# CONFORMAL TRANSFORMATIONS IN METRIC-AFFINE GRAVITY

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# ABSTRACT

CONFORMAL TRANSFORMATIONS IN METRIC-AFFINE GRAVITY

Conformal transformations play a widespread role in gravitation in regard to their cosmological and the other implications. In this thesis, the effects of conformal transformations on General Relativity comparatively in metric formulation and in metric-affine formulation are analyzed.

In the metric formulation of General Relativity ( pure metric theory of gravity ), conformal transformations, like gauge transformations, add a new degree of freedom to the system - the conformal factor. In this sense, they change a frame to a new one involving an additional degree of freedom. However, this new degree of freedom turns out to be a ghost field in pure metrical formulation i.e. Einstein-Hilbert action. This possesses a serious problem since ghosts are manifestly unphysical.

To overcome this problem, we explore conformal transformations in metric-affine formulation of General Relativity ( metric-affine theory of gravity ) in which the metric and connection are treated as independent variables from the scratch. In metric-affine formulation, there is no a priori relation between metric and connection, and thus, their transformations under conformal transformations do not need to exhibit the correlation present in pure metrical formulation. We thus exploit this fact by assigning different transformation rules for connection to have ghost-free Lagrangians. Firstly, we use the conformally invariant connection, while the metric changes as in metric formulation. After these transformations, there is no ghost field generated by conformal factor. Indeed, there appears no kinetic term of the scalar field (auxiliary field-nondynamical field). This result is not sufficient for us. Because the main goal of our study is the obtaining a conformally invariant theory for gravity with a dynamical scalar field. Then, we use the multiplicatively transforming connection. This transformation does not give the result corresponding to our aim. Finally, we find that if connection transforms additively yet differently than in metrical formulation, the ghost generated by the conformal factor disappears. Additionally, we discuss the physical implications of these transformation rules.

# ÖZET


METRİK-AFİN KÜTLE ÇEKİM KURAMINDA UYUMLU DÖNÜŞÜMLER

Uyumlu dönüşümler, kozmolojik ve diğer anlamları bakımından kütle çekim teorisinde çok yaygın bir rol oynamaktadır. Bu tezde, uyumlu dönüşümlerin genel görelilik üzerine etkisi, karşılaştırılmalı olarak metrik formulasyonunda ve metrik-afin formulasyonunda incelenmiştir.

Genel Göreliliğin metrik formulasyonunda (saf metrik kütleçekim teorisi), uyumlu dönüşümler, ayar dönüşümleri gibi, sisteme yeni bir serbestlik derecesi ekler - uyumlu faktör. Bir anlamda, bir çerçeveyi ek bir serbestlik derecesi içeren diğer bir çerçeveye dönüştürür. Fakat, bu yeni serbestlik derecesi saf metrik formulasyonunda bir hayalet alana dönüşür, yani Einstein-Hilbert aksiyonunda. Hayalet alanlar açık olarak fiziksel olmadıkları için bu durum ciddi bir probleme sahiptir.

Bu problemi çözebilmek için uyumlu dönüşümleri, metrik ve bağlantının birbirinden bağımsız değişkenler olarak davrandığı genel göreliliğin metrik-afin formulasyonunda inceledik (metrik-afin kütleçekim teorisi). Metrik-afin formulasyonunda, metrik ve bağlantı arasında önceden belirlenmiş bir ilişki yoktur ve bu yüzden uyumlu dönüşümler altındaki dönüşümleri, saf metrik formulasyonundaki gibi karşılıklı bir ilişki göstermek zorunda değildir. Böylece hayaletsiz bir Lagrangian elde etmek için bağlantıya farklı transformasyon kuralları tayin ederek bu gerçekten yararlandık. İlk olarak metrik, metrik formulasyonundaki gibi değişirken uyumlu dönüşümler altında değişmez bir bağlantı kullandık. Bu transformasyondan sonra uyumlu faktörden ortaya çıkan hayalet alan yok oldu. Aslında, skaler alanın kinetik terimi ortaya çıkmadı (yardımcı alan- dinamik olmayan alan). Bu sonuç bizim için yeterli değildir. Çünkü bizim çalışmamızın asıl amacı, kütle çekimi için dinamik bir skaler alan içeren, uyumlu dönüşümler altında değişmez bir teori elde etmektir. Bunun için, çarpımsal olarak dönüşen bağlantı kullandık. Bu dönüşüm, amacımıza karşılık gelen bir sonuç vermedi. Son olarak, bağlantı, toplamsal olarak fakat metrik formulasyonundakinden farklı bir biçimde dönüşürse, uyumlu faktör tarafından oluşturulan hayalet alanın yok olduğunu bulduk. Ek olarak, bu dönüşüm kurallarının fiziksel anlamlarını tartıştık.

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# CHAPTER 1

# INTRODUCTION

Einstein's General Relativity (GR) is a major scientific success of last century due to its extraordinary definition of gravity. Although, it is a comprehensive theory of gravity, spacetime and matter, there are several shortcomings which come out from cosmology and quantum field theory. For instance, the Big bang singularity, horizon problem and flatness (Guth, 1981) are issues for cosmology. Because of these problems, Standart Cosmological Model (Weinberg, 1972), depended on GR, can not describe the universe at extreme regimes. On the other hand, because spacetime can not be quantized, GR can not be considered as a fundamental theory. In a sense, there is no complete quantum definition of gravity. Accordingly, these defects lead to new theories of gravity which cover the GR and its positive results. Though there is a huge number of gravitational theories which are alternative to GR, there is no one which can solve the problems of GR completely. However, they are needed as the way to solve them.

One of the most successful examples of alternative theories is "Extended Theories of Gravity"(ETG). These kind of theories are the extension of the standard Einstein's theory (GR) by adding some correction terms like higher-order curvature invariants ($R^2, R^{\mu\nu}R_{\mu\nu}, R^{\alpha\beta\mu\nu}R_{\alpha\beta\mu\nu}$) or minimally or nonminimally coupled scalar fields ($\phi^2 R$) into dynamics. These corrections come from the effective action of quantum gravity (Buchbinder, 1992) and are needed to obtain the effective action of quantum gravity on nearly Planck scale. On the other hand, by conformal transformations ( rescaling ), it is possible that the gravitational theories, which contain the higher-order and nonminimally coupled terms, turn into Einstein's GR plus one or more than one minimally coupled scalar fields. (Maeda, 1989) (Capozziello, 1998) (Allemandi, 2006), (Pulice, 2010) In other words, it is possible to change frame via conformal transformations. The frames related to each other by conformal transformations are called conformal frames. These frames are mathematically equivalent. However, the physical equivalence of them has been a debate among the physicists. (Flanagan, 2004), (Faraoni, 1999) Although there are several conformal frames, two of them have special names, **Einstein frame** and **Jordan frame**. Einstein frame implies the frame that there is only minimal coupling terms. On the other hand, Jordan frame possesses the non-minimal couplings between gravitational fields and the scalar fields.

Conformal transformations mentioned above are the local unit transformations (rescaling of the distances). They were first considered by H. Weyl in gravitation. (Weyl, 1952) However, Weyl called these transformations gauge transformations. Following Weyl, Dicke (Dick, 1962), Hoyle and Narlikar (Hoyle, 1974) and Hoyle (Hoyle, 1975) discussed them. After these studies, people called these transformations conformal transformations and the invariance under them conformal invariance. Conformal transformations will be explained particularly in the next chapter.

As we can obtain the standard Einstein's theory from the ETG by conformal transformations, it is also possible to obtain a conformally invariant theory, which contains the nonminimally coupled scalar field, from the standard GR by conformal transformations. However, after conformal transformations, there appears an unphysical situation such as a ghost field. Let us explain this unphysical situation briefly. For a real scalar field the energy density involves several contributions like kinetic energy $\frac{1}{2}\dot{\phi}^2$ and gradient energy $\frac{1}{2}(\nabla\phi)^2$ and the potential energy $V(\phi)$. (Carroll, 2004) Though the potential energy is Lorentz invariant, the others are not by themselves. However, it is possible that they are combined into a Lorentz invariant form like

$$-\frac{1}{2}\eta^{\alpha\beta}(\partial_\alpha\phi)(\partial_\beta\phi) = \frac{1}{2}\dot{\phi}^2 - \frac{1}{2}(\nabla\phi)^2 \tag{1.1}$$

where $\eta^{\alpha\beta}$ is the flat space metric and its sign convention is (-, +, +, + ...) Thus, the Lagrangian takes the form as

$$\begin{aligned} L &= K - V \\ &= -\frac{1}{2}\eta^{\alpha\beta}(\partial_\alpha\phi)(\partial_\beta\phi) - V(\phi) \end{aligned} \tag{1.2}$$

In a Lagrangian formulation, if there appears a term like $\frac{1}{2}\eta^{\alpha\beta}(\partial_\alpha\phi)(\partial_\beta\phi$ for the metric sign convention (-, +, +, + ...), the scalar field $\phi$ possess a negative kinetic energy and it is called as **ghost field**. Such a field is unphysical and undesired situation. According to the minimum energy principle in a closed system with a positive defined kinetic energy, the total energy is minimized in the equilibrium but if there is a ghost field, the total energy can not be minimized and there is no stability in the system. Because of this unphysical situation, it is noneligible to apply the conformal transformations to the standard GR. To get rid of this problem, it is convenient to apply conformal transformations to different formulation of GR like metric-affine formulation (**Metric-Affine Gravity:MAG**) .

Metric-affine formulation is one of three variational formulations of GR, **metric formulation**, **metric-affine formulation** and **purely affine formulation**. (Poplawski, 2008) (Poplawski, 2009) (Ferraris, 1982) In metric formulation, as we know from the standard General Relativity, the metric is an only geometrical variable and the connection is the Levi-Civita connection of metric tensor. On the other hand, in the purely affine formulation, the geometrical variable of the gravitational field is the symmetric connection. In metric-affine formulation, metric and connection are considered as independent geomatrical variables. There is one more formulation known as **Palatini formulation** which is very similar to metric-affine formulation but not same with it. Because of the historical misunderstandings this formulation is called Palatini formulation. (Palatini, 1919) (Ferraris, 1982). Actually, it was introduced by Einstein. (Einstein, 1925) The difference between these two formulations is that in Palatini formulation the matter field does not interact with the connection whereas in metric-affine formulation it does. Although, this difference seems to be trivial, indeed it has an essential physical meaning. More detail can be found in (Sotiriou, 2010), (Sotiriou, 2009) In this thesis we will deal with the metric-affine formulation. Here there should be a question to ask : Why MAG instead of metric formulation? It can be easily anwered. As we mention before, although GR explains the gravitational interactions on macroscopic scale very well, it is not successful on microscopic scales. In other words, it can not work at quantum level. Because spacetime has Riemannian geometry in GR, there are restrictions to add quantum corrections. These restrictions are that the connection is assumed to be symmetric (torsion-free) and metric compatible (parallel transport leave the lenghts and angles invariant). Hence, GR is not most general geometric theory of gravity. There should be a more general geometry called as non-Riemannian. Non-Riemannian geometry possesses some new structures like torsion and nonmetricity in the most general setting. (Lecian, 2007) These new geometrical structures lead to a modification of the gravitational Lagrangian. Let us explain these quantities.

- **Torsion**

  In GR, it is said that curvature tells the matter how to move and the matter tells the space how to curve. At macroscopic level it is true. However at microscopic level, there is another physical quantitiy:spin (intrinsic angular momentum). In particle physics, the spinor fields are essential ingredients in the definiton of natural law. The spin current of matter fields produce the torsion of spacetime. Thus, to bring spinors into the curved spacetime, torsion should be taken into account. (Watanabe,

) The torsion tensor is defined by the anti-symmetric part of the connection

$$S^{\lambda}_{\alpha\beta} = \Gamma^{\lambda}_{\alpha\beta} - \Gamma^{\lambda}_{\beta\alpha} \tag{1.3}$$

- **Nonmetricity**

  Nonmetricity is induced by dilation and shear currents. Dilation field is the primordial scalar field. This field caused the inflation and the evolution of universe. On the other hand, the shear current can be related to the hadronic quadrupole excitation The non-metricity is defined as

$$Q_{\alpha\mu\nu} = -\nabla_{\alpha} g_{\mu\nu} \tag{1.4}$$

In the MAG, there are not limitations for these structures in contrast with standard GR. (Sotiriou, 2007) Because MAG allows the non-Rieamannian geometry, it is more general geomatrical theory of gravity. These three formulations mentioned above will be explained particularly in the Chapter 3. Here, the most important point is that after conformal transformations MAG gives a conformally invariant theory without ghosty problem.

In Chapter 2 we will give the conformal transformation rules of geometrical quantities and the effects of conformal transformations on matter sector. In Section 2.2 we also see that empty Minkowski space after conformal transformations create an extra non-zero energy momentum tensor to bend the spacetime. Then, in Section 2.3 and 2.4 we will explain the scale and conformal invariance by giving the examples.

In Chapter 3 we will explain the three different formulations of GR. In Section 3.1 we will apply the conformal transformations to the action of gravitational field (Einstein-Hilbert action) in standard GR and give the positive and negative results. As we mention above, this negative result is the ghosty problem. Following this, for solving this problem we will apply the conformal transformations in MAG in Section 3.2. We will see that the ghosty problem can be solved in MAG under some conditions. In Section 3.3 we will obtain the equations of motion in metric-affine formulation of GR.

Finally, we will conclude in Chapter 4.

# CHAPTER 2

# CONFORMAL TRANSFORMATIONS

Symmetry is essential to understand the physical process deeply. In the other words, it makes the physics more intelligible and because of this feature it has been used as a guide to develop physical theories. For instance, continuous symmetries explain the emergence of conservation laws and conserved quantities or the existence of the new particles and anti-particles are predicted in the ligth of the symmetries. Symmetries are considered in two types: Global and Local. Global symmetries have constant parameters whereas the local symmetries have space-time dependent parameters. If a physical theory is invariant under a global or local transformation, it has a symmetry (invariance) correponding to this transformation. All physical theories respect the global unit invariance. This means that the laws of these theories are the same in the cgs system and the mks system. However, it is widely accepted in modern physics that the fundamental transformations and symmetries should be local. Such transformations in physics were first proposed by Weyl in 1918 as conformal transformations. (Weyl, 1952) Because of this, conformal transformations are sometimes called as Weyl transformations. After Einstein introduced the gravity in the geometrical framework, Weyl wondered if electromagnetism and gravitation can be formulated in a single geometrical framework. It was shown that electromagnetism respect the local scale invariance. (Cunningham, 1909), (Bateman, 1910) However, local scale transformations does not preserve the length of vectors as they move in spacetime. In Riemannian geometry, the nonvanishing curvature denotes that the direction of a vector on parallel transport around loop changes compared to the original vector, while its norm remains uncahnged. Weyl put forward that the norm of a vector should change around the loop and this change depends on its spacetime location. The parallel transport of a vector implies a condition of integrability for the direction of this vector such as

$$\nabla_\mu \xi^\alpha = 0 \to R^\alpha_{\mu\nu\sigma}\xi^\sigma = 0 \tag{2.1}$$

while no such condition exists for its norm. Weyl wanted a similar integrability condition on the norm as well. In Riemannian geometry, the norm of a vector is given by

$$l^2 = g_{\mu\nu}\xi^\mu\xi^\nu \tag{2.2}$$

The total derivative of the expression (2.2) is

$$2ldl = (\partial_\alpha g_{\mu\nu}dx^\alpha)\xi^\mu\xi^\nu + g_{\mu\nu}d\xi^\mu\xi^\nu + g_{\mu\nu}\xi^\mu d\xi^\nu \tag{2.3}$$

The total derivative of the vector can be written as

$$\begin{aligned} d\xi^\mu &= -\Gamma^\mu_{\alpha\beta}\xi^\alpha dx^\beta \\ d\xi^\nu &= -\Gamma^\nu_{\alpha\beta}\xi^\alpha dx^\beta \end{aligned} \tag{2.4}$$

After substitution of these expressions into the (2.3) and rearrangement of the indices,

$$\begin{aligned} 2ldl &= \partial_\alpha g_{\mu\nu}dx^\alpha\xi^\mu\xi^\nu - \Gamma^\lambda_{\mu\alpha}g_{\nu\lambda}dx^\alpha\xi^\mu\xi^\nu - \Gamma^\lambda_{\nu\alpha}g_{\mu\lambda}dx^\alpha\xi^\mu\xi^\nu \\ &= \underbrace{\left(\partial_\alpha g_{\mu\nu} - \Gamma^\lambda_{\mu\alpha}g_{\nu\lambda} - \Gamma^\lambda_{\nu\alpha}g_{\mu\lambda}\right)}_{\nabla_\alpha g_{\mu\nu}}\xi^\mu\xi^\nu dx^\alpha \end{aligned} \tag{2.5}$$

In Riemannian geometry the covariant derivative of the metric tensor vanishes due to the metric compatibility. Thus, the change in the length of the vector is zero. On the other hand, Weyl realized that Riemannian geometry must be modified to allow the possibility of varying norm. For this purpose he assumed that the metricity condition of Riemannian space could be replaced by a less restrictive conformal condition

$$\nabla_\alpha g_{\mu\nu} \sim g_{\mu\nu} \tag{2.6}$$

and he considered an alteration

$$\widetilde{g}_{\mu\nu} = e^{2\epsilon\lambda(x)}g_{\mu\nu} \approx (1 + 2\epsilon\lambda(x))g_{\mu\nu} \tag{2.7}$$

Using the new metric $\widetilde{g}_{\mu\nu}$ the change of the vector magnitute

$$d\widetilde{l} = \epsilon(\nabla_\alpha \lambda)\widetilde{l} dx^\alpha \tag{2.8}$$

This means that for a vector transported around a closed loop by parallel displacement not only its direction but also its length can change. Thus, Weyl achieved the varying norm by rescaling the metric tensor $g_{\mu\nu} \to e^{2\epsilon\lambda(x)} g_{\mu\nu} = \Omega^2 g_{\mu\nu}$

Conformal transformation is the rescaling of a system by a spacetime dependent, nonvanishing positive function $\Omega(x)$ called conformal factor. These local unit trasformations can be applied to a system in two different ways. Because a line element defines distance, these two different ways can be shown by using it. One of them is the rescaling of all lengths of the system by multiplying them separately with conformal factor $\Omega(x)$

$$\begin{aligned} d\widetilde{s}^2 &= g_{\mu\nu}\Omega(x)dx^\mu \Omega(x)dx^\nu \\ &= \Omega^2(x) g_{\mu\nu} dx^\mu dx^\nu \end{aligned} \tag{2.9}$$

and the other way is the rescaling of the metric tensor $g_{\mu\nu}$ (meausurement tool) by multiplying it directly with the square of conformal factor while the all lengths are assumed to remain unchanged.

$$d\widetilde{s}^2 = \Omega^2(x) g_{\mu\nu} dx^\mu dx^\nu \tag{2.10}$$

The resulting transformed line elements are equal naturally. The second way is simply analogue to a meter-stick whose size depends on its location in space-time as in the figure 2.1

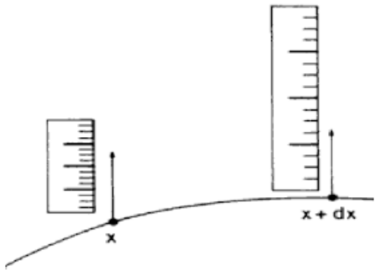


Figure 2.1. The change in meter-stick from x to x+dx.

Conformal transformation affects the distances between two points in the same coordinate system by a rate that differs from point to point on spacetime manifold without any direction specified. This means that it changes the rate isotropically, namely changes in spatial distance and changes in time interval at the same rate. However, the angle between any two vectors and the light cones are preserved. In the other words, the spacetimes $(M, g_{\mu\nu})$ and $(M, \Omega(x)g_{\mu\nu})$ have the same causal structure. (Wald, 1984)This is the reason why it is called conformal.

Conformal transformatons turn into scale transformations when we take the conformal factor constant $\Omega(x) = \Omega$. In a sense, conformal transformations are the localized scale transformations. This implies that meter-sticks and unit of clock are changed by multiplying the same number.

## 2.1. Conformal Transformations of Geometrical Quantities

In this section, we will give the conformal transformations of the geometrical quantities like the points, coordinates, tensors (Fulton, 1962) and the conformal transformations of some geometrical structures of general relativity like connection, Riemann curvature tensor, Ricci tensor and Ricci scalar (Dabrowski, 2008), (Faraoni, 1999), (Wald, 1984). Moreover, this section will show us the difference between coordinate transformations and the conformal transformations.

- Conformal transformations of the points

In this part, two different points in the same coordinate frame are denoted by $x$, $\bar{x}$ and the point which is measured in two different coordinate systems is $x$ and $x'$.

A point transformation generally is given by

$$\bar{x} = f^{\mu}(x) \tag{2.11}$$

which determines the components of the point $\bar{x}$ in $S$ when the components of the point x are known in the same coordinate system $S$.

The line element of a time-like curve is given by

$$d\tau^2(x) = g_{\mu\nu}(x)dx^{\mu}dx^{\nu} \tag{2.12}$$

The point transformation (2.11) leads to

$$d\bar{x}^{\mu} = \partial_{\alpha}\bar{x}^{\mu}dx^{\alpha} \tag{2.13}$$

which determines the difference of two infinitesimally close points $\bar{x}$ and $\bar{x} + d\bar{x}$ into which two nearby points $x$ and $x + dx$.

