LOWER-TOP AND UPPER-BOTTOM POINTS FOR ANY FORMULA IN TEMPORAL LOGIC

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ABSTRACT

LOWER-TOP AND UPPER-BOTTOM POINTS FOR ANY FORMULA IN TEMPORAL LOGIC

In temporal logic, which is a branch of modal logic, models are constructed on some kind of frames. Common properties of all these frames include totally ordered relations and these frames are bi-directional. These common properties provide the temporal logic time interpretation. By means of this interpretation temporal language has lots of application areas. The main aim of this study is to propose new technic which gets easier proof of some kind of valid formulas in the most popular temporal frame $\mathfrak{T}$ and to produce new valid formulas with the medium of this new technic. To be able to realize this main aim, first of all the frame $\mathfrak{T} = (\mathbb{N}, \leqslant, \geqslant, R_\circ, R_\ominus, R_U, R_S)$ for temporal language has been composed step by step in accordance with principles of modal logic. Then the new terms " lower-top and upper-bottom points for any temporal formula " has been defined in the model $\mathfrak{M} = (\mathfrak{T}, V)$ which is built over the frame $\mathfrak{T}$ and some propositions of this term have been obtained. At the end of the study it has been presented that proofs of some theorems have been done easier and it has been given possibility to produce the new theorems. Moreover a general investigation about the frame $\mathfrak{T}$ has been done and presented, furthermore it has been shown that the mirror image of the valid formulas do not have to be valid and it is also possible that the mirror image of non valid formulas can be valid.

ÖZET

ZAMAN MANTIĞINDA HERHANGİ BİR FORMÜL İÇİN ALT-TEPE VE ÜST-TABAN NOKTALAR

Modal lojiğin alt dalı olan zaman lojiğinde modeller çeşitli çatılar üzerinde inşa edilirler. Bu çatıların hepsinin ortak özellikleri elemanlarının tam bir sıralama bağıntısı ile inşa edilmesi ve çift yönlü olmasıdır. Bu ortak özellik zaman lojiğine zaman yorumu kazandırır. Bu yorum sayesinde zaman lojiği geniş bir uygulama alanına sahiptir. Bu çalışmada, zaman lojiğinde oldukça sık kullanılan $\mathfrak{T}$ çatısı için geçerli formüllerin ispatlarını kolaylaştıracak yeni bir teknik geliştirmek ve bu teknik sayesinde yeni geçerli formüller türetebilmek amaçlanmıştır. Bu amacı gerçekleştirebilmek için önce zamansal dil için $\mathfrak{T} = (\mathbb{N}, \leqslant, \geqslant, R_{\circ}, R_{\ominus}, R_U, R_S)$ çatısı modal lojikteki tanımlara bağlı kalınarak adım adım oluşturuldumuş; daha sonra bu $\mathfrak{T}$ çatısı üzerine kurulan $\mathfrak{M} = (\mathfrak{F}, V)$ modellerinde " herhangi bir formül için alt-tepe ve üst-taban noktalar " kavramları tanımlanıp bu kavramların bazı özelliklerine ulaşılmıştır. Görülmüştürki bu özellikler sayesinde $\mathfrak{T}$ çatısına ait bazı teoremlerin ispatları daha kolay yapılabilmiş ve yeni teoremlerin türetilmesine olanak sağlanmıştır. Ayrıca bu çalışmada aynı $\mathfrak{T}$ çatısının yapısı hakkında genel bir inceleme yapılmış ve geçerli formüllerin yansımalarının da daima geçerli olamayacağı, geçerli olmayanların da yansımalarının geçerli olabileceği gösterilmiştir.

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CHAPTER 1

INTRODUCTION TO TEMPORAL LOGIC

In this study, it is given some background about modal logic, since temporal logic is a brunch of modal logic. Before I start, I want to emphasize that only in this chapter most of definitions and information about modal languages were taken from the book "Modal Logic" written by Patrick Blackburn, Maarten de Rijke and Yde Venema , (2001).

Although we can't give an exact definition of modal logic, in literature there are three different slogans.

Firstly inspite of their simplicity, the modal languages with its operators are perfect way for talking about relational structures.

A relational structure consists of a set and relation collections on it. In fact it can be thought that every mathematical structure is a kind of relational structure.

Secondly in contrast to a classical logic the modal languages work internal and local on relational structures. Therefore it can be said that modal formulas are evaluated inside the structures at a special points. For this reason, the universal quantifiers are never use in modal logic.

The Third slogan of modal logic is that modal languages are not isolated formal systems.

Every modal language has corresponding classical language that describe the same class of structures. Both modal and classical languages talk about relational structures but they do so very differently. Whereas modal languages take an internal perspective, classical languages are the prime example of how to take an external perspective on relational structures. In spite of this we have a standard translation of any modal language into its corresponding classical language. This translation provides a bridge between the worlds of modal and classical logic. The resultant study is called correspondence theory between the worlds of modal and classical logic. The resultant study is called correspondence theory which is the cornerstone of the modal logic.

The modal logic is linked up with universal algebra via the apparatus of duality theory. In this framework, modal formulas are viewed as algebraic terms (Blackburn, Rijke and Venema 2001).

It is now time to meet the modal languages that we will be working with. First, we introduce the basic modal languages. We then define modal languages of arbitrary similarity type.

1.1. Modal Languages

Definition 1.1 *The Basic modal language is defined using a set of propositional letters* Φ *whose elements are usually denoted by* $p_1, p_2, p_3, ...$ *the constant truth* $\top$ *, boolean connectives* $\vee, \neg,$ *and the modal operator* $\Diamond$*. Modal formulas are denoted by Greek letters and are built in the following way:*

$$\phi := p_i | \top | \neg\phi | \psi \vee \phi | \Diamond\phi.$$

This definition means that a formula is either a propositional latter, the propositional constant truth (top), a negated formula, a disjunction of formulas, or a formula prefixed by a diamond, respectively. (Blackburn, Rijke and Venema 2001) .

Just as the familiar first-order existential and universal quantifiers are dual to each other such that $\forall xp := \neg\exists x\neg p$, we have a dual operator $\Box$ (box) of our diamond which is defined by $\Box\phi := \neg\Diamond\neg\phi$

We also make the use of classical abbreviations for conjunctions, implication, bi-implication and the constant falsum (bottom) $\perp$

$\phi \wedge \psi := \neg(\neg\phi \vee \neg\psi)$ $\qquad\qquad$ $\phi \to \psi := \neg\phi \vee \psi$

$\phi \leftrightarrow \psi := (\phi \to \psi) \wedge (\phi \leftarrow \psi)$ $\qquad\qquad$ $\perp := \neg\top$

The formal definitions of $\Diamond$ and $\Box$,which are unary operators, will be given in the modeling section. And this definition will be valid at temporal logic, but it will have a time meaning, as well.

- $\Diamond\phi$ can be read “it is possibly the case that ϕ ”.
- $\Box\phi$ means “it is not possible that not ϕ” that is “necessarily ϕ”

Up to now we have defined the basic modal language. From now on we will generalize it.

The purpose of generalizing the basic modal language is that it lacks of defining the binary operators.

Firstly we will define modal similarity type:

Definition 1.2 *A modal similarity type is a pair $\tau = (O, e)$ where O is a nonempty set, and e is a function such that $e : O \rightarrow \mathbb{N}$. The elements of O are called modal operators; we use $\triangle$ to denote the elements of O. The function e assigns to each operator $\triangle \in O$ a finite arity, indicating the number of arguments $\triangle$ can be applied to. (Blackburn, Rijke and Venema 2001)*

Definition 1.3 *A modal language $ML(\tau, \Phi)$ is built up using a modal similarity type $\tau = (O, e)$ and a set of proposition letters Φ. The set of modal formulas over τ and Φ is given by the rule .*

$$\phi := p|\top|\neg\phi|\psi \vee \phi| \triangle (\phi_1, ..., \phi_{e(\triangle)})$$

where p ranges over the elements of Φ. (Blackburn, Rijke and Venema 2001)

Notation of τ_0 is used for only basic modal language .

For binary modal operators, we often use the notation $\phi \triangle \psi$ instead of $\triangle (\phi, \psi)$. For example in general we use the notation $\phi\, \mathcal{U}\, \psi$ instead of $\mathcal{U}(\phi, \psi)$ for very important binary temporal operator $\mathcal{U}$ (until).

In the modal languages for every modal operator, there is one dual operator. For instance, it is known that $\Box$ is dual of $\Diamond$ and vice versa. Now we can define the dual operator of $\triangle$ notation of $\triangledown$.

Definition 1.4 *We now define dual operators for non-nullary triangles. For each $\triangle \in O$, the dual $\triangledown$ of $\triangle$ is defined as;*

$$\triangledown(\phi_1, ..., \phi_n) := \neg \triangle (\neg\phi_1, ..., \neg\phi_n)$$

(Blackburn, Rijke and Venema 2001)

Now we can define the basic temporal language which is our main structure.

1.2. Basic Temporal Language

The basic temporal language is built by using a set of unary operators
$O = \{<F>, <P>\}$.

The intended interpretation of a formula $\Diamond\phi$ is "ϕ will be true at some future time" and the intended interpretation of a formula $\diamondminus\phi$ is "ϕ was true at some past time". This language is called the basic temporal language.

On the other hand in the basic temporal language because of time meaning some more operators can be defined. For instance next ($\circ$) and its mirror image previous ($\ominus$) are very important operators in basic temporal language so our operator set for basic temporal language O can be enriched such on; $O = \{\Diamond, \diamondminus, \circ, \ominus\}$

A formula containing at least one of such operators is called a basic temporal formula. (Seow & Devanathan 1994)

Now let us define the all of basic temporal operators informally.

- $\Diamond\phi$:="ϕ will be true at some future time ." (Eventually)
- $\Box\phi := \neg\Diamond\neg\phi$:="ϕ will be true at every future time." (Henceforth)
- $\circ\ \phi$:="ϕ will be true at next time." (Next)
- $\diamondminus\phi$:="ϕ was true at some past time." (Once)
- $\boxminus\phi$:=$\neg\diamondminus\neg\phi$:="ϕ was true at every past time." (Has-always-been)
- $\ominus\phi$:="ϕ was true at previous time." (Previous) (Manna & Pnueli 1983)

Since we said that every operator had a dual operator in modal language $\circ$ and $\ominus$operators have dual operators. We can show that a dual of $\circ$ is same as again $\circ$. Firstly we know that $\circ\phi$ means "ϕ is true next state" so dual of $\circ\phi$ ($\neg\circ\neg\phi$) is "Next state is not state at which ϕ is not true", which amounts to "ϕ is true at next state". We can show similarly that the dual of $\ominus$ is again $\ominus$.

We can express many interesting assertions about time with this language. For example, $\diamondminus\phi \rightarrow \Box\diamondminus\phi$, says "whatever has happened will always have happened". If we insist that $\Diamond\phi \rightarrow \Diamond\Diamond\phi$ must always be true, it shows that we are thinking of time as dense: between any two instants there is always a third. (Blackburn, Rijke and Venema 2001)

Although the discussion in this study has contained many semantically suggestive phrases such as "true" and "intended interpretation", as yet it is not given them no mathematical content. The modal languages are interpreted in relational structure. In fact by the end of the chapter it will be have done this in two distinct way: at the level of models and at the level of frames. Both levels are important, though in different ways. The level of models is important, because this is the place where the fundamental notion of satisfaction (or truth) is defined. The level of frames is also important, since it supports the key logical notion of validity. (Blackburn, Rijke and Venema 2001)

Firstly we define frames, models and satisfaction relation for the basic modal language and then we will carry it to the basic temporal language.

1.3. Frames, Models and Satisfaction for The Basic Modal Language

Definition 1.5 *If we are given a nonempty set A and a positive integer n, we say that R is an $n - ary\ relation$ on A if $A^n \supseteq R$. R is unary if $n = 1$, binary if $n = 2$, and ternary if $n = 3$.*

Definition 1.6 *A frame for the basic modal language is a pair $\mathfrak{F} = (W, R)$ such that W is a non-empty set and R is a binary relation on W.(Blackburn, Rijke and Venema 2001)*

We refer to elements of W by many different names. In general we use state, point, position, time, world etc.

Now the definition of a model on the basic modal language can be obtained.

Definition 1.7 *A model for the basic modal language is a pair $\mathfrak{M} = (\mathfrak{F}, V)$, where $\mathfrak{F}$ is a frame for the basic modal language, and V is a function assigning to each proposition letter p in Φ a subset $V(p)$ of W.*

Formally, V is a map: $\Phi \to \mathcal{P}(W)$, where $\mathcal{P}(W)$ denotes the power set of W. Informally we think of $V(p)$ as the set of points in our model where p is true. The function V is called a valuation. For given any model $\mathfrak{M} = (\mathfrak{F}, V)$, we can say that $\mathfrak{M}$ is based on the frame $\mathfrak{F}$, or that $\mathfrak{F}$ is the frame underlying $\mathfrak{M}$. (Blackburn, Rijke and Venema 2001)

We may use the notation $\mathfrak{M} = (\mathfrak{F}, V(p), V(q), V(r), ...)$ instead of $\mathfrak{M} = (\mathfrak{F}, V)$ for any model.

The next definition explains how to interpret the basic modal languages in models by means of the following satisfaction definition.

Definition 1.8 *Suppose w is a state in a model $\mathfrak{M} = (\mathfrak{F}, V)$. Then we inductively define the notion of a formula ϕ being satisfied (or true) in $\mathfrak{M}$ at state w as follows:*

- $\mathfrak{M}, w \Vdash p$ (iff) $w \in V(p),\ p \in \Phi$.

  It means a propositional latter p is true at the state w in the model $\mathfrak{M}$.

- $\mathfrak{M}, w \Vdash \top$ always ($\mathfrak{M}, w \Vdash \bot$ never).

- $\mathfrak{M}, w \Vdash \neg\phi$ (iff) not $\mathfrak{M}, w \Vdash \phi$.

  It means the formula ϕ can be falsifiable or refutable at the state w in the model $\mathfrak{M}$. In addition we can use $\mathfrak{M}, w \nVdash \phi$ instead of $\mathfrak{M}, w \Vdash \neg\phi$.

- $\mathfrak{M}, w \Vdash \phi \vee \psi$ (iff) $\mathfrak{M}, w \Vdash \phi$ or $\mathfrak{M}, w \Vdash \psi$.

- $\mathfrak{M}, w \Vdash \Diamond\phi$ (iff) for some $v \in W$ with R_{wv}, we have $\mathfrak{M}, v \Vdash \phi$.

- $\mathfrak{M}, w \Vdash \Box\phi$ (iff) for all $v \in W$ such that R_{wv}, we have $\mathfrak{M}, v \Vdash \phi$.

Now it is convenient to extend the valuation V from proposition letters to arbitrary formulas so that $V(\phi)$ always denote the set of states at which ϕ is true:

$$V(\phi) = \{w : \mathfrak{M}, w \Vdash \phi\}$$

(Blackburn, Rijke and Venema 2001)

Definition 1.9 • *A formula ϕ is globally or universally true in a model $\mathfrak{M}$ (Notation:$\mathfrak{M} \Vdash \phi$) if it is satisfied at all points in $\mathfrak{M}$ (that is, if $\mathfrak{M}, w \Vdash \phi$, for all $w \in W$). A formula ϕ is satisfiable in a model $\mathfrak{M}$ if there is some state in $\mathfrak{M}$ at which ϕ is true; a formula is falsifiable or refutable in a model if its negation is satisfiable.*

A set Γ of formulas is globally true (satisfiable, respectively) in a model $\mathfrak{M}$ if $\mathfrak{M}, w \Vdash \Gamma$ for all states w in $\mathfrak{M}$ (some state w in $\mathfrak{M}$, respectively).

