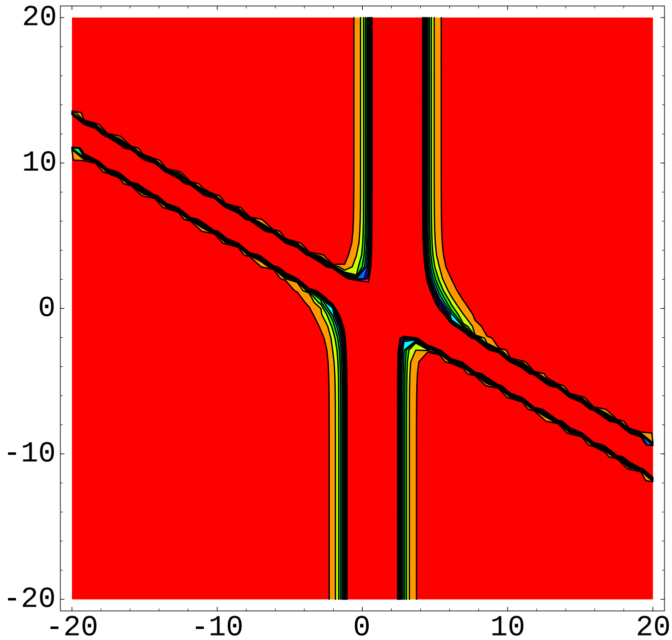

Figure 7.7: Solitons in the ocean

Chapter 8

CONCLUSIONS

In the last decades, it has been a very exciting time of developing the soliton theory, and quickly expanding wide field of applications to nonlinear phenomena of this abstract concept of integrable Hamiltonian systems with a specific structure of the phase space. Now it is difficult to find the subject of natural sciences where solitons and other non-perturbative approaches have not been applied yet. From microscale of the elementary particle physics and quantum theory until macroscale of the cosmology, solitons create a new paradigma of the nonlinear world, similar to the role of the harmonic oscillator in the linear world.

In the present thesis we considered some aspects of integrability in finite dimensional Hamiltonian systems, their relations with soliton equations and new type of resonance soliton dynamics. We found new hierarchy, mixing two famous soliton equations as the KdV and MKdV equations, corresponding recursion operator and soliton solution. The finite dimensional reduction of soliton equations for the stationary flows in the form of the Henon-Heiles system we extended with several terms and found corresponding separation of variables in the Hamilton-Jacobi formalism and the bi-Hamiltonian structure.

We constructed the Hirota bilinear representation for some systems of soliton equations with third order dispersion and one and two soliton solutions. We applied these bilinear representations to integrate 2+1 dimensional KdV model known as KPII, and found new resonance character of its soliton interactions. We hope that finite dimensional models considered in this thesis due to the exact solvability can help one to understand better the nature of transition from integrability to the chaos. The idea to use couple of equations from the AKNS hierarchy, which we explored in our study, can be applied also to multidimensional sytems with zero curvature structure. These type of systems are known as non-Maxwell gauge theories, or the Chern-Simons theories and have applications in many areas of physics and mathematics.

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APPENDIX A

HIROTA DERIVATIVES AND ITS PROPERTIES

In this Appendix we list some properties of the Hirota derivative operators D_t,D_x defined by equation

$$D_x^n(f \cdot g) = (\partial_{x_1} - \partial_{x_2})^n f(x_1)g(x_2)|_{x_2=x_1=x} \tag{A.1}$$

or in more general form

$$D_t^n D_x^m(f.g) = \left(\frac{\partial}{\partial t} - \frac{\partial}{\partial t'}\right)^n \left(\frac{\partial}{\partial x} - \frac{\partial}{\partial x'}\right)^n f(t,x)g(t',x')|_{t=t',x=x'}. \tag{A.2}$$

From the definition above we can find the general expression for n-th Hirota derivative

$$D_x^n(f.g) = (\partial_{x_1} - \partial_{x_2})^n f(x_1)g(x_2)|_{x_2=x_1=x} = \sum_{k=0}^{n} \left(\begin{array}{c} n \\ k \end{array}\right) \partial_{x_1}^k \partial_{x_2}^{(n-k)} f(x)g(x), \tag{A.3}$$

or

$$D_x^n(f.g) = \sum_{k=0}^{n} \left(\begin{array}{c} n \\ k \end{array}\right) f^{(n-k)}(x)g^{(k)}(x)(-1)^k. \tag{A.4}$$

