THE SOLUTION OF SOME DIFFERENTIAL EQUATIONS BY NONSTANDARD FINITE DIFFERENCE METHOD

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ABSTRACT

In this thesis, the nonstandard finite difference method is applied to construct the new finite difference equations for the first order nonlinear dynamic equation, second order singularly perturbed convection diffusion equation and nonlinear reaction diffusion partial differential equation. The new discrete representation for the first order nonlinear dynamic equation allows us to obtain stable solutions for all step-sizes. For singularly perturbed convection diffusion equation, the error analysis reveals that the nonstandard finite difference representation gives the better results for any values of the perturbation parameters. Finally, the new discretization for the last equation is obtained. The lemma related to the positivity and boundedness conditions required for the nonstandard finite difference scheme is stated. Numerical simulations for all differential equations are illustrated based on the parameters we considered.

ÖZET

Bu tezde, birinci mertebeden lineer olmayan dinamik, ikinci mertebeden tekil pertürbe konveksiyon difüzyon ve lineer olmayan kismi diferansiyel reaksiyon-difüzyon denklemlerine standart olmayan sonlu fark metodu uygulanarak yeni sonlu fark denklemleri oluşturuldu. Birinci mertebeden lineer olmayan dinamik denklem için yazilan yeni gösterim, her adimda kararli çözüm elde edilmesini sağladi. Tekil pertürbe konveksiyon-difüzyon denkleminin standart olmayan sonlu fark metodu ile çözümü, pertürbasyon parametresinin aldiği her değere karşilik sonlu fark metoduna göre daha iyi sonuç verdiği hata analizi ile gösterildi. Son olarak, reaksiyon-difüzyon denklemi için yeni bir gösterim elde edildi. Standart olmayan sonlu fark metodu için gereken pozitiflik ve sinirlilik koşullari lemma ile belirlendi. Tüm diferansiyel denklemlerin nümerik simülasyonlari parametrelere bağli olarak örneklendirildi.

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CHAPTER 1

INTRODUCTION

In this thesis, we study the nonstandard finite difference method for constructing discrete models of ordinary differential equation, second order differential equation for singularly perturbed problem and nonlinear partial differential equation.

In general, a given linear or nonlinear differential equation does not have a complete solution that can be expressed in terms of a finite number of elementary functions (Ross 1964, Humi and Miller 1988, Zwillinger 1989, Zwillinger 1992). A first attack on this situation is to seek approximate analytic solutions by means of various perturbation methods (Bender and Orszag 1978, Mickens 1981, Kevorkian and Cole 1981). However, such procedures only hold for limited ranges of the system parameters and/or the independent variables. For arbitrary values of the system parameters, only numerical integration techniques can provide accurate numerical solutions to the original differential equations. A major difficulty with numerical techniques is that a separate calculation must be formulated for each particular set of initial and/or boundary values. Consequently, obtaining a global picture of the general solution to the differential equations often requires a great deal of computation and time. However, for many problems being investigated in science and technology, there exist no alternatives to numerical methods. The process of numerical method is the replacement of a set of differential equations, both of whose independent and dependent variables are continuous, by a model for which these variables may be discrete. In general, in the model the independent variables have a one-to-one corresponds with the integers while the dependent variables can take real values.

One of the traditional technique to find an approximate solution for the given problem is the finite difference method. The short history of the finite difference method starts with the 1930s. Even though some ideas may be traced back further, we begin the fundamental theoretical paper by Courant, Friedrichs and Lewy (1928) on the solutions of the problems of mathematical physics by of finite differences. A finite difference approximation was first defined for the wave equation, and the CFL stability condition was shown to be necessary for convergence. Error bounds for difference approximations of elliptic problems were first derived by Gershgorin (1930) whose work was based on a discrete analogue of the maximum principle for Laplace's equation. This approach was pursued through the 1960s by, e.g., Collatz, Matzkin, Wasow, Bramble, and Hubbard, and various approximations of elliptic equations and associated boundary conditions were analyzed (Thomee 1999). The finite difference theory for general initial value problems and parabolic problems then had an intense period of development during 1950s and 1960s, when the concept of stability was explored in the Lax equivalence theorem and the Kreiss matrix lemmas. For hyperbolic equations and nonlinear conservation laws, the finite difference method has continued to play a dominating role up to the present time.

Now let us introduce the construction of discrete standard finite difference models that we will employ. We set $t_k = hk$ for k=0,...,n+1. $t_{k+1} = t_k + h$ and $t_{k-1} = t_k - h$, $h = \frac{t_{n+1}-t_0}{n}$.

$$\frac{dy}{dt} = \frac{y(t+h) - y(t)}{h} \approx \frac{y_{k+1} - y_k}{h}. \tag{1.1}$$

$$\frac{dy}{dt} = \frac{y(t) - y(t-h)}{h} \approx \frac{y_k - y_{k-1}}{h}. \tag{1.2}$$

$$\frac{dy}{dt} = \frac{y(t+h) - y(t-h)}{2h} \approx \frac{y_{k+1} - y_{k-1}}{2h}. \tag{1.3}$$

These representations of the first derivative are known, respectively, as the forward Euler, backward Euler, and central difference schemes. Composing the forward and backward Euler difference schemes, we get the following central approximations for the second derivative:

$$\frac{d^2y}{dt^2} = \frac{y(t+h) - 2y(t) + y(t-h)}{h^2} \approx \frac{y_{k+1} - 2y_k + y_{k-1}}{h^2}. \tag{1.4}$$

Standard finite difference rules does not lead to a unique discrete model. One of the questions is which of the standard finite difference schemes should be used to obtain numerical solutions for a differential equation? Another problem is the relationship between the solutions to a given discrete model and that of corresponding differential equation. This connection may be tenuous. This and related matters lead to the study of numerical instabilities (Mickens 1994).

Once a discrete model is selected, the calculation of a numerical solution requires the choice of a time and/or space step-size. How should this be done? For problems in the sciences and engineering, the value of the step-sizes must be determined such that the physical phenomena of interest can be resolved on the scale of the computational grid or lattice. However, suppose one is interested in the long-time or asymptotic-in-space behavior of the solution; can the step-sizes be taken as large as one wishes? Numerical instabilities may exist.

In this work, we study to eliminate the elementary numerical instabilities that can arise in the finite-difference models of differential equations. Our purpose will be the construction of discrete models whose solutions have the same qualitative properties as that of the corresponding differential equation for all step-sizes. We have not completely succeeded in this effort, but, progress has definitely been made.

The method that we employ to the differential equations is nonstandard finite difference model began with the 1989 publication of Mickens (Mickens 1989). Extensions and a summary of the known results up to 1994 are given in Mickens (Mickens 1994). This class of schemes and their formulation center on two issues: first, how should discrete representations for derivatives be determined, and second, what are the proper forms to be used for nonlinear terms.

