FREE VIBRATION ANALYSIS OF CURVED BEAMS WITH VARIABLE RADII OF CURVATURE ON ELASTIC FOUNDATIONS

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ABSTRACT

FREE VIBRATION ANALYSIS OF CURVED BEAMS WITH VARIABLE RADII OF CURVATURE ON ELASTIC FOUNDATIONS

Free out of plane vibration characteristics of curved beams with variable curvature on elastic foundation are studied by Finite Difference Method (FDM) since the mathematical model of the present system based on the coupled differential eigenvalue problem with variable coefficients. Firstly, vibrations of beams on different elastic foundations are reviewed. Then, out of plane vibrations of curved beams on different elastic foundations are investigated. FDM is detailed for this study. To solve the coupled differential eigenvalue problem, FDM is examined with several computer programs developed in Mathematica. The effects of curvature of the curved beams and elastic foundation parameters on natural frequencies are investigated. The accuracy of the present results obtained from the developed program is evaluated by comparing with FEM results found from model created in ANSYS.

ÖZET

ELASTİK ZEMİNDEKİ DEĞİŞKEN EĞRİLİK YARIÇAPLI EĞRİ ÇUBUKLARIN SERBEST TİTREŞİM ANALİZİ

Elastik zemindeki değişken eğrilik yarıçaplı eğri çubukların düzlem dışı titreşim karakteristikleri, mevcut sistem değişken katsayılı bağlaşık diferansiyel özdeğer problemine bağlı olduğu için, Sonlu Farklar Yöntemi (SFY) ile çalışılmıştır. Öncelikle, farklı elastik zeminlerdeki çubukların titreşimleri gözden geçirilmiştir. Sonra, farklı elastik zeminlerdeki eğri çubukların düzlem dışı titreşimleri araştırılmıştır. SFY bu çalışma için detaylandırılmıştır. Bağlaşık diferansiyel özdeğer problemini çözmek için, Mathematica'da geliştirimiş çeşitli programlar sınanmıştır. Eğri çubukların eğriliklerinin ve elastik zemin parametrelerinin doğal frekanslara etkileri araştırılmıştır. Geliştirilen programdan elde edilen çözümlerin hassasiyeti ANSYS de oluşturulan modelden bulunan Sonlu Elemanlar Metodu sonuçları ile karşılaştırılarak değerlendirilmiştir.

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A	cross-sectional area of the beam
b	breadth of the beam
B(s)	angular displacement about z axis as function of s
c_1, c_2	Reissner coefficients
C_β	the ratio of breadth/depth
d	viscous damping coefficient
D_0	Hatenyi coefficent
E	modulus of elasticity
f	natural frequency
F_y	external force in y direction
G	shear modulus
G_0	Pasternak coefficient
h	depth of the beam
i	mass polar moment of inertia per unit length
I_{xx}	area moment of inertia of the cross-section about xx axis
k_o	Winkler parameter
k_1	Second parameter for foundation
J	torsional constant of cross-section
m	mass per unit length of the beam
M_x, M_z	bending and twisting moments
N_z	internal force along z axis
$p(y, t)$	reaction of the foundation
R_0	radius parameter of the catenary curve
s	arc length of the curve
s_L	length of the curve
T	kinetic energy
T_z	external twisting moment about z axis
x	co-ordinate axis
$v(s,t)$	displacement in y direction
V	elastic strain energy
V(s)	displacement in y direction as function of s

V_y	shear force in y direction
$y(s,t)$	lateral displacement
α, α_r	slope of the curve at any point and point r
$\beta(s,t)$	angular displacement about z axis
δ	variation operator
κ	curvature in zx plane
λ	eigenvalue
ρ_0	radius of curvature in zx plane
τ	twist about z axis
(`)	differentiation with respect to s
(˙)	differentiation with respect to time

CHAPTER 1

GENERAL INTRODUCTION

Curved beams are used in many engineering applications such as turbomachinery blades, stiffeners in aircraft structures, curved girder bridges. Depending on the functionality of the curved beam, they can be in the shape of a space curve or a plane curve. They may also have constant or variable curvature / cross section.

For dynamic analysis, it is known that either Bernoulli-Euler beam theory or Timoshenko beam theory is used depending on the geometry of the beam and the desired vibration modes. If the beam is thin and lower modes of vibration is satisfactory, Bernoulli-Euler beam theory is used. On the other hand, if the beams have large cross-sectional dimensions in comparison with their lengths and higher vibration modes are required, the Timoshenko beam theory should be used.

While the out-of-plane and the in-plane vibrations of curved beams are coupled for unsymmetrical cross-section, they are not coupled for symmetrical cross-section. Many studies have been performed for out-of-plane vibrations of the curved beams, but only a few have been focusing on the vibration problem of curved beam with variable curvature on elastic foundation. The literature survey on present problem is presented in the next paragraphs according to the subjects. The literature survey on elastic foundations is based on the lecture notes by Yardimoglu (2012).

The elastic foundation models for the vibration of beams on elastic foundations are given in the textbook written by Karnovsky (2000). The first elastic foundation model is the Winkler (1867) model. Winkler model is known as one-parameter model. Since this is very old and simple, it is extensively used. In Winkler model, foundation is considered as closely spaced unconnected linear elastic vertical springs, i.e. adjacent springs are uncoupled. To overcome this poorness, Filonenko-Borodich (1940, 1945) proposed a model connecting the vertical springs by a thin elastic tensioned membrane placed over the springs (Das 2011). Hatenyi (1946) presented exact solutions of straight beams on Winkler foundations. Pasternak (1954) modified Winkler model by introducing a second parameter regarding coupling effect of the linear elastic springs, also known as shear interactions. This model is known as two-parameter model. The Reissner model (1958) regarding deflections of plates on a viscoelastic foundation is the generalization of Pasternak model. Kerr (1964) offered three-parameter model by placing a spring bed over Pasternak foundation. Vlasov and Leontiev (1966) considered the shear interactions in their foundation model. Each model may have also viscoelastic properties which can be provided by adding a viscous damping term.

Carefully selected samples of the literature on vibration of beams on different elastic foundations are presented as follows: Exact analytical vibration characteristics of the aforementioned subject are covered in some textbooks (Hatenyi 1946, Meirovitch 1967, Volterra and Gaines 1971). Yihua et al. (2009) analyzed the vibrations of Timoshenko beams on a nonlinear elastic foundation. A weak form Quadrature Element Method is used for the vibration analysis. The nonlinear foundation parameter stiffness is assumed as:

$$k_f = \alpha + \beta v^2$$

Eisenberger (1994) presented the exact vibration frequencies of beams resting on variable one- and two-parameter elastic foundation. His solution is based on dynamic stiffness matrix for the member including the effects of the variable foundation stiffness. Stiffness of the two-parameter elastic foundation is expressed as follows:

$$k_f(x) = -\frac{\partial}{\partial x}\left[k_1(x)\, v(x) \frac{\partial v(x)}{\partial x} \right] + k(x)\, v(x)$$

Avramidis and Morfidis (2006) formulated and analytically solved the bending of a Timoshenko beam resting on a Kerr-type three-parameter elastic foundation.