The property that the line element $d\tau(\bar{x})$ at the point $\bar{x}$ is related to the line element at $x$ by a scalar function $\Omega(x)$ determines the conformal transformation of a point,

$$d\tau^2(\bar{x}) = \Omega^2(x)d\tau^2(x) \tag{2.14}$$

Then, the relation between the line elements at the different points can be written explicitly as

$$g_{\mu\nu}(\bar{x})d\bar{x}^{\mu}d\bar{x}^{\nu} = \Omega^2(x)g_{\alpha\beta}(x)dx^{\alpha}dx^{\beta} \tag{2.15}$$

where $\Omega(x)$ is a positive function. By using the (2.13), (2.15) takes the form as

$$g_{\mu\nu}(\bar{x})\partial_{\alpha}\bar{x}^{\mu}\partial_{\beta}\bar{x}^{\nu} = \Omega^2(x)g_{\alpha\beta}(x) \tag{2.16}$$

It can be seen that (2.16) is equivalent to (2.14). Thus, it can be said that (2.16) describes the conformal point transformation. In addition to this, $\partial_\alpha \bar{x}^\mu = \delta^\mu_\alpha$ and $\partial_\beta \bar{x}^\nu = \delta^\nu_\beta$, then, (2.16) takes the form

$$g_{\alpha\beta}(\bar{x}) = \Omega^2(x) g_{\alpha\beta}(x) \tag{2.17}$$

It is important to distinguish the coordinate transformation and conformal transformation. We can see that (2.16) is different from the coordinate transformation. A coordinate transformation is given as

$$g_{\mu'\nu'}(x')\partial_\alpha x^{\mu'}\partial_\beta x^{\nu'} = g_{\alpha\beta}(x) \tag{2.18}$$

whereas the conformal transformation of a point is given by (2.16).

- Conformal transformations of the coordinates

  Coordinate transformation from $S$ to $S'$ is given by

$$x^{\mu'} = h^{\mu\prime}(x) \tag{2.19}$$

which is components of the points $x$ as seen by two different coordinate systems. Infinitesimally close points transform as

$$dx^{\mu'} = \partial_\alpha x^{\mu'} dx^\alpha \tag{2.20}$$

Then, the metric tensor

$$g_{\mu'\nu'}(x') = \partial_{\mu'} x^\alpha \partial_{\nu'} x^\beta g_{\alpha\beta}(x) \tag{2.21}$$

It can be seen from the (2.21) that the line element in the coordinate system $S$ is equal the line element in the coordinate system $S'$,

$$d\tau^2(x') = d\tau^2(x) \tag{2.22}$$

whereas in (2.14) they are not equal each other unless $\Omega^2(x) = 1$.

The relation (2.21) is a characteristic transformation of a tensor field $T^{\mu\nu\ldots}_{\alpha\beta\ldots}$ generally given as

$$T^{\mu'\nu'\ldots\ldots}_{\alpha'\beta'\ldots\ldots}(x') = \partial_\mu x^{\mu'} \partial_\nu x^{\nu'} \ldots\partial_{\alpha'} x^{\alpha} \partial_{\beta'} x^{\beta} \ldots T^{\mu\nu\ldots\ldots}_{\alpha\beta\ldots\ldots}(x) \tag{2.23}$$

It is needed to relate the coordinate transformation to point transformation for the definition of the conformal transformations of the coordinates corresponding to the conformal transformations of the points. A point transformation is related to a coordinate transformation by requiring the relationship

$$\bar{x}^{\mu\prime} \doteq x^{\mu} \tag{2.24}$$

This means that for a given relation of the components of the point $x$ in two different coordinate systems, a point $\bar{x}$ is associated with $x$ in such a way that the components of $\bar{x}$ with respect to $S'$ are the same as the components of $x$ with repect to $S$. The dot equal sign in (2.24) implies that this equality is valid only in the coordinate systems indicated in the equation.

By the definition of (2.24), the relation between coordinate transformation and point transformation is given as follow

$$\bar{x}^{\mu'} = h^{\mu'}(\bar{x}) = h^{\mu'}(f(x)) \doteq x^{\mu} \tag{2.25}$$

means that the function $h^\mu$ in (2.19) is the inverse transformation to (2.11); if (2.11) implies

$$x^\mu = F^\mu(\bar{x}) \tag{2.26}$$

Then, (2.19) is

$$x^{\mu\prime} = F^{\mu\prime}(\bar{x}) = h^{\mu\prime}(x) \tag{2.27}$$

This relation can also be given by

$$\bar{x}^\mu = f^\mu(x) \doteq f^\mu(\bar{x}') \tag{2.28}$$

from which follows, using (2.24)

$$\partial\bar{x}^\mu/\partial x^\alpha \doteq \partial\bar{x}^\mu/\partial\bar{x}^{\alpha'} \tag{2.29}$$

It can be written now that conformal transformation of the coordinates corresponding to the conformal transformation of the points as

$$x^\mu = f^\mu(x') \tag{2.30}$$

where $f^\mu$ is the same function as in (2.11) and is such that it implies (2.16). It can be seen that (2.14) and (2.22) are consistent equations. By substituting (2.29) into (2.16), we can obtain

$$g_{\mu\nu}(\bar{x})\frac{\partial\bar{x}^\mu}{\partial\bar{x}^{\alpha'}}\frac{\partial\bar{x}^\nu}{\partial\bar{x}^{\beta'}} \doteq \Omega^2(x)g_{\alpha\beta}(x) \tag{2.31}$$

Using (2.21) for $\bar{x}$, it gives

$$g_{\alpha'\beta'}(\bar{x}') \doteq \Omega^2(x) g_{\alpha\beta}(x) \tag{2.32}$$

Thus, with (2.16),

$$d\tau^2(\bar{x}') \doteq \Omega^2(x) d\tau^2(x) \tag{2.33}$$

This equation shows the consistency of (2.14) and (2.22):

$$d\tau^2(\bar{x}') = d\tau^2(\bar{x}) = \Omega^2(x) d\tau^2(x) \tag{2.34}$$

- Conformal Transformations of the tensor fields

A coordinate transformation of covariant component of a vector field $A_\mu(x)$ is written, by using (2.23), as

$$A_{\alpha'}(\bar{x}') = \bar{\partial}_{\alpha'} \bar{x}^\mu A_\mu(\bar{x}) \tag{2.35}$$

which, by means of (2.29), is

$$A_{\alpha'}(\bar{x}') \doteq \partial_\alpha \bar{x}^\mu A_\mu(\bar{x}) \tag{2.36}$$

Then, we can define a new vector field $A_\alpha(x)$ as

$$A_{\alpha'}(\bar{x}') \doteq \bar{A}_\alpha(x) \tag{2.37}$$

A similar relation for the contravariant components holds.

However, it now follows from (2.16) and (2.37) that one can not identify $\bar{A}_\alpha(x)$ with $A^\alpha(x)$ and $\bar{A}_\alpha(x)$ with $A_\alpha(x)$, but only either

$$\begin{aligned} \bar{A}^\alpha(x) &= A^\alpha(x) \\ \bar{A}_\alpha(x) &= \Omega^2(x)A_\alpha(x) \end{aligned} \tag{2.38}$$

or

$$\begin{aligned} \bar{A}_\alpha(x) &= A_\alpha(x) \\ \bar{A}^\alpha(x) &= (1/\Omega^2(x))A^\alpha(x) \end{aligned} \tag{2.39}$$

Identities (2.38) can be proven by substitution of the first equation (2.38) into (2.16); similarly for equation (2.39).

The result of (2.38) and (2.39) can be stated as follows: If the components of a field A(x) transform as a covariant vector under a conformal point transformation, then the contravariant components transform as an affine contravariant vector with weight factor $\Omega^{-2}$,

$$A^\mu(\bar{x}) = (1/\Omega^2)\partial_\alpha \bar{x}^\mu A^\alpha(x) \tag{2.40}$$

Conversely, if under a conformal point transformation we have a contravariant vector, then the corresponding covariant components transform like

$$A_\mu(\bar{x}) = \Omega^2 \bar{\partial}_\mu x^\alpha A_\alpha(x) \tag{2.41}$$

As a consequence, the length of a vector A(x) transforms under conformal point transformation (2.11) and (2.16) as

$$A_\mu(\bar{x})A^\mu(\bar{x}) = [1/\Omega^2(x)]A_\nu(x)A^\nu(x) \tag{2.42}$$

whereas the length of a contravariant vector transforms as

$$A_{\mu}(\bar{x})A^{\mu}(\bar{x}) = \Omega^2(x)A_{\nu}(x)A^{\nu}(x) \tag{2.43}$$

- Conformal transformations of the metric tensor

As we mention before, metric tensor is responsible for measuring the distances. The metric tensor is a rank-2 tensor, therefore we can use the transformation rule of tensors (2.23) for the metric. The metric tensor must be dimension of square of length according to its definition. Thus, conformal transformation of the metric tensor can be written according to

$$\widetilde{g}_{\mu\nu}(x) = \Omega^2(x)g_{\mu\nu}(x) \tag{2.44}$$

where $\widetilde{g}_{\mu\nu}$ refers to the transformed metric tensor. Then the Eq. (2.16) takes the form,

$$\widetilde{g}_{\mu\nu}(x) = \partial_{\mu}\bar{x}^{\alpha}\partial_{\nu}\bar{x}^{\beta}g_{\alpha\beta}(\bar{x}) \tag{2.45}$$

Even if this equation looks like a coordinate transformation, $x$ and $\bar{x}$ refer to two different points in the same coordinate system rather than to different coordinates of the same point.

The conformal point and coordinate transformations are seen to be combination of the conformal transformation of the metric (2.44), with equation of the type (2.45) and (2.21) characterizing the tensor nature of $g_{\mu\nu}$.

Conformal transformation of inverse metric is

$$\widetilde{g}^{\mu\nu} = \Omega^{-2}g^{\mu\nu} \tag{2.46}$$

and the determinant of metric $g = det[g_{\mu\nu}]$ transforms as

$$\sqrt{-\widetilde{g}} = \Omega^{D}\sqrt{-g} \tag{2.47}$$

- Conformal transformation of the connection

In Einstein's Theory of General Relativity, connection, namely Levi-Civita connection, depends on metric $g_{\mu\nu}$ and inverse metric $g^{\mu\nu}$ linearly and governs the curving of spacetime. Connection has not a tensorial structure. Its relation with the metric is given as follows

$$\Gamma^{\lambda}_{\alpha\beta} = \frac{1}{2} g^{\lambda\rho} \left( \partial_\alpha g_{\beta\rho} + \partial_\beta g_{\rho\alpha} - \partial_\rho g_{\alpha\beta} \right) , \tag{2.48}$$

By using (2.44), Levi-Civita connection takes the form

$$\begin{aligned}
\widetilde{\Gamma}^{\alpha}_{\mu\nu} &= \frac{1}{2}\widetilde{g}^{\alpha\rho}[\partial_\mu \widetilde{g}_{\nu\rho} + \partial_\nu \widetilde{g}_{\rho\mu} - \partial_\rho \widetilde{g}_{\mu\nu}] \\
&= \frac{1}{2}\Omega^{-2} g^{\alpha\rho}[\partial_\mu(\Omega^2 g_{\nu\rho}) + \partial_\nu(\Omega^2 g_{\rho\mu}) - \partial_\rho(\Omega^2 g_{\mu\nu})] \\
&= \frac{1}{2} g^{\alpha\rho}[\partial_\mu g_{\nu\rho} + \partial_\nu g_{\rho\mu} - \partial_\rho g_{\mu\nu}] \\
&+ \frac{1}{2}\Omega^{-2} g^{\alpha\rho}[2\Omega(\partial_\mu \Omega) g_{\nu\rho} + 2\Omega(\partial_\nu \Omega) g_{\rho\mu} - 2\Omega(\partial_\rho \Omega) g_{\mu\nu}] \\
&= \Gamma^{\alpha}_{\mu\nu} + \Omega^{-1}[\delta^{\alpha}_{\nu}(\partial_\mu \Omega) + \delta^{\alpha}_{\mu}(\partial_\nu \Omega) - (\partial^\alpha \Omega) g_{\mu\nu}] \\
&= \Gamma^{\alpha}_{\mu\nu} + \Delta^{\alpha}_{\mu\nu}
\end{aligned} \tag{2.49}$$

where

$$\begin{aligned}
\Delta^{\alpha}_{\mu\nu} &= \Omega^{-1}[\delta^{\alpha}_{\nu}(\nabla_\mu \Omega) + \delta^{\alpha}_{\mu}(\nabla_\nu \Omega) - (\nabla^\alpha \Omega) g_{\mu\nu}] \\
&= \delta^{\alpha}_{\nu}\partial_\mu \ln\Omega + \delta^{\alpha}_{\mu}\partial_\nu \ln\Omega - g_{\mu\nu}\partial^\alpha \ln\Omega \,.
\end{aligned} \tag{2.50}$$

where we use $\nabla$ instead of $\partial$ at the first line of Eq. (2.50) because $\Omega$ is a scalar function. It does not matter if it is partial derivative or covariant derivative for a scalar. $\Delta^{\alpha}_{\mu\nu}$ is a (1,2) tensorial structure. The other components of the connection appearing in Riemann curvature tensor $R^{\alpha}_{\mu\beta\nu}$ like $\widetilde{\Gamma}^{\alpha}_{\mu\beta}$, $\widetilde{\Gamma}^{\alpha}_{\beta\lambda}$, $\widetilde{\Gamma}^{\alpha}_{\nu\lambda}$, $\widetilde{\Gamma}^{\lambda}_{\mu\nu}$, $\widetilde{\Gamma}^{\lambda}_{\mu\beta}$ can be derived with the same way.

It can be deduce that if connection does not depend on metric in this way, there is nothing obvious about its transformation. This situation will be discussed in the next chapter (in the section of metric-affine formulation).

After giving the conformal transformation rules of the fundamental dynamics of gravity like the metric tensor and the connection, we can now give the conformal transformations of the other geometrical quantities like Riemann curvature tensor $R^{\alpha}_{\mu\beta\nu}$, Ricci tensor $R_{\mu\nu}$, Ricci scalar $R$, Einstein tensor $G_{\mu\nu}$ and d'Alembertian operator $\Box = g^{\mu\nu}\nabla_\mu\nabla_\nu$.

Firstly, we apply the conformal transformations to the Riemann curvature tensor via transformed Levi-Civita connection,

$$\begin{aligned}
\widetilde{R}^{\alpha}_{\mu\beta\nu} &= \partial_\beta\widetilde{\Gamma}^{\alpha}_{\mu\nu} - \partial_\nu\widetilde{\Gamma}^{\alpha}_{\mu\beta} + \widetilde{\Gamma}^{\alpha}_{\beta\lambda}\widetilde{\Gamma}^{\lambda}_{\mu\nu} - \widetilde{\Gamma}^{\alpha}_{\nu\lambda}\widetilde{\lambda}_{\mu\beta} \\
&= \partial_\beta(\Gamma^{\alpha}_{\mu\nu} + \Delta^{\alpha}_{\mu\nu}) - \partial_\nu(\Gamma^{\alpha}_{\mu\beta} + \Delta^{\alpha}_{\mu\beta}) \\
&+ (\Gamma^{\alpha}_{\beta\lambda} + \Delta^{\alpha}_{\beta\lambda})(\Gamma^{\lambda}_{\mu\nu} + \Delta^{\lambda}_{\mu\nu}) - (\Gamma^{\alpha}_{\nu\lambda} + \Delta^{\alpha}_{\nu\lambda})(\Gamma^{\lambda}_{\mu\beta} + \Delta^{\lambda}_{\mu\beta}) \qquad (2.51)
\end{aligned}$$

By adding and subtracting the same term $\Gamma^{\lambda}_{\beta\nu}\Delta^{\alpha}_{\lambda\mu}$ into (2.51) and using the definition of covariant derivatives of the tensorial fields, (2.51) takes the form as

$$\widetilde{R}^{\alpha}_{\mu\beta\nu} = R^{\alpha}_{\mu\beta\nu}(\Gamma) + \nabla_\beta\Delta^{\alpha}_{\mu\nu} - \nabla_\nu\Delta^{\alpha}_{\mu\beta} + \Delta^{\alpha}_{\beta\lambda}\Delta^{\lambda}_{\mu\nu} - \Delta^{\alpha}_{\nu\lambda}\Delta^{\lambda}_{\mu\beta} \qquad (2.52)$$

It can be seen that the Riemann curvature tensor changes as additively under conformal transformations. Then, substitute (2.50) into (2.52),

$$\begin{aligned}
\widetilde{R}^{\alpha}_{\mu\beta\nu} &= R^{\alpha}_{\mu\beta\nu}(\Gamma) + \Omega^{-1}(\nabla_\beta\nabla_\mu\Omega\delta^{\alpha}_{\nu} - \nabla_\nu\nabla_\mu\Omega\delta^{\alpha}_{\beta} + \nabla_\nu\nabla_\sigma\Omega g_{\beta\mu}g^{\alpha\sigma} - \nabla_\beta\nabla_\sigma\Omega g_{\mu\nu}g^{\alpha\sigma}) \\
&+ \Omega^{-2}(2\nabla_\beta\Omega\nabla_\sigma\Omega g_{\mu\nu}g^{\alpha\sigma} - 2\nabla_\nu\Omega\nabla_\sigma\Omega g_{\mu\beta}g^{\alpha\sigma} \\
&+ \nabla^\sigma\Omega\nabla_\sigma\Omega\delta^{\alpha}_{\nu}g_{\mu\beta} - \nabla^\lambda\Omega\nabla_\lambda\Omega\delta^{\alpha}_{\beta}g_{\mu\nu}) \\
&= R^{\alpha}_{\mu\beta\nu} - [\delta^{\alpha}_{\beta}\delta^{\lambda}_{\nu}\delta^{\rho}_{\mu} - \delta^{\alpha}_{\nu}\delta^{\lambda}_{\beta}\delta^{\rho}_{\mu} + g_{\mu\nu}\delta^{\lambda}_{\beta}g^{\alpha\rho} - g_{\mu\beta}\delta^{\lambda}_{\nu}g^{\alpha\rho}]\Omega^{-1}(\nabla_\lambda\nabla_\rho\Omega) \\
&+ 2[\delta^{\alpha}_{\beta}\delta^{\lambda}_{\nu}\delta^{\rho}_{\mu} - \delta^{\alpha}_{\nu}\delta^{\lambda}_{\beta}\delta^{\rho}_{\mu} + g_{\mu\nu}\delta^{\lambda}_{\beta}g^{\alpha\rho} - g_{\mu\beta}\delta^{\lambda}_{\nu}g^{\alpha\rho}]\Omega^{-2}(\nabla_\lambda\Omega)(\nabla_\rho\Omega) \\
&+ [g_{\mu\beta}\delta^{\alpha}_{\nu}g^{\lambda\rho} - g_{\mu\nu}\delta^{\alpha}_{\beta}g^{\lambda\rho}]\Omega^{-2}(\nabla_\lambda\Omega)(\nabla_\rho\Omega) \qquad (2.53)
\end{aligned}$$

Conracting the indices $\alpha$ and $\beta$ yields the Ricci tensor

$$\begin{aligned}
\widetilde{R}_{\mu\nu} &= R_{\mu\nu} - [\delta^{\alpha}_{\alpha}\delta^{\lambda}_{\nu}\delta^{\rho}_{\mu} - \delta^{\alpha}_{\nu}\delta^{\lambda}_{\alpha}\delta^{\rho}_{\mu} + g_{\mu\nu}\delta^{\lambda}_{\alpha}g^{\alpha\rho} - g_{\mu\alpha}\delta^{\lambda}_{\nu}g^{\alpha\rho}]\Omega^{-1}(\nabla_\lambda\nabla_\rho\Omega) \\
&+ 2[\delta^{\alpha}_{\alpha}\delta^{\lambda}_{\nu}\delta^{\rho}_{\mu} - \delta^{\alpha}_{\nu}\delta^{\lambda}_{\alpha}\delta^{\rho}_{\mu} + g_{\mu\nu}\delta^{\lambda}_{\alpha}g^{\alpha\rho} - g_{\mu\alpha}\delta^{\lambda}_{\nu}g^{\alpha\rho}]\Omega^{-2}(\nabla_\lambda\Omega)(\nabla_\rho\Omega) \\
&+ [g_{\mu\alpha}\delta^{\alpha}_{\nu}g^{\lambda\rho} - g_{\mu\nu}\delta^{\alpha}_{\alpha}g^{\lambda\rho}]\Omega^{-2}(\nabla_\lambda\Omega)(\nabla_\rho\Omega) \\
&= R_{\mu\nu} - [(D-2)\delta^{\lambda}_{\nu}\delta^{\rho}_{\mu} + g_{\mu\nu}g^{\lambda\rho}]\Omega^{-1}(\nabla_\lambda\nabla_\rho\Omega) \\
&+ [2(D-2)\delta^{\lambda}_{\nu}\delta^{\rho}_{\mu} - (D-3)g_{\mu\nu}g^{\lambda\rho}]\Omega^{-2}(\nabla_\lambda\Omega)(\nabla_\rho\Omega) \qquad (2.54)
\end{aligned}$$

where $\delta^{\alpha}_{\alpha} = D$, named as number of dimension. we obtain the Ricci scalar by contracting the transformed Ricci tensor with $\widetilde{g}^{\mu\nu}$

$$\begin{aligned}\widetilde{R} &= \Omega^{-2}R - 2(D-1)\Omega^{-3}g^{\lambda\rho}(\nabla_\lambda\nabla_\rho\Omega) \\ &- (D-1)(D-4)g^{\lambda\rho}\Omega^{-4}(\nabla_\lambda\Omega)(\nabla_\rho\Omega)\end{aligned} \tag{2.55}$$