- *A formula ϕ is valid at a state w in a frame $\mathfrak{F}$ (notation:$\mathfrak{F}, w \Vdash \phi$) if ϕ is true at w in every model $(\mathfrak{F}, V)$ based on $\mathfrak{F}$; ϕ is valid in a frame $\mathfrak{F}$ (notation:$\mathfrak{F} \Vdash \phi$) if it is valid at every state in $\mathfrak{F}$.*

  A formula ϕ is valid on a class of frames $\mathbf{F}$ (notation:$\mathbf{F} \Vdash \phi$) if it is valid on the class of all frames.

  The set of all formulas that are valid in a class of frames $\mathbf{F}$ is called the logic of $\mathbf{F}$ (notation: $\Lambda_{\mathbf{F}}$). (Blackburn, Rijke and Venema 2001)

Some examples;

Although the formula $\Diamond\Diamond\phi \to \Diamond\phi$ is valid in a frame $\mathfrak{T}_0 = (\mathbb{N}, \leqslant, \geqslant)$ where $\leqslant$ $(\geqslant)$ is the natural ordering, it is not valid in non transitive frames. Let us consider the following example.

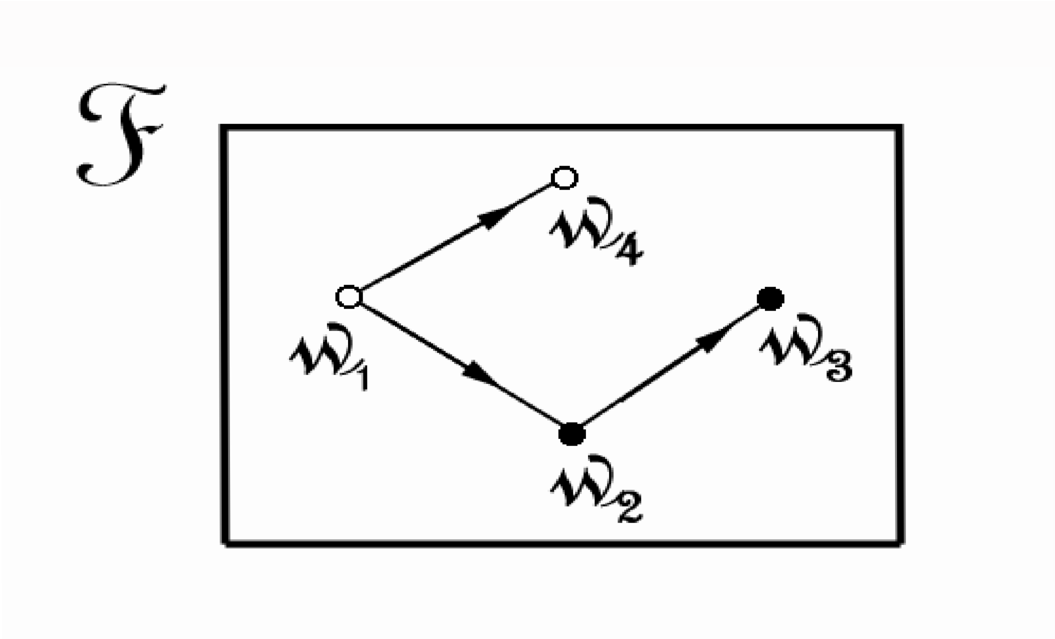

Figure 1.1: Example frame $\mathfrak{F}$ for a model

In this example $\mathfrak{M}$ is constructed on the frame $\mathfrak{F}$ such a way that
$V(\phi) = \{w_1, w_3\}$
$V(\psi) = \{w_2, w_4\}$
$\mathfrak{F} = (W, R)$ with $W = \{w_1, w_2, w_3, w_4\}$ and
$R = \{(w_1, w_2), (w_1, w_4), (w_2, w_3), (w_2, w_2), (w_3, w_3)\}$.

In this model it is clear that $\Diamond\Diamond\phi \to \Diamond\phi$ is satisfiable at the state w_3 but at w_1 does not, so it can be said that this formula is not universally true in this model.

Another example is related to the formula $\Diamond(\phi \wedge \psi) \to (\Diamond\phi \wedge \Diamond\psi)$ is valid on all frames. In order to observe that situation let us take any frame $\mathfrak{F}$ and state $w \in \mathfrak{F}$ and let V be any valuation on $\mathfrak{F}$. We have to show that if $(\mathfrak{F}, V), w \Vdash \Diamond(\phi \wedge \psi)$, then $(\mathfrak{F}, V), w \Vdash (\Diamond\phi \wedge \Diamond\psi)$ so, assume that $(\mathfrak{F}, V), w \Vdash \Diamond(\phi \wedge \psi)$. Then by the definition of $\Diamond$ there is a state v such that R_{wv} and $(\mathfrak{F}, V), v \Vdash \phi \wedge \psi$. But if $v \Vdash \phi \wedge \psi$ then $v \Vdash \phi$ and $v \Vdash \psi$. Hence $w \Vdash \Diamond\phi$ and $w \Vdash \Diamond\psi$. Other words, we have $w \Vdash \Diamond\phi \wedge \Diamond\psi$.

In this section it was given generally how to interpret the basic modal language in models. We should do same thing for the basic temporal language. However we know the basic temporal language has two unary operators F and P so this subject has to based on a concrete form. It will be done again firstly for the basic modal language.

1.3.1. Bidirectional Frames and Models

Let us denote the converse of a relation R by $\overline{R}$. It means $\forall xy(R_{xy} \leftrightarrow \overline{R}_{yx})$. We will call a frame of the form $\mathfrak{F} = (W, R, \overline{R})$, a bidirectional frame, and a model built over such a frame a bidirectional model. From now on we will only interpret the basic temporal language in bidirectional models such as we will use the relation of R for future operators and the relation of $\overline{R}$ which is converse of R, for past operators. Because of this reason in general R_F and R_P use instead of R and $\overline{R}$ respectively. That is, if $\mathfrak{M} = (W, R_F, R_P, V)$ is a bidirectional model then:

- $\mathfrak{M}, t \Vdash F\phi$ (iff) for some $s \in W$ $(R_{Fts} \wedge \mathfrak{M}, s \Vdash \phi)$
- $\mathfrak{M}, t \Vdash P\phi$ (iff) for some $s \in W$ $(R_{Pts} \wedge \mathfrak{M}, s \Vdash \phi)$ it means
  $\mathfrak{M}, t \Vdash P\phi$ (iff) for some $k \in W$ $(R_{Fst} \wedge \mathfrak{M}, s \Vdash \phi)$

In temporal logic the models are built over any one of these frames $(\mathbb{N}, \leqslant, \geqslant)$, $(\mathbb{N}, <, >)$, $(\mathbb{Z}, <, >)$, $(\mathbb{Z}, \leqslant, \geqslant)$, $(\mathbb{Q}, <, >)$, $(\mathbb{Q}, \leqslant, \geqslant)$, $(\mathbb{R}, <, >)$, $(\mathbb{R}, \leqslant, \geqslant)$.

In this study, it is chosen $(\mathbb{N}, \leqslant, \geqslant)$ as a frame. Furthermore it is enriched with two more relations $R_\circ$ and $R_\ominus$ for the operators Next $(\circ)$ and Previous $(\ominus)$respectively then we call this frame basic temporal frame notation $\mathfrak{T}_0$ such that
$\mathfrak{T}_0 = (\mathbb{N}, \leqslant, \geqslant, R_\circ, R_\ominus)$ where;
$\mathbb{N}$ is the set of natural numbers,
$\leqslant = \{(a, b) \in \mathbb{N} \times \mathbb{N} : a \leqslant b\}$,
$\geqslant = \{(a, b) \in \mathbb{N} \times \mathbb{N} : a \geqslant b\}$,
$R_\circ = \{(a, b) \in \mathbb{N} \times \mathbb{N} : b = a + 1\}$,
$R_\ominus = \{(a, b) \in \mathbb{N} \times \mathbb{N} : b = a - 1\}$.
This frame is very fundamental for this study.

Now the definition of unary operators in the temporal models can be obtained.

Suppose the model is $\mathfrak{M}_0 = (\mathfrak{T}_0, V)$ where $\mathfrak{T}_0 = (\mathbb{N}, \leqslant, \geqslant, R_\circ, R_\ominus)$,p is propositional letter, ϕ is any formula of this model and j, k are elements of $\mathbb{N}$.

- $\mathfrak{M}_0, j \Vdash p$ (iff) $j \in V(p)$

- $\mathfrak{M}_0, j \Vdash \bot$ never
- $\mathfrak{M}_0, j \Vdash \phi \vee \psi$ (iff) $\mathfrak{M}_0, j \Vdash \phi$ or $\mathfrak{M}_0, j \Vdash \psi$
- $\mathfrak{M}_0, j \Vdash \Diamond\phi$ (iff) for some $k \in \mathbb{N}$, with $j \leqslant k$ we have $\mathfrak{M}_0, k \Vdash \phi$
- $\mathfrak{M}_0, j \Vdash \diamondminus\phi$ (iff) for some $k \in \mathbb{N}$, with $j \geqslant k$ we have $\mathfrak{M}_0, k \Vdash \phi$
- $\mathfrak{M}_0, j \Vdash \Box\phi$ (iff) for all $k \in \mathbb{N}$ such that $j \leqslant k$ we have $\mathfrak{M}_0, k \Vdash \phi$
- $\mathfrak{M}_0, j \Vdash \boxminus\phi$ (iff) for all $k \in \mathbb{N}$ such that $j \geqslant k$ we have $\mathfrak{M}_0, k \Vdash \phi$
- $\mathfrak{M}_0, j \Vdash \circ\phi$ (iff) $\mathfrak{M}_0, j+1 \Vdash \phi$
- $\mathfrak{M}_0, j \Vdash \ominus\phi$ (iff) $\mathfrak{M}_0, j-1 \Vdash \phi$

Now we need to define the binary operators in models. In order to define the binary operators first we should define the frames, models and satisfaction for modal language of arbitrary similarity types and generalize these into bidirectional frames and models.

1.4. Models And Frames For Modal Languages Of Arbitrary Modal Similarity Type

Definition 1.10 *Let τ be a modal similarity type. A τ-frame is tuple $\mathfrak{F}$ consisting of the following ingredients:*

1. *A non-empty set W,*
2. *For each $n \geqslant 0$, and each n-ary modal operator $\triangle$ in the similarity type τ, an $(n+1)$-ary relation $R_{\triangle}$.*
   (As remember that there is a binary relation for every unary operator in basic modal language)

(Blackburn, Rijke and Venema 2001)

Here again frames are simply relational structures. These frames can be shown one of the following notations:

- $\mathfrak{F} = (W, R_{\Delta_1}, ..., R_{\Delta_n})$,
- $\mathfrak{F} = (W, R_{\Delta})_{\Delta \in \tau}$,
- $\mathfrak{F} = (W, \{R_{\Delta} : \Delta \in \tau\})$.

We turn such a frame into a model in exactly the same way that we did for the basic modal language (by adding a valuation). That is, a τ-model is a pair $\mathfrak{M} = (\mathfrak{F}, V)$ where $\mathfrak{F}$ is a τ-frame, and V is a valuation with domain Φ and range $\mathcal{P}(W)$, where W is universe of $\mathfrak{F}$.

The notation of a formula ϕ being satisfied (or true) at any state w in a model $\mathfrak{M} = (W, \{R_{\Delta} : \Delta \in \tau\}, V)$ (notation:$\mathfrak{M}, w \Vdash \phi$) is defined inductively. The clauses for the atomic and boolean cases are the same as for the basic modal language. As for the modal case, when $e_{(\Delta)} > 0$ we define

$$\begin{aligned}\mathfrak{M}, w \Vdash \Delta\ (\phi_1, ..., \phi_n) \text{ (iff) for some } v_1, ..., v_n \in W \text{ with } R_{\Delta} w v_1 ... v_n \\ \text{we have, for each } i, \mathfrak{M}, v_i \Vdash \phi_i.\end{aligned}$$

(Blackburn, Rijke and Venema 2001)

Now we should formulate the satisfaction clause for $\nabla(\phi_1, ..., \phi_n)$. For doing this we need duality properties such that $\nabla(\phi_1, ..., \phi_n) := \neg \Delta\ (\neg\phi_1, ..., \neg\phi_n)$. So we have,

$$\begin{aligned}\mathfrak{M}, w \Vdash \nabla(\phi_1, ..., \phi_n) \text{ (iff) for all } v_1, ..., v_n \in W \text{ such that } R_{\Delta} w v_1 ... v_n \\ \text{we have, for some } i, \mathfrak{M}, v_i \Vdash \phi_i\end{aligned}$$

1.5. Binary Operators in Temporal Logic

This section is very important because the basic temporal operators are not strong enough for dynamic systems. For example $\Box\phi$ says "ϕ will always be true" but we can not express that "ϕ will always be true until any future time" by only using the basic temporal operators. For that reason, we need the binary temporal operators $\mathcal{U}$ (until) and its mirror image $\mathcal{S}$ (since).

But firstly we should give 3-ary relations R_U and R_S (mirror image of R_U) for binary operators $\mathcal{U}$ and $\mathcal{S}$ such that;

- $R_U = \{(a, b, c) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N} : a \leqslant b < c\}$

- $R_S = \overline{R}_U = \{(a, b, c) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N} : a \geqslant b > c\}$

After that, we can extend our temporal frame from $\mathfrak{T}_0$ to $\mathfrak{T}$ in order to use the binary operators such that ;

$$\mathfrak{T} = (\mathbb{N}, \leqslant, \geqslant, R_\circ, R_\ominus, R_U, R_S)$$

By so far our formulas are constructed over the temporal model $\mathfrak{M} = (\mathfrak{T}, V)$. We will use the notation $\mathfrak{M}$ for this model. Let us interpret these new operators $\mathcal{U}$, $\mathcal{S}$ and their duals in the model $\mathfrak{M} = (\mathfrak{T}, V)$.