Out of plane vibration of curved beams on different elastic foundations are studied by following researchers: Rao (1971) presented three-dimensional vibrations of a ring on elastic foundation. Stiffness parameters of the elastic foundation are based on the bending-torsion motions of the curved beam. For this reason, the first parameter is related with the bending motion and the second parameter is about torsional motion. Panayotunakos and Theocaris (1980) made an analytical treatment for the determination of the natural frequencies of a circular Timoshenko beam on a Winkler foundation. Wang and Brannen (1982) studied the effects of Winkler-Pasternak foundations upon natural frequencies of finite circular curved beams vibrating out of their initial plane of curvature. Issa (1988) and Issa et al. (1990) examined the natural frequencies of curved Timoshenko beams on Winkler- and Pasternak-type foundations, respectively. Recently, Kim et al. (2007) presented the dynamic stiffness matrix for the spatially coupled free vibration analysis of thin-walled curved beams on Winkler- and Pasternak-type foundations. They used the power series method in their solution.

In this study, the effects of the variable curvature of the curved beams and elastic foundation parameters on natural frequencies are investigated. FDM is used to reduce to differential eigenvalue problem to discrete eigenvalue problem to solve numerically. A computer program is developed in Mathematica to determine the eigenvalues depending on cross-sectional and elastic foundation parameters. The accuracy of the present results obtained from the developed program is evaluated by comparing with FEM results found from model created in ANSYS.

CHAPTER 2

DIFFERENTIAL EQUATIONS AND THEIR SOLUTIONS

2.1. Introduction

This chapter is presented to describe the problem physically, to introduce the geometry of the curved beam which is the most critical object in this thesis. After presenting the aforementioned information, equation of motion of the curved beam with elastic foundation is obtained by using Newtonian method and Hamilton's principle. The benefit of the Hamilton's principle in this step is to obtain the boundary conditions without any difficulty that happens generally in Newtonian method for complicated problems such as the problem in this study. Therefore, boundary conditions are listed for different types of end conditions: clamped, pinned and free.

Elastic foundation models are summarized by using the very well written textbook written by Karnovsky and Lebed (2000) to provide the fundamental concepts to the readers.

The differential eigenvalue problem is solved numerically by using Finite Difference Method which is detailed in this chapter for vibration analysis.

2.2. Description of the Problem

In this study, the problem is the out-of-plane free vibrations of curved beam with variable curvature. A curved beam in the shape of catenary curve is selected. The boundary conditions of the curved beam are considered as clamped-free. The material of the curved beam is assumed as isotropic. Winkler and Pasternak foundation models are used for elastic foundations.

The cases of variable curvature and variable foundation parameters are investigated in order to find out the effects of these parameters on natural frequencies.

2.3. Geometry of Curved Beam

A catenary curve, its parameters shown in Figure 2.1 and equations are taken from Yardimoglu (2010).

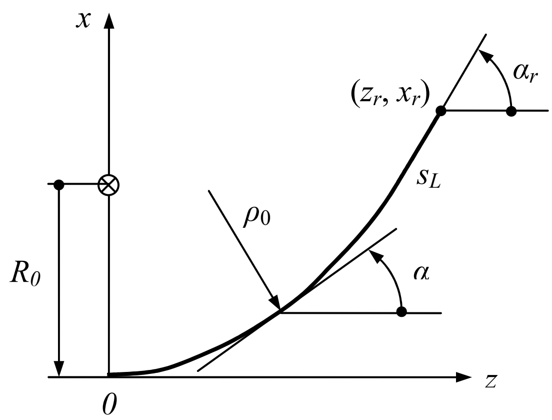


Figure 2.1. Parameters of catenary beam

The function of the catenary curve is given as follows:

$$x(z) = R_0[\cosh(z / R_0) - 1] \tag{2.1}$$

The slope α is obtained by differentiation of Equation 2.1 with respect to z as

$$\tan\alpha = dx(z) / dz = \sinh(z / R_0) \tag{2.2}$$

The tip co-ordinates of the curved beam (z_r, x_r) can be found as

$$z_r = R_0 \, arc\sinh(\tan\alpha_r) \tag{2.3}$$

$$x_r = R_0 \, (1 / \cos\alpha_r - 1) \tag{2.4}$$

Since the arc length s from origin to any point (z, x) on the curve is

$$s(z) = \int_0^s \sqrt{(1 + (dx(z) / dz)^2} \, dz \tag{2.5}$$

The following relationship between s and α is obtained:

$$s(\alpha) = R_0 \tan \alpha \tag{2.6}$$

Similarly, the arc length s_L from origin to point (z_r, x_r) can be expressed as

$$s_L = R_0 \tan \alpha_r \tag{2.7}$$

Radius of curvature at abscissa is found as

$$\rho_0(z) = \frac{\left[1 + (dx(z)/dz)^2\right]^{3/2}}{d^2x(z)/dz^2} = R_0 \cosh^2(z/R_0) \tag{2.8}$$

Eliminating the variable z in Equation 2.8 by using Equation 2.2, radius of curvature can be written in terms of α as follows:

$$\rho_0(\alpha) = R_0 / \cos^2 \alpha \tag{2.9}$$

Now, cos α can be expressed in terms of s by using Equation 2.6 as

$$\cos\alpha = R_0 / \sqrt{R_0^2 + s^2} \tag{2.10}$$

Therefore, radius of curvature can also be written in terms of s as follows:

$$\rho_0(s) = R_0 + s^2 / R_0 \tag{2.11}$$

2.4. Derivation of the Equations of Motions

In this section, derivations of equations of motion are presented by Newtonian Method and Hamilton's Principle. The advantages of Hamilton's principle are stated. Then, physical interpretations for boundary conditions are listed.

Newtonian Method: The following two vectorial equations are used:

$$\sum_i \vec{F}_i = m\vec{a} \tag{2.12}$$

$$\sum_i \vec{M}_i = I\vec{\alpha} \tag{2.13}$$

In this method, it is needed to neglect small quantities of higher order terms in order to obtain linear differential equations. The weakness of this method is expressing the boundary conditions since they are based on the understanding of the internal forces and moments.