After conformal transformation, d'Alembertian operator takes the form as

$$\begin{aligned}\widetilde{\Box}\phi &= \widetilde{g}^{\mu\nu}\widetilde{\nabla}_\mu\widetilde{\nabla_\nu}\phi \\ &= \Omega^{-2}g^{\mu\nu}[\widetilde{\nabla}_\mu(\nabla_\nu\phi)] \\ &= \Omega^{-2}g^{\mu\nu}[\partial_\mu(\nabla_\nu\phi) - \widetilde{\Gamma}^{\lambda}_{\mu\nu}(\nabla_\lambda\phi)] \\ &= \Omega^{-2}g^{\mu\nu}\{\partial_\mu(\nabla_\nu\phi) - [\Gamma^{\lambda}_{\mu\nu} + \Omega^{-1}(\delta^{\lambda}_{\nu}\nabla_\mu\Omega + \delta^{\lambda}_{\mu}\nabla_\nu\Omega - g_{\mu\nu}\nabla^{\lambda}\Omega)]\nabla_\lambda\phi\} \\ &= \Omega^{-2}g^{\mu\nu}\left[\nabla_\mu\nabla_\nu\phi - (\delta^{\alpha}_{\mu}\delta^{\beta}_{\nu} + \delta^{\beta}_{\mu}\delta^{\alpha}_{\nu} - g_{\mu\nu}g^{\alpha\beta})\Omega^{-1}(\nabla_\alpha\Omega)(\nabla_\beta\phi)\right] \\ &= \Omega^{-2}\left[\Box\phi + (D-2)g^{\alpha\beta}\Omega^{-1}(\nabla_\alpha\Omega)(\nabla_\beta\phi)\right]\end{aligned} \tag{2.56}$$

Einstein tensor is the geometrical part of the Einstein equations and it is explicitly written as

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R \tag{2.57}$$

After substitution of conformally transformed Ricci tensor and Ricci scalar into this equation, the conformally transformed Einstein tensor is obtained as

$$\begin{aligned}\widetilde{G}_{\mu\nu} &= \widetilde{R}_{\mu\nu} - \frac{1}{2}\widetilde{g}_{\mu\nu}\widetilde{R} \\ &= G_{\mu\nu} + \frac{D-2}{2\Omega^2}[4\partial_\mu\Omega\partial_\nu\Omega + (D-5)\partial_\beta\Omega\partial^\beta\Omega g_{\mu\nu}] \\ &- \frac{D-2}{\Omega}[\nabla_\mu\nabla_\nu\Omega - g_{\mu\nu}\Box\Omega]\end{aligned} \tag{2.58}$$

## 2.2. Conformal Transformations in the Matter Sector

Until now, we have given only the conformal transformations of the geometrical quantities. For the matter part of the gravity, the action is written as

$$S = \int \sqrt{-g} d^D x \mathcal{L}_m(g, \psi_m) \tag{2.59}$$

where the matter Lagrangian contains the metric tensor and the matter fields $\psi$

We assume that the matter Lagrangian changes under conformal transformations like

$$\widetilde{\mathcal{L}}_m = \Omega^{-D} \mathcal{L}_m \tag{2.60}$$

Then, we can write the transformed matter action as

$$\begin{aligned} \widetilde{S}_m &= \int \sqrt{-\widetilde{g}} d^D x \widetilde{\mathcal{L}}_m \\ &= \int \sqrt{-g} \Omega^D d^D x \Omega^{-D} \mathcal{L}_m \\ &= \int \sqrt{-g} d^D x \mathcal{L}_m \\ &= S_m \end{aligned} \tag{2.61}$$

It can be seen that the matter action is invariant under conformal transformations. This means that the matter part of the gravity can be studied in any conformal related frames as invariant quantity.

The energy momentum tensor $\mathcal{T}^m_{\mu\nu}$ is obtained from this action by taking the variation with respect to the metric tensor $g^{\mu\nu}$. Thus, the energy momentum tensor is in the form

$$\mathcal{T}^m_{\mu\nu} = \frac{2}{\sqrt{-g}} \frac{\delta}{\delta g^{\mu\nu}} (\sqrt{-g} \mathcal{L}_m) \tag{2.62}$$

After the conformal transformations, the energy momentum tensor takes the form,

$$\widetilde{\mathcal{T}}^m_{\mu\nu} = \frac{2}{\sqrt{-\widetilde{g}}}\frac{\delta}{\delta\widetilde{g}^{\mu\nu}}(\sqrt{-\widetilde{g}}\widetilde{\mathcal{L}}_m) \tag{2.63}$$

$$= \Omega^{-D}\frac{2}{\sqrt{-g}}\Omega^2\frac{\partial g^{\alpha\beta}}{\partial g^{\mu\nu}}\frac{\delta}{\delta g^{\alpha\beta}}(\sqrt{-g}\mathcal{L}_m)$$

$$= \Omega^{-D+2}\underbrace{\frac{\partial g^{\alpha\beta}}{\partial g^{\mu\nu}}}_{\delta^\alpha_\mu\delta^\beta_\nu}\underbrace{\frac{2}{\sqrt{-g}}\frac{\delta}{\delta g^{\alpha\beta}}(\sqrt{-g}\mathcal{L}_m)}_{\mathcal{T}^m_{\alpha\beta}} \tag{2.64}$$

Then,

$$\widetilde{\mathcal{T}}^m_{\mu\nu} = \Omega^{-D+2}\mathcal{T}^m_{\mu\nu} \tag{2.65}$$

If we take the trace of the Eq.(2.65), we obtain

$$\widetilde{\mathcal{T}}^m = \Omega^{-D}\mathcal{T}^m \tag{2.66}$$

The conservation law in the first frame:

$$\nabla_\mu\mathcal{T}^{\mu\nu}_m = 0 \tag{2.67}$$

The conservation law in the second frame leads to

$$\nabla_\mu\widetilde{\mathcal{T}}^{\mu\nu}_m = -\frac{\nabla^\nu\Omega}{\Omega}\widetilde{\mathcal{T}}_m \tag{2.68}$$

From (2.68) it is obvious that the transformed energy-momentum tensor is conserved only if the trace of it vanishes ($\widetilde{\mathcal{T}}_m = 0$). If the trace of an energy-momentum tensor in a frame vanishes, it can be easily seen from (2.66) that it is necessary that the trace of the energy-momentum tensor in the conformally related frame vanishes. This means that only traceless type of matter provide the energy conservation.

After giving the conformal transformations rules of the Einstein tensor and the energy-momentum tensor, we can now discuss how the conformal transformations effect the Einstein field equations. Firstly, the Einstein field equations are generally written in a untransformed frame in the form as

$$G^{\mu\nu} = \kappa^2 \mathcal{T}_m^{\mu\nu} \tag{2.69}$$

Conformally transformed Einstein field equations are

$$\widetilde{G}^{\mu\nu} = \kappa^2 \widetilde{\mathcal{T}}_m^{\mu\nu} \tag{2.70}$$

For the energy conservation, the application of Bianchi identity on this equation leads to

$$\nabla_\mu \widetilde{G}^{\mu\nu} = 0 \longrightarrow \nabla_\mu \widetilde{\mathcal{T}}_m^{\mu\nu} = 0 \tag{2.71}$$

By using (2.58) and (2.65)

$$\widetilde{G}_{\mu\nu} = G_{\mu\nu} + \mathcal{T}_{\mu\nu}^{\Omega} \tag{2.72}$$

(2.73)

or in the contravariant form,

$$\widetilde{G}^{\mu\nu} = \Omega^{-4}\left(G^{\mu\nu} + \mathcal{T}_{\Omega}^{\mu\nu}\right) \tag{2.74}$$

where

$$\begin{aligned} \mathcal{T}_{\mu\nu}^{\Omega} &= -\frac{D-2}{2\Omega^2}\left[4\nabla_\mu\Omega\nabla_\nu\Omega + (D-5)\nabla_\lambda\Omega\nabla^\lambda\Omega g^{\mu\nu}\right] \\ &- \frac{D-2}{\Omega}\left[\nabla_\mu\nabla_\nu\Omega - g_{\mu\nu}\Box\Omega\right] \end{aligned} \tag{2.75}$$

The field equations (2.70) take the form

$$G_{\mu\nu} + \mathcal{T}^{\Omega}_{\mu\nu}\Omega = \kappa^2\Omega^{-D-2}\mathcal{T}^{m}_{\mu\nu} \tag{2.76}$$

or, alternatively

$$G^{\mu\nu} = \kappa^2\Omega^{-D+2}\mathcal{T}^{\mu\nu}_{m} - \mathcal{T}^{\mu\nu}_{\Omega} \tag{2.77}$$

Imposition of Bianchi identity yields

$$\nabla_\mu G^{\mu\nu} = 0 = \kappa^2\nabla_\mu\left(\Omega^{-D+2}\mathcal{T}^{\mu\nu}_{m}\right) - \nabla_\mu\mathcal{T}^{\mu\nu}_{\Omega} \tag{2.78}$$

with (2.67),

$$\kappa^2(-D+2)\Omega^{-D+2}\frac{\nabla_\mu\Omega}{\Omega}\mathcal{T}^{\mu\nu}_{m} = \nabla_\mu\mathcal{T}^{\mu\nu}_{\Omega} \tag{2.79}$$

If we assume that there is no matter energy-momentum tensor $\mathcal{T}^{\mu\nu}_{m} = 0$, then the Einstein tensor $G^{\mu\nu}$ which is the geometrical part of the Einstein equations vanishes. In this case, we can said that the space-time is flat. However, in the conformally transformed frame, it can be easily seen from (2.58) that the Einstein tensor $\widetilde{G}^{\mu\nu}$ does not vanish.

$$\widetilde{G}^{\mu\nu} = -\widetilde{\mathcal{T}}^{\mu\nu}_{\Omega} \neq 0 \tag{2.80}$$

Consequently, it can be said that an empty Minkowski space after conformal transformations can create an extra non-zero energy momentum tensor composed of the conformal factor $\Omega$ to bend the space-time. (Dabrowski, 2008)

## 2.3. Scale Invariance

Invariance of a system under scale transformations is called scale invariance. The necessary and the sufficient condition for the scale invariance is that the system should have no fixed wavelength, mass or any other dimensionful coupling constants. For example, the Compton wavelength formula is given by

$$\lambda = \frac{h}{mc} \tag{2.81}$$

where c and h are constants. Studying in a system of units in which c and h are taken to be unit is convenient. Thus, three dimension, length, time and mass reduce to a single dimension and inverse of mass provides units of length and time,

$$\lambda = \frac{1}{m} \rightarrow [L] = \frac{1}{[M]} \tag{2.82}$$

If mass goes to zero, wavelength goes to infinity and we can stretch or contract the system freely. For the system have the nonzero mass, there is no scale invariance. Because mass is intrinsic quantity of the fundamental particles which constitute the matters and it is constant. If it is measured by using different meter-sticks or clocks, it should take different values. However, this is not possible. Thus, scale invariance of the system is broken by mass. (Yasunori, 2003)

On the other hand all physical theories having zero masses and no dimensionful coupling constants respect a scale invariance (global unit invariance), the fundamental symmetry in physics which prevent us from adding two physical quantities with different dimensions. The scaling of physical quantities are encoded in their weights. For instance, the fields of a theory under scale invariance transoform as

$$\phi_i \rightarrow \Omega^{w_i} \phi_i \tag{2.83}$$

where $\Omega$ is scale factor and $w$ is the weight of the physical quantity. Then, the scale invariance requires

$$S[\phi_i] = \widetilde{S}[\Omega^{w_i}\phi_i] \tag{2.84}$$

This equation can be used to find the weight of the field $\phi_i$. Let us consider the action of a scalar field

$$S[g,\phi] = \int d^Dx\sqrt{-g}\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi \tag{2.85}$$

Then the globally rescaled action is

$$\begin{aligned}
\widetilde{S}[\widetilde{\phi}] &= \int d^Dx\sqrt{-\widetilde{g}}\frac{1}{2}\widetilde{g}^{\mu\nu}\partial_\mu\widetilde{\phi}\partial_\nu\widetilde{\phi} \\
&= \int d^Dx\Omega^D\sqrt{-g}\frac{1}{2}\Omega^{-2}g^{\mu\nu}\partial_\mu(\Omega^w\phi)\partial_\nu(\Omega^w\phi) \\
&= \int d^Dx\Omega^{D-2+2w}\sqrt{-g}\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi
\end{aligned} \tag{2.86}$$

From the equivalence of these two action, the weight of the scalar field,

$$\begin{aligned}
\Omega^{D-2+2w} &= \Omega^0 \\
D-2+2w &= 0 \xrightarrow{weight} w = -\frac{D-2}{2}
\end{aligned} \tag{2.87}$$

Now, we will go on with a massless scalar field $\phi$ in $D = 4$ dimension as an example. Lagrangian density of a massless scalar field $\phi$ is

$$\mathcal{L} = \frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi \tag{2.88}$$

with the corresponding action in four dimensions

$$\begin{aligned}
S[g,\phi] &= \int d^4x\sqrt{-g}\mathcal{L} \\
&= \int d^4x\sqrt{-g}\Big\{\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi\Big\}
\end{aligned} \tag{2.89}$$

Let us apply the scale transformations to this system by using the transformations

$$\widetilde{g}_{\mu\nu} = \Omega^2 g_{\mu\nu}$$
$$\sqrt{-\widetilde{g}} = \Omega^4 \sqrt{-g}$$

while it is demanded that the scalar field $\phi$ transforms as

$$\widetilde{\phi} = \Omega^{-1}\phi \tag{2.90}$$

The transformed action is obtained as the following

$$\begin{aligned} \widetilde{S}[\widetilde{g},\widetilde{\phi}] &= \int d^4x(\Omega^4\sqrt{-g})\Big\{\frac{1}{2}(\Omega^{-2}g^{\mu\nu})\partial_\mu(\Omega^{-1}\phi)\partial_\nu(\Omega^{-1}\phi)\Big\} \\ &= \int d^4x\sqrt{-g}\Big\{\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi\Big\} \\ &= S[g,\phi] \end{aligned} \tag{2.91}$$

Because $\Omega$ is a constant, it is not effected by derivatives. It can be seen that the action (2.89) is invariant under scale transformations.

$$\widetilde{S}[\widetilde{g},\widetilde{\phi}] = S[g,\phi] \tag{2.92}$$

This equation implies that physics is independent of the global choice of unit system.

In addition to this, a theory can be accepted as scale invariance at energies so high that rest masses of the particles can be ignored. Furthermore, it can be said that at the beginning of the universe, before the particles get their masses by spontaneously symmetry breaking, there is completely scale invariance in the universe.

## 2.4. Conformal Invariance

Similar to scale invariance, the invariance of a system under conformal transformations is called conformal invariance. For the conformal invariance, the necessary condition is the same as for scale invariance. However, it is not sufficient condition for all systems. For example, whereas the electromagnetic fields are conformally invariant under this condition, massless scalar fields are not. For the scalar fields, in addition to this necessary condition, the sufficent condition is that the scalar field should couple to curvature R of the background spacetime directly (non-minimally). Furthermore, all fields whose energy-momentum tensors are traceless respect the conformal invariance. (Salehi, 2000) For instance, the Maxwell theory of electromagnetism is invariant under conformal transformations. This can be seen from the traceless energy-momentum tensor of the theory.

$$T_{\mu\nu} = F_{\mu\lambda}F_{\nu}^{\lambda} - \frac{1}{4}g_{\mu\nu}F_{\rho\sigma}F^{\rho\sigma} \tag{2.93}$$

Trace of (2.93) is

$$T_{\mu}^{\mu} = 0 \tag{2.94}$$

On the other hand, massless scalar fields do not have traceless energy-momentum tensor. Before seeing this, we will obtain the energy-momentum tensor of scalar fields by using the principle of least action. The most common way of obtaining the energy-momentum tensor is the taking variation of the corresponding action by varying the metric $g_{\mu\nu} \to g_{\mu\nu} + \delta g_{\mu\nu}$. The notation for energy-momentum tensor is introduced

$$\delta S = \int d^4\sqrt{-g}\frac{1}{2}T_{\mu\nu}\delta g^{\mu\nu} \tag{2.95}$$

$T_{\mu\nu}$ is identical to energy-momentum tensor and it is symmetric. (Landau, 1975) According to (2.95), we can obtain the energy-momentum tensor for scalar fields from the variation of the action (2.89) with respect to metric.

$$\delta S = \delta \int d^4x\sqrt{-g}\mathcal{L} = \int d^4x(\delta\sqrt{-g}\mathcal{L} + \sqrt{-g}\delta\mathcal{L}) \tag{2.96}$$

Here, variation of the $\sqrt{-g}$ is:

$$\delta\sqrt{-g} = -\frac{1}{2\sqrt{-g}}\delta g = -\frac{1}{2}\sqrt{-g}g_{\mu\nu}\delta g^{\mu\nu} \tag{2.97}$$

$$\delta\mathcal{L} = \frac{\partial\mathcal{L}}{\partial g^{\mu\nu}}\delta g^{\mu\nu} \tag{2.98}$$

After substitute (2.97) and (2.98) into (2.96), we can obtain the energy-momentum tensor:

$$\mathcal{T}_{\mu\nu} = 2\frac{\partial\mathcal{L}}{\partial g^{\mu\nu}} - g_{\mu\nu}\mathcal{L} \tag{2.99}$$

By using the Lagrangian density of the massless scalar field (2.88), $T_{\mu\nu}$ takes the form as

$$\mathcal{T}_{\mu\nu} = \partial_\mu\phi\partial_\nu\phi - \frac{1}{2}g_{\mu\nu}\partial_\lambda\phi\partial^\lambda\phi \tag{2.100}$$

Then, the trace of (2.100) is

$$T^\mu_\mu = -\partial_\mu\phi\partial^\mu\phi \neq 0 \tag{2.101}$$

Thus, it can be seen that the massless scalar fields do not exhibit conformal invariance. However, its generalization to

$$\mathcal{L} = -\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + \frac{1}{2}\zeta_D R\phi^2 \tag{2.102}$$

where $\zeta_D$ is the conformal coupling constant

$$\zeta_D = \frac{(D-2)}{4(D-1)} \tag{2.103}$$

is conformally invariant. (Demir, 2004)

Let us proceed with the conformal transformation of this system. By using the (2.44), the transformed Ricci scalar is obtained in four dimension as

$$\widetilde{R} = \Omega^{-2}(x)R - 6\Omega^{-3}(x)\nabla_\mu\nabla_\nu\Omega(x)g^{\mu\nu} \tag{2.104}$$

The scalar field must transform as follows

$$\widetilde{\phi} = \Omega^{-1}(x)\phi \tag{2.105}$$

Finally, by using the (2.104), (2.105) and (2.44)

$$\begin{aligned}
\sqrt{-\widetilde{g}}\widetilde{\mathcal{L}} &= \sqrt{-g}\Omega^4[\frac{1}{2}g^{\mu\nu}\Omega^{-6}\partial_\mu\Omega\partial_\nu\Omega\phi^2 + \frac{1}{2}g^{\mu\nu}\Omega^{-4}\partial_\mu\phi\partial_\nu\phi \\
&- \Omega^{-5}g^{\mu\nu}\phi\partial_\mu\Omega\partial_\nu\phi + \frac{1}{12}R\phi^2\Omega^{-4} - \frac{1}{2}\Omega^{-5}\nabla_\mu\nabla_\nu\Omega g^{\mu\nu}\phi^2] \\
&\equiv \sqrt{-g}[\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + \frac{1}{12}R\phi^2] + \sqrt{-g}Q
\end{aligned} \tag{2.106}$$

where in an inertial frame (Christoffel symbols are zero),

$$\begin{aligned}
Q &= -\frac{1}{2}g^{\mu\nu}\Big\{[-\Omega^{-2}\partial_\mu\Omega\partial_\nu\Omega + \Omega^{-1}\partial_\mu\partial_\nu\Omega]\phi^2 + 2\phi\partial_\nu\phi\Omega^{-1}\partial_\mu\Omega\Big\} \\
&= -\frac{1}{2}g^{\mu\nu}\Big\{\partial_\nu\left[(\partial_\mu\Omega)\,\Omega^{-1}\right]\phi^2 + \partial_\mu\Omega\Omega^{-1}\partial_\nu(\phi^2)\Big\} \\
&= -\frac{1}{2}g^{\mu\nu}\partial_\nu\left[(\partial_\mu\Omega)\,\Omega^{-1}\phi^2\right] \\
&= -\frac{1}{2}\partial_\mu\left[(\partial^\mu\Omega)\,\Omega^{-1}\phi^2\right]
\end{aligned} \tag{2.107}$$

The term in the square brackets transform as a vector-component. Let us define $\chi^\mu = (\partial^\mu\Omega)\Omega^{-1}\phi^2$ and using the following formula

$$\nabla_\mu\chi^\mu = \frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}\chi^\mu) \tag{2.108}$$

Then, the $\sqrt{-g}$ in front of the term is cancelled and a total derivative is obtained

$$\int d^4\chi\sqrt{-g}\partial_\mu\chi^\mu \to \int_V dV\partial_\mu(\sqrt{-g}\chi^\mu) = \int_S dS\sqrt{-g}\chi^\mu \tag{2.109}$$

As we see from (2.109), $\sqrt{-g}Q$ does not contribute to the action. Thus, it can be seen that the (2.106) is invariant under conformal transformations. (Tywoniuk, 2004)

In addition to this, null geodesics are invariant under conformal transformations,

$$\widetilde{g}_{\mu\nu}\frac{dx^\mu(\lambda)}{d\lambda}\frac{dx^\nu(\lambda)}{d\lambda} = \Omega^2(x)g_{\mu\nu}\frac{dx^\mu(\lambda)}{d\lambda}\frac{dx^\nu(\lambda)}{d\lambda} = 0 \tag{2.110}$$

This means that a null curve $x^\mu(\lambda)$(curves on the surface of the light cone)is not affected by conformal transformations. Because its tangent vector $\frac{dx^\mu(\lambda)}{d\lambda}$ does not change after conformal transformations. Thus, it can be said that the light cone is conformally invariant.