Until is the most important binary operator in temporal logic. In addition it has very strong future meaning. Its intended interpretation is that; $\phi\,\mathcal{U}\,\psi$ says "ϕ will always be true until ψ is true". As a formally;

let $\mathfrak{M}$ be any temporal model with j and k be any two time position;

$\mathfrak{M}, j \Vdash \phi\,\mathcal{U}\,\psi$ (iff) there is a $k,\ j \leqslant k$ such that $\mathfrak{M}, k \Vdash \psi$ and
for all i with $j \leqslant i < k$: $\mathfrak{M}, i \Vdash \phi$. (Pnueli 1986)

Until can be shown on the time line such that (Reynolds 1996):

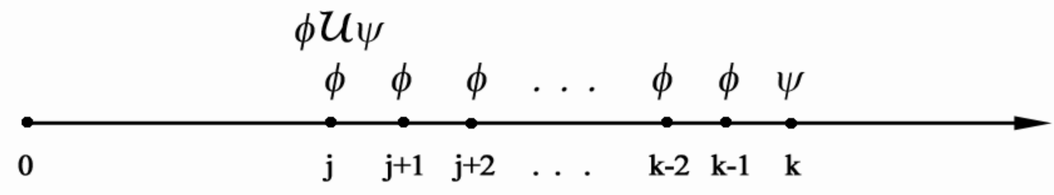

Figure 1.2: Until operator

Dual of Until ($\overline{\mathcal{U}}$) by using duality properties is defined as below;
($\overline{\mathcal{U}}(\phi, \psi) := \neg\mathcal{U}(\neg\phi, \neg\psi)$)such that ;

$\mathfrak{M}, j \Vdash \phi\,\overline{\mathcal{U}}\,\psi$ `iff` `not(there is a` $k,\ j \leqslant k$ `such that` $\mathfrak{M}, k \Vdash \neg\psi$
`and for all` i `with` $j \leqslant i < k$: $\mathfrak{M}, i \Vdash \neg\phi$
`iff` `for every` $k,\ j \leqslant k$ `with` $\mathfrak{M}, k \Vdash \psi$ `implies`
`there exist` $i,\ j \leqslant i < k$ `such that` $\mathfrak{M}, i \Vdash \phi$

Since which is another kind of binary temporal operator has very strong interpretation like Until. It is mirror image of $\mathcal{U}$ so it has past meaning. Its intended interpretation is that $\phi\ \mathcal{S}\ \psi$ says "ϕ was always true since ψ had been true". As a formally;

let $\mathfrak{M}$ be any temporal model and j and k be any two time position;

$$\mathfrak{M}, j \Vdash \phi\ \mathcal{S}\ \psi \text{ (iff) there exits } k,\ k \leqslant j \text{ with } \mathfrak{M}, k \Vdash \psi$$
$$\text{and for all } i,\ k < i \leqslant j \text{ we have } \mathfrak{M}, i \Vdash \phi\ .$$

Since can be indicated on the time line such that (Reynolds 1996):

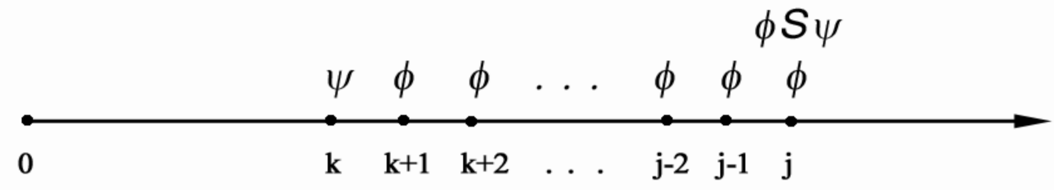

Figure 1.3: Since operator

Dual of Since can be described ($\overline{\mathcal{S}}$) like until by using the duality properties ($\overline{\mathcal{S}}(\phi,\psi) := \neg\mathcal{S}(\neg\phi,\neg\psi)$)such that ;

$$\begin{array}{lll} \mathfrak{M}, j \Vdash \phi\, \overline{\mathcal{S}}\, \psi & \texttt{iff} & \texttt{not(there is a}\ k,\ j \geqslant k\ \texttt{such that}\ \mathfrak{M}, k \Vdash \neg\psi \\ & & \texttt{and for all}\ i\ \texttt{with}\ j \geqslant i > k :\ \mathfrak{M}, i \Vdash \neg\phi \\ & \texttt{iff} & \texttt{for every}\ k,\ j \geqslant k\ \texttt{with}\ \mathfrak{M}, k \Vdash \psi\ \texttt{implies} \\ & & \texttt{there exist}\ i,\ j \geqslant i > k\ \texttt{such that} \mathfrak{M}, i \Vdash \phi \end{array}$$

Except for Until and Since, there are some binary operators for example Waiting-for ($\mathcal{W}$) and its mirror image Back-to ($\mathcal{B}$) (Dixon 2005). These binary operators are not of great importance for our study so their definitions are not mentioned.

CHAPTER 2

LOWER-TOP AND UPPER-BOTTOM POINTS FOR ANY FORMULA IN TEMPORAL LOGIC

In this section we will mention our new subject lower-top and upper-bottom point for any formula in the model $\mathfrak{M}$ build over the frame $\mathfrak{T} = (\mathbb{N}, \leqslant, \geqslant, R_\circ, R_\ominus, R_U, R_S)$. There is no doubt that it will be done level of models because this structure is about satisfiable or not for any temporal formula.

Definition 2.1 *Let ϕ be any temporal formula in a model $\mathfrak{M}$ on $\mathfrak{T}$; if any state $j \in \mathbb{N}$ satisfies $\Box\phi$ we say j is an "upper point" for the formula ϕ in this model and use the notation u_ϕ for the upper point of ϕ.*
Furthermore the state $u_\phi^\perp$ is "uper-bottom of ϕ" in the model $\mathfrak{M}$ where
$u_\phi^\perp = min\{j : j$ *is an upper point of* $\phi\}$

Formally;

$$\{u_\phi\} = \{j : \mathfrak{M}, j \Vdash \Box\phi\}$$
$$u_\phi^\perp = min\{u_\phi\}.$$

As a similarly we can define lower point of ϕ and lower-top of ϕ

Definition 2.2 *Let ϕ be any temporal formula in a model $\mathfrak{M}$ on $\mathfrak{T}$; if any state $j \in \mathbb{N}$ satisfies $\boxminus\phi$ we say j is an "lower point" for the formula ϕ in this model and use the notation l_ϕ for the lower point of ϕ.*
Furthermore the state $l_\phi^\top$ is "lower-top of ϕ" in the model where
$l_\phi^\top = max\{j : j$ *is an lower point of* $\phi\}$

Formally;

$$\{l_\phi\} = \{j : \mathfrak{M}, j \Vdash \boxminus\phi\}$$
$$l_\phi^\top = max\{l_\phi\}.$$

Let us show these definition on the time line .

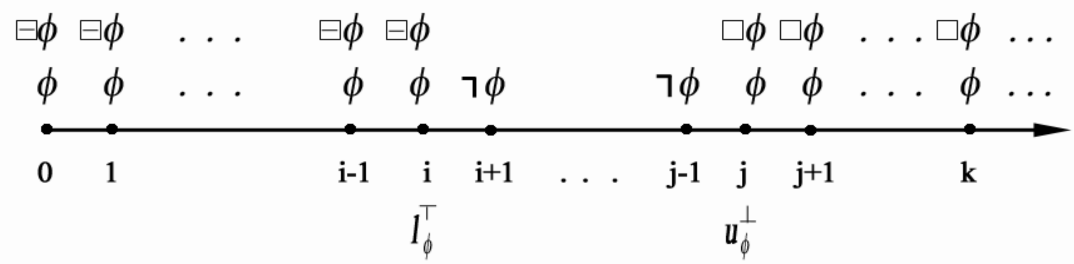

Figure 2.1: Lower-top $l_\phi^\top$ and upper-bottom $u_\phi^\perp$ point for a formula

Now we can give some propositions under these definitions. But we ought to emphasize that these propositions valid for only the temporal models which build on the frame $\mathfrak{T}$.

Proposition 2.1 *Any temporal formula ϕ in a model $\mathfrak{M}$ hasn't got lower-top point if any one of these two conditions is satisfied.*

- $\mathfrak{M}, 0 \nVdash \phi$ *(First state of this model $\mathfrak{M}$ doesn't satisfy ϕ.)*
- $\mathfrak{M} \Vdash \phi$ *(ϕ is universally true in this model $\mathfrak{M}$.)*

Proof: For first position, no state satisfies $\boxminus\phi$ because $\mathfrak{M}, 0 \nVdash \phi$. Then there is no lower point for the formula ϕ in this model. It means the set $\{l_\phi\}$ is empty so this set doesn't have a maximum element. So there is no lower-top point for the formula ϕ.

For the other position,

$$\text{for every } j \in \mathbb{N}, \mathfrak{M}, j \Vdash \phi$$

and so

$$\mathfrak{M}, j \Vdash \boxminus\phi \text{ for each } j \in \mathbb{N}.$$

Therefore $u_\phi = \mathbb{N}$. We know that the set of natural numbers doesn't have a maximum element. Thus ϕ doesn't have a lower-top point in this model.

Proposition 2.2 *If in a model $\mathfrak{M}$, the formula $\Diamond\neg\phi$ ($\neg\Box\phi$) is universally true then the formula ϕ doesn't have an upper-bottom point in this model.*

Proof: From the hypothesis

$$\text{for every } j \in \mathbb{N}\ ; \mathfrak{M}, j \Vdash \neg\Box\phi.$$

It means

$$\text{for every } j \in \mathbb{N};\ \mathfrak{M}, j \nVdash \Box\phi$$

Since $\{u_\phi\} = \{j : \mathfrak{M}; j \Vdash \Box\phi\}$, in the model $\mathfrak{M}$ there is no upper state for the formula ϕ. It means $\{u_\phi\} = \emptyset$. Finally since the empty set doesn't have the minimum element, ϕ doesn't have the upper-bottom point in this model.

Proposition 2.3 *Any temporal formula ϕ is universally true in a model $\mathfrak{M}$ iff $u_\phi^\perp = 0$ in this model.*

Proof: Let us suppose ϕ is universally true in a model $\mathfrak{M}$. It means;

$$\text{for every } j \in \mathbb{N}, \quad \mathfrak{M}, j \Vdash \phi.$$

Due to the fact that

$$\text{for every } j \in \mathbb{N},\ (\ \mathfrak{M}, k \Vdash \phi \text{ for each } k,\ j \leqslant k\ );$$
$$\text{for every } j \in \mathbb{N}, \quad \mathfrak{M}, j \Vdash \Box\phi$$

Then it is clear that

$$\{u_\phi\} = \{j : \mathfrak{M}, j \Vdash \Box\phi\} = \mathbb{N}$$

Finally

$$u_\phi^\perp = min\{u_\phi\} = min\mathbb{N} = 0$$

Conversely let us suppose beginning state of this model is upper-bottom of a formula ϕ. Other words the state 0 is first point satisfies $\Box\phi$ such that $\mathfrak{M}, 0 \Vdash \Box\phi$. Due to the definition of box ($\Box$)operator;

$$\text{for every k, } k \geqslant 0,\ \mathfrak{M}, k \Vdash \phi$$

Thus the formula ϕ is universally true in this model.

After that we consider the relations about temporal language operators $\circ$ and $\ominus$ with lower-top and upper-bottom points respectively. But it is useful to look at the time line interpretation in figure 2.1 before giving the corresponding properties.

It means any temporal formula in a model $\mathfrak{M}$ can not be satisfied next state of lower-top of ϕ and previous state of upper-bottom of ϕ.

Proposition 2.4 *Let ϕ be any temporal formula in a model $\mathfrak{M}$*

If ϕ has lower-top and upper-bottom point (different from zero)in this model then

- $\mathfrak{M}, u_\phi^\perp \nVdash \ominus\phi$
- $\mathfrak{M}, l_\phi^\top \nVdash \circ\phi$
- *There exists* $j,\ l_\phi^\top < j < u_\phi^\perp$ *such that* $\mathfrak{M}, j \nVdash \phi$

Proof: For the proof of first case let us assume

$$\mathfrak{M}, u_\phi^\perp \Vdash \ominus\phi. \text{ It is equal to } \mathfrak{M}, u_\phi^\perp - 1 \Vdash \phi.$$

And by the definition of upper-bottom of ϕ;

$$u_\phi^\perp = min\{j : \mathfrak{M}, j \Vdash \Box\phi\} \text{ so } \mathfrak{M}, u_\phi^\perp \Vdash \Box\phi.$$

Now we have

$$\mathfrak{M}, j \Vdash \phi \text{ for each } j \geqslant u_\phi^\perp, \quad \text{and } \mathfrak{M}, u_\phi^\perp - 1 \Vdash \phi$$

Then it is clear that

$$\mathfrak{M}, j \Vdash \phi \text{ for each } j \geqslant u_\phi^\perp - 1, \quad \text{so } \mathfrak{M}, u_\phi^\perp - 1 \Vdash \Box\phi.$$

Therefore $u_\phi^\perp - 1 \in \{u_\phi\}$. But this is contradiction with the minimality of $u_\phi^\perp$. Thus

$$\mathfrak{M}, u_\phi^\perp \nVdash \ominus\phi.$$

The proof of second case will be followed similarly. Let us assume

$$\mathfrak{M}, l_\phi^\top \Vdash \circ\phi. \text{ It is equal to } \mathfrak{M}, l_\phi^\top + 1 \Vdash \phi.$$

And by the definition of lower-top of ϕ

$$l_\phi^\top = max\{j : \mathfrak{M}, j \Vdash \boxminus\phi\} \text{ so } \mathfrak{M}, l_\phi^\top \Vdash \boxminus\phi.$$

Now we have

$$\mathfrak{M}, j \Vdash \phi \text{ for each } 0 \leqslant j \leqslant l_\phi^\top, \quad \text{and } \mathfrak{M}, l_\phi^\top + 1 \Vdash \phi$$

Then it is clear that

$$\mathfrak{M}, j \Vdash \phi \text{ for each } 0 \leqslant j \leqslant l_\phi^\top + 1, \quad \text{so } \mathfrak{M}, l_\phi^\top + 1 \Vdash \boxminus\phi.$$

Therefore $l^{\top}_{\phi} + 1 \in \{l_{\phi}\}$. But this is a contradiction with the maximality of $l^{\top}_{\phi}$. Thus

$$\mathfrak{M}, l^{\top}_{\phi} \nVdash \circ\phi.$$

For the third case let us assume

$$\text{for every } \ j, \ l^{\top}_{\phi} < j < u^{\perp}_{\phi}; \ \mathfrak{M}, j \Vdash \phi.$$

But this is a contradiction with previous of two properties

$$\mathfrak{M}, u^{\perp}_{\phi} \nVdash \ominus\phi \text{ and } \mathfrak{M}, l^{\top}_{\phi} \nVdash \circ\phi.$$

Thus

$$\mathfrak{M}, j \nVdash \phi \ \text{ for some } \ j, \ l^{\top}_{\phi} < j < u^{\perp}_{\phi}$$

Proposition 2.5 *Let ϕ and ψ be two temporal formula in any model $\mathfrak{M}$.*
If there exists $u^{\perp}_{\phi}$ and $u^{\perp}_{\psi}$ then $u^{\perp}_{\phi\wedge\psi} = max\{u^{\perp}_{\phi}, u^{\perp}_{\psi}\}$

Proof: Suppose $u^{\perp}_{\phi}$ and $u^{\perp}_{\psi}$ be upper-bottom of ϕ and ψ respectively such that;

$$u^{\perp}_{\phi} = min\{j : \mathfrak{M}, j \Vdash \Box\phi\} \text{ and } u^{\perp}_{\psi} = min\{j : \mathfrak{M}, j \Vdash \Box\psi\}$$

It means

$$\text{for every } j, \ j \geqslant u^{\perp}_{\phi}, \ \mathfrak{M}, j \Vdash \phi \text{ and for every } i, \ i \geqslant u^{\perp}_{\psi}, \ \mathfrak{M}, i \Vdash \psi.$$

Let us consider two results;

$$\text{for every } t, \ t \geqslant max\{u^{\perp}_{\phi}, u^{\perp}_{\psi}\}, \ \mathfrak{M}, t \Vdash \phi \wedge \psi.$$

By the definition of the box operator

$$\mathfrak{M}, max\{u^{\perp}_{\phi}, u^{\perp}_{\psi}\} \Vdash \Box(\phi \wedge \psi)$$

Now we should verify the state $max\{u^{\perp}_{\phi}, u^{\perp}_{\psi}\}$ is a minimum point that satisfies the formula $\Box(\phi \wedge \psi)$ in order to prove $max\{u^{\perp}_{\phi}, u^{\perp}_{\psi}\} = u^{\perp}_{\phi\wedge\psi}$
Assume that

$$\mathfrak{M}, max\{u^{\perp}_{\phi}, u^{\perp}_{\psi}\} - 1 \Vdash \Box(\phi \wedge \psi).$$

Since

$$\mathfrak{M}, max\{u^{\perp}_{\phi}, u^{\perp}_{\psi}\} - 1 \Vdash (\phi \wedge \psi),$$

we have

$$\mathfrak{M}, max\{u^{\perp}_{\phi}, u^{\perp}_{\psi}\} - 1 \Vdash \phi \text{ and } \mathfrak{M}, max\{u^{\perp}_{\phi}, u^{\perp}_{\psi}\} - 1 \Vdash \psi.$$

Now we should consider two conditions;

firstly

$$\text{if } u^{\perp}_{\phi} \geqslant u^{\perp}_{\psi} \text{ then } max\{u^{\perp}_{\phi}, u^{\perp}_{\psi}\} = u^{\perp}_{\phi}.$$

Since

$$\mathfrak{M}, max\{u^{\perp}_{\phi}, u^{\perp}_{\psi}\} - 1 \Vdash \phi$$

it can be written

$$\mathfrak{M}, u^{\perp}_{\phi} - 1 \Vdash \phi.$$

By the previous theorem ($(\mathfrak{T}, V), u^{\perp}_{\phi} \nVdash \ominus\phi$) it is contradiction.