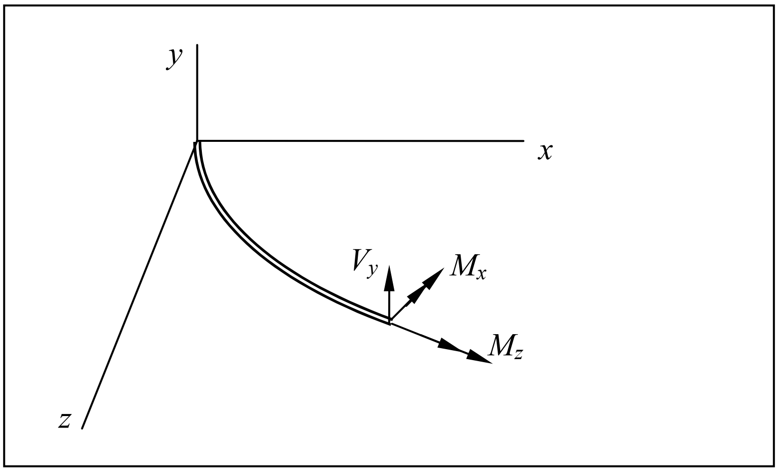


Figure 2.2. A curved beam with internal forces and moments

By using Equations 2.12 and 2.13, force and moment equilibrium equations of the curved beam can be obtained as follows (Love 1944):

$$\frac{dV_y}{ds} + F_y = m\ddot{v} \tag{2.14.a}$$

$$\frac{dM_x}{ds} + \frac{M_z}{\rho_0} - V_y = 0 \tag{2.14.b}$$

$$\frac{dM_z}{ds} - \frac{M_x}{\rho_0} + T_z = i\ddot{\beta} \tag{2.14.c}$$

where $m = \rho A$ is mass per unit length, (2.15.a)

$i = \rho J$ is mass polar moment inertia of unit length. (2.15.b)

In Equations 2.14.a, c $v(s,t)$ and $\beta(s,t)$ are transverse and angular displacements of the curved beam, respectively.

It should be noted that external force in y direction F_y and external twisting moment about z axis T_z in Equations 2.14.a-c can be treated as elastic foundation effects. A and J in Equations 2.15.a, b are cross-sectional area and torsional constant of cross-section, respectively. Detailed information on torsional constant can be found in textbooks about Mechanics of Solids (Popov and Balan 1998).

Bending and twisting moments in Equations 2.14.b and 2.14.c are given as

$$M_x = EI_{xx}\,\kappa \tag{2.16.a}$$

$$M_z = GJ\,\tau \tag{2.16.b}$$

where

$$\kappa = (\frac{\beta}{\rho_0} - \frac{\partial^2 v}{\partial s^2}) \tag{2.17.a}$$

and

$$\tau = (\frac{d\beta}{ds} + \frac{1}{\rho_0}\frac{\partial v}{\partial s}) \tag{2.17.b}$$

Hamilton's Method: This method uses the kinetic energy T and the potential energy V to derive the equation of motion and associated boundary conditions. Hamilton's principle can be stated as follows; "*Of all possible time histories of displacement states that satisfy the compatibility equations and the constraints or the kinematic boundary conditions and that also satisfy the conditions at initial and final times t_1 and t_2, the history to the actual solution makes the Langrangian a minimum*" (Meirovitch 1967). This principle is defined mathematically as follows:

$$\delta \int_{t_1}^{t_2} (T - V)\,dt = 0 \tag{2.18}$$

Kinetic and elastic strain energies of present problem are given as follows;

$$T=\frac{1}{2}\int_0^{S_L}(m\dot{v}^2+i\,\dot{\beta}^2)\,ds \tag{2.19.a}$$

$$V=\frac{1}{2}\int_0^{S_L}(M_x\,\kappa+M_z\,\tau-F_y\,v-T_z\,\beta)\,ds \tag{2.19.b}$$

Substituting Equations 2.15.a,b into Equation 2.19.a and Equations 2.16.a,b into Equation 2.19.b, kinetic and elastic strain energies become

$$T=\frac{1}{2}\int_0^{S_L}(\rho\,A\dot{v}^2+\rho\,J\,\dot{\beta}^2)\,ds \tag{2.20.a}$$

$$V=\frac{1}{2}\int_0^{S_L}(EI_{xx}\,\kappa^2+GJ\,\tau^{\,2}-F_y\,v-T_z\,\beta)\,ds \tag{2.20.b}$$

Using the Equations 2.20.a,b along with Equations 2.17.a,b in Equation 2.18, governing differential equations for vibrations of curved beams having variable cross-section and associated boundary conditions can be obtained as follows:

$$\begin{aligned}&\frac{\partial^2}{\partial s^2}\left[E\,I_x\left(\frac{\beta}{\rho_0(s)}-\frac{\partial^2 v}{\partial s^2}\right)\right]+\\&\frac{\partial}{\partial s}\left[\frac{G\,J}{\rho_0(s)}\left(\frac{\partial\beta}{\partial s}+\frac{1}{\rho_0(s)}\frac{\partial v}{\partial s}\right)\right]=\rho\,A\,\ddot{v}-F_y\end{aligned} \tag{2.21.a}$$

$$\begin{aligned}&\frac{E\,I_x}{\rho_0(s)}\left(\frac{\partial^2 v}{\partial s^2}-\frac{\beta}{\rho_0(s)}\right)+\\&\frac{\partial}{\partial s}\left[G\,J\left(\frac{\partial\beta}{\partial s}+\frac{1}{\rho_0(s)}\frac{\partial v}{\partial s}\right)\right]=\rho\,J\ddot{\beta}-T_z\end{aligned} \tag{2.21.b}$$

$$\frac{E\,I_x}{\rho_0(s)}\left(\frac{\partial^2 v}{\partial s^2}-\frac{\beta}{\rho_0(s)}\right)\delta v'\Bigg|_0^{S_L}=0 \tag{2.22.a}$$

$$G\,J\left(\frac{\partial\beta}{\partial s}+\frac{1}{\rho_0(s)}\frac{\partial v}{\partial s}\right)\delta\beta\,\Bigg|_0^{S_L}=0 \tag{2.22.b}$$

$$\left\{\begin{array}{l}\dfrac{\partial}{\partial s}\left[E\,I_x\left(\dfrac{\beta}{\rho_0(s)}-\dfrac{\partial^2 v}{\partial s^2}\right)\right]+\\ \dfrac{G\,J}{\rho_0(s)}\left(\dfrac{\partial\beta}{\partial s}+\dfrac{1}{\rho_0(s)}\dfrac{\partial v}{\partial s}\right)\end{array}\right\}\delta v\,\Bigg|_0^{S_L}=0 \qquad (2.22.c)$$

In Equation 2.22.a, prime represents the differentiation with respect to *s*.

Physical interpretations for boundary conditions corresponding to Equations 2.22.a-c are as follows, respectively:

a) Either bending moment is zero (pinned/free), or slope is zero (fixed),

b) Either twisting moment is zero (pinned/free), or rotation is zero (fixed),

c) Either shear force is zero (free), or displacement is zero (pinned/fixed).