Nonstandard finite difference scheme has been constructed by Ronald E. Mickens for some class of differential equations. One of them is the first order ordinary differential equations given as follows:

$$\frac{dy}{dt} = f(y) \tag{1.5}$$

which is called as decay equation when f(y)=$-\lambda y$, $\lambda > 0$ (Mickens 1994). Exponential decay occurs in a wide variety of situations. Most of these fall into the domain of the natural sciences. Any application of mathematics to the social sciences or humanities is risky and uncertain, because of the extraordinary complexity of human behavior. However, a few broadly exponential phenomena have been identified there as well. When $f(y) = y(1-y)$ then equation (1.5) becomes the logistic differential equation with two-fixed points (Mickens 1994):

$$\frac{dy}{dt} = y(1-y). \tag{1.6}$$

When $f(y) = y(1-y^2)$ then equation (1.5) is called as the cubic differential equation with three-fixed points (Mickens 1999a):

$$\frac{dy}{dt} = y(1-y^2). \tag{1.7}$$

When $f(y) = \sin(\pi y)$ then equation (1.5) is considered as a sine equation (Mickens 1999a):

$$\frac{dy}{dt} = \sin(\pi y). \tag{1.8}$$

Next; the following second order ordinary differential equations have been also solved by Mickens:

$$\frac{d^2y}{dt^2} + y + f(y^2)\frac{dy}{dt} + g(y^2)y = 0. \tag{1.9}$$

A large class of one-dimensional, nonlinear oscillators can be modeled by this differential equation(Mickens 1994). When $f(y^2) = 0$ and $g(y^2) = 0$, then equation (1.9) is the harmonic oscillator equation given as follows (Mickens 1994):

$$\frac{d^2y}{dt^2} + y = 0. \tag{1.10}$$

The form for $g(y^2)y$=0 is consistent with the analysis of the van der Pol equation. The van der Pol equation (1.9) corresponds to a non-linear oscillatory system that has both input and output sources of energy. This equation is given by the expression (Mickens 1997b):

$$\frac{d^2y}{dt^2} + y = \epsilon(1-y^2)\frac{dy}{dt}, \quad \epsilon > 0. \tag{1.11}$$

When $f(y^2) = 0$, then equation (1.9) is the equation of motion for a conservative oscillator. The periodic solutions of conservative oscillators have the property that the amplitude of the oscillations are constants. This property is used as the characteristic defining a conservative oscillator. Without loss of generality, they only considered the Duffing equation (Mickens et al. 1989, Mickens 1988):

$$\frac{d^2y}{dt^2} + y + \epsilon y^3 = 0, \quad \epsilon > 0. \tag{1.12}$$

Finally, some partial differential equations are also studied by Mickens. For example; nonlinear diffusion describes important phenomena in many systems in the physical, biological, and engineering sciences. In addition to giving solutions that behave like the usual diffusion processes, i.e., waves with an effective infinite speed of propagation, nonlinear diffusion can lead to solutions which exhibit shock-like, decreasing maximum amplitude, waves. An example of such an equation is the Boltzmann equation (Mickens 2000):

$$u_t = (uu_x)_x, \quad \text{where} \ \ u = u(x,t). \tag{1.13}$$

Burgers-Fisher partial differential equation (Mickens 1999b) is

$$u_t + auu_x = Du_{xx} + \lambda u(1-u) \tag{1.14}$$

where(a,D,λ) are non-negative parameters. This equation, with λ=0, has been used to investigate sound waves in a viscous medium. However, it was originally introduced by Burgers (Burgers 1948) to model one-dimensional turbulence and can also be applied to waves in fluid-filled viscous elastic tubes and magnetohydrodynamic waves in a medium with finite electrical (Debnath 1997) conductivity. With all three parameters positive, equation (1.14) corresponds to Burgers equation having non-linear reaction. An alternative view of equation (1.14) is to consider it as a modified Fisher equation (Murray 1989) with a=0:

$$u_t = Du_{xx} + \lambda u(1-u). \tag{1.15}$$

D=0 is the diffusionless Burgers equation (Mickens 1997c) with nonlinear reaction:

$$u_t + auu_x = \lambda u(1-u). \tag{1.16}$$

a=0 and λ=0 are the linear diffusion equation (Mickens 1997c):

$$u_t = Du_{xx}. \tag{1.17}$$

D=0 and λ=0 are the diffusionless Burgers equation (Mickens 1997c):

$$u_t + auu_x = 0. \tag{1.18}$$

Many interesting systems in acoustics and fluid dynamics may be mathematically modeled by partial differential equations where linear advection and/or non-linear reaction are the dominant effects. For two space dimensions, the PDE's take the form (Mickens 1997a):

$$u_t + au_x + bu_y = 0 \tag{1.19}$$

$$u_t + au_x + bu_y = u(1-u) \tag{1.20}$$

where a and b are positive constants.

The outline of this thesis is given as below:

In Chapter 2, we explain when the numerical instabilities occur in the computation. We introduce logistic differential equation and construct several discrete models. Then we compare the properties of the solutions to the difference equations to the corresponding properties of the original differential equation.

In Chapter 3, we define the exact finite difference scheme. Then, we give information to understand the general rules for the construction of nonstandard finite difference scheme for differential equations.

In Chapter 4, we construct a new finite difference scheme for nonlinear dynamic ordinary differential equation. Then, standard and nonstandard finite difference schemes are introduced and analyzed for the first order ordinary differential equation.

In Chapter 5, we introduce convection-diffusion problem. Standard and nonstandard finite difference schemes are described and analyzed for the given problem.

In Chapter 6, we consider nonlinear reaction-diffusion partial differential equation. A new nonstandard finite difference scheme is constructed and analyzed for the given problem.

CHAPTER 2

NUMERICAL INSTABILITIES

In this chapter, we construct several discrete models and compare the pro-perties of the solutions to the difference equations to the corresponding properties of the original differential equation in order to explain when the numerical instabilities occur in the computation. For this purpose, we consider the logistic differential equation. Any discrepancies found are indications of numerical instabilities.

2.1 Numerical Instabilities

A discrete model of a differential equation is said to have numerical instabilities if there exist solutions to the finite difference equation that do not correspond to any of the possible solutions of the differential equation. It is uncertain if an exact definition can ever be stated for the general concept of numerical instabilities. This is because it is always possible, in principle, for new forms of numerical instabilities to arise when new nonlinear differential equations are discretely modeled. Numerical instabilities are an indication that the discrete equations are not able to model the correct mathematical properties of the solutions to the differential equations of interest.

The most important reason for the existence of numerical instabilities is that the discrete models of differential equations have a larger parameter space than the corresponding differential equations. This can be easily demonstrated by the following argument. Assume that a given dynamic system is described in terms of the differential equation

$$\frac{dy}{dt} = f(y, \lambda) \tag{2.1}$$

where λ denotes n-dimensional parameter vector that defines the system. A discrete model for equation (2.1) takes the form

$$y_{k+1} = F(y_k, \lambda, h) \tag{2.2}$$

where $h = \Delta t$ is the time step-size. Note that the function F contains (n+1) parameters; this is because h can now be regarded as an additional parameter. The solutions to equation (2.1) and equation (2.2) can be written, respectively as $y(t,\lambda)$ and $y_k(\lambda,h)$. Even if $y(t,\lambda)$ and $y_k(\lambda,h)$ are close to each other for a particular value of h, say $h=h_1$. If h is changed to a new value, say $h=h_2$, the possibility exists that $y_k(\lambda,h_2)$ differs greatly from $y_k(\lambda,h_1)$ both qualitatively and quantitatively.

