IN-PLANE FREE VIBRATION ANALYSIS OF LAMINATED CURVED BEAMS WITH VARIABLE CURVATURE

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ABSTRACT

IN-PLANE FREE VIBRATION ANALYSIS OF LAMINATED CURVED BEAMS WITH VARIABLE CURVATURE

In this study, in plane free vibration characteristics of laminated curved beams with variable curvatures are studied. The present problem is modeled by differential eigenvalue problem with variable coefficients. FDM (Finite Difference Method) is used to solve the differential eigenvalue problem. A computer program is developed in Mathematica and this program is verified by using results available in the literature. The effects of curvature and lamination parameters of the curved beams on natural frequencies are investigated.

ÖZET

DEĞİŞKEN EĞRİLİK YARIÇAPLI TABAKALI KOMPOZİT EĞRİ ÇUBUKLARIN SERBEST TİTREŞİM ANALİZİ

Bu çalışmada, değişken eğrilik yarıçaplı tabakalı kompozit eğri çubukların düzlem içi titreşim karakteristikleri çalışılmıştır. Mevcut problem değişken katsayılı diferansiyel özdeğer problemi ile modellenmiştir. Diferansiyel özdeğer probleminin çözümü için SFY (Sonlu Farklar Yöntemi) kullanılmıştır. Mathematica`da bir bilgisayar programı geliştirilmiş ve bu program literaturde mevcut sonuçlar ile doğrulanmıştır. Eğri çubuğun eğrilik ve tabaka parametrelerinin doğal frekanslara etkileri araştırılmıştır.

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a	acceleration
A	cross-section
b	width of the beam
B	Bending Stiffness
E, E_1, E_2	Young's modulus
M_y	internal moment about y- axis
h	depth of the beam d
m	mass per unit length
n	number of grids
s, s_L	spatial variable (i.e., circumferential coordinate), circumferential length of the beam
t	time
T	kinetic energy
N	internal normal force
T	internal tangential force
u	radial displacement
V	elastic strain energy
w	tangential displacement
(`)	derivative with respect to "s"
(.)	derivative with respect to "t"
ω, λ	natural frequency, natural frequency parameter
ρ	density of material of beam
ρ_0, $\rho_0(s)$	constant radius of curvature, variable radius of curvature
κ_0, $\kappa(s)$, k_0, $k(s)$	curvature; constant, variable, at s=0, variation function
κ_1'	curvature after displacement occurs

CHAPTER 1

GENERAL INTRODUCTION

Curved beams are used in many engineering applications such as stiffeners in airplane/ship/roof structures. They can be classified depending on their geometrical properties. A curved beam can be in the shape of a space curve or a plane curve, have variable curvature and cross-section.

Many investigators studied vibrations of the isotropic curved beams, but only a few researchers studied laminated curved beams. The analysis of laminated curved beams with variable curvature is even rather limited.

Qatu (1993) derived a complete and consistent set of equations for the analysis of laminated composite curved thin and thick beams. Natural frequencies for simply-supported curved beams are obtained by exact solutions.

Lin and Hsieh (2007) presented the closed form general solutions for laminated curved beams of variable curvatures under in plane static loading. The quantities such as axial force, shear force, radial and tangential displacements are expressed as functions of angle of tangent slope. Applications of elliptic, parabola, catenary, cycloid, and exponential spiral laminated curved beams are shown.

Lin and Lin (2011) derived the finite deformation of 2-D laminated curved beams with variable curvatures. The analytical solutions of laminated curved beams of circular and spiral are presented.

In general, the out-of-plane and the in-plane vibrations of curved beams are coupled. However, if the cross-section of the curved beam is uniform and doubly symmetric, then the out-of-plane and the in-plane vibrations are independent (Ojalvo 1962).

On the other hand, in-plane vibrations of curved beams have two types of motions: (1) bending, (2) extensional. Aforementioned motions are coupled. In order to uncouple the equations for in plane vibration, inextensionality condition can be used. This condition requires zero axial strain in neutral axis.

In this study, in plane free vibration characteristics of laminated curved beams in the shape of catenary are studied by Finite Difference Method since the mathematical model of the present problem is based on the coupled differential eigenvalue problem with variable coefficients.

A computer program is developed in Mathematica and this program is verified by using results available in the literature. The effects of curvature and lamination parameters of the curved beams on natural frequencies are investigated.

CHAPTER 2

THEORETICAL VIBRATION ANALYSIS

2.1. Introduction

In this chapter, first of all, the problem is described mathematically. The selected geometry of axis of the curved beam is detailed with mathematical formulations. Equation of motion is derived by vectorial and analytical methods for in-plane vibrations of curved beam with variable curvature. Laminated composite curved beam formulations are presented. Finally, Finite Difference Method is summarized for finding the natural frequencies by solving eigenvalue problem.

2.2. Description of the Problem

The titled problem is based on *Differential Eigenvalue Problem* with variable coefficients. *Differential Eigenvalue Problem* can be reduced to *Discrete Eigenvalue Problem* by Finite Difference Method.

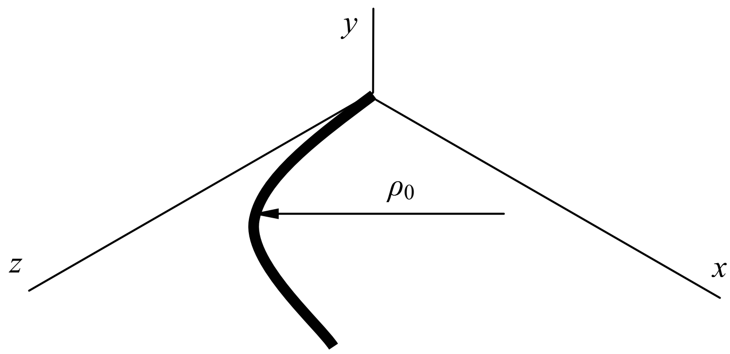


Figure 2.1. A planar curved beam with variable radius of curvature

2.3. Geometry of Curved Beam

A catenary curve, its parameters shown in Figure 2.2 and equations are taken from Yardimoglu (2010).