If the boundary conditions of the curved beam are homogeneous as in Equations 2.22.a-c, the solutions of Equations 2.21.a,b are assumed as

$$v(s,t)=V(s)\,T(t) \qquad (2.23.a)$$

$$\beta(s,t)=B(s)\,T(t) \qquad (2.23.b)$$

where $V(s)$ and $B(s)$ are linear and angular displacements as function of *s*, respectively. Time dependent function can be chosen as $T(t)=\exp(i\omega t)$ in which ω is the circular natural frequency of the harmonic vibrations. Thus, Equations 2.21.a,b are reduced to following coupled differential eigenvalue problem in terms of $V(s)$ and $B(s)$:

$$\begin{aligned}&\frac{d^2}{ds^2}\left[E\,I_x\left(\frac{B(s)}{\rho_0(s)}-\frac{d^2V(s)}{ds^2}\right)\right]+\\&\frac{d}{ds}\left[\frac{G\,J}{\rho_0(s)}\left(\frac{dB(s)}{ds}+\frac{1}{\rho_0(s)}\frac{dV(s)}{ds}\right)\right]\\&+F_y=-\omega^2\rho\,A\,V(s)\end{aligned} \qquad (2.24.a)$$

$$\begin{aligned}&\frac{E\,I_x}{\rho_0(s)}\left(\frac{d^2V(s)}{ds^2}-\frac{B(s)}{\rho_0(s)}\right)\\&+\frac{d}{ds}\left[G\,J\left(\frac{dB(s)}{ds}+\frac{1}{\rho_0(s)}\frac{dV(s)}{ds}\right)\right]\\&+T_z=-\omega^2\rho\,J\,B(s)\end{aligned} \qquad (2.24.b)$$

The boundary conditions as functions of s can be obtained easily by replacing d instead of ∂ in Equations 2.22.a-c since there is no time dependent term in BCs.

2.5. Elastic Foundation Models

The differential equation of the vibration of a beam on an elastic foundation may be written with operator notation as

$$L[y(s,t)] + p(y,t) = 0 \tag{2.25}$$

where $L[y(s,t)]$ is an operaor acting on the lateral displacement $y(s,t)$ and $p(y,t)$ is the reaction of the foundation. Several function for $p(y,t)$ are presented in Table 2.1.

Table 2.1 Elastic foundation models

Foundation Model	***p*(*y*,*t*) or DE to find *p*(*y*,*t*)**
Winkler foundation (Winkler 1867)	$p = k_0 y$
Viscoelastic Winkler foundation	$p = k_0 y + d\dfrac{\partial y}{\partial t}$
Hetenyi foundation (Hetenyi 1946)	$p = k_0 y + D_0 \nabla^2 \nabla^2 y$
Viscoelastic Hetenyi foundation	$p = k_0 y + D_0 \nabla^2 \nabla^2 y + d\dfrac{\partial y}{\partial t}$
Pasternak foundation (Pasternak 1954)	$p = k_0 y - G_0 \nabla^2 y$
Viscoelastic Pasternak foundation	$p = k_0 y - G_0 \nabla^2 y + d\dfrac{\partial y}{\partial t}$
Generalized foundation (Pasternak 1954)	$p = k_0 y, \quad m = k_1 \dfrac{dy}{dn}$
Reissner foundation (Reissner 1958)	$c_1 y - c_2 \nabla^2 y = p - \dfrac{c_2}{c_1} \nabla^2 p$

The notations used in Table 2.1 are listed in the "List of Symbols".

2.6. Finite Difference Method (FDM)

The Finite Difference Method is a numerical method for solution of differential equations by using approximate derivatives (Hildebrand 1987).

Since the differential equations having variable coefficients are analytically unsolvable except for equations having special combinations of coefficients, the Finite Difference Method can be used.

Figure 2.3 shows a one dimensional domain divided by six dub domains. Each point called as grid is represented by numbers. Roman numerals are used to represent the sub domains.

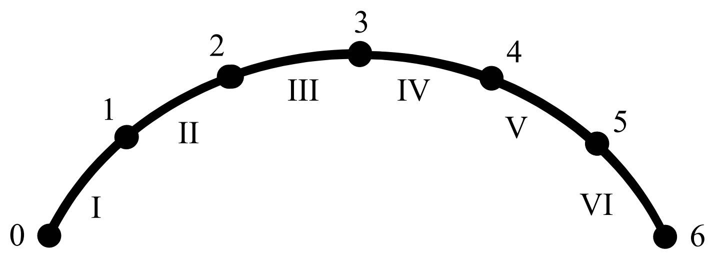


Figure 2.3. A curved domain divided into six sub domains for approximation

In the Finite Difference Method, the derivatives of dependent variables in the differential equations are replaced by the finite difference approximations at mesh points and these equations are enforced at each mesh points. Therefore, n simultaneous algebraic equations are obtained.

The finite differences can be seen in three forms as follows: forward, backward, and central differences. However, the central difference yields a more accurate approximation. Truncation error depends on the approximation order in FDM. Also, grid spacing is very critical parameter in this method. It should be determine very carefully.

In this study, central difference approximations which are detailed in Table 2.2 are used.

Table 2.2. Finite difference equations

Term	Central Difference Expressions for required derivatives of $V(s)$ and $B(s)$
$\frac{dV}{ds}$	$\frac{V(i+1)-V(i-1)}{2h}$
$\frac{d^2V}{ds^2}$	$\frac{V(i+1)-2V(i)+V(i-1)}{h^2}$
$\frac{d^3V}{ds^3}$	$\frac{V(i+2)-2V(i+1)+2V(i-1)-V(i-2)}{2h^3}$
$\frac{d^4V}{ds^4}$	$\frac{V(i+2)-4V(i+1)+6V(i)-4V(i-1)+V(i-2)}{h^4}$
$\frac{dB}{ds}$	$\frac{B(i+1)-B(i-1)}{2h}$
$\frac{d^2B}{ds^2}$	$\frac{B(i+1)-2B(i)+B(i-1)}{h^2}$

2.7. Natural Frequencies by FDM

The differential eigenvalue problem given by Equation 2.24.a,b and associated boundary conditions are reduced to discrete eigenvalue problem which can be written as follows:

$$[A]\{X\}=\lambda\,[B]\,\{X\} \tag{2.26}$$

Solutions of the generalized eigenvalue problem given by Equation 2.26 can be obtained by any mathematical software such as Matlab, Mathematica or Maple.