Consequently, theories of massless fermions and vector fields ( such as electromagnetic field ) are conformally invariant. However, massless scalar fields are not conformal invariant unless they couple to scalar curvature R.

Conformal invariance is essential to investigate a theory at the requested unit systems. If a theory is conformally invariant, it can be studied at all conformally related frames. This leads to the mathematical simplicity of the calculations.

# CHAPTER 3

# CONFORMAL TRANSFORMATIONS IN GENERAL RELATIVITY

General Relativity is a theory which explains the gravity and relation between the geometric structures of spacetime and matter. Although, it gave the consistent results with the observations at the first times, there appeared some shortcomings from the cosmology and the quantum field theory by the improvements in the observation techniques. To overcome these shortcomings, a huge number of generalizations of the classical Einstein-Hilbert Lagrangian formulation of GR have been put forward. In this chapter, we will consider some alternative variational principles, based on different choices of the gravitational field variables like metric, connection or both. These alternative variational principles are known as different formulations of GR or modified theories of GR in terms of different gravitational fields. One of them is the metric formulation of GR as we know from the Einstein's theory of GR. Second one is the metric-affine formulation of GR. Last of them is the purely affine formulation of GR. These three formulations are dynamically equivalent formulations in Riemannian spacetime and this can be seen after obtaining the equations of motion. Let us explain these three formulations briefly.

- **Metric formulation of GR**

  In metric formulation of GR, the only independent variable which represents the gravitational field is the metric tensor. The other geometrical quantities like connection, Riemann tensor, Ricci tensor and Ricci scalar depend on metric and derivatives of metric. Thus, Lagrangian density of the gravitational fields can be generally written as:

$$\mathcal{L}_M(g, \partial g, \partial^2 g) = R(g, \partial g, \partial^2 g)[-det(g_{\alpha\beta})]^{1/2} \tag{3.1}$$

  Then, by taking the variation of the corresponding action with respect to the metric $g^{\alpha\beta}$, we obtain ten vacuum Einstein equations:

$$G_{\mu\nu}(g) = R_{\mu\nu}(g) - \frac{1}{2}g_{\mu\nu}R(g) = 0 \tag{3.2}$$

There are also two important properties of the metric formulation as we mention in Introduction. These are the 'metric compatibility' and 'torsion-free connection'. The first one means that the covariant derivative of metric is zero

$$\nabla_\alpha g_{\mu\nu} = 0 \tag{3.3}$$

and the latter one means that the connection is symmetric in its lower indices.

$$\Gamma^\lambda_{\alpha\beta} = \Gamma^\lambda_{\beta\alpha} \tag{3.4}$$

- **Metric-Affine formulation of GR**

  In metric-affine formulation of GR, the metric tensor and the connection are considered as independent variables. There is no *a priori* relation between connection and metric. Therefore, the gravitational fields can be represented by the metric and connection.Then, the Lagrangian density can be written as:

$$\mathcal{L}_{MA}(g,\Gamma) = g^{\mu\nu}R_{\mu\nu}(\Gamma,\partial\Gamma)[-det(g_{\mu\nu})]^{1/2} \tag{3.5}$$

Then, the equations of motion are obtained by taking the variation of corresponding action separately with respect to metric tensor $g^{\mu\nu}$ and the connection $\Gamma^\alpha_{\mu\nu}$. Respectively, these equations are:

$$G_{\mu\nu}(g,\Gamma,\partial\Gamma) = R_{\mu\nu}(\Gamma,\partial\Gamma) - \frac{1}{2}g_{\mu\nu}R(\Gamma,\partial\Gamma) = 0 \tag{3.6}$$

$$\nabla^\Gamma_\sigma g_{\mu\nu} = 0 \tag{3.7}$$

where $\nabla^\Gamma_\sigma$ denotes the covariant derivative of the general connection $\Gamma$. From (3.7) it follows that the general connection $\Gamma$ coincides with the Levi-Civita connection of g. In a sense, there appears a relation between connection and metric, *a posteriori*. Thus, the equations in (3.6) are the usual vacuum Einstein equations. It can be seen that the metric formulation and the metric-affine formulation are dynamically equivalent formulations in Riemannian geometry. However, in non-Riemannian geometry these formulations are not identical. Because there are new structures coming from non-Riemannian geometry like torsion, nonmetricity and anti-symmetric Ricci tensor.

- **Purely Affine formulation of GR**

  In purely affine formulation of GR, the only dynamical variable is a symmetric linear connection $\Gamma^{\alpha}_{\mu\nu}$. The Lagrangian density of gravitational fields is represented in most general form as $\mathcal{L}_{PA}(\Gamma, \partial\Gamma)$ and the first example of the purely affine action has been introduced by Einstein and Eddington:

$$\mathcal{L}_{PA} = \mathcal{L}_{EE}(\Gamma, \partial\Gamma) = \sqrt{|detR_{(\mu\nu)}(\Gamma)|} \tag{3.8}$$

  where the metric structure is determined by prescription:

$$g^{\mu\nu}\sqrt{|g|} = \frac{\partial \mathcal{L}_{EE}}{\partial R_{(\mu\nu)}(\Gamma)} \tag{3.9}$$

  For the (3.8), the metric turns out to be proportional to the Ricci tensor of general connection $\Gamma^{\alpha}_{\mu\nu}$. Then, the variation of the corresponding action of (3.8) with respect to the connection gives the second order Euler-Lagrange equation. By inserting of (3.9),

$$\nabla_{\nu}(g^{\alpha\beta}\sqrt{|g|}) = 0 \tag{3.10}$$

  As we mention in the metric-affine formulation, the last equation can be satisfied only if $\Gamma^{\alpha}_{\mu\nu} = \check{\Gamma}^{\alpha}_{\mu\nu}$. Here $\check{\Gamma}^{\alpha}_{\mu\nu}$ denotes the Levi-Civita connection. Metric tensor $g_{\mu\nu}$ is necessarily proportional to its own Ricci tensor. Thus, solutions of vacuum equations of motion which are generated by the (3.8) are the same as the solutions of the vacuum Einstein equations. It can be said that the purely affine formulation is dynamically equivalent to metric formulation.

In next sections, we will investigate the effects of conformal transformations on the metric formulation and metric-affine formulation of GR.

## 3.1. Metric Formulation

The aim of this section is to indicate how the conformal transformations affect the dynamical variables of general relativity via only transformation of the metric tensor (2.44). In General Relativity, we apply conformal transformations only to metric because there is a linear relation between the metric tensor and connection coefficient, namely Levi-Civita connection. Then all the other dynamics like Riemann curvature tensor, Ricci tensor and Ricci scalar, which depend on metric tensor and Levi-Civita connection, can be obtain from these two transformed geometric variables. Conformal transformations of them have been given in the previous chapter by using metric formulation.

The Einstein-Hilbert action in an arbitrary dimension D reads in $(g, \Gamma)$ frame as

$$S_{EH}\left[g\right] = \int d^D x\, \sqrt{-g}\left[\frac{1}{2} M_\star^{D-2} R - \Lambda_\star + \mathfrak{L}_m\left(g, \Psi\right)\right] \tag{3.11}$$

where $M_\star$ is the fundamental scale of gravity in $D$ dimensions, $\Lambda_\star$ is the cosmological term, and $\mathfrak{L}_m$ is the Lagrangian of the matter and radiation fields, collectively denoted by $\Psi$. By using the conformally transformed geometrical quantities of gravity, Let us form the transformed Einstein-Hilbert action in $\left(\widetilde{g}, \widetilde{\Gamma}\right)$ frame. For the metric $(-, +, \ldots, +)$ convention, Einstein-Hilbert action takes the form as

$$\begin{aligned}
S_{EH}\left[g, \overline{\phi}\right] &= \int d^D x\, \sqrt{-g}\Bigg\{\frac{1}{2} M_\star^{D-2}\left[\Omega^{D-2} R - 2(D-1)\Omega^{D-3} g^{\alpha\beta}\nabla_\alpha\nabla_\beta\Omega\right. \\
&\quad \left. -\, (D-1)(D-4)\Omega^{D-4} g^{\alpha\beta}\nabla_\alpha\Omega\nabla_\beta\Omega\right] - \Lambda_\star\Omega^D + \widetilde{\mathfrak{L}}_m\left(\widetilde{g}, \widetilde{\Psi}\right)\Bigg\} \\
&= \int d^D x\, \sqrt{-g}\Bigg\{\frac{1}{2} M_\star^{D-2}\left[\Omega^{D-2} R - 2(D-1)\Omega^{D-2} g^{\alpha\beta}\nabla_\alpha\nabla_\beta \ln\Omega\right. \\
&\quad -2(D-1)\Omega^{D-4} g^{\alpha\beta}\nabla_\alpha\Omega\nabla_\beta\Omega - (D-1)(D-4)\Omega^{D-4} g^{\alpha\beta}\nabla_\alpha\Omega\nabla_\beta\Omega\Big] \\
&\quad -\Lambda_\star\Omega^D + \widetilde{\mathfrak{L}}_m\left(\widetilde{g}, \widetilde{\Psi}\right)\Bigg\} \\
&= \int d^D x\, \sqrt{-g}\Bigg\{\frac{1}{2} M_\star^{D-2}\left[\Omega^{D-2} R - 2(D-1)\Omega^{D-2}\Box\ln\Omega\right. \\
&\quad \left. -(D-1)(D-2)\Omega^{D-4} g^{\alpha\beta}\nabla_\alpha\Omega\nabla_\beta\Omega\right] - \Lambda_\star\Omega^D + \widetilde{\mathfrak{L}}_m\left(\widetilde{g}, \widetilde{\Psi}\right)\Bigg\}
\end{aligned}$$

where $\Box = g^{\alpha\beta}\nabla_\alpha\nabla_\beta$

$$\begin{aligned}
&\equiv \int d^D x\,\sqrt{-g}\left[\frac{1}{2}g^{\mu\nu}\left(\partial_\mu\overline{\phi}\right)\left(\partial_\nu\overline{\phi}\right)+\frac{1}{2}\zeta_D\overline{\phi}^2 R\right. \\
&- \left.\lambda_D\left(\zeta_D\overline{\phi}^2\right)^{\frac{D}{D-2}}+\widetilde{\mathfrak{L}}_m\left(\widetilde{g},\widetilde{\Psi}\right)\right] \qquad (3.12)
\end{aligned}$$

where the two dimensionless constants

$$\zeta_D = \frac{D-2}{4(D-1)}\,,\ \lambda_D = \frac{\Lambda_\star}{M_\star^D} \qquad (3.13)$$

designate, respectively, the conformal coupling of $\overline{\phi}$ to $R$ and the self-coupling of $\overline{\phi}$. The scalar field $\overline{\phi}$

$$\overline{\phi} = \frac{1}{\sqrt{\zeta_D}}\ \left(M_\star\Omega\right)^{\frac{(D-2)}{2}} \qquad (3.14)$$

derives from the conformal factor $\Omega$ in order to have canonical kinetic term. The quantity $\widetilde{\mathfrak{L}}_m\left(g,\widetilde{\Psi}\right)$ in (3.12) is the transformed matter Lagrangian, where each matter field $\Psi$ transforms, together with the metric, by an appropriate conformal weight.

This new action executes local conformal invariance (Weyl invariance) under the transformations

$$g_{\alpha\beta}\longrightarrow\psi^2\,g_{\alpha\beta}\ ,\ \ \overline{\phi}\longrightarrow\psi^{-\frac{(D-2)}{2}}\,\overline{\phi} \qquad (3.15)$$

where inhomogeneous terms generated by the kinetic term of $\overline{\phi}$ are neutralized by the terms generated by the transformation of the curvature scalar $R$.This happens thanks to the special, conformal value of $\zeta_D$. Therefore, the transformed action (3.12) provides a locally conformal-invariant representation of the original Einstein-Hilbert action (3.11). Notably, the original action (3.11) exhibits no sign of conformal invariance but the transformed one does and the reason behind it is the dressing of $M_\star$ and $\Lambda_\star$ by the transformation field $\Omega$ (Bekenstein, 1980), (Deser, 1970)

As we see from the (3.12), conformal transformations change the Einstein frame to the Jordan frame. Thus, a scalar field coupling non-minimally to the gravitational field occurs in the system. However, there is a point to notice about (3.12). The scalar field $\overline{\phi}$ (which is a function of the conformal factor $\Omega$) is a ghost (Demir, 2004), (Aslan, 2006), (Metaxas, 2009), (Gibbons, 1978). This is an unavoidable feature if gravity is to be an attractive force. Its ghosty nature follows from its non-positive kinetic term, and it signals that the system has no lower bound for energy. Such systems are inherently unphysical, and there seems to be no way of avoiding it unless some nonlinearities are added as extra features (Gabadadze, 2005).

## 3.2. Metric-Affine Formulation

As we mention before, metric-affine formulation (similar to Palatini formalism) implies that metric and connection are independent geometric variables (Peldan, 1994), (Magnano, 2005), (Dabrowski, 2008), as they indeed are. The related theory to this formulation is called metric-affine gravity. One of the most important consequences of this formulation is that conformal transformation of metric tensor gives rise to no direct change in connection, as happens in metric formulation of GR. Thus, there is no telling of how the general connection

$$\Gamma^{\lambda}_{\alpha\beta} \neq \Gamma^{\lambda}_{\alpha\beta}(g) \tag{3.16}$$

should transform under a rescaling of distances. In fact, the fact that connection has nothing to do with measuring the distances can be taken to imply that the connection $\Gamma^{\lambda}_{\alpha\beta}$ is completely inert under (2.44). However, it is still possible that connection transforms in some way, not necessarily like (2.49). Stating in a clearer fashion, there arise two main categories to be explored:

- The connection $\Gamma$ can be conformal-invariant: $\Gamma^{\lambda}_{\alpha\beta} \rightarrow \Gamma^{\lambda}_{\alpha\beta}$ despite (2.44) (Weyl, 1950),
- The connection $\Gamma$ can transform in various ways: Multiplicatively, additively or both while metric transforms as in (2.44).

Each of these two possibilities gives rise to novel effects not found in metrical GR, as indicated by the dependence of the Riemann curvature tensor on the connection Therefore in this section we shall analyze conformal transformations in two separate cases in regard to the transformation properties of the connection. In course of the analysis, the main goal will be to find appropriate transformation rules for $\Gamma^{\lambda}_{\alpha\beta}$ so that the resulting scalar field theory (in terms of the conformal factor $\Omega$) assumes physically sensible properties like emergent conformal invariance and absence of ghosts. Indeed, the main problem with the metrical GR discussed above is the unavoidable presence of a ghosty scalar in the spectrum. We will find that metric-affine gravity is capable of realizing conformal invariance and accommodating non-ghost scalar degrees of freedom.

In the metric-affine gravity, the Einstein-Hilbert action can be written as

$$S_{EH}\left[g,\Gamma\right] = \int d^D x\,\sqrt{-g}\left\{\frac{1}{2}M_{\star}^{D-2}g^{\mu\nu}\mathbb{R}_{\mu\nu}\left(\Gamma\right) - \Lambda_{\star} + \mathcal{L}\left(\Gamma-\check{\Gamma}, g, \psi_{\text{matter}}\right)\right\} \tag{3.17}$$

in a general $(g,\Gamma)$ frame. Here, $\psi_{\text{matter}}$ collectively denotes the matter fields, and $\mathcal{L}$ is composed of

$$\mathcal{L} = \mathcal{L}_{\text{geo}}\left(g,\mathcal{D}\right) + \mathcal{L}_{\text{ matter}}\left(g,\mathcal{D},\psi_{\text{matter}}\right) \tag{3.18}$$

which respectively stand for the geometrical and matter sector contributions. The geometrical sector consists of the rank (1,2) tensor field

$$\mathcal{D}^{\lambda}_{\alpha\beta} = \Gamma^{\lambda}_{\alpha\beta} - \check{\Gamma}^{\lambda}_{\alpha\beta} \tag{3.19}$$

as an additional geometrodynamical tensorial quantity. This variable is highly natural to consider since in the presence of the metric $g_{\alpha\beta}$ one naturally defines its compatible connection *i. e.* the Levi-Civita connection. Then difference between $\Gamma^{\lambda}_{\alpha\beta}$ and Levi-Civita connection becomes a tensorial quantity to be taken into account.

Here it is useful to clarify the meaning of $\mathcal{L}_{\text{geo}}\left(g,\mathcal{D}\right)$ in terms of the known dynamical quantities akin to non-Riemannian geometries. Non-Riemannian geometries are characterized by torsion tensor,

$$\mathbb{S}^{\lambda}_{\alpha\beta} = \mathcal{D}^{\lambda}_{\alpha\beta} - \mathcal{D}^{\lambda}_{\beta\alpha} \tag{3.20}$$

non-metricity tensor,

$$\mathbb{Q}_{\lambda}^{\alpha\beta} = \mathcal{D}_{\lambda\rho}^{\alpha} g^{\rho\beta} + \mathcal{D}_{\lambda\rho}^{\beta} g^{\alpha\rho} \tag{3.21}$$

Ricci curvature tensor,

$$\mathbb{R}_{\mu\nu}(\Gamma) = \mathbb{R}_{\mu\alpha\nu}^{\alpha}(\Gamma) \tag{3.22}$$

and the anti-symmetric Ricci curvature tensor

$$\mathbb{R}'_{\beta\nu} = \mathbb{R}_{\alpha\beta\nu}^{\alpha}(\Gamma) \tag{3.23}$$

All these tensor fields make up the geometrical sector of the theory. Clearly, torsion vanishes for theories with symmetric connections, and this is also the case throughout the present work. Moreover, $\mathbb{R}'_{\beta\nu}$ is an anti-symmetric tensor field whose curvature scalar vanishes identically. This tensor can give contributions to Lagragrangian at the quadratic and higher levels. The Lagrangian $\mathcal{L}_{\text{geo}}(g, \mathcal{D})$ includes all these tensorial contributions through the $\mathcal{D}_{\alpha\beta}^{\lambda}$ dependence

$$\mathcal{L}_{\text{geo}}(g, \mathcal{D}) = \mathcal{L}_{\text{geo}}(g, \mathbb{S}, \mathbb{Q}, \mathbb{R}, \mathbb{R}') \ , \tag{3.24}$$

throughout the text. It is clear that, $\mathcal{L}_{\text{geo}}$ can involve arbitrary powers and derivatives of the tensorial connection $\mathcal{D}_{\alpha\beta}^{\lambda}$.

The Lagrangian $\mathcal{L}$, through its $\Gamma$ or $\mathcal{D}$ dependence, gives rise to important modifications in the equations of motion (Burton, 1998) so that $\Gamma = \check{\Gamma}$ limit (which is what is behind the Palatini formulation) does not necessarily hold. The contributions of $\mathcal{L}_{\text{geo}}(g, \mathcal{D})$ and $\mathfrak{L}_m(g, \mathcal{D}, \Psi)$ generically avoid the limit $\Gamma = \check{\Gamma}$. We will discuss this point in the following section.

In two subsections to follow, we will not explicitly analyze $\mathcal{L}$; our analysis will take into account the minimal structure only for explicating the implications of the conformal transformations. This is done for the purpose of definiteness and simplicity. We will turn on $\mathcal{L}_{\text{geo}}$ in Sec. 3.3 for discussing the equations of motion for $\mathcal{D}_{\alpha\beta}^{\lambda}$.

### 3.2.1. Conformal-Invariant Connection

We start the analysis by first considering a conformal-invariant connection by which we mean that connection is inert to rescalings of the metric. Therefore, along with the transformation of metric (2.44), the connection transforms as (Weyl, 1950)

$$\widetilde{\Gamma}^{\lambda}_{\alpha\beta} = \Gamma^{\lambda}_{\alpha\beta} \tag{3.25}$$

and hence,

$$\widetilde{\mathbb{R}}^{\alpha}_{\mu\beta\nu}\left(\widetilde{\Gamma}\right) = \mathbb{R}^{\alpha}_{\mu\beta\nu}\left(\Gamma\right) \, , \; \widetilde{\mathbb{R}}_{\mu\nu}\left(\widetilde{\Gamma}\right) = \mathbb{R}_{\mu\nu}\left(\Gamma\right) \tag{3.26}$$

Since Riemann tensor (2.51) does not involve the metric tensor unless the connection does. The only non-trivial transformation occurs for the Ricci scalar

$$\widetilde{g}^{\mu\nu}\widetilde{\mathbb{R}}_{\mu\nu}\left(\widetilde{\Gamma}\right) = \Omega^{-2} g^{\mu\nu}\mathbb{R}_{\mu\nu}\left(\Gamma\right) \tag{3.27}$$

which is nothing but an overall dressing by $\Omega^{-2}$. In particular, no derivatives of $\Omega$ are involved in the transformations of curvature tensor. This implies that $\Omega$ can develop no kinetic term. Indeed, under the transformation (3.27), the action (3.17) with conformal-invariant connection goes over

$$S_{EH}\left[\widetilde{g}, \widetilde{\Gamma}, \overline{\phi}\right] = \int d^D x \, \sqrt{-g}\left[\frac{1}{2}\zeta_D \overline{\phi}^2 g^{\mu\nu}\mathbb{R}_{\mu\nu}\left(\Gamma\right) - \lambda_D\left(\zeta_D \overline{\phi}^2\right)^{\frac{D}{D-2}}\right] \tag{3.28}$$

in $\left(\widetilde{g}, \widetilde{\Gamma}\right)$ frame and in the absence of the geometrical and matter parts $\mathcal{L}$. Obviously, this action is locally conformal invariant under

$$g_{\alpha\beta} \longrightarrow \psi^2 \, g_{\alpha\beta} \; , \;\; \Gamma^{\lambda}_{\alpha\beta} \longrightarrow \Gamma^{\lambda}_{\alpha\beta} \; , \;\; \overline{\phi} \longrightarrow \psi^{-\frac{(D-2)}{2}} \, \overline{\phi} \tag{3.29}$$

as was the case for metrical gravity, defined in (3.15). Therefore, though the original action (3.17) exhibits no sign of conformal invariance and hence the new action (3.28) arises, this transformed action exhibits manifest conformal invariance. The reason is as in the metrical gravity; the conformal factor $\Omega$ dresses the fixed scales ($M_\star$ and $\Lambda_\star$) in (3.17) to make them as effective fields transforming nontrivially under local rescalings of the fields (Bekenstein, 1980).