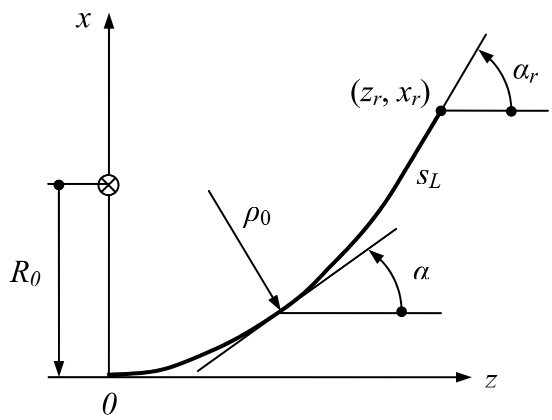


Figure 2.2. Parameters of catenary beam (Source: Yardimoglu 2010)

The function of the catenary curve is given as follows:

$$x(z) = R_0[\cosh(z / R_0) - 1] \tag{2.1}$$

The slope α is obtained by differentiation of Equation 2.1 with respect to z as

$$\tan\alpha = dx(z) / dz = \sinh(z / R_0) \tag{2.2}$$

The tip co-ordinates of the curved beam (z_r, x_r) can be found as

$$z_r = R_0\, arc\sinh(\tan\alpha_r) \tag{2.3}$$

$$x_r = R_0\,(1 / \cos\alpha_r - 1) \tag{2.4}$$

Since the arc length s from origin 0 to any point (z, x) on the curve is

$$s(z) = \int_0^s \sqrt{(1 + (dx(z) / dz)^2}\, dz = R_0 \tan\alpha \tag{2.5}$$

Equation 2.5 provides a relationship between s and α. On the other hand, radius of curvature at abscissa is found as

$$\rho_0(z) = \frac{\left[1 + (dx(z)/dz)^2\right]^{3/2}}{d^2x(z)/dz^2} = R_0 \cosh^2(z/R_0) \tag{2.6}$$

Eliminating the variable z in Equation 2.6 by using Equation 2.2, radius of curvature can be written in terms of α as follows:

$$\rho_0(\alpha) = R_0 / \cos^2 \alpha \tag{2.7}$$

Now, cos α can be expressed in terms of s by using Equation 2.5 as

$$\cos\alpha = R_0 / \sqrt{R_0^2 + s^2} \tag{2.8}$$

Therefore, radius of curvature can also be written in terms of s as follows:

$$\rho_0(s) = R_0 + s^2 / R_0 \tag{2.9}$$

2.4. Derivation of the Equation of Motion

2.4.1. Newtonian Method

This is based on the following two vectorial equations:

$$\sum_i \vec{F}_i = m\vec{a} \tag{2.10}$$

$$\sum_i \vec{M}_i = I\vec{\alpha} \tag{2.11}$$

In this method, it is needed to neglect small quantities of higher orders terms in order to obtain linear differential equations. Moreover, expressing the boundary conditions are based on the understanding of the internal forces and moments.

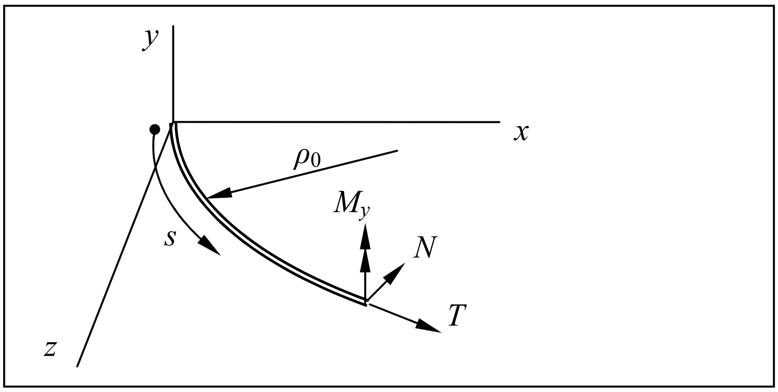


Figure 2.3. A curved beam with internal forces and moments

By using Equations 2.10 and 2.11, force and moment equilibrium equations of the curved beam can be obtained as follows (Love 1944):

$$\frac{dN}{ds} + T\kappa_1' = m\,\ddot{u} \tag{2.12}$$

$$\frac{dT}{ds} - N\,\kappa_1' = m\,\ddot{w} \tag{2.13}$$

$$\frac{dM_y}{ds} + N = 0 \tag{2.14}$$

where N, T and M_y are internal forces and moments,
κ_1' is dynamic curvature in x-z plane,
u, w are displacements in x and z directions,
$m = \rho\,A$ is mass per unit length,

in which A is area of the cross-section.

The dynamic curvatures is given by

$$\kappa_1' = \kappa_0' + \frac{d}{ds}\left(\frac{du}{ds} + \kappa_0'\,w\right) \tag{2.15}$$

Axial force T and bending moment M_y in Equations 2.12-14 are given as

$$T = E\,A\,\varepsilon \tag{2.16}$$

$$M_y = B(\kappa_1' - \kappa_0') \tag{2.17}$$

ε in Equation 2.16 is tangential strain due to tension. It is expressed as

$$\varepsilon = \frac{dw}{ds} - \kappa_0' u \tag{2.18}$$

B in Equation 2.17 is bending rigidity of curved beam material.