CHAPTER 3

NUMERICAL RESULTS AND DISCUSSION

3.1. Introduction

In this chapter, numerical applications are carried out for curved beams in the shape of catenary with different parameters. The beam is in *z*-*x* plane as shown in Figures 2.1 or 2.2 and has clamped-free boundary conditions. The main numerical data are as follows: b=h=0.01 m, E=200 GPa, G=80 GPa, ρ=7850 kg/m^3, s_L=0.12 m. Other data are given in table and figure legends. The numerical results are found and compared with the FEM results obtained from the model created in ANSYS. The results given in tabular and graphical forms are discussed.

3.2. Verifications and Selecting the Number of Nodes for FDM

In this section, as first step, in order to determine the proper number of node in FDM, the first natural frequencies are found for different number of node and shown in Figure 3.1.

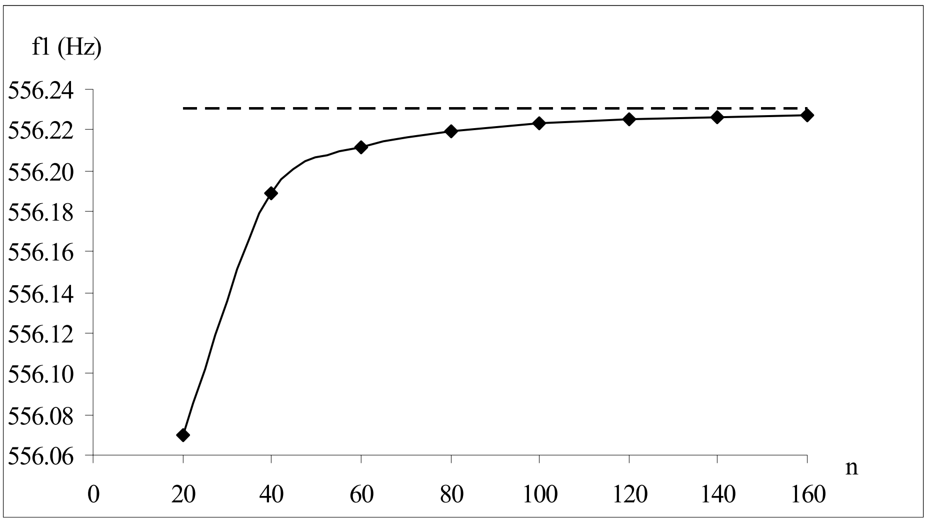


Figure 3.1. Convergence of first natural frequencies for R_0=50 mm

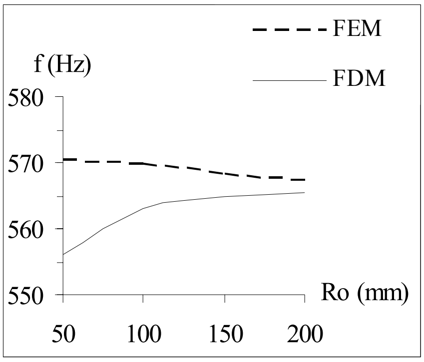
Considering the tendancy of the curve shown in Figure 3.1, the proper number of node for FDM is selected as 100 for all calculations.

To evaluate the results of FDM for the case of variable curvature, four different models depending on R_0 are created in ANSYS by using 30 Beam44 elements with consistent mass matrix option. The proper number of finite element is determined after observing the convergences.

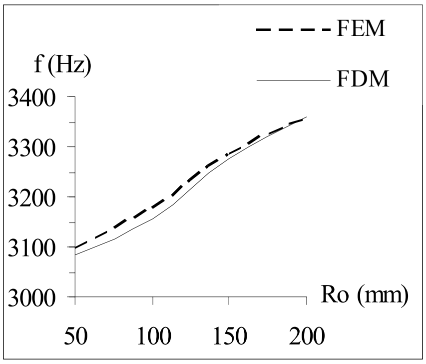
The natural frequencies found for different R_0 by FDM and Finite Element Models are tabulated in Table 3.1 and plotted in Figures 3.2.a-d and Figures 3.3.a-d as natural ferqunecies versus R_0. It can be seen from these figures that FDM and FEM have very similar results except first mode.

Table 3.1. Comparison of natural frequencies found for different R_0 by FDM with FEM results

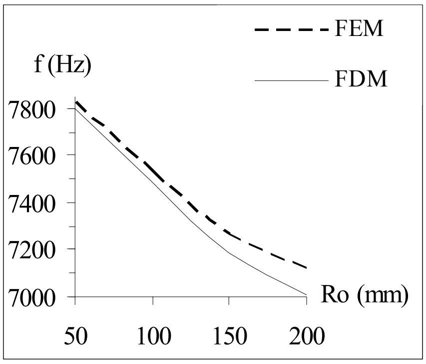
		R_0=50 mm	R_0=100 mm	R_0=150 mm	R_0=200 mm
f_1 (Hz)	FDM	556.2	562.9	564.7	565.4
	FEM	570.2	569.7	568.2	567.3
f_2 (Hz)	FDM	3084.5	3157.9	3275.8	3358.3
	FEM	3095.6	3175.2	3284.5	3357.4
f_3 (Hz)	FDM	7802.6	7481.7	7186.7	7009.3
	FEM	7821.6	7523.5	7267.1	7118.6
f_4 (Hz)	FDM	9556.1	9810.8	9891.1	9913.9
	FEM	9386.2	9621.1	9694.0	9714.5
f_5 (Hz)	FDM	18942	19161	19228	19275
	FEM	18291	18492	18569	18614
f_6 (Hz)	FDM	21119	20547	20330	20217
	FEM	21486	20989	20789	20690
f_7 (Hz)	FDM	31422	31799	31920	31982
	FEM	29736	30013	30116	30166
f_8 (Hz)	FDM	34217	33657	33486	33404
	FEM	35027	34584	34433	34363