The secondly

$$\text{if } u^{\perp}_{\phi} < u^{\perp}_{\psi} \text{ then } max\{u^{\perp}_{\phi}, u^{\perp}_{\psi}\} = u^{\perp}_{\psi}.$$

Similarly since

$$\mathfrak{M}, max\{u^{\perp}_{\phi}, u^{\perp}_{\psi}\} - 1 \Vdash \psi$$

it can be written

$$\mathfrak{M}, u^{\perp}_{\psi} - 1 \Vdash \psi.$$

Again we have a contradiction because of the same reason. Thus

$$u^{\perp}_{\phi\wedge\psi} = max\{u^{\perp}_{\phi}, u^{\perp}_{\psi}\}$$

Proposition 2.6 *Let ϕ and ψ be two temporal formula in a model $\mathfrak{M}$. If there exists $l^{\top}_{\phi}$ and $l^{\top}_{\psi}$ in this model then $l^{\top}_{\phi\wedge\psi} = min\{l^{\top}_{\phi}, l^{\top}_{\psi}\}$.*

Proof: This proof similar to previous proof. Let us show it on the time line:

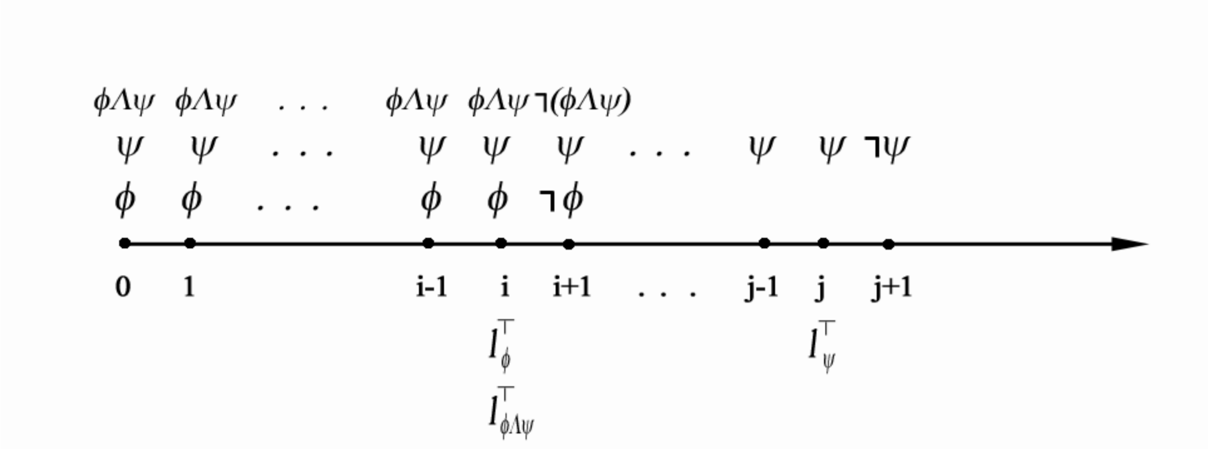

Figure 2.2: $l^{\top}_{\phi\wedge\psi} = min\{l^{\top}_{\phi}, l^{\top}_{\psi}\}$

Proposition 2.7 *If any temporal formula ϕ has an upper-bottom point ($u^{\perp}_{\phi}$) in a model $\mathfrak{M}$ (it means $\{u_\phi\} \neq \emptyset$) then the formula $\Diamond\phi$ is universally true in this model. This theorem can be represented formally such that*

$$\textit{if } \mathfrak{M}, j \Vdash \Box\phi \textit{ for any state } j \in \mathbb{N} \textit{ then } \mathfrak{M} \Vdash \Diamond\phi.$$

Proof: From the hypothesis

$$\mathfrak{M}, j \Vdash \Box\phi \text{ for any point } j \in \mathbb{N}. \text{ It means } \mathfrak{M}, k \Vdash \phi \text{ for every } j,\ j \leqslant k.$$

Furthermore it is clear

$$\mathfrak{M}, k \Vdash \Diamond\phi \text{ for every } k,\ j \leqslant k,$$

due to the valid formula $\phi \to \Diamond\phi$ (theorem 3.1) on $\mathfrak{T}$

On the other hand, since $\mathfrak{M}, j \Vdash \phi$, we have

$$\mathfrak{M}, t \Vdash \Diamond\phi \text{ for every } t,\ t \leqslant j.$$

Consider these two results we can say

$$\mathfrak{M}, j \Vdash \Diamond\phi \text{ for each state } j \in \mathbb{N}.$$

Thus if $\mathfrak{M}, j \Vdash \Box\phi$ for any state j in the model $\mathfrak{M}$ then the formula $\Diamond\phi$ is universally true in this model.

Proposition 2.8 *Any temporal formula ϕ has an upper-bottom point ($u_\phi^\perp$) in a model $\mathfrak{M}$ iff the formula $\Diamond\Box\phi$ is universally true ($\mathfrak{M} \Vdash \Diamond\Box\phi$) in this model.*
Formally ;

$$\mathfrak{M}, j \Vdash \Box\phi \textit{ for an any state in this model iff } \mathfrak{M} \Vdash \Diamond\Box\phi$$

Proof: Suppose any temporal formula ϕ has an upper-bottom point in a model $\mathfrak{M}$. It means $\{u_\phi\} \neq \emptyset$ and then $\mathfrak{M}, j \Vdash \Box\phi$ for an any state in this model.
Since the valid formula $\Box\phi \leftrightarrow \Box\Box\phi$ (idempotent properties (theorem 3.3) on $\mathfrak{T}$, we have

$$\mathfrak{M}, j \Vdash \Box\Box\phi \text{ for some state } j \in \mathbb{N}.$$

Now let us rename the formula $\Box\phi := \psi$ then we have,

$$\mathfrak{M}, j \Vdash \Box\psi \text{ for some state } j \in \mathbb{N}.$$

Since the proposition 2.7, it can be written

$$\Diamond\psi \text{ is universally true in the model } \mathfrak{M}$$

Thus if we write $\Box\phi$ instead of ψ then

$$\Diamond\Box\phi \text{ is universally true in the model } \mathfrak{M}$$

Conversely let us suppose

$$\Diamond\Box\phi \text{ is universally true in the model } \mathfrak{M}$$

Then

$$\text{for an any point } j \in \mathbb{N},\ \ \mathfrak{M}, j \Vdash \Diamond\Box\phi.$$

Because of definition of the diamond operator

$$\mathfrak{M}, k \Vdash \Box\phi \text{ for some } k,\ \ j \leqslant k.$$

It is clear that the state $k \in \mathbb{N}$ is an upper point of ϕ so $\{u_\phi\} \neq \emptyset$. Finally the set $\{u_\phi\} \subseteq \mathbb{N}$ has a minimum element which is upper-bottom of ϕ.

Our this definitions (upper-bottom and lower-top point for an any formula) helps us to prove some theorems in next section.

CHAPTER 3

THEOREMS

In this chapter it will be shown each given formulas are valid in the frame $\mathfrak{T}$.

Theorem 3.1 $\mathfrak{T} \Vdash \phi \rightarrow \Diamond\phi$

Proof: Let $\mathfrak{M}$ be a model over the frame $\mathfrak{T}$ and let j be an any state of $\mathfrak{M}$. Assume that

$$\mathfrak{M}, j \Vdash \phi \quad \text{but} \quad \mathfrak{M}, j \Vdash \neg\Diamond\phi.$$

Since $\mathfrak{M}, j \Vdash \neg\Diamond\phi$,

$$\mathfrak{M}, j \Vdash \Box\neg\phi.$$

Using $\Box$ definition we can write

$$\mathfrak{M}, t \Vdash \neg\phi \text{ for all } t, \ t \geq j.$$

If we choose $\underline{t = j}$ we have

$$\mathfrak{M}, j \Vdash \neg\phi \text{ and } \mathfrak{M}, j \Vdash \phi$$

wich have a contradiction.
Thus;

$$\mathfrak{M}, j \Vdash \Diamond\phi.$$

Theorem 3.2 $\mathfrak{T} \Vdash \circ\phi \rightarrow \Diamond\phi$

Proof: Let $\mathfrak{M}$ be a model over the frame $\mathfrak{T}$ and let j be any state of $\mathfrak{M}$.
Assume that

$$\mathfrak{M}, j \Vdash \circ\phi.$$

Since $\mathfrak{M}, j \Vdash \circ\phi$,

$$\mathfrak{M}, j+1 \Vdash \phi.$$

It can be said $\mathfrak{M}, k \Vdash \phi$ for some $k \geq j$. it is clear that this is the definition of

$$\mathfrak{M}, j \Vdash \Diamond\phi.$$

Theorem 3.3 $\mathfrak{T} \Vdash \Box\phi \leftrightarrow \Box\Box\phi$

Proof: Let $\mathfrak{M}$ be a model over the frame $\mathfrak{T}$ and let j be any state of $\mathfrak{M}$.

$$\begin{aligned} \mathfrak{M}, j \Vdash \Box\phi \quad & \texttt{iff} \quad \mathfrak{M}, k \Vdash \phi \ \texttt{for every}\, k,\ k \geq j. \\ & \texttt{iff} \quad \mathfrak{M}, t \Vdash \Box\phi \ \texttt{for every}\, t,\ k \geq t \geq j. \\ & \texttt{iff} \quad \mathfrak{M}, j \Vdash \Box\Box\phi. \end{aligned}$$

Theorem 3.4 $\mathfrak{T} \Vdash \Diamond\phi \leftrightarrow \Diamond\Diamond\phi$

Proof: Let $\mathfrak{M}$ be a model over the frame $\mathfrak{T}$ and let j be any state of $\mathfrak{M}$.

$$\begin{aligned} \mathfrak{M}, j \Vdash \Diamond\phi \quad & \texttt{iff} \quad \mathfrak{M}, k \Vdash \phi \ \texttt{for some}\, k,\ k \geq j. \\ & \texttt{iff} \quad \mathfrak{M}, k \Vdash \phi \ \texttt{for some}\, k,\ k \geq t \geq j. \\ & \texttt{iff} \quad \mathfrak{M}, t \Vdash \Diamond\phi \ \texttt{for some}\, t,\ t \geq j. \\ & \texttt{iff} \quad \mathfrak{M}, j \Vdash \Diamond\Diamond\phi. \end{aligned}$$

Theorem 3.5 $\mathfrak{T} \Vdash \Diamond\neg\phi \leftrightarrow \neg\Box\phi$

Proof: Let $\mathfrak{M}$ be a model over the frame $\mathfrak{T}$ and let j be any state of $\mathfrak{M}$.

$$\begin{aligned} \mathfrak{M}, j \Vdash \neg\Box\phi \quad & \texttt{iff} \quad \texttt{not}\ \mathfrak{M}, j \Vdash \Box\phi. \\ & \texttt{iff} \quad \texttt{not}\ [\mathfrak{M}, k \Vdash \phi \ \texttt{for every}\, k,\ k \geq j\,]. \\ & \texttt{iff} \quad \mathfrak{M}, k \Vdash \neg\phi \ \texttt{for some}\ k,\ k \geq j. \\ & \texttt{iff} \quad \mathfrak{M}, j \Vdash \Diamond\neg\phi. \end{aligned}$$

Theorem 3.6 $\mathfrak{T} \Vdash \Diamond\phi \rightarrow \boxminus\Diamond\phi$

Proof: Let $\mathfrak{M}$ be a model over the frame $\mathfrak{T}$ and let j be any state of $\mathfrak{M}$. Suppose that

$$\mathfrak{M}, j \Vdash \Diamond\, \phi \text{ for some } j \geqslant 0,$$

but

$$\mathfrak{M}, k \nVdash \Diamond\phi \text{ for some } k,\ \ k \leqslant j.$$

Since $\mathfrak{M}, j \Vdash \Diamond\, \phi$,

$$\mathfrak{M}, t \Vdash \phi \text{ for some } t,\ \ t \geqslant j,$$

and since $\mathfrak{M}, k \nVdash \Diamond\, \phi$, which is equivalent to $\mathfrak{M}, k \Vdash \Box\neg\phi$,

$$\mathfrak{M}, s \Vdash \neg\phi \text{ for all } s,\ \ k \leqslant s.$$

In the case of $k \leqslant j \leqslant t$, we can say $\mathfrak{M}, t \Vdash \neg\phi$ but we have $\mathfrak{M}, t \Vdash \phi$ which is contradiction. Thus,

$$\text{If } \mathfrak{M}, j \Vdash \Diamond\, \phi \text{ then } \mathfrak{M}, k \Vdash \Diamond\, \phi \text{ for all } k,\ \ k \leqslant j.$$

Theorem 3.7 $\mathfrak{T} \Vdash \Box(\phi \to \psi) \to (\Diamond\phi \to \Diamond\psi)$

Proof: Let $\mathfrak{M}$ be a model over the frame $\mathfrak{T}$ and let j be any state of $\mathfrak{M}$.
Assume that

$$\mathfrak{M}, j \Vdash \Box(\phi \to \psi).$$

It means

$$\text{for every } k,\ \ k \geq j, \text{ if } \mathfrak{M}, k \Vdash \phi \text{ then } \mathfrak{M}, k \Vdash \psi.$$