2.4.2. Hamilton's Method

The principle is defined as follows (Meirovitch 1967):

$$\delta \int_{t_1}^{t_2} (T - V)\,dt = 0 \tag{2.19}$$

where T and V are the kinetic and strain energies, respectively. For the present problem, they are given as follows;

$$T = \frac{1}{2}\int_0^{S_L} (m\dot{u}^2 + m\dot{w}^2)\,ds \tag{2.20}$$

$$V = \frac{1}{2}\int_0^{S_L} M_y(\kappa_1' - \kappa_0')\,ds \tag{2.21}$$

By using Equation 2.17 along with Equation 2.15 in Equation 2.21, the following strain energy expression is obtained:

$$V = \frac{1}{2}\int_0^{S_L} (B[\frac{d^2u}{ds^2} + \frac{d}{ds}(\kappa_0' w)]^2\,ds \tag{2.22}$$

If central line of curved beam is assumed as unextended, the inextensionality condition is obtained from Equation 2.18 as

$$\frac{dw}{ds} = u\kappa_0' \tag{2.23}$$

By substituting Equations 2.20 and 2.22 along with Equation (2.23) in Equation 2.19, governing differential equations for vibrations of curved beams having variable radius of curvature are obtained as follows:

$$\begin{aligned} & f_6(s)\frac{\partial^6 w}{\partial s^6} + f_5(s)\frac{\partial^5 w}{\partial s^5} + f_4(s)\frac{\partial^4 w}{\partial s^4} + \\ & f_3(s)\frac{\partial^3 w}{\partial s^3} + f_2(s)\frac{\partial^2 w}{\partial s^2} + f_1(s)\frac{\partial w}{\partial s} + \\ & f_0(s) - \rho A\frac{\partial^2 w}{\partial t^2} - \frac{2\rho A}{\kappa_0'(s)^3}\frac{\partial \kappa_0'}{\partial s}\frac{\partial^3 w}{\partial s \partial t^2} + \\ & \frac{\rho A}{\kappa_0'(s)^2}\frac{\partial^4 w}{\partial s^2 \partial t^2} = 0 \end{aligned} \tag{2.24}$$

where

$$f_0(s) = B\kappa_0'(s)\frac{\partial^2 \kappa_0'}{\partial s^2} - \frac{B}{\kappa_0'(s)^2}\frac{\partial \kappa_0'}{\partial s}\frac{\partial^3 \kappa_0'}{\partial s^3} + \frac{B}{\kappa_0'(s)}\frac{\partial^4 \kappa_0'}{\partial s^4} \tag{2.25.a}$$

$$\begin{aligned} f_1(s) = {} & 2B\kappa_0'(s)\frac{\partial \kappa_0'}{\partial s} - \frac{6B}{\kappa_0'(s)^3}\frac{\partial \kappa_0'^3}{\partial s} \\ & - \frac{144B}{\kappa_0'(s)^7}\frac{\partial \kappa_0'^5}{\partial s} + \frac{3B}{\kappa_0'(s)^2}\frac{\partial \kappa_0'}{\partial s}\frac{\partial^2 \kappa_0'}{\partial s^2} \\ & + \frac{276B}{\kappa_0'(s)^6}\frac{\partial \kappa_0'^3}{\partial s}\frac{\partial^2 \kappa_0'}{\partial s^2} - \frac{96B}{\kappa_0'(s)^5}\frac{\partial \kappa_0'}{\partial s}\frac{\partial^2 \kappa_0'^2}{\partial s^2} \\ & + \frac{3B}{\kappa_0'(s)}\frac{\partial^3 \kappa_0'}{\partial s^3} - \frac{68B}{\kappa_0'(s)^5}\frac{\partial \kappa_0'^2}{\partial s}\frac{\partial^3 \kappa_0'}{\partial s^3} \\ & + \frac{20B}{\kappa_0'(s)^4}\frac{\partial^2 \kappa_0'}{\partial s^2}\frac{\partial^3 \kappa_0'}{\partial s^3} + \frac{11B}{\kappa_0'(s)^4}\frac{\partial \kappa_0'}{\partial s}\frac{\partial^4 \kappa_0'}{\partial s^4} \\ & - \frac{B}{\kappa_0'(s)^3}\frac{\partial^5 \kappa_0'}{\partial s^5} \end{aligned} \tag{2.25.b}$$

$$f_2(s) = B\kappa_0'(s)^2 + \frac{3B}{\kappa_0'(s)^2}\frac{\partial\kappa_0'^2}{\partial s} + \frac{144B}{\kappa_0'(s)^6}\frac{\partial\kappa_0'^4}{\partial s}$$
$$+\frac{3B}{\kappa_0'(s)}\frac{\partial^2\kappa_0'}{\partial s^2} - \frac{204B}{\kappa_0'(s)^5}\frac{\partial\kappa_0'^2}{\partial s}\frac{\partial^2\kappa_0'}{\partial s^2}$$
$$+\frac{30B}{\kappa_0'(s)^4}\frac{\partial^2\kappa_0'^2}{\partial s^2} + \frac{44B}{\kappa_0'(s)^4}\frac{\partial\kappa_0'}{\partial s}\frac{\partial^3\kappa_0'}{\partial s^3} \qquad (2.25.c)$$
$$-\frac{5B}{\kappa_0'(s)^3}\frac{\partial^4\kappa_0'}{\partial s^4}$$