2.2 Logistic Differential Equation

We will consider the following logistic differential equation

$$\frac{dy}{dt} = y(1-y) \tag{2.3}$$

which we can solve exactly

$$y(t) = \frac{y_0}{y_0 + (1-y_0)e^{-t}} \tag{2.4}$$

where the initial condition is

$$y_0 = y(0). \tag{2.5}$$

Figure 2.1 and Figure 2.2 illustrate the general nature of the various solution behaviors for $y_0 > 0$ and $y_0 < 0$, respectively. If $y_0 > 0$, then all solutions monotonically approach the stable fixed-point at y(t)=1. If $y_0 < 0$, then the solution at first decreases to $-\infty$ at the singular point

$$t = t^* = \ln[\frac{1+\mid y_0 \mid}{\mid y_0 \mid}] \tag{2.6}$$

after which, for $t > t^*$, it decreases monotonically to the fixed-point at y(t)=1. Note that y(t)=0 is an unstable fixed-point.

Our first discrete model is constructed by using a central difference scheme for the derivative:

$$\frac{y_{k+1} - y_{k-1}}{2h} = y_k(1-y_k). \tag{2.7}$$

Since equation (2.7) is a second-order difference equation, while equation (2.3) is a first order differential equation, the value of $y_1 = y(h)$ must be determined by some procedure. We do this by use of the Euler result

$$y_1 = y_0 + hy_0(1-y_0). \tag{2.8}$$

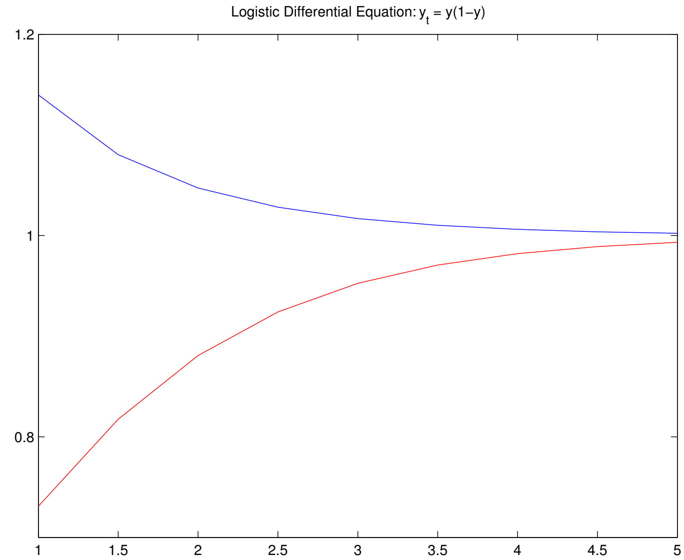

Figure 2.1. Exact solution of the problem (2.3) for $y_0 > 0$.

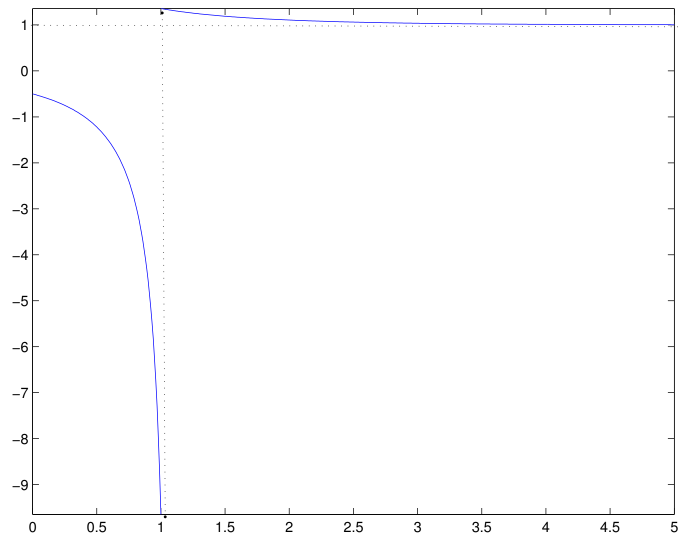

Figure 2.2. Exact solution of the problem (2.3) for $y_0 < 0$.

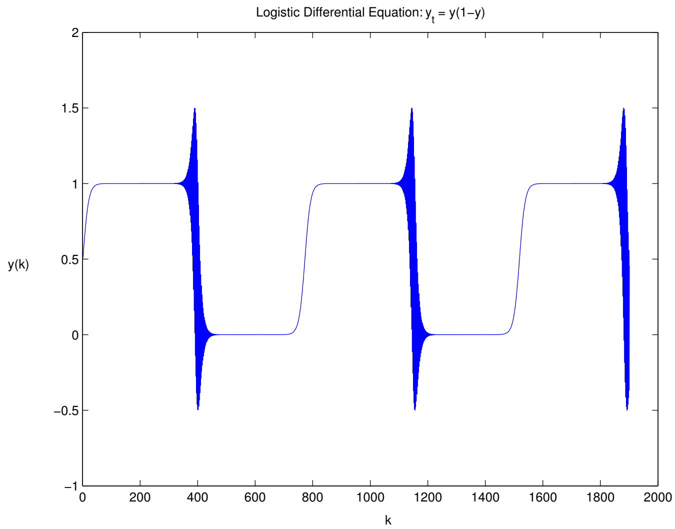

Figure 2.3. The central difference scheme given in equation (2.7) with $y_0 = 0.5$, $h = 0.1$.

A typical plot of the numerical solution to equation (2.7) is shown in Figure 2.3. This type of plot is obtained for any value of the step-sizes.

This result can be revealed by the linear stability analysis of the two fixed-points of the equation (2.7). First of all, note that equation (2.7) has two constant solutions or fixed-points. They are

$$y_k = \bar{y}^{(0)} = 0, \quad \text{and} \quad y_k = \bar{y}^{(1)} = 1. \tag{2.9}$$

To investigate the stability of $y_k = \bar{y}^{(0)}$, we set

$$y_k = \bar{y}^{(0)} + \epsilon_k, \quad \mid \epsilon \mid \leq 1, \tag{2.10}$$

substitute this result into equation (2.7) and neglect all but the linear terms. Doing this gives

$$\frac{\varepsilon_{k+1} - \varepsilon_{k-1}}{2h} = \epsilon_k. \tag{2.11}$$

The solution to this second-order difference equation is

$$\epsilon_k = A(r_+)^k + B(r_-)^k \tag{2.12}$$

where A and B are arbitrary, but, small constants; and

$$r_\pm(h) = h \pm \sqrt{1 + h^2}. \tag{2.13}$$

From equation (2.13), it can be concluded that the first term on the right-side of equation (2.12) is exponentially increasing, while the second term oscillates with an exponentially decreasing amplitude. A small perturbation to the fixed-point at $\bar{y}^{(1)} = 1$ can be represented as