We can write that

$$\text{for some } t,\ \ t \geq j, \text{ if } \mathfrak{M}, t \Vdash \phi \text{ then } \mathfrak{M}, t \Vdash \psi.$$

Then

$$\text{if } \mathfrak{M}, j \Vdash \Diamond\phi \text{ then } \mathfrak{M}, j \Vdash \Diamond\psi.$$

Finally $\mathfrak{M}, j \Vdash \Diamond\phi \to \Diamond\psi$

Theorem 3.8 $\mathfrak{T} \Vdash \Box(\phi \wedge \psi) \leftrightarrow (\Box\phi \wedge \Box\psi)$

Proof: Let $\mathfrak{M}$ be a model over the frame $\mathfrak{T}$ and let j be any state of $\mathfrak{M}$.

$$\begin{array}{lll} \mathfrak{M}, j \Vdash \Box(\phi \wedge \psi) & \texttt{iff} & \texttt{for all } k,\ k \geq j,\ \mathfrak{M}, k \Vdash \phi \wedge \psi. \\ & \texttt{iff} & \texttt{for all } k,\ k \geq j,\ \mathfrak{M}, k \Vdash \phi \texttt{ and } \mathfrak{M}, k \Vdash \psi. \\ & \texttt{iff} & \mathfrak{M}, j \Vdash \Box\phi \texttt{ and } \mathfrak{M}, j \Vdash \Box\psi. \\ & \texttt{iff} & \mathfrak{M}, j \Vdash \Box\phi \wedge \Box\psi. \end{array}$$

Theorem 3.9 $\mathfrak{T} \Vdash \Diamond(\phi \vee \psi) \leftrightarrow (\Diamond\phi \vee \Diamond\psi)$

Proof: Let $\mathfrak{M}$ be a model over the frame $\mathfrak{T}$ and let j be any state of $\mathfrak{M}$.

$$\begin{array}{lll} \mathfrak{M}, j \Vdash \Diamond(\phi \vee \psi) & \texttt{iff} & \texttt{for some } k,\ k \geq j,\ \mathfrak{M}, k \Vdash (\phi \vee \psi). \\ & \texttt{iff} & \texttt{for some } k,\ k \geq j,\ \mathfrak{M}, k \Vdash \phi \texttt{ or } \mathfrak{M}, k \Vdash \psi. \\ & \texttt{iff} & \mathfrak{M}, j \Vdash \Diamond\phi \textit{ or } \mathfrak{M}, j \Vdash \Diamond\psi. \\ & \texttt{iff} & \mathfrak{M}, j \Vdash \Diamond\phi \vee \Diamond\psi. \end{array}$$

Theorem 3.10 $\mathfrak{T} \Vdash (\Box\phi \vee \Box\psi) \rightarrow \Box(\phi \vee \psi)$

Proof: Let $\mathfrak{M}$ be a model over the frame $\mathfrak{T}$ and let j be any state of $\mathfrak{M}$.
Assume that

$$\mathfrak{M}, j \Vdash \Box\phi \vee \Box\psi$$

but

$$\mathfrak{M}, j \Vdash \neg\Box(\phi \vee \psi).$$

Since $\mathfrak{M}, j \Vdash \neg\Box(\phi \vee \psi)$,

$$\mathfrak{M}, j \Vdash \Diamond\neg(\phi \vee \psi).$$
$$\text{Then } \mathfrak{M}, j \Vdash \Diamond(\neg\phi \wedge \neg\psi).$$

Since $\mathfrak{M}, j \Vdash \Box\phi \vee \Box\psi$,

$$\mathfrak{M}, j \Vdash \Box\phi \text{ or } \mathfrak{M}, j \Vdash \Box\psi.$$

Since $\mathfrak{M}, j \Vdash \Diamond(\neg\phi \wedge \neg\psi)$,

$$\mathfrak{M}, k \Vdash \neg\phi \wedge \neg\psi \text{ for some } k,\ \ k \geq j.$$

$$\text{Then } (\mathfrak{M}, k \Vdash \neg\phi \text{ and } \mathfrak{M}, k \Vdash \neg\psi)\ \text{ for some } k,\ k \geq j\ .$$

Because of $\mathfrak{M}, j \Vdash \Box\phi$,

$$\mathfrak{M}, k \Vdash \phi \quad \text{for all } k,\ k \geq j$$

but

$$\mathfrak{M}, k \Vdash \neg\phi \ \text{ for some } k,\ k \geq j.$$

It is contradiction.

Since $\mathfrak{M}, j \Vdash \Box\psi$,

$$\mathfrak{M}, k \Vdash \psi \text{ for all } k,\ k \geq j$$

but

$$\mathfrak{M}, k \Vdash \neg\psi \ \text{ for some } k,\ k \geq j$$

We have again a contradiction.

Thus $\mathfrak{M}, j \Vdash \Box(\phi \vee \psi)$

Theorem 3.11 $\mathfrak{T} \Vdash \Diamond(\phi \wedge \psi) \rightarrow (\Diamond\phi \wedge \Diamond\psi)$

Proof: Let $\mathfrak{M}$ be a model over the frame $\mathfrak{T}$ and let j be any state of $\mathfrak{M}$. Since $\mathfrak{M}, j \Vdash \Diamond(\phi \wedge \psi)$,

$$\mathfrak{M}, k \Vdash (\phi \wedge \psi) \text{ for some } k,\ \ k \geq j$$

It means

$$(\ \mathfrak{M}, k \Vdash \phi \text{ and } \mathfrak{M}, k \Vdash \psi)\text{for some } k,\ k \geq j$$

It is clear that,

$$\mathfrak{M}, k \Vdash \phi \text{ for some } k,\ \ k \geq j \text{ and } \mathfrak{M}, k \Vdash \psi \text{ for some } k,\ \ k \geq j.$$

Therefore we can write

$$\mathfrak{M}, k \Vdash \Diamond\phi \text{ and } \mathfrak{M}, k \Vdash \Diamond\psi$$

Finally $\mathfrak{M}, k \Vdash \Diamond\phi \wedge \Diamond\psi$

Theorem 3.12 $\mathfrak{T} \Vdash (\Box\phi \wedge \Diamond\psi) \to \Diamond(\phi \wedge \psi)$

Proof: Let $\mathfrak{M}$ be a model over the frame $\mathfrak{T}$ and let j be any state of $\mathfrak{M}$. Assume that

$$\mathfrak{M}, j \Vdash \Box\phi \wedge \Diamond\psi.$$

Since $\mathfrak{M}, j \Vdash \Box\phi \wedge \Diamond\psi$,

$$\mathfrak{M}, j \Vdash \Box\phi \text{ and } \mathfrak{M}, j \Vdash \Diamond\psi.$$

It means $\mathfrak{M}, k \Vdash \phi$ for all k, $k \geq j$ and $\mathfrak{M}, t \Vdash \psi$ for some t, $t \geq j$.
If we choose $\underline{k = t}$, we have $\mathfrak{M}, t \Vdash \phi$ for some t, $t \geq j$.
Since $\mathfrak{M}, t \Vdash \phi$ for some t, $t \geq j$ and $\mathfrak{M}, t \Vdash \psi$ for some t, $t \geq j$,

$$\mathfrak{M}, t \Vdash \phi \wedge \psi \text{ for some } t,\ t \geq j.$$

Finally this is definition of $\mathfrak{M}, j \Vdash \Diamond(\phi \wedge \psi)$.

Theorem 3.13 $\mathfrak{T} \Vdash \circ\,(\phi \wedge \psi) \leftrightarrow (\circ\,\phi \wedge \circ\,\psi)$

Proof: Let $\mathfrak{M}$ be a model over the frame $\mathfrak{T}$ and let j be any state of $\mathfrak{M}$.

$$\begin{aligned}
\mathfrak{M}, j \Vdash \circ\,(\phi \wedge \psi) \quad & \texttt{iff} \quad \mathfrak{M}, j+1 \Vdash \phi \wedge \psi. \\
& \texttt{iff} \quad \mathfrak{M}, j+1 \Vdash \phi \;\; \texttt{and} \;\; \mathfrak{M}, k+1 \Vdash \psi. \\
& \texttt{iff} \quad \mathfrak{M}, j \Vdash \circ\,\phi \;\; \texttt{and} \;\; \mathfrak{M}, j \Vdash \circ\,\psi. \\
& \texttt{iff} \quad \mathfrak{M}, j \Vdash \circ\,\phi \wedge \circ\,\psi.
\end{aligned}$$

Theorem 3.14 $\mathfrak{T} \Vdash \circ\,(\phi \vee \psi) \leftrightarrow (\circ\,\phi \vee \circ\psi)$

Proof: Let $\mathfrak{M}$ be a model over the frame $\mathfrak{T}$ and let j be any state of $\mathfrak{M}$.

$$\begin{aligned}
\mathfrak{M}, j \Vdash \circ(\phi \vee \psi) \quad & \texttt{iff} \quad \mathfrak{M}, j+1 \Vdash (\phi \vee \psi). \\
& \texttt{iff} \quad \mathfrak{M}, j+1 \Vdash \phi \;\; \texttt{or} \;\; \mathfrak{M}, j+1 \Vdash \psi. \\
& \texttt{iff} \quad \mathfrak{M}, j \Vdash \circ\,\phi \;\; \texttt{or} \;\; \mathfrak{M}, j \Vdash \circ\,\psi. \\
& \texttt{iff} \quad \mathfrak{M}, j \Vdash \circ\,\phi \vee \circ\,\psi.
\end{aligned}$$

Theorem 3.15 $\mathfrak{T} \Vdash \circ(\phi \to \psi) \leftrightarrow (\circ\phi \to \circ\psi)$

Proof: Let $\mathfrak{M}$ be a model over the frame $\mathfrak{T}$ and let j be any state of $\mathfrak{M}$.

$$\begin{array}{lll} \mathfrak{M}, j \Vdash \circ(\phi \to \psi) & \texttt{iff} & \mathfrak{M}, j+1 \Vdash \phi \to \psi. \\ & \texttt{iff} & \texttt{if}\, \mathfrak{M}, j+1 \Vdash \phi \;\texttt{then}\; \mathfrak{M}, j+1 \Vdash \psi. \\ & \texttt{iff} & \texttt{if}\, \mathfrak{M}, j \Vdash \circ\,\phi \;\texttt{then}\, \mathfrak{M}, j \Vdash \circ\,\psi. \\ & \texttt{iff} & \mathfrak{M}, j \Vdash \circ\,\phi \to \circ\,\psi. \end{array}$$

Theorem 3.16 $\mathfrak{T} \Vdash \circ(\phi \leftrightarrow \psi) \leftrightarrow (\circ\phi \leftrightarrow \circ\psi)$

Proof: Let $\mathfrak{M}$ be a model over the frame $\mathfrak{T}$ and let j be any state of $\mathfrak{M}$.

$$\begin{array}{lll} \mathfrak{M}, j \Vdash \circ(\phi \leftrightarrow \psi) & \texttt{iff} & \mathfrak{M}, j+1 \Vdash \phi \leftrightarrow \psi. \\ & \texttt{iff} & (\mathfrak{M}, j+1 \Vdash \phi \;\texttt{iff}\; \mathfrak{M}, j+1 \Vdash \psi). \\ & \texttt{iff} & (\mathfrak{M}, j \Vdash \circ\phi \;\texttt{iff}\; \mathfrak{M}, j \Vdash \circ\,\psi. \\ & \texttt{iff} & \mathfrak{M}, j \Vdash \circ\,\phi \leftrightarrow \circ\,\psi. \end{array}$$

Theorem 3.17 $\mathfrak{T} \Vdash \circ\Box\phi \leftrightarrow \Box \circ \phi$

Proof: Let $\mathfrak{M}$ be a model over the frame $\mathfrak{T}$ and let j be any state of $\mathfrak{M}$.

$$\begin{array}{lll} \mathfrak{M}, j \Vdash \circ\,\Box\,\phi & \texttt{iff} & \mathfrak{M}, j+1 \Vdash \Box\phi. \\ & \texttt{iff} & \mathfrak{M}, k \Vdash \phi \;\texttt{for all}\; k,\; k \geq j+1. \\ & \texttt{iff} & \mathfrak{M}, k-1 \Vdash \phi \;\texttt{for all}\; k-1,\; k-1 \geq j. \\ & \texttt{iff} & \mathfrak{M}, j \Vdash \Box \circ \phi. \end{array}$$

Theorem 3.18 $\mathfrak{T} \Vdash \circ \Diamond \phi \leftrightarrow \Diamond \;\circ\; \phi$

Proof: Let $\mathfrak{M}$ be a model over the frame $\mathfrak{T}$ and let j be any state of $\mathfrak{M}$.

$$\begin{array}{lll} \mathfrak{M}, j \Vdash \circ\Diamond\phi & \texttt{iff} & \mathfrak{M}, j+1 \Vdash \Diamond\,\phi. \\ & \texttt{iff} & \mathfrak{M}, k \Vdash \phi \;\texttt{for some}\; k,\; k \geq j+1. \\ & \texttt{iff} & \mathfrak{M}, k-1 \Vdash \phi \;\texttt{for some}\; k-1,\; k-1 \geq j. \\ & \texttt{iff} & \mathfrak{M}, j \Vdash \Diamond \;\circ\; \phi. \end{array}$$

Theorem 3.19 $\mathfrak{T} \Vdash \Diamond \Box \phi \to \Box \Diamond \phi$.

Proof: Let $\mathfrak{M}$ be a model over the frame $\mathfrak{T}$ and let j be any state of $\mathfrak{M}$. Assume that

$$\mathfrak{M}, j \Vdash \Diamond\Box\phi \text{ for an any state } j \in \mathbb{N}.$$

It is clear the formula ϕ has an upper-bottom point. Then because of the proposition 2.7 the formula $\Diamond\phi$ is universally true in this model. Now especially it can be said

$$\mathfrak{M}, k \Vdash \Diamond\phi \text{ for each } k,\ \ k \geqslant j.$$

Finally by using the definition of box operator

$$\mathfrak{M}, j \Vdash \Box\Diamond\phi.$$

Note : Of course we showed $\mathfrak{T} \Vdash \Diamond\Box\phi \to \Box\Diamond\phi$. We can think this question that the McKinsey formula $\Box\Diamond\phi \to \Diamond\Box\phi$ is valid or not in this frame $\mathfrak{T}$. Let us show this formula is not valid in the frame $\mathfrak{T}$. We should obtain a counter model $\mathfrak{M}$ on the frame $\mathfrak{T}$ such that $V(\phi)$ is a collection of odd numbers. It seems easily for any state $j \in \mathbb{N}$, $\mathfrak{M}, j \Vdash \Box\Diamond\phi$ but we know that for the states j or $j+1$ the formula ϕ is not true so for this model there is no state j such that $\mathfrak{M}, j \Vdash \Box\phi$ (other words $\mathfrak{M} \nVdash \Box\phi$). Finally due to the fact that there is a model $\mathfrak{M}$ over the frame $\mathfrak{T}$ such that $\mathfrak{M}, j \nVdash \Diamond\Box\phi$ for any state $j \in \mathbb{N}$, McKinsey formula ($\Box\Diamond\phi \to \Diamond\Box\phi$) is not valid in this frame.