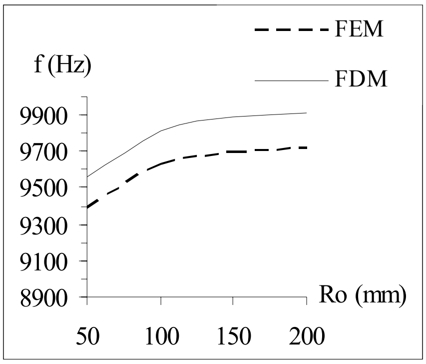
a) First frequency



b) Second frequency

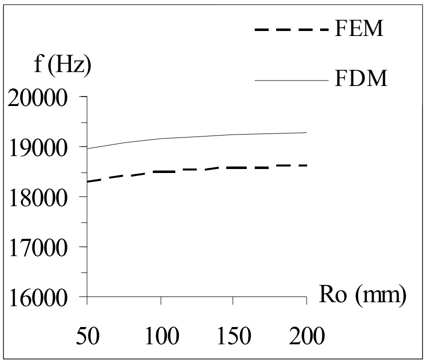


c) Third frequency

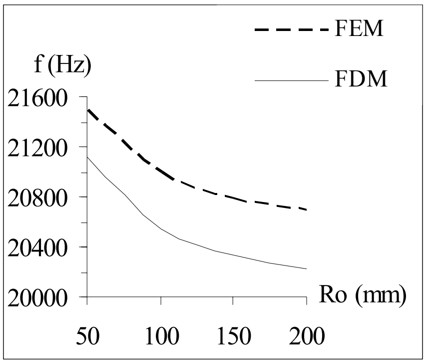


d) Fourth frequency

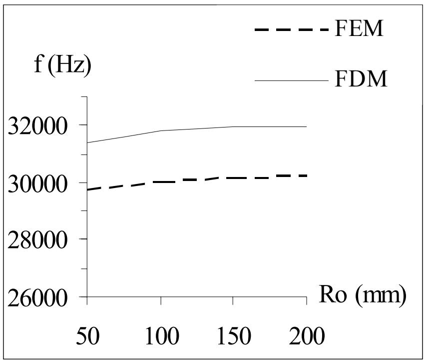
Figure 3.2. Comparisons of natural frequencies found for different R_0 by FDM with FEM results (f_1-f_4).



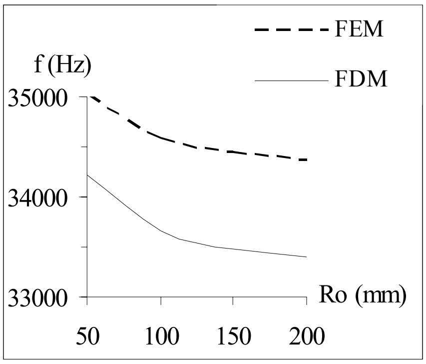
a) Fifth frequency



b) Sixth frequency



c) Seventh frequency



d) Eighth frequency

Figure 3.3. Comparisons of natural frequencies found for different R_0 by FDM with FEM results (f_5-f_8).

3.3. Applications for Curved Beams on Elastic Foundations

Effects of curvature and foundation parameters on natural frequencies are studied and the results are given in Table 3.2-11. Winkler foundation is considered in Tables 3.2-6. Second foundation parameter, which is called Pasternak parameter, k_1=2000 N is taken in other tables.

Table 3.2. Effects of parameter R_0 on natural frequencies for k_0=20 MN/m^2/m

	R_0=100 mm	R_0=200 mm
f_1 (Hz)	568.557	571.074
f_2 (Hz)	3158.91	3359.2
f_3 (Hz)	7481.77	7009.28
f_4 (Hz)	9811.1	9914.23
f_5 (Hz)	19160.6	19274.7
f_6 (Hz)	20546.6	20217
f_7 (Hz)	31798.8	31982.2
f_8 (Hz)	33656.5	33403.8

Table 3.3. Effects of parameter R_0 on natural frequencies for k_0=50 MN/m^2/m

	R_0=100 mm	R_0=200 mm
f_1 (Hz)	577.006	579.487
f_2 (Hz)	3160.37	3360.59
f_3 (Hz)	7481.84	7009.32
f_4 (Hz)	9811.56	9914.7
f_5 (Hz)	19160.9	19274.9
f_6 (Hz)	20546.6	20217
f_7 (Hz)	31798.9	31982.3
f_8 (Hz)	33656.5	33403.8

Table 3.4. Effects of parameter R_0 on natural frequencies for k_0=100 MN/m^2/m

	R_0=100 mm	R_0=200 mm
f_1 (Hz)	590.82	593.244
f_2 (Hz)	3162.8	3362.92
f_3 (Hz)	7481.95	7009.39
f_4 (Hz)	9812.33	9915.49
f_5 (Hz)	19161.3	19275.3
f_6 (Hz)	20546.6	20217.1
f_7 (Hz)	31799.2	31982.6
f_8 (Hz)	33656.6	33403.8

Table 3.5. Effects of parameter R_0 on natural frequencies for k_0=150 MN/m^2/m

	R_0=100 mm	R_0=200 mm
f_1 (Hz)	604.318	606.689
f_2 (Hz)	3165.24	3365.25
f_3 (Hz)	7482.06	7009.46
f_4 (Hz)	9813.10	9916.27
f_5 (Hz)	19161.7	19275.7
f_6 (Hz)	20546.6	20217.1
f_7 (Hz)	31799.4	31982.8
f_8 (Hz)	33656.6	33403.8

Table 3.6. Effects of parameter R_0 on natural frequencies for k_0=200 MN/m^2/m

	R_0=100 mm	R_0=200 mm
f_1 (Hz)	617.521	619.843
f_2 (Hz)	3167.67	3367.58
f_3 (Hz)	7482.17	7009.53
f_4 (Hz)	9813.87	9917.06
f_5 (Hz)	19162.1	19276.1
f_6 (Hz)	20546.6	20217.1
f_7 (Hz)	31799.7	31983.1
f_8 (Hz)	33656.6	33403.8

It is clear from Tables 3.2-6 that first, second, fourth, fifth and seventh natural frequencies increase when R_0 is increased from 100 mm to 200 mm. Also, it can be seen from the same tables that when k_0 is increased from 20 MN/m^2/m to 200 MN/m^2/m, only the first and second natural frequencies increase significantly, but other natural frequencies higher than second is changed in decimal places.

Table 3.7. Effects of parameter R_0 on natural frequencies for k_0=20 MN/m^2/m

	R_0=100 mm	R_0=200 mm
f_1 (Hz)	565.524	567.92
f_2 (Hz)	3167.27	3367.34
f_3 (Hz)	7482.89	7009.97
f_4 (Hz)	9821.02	9924.33
f_5 (Hz)	19171.7	19285.0
f_6 (Hz)	20547.0	20218.1
f_7 (Hz)	31810.6	31993.9
f_8 (Hz)	33656.7	33404.0

Table 3.8. Effects of parameter R_0 on natural frequencies for k_0=50 MN/m^2/m

	R_0=100 mm	R_0=200 mm
f_1 (Hz)	574.018	576.379
f_2 (Hz)	3168.73	3368.74
f_3 (Hz)	7482.96	7010.02
f_4 (Hz)	9821.48	9924.8
f_5 (Hz)	19171.9	19285.3
f_6 (Hz)	20547.0	20218.1
f_7 (Hz)	31810.7	31994.0
f_8 (Hz)	33656.7	33404.0

Table 3.9. Effects of parameter R_0 on natural frequencies for k_0=100 MN/m^2/m

	R_0=100 mm	R_0=200 mm
f_1 (Hz)	587.902	590.209
f_2 (Hz)	3171.16	3371.06
f_3 (Hz)	7483.07	7010.08
f_4 (Hz)	9822.25	9925.59
f_5 (Hz)	19172.3	19285.6
f_6 (Hz)	20547.0	20218.1
f_7 (Hz)	31811.0	31994.3
f_8 (Hz)	33656.7	33404.0