$$f_3(s) = -\frac{72B}{\kappa_0'(s)^5}\frac{\partial\kappa_0'^3}{\partial s} + \frac{66B}{\kappa_0'(s)^4}\frac{\partial\kappa_0'}{\partial s}\frac{\partial^2\kappa_0'}{\partial s^2}$$
$$-\frac{10B}{\kappa_0'(s)^3}\frac{\partial^3\kappa_0'}{\partial s^3} \qquad (2.25.d)$$

$$f_4(s) = 2B + \frac{24B}{\kappa_0'(s)^4}\frac{\partial\kappa_0'^2}{\partial s} - \frac{10B}{\kappa_0'(s)^3}\frac{\partial^2\kappa_0'}{\partial s^2} \qquad (2.25.e)$$

$$f_5(s) = -\frac{6B}{\kappa_0'(s)^3}\frac{\partial\kappa_0'}{\partial s} \qquad (2.25.f)$$

$$f_6(s) = \frac{B}{\kappa_0'(s)^2} \qquad (2.25.g)$$

Physical interpretations of boundary conditions are as follows:

a) Either bending moment is zero (pinned or free), or slope is zero (clamped).

b) Either shear force is zero (free), or displacement is zero (pinned or clamped).

c) Either bending moment is zero (pinned or free), or displacement is zero (pinned or clamped).

2.5. Fiber-Reinforced Laminated Curved Beam

In order to obtain the bending rigidity for a fiber-reinforced laminated curved beam shown in Figure 2.4, the following equations are needed (Kaw 2006).

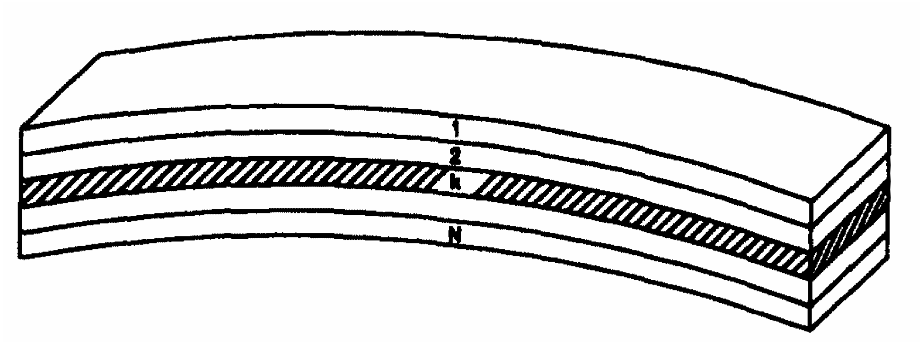


Figure 2.4. Curved beam laminated in radial direction
(Source: Fraternali and Bilotti 1997)

The stress–strain equation for k^{th} layer along tangential direction is

$$\sigma^k = \overline{Q}_{11}^k \varepsilon^k \tag{2.26}$$

where $\overline{Q}_{11}^k$ is the elastic stiffness coefficient for the material and given as

$$\overline{Q}_{11}^k = Q_{11}^k \cos^4 \gamma^k + 2(Q_{12}^k + 2Q_{66}^k)\cos^2 \gamma^k \sin^2 \gamma^k + Q_{22}^k \sin^4 \gamma^k \tag{2.27}$$

in which

$$Q_{11}^k = \frac{E_1^k}{1 - \nu_{12}^k \nu_{21}^k} \tag{2.28.a}$$

$$Q_{12}^k = \frac{\nu_{12}^k E_2^k}{1 - \nu_{12}^k \nu_{21}^k} \tag{2.28.b}$$

$$Q_{66}^k = G_{12}^k \tag{2.28.c}$$

$$Q_{22}^k = \frac{E_2^k}{1 - \nu_{12}^k \nu_{21}^k} \tag{2.28.d}$$

and γ^k is the angle between tangential direction and fiber direction shown in Figure 2.5.

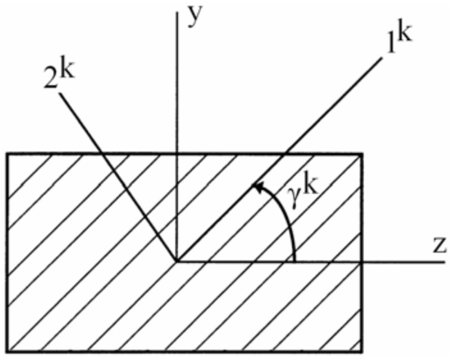


Figure 2.5. Local and global axes of an angle lamina.

(Source: Kaw 2006)

Strain due to bending at a distance of x defined by

$$\varepsilon_b = x(\kappa_1' - \kappa_0') \tag{2.29}$$

The moment is the integral of the stress over the beam thickness h

$$M_y = b\int_{-h/2}^{h/2} \sigma\, x\, dx \tag{2.30}$$

where b is width of the beam. The resultant moment of the laminate is obtained by integrating the stress in each layer through the thickness

$$M_y = b\sum_{k=1}^{N}\int_{h_{k-1}}^{h_k} \sigma^k\, x\, dx \tag{2.31}$$

Substituting Equation 2.26 along with Equation 2.29 into Equation 2.30 and carrying out the integration over the thickness piecewise, from layer to layer, yields:

$$M_y = D_{11}(\kappa_1' - \kappa_0') \tag{2.32}$$

where D_{11} is the stiffness coefficient arising from the piecewise integration and expressed as

$$D_{11} = \frac{1}{3} b \sum_{k=1}^{N} \overline{Q}_{11}^{k} (x_k^3 - x_{k-1}^3) \tag{2.33}$$

It should be noted that D_{11} corresponds to B appeared in coefficients of equation of motion given by Equations 2.25a-g.