Theorem 3.20 $\mathfrak{T} \Vdash \Box\Diamond\Box\phi \leftrightarrow \Diamond\Box\phi$

Proof: Let $\mathfrak{M}$ be a model over the frame $\mathfrak{T}$ and let j be any state of $\mathfrak{M}$. Assume that $\mathfrak{M}, j \Vdash \Box\Diamond\Box\phi$. Then

$$\mathfrak{M}, j \Vdash \Diamond\Box\phi \text{ for all } k \geq j.$$

If we choose $\underline{k = j}$ we have

$$\mathfrak{M}, j \Vdash \Diamond\Box\phi.$$

In order to prove other side of the theorem assume that

$$\mathfrak{M}, j \Vdash \Diamond\Box\phi$$

It means the formula ϕ has got a upper-bottom point in this model.

By the proposition 2.8 the formula $\Diamond\Box\phi$ is universally true in this model so we can say

$$\mathfrak{M}, j \Vdash \Diamond\Box\phi \text{ for every state } j,\ \ j \in \mathbb{N}$$

Finally it is obvious that

$$\mathfrak{M}, j \Vdash \Diamond\Box\phi \text{ for every state } k,\ \ k \geqslant j$$

Clearly this is the definition of Box operator such that;

$$\mathfrak{M}, j \Vdash \Box\Diamond\Box\phi$$

Theorem 3.21 $\mathfrak{T} \Vdash \Diamond\Box\Diamond\phi \leftrightarrow \Box\Diamond\phi$

Proof: Let $\mathfrak{M}$ be a model over the frame $\mathfrak{T}$ and let j be any state of $\mathfrak{M}$.

Assume that $\mathfrak{M}, j \Vdash \Diamond\Box\Diamond\phi$. Let us use the substitution $\Diamond\phi := \psi$ then we have

$$\mathfrak{M}, j \Vdash \Diamond\Box\psi\ .$$

By the theorem 3.19;

$$\mathfrak{M}, j \Vdash \Box\Diamond\psi\ .$$

Then use the the formula $\Diamond\phi$ instead of ψ, it can be written;

$$\mathfrak{M}, j \Vdash \Box\Diamond\Diamond\phi$$

If we use the definition of Box operator,

$$\mathfrak{M}, k \Vdash \Diamond\Diamond\phi \text{ for every } k,\ k \geqslant j$$

By the Theorem 3.4 ($\Diamond\phi \leftrightarrow \Diamond\Diamond\phi$ is valid in the frame $\mathfrak{T}$);

$$\mathfrak{M}, k \Vdash \Diamond\phi \text{ for every } k,\ k \geqslant j$$

Thus this is the definition of

$$\mathfrak{M}, j \Vdash \Box\Diamond\phi$$

On the other hand assume that

$$\mathfrak{M}, j \Vdash \Box\Diamond\phi.$$

If we use the substitution $\Box\Diamond\phi := \psi$ we have

$$\mathfrak{M}, j \Vdash \psi.$$

Since the theorem 3.1 ($\phi \rightarrow \Diamond\phi$ is valid in the frame $\mathfrak{T}$);

$$\mathfrak{M}, j \Vdash \Diamond\psi$$

Finally use the substitutions $\psi := \Box\Diamond\phi$;

$$\mathfrak{M}, j \Vdash \Diamond\Box\Diamond\phi$$

Theorem 3.22 $\mathfrak{T} \Vdash \Box\phi \rightarrow (\phi \wedge \circ\Box\phi)$

Proof: Let $\mathfrak{M}$ be a model over the frame $\mathfrak{T}$ and let j be any state of $\mathfrak{M}$.

$$\begin{aligned} \mathfrak{M}, j \Vdash \Box\,\phi \quad & \texttt{iff} \quad \mathfrak{M}, j \Vdash \phi \;\; \texttt{and} \;\; \mathfrak{M}, j+1 \Vdash \Box\phi. \\ & \texttt{iff} \quad \mathfrak{M}, j \Vdash \phi \;\; \texttt{and} \;\; \mathfrak{M}, j \Vdash \circ\,\Box\,\phi. \\ & \texttt{iff} \quad \mathfrak{M}, j \Vdash \phi \wedge \circ\,\Box\,\phi. \end{aligned}$$

Theorem 3.23 $\mathfrak{T} \Vdash \Diamond\phi \leftrightarrow (\phi \vee \circ\Diamond\phi)$

Proof: Let $\mathfrak{M}$ be a model over the frame $\mathfrak{T}$ and let j be any state of $\mathfrak{M}$.

$$\begin{aligned} \mathfrak{M}, j \Vdash \Diamond\phi \quad & \texttt{iff} \quad \mathfrak{M}, j \Vdash \phi \;\; \texttt{or} \;\; \mathfrak{M}, j+1 \Vdash \Diamond\,\phi. \\ & \texttt{iff} \quad \mathfrak{M}, j \Vdash \phi \;\; \texttt{or} \;\; \mathfrak{M}, j \Vdash \circ\,\Diamond\,\phi. \\ & \texttt{iff} \quad \mathfrak{M}, j \Vdash \phi \vee \circ\,\Diamond\,\phi. \end{aligned}$$

Theorem 3.24 $\mathfrak{T} \Vdash (\phi \wedge \Diamond\neg\phi) \rightarrow \Diamond(\phi \wedge \circ\neg\phi)$

Proof: Let $\mathfrak{M}$ be a model over the frame $\mathfrak{T}$ and let j be any state of $\mathfrak{M}$.
Assume that $\mathfrak{M}, j \Vdash \phi \wedge \Diamond\neg\phi$. Then,

$$\mathfrak{M}, j \Vdash \phi \text{ and } \mathfrak{M}, j \Vdash \Diamond\neg\phi.$$

Since $\mathfrak{M}, j \Vdash \Diamond\neg\phi$,

$$\mathfrak{M}, k \Vdash \neg\phi \text{ for some } k,\;\; k \geq j$$

It is clear that $k \neq j$ because we have $\mathfrak{M}, j \Vdash \phi$.

Let k be the first state satisfies the state formula $\neg\phi$ after j.

It means

$$\mathfrak{M}, t \Vdash \phi \text{ for every } t,\ j \leq t < k,$$

so we can say $\mathfrak{M}, k-1 \Vdash \phi$ for some $k-1,\ k-1 \geq j$. In addition to this; since $\mathfrak{M}, k \Vdash \neg\phi$,

$$\mathfrak{M}, k-1 \Vdash \circ\neg\phi.$$

All in all consider we have

$$\mathfrak{M}, k-1 \Vdash \phi \text{ and } \mathfrak{M}, k-1 \Vdash \circ\neg\phi \text{ for some } k-1,\ k-1 \geq j.$$

It means $\mathfrak{M}, k-1 \Vdash \phi \wedge \circ\neg\phi$ for some $k-1 \geq j$.

Finally $\mathfrak{M}, j \Vdash \Diamond(\phi \wedge \circ\neg\phi)$

Theorem 3.25 $\mathfrak{T} \Vdash [(\neg\phi)U\phi] \leftrightarrow \Diamond\phi$

Proof: Let $\mathfrak{M}$ be a model over the frame $\mathfrak{T}$ and let j be any state of $\mathfrak{M}$.

Assume that $\mathfrak{M}, j \Vdash (\neg\phi)\ U\ \phi$ but $\mathfrak{M}, j \nVdash \Diamond\phi$.

Since $\mathfrak{M}, j \Vdash (\neg\phi)\ U\ \phi$

$$\mathfrak{M}, k \Vdash \phi \text{ for some } k,\ k \geq j \text{ and } \mathfrak{M}, i \Vdash \neg\phi \text{ for all } i,\ k > i \leq j$$

On the other hand since $\mathfrak{M}, j \nVdash \Diamond\phi$,

$$\mathfrak{M}, j \Vdash \Box\neg\phi.$$

It means $\mathfrak{M}, t \Vdash \neg\phi$ for all $t,\ t \geq j$.

If we choose $\underline{t = k}$ then we have $\mathfrak{M}, k \Vdash \neg\phi$. But we have $\mathfrak{M}, k \Vdash \phi$ so we have contradiction.

Thus $\mathfrak{M}, j \Vdash \Diamond\phi$.

Now let us assume $\mathfrak{M}, j \Vdash \Diamond\phi$ to prove the other side of this theorem.

Since $\mathfrak{M}, j \Vdash \Diamond\phi$,

$$\mathfrak{M}, k \Vdash \phi \text{ for some } k,\ k \geq j.$$

Let k be the first state satisfies the state formulla ϕ such that $k \geq j$.

It means

$$\mathfrak{M}, i \Vdash \neg\phi \text{ for every } i,\ \ j \leq i < k$$

All in all consider we have

$$\mathfrak{M}, j \Vdash \phi \text{ for some } k,\ k \geq j \text{ and } \mathfrak{M}, i \Vdash \neg\phi \text{ for every } i,\ \ j \leq i < k$$

It is clear that this formula is the definition of $\mathfrak{M}, j \Vdash (\neg\phi)\ U\ \phi$

Theorem 3.26 $\mathfrak{T} \Vdash (\Box\phi \wedge \Diamond\psi) \rightarrow (\phi\ U\ \psi)$

Proof: Let $\mathfrak{M}$ be a model over the frame $\mathfrak{T}$ and let j be any state of $\mathfrak{M}$.
Assume that $\mathfrak{M}, j \Vdash \Box\phi \wedge \Diamond\psi$.
Then we have

$$\mathfrak{M}, j \Vdash \Box\phi \text{ and } \mathfrak{M}, j \Vdash \Diamond\psi.$$

Since $\mathfrak{M}, j \Vdash \Diamond\psi$,

$$\mathfrak{M}, k \Vdash \psi \text{ for some } k,\ \ k \geq j.$$

Since $\mathfrak{M}, j \Vdash \Box\phi$,

$$\mathfrak{M}, t \Vdash \phi \text{ for all } t,\ \ t \geq j.$$

From this formulla we can produce $\mathfrak{M}, i \Vdash \phi$ for every i, $k > i \geq j$.

All in all consider we have

$\mathfrak{M}, k \Vdash \psi$ for some k, $k \geq j$ and $\mathfrak{M}, i \Vdash \phi$ for every i, $k > i \geq j$.

Thus $\mathfrak{M}, j \Vdash \phi\ U\ \psi$.

Theorem 3.27 $\mathfrak{T} \Vdash \psi \rightarrow (\phi U \psi)$

Proof: Let $\mathfrak{M}$ be a model over the frame $\mathfrak{T}$ and let j be any state of $\mathfrak{M}$.
Suppose that $\mathfrak{M}, j \Vdash \psi$ but not $\mathfrak{M}, j \Vdash \phi\ U\ \psi$.
Since $\mathfrak{M}, j \nVdash \phi\ U\ \psi$,

not ($\mathfrak{M}, k \Vdash \psi$ for some j, $j \leq k$ and for all i ,$j \leq i < k$, $\mathfrak{M}, i \Vdash \phi$).

$\mathfrak{M}, k \Vdash \psi$ for all j, $j \leq k$ implies for some i ,$j \leq i < k$ and $\mathfrak{M}, i \nVdash \phi$.

Since we have $\mathfrak{M}, j \Vdash \psi$, there must be at least an i ,$j \leq i < j$, $\mathfrak{M}, k \nVdash \phi$.
But there is not so we have a contradiction.
Thus $\mathfrak{M}, j \Vdash \phi\ U\ \psi$.

Theorem 3.28 $\mathfrak{T} \Vdash (\phi U \psi) \leftrightarrow [\phi U(\phi U \psi)]$

Proof: Let $\mathfrak{M}$ be a model over the frame $\mathfrak{T}$ and let j be any state of $\mathfrak{M}$.
Suppose

$$\mathfrak{M}, j \Vdash \phi U \psi.$$

Let us use the substations $\varphi := \phi U \psi$
By the theorem 27 ($\mathfrak{T} \Vdash \psi \rightarrow \phi U \psi$), we have $\mathfrak{M}, j \Vdash \phi U \varphi$ from $\mathfrak{M}, j \Vdash \varphi$.
Finally by means of $\phi U \psi := \varphi$,

$$\mathfrak{M}, j \Vdash \phi U(\phi U \psi)\ .$$

Now suppose

$$\mathfrak{M}, j \Vdash \phi U(\phi U \psi).$$

It means

$$\mathfrak{M}, k \Vdash \phi U \psi \text{ for some } k,\ \ k \leq j \text{ such that } \mathfrak{M}, i \Vdash \phi \text{ for all } i,\ j \leq i < k.$$

Since $\mathfrak{M}, k \Vdash \phi U \psi$ for some $k,\ \ j \leq k$,

$$\mathfrak{M}, t \Vdash \psi \text{ for some } k,\ k \leq t \text{ such that } \mathfrak{M}, n \Vdash \phi \text{ for all n},\ k \leq n < t\ .$$

Since we have

$$\mathfrak{M}, i \Vdash \phi \text{ for all } i,\ \ j \leq i < k \text{ and } \mathfrak{M}, n \Vdash \phi \text{ for all } n,\ \ k \leq n < t\ .$$

It is clear

$$\mathfrak{M}, m \Vdash \phi \text{ for all } m,\ \ j \leq m < t.$$

All in all consider we have

$$\mathfrak{M}, t \Vdash \psi \text{ for some } t,\ j \leq t \text{ and } \mathfrak{M}, m \Vdash \phi \text{ for all } m,\ \ j \leq m < t$$

Thus this is the definition of

$$\mathfrak{M}, j \Vdash \phi\ U\ \psi.$$

Theorem 3.29 $\mathfrak{T} \Vdash [\Box \phi \wedge (\psi U \varphi)] \rightarrow [(\phi \wedge \psi) U(\phi \wedge \varphi)]$

Proof: Let $\mathfrak{M}$ be a model over the frame $\mathfrak{T}$ and let j be any state of $\mathfrak{M}$. Assume that

$$\mathfrak{M}, j \Vdash \Box\phi \text{ and } \mathfrak{M}, j \Vdash \psi U \varphi.$$

It means that

$$\mathfrak{M}, k \Vdash \phi \text{ for all } k,\ j \le k \text{ and (} \mathfrak{M}, t \Vdash \varphi \text{ for some } t,\ j \le t \text{ and } \mathfrak{M}, i \Vdash \psi \text{ for all } i,\ j \le i < t).$$

It can be written,

$$(\mathfrak{M}, k \Vdash \phi \text{ for all } k,\ j \le k \text{ and } \mathfrak{M}, t \Vdash \varphi \text{ for some } t,\ j \le t) \text{ and } (\mathfrak{M}, k \Vdash \phi \text{ for all } k,\ j \le k \text{ and } \mathfrak{M}, i \Vdash \psi \text{ for all } i,\ j \le i < t).$$

Since $\mathfrak{M}, k \Vdash \phi$ for all k, $j \le k$ and $\mathfrak{M}, t \Vdash \varphi$ for some t, $j \le t$,

$$\mathfrak{M}, t \Vdash \phi \wedge \varphi \text{ for some } t,\ j \le t.$$

And since ($\mathfrak{M}, k \Vdash \phi$ for all k, $j \le k$ and $\mathfrak{M}, i \Vdash \psi$ for all i, $j \le i < t$),

$$\mathfrak{M}, i \Vdash \phi \wedge \psi \text{ for all } i,\ j \le i < t.$$

Now we have

$$\mathfrak{M}, t \Vdash \phi \wedge \varphi \text{ for some } t,\ j \le t \text{ and } \mathfrak{M}, i \Vdash \phi \wedge \psi \text{ for all } i,\ j \le i < t.$$