Apart from this emergent conformal invariance, the action (3.28) possesses a highly important aspect not found in metrical GR: It is that $\overline{\phi}$ is not a ghost at all. It is a non-dynamical scalar field having vanishing kinetic energy, and thus, the impasse caused by the ghosty scalar field encountered in metrical GR is resolved. The non-dynamical nature of $\overline{\phi}$ continues to hold even if the matter sector is included. This result stems from the affine nature of the gravitational theory under concern, and especially from the invariance of the connection under conformal transformations.

At this point it proves useful to discuss the 'non-dynamical' nature of the scalar field $\overline{\phi}$ in the action (3.28). At the level of the transformations employed and the Einstein-Hilbert action the non-dynamical nature of the conformal factor (and hence, the $\overline{\phi}$) is unavoidable. However, one immediately notices that this 'non-dynamical' structure depends sensitively on the quantum fluctuations. Indeed, if quantum fluctuations are included into (3.28) the scalar field $\overline{\phi}$ is found to develop a kinetic term via the graviton loops (Shapiro, 1995), (Shapiro, 1997). We shall keep analysis at the classical level throughout the work. However, one is warned of such delicate effects which can come from quantum corrections or higher order geometrical invariants.

### 3.2.2. Conformal-Variant Connection

As an alternative to conformal-invariant connection, in this subsection we investigate different scenarios where $\Gamma^\lambda_{\alpha\beta}$ exhibits nontrivial changes along with the transformation of the metric in (2.44).

As a possible transformation property, we first discuss the multiplicative transformation of connection. Namely, connection transforms similar to the metric itself

$$\widetilde{\Gamma}^\lambda_{\alpha\beta} = f(\Omega)\Gamma^\lambda_{\alpha\beta} \tag{3.30}$$

where $f(\Omega)$ is a generic function of the conformal factor. Inserting this transformed connection into (2.51), one straightforwardly determines the transformed Riemann tensor

$$\begin{aligned}
\widetilde{\mathbb{R}}^{\alpha}_{\mu\beta\nu}\left(\widetilde{\Gamma}\right) &= \partial_\beta\widetilde{\Gamma}^{\alpha}_{\mu\nu} - \partial_\nu\widetilde{\Gamma}^{\alpha}_{\mu\beta} + \widetilde{\Gamma}^{\alpha}_{\beta\lambda}\widetilde{\Gamma}^{\lambda}_{\mu\nu} - \widetilde{\Gamma}^{\alpha}_{\nu\lambda}\widetilde{\Gamma}^{\lambda}_{\mu\beta} \\
&= \partial_\beta(f(\Omega)\Gamma^{\alpha}_{\mu\nu}) - \partial_\nu(f(\Omega)\Gamma^{\alpha}_{\mu\beta}) + f^2(\Omega)\Gamma^{\alpha}_{\beta\lambda}\Gamma^{\lambda}_{\mu\nu} - f^2(\Omega)\Gamma^{\alpha}_{\nu\lambda}\Gamma^{\lambda}_{\mu\beta} \\
&= \Gamma^{\alpha}_{\mu\nu}\partial_\beta f(\Omega) + f(\Omega)\partial_\beta\Gamma^{\alpha}_{\mu\nu} - \Gamma^{\alpha}_{\mu\beta}\partial_\nu f(\Omega) - f(\Omega)\partial_\nu\Gamma^{\alpha}_{\mu\beta} \\
&+ f^2(\Omega)\Gamma^{\alpha}_{\beta\lambda}\Gamma^{\lambda}_{\mu\nu} - f^2(\Omega)\Gamma^{\alpha}_{\nu\lambda}\Gamma^{\lambda}_{\mu\beta} + f(\Omega)\Gamma^{\alpha}_{\beta\lambda}\Gamma^{\lambda}_{\mu\nu} - f(\Omega)\Gamma^{\alpha}_{\beta\lambda}\Gamma^{\lambda}_{\mu\nu} \\
&+ f(\Omega)\Gamma^{\alpha}_{\nu\lambda}\Gamma^{\lambda}_{\mu\beta} - f(\Omega)\Gamma^{\alpha}_{\nu\lambda}\Gamma^{\lambda}_{\mu\beta} \\
&= f(\Omega)\,\mathbb{R}^{\alpha}_{\mu\beta\nu}(\Gamma) + \partial_\beta f(\Omega)\Gamma^{\alpha}_{\mu\nu} - \partial_\nu f(\Omega)\Gamma^{\alpha}_{\mu\beta} \\
&+ f(\Omega)\left(f(\Omega) - 1\right)\left[\Gamma^{\alpha}_{\beta\lambda}\Gamma^{\lambda}_{\mu\nu} - \Gamma^{\alpha}_{\nu\lambda}\Gamma^{\lambda}_{\mu\beta}\right] \qquad (3.31)
\end{aligned}$$

and by contraction, the transformed Ricci scalar

$$\begin{aligned}
\widetilde{g}^{\mu\nu}\widetilde{\mathbb{R}}_{\mu\nu}\left(\widetilde{\Gamma}\right) &= \Omega^{-2}\Bigg\{ f(\Omega)\,\mathbb{R}(\Gamma) + \partial_\alpha f(\Omega)\, g^{\mu\nu}\Gamma^{\alpha}_{\mu\nu} - \partial_\nu f(\Omega)\, g^{\mu\nu}\Gamma^{\alpha}_{\mu\alpha}. \\
&+ f(\Omega)\left(f(\Omega) - 1\right)\left[\Gamma^{\alpha}_{\alpha\lambda}g^{\mu\nu}\Gamma^{\lambda}_{\mu\nu} - \Gamma^{\alpha}_{\nu\lambda}g^{\mu\nu}\Gamma^{\lambda}_{\mu\alpha}\right]\Bigg\}. \qquad (3.32)
\end{aligned}$$

It is straightforward to check that the $\Gamma$–dependent terms at the right-hand side form a true scalar under general coordinate transformations.

$$\begin{aligned}
\rightarrow\quad & \Omega^{-2}\partial_\alpha f(\Omega)\Bigg[\frac{\partial x^\mu}{\partial x^{\mu'}}\frac{\partial x^\nu}{\partial x^{\nu'}}g^{\mu'\nu'}\left(\frac{\partial x^\alpha}{\partial x^{\alpha'}}\frac{\partial x^{\mu'}}{\partial x^\mu}\frac{\partial x^{\nu'}}{\partial x^\nu}\Gamma^{\alpha'}_{\mu'\nu'} + \frac{\partial x^\alpha}{\partial x^{\alpha'}}\frac{\partial^2 x^{\alpha'}}{\partial x^\mu\partial x^\nu}\right) \\
& -\frac{\partial x^\mu}{\partial x^{\mu'}}\frac{\partial x^\alpha}{\partial x^{\alpha'}}g^{\mu'\alpha'}\left(\frac{\partial x^\lambda}{\partial x^{\lambda'}}\frac{\partial x^{\lambda'}}{\partial x^\lambda}\frac{\partial x^{\mu'}}{\partial x^\mu}\Gamma^{\lambda'}_{\lambda'\mu'} + \frac{\partial x^\lambda}{\partial x^{\lambda'}}\frac{\partial^2 x^{\lambda'}}{\partial x^\lambda\partial x^\mu}\right)\Bigg] \\
& +\Omega^{-2}(f^2 - f)\Bigg[\left(\frac{\partial x^\alpha}{\partial x^{\alpha'}}\frac{\partial x^{\alpha'}}{\partial x^\alpha}\frac{\partial x^{\lambda'}}{\partial x^\lambda}\Gamma^{\alpha'}_{\alpha'\lambda'} + \frac{\partial x^\alpha}{\partial x^{\alpha'}}\frac{\partial^2 x^{\alpha'}}{\partial x^\alpha\partial x^\lambda}\right) \\
& \cdot\frac{\partial x^\mu}{\partial x^{\mu'}}\frac{\partial x^\nu}{\partial x^{\nu'}}g^{\mu'\nu'}\left(\frac{\partial x^\lambda}{\partial x^{\lambda'}}\frac{\partial x^{\mu'}}{\partial x^\mu}\frac{\partial x^{\nu'}}{\partial x^\nu}\Gamma^{\lambda'}_{\mu'\nu'} + \frac{\partial x^\lambda}{\partial x^{\lambda'}}\frac{\partial^2 x^{\lambda'}}{\partial x^\mu\partial x^\nu}\right) \\
& -\left(\frac{\partial x^\alpha}{\partial x^{\alpha'}}\frac{\partial x^{\nu'}}{\partial x^\nu}\frac{\partial x^{\lambda'}}{\partial x^\lambda}\Gamma^{\alpha'}_{\nu'\lambda'} + \frac{\partial x^\alpha}{\partial x^{\alpha'}}\frac{\partial^2 x^{\alpha'}}{\partial x^\nu\partial x^\lambda}\right) \\
& \cdot\frac{\partial x^\mu}{\partial x^{\mu'}}\frac{\partial x^\nu}{\partial x^{\nu'}}g^{\mu'\nu'}\left(\frac{\partial x^\lambda}{\partial x^{\lambda'}}\frac{\partial x^{\mu'}}{\partial x^\mu}\frac{\partial x^{\alpha'}}{\partial x^\alpha}\Gamma^{\lambda'}_{\mu'\alpha'} + \frac{\partial x^\lambda}{\partial x^{\lambda'}}\frac{\partial^2 x^{\lambda'}}{\partial x^\mu\partial x^\alpha}\right)\Bigg]
\end{aligned}$$

$$
\begin{aligned}
= \;& \Omega^{-2}\partial_{\alpha} f(\Omega)\Bigg[\frac{\partial x^{\alpha}}{\partial x^{\alpha'}} g^{\mu'\nu'}\Gamma^{\alpha'}_{\mu'\nu'} + \frac{\partial x^{\alpha}}{\alpha'}\frac{\partial x^{\mu}}{\partial x^{\mu'}}\frac{\partial x^{\nu}}{\partial x^{\nu'}}\frac{\partial}{\partial x^{\nu}}\left(\frac{\partial x^{\alpha'}}{\partial x^{\mu}}\right) g^{\mu'\alpha'} \\
& -\frac{\partial x^{\alpha}}{\partial x^{\alpha'}} g^{\mu'\alpha'}\Gamma^{\lambda'}_{\lambda'\mu'} - \frac{\partial x^{\alpha}}{\partial x^{\alpha'}}\frac{\partial x^{\lambda}}{\partial x^{\lambda'}} g^{\mu'\nu'}\frac{\partial x^{\mu}}{\partial x^{\mu'}}\frac{\partial^2 x^{\lambda'}}{\partial x^{\lambda}\partial x^{\mu}}\Bigg] \\
& +\Omega^{-2}(f^2-f)\left[\left(\frac{\partial x^{\lambda'}}{\partial x^{\lambda}}\Gamma_{\alpha'\lambda'}\right)\left(g^{\mu'\nu'}\frac{\partial x^{\lambda}}{\partial x^{\lambda'}}\Gamma^{\lambda'}_{\mu'\nu'} + \frac{\partial x^{\lambda}}{\partial x^{\lambda'}}\frac{\partial x^{\mu}}{\partial x^{\mu'}} g^{\mu'\nu'}\frac{\partial x^{\nu}}{\partial x^{\nu'}}\frac{\partial^2 x^{\lambda'}}{\partial x^{\mu}\partial x^{\nu}}\right)\right] \\
& -\Omega^{-2}(f^2-f)\Bigg[\left(\frac{\partial x^{\alpha}}{\partial x^{\alpha'}}\frac{\partial x^{\lambda'}}{\partial x^{\lambda}}\Gamma^{\alpha'}_{\nu'\lambda'} + \frac{\partial x^{\alpha}}{\partial x^{\alpha'}}\frac{\partial x^{\nu}}{\partial x^{\nu'}}\frac{\partial^2 x^{\alpha'}}{\partial x^{\nu}\partial x^{\lambda}}\right) \\
& \cdot\left(g^{\mu'\nu'}\frac{\partial x^{\lambda}}{\partial x^{\lambda'}}\frac{\partial x^{\alpha'}}{\partial x^{\alpha}}\Gamma^{\lambda'}_{\mu'\alpha'} + g^{\mu'\nu'}\frac{\partial x^{\mu}}{\partial x^{\mu'}}\frac{\partial x^{\lambda}}{\partial x^{\lambda'}}\frac{\partial^2 x^{\lambda'}}{\partial x^{\mu}\partial x^{\alpha}}\right)\Bigg] \\
= \;& \Omega^{-2}\frac{\partial f}{\partial x^{\alpha'}}\frac{\partial x^{\alpha'}}{\partial x^{\alpha}}\frac{\partial x^{\alpha}}{\partial x^{\alpha'}} g^{\mu'\nu'}\Gamma^{\alpha'}_{\mu'\nu'} - \Omega^{-2}\frac{\partial f}{\partial x^{\alpha'}}\frac{\partial x^{\alpha'}}{\partial x^{\alpha}}\frac{\partial x^{\alpha}}{\partial x^{\alpha'}} g^{\mu'\alpha'}\Gamma^{\lambda'}_{\mu'\lambda'} \\
& +\Omega^{-2}(f^2-f)\left[\Gamma^{\alpha'}_{\alpha'\lambda'} g^{\mu'\nu'}\Gamma^{\lambda'}_{\mu'\nu'} - \Gamma^{\alpha'}_{\nu'\lambda'} g^{\mu'\nu'}\Gamma^{\lambda'}_{\mu'\alpha'}\right] \\
= \;& \Omega^{-2}\partial_{\alpha'} f(\Omega) g^{\mu'\nu'}\Gamma^{\alpha'}_{\mu'\nu'} - \Omega^{-2}\partial_{\nu'} f(\Omega) g^{\mu'\nu'}\Gamma^{\alpha'}_{\alpha'\mu'} \\
& +\Omega^{-2}(f^2-f)\left[\Gamma^{\alpha'}_{\alpha'\lambda'} g^{\mu'\nu'}\Gamma^{\lambda'}_{\mu'\nu'} - \Gamma^{\alpha\prime}_{\nu'\lambda'} g^{\mu'\nu'}\Gamma^{\lambda'}_{\mu'\alpha'}\right]
\end{aligned}
$$

This conformal transformation rule for Ricci scalar dictates what form the gravitational action (3.17) in $(g, \Gamma)$ frame takes in $(\widetilde{g}, \widetilde{\Gamma})$ frame. It is clear that the transformed action will involve $\Omega$ as well as its partial derivatives. Therefore, contrary to the previous case of conformal-invariant connection, $\Omega$ is a dynamical field. However, it does not possess a true kinetic term in the sense of a scalar field theory. Its derivative interactions are always accompanied by the connection, $\Gamma^{\lambda}_{\alpha\beta}$.

As another transformation property of the connection, we now turn to analysis of additive transformation of $\Gamma^{\lambda}_{\alpha\beta}$. We thus consider the generic transformation rule

$$
\widetilde{\Gamma}^{\lambda}_{\alpha\beta} = \Gamma^{\lambda}_{\alpha\beta} + \Delta^{\lambda}_{\alpha\beta}(\Omega) \tag{3.33}
$$

where $\Delta^{\lambda}_{\alpha\beta}(\Omega)$, being the difference between $\widetilde{\Gamma}^{\lambda}_{\alpha\beta}$ and $\Gamma^{\lambda}_{\alpha\beta}$, is a rank (1,2) tensor field. It is a tensorial connection. This transformation of the connection is understood to run simultaneously with the transformation of the metric in (2.44). Then, as follows from (2.51), the Riemann tensor transforms as

$$
\widetilde{\mathbb{R}}^{\alpha}_{\mu\beta\nu}\left(\widetilde{\Gamma}\right) = \mathbb{R}^{\alpha}_{\mu\beta\nu}(\Gamma) + \nabla_{\beta}\Delta^{\alpha}_{\mu\nu} - \nabla_{\nu}\Delta^{\alpha}_{\mu\beta} + \Delta^{\alpha}_{\lambda\beta}\Delta^{\lambda}_{\mu\nu} - \Delta^{\alpha}_{\lambda\nu}\Delta^{\lambda}_{\beta\mu} \tag{3.34}
$$

where the $\Delta$–dependent part at the right-hand side, though seems so, is not a true curvature tensor; it is not generated by any of the covariant derivatives induced by $\Gamma^{\lambda}_{\alpha\beta}$ or $\Gamma^{\lambda}_{\alpha\beta}$. This extra $\Delta$–dependent piece is just a rank (1,3) tensor field induced by $\Delta^{\lambda}_{\alpha\beta}$ alone.

In accordance with the transformation of Riemann tensor in (3.34), the Ricci scalar transforms as

$$\widetilde{g}^{\mu\nu}\widetilde{\mathbb{R}}_{\mu\nu}\left(\widetilde{\Gamma}\right) = \Omega^{-2}g^{\mu\nu}\left\{\mathbb{R}_{\mu\nu}\left(\Gamma\right) + \nabla_{\alpha}\Delta^{\alpha}_{\mu\nu} - \nabla_{\nu}\Delta^{\alpha}_{\mu\alpha} + \Delta^{\alpha}_{\lambda\alpha}\Delta^{\lambda}_{\mu\nu} - \Delta^{\alpha}_{\lambda\nu}\Delta^{\lambda}_{\alpha\mu}\right\} . \tag{3.35}$$

This transformation rule is rather generic for connections which transform additively (Demir, 2004). Nevertheless, it is necessary to determine physically admissible forms of $\Delta^{\lambda}_{\alpha\beta}$ so that the conformal factor $\Omega$ assumes appropriate dynamics in regard to absence of ghosts and emerging of a new conformal invariance in the sense of (3.29).