$$y_k = \bar{y}^{(1)} + \eta_k, \quad \mid \eta_k \mid \leq 1. \tag{2.14}$$

The linear perturbation equation for η_k is

$$\frac{\eta_{k+1} - \eta_{k-1}}{2h} = -\eta_k, \tag{2.15}$$

whose solution is

$$\eta_k = C(S_+)^k + D(S_-)^k, \tag{2.16}$$

where C and D are small arbitrary constants, and

$$S_\pm(h) = -h \pm \sqrt{1 + h^2}. \tag{2.17}$$

Thus, the first term on the right-side of equation (2.16) exponentially decreases, while the second term oscillates with an exponentially increasing amplitude. Putting these results together, it follows that the central difference scheme has exactly the same two fixed-points as the logistic differential equation. However, while y(t)=0 is (linearly) unstable for the differential equation, both fixed points are linearly unstable for the central difference scheme. The results of the linear stability analysis, as given in equation (2.12) and equation (2.16), are shown in Figure 2.3. For initial value y_0, such that $0 < y_0 < 1$, the values of y_k increase and exponentially approach the fixed-point $\bar{y}^{(1)} = 1$; y_k then begins to oscillate with an exponentially increasing amplitude about $\bar{y}^{(1)} = 1$ until it reaches the neighborhood of the fixed point $\bar{y}^{(0)} = 0$. After an initial exponential decrease to $\bar{y}^{(0)} = 0$, the y_k value then begin their increase back to the fixed-point at $\bar{y}^{(1)} = 1$.

The major conclusion is that the use of central difference scheme

$$\frac{y_{k+1} - y_{k-1}}{2h} = f(y_k) \tag{2.18}$$

for the scalar first-order differential equation

$$\frac{dy}{dt} = f(y) \tag{2.19}$$

forces all the fixed-points to become unstable. Consequently, the central difference discrete derivative should never be used for this class of ordinary differential equation.

However,before leaving the use of central difference scheme, let us consider the following discrete model for the logistic equation:

$$\frac{y_{k+1} - y_{k-1}}{2h} = y_{k-1}(1 - y_{k+1}). \tag{2.20}$$

Our major reason for studying this model is that an exact analytic solution exists for equation (2.20). Observe that the function

$$f(y) = y(1 - y) \tag{2.21}$$

is modeled locally on the lattice in equation (2.7), while it is modeled nonlocally in equation (2.20), i.e., at lattice points k-1 and k+1.

The substitution

$$y_k = \frac{1}{x_k}, \tag{2.22}$$

transforms equation (2.20) to the expression

$$x_{k+1} - (\frac{1}{1+2h})x_{k-1} = \frac{2h}{1+2h}. \tag{2.23}$$

Note that equation (2.20) is a nonlinear, second-order difference equation, while equation (2.23) is a linear, inhomogeneous equation with constant coefficients. Solving equation (2.23) gives the general solution

$$x_k = 1 + [A + B(-1)^k](1 + 2h)^{-k/2}, \tag{2.24}$$

where A and B are arbitrary constants. Therefore, y_k is

$$y_k = \frac{1}{1 + [A + B(-1)^k](1 + 2h)^{-k/2}}. \tag{2.25}$$

For y_0 such that $0 < y_0 < 1$, and y_1 selected such that $y_1 = y_0 + hy_0(1 - y_0)$, the solutions to equation (2.25) have the structure indicated in Figure 2.4. Observe that the numerical solution has the general properties of the solution to the logistic differential equation, see Figure 2.1 and Figure 2.2, except that small oscillations occur about the smooth solution.

The direct forward Euler discrete model for the Logistic differential equation is

$$\frac{y_{k+1} - y_k}{h} = y_k(1 - y_k). \tag{2.26}$$

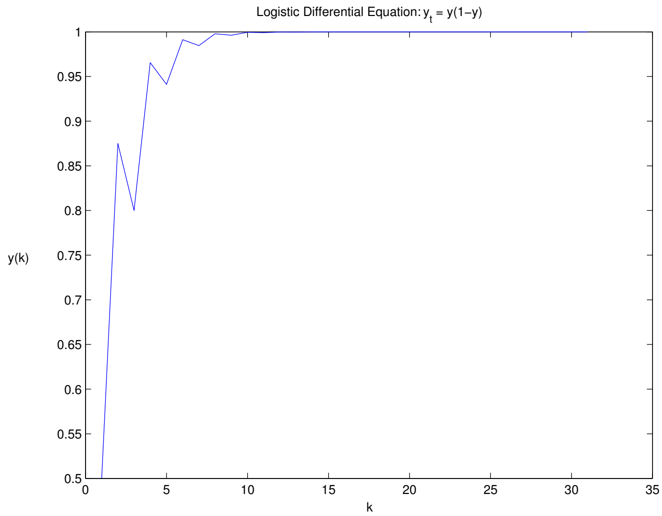

Figure 2.4. The central difference scheme with nonlocal representation given in equation (2.20) with $y_0 = 0.5$, $h = 0.5$

This first-order difference equation has two constant solutions or fixed-points at $\bar{y}^{(0)} = 0$ and $\bar{y}^{(1)} = 1$. Perturbations about these fixed-points, i.e.,

$$y_k = \bar{y}^{(0)} + \epsilon_k = \epsilon_k, \quad | \epsilon_k | \leq 1, \tag{2.27}$$

$$y_k = \bar{y}^{(1)} + \eta_k = 1 + \eta_k, \quad | \eta_k | \leq 1, \tag{2.28}$$

give the following solutions for ϵ_k and η_k:

$$\epsilon_k = \epsilon_0 (1 + h)^k, \tag{2.29}$$

$$\eta_k = \eta_0 (1 - h)^k. \tag{2.30}$$

The expression for ϵ_k shows that $\bar{y}^{(0)}$ is unstable for all $h > 0$. However, the linear stability properties of the fixed-point $\bar{y}^{(1)}$ depend on the value of the step-size. For example:

$0 < h < 1 : \bar{y}^{(1)}$ is linearly stable; perturbations decrease exponentially.

$1 < h < 2 : \bar{y}^{(1)}$ is linearly stable; however, the perturbations decrease exponentially with an oscillating amplitude.

$h > 2 : \bar{y}^{(1)}$ is linearly unstable; the perturbations oscillate with an exponentially increasing amplitude.

Our conclusion is that the forward Euler scheme gives the correct linear stability properties only if $0 < h < 1$. For this interval of step-size values, the qualitative properties of the solutions to the differential and difference equations are the same. Consequently, for $0 < h < 1$, there are no numerical instabilities.

Figure 2.5, Figure 2.6 and Figure 2.7 present numerical solutions of the equation (2.26) by the forward Euler scheme for the initial condition $y_0 = 0.5$. For all cases, the step sizes are taken as h=0.1, 1.5 and 2.5 respectively.

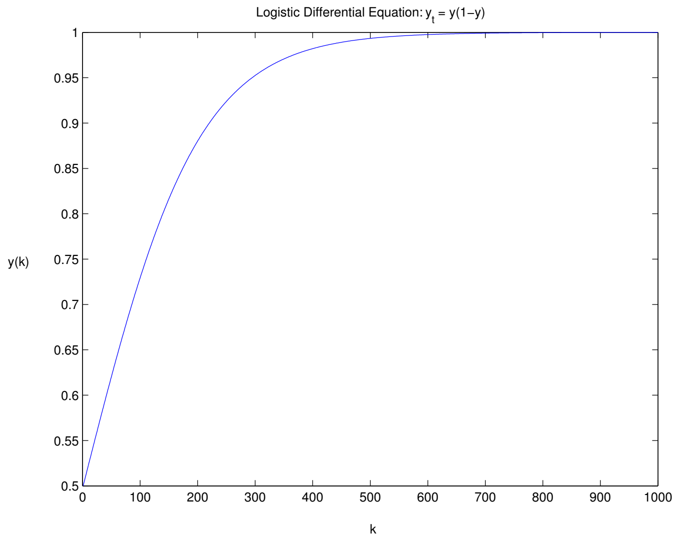

Figure 2.5. The forward Euler scheme given in equation (2.26) with $y_0 = 0.5$, $h = 0.1$.