Thus this is the definition of

$$\mathfrak{M}, j \Vdash (\phi \wedge \psi)\, U\, (\phi \wedge \varphi).$$

Theorem 3.30 $\mathfrak{T} \Vdash \circ(\phi U \psi) \leftrightarrow (\circ\phi U \circ \psi)$

Proof: Let $\mathfrak{M}$ be a model over the frame $\mathfrak{T}$ and let j be any state of $\mathfrak{M}$.

$$\begin{array}{lll}
\mathfrak{M}, j \Vdash \circ(\phi\, U\, \psi) & \texttt{iff} & \mathfrak{M}, j+1 \Vdash \phi\, U\, \psi. \\
& \texttt{iff} & \mathfrak{M}, k+1 \Vdash \psi \texttt{ for some } k+1,\ j+1 \le k+1 \\
& & \texttt{and } \mathfrak{M}, i+1 \Vdash \phi \texttt{ for all } i+1, \\
& & j+1 \le i+1 < k+1. \\
& \texttt{iff} & \mathfrak{M}, k \Vdash \circ\psi \texttt{ for some } j \le k \\
& & \texttt{and } \mathfrak{M}, i \Vdash \circ\phi \texttt{ for all } i,\ j \le i < k. \\
& \texttt{iff} & \mathfrak{M}, j \Vdash \circ\phi\, U\, \circ \psi.
\end{array}$$

Theorem 3.31 $\mathfrak{T} \Vdash [(\phi \wedge \psi) U \varphi] \leftrightarrow [(\phi U \varphi) \wedge (\psi U \varphi)]$

Proof: Let $\mathfrak{M}$ be a model over the frame $\mathfrak{T}$ and let j be any state of $\mathfrak{M}$.

$$
\begin{array}{lll}
\mathfrak{M}, j \Vdash (\phi \wedge \psi)\, U\, \varphi & \texttt{iff} & \mathfrak{M}, k \Vdash \varphi \texttt{ for some } k,\ j \leq k \\
& & \text{and } [\mathfrak{M}, i \Vdash \phi \text{ and } \mathfrak{M}, i \Vdash \psi \texttt{ for all } i, \\
& & j \leq i < k]. \\
& \texttt{iff} & \mathfrak{M}, k \Vdash \varphi \texttt{ for some } k,\ j \leq k, \\
& & \text{and } [\mathfrak{M}, i \Vdash \phi \texttt{ for all } i,\ j \leq i < k \\
& & \text{and } \mathfrak{M}, i \Vdash \psi \texttt{ for all } i,\ j \leq i < k]. \\
& \texttt{iff} & [\mathfrak{M}, k \Vdash \varphi \texttt{ for some } k,\ j \leq k \\
& & \text{and } \mathfrak{M}, i \Vdash \phi \texttt{ for all } i,\ j \leq i < k] \\
& & \text{and } [\mathfrak{M}, k \Vdash \varphi \texttt{ for some } k,\ j \leq k \\
& & \text{and } \mathfrak{M}, i \Vdash \psi \texttt{ for all } i,\ j \leq i < k]. \\
& \texttt{iff} & \mathfrak{M}, j \Vdash \phi\, U\, \varphi \text{ and } \mathfrak{M}, j \Vdash \psi\, U\, \varphi. \\
& \texttt{iff} & \mathfrak{M}, j \Vdash (\phi\, U\, \varphi) \wedge (\psi\, U\, \varphi).
\end{array}
$$

Theorem 3.32 $\mathfrak{T} \Vdash [\phi\, U\, (\psi \vee \varphi)] \leftrightarrow [(\phi\, U\, \psi) \vee [(\phi\, U\, \varphi)]]$

Proof: Let $\mathfrak{M}$ be a model over the frame $\mathfrak{T}$ and let j be any state of $\mathfrak{M}$.

$$
\begin{aligned}
\mathfrak{M}, j \Vdash \phi\, U\, (\psi \vee \varphi) \quad \texttt{iff} \quad & \mathfrak{M}, k \Vdash \psi \vee \varphi \texttt{ for some } k,\ j \leq k \\
& \texttt{and } \mathfrak{M}, i \Vdash \phi \texttt{ for every } i,\ j \leq i < k. \\
\texttt{iff} \quad & [\mathfrak{M}, k \Vdash \psi \texttt{ or } \mathfrak{M}, k \Vdash \varphi] \texttt{ for some } k,\ j \leq k \\
& \mathfrak{M}, i \Vdash \phi \texttt{ for every } i,\ j \leq i < k. \\
\texttt{iff} \quad & [\mathfrak{M}, k \Vdash \psi \texttt{ for some } k,\ j \leq k \\
& \texttt{or } \mathfrak{M}, k \Vdash \varphi \texttt{ for some } k,\ j \leq k] \\
& \texttt{and } \mathfrak{M}, i \Vdash \phi \texttt{ for every } i,\ j \leq i < k. \\
\texttt{iff} \quad & [\mathfrak{M}, k \Vdash \psi \texttt{ for some } k,\ j \leq k \\
& \texttt{and } \mathfrak{M}, i \Vdash \phi \texttt{ for every } i,\ j \leq i < k] \\
& \texttt{or } [\mathfrak{M}, k \Vdash \varphi \texttt{ for some } k,\ j \leq k \\
& \texttt{and } \mathfrak{M}, i \Vdash \phi \texttt{ for every } i,\ j \leq i < k]. \\
\texttt{iff} \quad & \mathfrak{M}, j \Vdash \phi\, U\, \psi \texttt{ or } \mathfrak{M}, j \Vdash \phi\, U\, \varphi. \\
\texttt{iff} \quad & \mathfrak{M}, j \Vdash (\phi\, U\, \psi) \vee (\phi\, U\, \varphi).
\end{aligned}
$$

Theorem 3.33 $\mathfrak{T} \Vdash (\Diamond\phi \vee \Diamond\psi) \rightarrow \{[(\neg\phi)U\psi] \vee [(\neg\psi)U\phi]\}$

Proof: Let $\mathfrak{M}$ be a model over the frame $\mathfrak{T}$ and let j be any state of $\mathfrak{M}$.
Suppose

$$\mathfrak{M}, j \Vdash \Diamond\phi \vee \Diamond\psi.$$

Then by the theorem 3.9 [$\mathfrak{T} \Vdash \Diamond(\phi \vee \psi) \leftrightarrow (\Diamond\phi \vee \Diamond\psi)$] we have

$$\mathfrak{M}, j \Vdash \Diamond(\phi \vee \psi).$$

So there are some states bigger equal than j which satisfies $\phi \vee \psi$.
Let $j \leq k$ be the first of them such that;

$$\mathfrak{M}, k \Vdash \phi \vee \psi \text{ for some } k,\ j \leq k.$$

It means $\mathfrak{M}, i \nVdash \phi \vee \psi$ for every $i,\ j \leq i < k$.
Then

$$[\mathfrak{M}, i \nVdash \phi \wedge \mathfrak{M}, i \nVdash \psi] \text{ for every } i,\ j \leq i < k.$$

We have

$$(\mathfrak{M}, k \Vdash \phi \text{ for some } k,\ j \leq k \text{ and } \mathfrak{M}, i \nVdash \psi \text{ for all } i,\ j \leq i < k) \quad \text{or}$$
$$(\mathfrak{M}, k \Vdash \psi \text{ for some } k\ \ j \leq k \text{ and } \mathfrak{M}, i \nVdash \phi \text{ for all } i,\ j \leq i < k).$$

Then $\mathfrak{M}, j \Vdash (\neg\psi)U\phi$ or $\mathfrak{M}, j \Vdash (\neg\phi)U\psi$

Thus $\mathfrak{M}, j \Vdash [(\neg\psi)U\phi] \vee [(\neg\phi)U\psi]$.

Theorem 3.34 $\mathfrak{T} \Vdash [\phi U(\psi \wedge \varphi)] \rightarrow [(\phi U\psi) \wedge (\phi U\varphi)]$

Proof: Let $\mathfrak{M}$ be a model over the frame $\mathfrak{T}$ and let j be any state of $\mathfrak{M}$.
Assume that

$$\mathfrak{M}, j \Vdash \phi U(\psi \wedge \varphi).$$

It means

$$\mathfrak{M}, k \Vdash \psi \wedge \varphi \text{ for some } k,\ j \leq k \text{ and } \mathfrak{M}, i \Vdash \phi \text{ for all } i, j \leq i < k.$$

Then

$$(\mathfrak{M}, k \Vdash \psi \text{ for some } k,\ j \leq k \text{ and } \mathfrak{M}, k \Vdash \varphi \text{ for some } k,\ j \leq k)$$
$$\text{and } \mathfrak{M}, i \Vdash \phi \text{ for all } i, j \leq i < k.$$

Then

$$[\mathfrak{M}, k \Vdash \psi \text{ for some } k,\ j \leq k \text{ and } \mathfrak{M}, i \Vdash \phi \text{ for all } i, j \leq i < k]$$
$$\text{and } [\mathfrak{M}, k \Vdash \varphi \text{ for some } k, j \leq k \text{ and } \mathfrak{M}, i \Vdash \phi \text{ for all } i,\ j \leq i < k].$$

Now we can write

$$(\mathfrak{M}, j \Vdash \phi U\psi) \wedge (\mathfrak{M}, j \Vdash \phi U\varphi).$$

Thus

$$\mathfrak{M}, j \Vdash (\phi U\psi) \wedge (\phi U\varphi).$$

Theorem 3.35 $\mathfrak{T} \Vdash [(\phi \wedge \psi)U\varphi] \leftrightarrow [(\phi U\varphi) \wedge (\psi U\varphi)]$

Proof: Let $\mathfrak{M}$ be a model over the frame $\mathfrak{T}$ and let j be any state of $\mathfrak{M}$.

$$
\begin{array}{lll}
\mathfrak{M}, j \Vdash (\phi \wedge \psi)\ U\ \varphi & \texttt{iff} & \mathfrak{M}, k \Vdash \varphi \texttt{ for some } k,\ j \leq k \texttt{ and} \\
& & \mathfrak{M}, i \Vdash \phi \wedge \psi \texttt{ for all } i\ , j \leq i < k. \\
& \texttt{iff} & \mathfrak{M}, k \Vdash \varphi \texttt{ for some } k,\ j \leq k \texttt{ and} \\
& & [\mathfrak{M}, i \Vdash \phi \texttt{ for all } i,\ j \leq i < k \texttt{ and} \\
& & \mathfrak{M}, i \Vdash \psi \texttt{ for all } i,\ j \leq i < k]. \\
& \texttt{iff} & [\mathfrak{M}, k \Vdash \varphi \texttt{ for some } k,\ j \leq k \texttt{ and} \\
& & \mathfrak{M}, i \Vdash \phi \texttt{ for all } i,\ j \leq i < k] \texttt{ and} \\
& & [\mathfrak{M}, k \Vdash \varphi \texttt{ for some } k,\ j \leq k \texttt{ and} \\
& & \mathfrak{M}, i \Vdash \psi \texttt{ for all } i, j \leq i < k]. \\
& \texttt{iff} & \mathfrak{M}, j \Vdash \phi\ U\ \varphi \texttt{ and } \mathfrak{M}, j \Vdash \psi\ U\ \varphi. \\
& \texttt{iff} & \mathfrak{M}, j \Vdash \phi\ U\ \varphi \wedge \psi\ U\ \varphi.
\end{array}
$$

Theorem 3.36 $\mathfrak{T} \Vdash [(\phi U \varphi) \vee (\psi U \varphi)] \rightarrow [(\phi \vee \psi) U \varphi]$

Proof: Let $\mathfrak{M}$ be a model over the frame $\mathfrak{T}$ and let j be any state of $\mathfrak{M}$.

$$
\begin{array}{lll}
\texttt{If}\mathfrak{M}, j \Vdash (\phi\ U\ \varphi) \vee (\psi\ U\ \varphi) & \texttt{then} & \mathfrak{M}, j \Vdash \phi\ U\ \varphi \texttt{ or } \mathfrak{M}, j \Vdash \psi\ U\ \varphi. \\
& \texttt{Then} & [\mathfrak{M}, k \Vdash \varphi \texttt{ for some } k,\ j \leq k \texttt{ and} \\
& & \mathfrak{M}, i \Vdash \phi \texttt{ for all } i,\ j \leq i < k] \texttt{ or} \\
& & [\mathfrak{M}, k \Vdash \varphi \texttt{ for some } k,\ j \leq k \texttt{ and} \\
& & \mathfrak{M}, i \Vdash \psi \texttt{ for all } i,\ j \leq i < k]. \\
& \texttt{Then} & \mathfrak{M}, k \Vdash \varphi \texttt{ for some } k,\ j \leq k \texttt{ and} \\
& & [\mathfrak{M}, i \Vdash \phi \texttt{ for all } i,\ j \leq i < k \texttt{ or} \\
& & \mathfrak{M}, i \Vdash \psi \texttt{ for all } i,\ j \leq i < k]. \\
& \underline{\texttt{Then}} & \mathfrak{M}, k \Vdash \varphi \texttt{ for some } k,\ j \leq k \texttt{ and} \\
& & \mathfrak{M}, i \Vdash \phi \vee \psi \texttt{ for all } i,\ j \leq i < k. \\
& \texttt{Then} & \mathfrak{M}, j \Vdash (\phi \vee \psi) U \varphi.
\end{array}
$$

Theorem 3.37 $\mathfrak{T} \Vdash [(\phi \to \psi)U\varphi] \to [(\phi U\varphi) \to (\psi U\varphi)]$

Proof: Firstly by the rule of first order logic EXP $[\phi \to (\psi \to \varphi) := (\phi \wedge \psi) \to \varphi]$ (Mendelson 1997). It is enough to show $\{[(\phi \to \psi)U\varphi] \wedge (\phi U\varphi)\} \to (\psi U\varphi)$ is valid formula in the frame $\mathfrak{T}$

Let $\mathfrak{M}$ be an any model over the frame $\mathfrak{T}$ and let j be any state of $\mathfrak{M}$.

Let us suppose

$$\mathfrak{M}, j \Vdash [(\phi \to \psi)U\varphi] \wedge (\phi U\varphi).$$

By the theorem 35 ($\mathfrak{T} \Vdash [\phi U(\psi \wedge \varphi)] \leftrightarrow [(\phi U\psi) \wedge (\phi U\varphi)]$),

we have

$$\mathfrak{M}, j \Vdash [(\phi \to \psi) \wedge \phi]U\varphi.$$

Finally by means of rule of first order logic MP $[(\phi \to \psi),\ \phi \vdash \psi]$ (Mendelson 1997) , we have

$$\mathfrak{M}, j \Vdash \psi\ U\ \varphi.$$

Theorem 3.38 $\mathfrak{T} \Vdash \{(\phi U\psi) \wedge [(\neg\psi)U\varphi]\} \to (\phi U\varphi)$

Proof: Let $\mathfrak{M}$ be a model over the frame $\mathfrak{T}$ and let j be any state of $\mathfrak{M}$.