At this stage, right question to ask is this: 'How is $\Delta^{\lambda}_{\alpha\beta}$ related to $\Omega$ ?' To answer this question, one has to check out a series of possibilities. Being a rank (1,2) tensor field, $\Delta^{\lambda}_{\alpha\beta}$ can assume a number of forms like $V^{\lambda}g_{\alpha\beta}$ or $\delta^{\lambda}_{\alpha}V_{\beta}$ or $V^{\lambda}T_{\alpha\beta}$, with $V_{\alpha}$ being a vector field and $T_{\alpha\beta}$ a symmetric tensor field. If the transformation of connection (3.33) is to coexist with that of the metric in (2.44), then $V_{\alpha}$, $T_{\alpha\beta}$ or any other structure must be related to gradients of $\Omega$ so that $\Delta^{\lambda}_{\alpha\beta}$ vanishes when $\Omega$ is unity or, more precisely, constant. Therefore, one can identify $V_{\alpha}$ with $\partial_{\alpha}\Omega$, and $T_{\alpha\beta}$ with $\nabla_{\alpha}\partial_{\beta}\Omega$ or $\partial_{\alpha}\Omega\ \partial_{\beta}\Omega$. Consequently, $\Delta^{\lambda}_{\alpha\beta}$ should be composed of $\partial^{\lambda}\Omega\ g_{\alpha\beta}$, $\delta^{\lambda}_{\alpha}\partial_{\beta}\Omega$ or relevant higher derivatives of $\Omega$ or higher powers of $\partial_{\alpha}\Omega$. Hence, at the linear level, $\Delta^{\lambda}_{\alpha\beta}$ must be of the form

$$\Delta^{\lambda}_{\alpha\beta} \quad = c_1\left(\delta^{\lambda}_{\alpha}\partial_{\beta}\ln\Omega + \delta^{\lambda}_{\beta}\partial_{\alpha}\ln\Omega\right) + c_2 g_{\alpha\beta}\partial^{\lambda}\ln\Omega \tag{3.36}$$

where $c_1$ and $c_2$ are real constants. In here, one notices that a very similar form of this connection was also found in (Park, 1997),**?**, in spacetimes with non-vanishing torsion. One readily notices that the tensorial structures involved here are the same as the ones appearing in the transformation of the Levi-Civita connection under conformal transformations. This is seen from direct comparison of (3.36) with (2.50). The difference is the generality of (3.36) in terms of the constants $c_1$ and $c_2$ since $c_1 = -c_2 = 1$ in the transformation (2.50) of the Levi-Civita connection. Under the transformation (3.36), the

Ricci scalar $\widetilde{\mathbb{R}}$ in (3.35) takes the form as

$$\begin{aligned}
\widetilde{\mathbb{R}} &= \Omega^{-2}R + \Omega^{-2}g^{\mu\nu}\Big\{c_1\delta^\alpha_\mu\nabla_\alpha\partial_\nu\ln\Omega + c_1\delta^\alpha_\nu\nabla_\alpha\partial_\mu\ln\Omega + c_2 g_{\mu\nu}\nabla_\alpha\partial^\alpha\ln\Omega\Big\} \\
&- \Omega^{-2}g^{\mu\nu}\Big\{c_1\delta^\alpha_\mu\nabla_\nu\partial_\alpha\ln\Omega + c_1\delta^\alpha_\alpha\nabla_\nu\partial_\mu\ln\Omega + c_2 g_{\mu\alpha}\nabla_\nu\partial^\alpha\ln\Omega\Big\} \\
&+ \Omega^{-2}g^{\mu\nu}\Big\{c_1^2[\delta^\alpha_\alpha\delta^\lambda_\mu\partial_\lambda\ln\Omega\partial_\nu\ln\Omega + \delta^\alpha_\alpha\delta^\lambda_\nu\partial_\lambda\ln\Omega\partial_\mu\ln\Omega \\
&+ \delta^\lambda_\mu\delta^\alpha_\lambda\partial_\alpha\ln\Omega\partial_\nu\ln\Omega + \delta^\alpha_\lambda\delta^\lambda_\nu\partial_\alpha\ln\Omega\partial_\mu\ln\Omega] \\
&+ c_1c_2[\delta^\alpha_\alpha g_{\mu\nu}\partial_\lambda\ln\Omega\partial^\lambda\ln\Omega + \delta^\alpha_\lambda g_{\mu\nu}\partial_\alpha\ln\Omega\partial^\lambda\ln\Omega] \\
&+ c_1c_2[\delta^\lambda_\mu g_{\alpha\lambda}\partial_\nu\ln\Omega\partial^\alpha\ln\Omega + \delta^\lambda_\nu g_{\alpha\lambda}\partial_\mu\ln\Omega\partial^\alpha\ln\Omega] \\
&+ c_2^2 g_{\alpha\lambda}g_{\mu\nu}\partial^\alpha\ln\Omega\partial^\lambda\ln\Omega\Big\} \\
&- \Omega^{-2}g^{\mu\nu}\Big\{c_1^2[\delta^\alpha_\nu\delta^\lambda_\mu\partial_\lambda\ln\Omega\partial_\alpha\ln\Omega + \delta^\alpha_\nu\delta^\lambda_\alpha\partial_\lambda\ln\Omega\partial_\mu\ln\Omega \\
&+ \delta^\alpha_\lambda\delta^\lambda_\mu\partial_\alpha\ln\Omega\partial_\nu\ln\Omega + \delta^\lambda_\alpha\delta^\alpha_\lambda\partial_\nu\ln\Omega\partial_\mu\ln\Omega] \\
&+ c_1c_2[\delta^\alpha_\nu g_{\mu\alpha}\partial_\lambda\ln\Omega\partial^\lambda\ln\Omega + \delta^\alpha_\lambda g_{\mu\alpha}\partial_\nu\ln\Omega\partial^\lambda\ln\Omega] \\
&+ c_1c_2[\delta^\lambda_\mu g_{\nu\lambda}\partial_\alpha\ln\Omega\partial^\alpha\ln\Omega + \delta^\lambda_\alpha g_{\nu\lambda}\partial_\mu\ln\Omega\partial^\alpha\ln\Omega] \\
&+ c_2^2 g_{\mu\alpha}g_{\nu\lambda}\partial^\alpha\ln\Omega\partial^\lambda\ln\Omega\Big\} \\
&= \Omega^{-2}\Big\{(c_2-c_1)(D-1)\Big\}\nabla_\alpha\partial^\alpha\ln\Omega \\
&+ \Omega^{-2}\Big\{(d-1)c_1^2 + (D^2-D)c_1c_2 + (D-1)c_2^2\Big\}\partial_\alpha\ln\Omega\partial^\alpha\ln\Omega \qquad (3.37)
\end{aligned}$$

After substitution of (3.37) into the metric-affine action (3.17), we obtain the following equation for transformed action

$$\begin{aligned}
S_{EH}\left[\widetilde{g},\widetilde{\Gamma}\right] &= \int d^Dx\,\sqrt{-g}\Bigg\{\frac{1}{2}M_\star^{D-2}\Omega^{D-2}R \\
&+ \frac{1}{2}M^{D-2}\Omega^{D-4}[(D-1)(c_1^2+c_2^2+Dc_1c_2+2c_1-2c_2)]\,g^{\alpha\beta}\nabla_\alpha\Omega\nabla_\beta\Omega \\
&- \Omega^D\Lambda\Bigg\}
\end{aligned}$$

Then, we introduce a new parameter for abbreviation

$$
\begin{aligned}
\kappa_D &= \frac{c_1^2 + c_2^2 + Dc_1c_2 + 2c_1 - 2c_2)}{D-2} \\
&= \frac{(c_1+c_2)^2 - 2c_1c_2 + Dc_1c_2 + 2c_1 - 2c_2}{D-2} \\
&= \frac{(c_1+c_2)^2 + (D-2)c_1c_2 + 2(c_1-c_2)}{D-2}
\end{aligned} \tag{3.38}
$$

The action takes the form as

$$
\begin{aligned}
S_{EH}\left[\widetilde{g}, \widetilde{\Gamma}\right] &= \int d^D x\sqrt{-g}\Bigg\{\frac{1}{2}M_\star^{D-2}\Omega^{D-2}R \\
&+ \frac{1}{2}M_\star^{D-2}(D-1)(D-2)\underbrace{\frac{\kappa_D}{\mid \kappa_D \mid}}_{Sign\kappa_D} \mid \kappa_D \mid \Omega^{D-4} g^{\alpha\beta}\partial_\alpha\Omega\partial_\beta\Omega - \Omega^D\Lambda\Bigg\}
\end{aligned}
$$

By using two new definitions, a canonical scalar field and a self coupling constant, respectively

$$
\overline{\phi} = \frac{1}{\sqrt{\zeta_D'}}\,(M_\star\Omega)^{\frac{(D-2)}{2}} \tag{3.39}
$$

$$
\zeta_D' = \frac{D-2}{4(D-1)\mid \kappa_D \mid} = \frac{\zeta_D}{\mid \kappa_D \mid} \tag{3.40}
$$

Finally, transformed action is obtained as

$$
\begin{aligned}
S_{EH}\left[\widetilde{g}, \widetilde{\Gamma}, \overline{\phi}\right] &= \int d^D x\sqrt{-g}\Bigg\{\frac{1}{2}\text{Sign}\,(\kappa_D)\, g^{\mu\nu}\partial_\mu\overline{\phi}\partial_\nu\overline{\phi} \\
&+\frac{1}{2}\zeta_D'\overline{\phi}^2 g^{\mu\nu}\mathbb{R}_{\mu\nu}\,(\Gamma) - \lambda_D\left(\zeta_D'\overline{\phi}^2\right)^{\frac{D}{D-2}}\Bigg\}
\end{aligned} \tag{3.41}
$$

The action (3.41) is to be contrasted with the transformed action (3.12) in metrical gravity. The differences between the two are spectacular, and it could prove useful to discuss them here in detail:

- One first notes that, the action (3.41) is invariant under the emergent conformal transformations

$$
\begin{aligned}
& g_{\alpha\beta} \longrightarrow \psi^2\, g_{\alpha\beta} \\
& \Gamma^{\lambda}_{\alpha\beta} \longrightarrow \Gamma^{\lambda}_{\alpha\beta} + \Delta^{\lambda}_{\alpha\beta}(\psi) \\
& \overline{\phi} \longrightarrow \psi^{-\frac{(D-2)}{2}}\, \overline{\phi}
\end{aligned}
\tag{3.42}
$$

  similar to what we have found in (3.15) for the metrical GR. This invariance guarantees that all the fixed scales in (3.17) are appropriately dressed by the conformal factor $\Omega$ (Bekenstein, 1980).

- The conformal coupling $\zeta_D$ in (3.12) of the pure metric gravity changes to $\zeta_D / \left|\kappa_D\right|$ in the metric-affine action under concern. The presence of $\kappa_D$ reflects the generality of the transformation of the connection, as noted in (3.36). This is a highly important result since it generalizes the very concept of 'conformal coupling' between scalar fields and curvature scalar by changing $\zeta_D$ to $\zeta_D'$. This modification can have observable consequences in cosmological (Bauer, 2008), (Faraoni, 1999), (Sokolowski, ) as well as collider observables (Giudice, 2001), (Aslan, 2006) of the metric-affine gravity.

- In complete contrast to (3.12), the scalar field $\overline{\phi}$ in (3.41) obtains an indefinite kinetic term. The sign of the kinetic term is determined by the sign of $\kappa_D$. One here notes two physically distinct cases:

  1. If $\kappa_D > 0$ then $\overline{\phi}$ is a scalar ghost as in the metrical GR. In (3.12) $\kappa_D = 1$ (since $c_1 = 1$ and $c_2 = -1$ for the change of Levi-Civita connection (2.50) under conformal transformations), and $\overline{\phi}$ is necessarily a ghost if gravity is to stay as an attractive force.
  2. If, however, $\kappa_D < 0$ then $\overline{\phi}$ becomes a true scalar field with no problems like ghosty behavior. One notices from (3.41) that this very regime is realized with no modification in the attractive nature of the gravitational force. Gravity is attractive and $\overline{\phi}$ is a non-ghost, true scalar field. This result follows form the generality of the transformation of $\Gamma^{\lambda}_{\alpha\beta}$ in (3.36) compared to that of the Levi-Civita connection. The real constants $c_1$ and $c_2$ gives enough freedom to make

$\kappa_D$ negative for having a canonical scalar field theory, and this happens for

$$c_2 \;>\; 1-\frac{1}{2}(D-2)c_1-\frac{1}{2}\sqrt{D(D-4)c_1^2-4Dc_1+4}$$

and

$$c_2 \;<\; 1-\frac{1}{2}(D-2)c_1+\frac{1}{2}\sqrt{D(D-4)c_1^2-4Dc_1+4}$$

where $c_1$ is restricted to lie outside the interval $\left[\frac{4D-2\sqrt{D}}{D(D-4)}, \frac{4D+2\sqrt{D}}{D(D-4)}\right]$ for $D > 4$. One can see that for any dimension $D \geq 4$ there exist wide ranges of values of $c_1$ for which $c_2$ takes on admissible negative or positive real values. In particular, for $D = 4$ we find $c_1 < \frac{1}{4}$. Similar considerations pertaining to the metric-scalar-torsion system can be found in (Helayel-Neto, 2000).

3. The fact that the metric-affine gravity offers a true scalar field $\overline{\phi}$ elevates the arguments on the cosmological constant problem in (Polyakov, 2001), (Jackiw, 2005) to a more physical status since one then does not need to multiply the scalar field by the imaginary unit to make sense of the resulting scalar field theory. For $\kappa_D < 0$ and $\tilde{g}_{\mu\nu} = \eta_{\mu\nu}$, the affine-gravitational action (3.41) can realize infrared fixed point for $\overline{\phi}$ with no artificial changes in the sign of its kinetic term.

- The geometrical part of $\mathcal{L}\left(g, \mathcal{D}, \Psi\right)$, which only consist of the metric and $\mathcal{D}^{\lambda}_{\alpha\beta} = \Gamma^{\lambda}_{\alpha\beta} - \check{\Gamma}^{\lambda}_{\alpha\beta}$, will also transform under conformal transformation (2.44) with additively conformal-variant connection (3.33). Under the conformal transformations (3.42), $\mathcal{D}$ changes as

$$\widetilde{\mathcal{D}}^{\lambda}_{\alpha\beta} \;=\; \mathcal{D}^{\lambda}_{\alpha\beta} + (c_2+1)g_{\alpha\beta}\partial^{\lambda}\ln\psi + (c_1-1)\left(\delta^{\lambda}_{\alpha}\partial_{\beta}\ln\psi + \delta^{\lambda}_{\beta}\partial_{\alpha}\ln\psi\right)$$

as expected from transformation properties of $\Gamma^{\lambda}_{\alpha\beta}$ and $\check{\Gamma}^{\lambda}_{\alpha\beta}$. This gives geometrodynamical terms and couplings of $\mathcal{D}$ with the emergent scalar field $\psi$.

The analysis above ensures that additively transforming connections, such as the one (3.36), gives rise to a physically sensible mechanism where gravitational sector as well as the emergent scalar field from conformal transformation are both physical. Removal of the ghosty degree of freedom in metrical GR is a highly important aspect of the metric-affine gravity. Essentially, freeing connection from metric enables one to reach a physically consistent picture in regard to conformal frame changes in the gravitational action.

## 3.3. Equations of Motion

We have found that metric-affine gravity provides a means of generating non-ghost scalar field $\overline{\phi}$ by executing a more general transformation property as indicated in (3.36). However, we know that equations of motion relate $\Gamma^{\lambda}_{\alpha\beta}$ to Levi-Civita connection, and it is questionable if one can indeed realize such generalized transformation properties. For a detailed analysis of the problem, we will proceed systematically by examining different forms of geometrodynamical action densities.

- First of all, one notes that the affine gravitational action (3.17) becomes a highly conservative one for $\mathcal{L} = 0$. In this case, variation of action with respect to the connection $\Gamma^{\lambda}_{\alpha\beta}$ gives

  $$\nabla^{\Gamma}_{\lambda}\left(\sqrt{-g}g^{\alpha\beta}\right) = 0 \tag{3.43}$$

  where the covariant derivative of the tensor densitiy is defined as

  $$\nabla^{\Gamma}_{\lambda}\sqrt{-g} = \partial_{\lambda}\sqrt{-g} - \Gamma^{\alpha}_{\alpha\lambda}\sqrt{-g} \tag{3.44}$$

  Then the equation (3.43) is solved uniquely for

  $$\Gamma^{\lambda}_{\alpha\beta} = \check{\Gamma}^{\lambda}_{\alpha\beta}. \tag{3.45}$$

  Therefore, the action (3.17) is equivalent to the action for metrical gravity in (3.11). The main advantage of metric-affine gravity (actually the Palatini formalism itself) is that one arrives at the equations of GR with no need to extrinsic curvature (which is needed in metrical gravity). In sum, with $\mathcal{L} = 0$, (3.17) gives an equivalent description of (3.11). We will elaborate more on this point below.

- There can, however, be various sources of departure from the conservative action (3.17). These sources of departure are contained in $\mathcal{L}$. Let us first examine $\mathcal{L}_{\text{geo}}\left(g, \mathcal{D}\right)$ which involves metric and the tensorial connection $\mathcal{D}^{\lambda}_{\alpha\beta}$. The tensorial connection $\mathcal{D}^{\lambda}_{\alpha\beta}$ gives rise to novel geometrodynamical structures not necessarily governed by the curvature tensor $\mathbb{R}^{\alpha}_{\mu\beta\nu}\left(\Gamma\right)$ and its contractions and higher powers (though such sources of $\mathcal{D}^{\lambda}_{\alpha\beta}$ are to be also included in $\mathcal{L}_{\text{geo}}\left(g, \mathcal{D}\right)$). Indeed, the action can be added various new terms involving appropriate powers of $\mathcal{D}^{\lambda}_{\alpha\beta}$ as long as general covariance is respected. One notices that only even powers of $\mathcal{D}^{\lambda}_{\alpha\beta}$ can arise in the action (Pirinccioglu, ), (Demir, 2009). Needless to say, presence of additional terms involving $\mathcal{D}^{\lambda}_{\alpha\beta}$ changes the equation of motion for $\Gamma^{\lambda}_{\alpha\beta}$. In particular, its dynamical equivalence to Levi-Civita connection, in the sense of (3.43), gets lost.

  For explicating these points we go back to (3.17) and switch on $\mathcal{L}_{\text{geo}}\left(g, \mathcal{D}\right)$ after which the $\mathcal{D}^{\lambda}_{\alpha\beta}$ dependence of the action takes the form For explicating these points we go back to (3.17) and switch on $\mathcal{L}_{\text{geo}}\left(g, \mathcal{D}\right)$ after which the $\mathcal{D}^{\lambda}_{\alpha\beta}$ dependence of the action takes the form

$$\begin{aligned} S_{EH}\left[g, \mathcal{D}\right] &= \int d^{D}x\,\sqrt{-g}\Bigg\{\frac{1}{2}M_{\star}^{D-2}g^{\mu\nu}\underbrace{\left[R_{\mu\nu}\left(\check{\Gamma}\right)+\mathcal{R}_{\mu\nu}\left(\mathcal{D}\right)\right]}_{\mathbb{R}_{\mu\nu}(\Gamma)} \\ &- \Lambda_{\star}+\mathcal{L}_{\text{geo}}\left(g, \mathcal{D}\right)\Bigg\} \end{aligned} \tag{3.46}$$

  where we discarded $\mathcal{L}\left(g, \mathcal{D}, \psi\right)$ momentarily, to analyze the effects of geometrical part of $\mathcal{L}$ in isolation. Actually, as we have mentioned before in (3.24), $\mathcal{L}_{\text{geo}}\left(\mathcal{D}\right)$ can always be expressed in terms of torsion (which vanishes in our case), non-metricity, and curvature tensors. We here prefer to use generic function $\mathcal{L}_{\text{geo}}\left(\mathcal{D}\right)$ instead of expressing it in terms of those tensor structures in (3.24). From (3.34) it follows that

$$\mathcal{R}_{\mu\nu}\left(\mathcal{D}\right) = \nabla_{\alpha}\mathcal{D}^{\alpha}_{\mu\nu}-\nabla_{\nu}\mathcal{D}^{\alpha}_{\mu\alpha}+\mathcal{D}^{\alpha}_{\lambda\alpha}\mathcal{D}^{\lambda}_{\mu\nu}-\mathcal{D}^{\alpha}_{\lambda\nu}\mathcal{D}^{\lambda}_{\alpha\mu} \tag{3.47}$$

  in the action (3.46). Let us calculate the variation of (3.46) step by step. Firstly, we will calculate the variation of Ricci tensor with respect to $\mathcal{D}^{\lambda}_{\alpha\beta}$

$$
\begin{aligned}
\frac{\delta \mathcal{R}_{\mu\nu}(\mathcal{D}(x))}{\delta \mathcal{D}^{\lambda}_{\alpha\beta}(z)} &= \nabla_{\alpha}\left(\frac{\delta \mathcal{D}^{\alpha}_{\mu\nu}(x)}{\delta \mathcal{D}^{\lambda}_{\alpha\beta}}\right) - \nabla_{\nu}\left(\frac{\delta \mathcal{D}^{\alpha}_{\mu\alpha}(x)}{\delta \mathcal{D}^{\lambda}_{\alpha\beta}}\right) \\
&+ \frac{\delta \mathcal{D}^{\alpha}_{\lambda\alpha}(x)}{\delta \mathcal{D}^{\lambda}_{\alpha\beta}(z)}\mathcal{D}^{\lambda}_{\mu\nu}(x) + \mathcal{D}^{\alpha}_{\lambda\alpha}(x)\frac{\delta \mathcal{D}^{\lambda}_{\mu\nu}(x)}{\delta \mathcal{D}^{\lambda}_{\alpha\beta}(z)} \\
&- \frac{\delta \mathcal{D}^{\alpha}_{\lambda\nu}(x)}{\delta \mathcal{D}^{\lambda}_{\alpha\beta}(z)}\mathcal{D}^{\lambda}_{\alpha\mu}(x) - \mathcal{D}^{\alpha}_{\lambda\nu}(x)\frac{\delta \mathcal{D}^{\lambda}_{\alpha\mu}(x)}{\delta \mathcal{D}^{\lambda}_{\alpha\beta}(z)} \\
&= \delta^{\alpha}_{\lambda}\delta^{\alpha}_{\lambda}\delta^{\beta}_{\alpha}\mathcal{D}^{\lambda}_{\mu\nu}(x)\delta^{D}(x-z) + \delta^{\lambda}_{\lambda}\delta^{\alpha}_{\mu}\delta^{\beta}_{\nu}\mathcal{D}^{\alpha}_{\lambda\nu}(x)\delta^{D}(x-z) \\
&- \delta^{\alpha}_{\lambda}\delta^{\alpha}_{\lambda}\delta^{\beta}_{\nu}\mathcal{D}^{\lambda}_{\alpha\mu}(x)\delta^{D}(x-z) \\
&- \delta^{\lambda}_{\lambda}\delta^{\alpha}_{\alpha}\delta^{\beta}_{\mu}\mathcal{D}^{\alpha}_{\lambda\nu}(x)\delta^{D}(x-z) \qquad (3.48)
\end{aligned}
$$

By using (3.48), variation of the action with respect to $\mathcal{D}^{\lambda}_{\alpha\beta}(z)$ gives the equations of motion

$$
\begin{aligned}
&\delta^{\beta}_{\lambda} g^{\mu\nu}(z)\mathcal{D}^{\alpha}_{\mu\nu} + g^{\alpha\beta}(z)\mathcal{D}^{\nu}_{\lambda\nu}(z) - g^{\mu\beta}(z)\mathcal{D}^{\alpha}_{\lambda\mu}(z) \\
&- g^{\beta\nu}(z)\mathcal{D}^{\alpha}_{\lambda\nu}(z) + \mathcal{G}^{\alpha\beta}_{\lambda}(g,\mathcal{D}) = 0 \qquad (3.49)
\end{aligned}
$$

where $\mathcal{G}^{\alpha\beta}_{\lambda}(g,\mathcal{D})$ stands for the variation of the geometrical part $\mathcal{L}_{\text{geo}}(g,\mathcal{D})$ with respect to $\mathcal{D}^{\lambda}_{\alpha\beta}(z)$.