Our next model of the logistic differential equation is constructed by using a forward Euler for the first-derivative and a nonlocal expression for the function $f(y) = y(1 - y)$. This model is

$$\frac{y_{k+1} - y_k}{h} = y_k(1 - y_{k+1}). \tag{2.31}$$

This first-order, nonlinear difference equation can be solved exactly by using the variable change

$$y_k = \frac{1}{x_k}, \tag{2.32}$$

to obtain

$$x_{k+1} - \frac{1}{1+h}x_k = \frac{h}{1+h}, \tag{2.33}$$

whose general solution is

$$x_k = 1 + A(1+h)^{-k}, \tag{2.34}$$

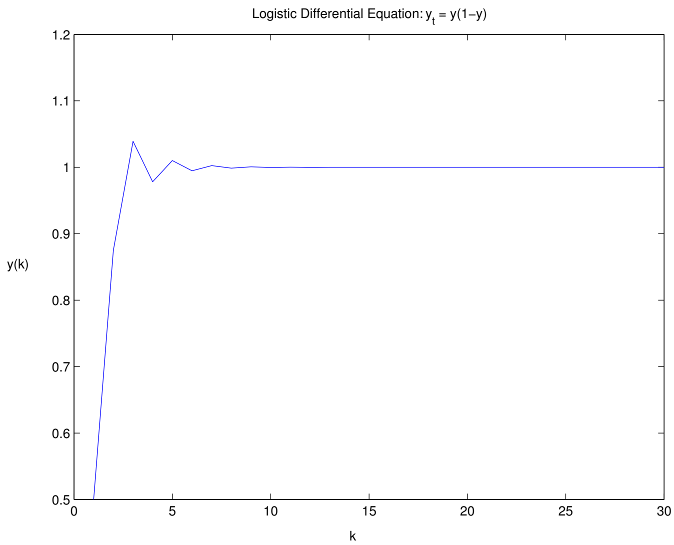

Figure 2.6. The forward Euler scheme given in equation (2.26) with $y_0 = 0.5$, $h = 1.5$.

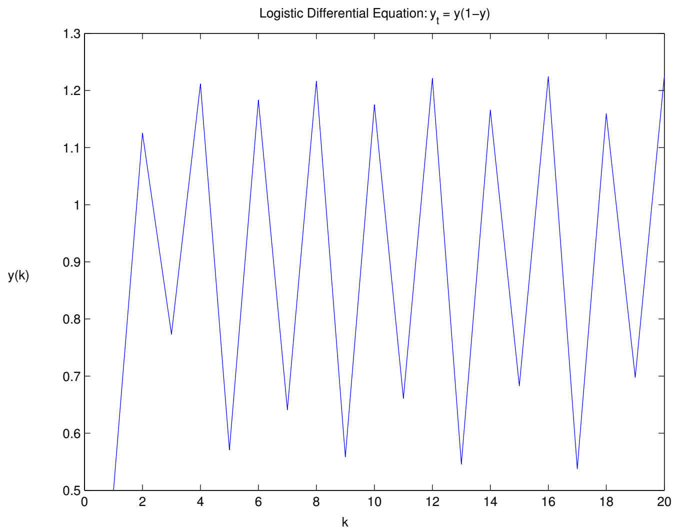

Figure 2.7. The forward Euler scheme given in equation (2.26) with $y_0 = 0.5$, $h = 2.5$.

where A is an arbitrary constant. Imposing the initial condition

$$x_0 = \frac{1}{y_0} \tag{2.35}$$

gives

$$A = \frac{1 - y_0}{y_0}, \tag{2.36}$$

and

$$y_k = \frac{y_0}{y_0 + (1 - y_0)(1 + h)^{-k}}. \tag{2.37}$$

Examination of equation (2.37) shows that, for $h > 0$, its qualitative properties are the same as the corresponding exact solution to the Logistic differential equation, namely, equation (2.4). Hence, the forward Euler, nonlocal discrete model has no numerical instabilities for any step-size. Figure 2.8, 2.9 and Figure 2.10 are illustrated that the numerical solution of the equation (2.3) by using the nonlocal representation given in equation (2.31) for h=0.1, 1.5 and 2.5 respectively. For all cases, the initial condition is $y_0 = 0.5$.

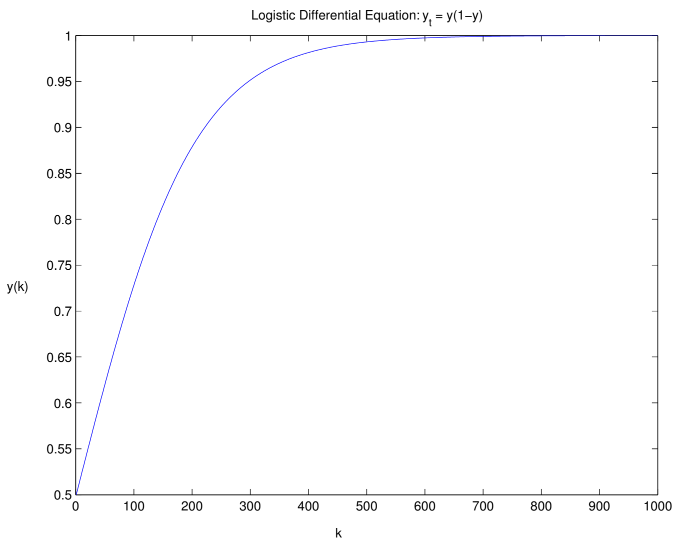

Figure 2.8. The forward Euler scheme with nonlocal representation given in equation (2.31) with $y_0 = 0.5$, $h = 0.1$.

To illustrate the construction of discrete finite difference models of differential equations, we begin with the scalar ordinary equation

$$\frac{dy}{dt} = f(y) \tag{2.38}$$

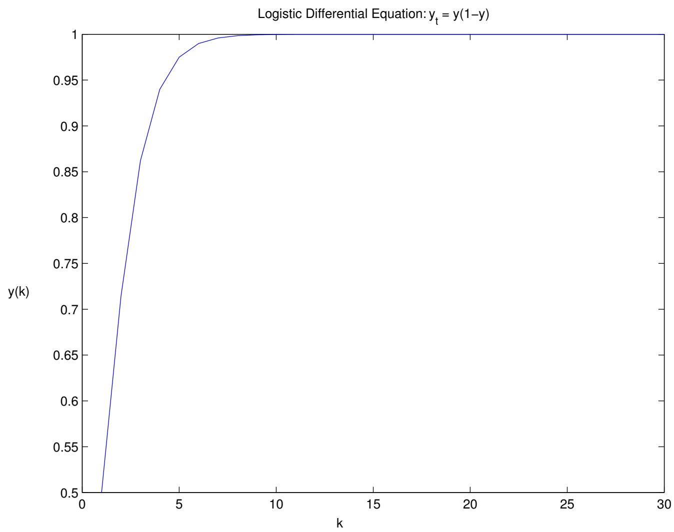

Figure 2.9. The forward Euler scheme with nonlocal representation given in equation (2.31) with $y_0 = 0.5$, $h = 1.5$.