Assume that

$$\mathfrak{M}, j \Vdash (\phi U\psi) \wedge [(\neg\psi)U\varphi].$$

Since $\mathfrak{M}, j \Vdash \phi U\psi$,

$$\mathfrak{M}, k \Vdash \psi \text{ for some } k,\ j \leq k \text{ and } \mathfrak{M}, i \Vdash \phi \text{ for all } i,\ j \leq i < k.$$

Since $\mathfrak{M}, j \Vdash (\neg\psi)U\varphi$,

$$\mathfrak{M}, t \Vdash \varphi \text{ for some } t,\ j \leq t \text{ and } \mathfrak{M}, s \Vdash \neg\psi \text{ for all } s,\ j \leq s < t.$$

But k must be bigger or equal than than t.

It can be written

$$\mathfrak{M}, s \Vdash \phi \text{ for every state } s,\ j \leq s < t,$$

and we have

$$\mathfrak{M}, t \Vdash \varphi \text{ for some } t,\ j \leq t.$$

Thus $\mathfrak{M}, j \Vdash \phi U \varphi$.

Theorem 3.39 $\mathfrak{T} \Vdash [\phi U(\psi \wedge \varphi)] \rightarrow [(\phi U \psi) U \varphi]$

Proof: Let $\mathfrak{M}$ be a model over the frame $\mathfrak{T}$ and let j be any state of $\mathfrak{M}$. Assume that

$$\mathfrak{M}, j \Vdash \phi U(\psi \wedge \varphi).$$

Since $\mathfrak{M}, j \Vdash \phi U(\psi \wedge \varphi)$,

$$\mathfrak{M}, k \Vdash \psi \wedge \varphi \text{ for some } k,\ j \leq k \text{ and } \mathfrak{M}, i \Vdash \phi \text{ for all } i,\ j \leq i < k.$$

Then we can say

$$[\mathfrak{M}, k \Vdash \psi \text{ for some } k,\ j \leq k \text{ and } \mathfrak{M}, i \Vdash \phi \text{ for all } i,\ j \leq i < k]$$
$$\text{and } \mathfrak{M}, k \Vdash \varphi \text{ for some } k,\ j \leq k.$$

Then we have

$$[\mathfrak{M}, i \Vdash \phi U \psi \text{ for all } i,\ j \leq i < k] \text{ and } \mathfrak{M}, k \Vdash \varphi \text{ for some } k,\ j \leq k.$$

Thus $\mathfrak{M}, j \Vdash (\phi U \psi) U \varphi$.

Theorem 3.40 $\mathfrak{T} \Vdash (\phi U \psi) \rightarrow (\phi \vee \psi)$

Proof: Let $\mathfrak{M}$ be a model over the frame $\mathfrak{T}$ and let j be any state of $\mathfrak{M}$. Assume that

$$\mathfrak{M}, j \Vdash \phi U \psi \text{ but } \mathfrak{M}, j \nVdash \phi \vee \psi.$$

Since $\mathfrak{M}, j \nVdash \phi \vee \psi$,

$$\mathfrak{M}, j \Vdash \neg\phi \text{ and } \mathfrak{M}, j \Vdash \neg\psi.$$

Since $\mathfrak{M}, j \Vdash \phi U \psi$,

$$\mathfrak{M}, k \Vdash \psi \text{ for some } k,\ j \leq k \text{ and } \mathfrak{M}, i \Vdash \phi \text{ for all } i,\ j \leq i < k.$$

In this situation, we have to consider two different cases;

Case 1: $\underline{k = j}$. We have $\mathfrak{M}, j \Vdash \psi$ but we have $\mathfrak{M}, j \Vdash \neg\psi$ so it is contradiction.

Case 2: $\underline{k \geq j}$. We have $\mathfrak{M}, j \Vdash \phi$ but we have $\mathfrak{M}, j \Vdash \neg\phi$ so again contradiction.

Thus $\mathfrak{M}, j \Vdash (\phi U \psi) \to (\phi \vee \psi)$.

Theorem 3.41 $\mathfrak{T} \Vdash [(\phi U \psi) U \varphi] \to [(\phi \vee \psi) U \varphi]$

Proof: Let $\mathfrak{M}$ be a model over the frame $\mathfrak{T}$ and let j be any state of $\mathfrak{M}$. Suppose

$$\mathfrak{M}, j \Vdash (\phi U \psi) U \varphi.$$

Since hypothesis,

$$\mathfrak{M}, k \Vdash \varphi \text{ for some } k,\ j \leq k \text{ and } \mathfrak{M}, i \Vdash \phi U \psi \text{ for all } i,\ j \leq i < k$$

By the theorem 3.40 ($\mathfrak{T} \Vdash (\phi U \psi) \to (\phi \vee \psi)$) we can say,

$$\mathfrak{M}, k \Vdash \varphi \text{ for some } k,\ j \leq k \text{ and } \mathfrak{M}, i \Vdash \phi \vee \psi \text{ for all } i,\ j \leq i < k.$$

Thus $\mathfrak{M}, j \Vdash (\phi \vee \psi) U \varphi$.

Theorem 3.42 $\mathfrak{T} \Vdash [\phi U (\psi U \varphi)] \to [(\phi \vee \psi) U \varphi]$

Proof: Let $\mathfrak{M}$ be a model over the frame $\mathfrak{T}$ and let j be any state of $\mathfrak{M}$. Suppose

$$\mathfrak{M}, j \Vdash \phi U (\psi U \varphi).$$

Since the hypothesis,

$$\mathfrak{M}, k \Vdash \psi U \varphi \text{ for some } k,\ j \leq k \text{ and } \mathfrak{M}, i \Vdash \phi \text{ for all } i,\ j \leq i < k$$

Since $\mathfrak{M}, k \Vdash \psi U \varphi$ for some $k,\ j \leq k$,

$$\mathfrak{M}, t \Vdash \varphi \text{ for some } t,\ k \leq t \text{ and } \mathfrak{M}, s \Vdash \psi \text{ for all } s,\ k \leq s < t$$

Since we have $\mathfrak{M}, i \Vdash \phi$ for all $i,\ j \leq i < k$ and $\mathfrak{M}, s \Vdash \psi$ for all $s, k \leq s < t$,

$$\mathfrak{M}, l \Vdash \phi \vee \psi \text{ for all } l,\ j \leq l < t.$$

Then we have $\mathfrak{M}, l \Vdash \phi \vee \psi$ for all $l,\ j \leq l < t$ and $\mathfrak{M}, t \Vdash \varphi$ for some $t,\ j \leq t$. Thus

$$\mathfrak{M}, j \Vdash (\phi \vee \psi) U \varphi.$$

Theorem 3.43 $\mathfrak{T} \Vdash [(\phi U \psi) U \psi] \leftrightarrow (\phi U \psi)$

Proof: Let $\mathfrak{M}$ be a model over the frame $\mathfrak{T}$ and let j be any state of $\mathfrak{M}$. Suppose

$$\mathfrak{M}, j \Vdash (\phi U \psi) U \psi.$$

Since the hypothesis;

$$\mathfrak{M}, k \Vdash \psi \text{ for some } k,\ j \leq k \text{ and } \mathfrak{M}, i \Vdash \phi U \psi \text{ for all } i,\ j \leq i < k.$$

In this situation, we have to consider two different cases;

Case 1: $\underline{k = j}$. we have $\mathfrak{M}, j \Vdash \psi$. By the theorem 3.27 it can be said $\mathfrak{M}, j \Vdash \phi U \psi$

Case 2: $\underline{j < k}$. We can choose $i = j$ so it is clear that $\mathfrak{M}, j \Vdash \phi U \psi$.

Thus $\mathfrak{M}, j \Vdash (\phi U \psi) U \psi \rightarrow \phi U \psi$.
Now let us suppose

$$\mathfrak{M}, j \Vdash \phi U \psi.$$

Since the hypothesis;

$$\mathfrak{M}, k \Vdash \psi \text{ for some } k,\ j \leq k \text{and } \mathfrak{M}, i \Vdash \phi \text{ for all } i,\ j \leq i < k.$$

Since we have

$$[\mathfrak{M}, s \Vdash \phi \text{ for all } s,\ i \leq s < k \text{ and } \mathfrak{M}, k \Vdash \psi \text{ for some } k,\ j \leq k] \text{ for all } i,\ j \leq i < k.$$

Then we can say $\mathfrak{M}, i \Vdash \phi U \psi$ for all $i,\ j \leq i < k$.
All in all consider ,since we have

$$\mathfrak{M}, k \Vdash \psi \text{ for some } j \leq k \text{ and } \mathfrak{M}, i \Vdash \phi\ U\ \psi \text{ for all } i, j \leq i < k.$$

Thus $\mathfrak{M}, j \Vdash (\phi U \psi) U \psi$.

If you look at carefully nearly all of given valid formulas in $\mathfrak{T}$ include only future operators ($\circ,\ \diamond,\ \Box,\ U$). But mirror images of all these formulas can be derived.

Let us consider theorem 3.26. It was showed the formula $(\Box\phi \wedge \diamond\psi) \rightarrow (\phi\ U\ \psi)$ is valid in the frame $\mathfrak{T}$. Mirror image of this formula is $(\boxminus\phi \wedge \diamondminus\psi) \rightarrow (\phi\ S\ \psi)$. Let us show this formula is valid in $\mathfrak{T}$ by means of the time line.

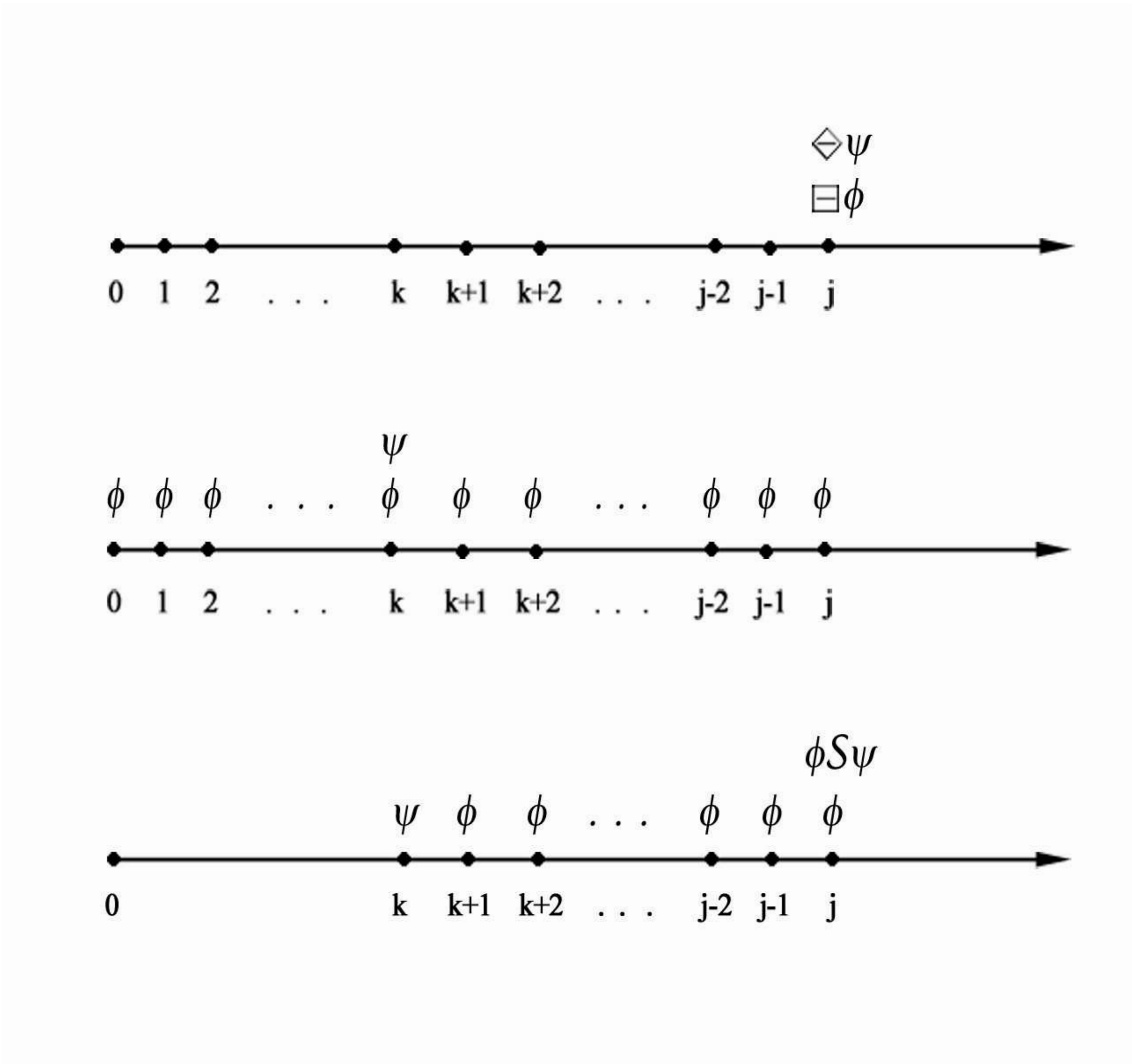

Figure 3.1: $\mathfrak{T} \Vdash (\boxminus\phi \wedge \diamondminus\psi) \rightarrow (\phi\ S\ \psi)$

But it can not be said that the mirror image of every valid formula in the frame $\mathfrak{T}$ is again valid. We can give an example to show it. Of course, it is showed that the McKinsey formula $(\Box\Diamond\phi \rightarrow \Diamond\Box\phi)$ is not valid in $\mathfrak{T}$ but the formula $(\Diamond\Box\phi \rightarrow \Box\Diamond\phi)$ is valid so the formula $(\Diamond\Box\phi \leftrightarrow \Box\Diamond\phi)$ is not valid in the frame $\mathfrak{T}$. But the mirror image of this formula $(\diamondminus\boxminus\,\phi \leftrightarrow \boxminus\diamondminus\phi)$ is valid in this frame.

CHAPTER 4

CONCLUSION

The goal of this study is to analyze the most important frame $\mathfrak{T}$ in temporal logic. For this purpose the temporal languages has been studied with considering the modal languages in the first chapter then the modeling in temporal logic has been constructed on the frame $\mathfrak{T}$ step by step.

Although the temporal logic has many applications in computer programming and dynamic systems, we have not mentioned about this. Furthermore, the most of authors use some different operators belonging to temporal language, because they need new interpretations in their systems. In contrary, we have studied different approach. We have only used the operators which are allowed by modal languages except next ($\circ$) and its mirror image ($\ominus$) .

In this study we have improved our new structure upper-bottom and down-top points concept for any temporal formula. By means of these definitions we have reached some propositions. Then we have used these propositions in the section of theorem for simplification proofs of some theorems. Especially they help us to show the formulas ,which are including the operators Box ($\Box$) and Diamond ($\Diamond$), are valid. Furthermore the valuation set of an any formula can be generally described by using these definitions. We also give the proofs of some valid temporal formulas with using only simple derivation and the time line interpretation. Furthermore we have shown how we can find mirror image of any temporal formula.

In conclusion I want to emphasize that all of this study is true only in the frame $\mathfrak{T} = (\mathbb{N}, \leqslant, \geqslant, R_\circ, R_\ominus, R_U, R_S)$. It means if somebody want to change this frame then the most of valid formulas or propositions might be falsified.

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