Table 3.10. Effects of parameter R_0 on natural frequencies for k_0=150 MN/m^2/m

	R_0=100 mm	R_0=200 mm
f_1 (Hz)	601.465	603.721
f_2 (Hz)	3173.58	3373.38
f_3 (Hz)	7483.18	7010.15
f_4 (Hz)	9823.01	9926.37
f_5 (Hz)	19172.7	19286.0
f_6 (Hz)	20547.0	20218.2
f_7 (Hz)	31811.2	31994.5
f_8 (Hz)	33656.7	33404.0

Table 3.11. Effects of parameter R_0 on natural frequencies for k_0=200 MN/m^2/m

	R_0=100 mm	R_0=200 mm
f_1 (Hz)	614.730	616.938
f_2 (Hz)	3176.01	3375.7
f_3 (Hz)	7483.29	7010.22
f_4 (Hz)	9823.78	9927.16
f_5 (Hz)	19173.1	19286.4
f_6 (Hz)	20547.0	20218.2
f_7 (Hz)	31811.5	31994.8
f_8 (Hz)	33656.7	33404.0

Similarly, it is clear from Tables 3.7-11 which are given for k_1=2000 N that first, second, fourth, fifth and seventh natural frequencies increase when R_0 is increased from 100 mm to 200 mm. Also, it can be seen from the same tables that when k_0 is increased from 20 MN/m^2/m to 200 MN/m^2/m, only the first and second natural frequencies increase significantly, but other natural frequencies higher than second is changed in decimal places.