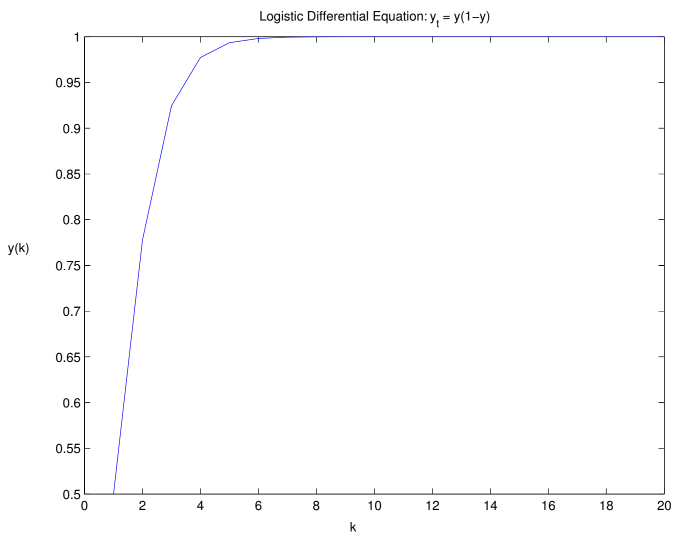

Figure 2.10. The forward Euler scheme with nonlocal representation given in equation (2.31) with $y_0 = 0.5$, $h = 2.5$.

where $f(y)$ is, in general, a nonlinear function of y. For a uniform lattice, with step-size, $\Delta t = h$, we replace the independent variable t by

$$t \to t_k = hk, \tag{2.39}$$

where k is an integer, i.e.,

$$t \in \{..., -2, -1, 0, 1, 2, 3, ...\}. \tag{2.40}$$

The dependent variable $y(t)$ is replaced by

$$y(t) \to y_k, \tag{2.41}$$

where y_k is approximation of $y(t_k)$. Likewise, the function $f(y)$ is replaced by

$$f(y) \to f_k, \tag{2.42}$$

where f_k is the approximation to $f[y(t_k)]$. The simplest possibility for f_k is

$$f_k = f(y_k). \tag{2.43}$$

2.3 Discussion

Comparing the four finite-difference schemes that were used to model the Logistic differential equation, the nonlocal forward Euler method clearly gave the best results. For all values of the step-size it has solutions that are in qualitative agreement with the corresponding solutions of the differential equation. The other discrete models had, for certain values of step-size, numerical instabilities.

CHAPTER 3

NONSTANDARD FINITE DIFFERENCE SCHEMES

In this chapter, we first review the finite difference schemes by considering first order scalar ordinary differential equations. Next, we define the exact-finite difference scheme. Finally, we present the rules of nonstandard finite difference scheme.

3.1 General Finite Difference Schemes

We would like to make several comments related to the discrete modeling of the scalar ordinary differential equation

$$\frac{dy}{dt} = f(y, \lambda) \tag{3.1}$$

where λ is an n-parameter vector. The most general finite-difference model for equation (3.1) that is of first-order in the discrete derivative takes the following form

$$\frac{y_{k+1} - y_k}{\phi(h, \lambda)} = F(y_k, y_{k+1}, \lambda, h). \tag{3.2}$$

The discrete derivative, on the left-side, is a generalization of that which is normally used, namely,

$$\frac{dy}{dt} \longrightarrow \frac{y_{k+1} - y_k}{h}. \tag{3.3}$$

From equation (3.2), we have

$$\frac{dy}{dt} \longrightarrow \frac{y_{k+1} - y_k}{\phi(h, \lambda)}, \tag{3.4}$$

where the denominator function $\phi(h, \lambda)$ has the property

$$\phi(h, \lambda) = h + O(h^2)$$

$$\lambda = \text{fixed}, \qquad \text{h} \rightarrow 0. \tag{3.5}$$

This form for the discrete derivative is based on the traditional definition of the derivative which can be generalized as follows:

$$\frac{dy}{dt} = \lim_{h \to 0} \frac{y[t + \psi_1(h)] - y(t)}{\psi_2(h)}, \tag{3.6}$$

where

$$\psi_i(h) = h + O(h^2), \qquad h \to 0; \qquad i = 1, 2. \tag{3.7}$$

Examples of functions $\psi(h)$ that satisfy this condition are

$$\psi(h) = \text{h}, \quad \text{sinh}, \quad e^h - 1, \quad \frac{1 - e^{-\lambda h}}{\lambda}, \quad \text{etc.}$$

Note that in taking the $\lim \text{h} \to 0$ to obtain the derivative, the use of any of these ψ_h will lead to the usual result for the first derivative

$$\frac{dy}{dt} = \lim_{h \to 0} \frac{y[t + \psi_1(h)] - y(t)}{\psi_2(h)} = \lim_{h \to 0} \frac{y(t + h) - y(t)}{h}. \tag{3.8}$$

However, for h finite, these discrete derivatives will differ greatly from those conventionally given in the literature, such as equation (3.3). This fact not only allows for the construction of a larger class of finite-difference models, but also provides for more ambiguity in the modeling process.

3.2 Exact Finite-Difference Schemes

We consider only first-order, scalar ordinary differential equations. However, the results can be generalized to coupled systems of first-order ordinary differential equations.

It should be acknowledged that the early work of Potts (Potts 1982) played a fundamental role in interesting the concept of exact finite difference schemes.

Consider the general first-order differential equation

$$\frac{dy}{dt} = f(y, t, \lambda), \qquad y(t_0) = y_0, \tag{3.9}$$

where $f(y, t, \lambda)$ is such that equation (3.9) has a unique solution over the interval, $0 \leq t < T$ and for λ in the interval, $\lambda_1 \leq \lambda \leq \lambda_2$. This solution can be written as

$$y(t) = \phi(\lambda, y_0, t_0, t), \tag{3.10}$$

with

$$\phi(\lambda, y_0, t_0, t_0) = y_0. \tag{3.11}$$

Now consider a discrete model of equation (3.9)

$$y_{k+1} = g(\lambda, h, y_k, t_k), \qquad t_k = hk. \tag{3.12}$$

Its solution can be expressed in the form

$$y_k = \phi(\lambda, h, y_0, t_0, t_k), \tag{3.13}$$

with

$$\phi(\lambda, h, y_0, t_0, t_0) = y_0. \tag{3.14}$$

Definition 3.2.1. Equation (3.9) and equation (3.12) are said to have same general solution if and only if

$$y_k = y(t_k)$$

for arbitrary values of h.

Definition 3.2.2. An exact difference scheme is one for which the solution to the difference equation has the same general solution as the associated differential equation.