2.6. Natural Frequencies by Finite Difference Method

Differential Eigenvalue Problem can be described by

$$L[x] = \omega^2 M[x] \tag{2.34}$$

where $L[x]$ and $M[x]$ are linear differential operators having variable coefficients of the derivatives and ω is eigenvalues.

Solution of Equation 2.34 can be obtained by the FDM (Hildebrand 1987). In the FDM, the derivatives of dependent variables in Equation 2.34 are replaced by the finite differences at mesh points shown in Figure 2.6.

There are three types of finite differences: forward, backward, and central. However, the central difference provides more accurate approximation. Accuracy of the solution by FDM is based on truncation error and grid spacing which is depends on the approximation order and selecting procedure, respectively. The most critical one is the grid spacing. It is selected by observing the convergence of desired results.

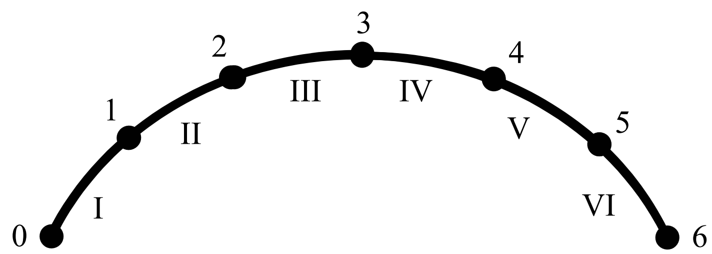


Figure 2.6. A curved domain divided into six subdomains.

Therefore, by using the central difference approximations for derivatives listed in Table A.1, n simultaneous algebraic equations are obtained. Also, boundary conditions are considered in these n simultaneous algebraic equations. Thus, differential eigenvalue problem is reduced to discrete eigenvalue problem which can be written as follows:

$$[A]\{X\} = \omega^2 \, [B] \, \{X\} \tag{2.35}$$

Solutions of the generalized eigenvalue problem given by Equation 2.35 can be calculated by a mathematical software such as Matlab, Mathematica or Maple.

CHAPTER 3

NUMERICAL RESULTS AND DISCUSSION

3.1. Introduction

In this chapter, the following numerical applications are presented:

a) isotropic curved beams with constant curvature,

b) isotropic curved beams with variable curvature,

c) fiber-reinforced laminated curved beams with constant curvature,

d) fiber-reinforced laminated curved beams with variable curvature.

The present numerical results are compared with the results available in the existing literature.

3.2. Comparisons for Isotropic Curved Beams

3.2.1. Isotropic Curved Beams with Constant Curvature

In plane vibration analysis of isotropic curved beams with constant curvature can be solved analytically, but it is considered here to test the FDM algorithm and to show the accuracy and precision of the symbolic program developed in Mathematica.

Convergence of first natural frequency for ρ_0=50 mm is plotted in Figure 3.1. It is seen from Figure 3.1 that n =100 can be selected for all calculation in this chapter.

Comparisons of natural frequency parameters of fixed-fixed curved beams by FDM with analytical results of Archer (1960) can be done by using the Table 3.1. It can be said that, the present results for this case have good agreement with analytical results of Archer (1960).

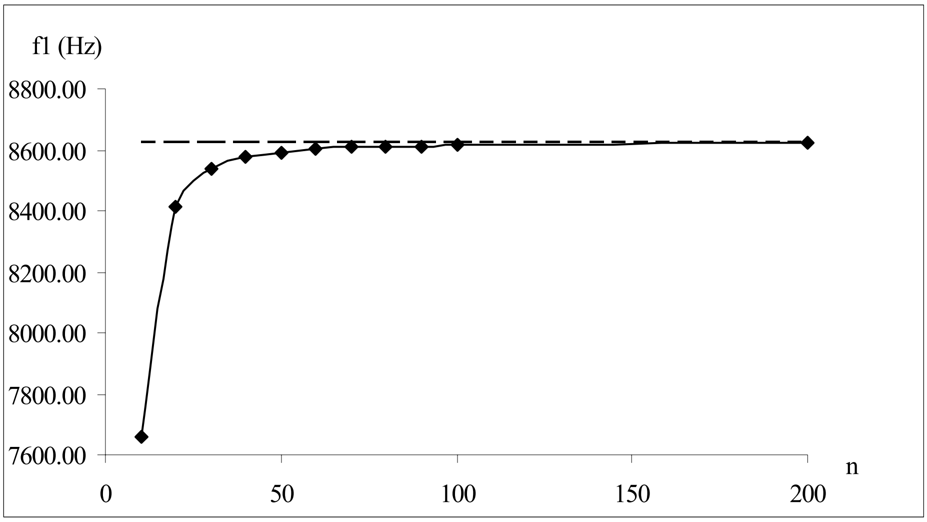
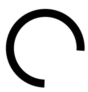
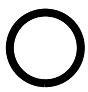


Figure 3.1. Convergence of first natural frequency for ρ_0=50 mm

Table 3.1. Comparison of present natural frequency parameters of a curved beams with the analytical results of Archer (1960)

Mode	Opening Angle		Present λ (n =100)	λ (Archer 1960)
1		π	19.22	19.22
2			93.06	93.15
3			320.5	321.5
4			754.86	756.3
1		3π/2	1.945	1.946
2			12.84	12.85
3			49.46	49.58
4			128.06	126.6
1		2π	0.321	0.3208
2			2.54	2.545
3			11.42	11.46
4			32.98	33.06

Also, it can be seen from Table 3.1 that, when the opening angle increases, the natural frequencies decreases due to the reducing stiffness properties.