- From (3.49) one immediately observes that, for $\mathcal{L}_{\text{geo}}(g,\mathcal{D}) = 0$ (in addition to assumed vanishing of the matter contribution), the tensorial connection identically vanishes, $\mathcal{D}^{\lambda}_{\alpha\beta} = 0$. This implies that the general connection $\Gamma^{\lambda}_{\alpha\beta}$ equals the Levi-Civita connection $\check{\Gamma}^{\lambda}_{\alpha\beta}$. In such a case, of course, $\Gamma^{\lambda}_{\alpha\beta}$ is expected to exhibit the same transformation properties as $\check{\Gamma}^{\lambda}_{\alpha\beta}$. Consequently, the general conformal transformation property (3.36) as well as the conclusions drawn from it will not hold for minimal Lagrangians, like (3.17) with $\mathcal{L} = 0$. In this sense, analysis of the previous section, though designed to show how varying conformal transformation properties of $\Gamma^{\lambda}_{\alpha\beta}$ modify the ghosty nature of $\overline{\phi}$, is physically sensible yet incomplete for it does not take into account the effects of non-vanishing $\mathcal{L}$ effects.
- We have just concluded that we need non-vanishing $\mathcal{L}$ for maintaining the independence of $\Gamma^{\lambda}_{\alpha\beta}$ from $\check{\Gamma}^{\lambda}_{\alpha\beta}$. Now it proves useful to check some reasonable

forms of $\mathcal{L}_{\text{geo}}\left(g,\mathcal{D}\right)$ in light of the equations of motion (3.49). Leaving aside the single-derivative terms as well as quadratic ones whose special forms are already contained in the curvature tensor, the lowest-order terms which can contribute to geometrical part take the form

$$\begin{aligned}\mathcal{L}_{\text{geo}}\left(g,\mathcal{D}\right) &= A^{\alpha\beta\mu\nu\chi\xi\eta\kappa}_{\lambda\rho\zeta\epsilon}\left(g\right)\mathcal{D}^{\lambda}_{\alpha\beta}\mathcal{D}^{\rho}_{\mu\nu}\mathcal{D}^{\zeta}_{\chi\xi}\mathcal{D}^{\epsilon}_{\eta\kappa}\left(\mathcal{D}\right)\\ &+ B^{\alpha\beta\mu\nu\rho\theta}_{\lambda\zeta}\left(g\right)\nabla_{\mu}\mathcal{D}^{\lambda}_{\alpha\beta}\nabla_{\nu}\mathcal{D}^{\zeta}_{\rho\theta}+\cdots\end{aligned} \tag{3.50}$$

where $A$ and $B$ are tensorial structures composed of the metric tensor. They are supposed to contain all possible combinatorics of the indices. It is clear that, after computing $\mathcal{G}^{\alpha\beta}_{\lambda}\left(g,\mathcal{D}\right)$ from this combination, the equations of motion (3.49) will yield non-vanishing $\mathcal{D}^{\lambda}_{\alpha\beta}$ even without including its derivatives. Let us calculate the $\mathcal{G}^{\alpha\beta}_{\lambda}\left(g,\mathcal{D}\right)$ by taking the variation of $\mathcal{L}_{\text{geo}}\left(g,\mathcal{D}\right)$ with respect to $\mathcal{D}^{\lambda}_{\alpha\beta}$.

$$\begin{aligned}\mathcal{G}^{\alpha\beta}_{\lambda}\left(g,\mathcal{D}\right) &= \frac{\delta\mathcal{L}_{\text{geo}}\left(g,\mathcal{D}(x)\right)}{\delta\mathcal{D}^{\lambda}_{\alpha\beta}(z)}\\ &= A^{\alpha\beta\mu\nu\chi\xi\eta\kappa}_{\lambda\rho\zeta\epsilon}\left(g\right)\left\{\frac{\delta\mathcal{D}^{\lambda}_{\alpha\beta}(x)}{\delta\mathcal{D}^{\lambda}_{\alpha\beta}(z)}\mathcal{D}^{\rho}_{\mu\nu}\mathcal{D}^{\zeta}_{\chi\xi}\mathcal{D}^{\epsilon}_{\eta\kappa}+\frac{\delta\mathcal{D}^{\rho}_{\mu\nu}(x)}{\delta\mathcal{D}^{\lambda}_{\alpha\beta}(z)}\mathcal{D}^{\lambda}_{\alpha\beta}\mathcal{D}^{\zeta}_{\chi\xi}\mathcal{D}^{\epsilon}_{\eta\kappa}\right.\\ &+ \left.\frac{\delta\mathcal{D}^{\zeta}_{\chi\xi}(x)}{\delta\mathcal{D}^{\lambda}_{\alpha\beta}(z)}\mathcal{D}^{\rho}_{\mu\nu}\mathcal{D}^{\lambda}_{\alpha\beta}\mathcal{D}^{\epsilon}_{\eta\kappa}+\frac{\delta\mathcal{D}^{\epsilon}_{\eta\kappa}(x)}{\delta\mathcal{D}^{\lambda}_{\alpha\beta}(z)}\mathcal{D}^{\rho}_{\mu\nu}\mathcal{D}^{\zeta}_{\chi\xi}\mathcal{D}^{\lambda}_{\alpha\beta}\right\}\\ &+ B^{\alpha\beta\mu\nu\rho\theta}_{\lambda\zeta}\left(g\right)\left\{\left(\nabla_{\mu}\frac{\delta\mathcal{D}^{\lambda}_{\alpha\beta}(x)}{\delta\mathcal{D}^{\lambda}_{\alpha\beta}(z)}\right)\nabla_{\nu}\mathcal{D}^{\zeta}_{\rho\theta}+\right.\\ &+ \left.\left(\nabla_{\nu}\frac{\delta\mathcal{D}^{\zeta}_{\rho\theta}(x)}{\delta\mathcal{D}^{\lambda}_{\alpha\beta}(z)}\right)\nabla_{\mu}\mathcal{D}^{\lambda}_{\alpha\beta}\right\}\\ &= A^{\alpha\beta\mu\nu\chi\xi\eta\kappa}_{\lambda\rho\zeta\epsilon}\left(g\right)\left\{\delta^{\lambda}_{\lambda}\delta^{\alpha}_{\alpha}\delta^{\eta}_{\beta}\delta^{D}(x-z)\mathcal{D}^{\rho}_{\mu\nu}\mathcal{D}^{\zeta}_{\chi\xi}\mathcal{D}^{\epsilon}_{\eta\kappa}\right.\\ &+ \delta^{\rho}_{\lambda}\delta^{\alpha}_{\mu}\delta^{\beta}_{\nu}\delta^{D}(x-z)\mathcal{D}^{\lambda}_{\alpha\beta}\mathcal{D}^{\zeta}_{\chi\xi}\mathcal{D}^{\epsilon}_{\eta\kappa}+\delta^{\zeta}_{\lambda}\delta^{\alpha}_{\chi}\delta^{\beta}_{\xi}\delta^{D}(x-z)\mathcal{D}^{\rho}_{\mu\nu}\mathcal{D}^{\lambda}_{\alpha\beta}\mathcal{D}^{\epsilon}_{\eta\kappa}\\ &+ \left.\delta^{\epsilon}_{\lambda}\delta^{\alpha}_{\nu}\delta^{\beta}_{\kappa}\delta^{D}(x-z)\mathcal{D}^{\rho}_{\mu\nu}\mathcal{D}^{\zeta}_{\chi\xi}\mathcal{D}^{\lambda}_{\alpha\beta}\right\}\\ &+ B^{\alpha\beta\mu\nu\rho\theta}_{\lambda\zeta}\left\{\nabla_{\mu}\left(\delta^{\lambda}_{\alpha}\delta^{\alpha}_{\alpha}\delta^{\beta}_{\beta}\delta^{D}(x-z)\right)\left(\nabla_{\nu}\mathcal{D}^{\zeta}_{\rho\theta}\right)\right.\\ &+ \left.\nabla_{\nu}\left(\delta^{\zeta}_{\lambda}\delta^{\alpha}_{\rho}\delta^{\beta}_{\theta}\delta^{D}(x-z)\right)\left(\nabla_{\mu}\mathcal{D}^{\lambda}_{\alpha\beta}\right)\right\}\end{aligned} \tag{3.51}$$

After some arrangement of (3.51), the equations of motion (3.49) take the form

$$
\begin{aligned}
& \mathcal{D}^{\sigma}_{\rho\theta}\Big[g^{\rho\theta}\delta^{\beta}_{\lambda}\delta^{\alpha}_{\sigma}+g^{\alpha\beta}\delta^{\theta}_{\sigma}\delta^{\rho}_{\lambda}-g^{\theta\beta}\delta^{\alpha}_{\sigma}\delta^{\rho}_{\lambda}-g^{\beta\theta}\delta^{\alpha}_{\sigma}\delta^{\rho}_{\lambda} \\
& +\mathcal{D}^{\zeta}_{\chi\xi}\mathcal{D}^{\epsilon}_{\eta\kappa}\left(A^{\alpha\beta\rho\theta\chi\xi\eta\kappa}_{\lambda\sigma\zeta\epsilon}+A^{\rho\theta\alpha\beta\chi\xi\eta\kappa}_{\lambda\sigma\zeta\epsilon}\right) \\
& +\mathcal{D}^{\zeta}_{\mu\nu}\mathcal{D}^{\epsilon}_{\chi\xi}\left(A^{\rho\theta\mu\nu\alpha\beta\chi\xi}_{\sigma\zeta\lambda\epsilon}+A^{\rho\theta\mu\nu\chi\xi\alpha\beta}_{\sigma\zeta\epsilon\lambda}\right)\Big] \\
- \quad & \nabla_{\rho}\nabla_{\theta}\mathcal{D}^{\sigma}_{\mu\nu}\left(B^{\rho\theta\alpha\beta\mu\nu}_{\lambda\sigma}+B^{\rho\theta\mu\nu\alpha\beta}_{\lambda\sigma}\right)=0\,.
\end{aligned}
\tag{3.52}
$$

These equations automatically suggest that $\mathcal{D}^{\lambda}_{\alpha\beta}\neq 0$ (or $\Gamma^{\lambda}_{\alpha\beta}\neq\check{\Gamma}^{\lambda}_{\alpha\beta}$) even if $\mathcal{L}_{\text{geo}}(g,\mathcal{D})$ does not include its derivatives (the coefficients $B$ vanish). If derivative terms vanish, then $\mathcal{D}^{\lambda}_{\alpha\beta}$ is obtained in terms of the metric tensor with, however, a general structure which should resemble (3.36) in any case. The details of the structure depend on how the coefficients $A^{\alpha\beta\mu\nu\chi\xi\eta\kappa}_{\lambda\rho\zeta\epsilon}$ are organized in terms of the metric tensor.

On the other hand, if the derivative terms are included then $\mathcal{D}^{\lambda}_{\alpha\beta}$ becomes a dynamical field. In this case, again, one obtains a non-trivial $\Gamma^{\lambda}_{\alpha\beta}$ not equaling $\check{\Gamma}^{\lambda}_{\alpha\beta}$.

From this analysis we conclude that, the analysis of the previous section, which has clearly shown how $\overline{\phi}$ becomes a non-ghost scalar for a general $\Gamma^{\lambda}_{\alpha\beta}$ transforming as in (3.36), in general, the connection $\Gamma^{\lambda}_{\alpha\beta}$ does not reduce to $\check{\Gamma}^{\lambda}_{\alpha\beta}$, and a conformal transformation property as in (3.36) can result in a multitude of ways.

- Another source of departure from (3.17) is the matter Lagrangian $\mathcal{L}(g,\mathcal{D},\psi)$. By switching on this Lagrangian one can still find additional structures which cause $\Gamma^{\lambda}_{\alpha\beta}$ to be independent of $\check{\Gamma}^{\lambda}_{\alpha\beta}$. Then the main difference from the previous analysis will be the dependence of the $\Gamma^{\lambda}_{\alpha\beta}$ on the matter fields themselves – a situation not discussed before. The question of how $\mathcal{L}(g,\mathcal{D},\psi)$ involves $\Gamma^{\lambda}_{\alpha\beta}$ is easy to answer given that, rather generically, connection-dependent terms arise in scalar and spinor field theories already at the renormalizable level (Deser, 1976), (Borunda, 2008). In such cases it could be difficult to arrange general conformal transformations of the form (3.36) yet one should keep such matter sector sources in mind in analyzing the conformal transformation properties in non-Riemannian geometries.

# CHAPTER 4

# CONCLUSION

In this work we have discussed conformal transformations in metric-affine gravity. The analysis is a comparative one between the metric formulation and the metric-affine formulation of GR. The main result of the analysis is that metric-affine gravity admits, under general additive transformations of the connection with two new parameters $c_1$ and $c_2$ , conformally-related frames in which both gravitational and scalar sectors behave physically. (Ateş, 2010) The transformed frame consists of no ghost field, and exhibits emergent conformal invariance (sometimes called Weyl-Stückelberg invariance). The results can have far-reaching consequences for collider experiments (Giudice, 2001,A), cosmological evolution (Bauer, 2008) as well as the electroweak breaking (Demir, 2004).

We have also analyzed equations of motion under general circumstances allowed by general covariance, and concluded that general Lagrangians allow for generalized conformal transformations of the connection without spoiling the essence of the theory in the transformed frame.

The affine gravitational action (3.17) can give rise to novel effects not found in the minimal version (the Einstein-Hilbert action). The conformally-reached frame can have various modifications in gravitational, matter as well as conformal factor (*i.e.* the $\Omega$ related to $\overline{\phi}$) dynamics. The fact that the metric-affine gravity can accommodate correct gravitational dynamics plus non-ghost scalar degree of freedom under conformal transformations is an important aspect. This feature can have important implications in cosmological and other settings since transformation of system to a conformal frame now involves no ghosty degree of freedom.

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# APPENDIX A

# GEOMETRICAL QUANTITIES IN GR

In General Relativity there are some geometrical quantities like metric tensor, connection, Riemann curvature tensor, Ricci tensor, Ricci scalar to construct a definition about the curvature of the space-time. We can give some brief explanations about them in this part.

Metric tensor is a very important object in curved space. The main feature of it is that it measures the shortest distances between two points and symbolized with $g_{\mu\nu}$. It is a symmetric rank $(0, 2)$ tensor.

Connection is also very important geometrical dynamic. It connects the tangent spaces on the manifold and responsible to warp the space. Because of these features, it appears in generalized partial derivation, namely covariant derivative. In flat space, partial derivatives determine how a quantity changes in spatial and temporal directions. But in curved space in addition to partial derivative, there must be a correction term to show the warping feature of spacetime. Covariant derivative of a contravector can be written as

$$\nabla_\mu A^\nu = \partial_\mu A^\nu + \Gamma^\nu_{\mu\lambda} A^\lambda \tag{A.1}$$

On the other hand, covariant derivative of a covector is

$$\nabla_\mu A_\nu = \partial_\mu A_\nu - \Gamma^\lambda_{\mu\nu} A_\lambda \tag{A.2}$$

and the general covariant derivative of a tensor $A^{\mu\nu....}_{\alpha\beta....}$

$$\nabla_\kappa A^{\mu\nu....}_{\alpha\beta....} = \partial_\kappa A^{\mu\nu....}_{\alpha\beta....} - \Gamma^\lambda_{\kappa\alpha} A^{\mu\nu....}_{\lambda\beta....} - ..... + \Gamma^\mu_{\kappa\lambda} A^{\lambda\nu....}_{\alpha\beta....} + ..... \tag{A.3}$$

If connection depend on metric tensor, it is called Levi-Civita connection (sometimes it can be called the other name like Christoffel symbols). Connection is not a tensor. It can be checked by changing coordinate system. The result does not give the tensorial changing.

$$\Gamma^{\lambda'}_{\mu'\nu'} = \frac{\partial x^{\lambda'}}{\partial x^{\lambda}} \frac{\partial x^{\mu}}{\partial x^{\mu'}} \frac{\partial x^{\nu}}{\partial x^{\nu'}} \Gamma^{\lambda}_{\mu\nu} + \frac{\partial x^{\lambda'}}{\partial x^{\lambda}} \frac{\partial^2 x^{\lambda}}{\partial x^{\mu'} \partial x^{\nu'}} \tag{A.4}$$

Riemann curvature tensor carries all information about the curved spacetime structure. Similar to Riemann curvature tensor, Ricci tensor obtained by contraction of the upper indice and the one of the lower indice of the Riemannian curvature tensor $R^{\rho}_{\alpha\rho\beta} = R_{\alpha\beta}$ and the Ricci scalar obtained by the contraction of Ricci tensor with the metric tensor $g^{\alpha\beta} R_{\alpha\beta} = R$ give the information about the warping structure of spacetime. The last one takes part in the Lagrangian because of its scalar form.

Riemann curvature can be derived by using three different ways. One of them is the commutator of covariant derivatives, the other one is the parallel transport around a closed loop and the last of them is the repeated derivatives of connection. Let us derive the Riemann curvature tensor from these different ways respectively.

- **Commutator of covariant derivatives**

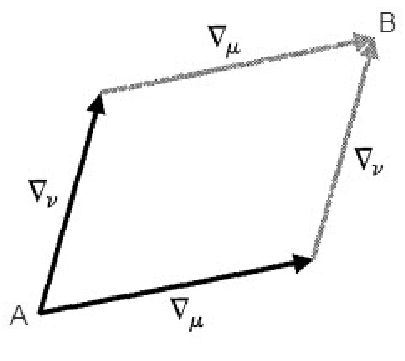


Figure A.1. The commutator of two covariant derivatives.

Consider a vector goes from a point to another point in the curved spacetime by using the two different way as in fig. When the vector goes to point B by using the way 1, firstly the covariant derivative $\nabla_\beta$, then the covariant derivative $\nabla_\nu$ should be applied to the vector $V^\alpha$. When the vector choose way 2 to go to point B, in this case firstly $\nabla_\nu$, then $\nabla_\beta$ is applied

$$
\begin{aligned}
[\nabla_\beta, \nabla_\nu]V^\alpha &= \nabla_\beta(\nabla_\nu V^\alpha) - \nabla_\nu(\nabla_\beta V^\alpha) \\
&= \partial_\beta(\nabla_\nu V^\alpha) + \Gamma^\alpha_{\beta\sigma}(\nabla_\nu V^\sigma) - \Gamma^\kappa_{\beta\nu}(\nabla_\kappa V^\alpha) \\
&- \partial_\nu(\nabla_\beta V^\alpha) - \Gamma^\alpha_{\nu\sigma}(\nabla_\beta V^\sigma) + \Gamma^\kappa_{\nu\beta}(\nabla_\kappa V^\alpha) \\
&= \partial_\beta(\partial_\nu V^\alpha + \Gamma^\alpha_{\nu\rho}V^\rho) + \Gamma^\alpha_{\beta\sigma}(\partial_\nu V^\sigma + \Gamma^\sigma_{\nu\rho}V^\rho) \\
&- \Gamma^\kappa_{\beta\nu}(\partial_\kappa V^\alpha + \Gamma^\alpha_{\kappa\rho}V^\rho) - \partial_\nu(\partial_\beta V^\alpha + \Gamma^\alpha_{\beta\rho}V^\rho) \\
&- \Gamma^\alpha_{\nu\sigma}(\partial_\beta V^\sigma + \Gamma^\sigma_{\beta\rho}V^\rho) + \Gamma^\kappa_{\nu\beta}(\partial_\kappa V^\alpha + \Gamma^\alpha_{\kappa\rho}V^\rho) \\
&= \partial_\beta\partial_\nu V^\alpha + \underbrace{\partial_\beta(\Gamma^\alpha_{\nu\rho}V^\rho)} + \overbrace{\Gamma^\alpha_{\beta\sigma}(\partial_\nu V^\sigma)} + \Gamma^\alpha_{\beta\sigma}\Gamma^\sigma_{\nu\rho}V^\rho - \Gamma^\kappa_{\beta\nu}(\partial_\kappa V^\alpha) \\
&- \Gamma^\kappa_{\beta\nu}\Gamma^\alpha_{\kappa\rho}V^\rho - \partial_\nu\partial_\beta V^\alpha - \overbrace{\partial_\nu(\Gamma^\alpha_{\beta\rho}V^\rho)} - \underbrace{\Gamma^\alpha_{\nu\sigma}(\partial_\beta V^\sigma)} - \Gamma^\alpha_{\nu\sigma}\Gamma^\sigma_{\beta\rho}V^\rho \\
&+ \Gamma^\kappa_{\nu\beta}(\partial_\kappa V^\alpha) + \Gamma^\kappa_{\nu\beta}\Gamma^\alpha_{\kappa\rho}V^\rho \\
&= (\partial_\beta\Gamma^\alpha_{\nu\rho} - \partial_\nu\Gamma^\alpha_{\beta\rho} + \Gamma^\alpha_{\beta\sigma}\Gamma^\sigma_{\nu\rho} - \Gamma^\alpha_{\nu\sigma}\Gamma^\sigma_{\beta\rho})V^\rho - 2\Gamma^\kappa_{[\beta\nu]}\nabla_\kappa V^\alpha \\
&= R^\rho_{\alpha\beta\nu}V^\alpha - T^\kappa_{\beta\nu}\nabla_\kappa V^\alpha \qquad \text{(A.5)}
\end{aligned}
$$

where

$$T^\kappa_{\beta\nu} = 2\Gamma^\kappa_{[\beta\nu]} = \Gamma^\kappa_{\beta\nu} - \Gamma^\kappa_{\nu\beta} \qquad \text{(A.6)}$$

is torsion tensor. If the connection is torsion free, this term vanishes.