By using these two definitions, the following theorem can be stated.

Theorem 3.2.1. The differential equation

$$\frac{dy}{dt} = f(y, t, \lambda), \qquad y(t_0) = y_0, \tag{3.15}$$

has an exact finite-difference scheme given by the expression

$$y_{k+1} = \phi[\lambda, y_k, t_k, t_{k+1}], \tag{3.16}$$

where ϕ is that of equation (3.10).

Proof The group property of the solutions to equation (3.15) gives

$$y(t+h) = \phi[\lambda, y(t), t, t+h] \tag{3.17}$$

If now make the identifications

$$t \to t_k, \qquad y(t) \to y_k, \tag{3.18}$$

then equation (3.17) becomes

$$y_{k+1} = \phi(\lambda, y_k, t_k, t_{k+1}). \tag{3.19}$$

This is the requirement for ordinary difference equation that has the same general solution as equation (3.9).

Comments.

(i) If all solutions of equation (3.15) exist for all time, i.e., $t = \infty$, then equation (3.17) holds for all t and h. Otherwise, the relation is assumed to hold whenever the right-side is well defined.

(ii) The theorem is only an existence theorem. It basically says that if an differential equation has a solution, then an exact finite-difference scheme exists. In general, no guidance is given as to how to actually construct such a scheme.

(iii) A major implication of the theorem is that the solution of the difference equation is exactly equal to the solution of the ordinary differential equation on the computational grid for fixed, but, arbitrary step-size h.

(iv) The theorem can be easily generalized to systems of coupled, first-order differential equations.

The question now arises as to whether exact finite difference schemes exist for partial differential equation. The answer is (probably) no. This negative result is a consequence of the fact that given an arbitrary partial differential equation there exists no clear, unambiguous accepted definition of a general solution to the equation. However, we should expect that certain classes of partial differential equations will have exact difference models. Note that in this case some type of functional relation should exist between the various (space and time) step-sizes.

The discovery of exact discrete models for particular ordinary and partial differential equations is of great importance, primarily because it allows us to gain insights into the better construction of finite-difference schemes. They also provide the computational investigator with useful benchmarks for comparison with the standard procedures.

3.3 Example of Exact Finite Difference Schemes

In this section, we will use the theorem of the last section "in reverse" to construct exact finite difference schemes for several ordinary and partial differential differential equations for which exact general solutions are explicitly known. These schemes have the property that their solutions do not have numerical instabilities.

For nonlinear differential equations, the steps should be applied to construct exact finite-difference schemes.

(i) Consider a system of N coupled, first order, ordinary differential equations

$$\frac{dY}{dt} = F(Y, t, \lambda), \quad Y(t_0) = Y_0, \tag{3.20}$$

where Y,F are N-dimensional column vectors whose i-th components are

$$(Y)_i = y^i(t), \tag{3.21}$$

$$(F)_i = f^i[y^{(1)}, y^{(2)}, ..., y^{(N)}; t, \lambda]. \tag{3.22}$$

(ii) Denote the general solution to equation (3.20) by

$$Y(t) = \phi(\lambda, Y_0, t_0, t) \tag{3.23}$$

where

$$y^i(t) = \phi^i[\lambda, y_0^1, y_0^2, ..., y_0^N, t_0, t]. \tag{3.24}$$

(iii)The exact difference equation corresponding to the differential equation is obtained by making the following substitutions in equation (3.23):

$$Y(t) \to Y_{k+1}, \quad Y_0 = Y(t_0) \to Y_k, \quad t_0 \to t_k, \quad t \to t_{k+1}. \tag{3.25}$$

Next, we will give the following example for the exact finite difference schemes.

Consider the general logistic differential equation

$$\frac{dy}{dt} = \lambda_1 y - \lambda_2 y^2, \quad y(t_0) = y_0, \tag{3.26}$$

where λ_1 and λ_2 are constants. The solution to the initial value problem of equation (3.26) is given by the following expression

$$y(t) = \frac{\lambda_1 y_0}{(\lambda_1 - y_0\lambda_2)e^{-\lambda_1(t-t_0)} + \lambda_2 y_0}. \tag{3.27}$$

Making the substitution of equation (3.25) gives

$$y(k+1) = \frac{\lambda_1 y_k}{(\lambda_1 - \lambda_2 y_k)e^{-\lambda_1 h} + \lambda_2 y_k}. \tag{3.28}$$

Additional algebraic manipulation we can obtain the exact difference scheme for the Logistic differential equation

$$\frac{y_{k+1} - y_k}{\frac{e^{\lambda_1 h - 1}}{\lambda_1}} = \lambda_1 y_k - \lambda_2 y_{k+1} y_k. \tag{3.29}$$

3.4 Nonstandard Modeling Rules

In particular, we concentrate on the exact finite difference scheme for the general logistic differential equation. The following observations are important:

(i) Exact finite-difference schemes generally require that nonlinear terms be modeled nonlocally. Thus, for the logistic equation the y^2 term is evaluated at two different grid points.

$$y^2 \longrightarrow y_{k+1}y_k.$$

However, for finite, fixed, nonzero values of step-sizes, the two representations of the squared terms are not equal,i.e.,

$$y_{k+1}y_k \neq (y_k)^2.$$

Therefore, a seemingly trivial modification in the modeling nonlinear terms can lead to major changes in the solution behaviors of the difference equations.

(ii)The discrete derivatives for both differential equations have denominator functions that are more complicated than those used in the standard modeling procedure. For example, the time-derivative in the Logistic equation is replaced by the following discrete representation

$$\frac{dy}{dt} = \frac{y_{k+1} - y_k}{(\frac{e^{\lambda_1 h - 1}}{\lambda_1})}.$$

Thus, the denominator function depends on both the parameter λ_1 and the step-size $h = \Delta t$.

(iii)The order of discrete derivatives in the exact finite difference schemes is always equal to the corresponding order of derivatives of the differential equation.Consider the following finite difference scheme for the logistic equation:

$$\frac{y_{k+1} - y_k}{h} = y_k(1 - y_k).$$

This discrete representation is expected to have numerical instabilities for two reasons:

(a)The denominator function is incorrect.

(b)The nonlinear term is modeled locally on the grid.

Now, we present the rules for the construction of discrete models.

Rule 1: The orders of the discrete derivatives must be exactly equal to the orders of the corresponding derivatives of the differential equations.

Rule 2: Denominator functions for the discrete derivatives must, in general, be expressed in terms of more complicated functions of the step-sizes than those conventionally used.

Rule 3: Nonlinear terms must, in general, be modeled nonlocally on the computational grid or lattice.

Rule 4: Special solutions of the differential equations should be special(discrete) solutions of the finite-difference models.

Rule 5: The finite-difference equations should not have solutions that do not correspond exactly to solutions of the differential equations.

3.5 Discussion

A major advantage of having an exact difference equation model for a differential equation is that questions related to the usual considerations of consistency, stability and convergence need not arise. However, it is essentially impossible to construct an exact discrete model for an arbitrary differential equation. This is because to do so would be tantamount to knowing the general solution of the original differential equation. However, the situation is not hopeless. The above five modeling rules can be applied to the construction of finite-difference schemes. While these discrete models, in general, will not be exact schemes, they will possess certain very desirable properties. In particular, we may hope to eliminate a number of the problems related to numerical instabilities.