3.2.2. Isotropic Curved Beams with Variable Curvature

In this section, numerical applications are carried out for isotropic fixed-fixed curved beams with different curvature parameters to see the curvature parameter effect on natural frequencies. The main numerical data used in this chapter are as follows: b=h=0.01 m, E=200 GPa, ρ=7850 kg/m^3, s_L=0.12 m. Other data are given in tables.

Table 3.2. Natural frequencies found for different R_0

	R_0=0.05 m	R_0=0.1 m	R_0=0.15 m	R_0=0.2 m
f_1 (Hz)	8615.06	9418.99	9678.28	9783.76
f_2 (Hz)	17149.	17743.9	17845.8	17867.4
f_3 (Hz)	30650.	31599.7	31875.8	31989.6
f_4 (Hz)	45783.	46414.4	46519	46542.7
f_5 (Hz)	65438.8	66409.2	66692.8	66811.4
f_6 (Hz)	86900.9	87550.1	87661.9	87688.9
f_7 (Hz)	112710.	113700	113992	114115
f_8 (Hz)	140379.	141056	141177	141209

It can be seen from the Tables 3.2 that, when the curvature parameter R_0 increases, natural frequency parameters decreases. It should be stated that when R_0 increases, curved beam becomes closer to straight beam.

3.3. Comparisons for Fiber-Reinforced Laminated Curved Beams

3.3.1. Applications for Curved Beams with Constant Curvature

Some laminate codes used in this study are illustrated in Figure 3.2.a-d.

[0/–45/90/60/30] denotes the code for the laminate shown in Figure 3.2.a. It consists of five plies, each of which has a different angle to the reference *x*-axis. A slash separates each lamina. The code also implies that each ply is made of the same material and is of the same thickness.

$[0/–45/90_2/60/0]$ denotes the laminate shown in Figure 3.2.b, which consists of six plies. Because two 90° plies are adjacent to each other, 90_2 denote them, where the subscript 2 is the number of adjacent plies of the same angle.

$[0/-45/60]_s$ denotes the laminate consisting six plies as shown in Figure 3.2.c. The plies above the midplane are of the same orientation, material, and thickness as the plies below the midplane, so this is a symmetric laminate. The top three plies are written in the code, and the subscript s outside the brackets represents that the three plies are repeated in the reverse order.

$[0/-45/\overline{60}]_s$ denotes this laminate shown in Figure 3.2.d, which consists of five plies. The number of plies is odd and symmetry exists at the midsurface; therefore, the 60° ply is denoted with a bar on the top (Kaw 2006).

0
–45
90
60
30

(a)

0
–45
90
90
60
0

(b)

0
–45
60
60
–45
0

(c)

0
–45
60
–45
0

(d)

Figure 3.2. Laminate code examples

In this section, numerical applications are carried out for laminated fixed-fixed curved beams with lamination code $[0/-45/60]_s$ and with constant curvature. The main numerical data for geometry are as follows: b=0.01 m, h=0.012 m, s_L=0.12 m,

The composite material made of T300/5208 Graphite/Epoxy is used. Its engineering constants are given as follows by Tsai (1980): E_1=181 GPa, E_2=10.3 GPa, G_{12}=7.17 GPa, $\upsilon_{12}=\upsilon_{21}$=0.28, ρ=1600 kg/m^3. Other data are given in tables.

Table 3.3. Natural frequencies found for different ρ_0

	ρ_0=0.05 m	ρ_0=0.1 m	ρ_0=0.15 m	ρ_0=0.2 m
f_1 (Hz)	18645.7	21890.9	22614.1	22878.4
f_2 (Hz)	38181.1	40845.7	41368.4	41553.6
f_3 (Hz)	69718.1	73661.	74501.	74805.9
f_4 (Hz)	104863.	107841.	108410.	108610.
f_5 (Hz)	150928.	155057.	155934.	156253.
f_6 (Hz)	200952.	204053.	204642.	204849.
f_7 (Hz)	261476.	265681.	266576.	266901.
f_8 (Hz)	326063.	329226.	329826.	330037.