- **Parallel transport around the closed loop**

  The parallel displacement of a vector along an infinitesimal closed loop gives the information about the geometry of spacetime. If the vector after parallel transport remain unchanged, it can be said that the spacetime is flat. However, if the vector transforms after this displacement, the space is curved and the transformation of the vector depends on the total curvature. Let us see this by formulations.

  The change in a vector $A_\mu$ after parallel displacements can be written as

$$\Delta A_\mu = \oint \delta A_\mu \qquad \text{(A.7)}$$

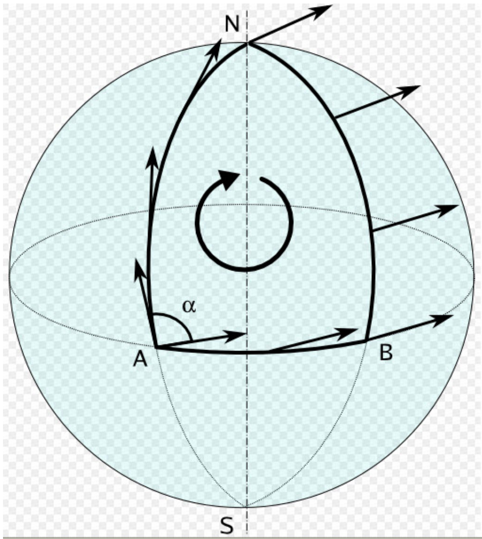


Figure A.2. Parallel displacement in curved space.

and

$$\delta A_{\mu} = \Gamma^{\nu}_{\mu\lambda} A_{\nu} dx^{\lambda} \tag{A.8}$$

After substitution of (A.8) into (A.7),

$$\Delta A_{\mu} = \oint \Gamma^{\lambda}_{\mu\nu} A_{\lambda} dx^{\nu} \tag{A.9}$$

The term $A_{\lambda}$ is determined uniquely by its value at points inside the closed loop via equation (A.8). By the derivative

$$\frac{\partial A_{\lambda}}{\partial x^{\nu}} = \Gamma^{\rho}_{\nu\lambda} A_{\rho} \tag{A.10}$$

By Stoke's theorem

$$\oint A_{\lambda} dx^{\lambda} = \int df^{\mu\lambda} \frac{\partial A_{\lambda}}{\partial x^{\mu}} = \frac{1}{2} \int df^{\mu\lambda} \left( \frac{\partial A_{\mu}}{\partial x^{\lambda}} - \frac{\partial A_{\lambda}}{\partial x^{\mu}} \right) \tag{A.11}$$

After applying this theorem to the (A.9) ,

$$\begin{aligned}\Delta A_\mu &= \frac{1}{2}\left\{\frac{\partial(\Gamma^\lambda_{\mu\beta}A_\lambda)}{\partial x^\nu} - \frac{\partial(\Gamma^\lambda_{\mu\nu}A_\lambda)}{\partial x^\beta}\right\}\Delta f^{\nu\beta} \\ &= \frac{1}{2}\left\{\frac{\partial(\Gamma^\lambda_{\mu\beta})}{\partial x^\nu}A_\lambda - \frac{\partial(\Gamma^\lambda_{\mu\nu})}{\partial x^\beta}A_\lambda + \Gamma^\lambda_{\mu\beta}\frac{\partial A_\lambda}{\partial x^\nu} - \Gamma^\lambda_{\mu\nu}\frac{\partial A_\lambda}{\partial x^\beta}\right\}\Delta f^{\nu\beta} \quad (A.12)\end{aligned}$$

where $\Delta f^{\nu\beta}$ is the infinitesimal area enclosed by the closed curve. By substituting of (A.10) into the last equation, the total change $_\mu$ is

$$\Delta A_\mu = \frac{1}{2}R^\lambda_{\mu\nu\beta}A^{\nu\beta}_\lambda \quad (A.13)$$

where $R^\lambda_{\mu\nu\beta}$ is the Riemann curvature tensor. If the spacetime is flat, Riemann tensor is zero. Thus, the total change in the vector $A_\mu$ is zero. There is no difference between the first vector and the parallel transported vector.

- **Repeated derivatives**

In an inertial frame, the equation of motion for a free relativistic particles can be written as

$$\frac{d\zeta^\alpha(\tau)}{d\tau} = 0 \quad (A.14)$$

and this equation give us the straight line trajectory of the particle in the flat spacetime.

Let us pass from inertial coordinate system $\zeta(\tau)$ to any general coordinate system $x^\mu$ and consider that these two coordinate systems are related by invertible functional relation

$$\begin{aligned}\zeta^\alpha &= \zeta^\alpha(x) \quad (A.15) \\ x^\mu &= x^\mu(\zeta)\end{aligned}$$

then, the (A.14) takes the form as,

$$\frac{d}{d\tau}(\frac{\partial\zeta^\alpha}{\partial x^\mu}\frac{\partial x^\mu}{\partial\tau}) = \frac{\partial^2\zeta^\alpha}{\partial x^\mu\partial x^\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}+\frac{\partial\zeta^\alpha}{\partial x^\mu}\frac{d^2x^\mu}{d\tau^2} = \delta^\lambda_\mu\frac{d^2x^\mu}{d\tau^2}+\frac{\partial x^\lambda}{\partial\zeta^\alpha}\frac{\partial^2\zeta^\alpha}{\partial x^\mu\partial x^\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} = 0$$

$$\frac{d^2x^\lambda}{d\tau^2} + \Gamma^\lambda_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} = 0 \tag{A.16}$$

where

$$\Gamma^\lambda_{\mu\nu} = \frac{\partial x^\lambda}{\partial\zeta^\alpha}\frac{\partial^2\zeta^\alpha}{\partial x^\mu\partial x^\nu} \tag{A.17}$$

Eq. (A.16) is called geodesic equation of the particle in curved spacetime. If $\Gamma^\lambda_{\mu\nu} = 0$, this equation turns into the equation of motion of a particle in flat space.

By general coordinate transformation, as we mention before, the connection transforms as

$$\Gamma^{\lambda'}_{\mu'\nu'} = \frac{\partial x^{\lambda'}}{\partial x^\lambda}\frac{\partial x^\mu}{\partial x^{\mu'}}\frac{\partial x^\nu}{\partial x^{\nu'}}\Gamma^\lambda_{\mu\nu} + \frac{\partial x^{\lambda'}}{\partial x^\lambda}\frac{\partial^2 x^\lambda}{\partial x^{\mu'}\partial x^{\nu'}}$$

It can be seen that the connection is not a tensorial structure,because there appears an inhomogenous term. By invertible functional relation (A.15), the inhomogeneous term can be written as

$$\frac{\partial^2 x^{\lambda'}}{\partial x^\mu\partial x^\nu} = \frac{\partial x^{\lambda'}}{\partial x^\lambda}\Gamma^\lambda_{\mu\nu} - \frac{\partial x^{\mu'}}{\partial x^\mu}\frac{\partial x^{\nu'}}{\partial x^\nu}\Gamma^{\lambda'}_{\mu'\nu'} \tag{A.18}$$

For eliminating the inhomogeneous term , we can take the derivative of (A.18) with respect to $x^\beta$, and then by changing $\beta \leftrightarrow \nu$ and substracting the first one from the second one,

$$\frac{\partial^3 x^{\lambda'}}{\partial x^\beta\partial x^\mu\partial x^\nu} - \frac{\partial^3 x^{\lambda'}}{\partial x^\nu\partial x^\mu\partial x^\beta} = 0 \tag{A.19}$$

Then, the right hand side by taking the derivative of inhomogeneous terms

$$
\begin{aligned}
0 &= \frac{\partial x^{\lambda'}}{\partial x^{\lambda}}\left(\underbrace{\frac{\partial\Gamma^{\lambda}_{\mu\nu}}{\partial x^{\beta}} - \frac{\partial\Gamma^{\lambda}_{\mu\beta}}{\partial x^{\nu}} + \Gamma^{\lambda}_{\beta\theta}\Gamma^{\theta}_{\mu\nu} - \Gamma^{\lambda}_{\nu\theta}\Gamma^{\theta}_{\mu\beta}}_{R^{\lambda}_{\mu\nu\beta}}\right) \\
&- \frac{\partial x^{\mu'}}{\partial x^{\mu}}\frac{\partial x^{\nu'}}{\partial x^{\nu}}\frac{\partial x^{\beta'}}{\partial x^{\beta}}\left(\underbrace{\frac{\partial\Gamma^{\lambda'}_{\mu'\nu'}}{\partial x^{\beta'}} - \frac{\partial\Gamma^{\lambda'}_{\mu'\beta'}}{\partial x^{\nu'}} + \Gamma^{\lambda'}_{\beta'\theta'}\Gamma^{\theta'}_{\mu'\nu'} - \Gamma^{\lambda'}_{\nu'\theta'}\Gamma^{\theta'}_{\mu'\beta'}}_{R^{\lambda'}_{\mu'\nu'\beta'}}\right) \quad \text{(A.20)}
\end{aligned}
$$

and,

$$
R^{\lambda'}_{\mu'\nu'\beta'} = \frac{\partial x^{\mu}}{\partial x^{\mu'}}\frac{\partial x^{\nu}}{\partial x^{\nu'}}\frac{\partial x^{\beta}}{\partial x^{\beta'}}\frac{\partial x^{\lambda'}}{\partial x^{\lambda}}R^{\lambda}_{\mu\nu\beta} \quad \text{(A.21)}
$$

This is clearly tensor transformation law. Here, $R^{\lambda}_{\mu\nu\beta}$ is called Riemann curvature tensor as we know.

# APPENDIX B

# EINSTEIN FIELD EQUATIONS IN RIEMANNIAN SPACE

In this part, we will derive the equations of motion for two different formulations of GR, metric formulation and metric-affine formulation. These two formulations are dynamically equivalent formulations in Riemannian spacetime. This can be seen from the equations of motion.

**In metric formulation**

Action of gravitational fields in empty space is given by

$$S[g] = \int d^4x\sqrt{-g}\Big\{\frac{1}{2}M_\star^2 R\Big\} \tag{B.1}$$

In this formulation, the only geometrical variable is metric tensor. The principle of least action tells us that the small perturbations around this metric tensor should be zero. Following the variation of the action with respect to metric tensor $g^{\mu\nu}$:

$$\frac{\delta S}{\delta g^{\mu\nu}}\delta g^{\mu\nu} = \int d^4x\Big\{\frac{\delta(\sqrt{-g})}{\delta g^{\mu\nu}}\frac{M_\star^2}{2}R + \sqrt{-g}\frac{M_\star^2}{2}\frac{\delta(g^{\mu\nu}R_{\mu\nu})}{\delta g^{\mu\nu}}\Big\}\delta g^{\mu\nu} \tag{B.2}$$

where

$$g = \exp[tr\ln(g_{\mu\nu})] \tag{B.3}$$

and

$$
\begin{aligned}
\delta\sqrt{-g} &= -\exp[tr\ln(g_{\mu\nu})]^{1/2} \\
&= i\exp[tr\ln(g_{\mu\nu})]^{1/2} \\
&= i\frac{1}{2}\exp[tr\ln(g_{\mu\nu})]^{-1/2}\exp[tr\ln(g_{\mu\nu})](g_{\mu\nu})^{-1}\delta g_{\mu\nu} \\
&= i\frac{1}{2}\frac{1}{\sqrt{g}}gg^{\mu\nu}\delta g_{\mu\nu} \\
&= -i\frac{1}{2}\frac{1}{\sqrt{g}}gg_{\mu\nu}\delta g^{\mu\nu} \\
&= -i\frac{1}{2}\sqrt{g}g_{\mu\nu}\delta g^{\mu\nu} \\
&= -\frac{1}{2}\sqrt{-g}g_{\mu\nu}\delta g^{\mu\nu}
\end{aligned} \tag{B.4}
$$

$$\frac{\delta\sqrt{-g}}{\delta g^{\mu\nu}} = -\frac{1}{2}\sqrt{-g}g_{\mu\nu} \tag{B.5}$$

For obtaining the variation of the Ricci scalar $R_{\mu\nu}$, we begin with the variation of the Riemann tensor:

$$\delta R^{\rho}_{\mu\sigma\nu} = \partial_{\sigma}\delta\Gamma^{\rho}_{\mu\nu} - \partial_{\nu}\delta\Gamma^{\rho}_{\mu\sigma} + \delta(\Gamma^{\rho}_{\sigma\lambda})\Gamma^{\lambda}_{\mu\nu} + \delta(\Gamma^{\lambda}_{\mu\nu})\Gamma^{\rho}_{\sigma\lambda} - \delta(\Gamma^{\rho}_{\nu\lambda})\Gamma^{\lambda}_{\mu\sigma} - \delta(\Gamma^{\lambda}_{\mu\sigma})\Gamma^{\rho}_{\nu\lambda} \tag{B.6}$$

Since $\delta\Gamma$ is the difference of two connection, it is a tensor and we can calculate its covariant derivative as:

$$\nabla_{\lambda}\delta\Gamma^{\rho}_{\mu\nu} = \partial_{\lambda}\delta\Gamma^{\rho}_{\mu\nu} + \Gamma^{\rho}_{\lambda\sigma}\delta\Gamma^{\lambda}_{\mu\nu} - \Gamma^{\sigma}_{\lambda\mu}\delta\Gamma^{\rho}_{\sigma\nu} - \Gamma^{\sigma}_{\lambda\nu}\delta\Gamma^{\rho}_{\sigma\mu} \tag{B.7}$$

By adding and substracting the $\Gamma^{\lambda}_{\nu\sigma}\delta\Gamma^{\rho}_{\mu\lambda}$ into (B.6), it can be easily seen that (B.6) is equal to the difference of two such terms:

$$\delta R^{\rho}_{\mu\sigma\nu} = \nabla_{\sigma}(\delta\Gamma^{\rho}_{\mu\nu}) - \nabla_{\nu}(\delta\Gamma^{\rho}_{\mu\sigma}) \tag{B.8}$$

Now, we can obtain the variation of the Ricci tensor by contracting two indices of the variation of Riemann tensor:

$$\delta R^{\rho}_{\mu\rho\nu} = \delta R_{\mu\nu} = \nabla_{\rho}(\delta\Gamma^{\rho}_{\mu\nu}) - \nabla_{\nu}(\delta\Gamma^{\rho}_{\mu\rho}) \tag{B.9}$$

Then, Ricci scalar is:

$$R = g^{\mu\nu}R_{\mu\nu} \tag{B.10}$$

and variation of Ricci scalar is:

$$\begin{aligned} \delta R &= \delta g^{\mu\nu}R + g^{\mu\nu}\delta R \\ &= \delta g^{\mu\nu}R + \nabla_{\sigma}(g^{\mu\nu}\delta\Gamma^{\sigma}_{\mu\nu} - g^{\mu\sigma}\delta\Gamma^{\rho}_{\rho\mu}) \end{aligned} \tag{B.11}$$

As we mention before, covariant derivative of the metric tensor is zero in the metric formulation of GR (metric-compability). Thus, we used this property in the last line of the last equation. The $\nabla_{\sigma}(g^{\mu\nu}\delta\Gamma^{\sigma}_{\mu\nu} - g^{\mu\sigma}\delta\Gamma^{\rho}_{\rho\mu})$ is a total derivative and thus by Stokes' theorem yields a boundary term when it is integrated. Because the variation of the metric $\delta g^{\mu\nu}$ vanishes at infinity, there is no contributions from this term to the action. Thus, we obtain:

$$\frac{\delta R}{\delta g^{\mu\nu}} = R_{\mu\nu} \tag{B.12}$$

Finally, we substitute (B.12) and (B.5) into (B.2)

$$\delta S = \int d^4x \frac{1}{2}M_{\star}^2\Big\{ -\frac{1}{2}g_{\mu\nu}R + R_{\mu\nu}\Big\}\sqrt{-g}\delta g^{\mu\nu} \tag{B.13}$$

According to principle of least action,

$$\delta S = 0 \tag{B.14}$$

Then, the basic equations of the general relativity:

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 0 \tag{B.15}$$

These equations are called Einstein equations. Here, because the space is empty, $R_{\mu\nu} = 0$. This situation does not imply that the spacetime is flat. For the flat spacetime, it is needed $R^{\rho}_{\mu\sigma\nu} = 0$ as a sufficient condition.

**In metric-affine formulation**

As we mention before, metric formulation and the metric-affine formulation are dynamically equivalent formulations only in Riemannian geometry. Now, we will show this equivalence by deriving the equations of motion in metric-affine formulation.

Let us begin with the action in metric-affine formulation.

$$S[g,\Gamma] = \int d^4x\sqrt{-g}\Big\{\frac{1}{2}M_\star^2 g^{\mu\nu}R_{\mu\nu}(\Gamma)\Big\} \tag{B.16}$$

Here, the connection is general connection which does not depend on metric. In this formulation, because the metric and the connection are considered as independent variables, equations of motion are obtained by taking the variation of action seperately with respect to the metric and the connection. The first variation gives the Einstein equations as in metric formulation. Let us calculate the second variation.

$$\delta S[g,\Gamma] = \int d^4\sqrt{-g}\Big\{\frac{1}{2}M_\star^2 g^{\mu\nu}\delta R_{\mu\nu}(\Gamma)\Big\} \tag{B.17}$$

where

$$R_{\mu\nu}(\Gamma) = \partial_\alpha\Gamma^\alpha_{\mu\nu} - \partial_\nu\Gamma^\alpha_{\alpha\mu} + \Gamma^\alpha_{\mu\nu}\Gamma^\beta_{\beta\alpha} - \Gamma^\alpha_{\mu\beta}\Gamma^\beta_{\nu\alpha} \tag{B.18}$$

and

$$\delta R_{\mu\nu}(\Gamma) = \partial_\alpha\delta\Gamma^\alpha_{\mu\nu} - \partial_\nu\delta\Gamma^\alpha_{\alpha\mu} + \delta\Gamma^\alpha_{\mu\nu}\Gamma^\beta_{\beta\alpha} + \Gamma^\alpha_{\mu\nu}\delta\Gamma^\beta_{\beta\alpha} - \delta\Gamma^\alpha_{\mu\beta}\Gamma^\beta_{\nu\alpha} - \Gamma^\alpha_{\mu\beta}\delta\Gamma^\beta_{\nu\alpha} \tag{B.19}$$

Although the connection is not a tensor itself, the difference between two connections is a tensor. Thus the variation of the connection is also a tensor. (Carroll, 2004) By using this fact the variation of Ricci tensor can be written as

$$\delta R_{\mu\nu}(\Gamma) = \nabla_{\alpha}^{\Gamma}\left(\delta\Gamma_{\mu\nu}^{\alpha}\right) - \nabla_{\nu}^{\Gamma}\left(\delta\Gamma_{\alpha\mu}^{\alpha}\right) \tag{B.20}$$

Then the variation of

$$\delta S[g,\Gamma] = \int d^4\sqrt{-g}\Big\{\frac{1}{2}M_{\star}^2 g^{\mu\nu}\left[\nabla_{\alpha}^{\Gamma}\left(\delta\Gamma_{\mu\nu}^{\alpha}\right) - \nabla_{\nu}^{\Gamma}\left(\delta\Gamma_{\alpha\mu}^{\alpha}\right)\right]\Big\} \tag{B.21}$$

After integating by parts and throwing away the surface terms by setting $\delta_{\mu\nu}^{\alpha} = 0$ on the boundary, Eq. (B.21) takes the form,

$$\begin{aligned}\delta S[g,\Gamma] &= -\frac{1}{2}M_{\star}^2\int d^4x\left[\nabla_{\alpha}^{\Gamma}\left(\sqrt{-g}g^{\mu\nu}\right)\delta\Gamma_{\mu\nu}^{\alpha} - \nabla_{\nu}^{\Gamma}\left(\sqrt{-g}g^{\mu\nu}\right)\delta\Gamma_{\alpha\mu}^{\alpha}\right] \\ &= -\frac{1}{2}M_{\star}^2\int d^4x\delta\Gamma_{\lambda\mu}^{\sigma}\left[\delta_{\nu}^{\lambda}\nabla_{\sigma}^{\Gamma}\left(\sqrt{-g}g^{\mu\nu}\right) - \delta_{\sigma}^{\lambda}\nabla_{\nu}^{\Gamma}\left(\sqrt{-g}g^{\mu\nu}\right)\right]\end{aligned} \tag{B.22}$$

According to least action principle $\delta S = 0$,

$$\delta_{\nu}^{\lambda}\nabla_{\sigma}^{\Gamma}\left(\sqrt{-g}g^{\mu\nu}\right) - \delta_{\sigma}^{\lambda}\nabla_{\nu}^{\Gamma}\left(\sqrt{-g}g^{\mu\nu}\right) = 0 \tag{B.23}$$

Contracting on the indices $\lambda$ and $\nu$ yields

$$\nabla_{\sigma}^{\Gamma}\left(\sqrt{-g}g^{\mu\nu}\right) = 0 \tag{B.24}$$

This equation can be solved uniquely for $\Gamma_{\alpha\beta}^{\lambda} = \check{\Gamma}_{\alpha\beta}^{\lambda}$ (Levi-Civita connection). Thus, it can be said that the metric-affine formulation in Riemannian space, where there is no quantum effects, gives the same equations of motion as in the metric formulation. However, in non-Riemannian space, as we mentioned before, there is an extra term from D tensorial structure and this term leads to an extra term in equations of motion like

$$\nabla^{\Gamma}_{\lambda}(\sqrt{-g}g^{\alpha\beta}) + Extra(\Gamma) = 0 \tag{B.25}$$

This shows us $\Gamma^{\lambda}_{\alpha\beta} \neq \check{\Gamma}^{\lambda}_{\alpha\beta}$.