For the first few derivatives we have explicitly

$$\begin{aligned} D_x(f.g) &= f'g - g'f \ , \\ D_x^2(f.g) &= f''g - 2f'g' + g''f \ , \\ D_x^3(f.g) &= f'''g - 3f''g' + 3g''f' - g'''f \ , \\ D_x^4(f.g) &= f^{(IV)}g - 4f'''g' + 6f''g'' - 4f'g''' + fg^{(IV)} \ , \\ &\ldots \end{aligned} \tag{A.5}$$

The following properties are easily seen from the definition

1. $D_x^m(f.1) = (\frac{\partial}{\partial x})^m f$

2. $D_x^m(f.g) = (-1)^m D_x^m(g.f)$

3. $D_x^m(f.f) = 0$ for odd m.

4. $D_x^2(f.f) = 2f''f - 2f'^2$

5. $D_x^m(f.g) = D_x^{m-1}(f_x.g - f.g_x)$

6. $D_x D_t(f.f) = 2D_x(f_t.f) = 2D_t(f_x.f)$ for even m.

7. $D_x^m(e^{p_1 x}.e^{p_2 x}) = (p_1 - p_2)^m e^{(p_1+p_2)x}$

8. $D_x^m(e^{\Omega_1 t + p_1 x}.e^{\Omega_2 t + p_2 x}) = (p_1 - p_2)^m e^{(\Omega_1+\Omega_2)t+(p_1+p_2)x}$

9. $D_t^n(e^{\Omega_1 t + p_1 x}.e^{\Omega_2 t + p_2 x}) = (\Omega_1 - \Omega_2)^n e^{(\Omega_1+\Omega_2)t+(p_1+p_2)x}$

10. Let $P(D_t, D_x)$ be a polynomial of D_t and D_x, we have
$P(D_t, D_x)(e^{\Omega_1 t + p_1 x}.e^{\Omega_2 t + p_2 x}) = P(\Omega_1 - \Omega_2, p_1 - p_2)e^{(\Omega_1+\Omega_2)t+(p_1+p_2)x}$

11. $e^{(\varepsilon D_x)}(f(x).g(x)) = \sum_{k=0}^{\infty} \frac{\varepsilon^k}{k!} D_x^n(f.g) = \sum_{k=0}^{\infty} \frac{\varepsilon^k}{k!} \sum_{n=0}^{k} \begin{pmatrix} n \\ k \end{pmatrix} (-1)^{k-n} f^{(n)}(x) g^{(k-n)}(x)$

$$= \sum_{n=0}^{\infty} \frac{f^{(n)}(x)}{n!} \varepsilon^n \sum_{m=0}^{\infty} \frac{g^{(m)}(x)}{m!} (-\varepsilon)^m$$

where k = m+n. As a result we get that this is equal to

$$= f(x+\varepsilon)g(x-\varepsilon)$$

12. $D_x(fg.h) = (\frac{\partial f}{\partial x})gh + fD_x(g.h)$

13. $D_x^2(fg.h) = (\frac{\partial^2 f}{\partial x^2})gh + 2(\frac{\partial f}{\partial x})D_x(g.h) + fD_x^2(g.h)$

14. $D_x^m((e^{px}f).(e^{px}g)) = e^{2px} D_x^m(f.g)$

The following formulas are useful for transforming nonlinear differential equations into bilinear forms.

15. $\frac{\partial}{\partial x}(\frac{g}{f}) = \frac{D_x(f.g)}{f^2}$

16. $\frac{\partial^2}{\partial x^2}(\frac{g}{f}) = \frac{D_x^2(g.f)}{f^2} - \frac{g}{f}\frac{D_x^2(g.f)}{f^2}$

17. $\frac{\partial^3}{\partial x^3}(\frac{g}{f}) = \frac{D_x^3(g.f)}{f^2} - 3[\frac{D_x^2(g.f)}{f^2}\frac{D_x^2(f.f)}{f^2}]$

18. $\frac{\partial^2}{\partial x^2}(logf) = \frac{D_x^2(f.f)}{2f^2}$

19. $\frac{\partial^4}{\partial x^4}(logf) = \frac{D_x^4(f.f)}{2f^2} - 6[\frac{D_x^2(f.f)}{2f^2}]^2$