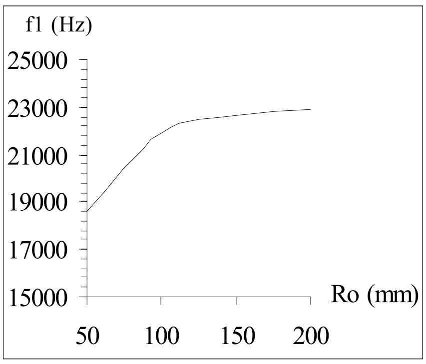


Figure 3.3. First natural frequencies

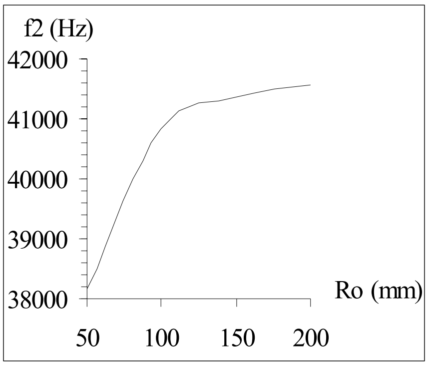


Figure 3.4. Second natural frequencies for different ρ_0

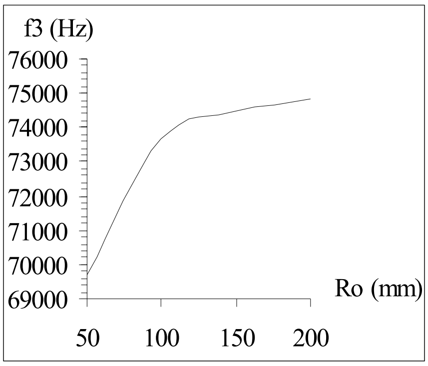


Figure 3.5. Third natural frequencies for different ρ_0

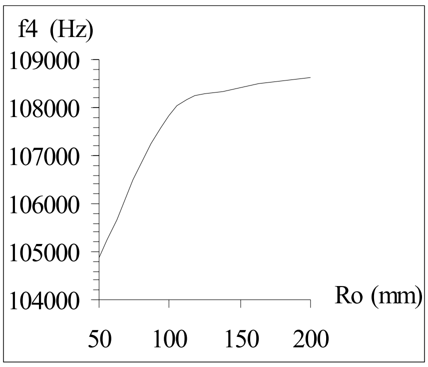


Figure 3.6. Fourth natural frequencies for different ρ_0

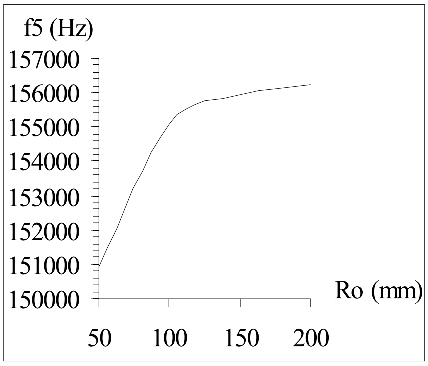


Figure 3.7. Fifth natural frequencies for different ρ_0

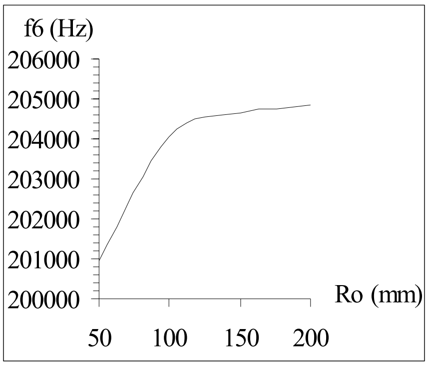


Figure 3.8. Sixth natural frequencies for different ρ_0

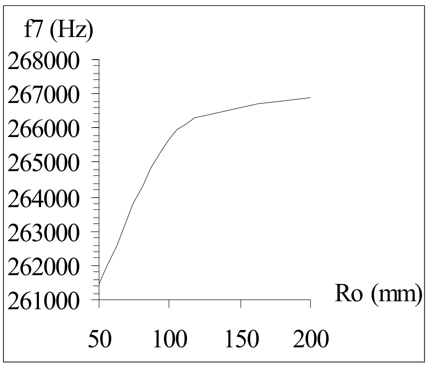


Figure 3.9. Seventh natural frequencies for different ρ_0

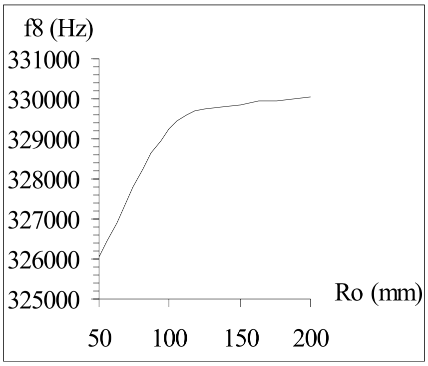


Figure 3.10. Eighth natural frequencies for different ρ_0

3.3.2. Applications for Curved Beams with Variable Curvature

The results in this section are obtained due to the title of this thesis. Numerical applications are carried out for laminated curved beams with lamination code $[0/-45/60]_s$ and with variable curvature. The main numerical data are as follows: b=0.01m, h=0.002 m, s_L=0.12 m, E_1=132 GPa, E_2=10.8 GPa, G_{12}=5.65 GPa, υ_{12}= υ_{21}=0.24, ρ=3250 kg/m^3. Other data are given in tables.

Table 3.4. Natural frequencies found for different R_0

	R_0=0.05 m	R_0=0.1 m	R_0=0.15 m	R_0=0.2 m
f_1 (Hz)	20149.8	22030.1	22636.6	22883.3
f_2 (Hz)	40109.8	41501.2	41739.5	41790.1
f_3 (Hz)	71687.3	73908.6	74554.3	74820.7
f_4 (Hz)	107082.	108559.	108803.	108859
f_5 (Hz)	153055.	155325.	155988.	156265
f_6 (Hz)	203253.	204771.	205033.	205096
f_7 (Hz)	263617.	265934.	266617.	266904
f_8 (Hz)	328332.	329916.	330200.	330274