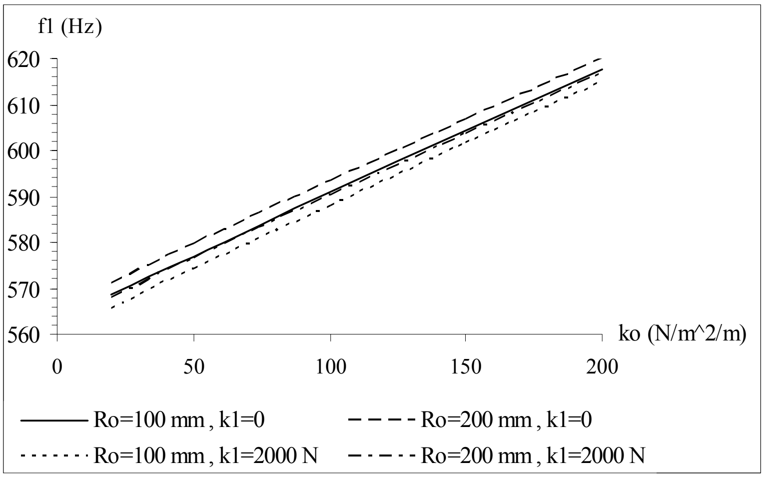


Figure 3.4. First natural frequencies

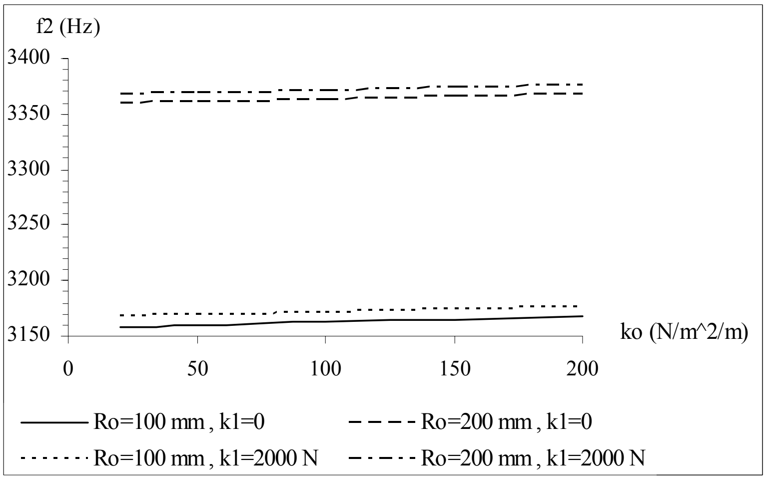


Figure 3.5. Second natural frequencies

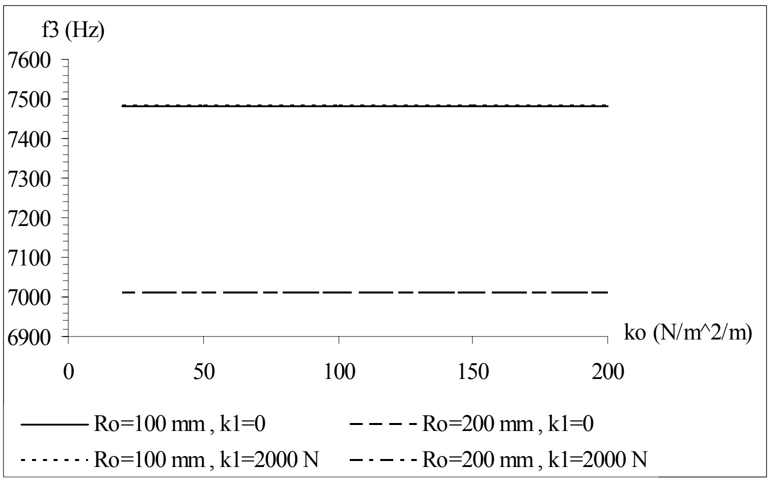


Figure 3.6. Third natural frequencies

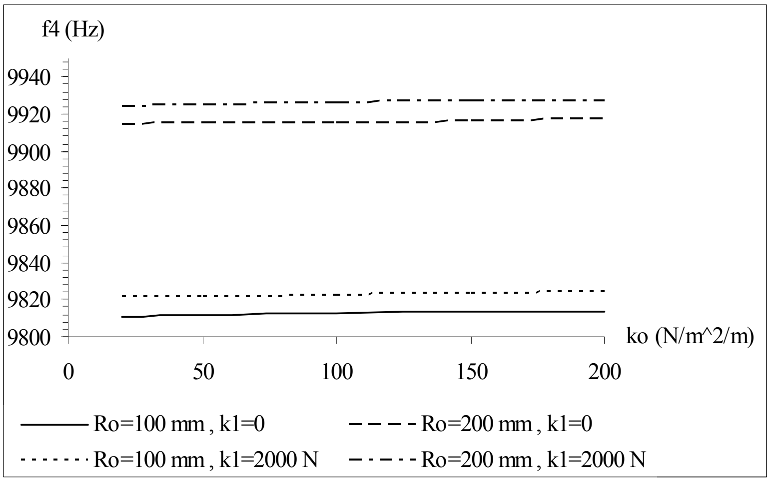


Figure 3.7. Fourth natural frequencies

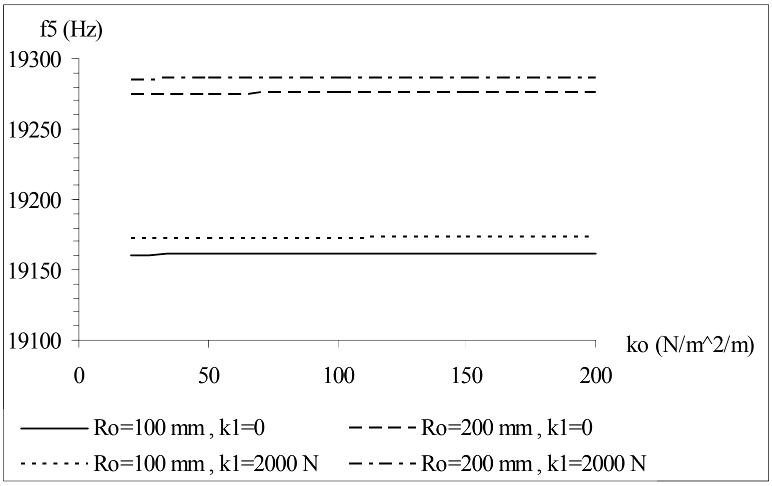


Figure 3.8. Fifth natural frequencies

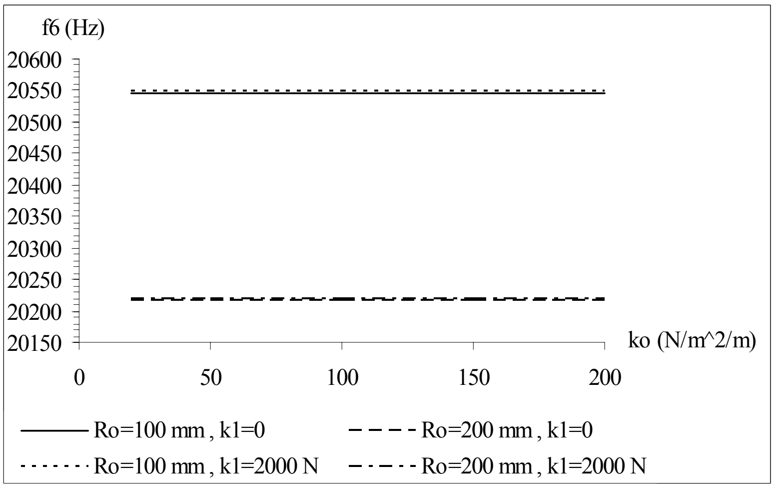


Figure 3.9. Sixth natural frequencies

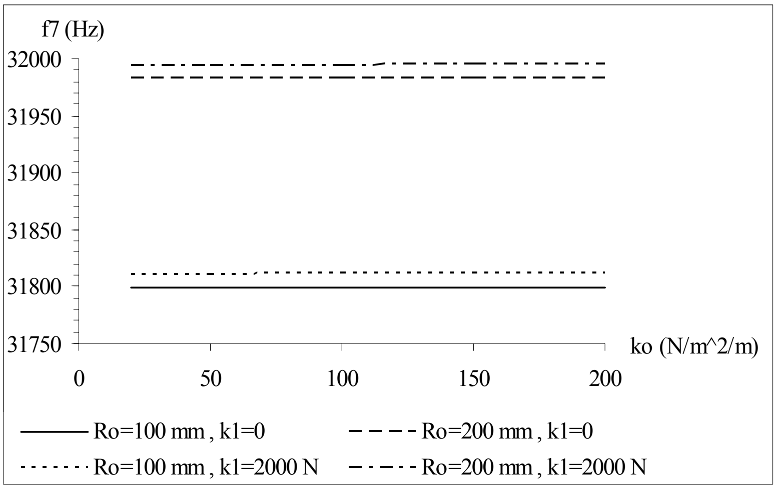


Figure 3.10. Seventh natural frequencies

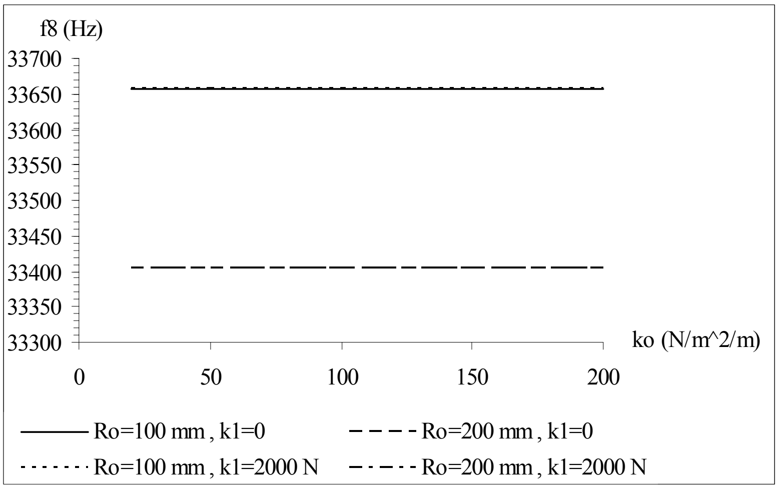


Figure 3.11. Eighth natural frequencies

The numerical results presented in Tables 3.2-11 are plotted in Figures 3.4-11 in order to see the effects of foundation parameters and curvature parameter on natural frequencies. The major effect of foundation parameters on natural frequencies can be seen from Figure 3.4 which shows the variation of first natural frequencies depending on all parameters in this study. On the other hand, it is very interesting that third, sixth and eighth natural frequencies decrease when R_0 is increased from 100 mm to 200 mm.

CHAPTER 4

CONCLUSIONS

In this study, the differential equations governing the free out-of plane vibrations of curved beams with variable curvature on elastic foundation are presented. Since the coefficients of the derived differential equations are not constant, it is generally very difficult to express the exact solution.

In the existing literature, for the free out-of-plane vibrations of curved beams, most of the researchers investigated the symmetrical boundary conditions such as both ends fixed, pinned or free conditions.

With this study, as far as the author is aware, for the first time, the natural frequencies for out-of plane vibrations of curved beams with variable curvature on elastic foundation are studied and presented.

In order to validate the developed computer program to solve the differential eigenvalue problem based on FDM, the solid models are created for Finite Element analysis. The results, found out from FDM, are compared with the results from FEM (Finite Element Method). The effects curvature and foundation parameters on natural frequencies are found for the curved beams in the shape of catenary with selected geometries.

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