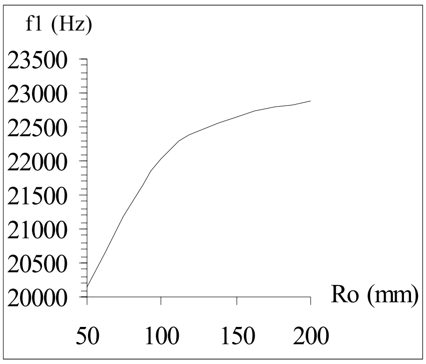


Figure 3.11. First natural frequencies

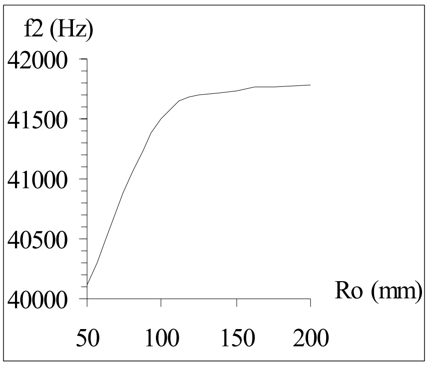


Figure 3.12. Second natural frequencies

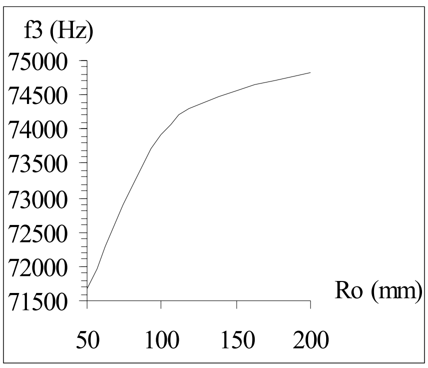


Figure 3.13. Third natural frequencies

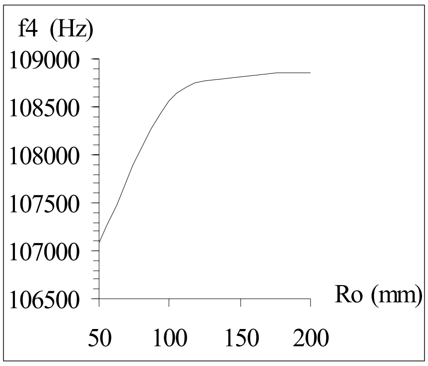


Figure 3.14. Fourth natural frequencies

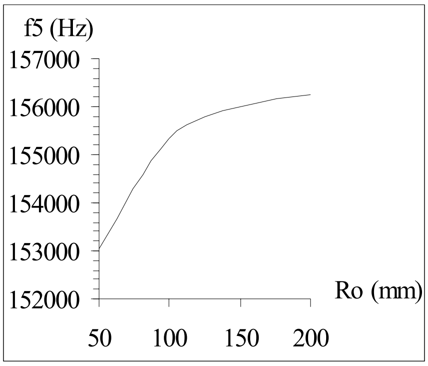


Figure 3.15. Fifth natural frequencies

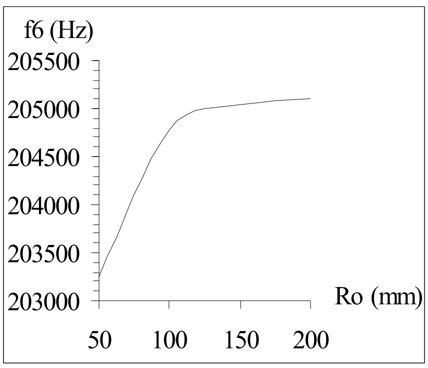


Figure 3.16. Sixth natural frequencies

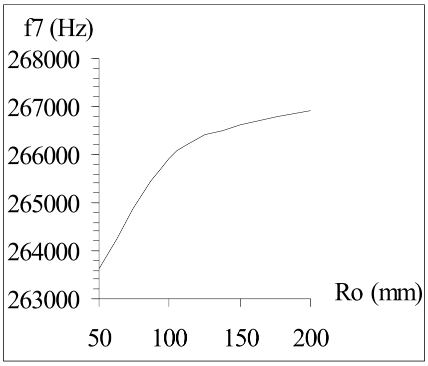


Figure 3.17. Seventh natural frequencies

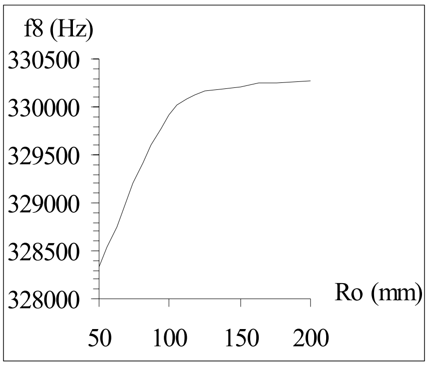


Figure 3.18. Eighth natural frequencies

CHAPTER 4

CONCLUSIONS

In this study, in plane free vibration characteristics of fiber-reinforced laminated curved beams with variable curvatures are studied. Equations of motions are derived by using Newtonian and Hamiltonian methods. The present problem is modeled by two coupled Differential Eigenvalue Problem with variable coefficients. By using inextensionality conditions, two coupled equations are reduced to one Differential Eigenvalue Problem. Central Difference approach is selected in Finite Difference Method to obtain the Discrete Eigenvalue problem from the Differential Eigenvalue Problem.

As a variable curvature, catenary function is selected. Various laminations are considered. The effects of curvature and lamination parameters of the curved beams on natural frequencies are investigated.

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APPENDIX A

CENTRAL DIFFERENCES

Table A.1. Central differences approximations for derivatives

Term	Central Difference Expressions
$\dfrac{dw}{ds}$	$\dfrac{w(i+1)-w(i-1)}{2h}$
$\dfrac{d^2w}{ds^2}$	$\dfrac{w(i+1)-2w(i)+w(i-1)}{h^2}$
$\dfrac{d^3w}{ds^3}$	$\dfrac{w(i+2)-2w(i+1)+2w(i-1)-w(i-2)}{2h^3}$
$\dfrac{d^4w}{ds^4}$	$\dfrac{w(i+2)-4w(i+1)+6w(i)-4w(i-1)+w(i-2)}{h^4}$
$\dfrac{d^5w}{ds^5}$	$\dfrac{w(i+3)-4w(i+2)+5w(i+1)-5w(i-1)+4w(i-2)-w(i-3)}{h^5}$
$\dfrac{d^6w}{ds^6}$	$\dfrac{w(i+3)-6w(i+2)+15w(i+1)-20w(i)+15w(i-1)-6w(i-2)+w(i-3)}{h^6}$