CHAPTER 4

FIRST ORDER ORDINARY DIFFERENTIAL EQUATIONS

This chapter provides a technique for constructing finite-difference models of a single-scalar differential equation. We study the following first order differential equation,

$$\frac{dy}{dt} = f(y). \tag{4.1}$$

Our analysis is done under the assumption that

$$f(y) = 0 \tag{4.2}$$

has only simple zeros. Our purpose is to construct discrete models of equation (4.1) that do not exhibit elementary numerical instabilities. For equation (4.1) numerical instabilities occur whenever the linear stability properties of any of the fixed-points for the difference scheme differs from that of the differential equation.

Our goal is to prove, for equation (4.1), that it is possible to construct a new finite-difference scheme that have the correct linear stability properties for finite all step-sizes.

4.1 A New Finite Difference Scheme

Denote the fixed-points of equation (4.1) by

$$\{\bar{y}^{(i)}; i = 1, 2, ..., I\}, \tag{4.3}$$

where I may be unbounded. The fixed-points are the solutions to the equation

$$f(\bar{y}) = 0. \tag{4.4}$$

Define R_i as

$$R_i = \frac{df[\bar{y}^{(i)}]}{dy}, \tag{4.5}$$

and R^* as

$$R^* = Max| \ R_i \ |; \quad \imath = 1, 2, ..., I. \tag{4.6}$$

Linear stability analysis applied to the i-th fixed-point gives the following results:

(i) If $R_i > 0$, the fixed-point $y(t) = \bar{y}^{(i)}$ is linearly unstable.

(ii) If $R_i < 0$, the fixed-point $y(t) = \bar{y}^{(i)}$ is linearly stable.

Consider the following finite-difference scheme for equation (4.1)

$$\frac{y_{k+1} - y_k}{\left[\frac{\phi(hR^*)}{R^*}\right]} = f(y_k) \tag{4.7}$$

where $\phi(z)$ has the two properties

$$\phi(z) = z + O(z^2), \qquad z \to 0 \tag{4.8a}$$

$$0 < \phi(z) < 1, \qquad z > 0. \tag{4.8b}$$

Theorem 4.1.1. The finite difference scheme of equation (4.7) has fixed-points with exactly the same linear stability properties as the differential equation

$$\frac{dy}{dt} = f(y) \quad \text{for all} \quad h > 0. \tag{4.9}$$

This theorem demonstrates that it is possible to construct discrete models for a single scalar ordinary differential equation such that elementary numerical instabilities do not occur in their solutions. This result is related to the fact that most elementary numerical instabilities exist from a given fixed-point having the opposite linear stability properties in the difference scheme to the differential equation. The above construction shows that to achieve the correct linear stability behavior, a generalized definition must be used. Standard finite-difference modeling procedures do not have the correct linear stability behavior for all step-sizes.

The above finite-difference scheme uses the following denominator function for the discrete first-derivative

$$D(h, R^*) = \frac{\phi(hR^*)}{R^*} \tag{4.10}$$

where ϕ and R^* are given by equations (4.6) and (4.8). This form replaces the simple h function found in the standard finite-difference schemes

$$\frac{dy}{dt} \to \frac{y_{k+1} - y_k}{h}. \tag{4.11}$$

Note that in the limits, the generalized discrete derivative reduces to the first derivative,

$$\frac{y_{k+1} - y_k}{\left[\frac{\phi(hR^*)}{R^*}\right]} = \frac{dy}{dt}. \tag{4.12}$$

4.2 A New Finite Difference Scheme for Nonlinear Dynamic Equation

We construct a new finite difference discretization for the scalar first order nonlinear differential equation. We illustrate the power of the new finite difference scheme to eliminate the numerical instability. We consider the following general nonlinear first order dynamic equation with the initial condition:

$$\frac{dy}{dt} = y(1 - y^n), \qquad y(0) = 0.5 \tag{4.13}$$

where n is a positive integer. We first develop the denominator function $D(h, R^*)$, then nonlocally new representation for $y(1 - y^n)$. The equation (4.13) can be solved easily,

$$y(x) = \frac{1 + (2^n - 1)e^{n(1-x)}}{n}. \tag{4.14}$$

This equation can be reduced to a well known equation for a variable n, such as when n=1 the equation (4.13) becomes a logistic differential equation which we discussed in chapter 2

$$\frac{dy}{dt} = y(1 - y) \tag{4.15}$$

for this case, the nonlinear part is

$$f(y) = y(1 - y) \tag{4.16}$$

which has two fixed points given in equation (4.17) at

$$\bar{y}^{(1)} = 0, \qquad \bar{y}^{(2)} = 1 \tag{4.17}$$

and

$$R_1 = 1, \qquad R_2 = -1, \qquad R^* = 1. \tag{4.18}$$

Using $\phi(z) = 1 - e^{-z}$, we obtain, after substituting the equation (4.18) into the equation (4.7), the following discrete model of equation (4.15)

$$\frac{y_{k+1} - y_k}{1 - e^{-h}} = y_k(1 - y_k). \tag{4.19}$$

The equation (4.13) has a cubic nonlinearity when n=2

$$\frac{dy}{dt} = y(1 - y^2) \tag{4.20}$$

the cubic nonlinear part is

$$f(y) = y(1-y^2). \tag{4.21}$$

The dynamic equation for this choice of n produces three fixed points

$$\bar{y}^{(1)} = 0, \quad \bar{y}^{(2)} = 1, \quad \bar{y}^{(3)} = -1 \tag{4.22}$$

and

$$R_1 = 1, \quad R_2 = R_3 = -2, \quad R^* = 2. \tag{4.23}$$

The substitution of equations (4.21), (4.23) and $\phi(z) = 1 - e^{-z}$ into equation (4.7) gives the following new finite difference equation for the equation (4.20)

$$\frac{y_{k+1} - y_k}{\left(\frac{1-e^{-2h}}{2}\right)} = y_k(1 - y_k^2). \tag{4.24}$$

In the same manner, for n=3 equation (4.13) reduces to the following equation with four fixed points

$$\frac{dy}{dt} = y(1-y^3). \tag{4.25}$$

One can easily show that, the fourth order polynomial,

$$f(y) = y(1-y^3) \tag{4.26}$$

has a four fixed points given in equation (4.27)

$$\bar{y}^{(1)} = 0, \quad \bar{y}^{(2)} = 1, \quad \bar{y}^{(3)} = \frac{-1+\sqrt{3}i}{2}, \quad \bar{y}^{(4)} = \frac{-1-\sqrt{3}i}{2} \tag{4.27}$$

and

$$R_1 = 1, \quad R_2 = R_3 = R_4 = -3, \quad R^* = 3. \tag{4.28}$$

The substitution of equations (4.26), (4.28) and $\phi(z) = 1 - e^{-z}$ into equation (4.7) gives the following new finite difference equation for the equation (4.25)

$$\frac{y_{k+1} - y_k}{\left(\frac{1-e^{-3h}}{3}\right)} = y_k(1 - y_k^3). \tag{4.29}$$

In general, for any $n > 0$, we use the following nonstandard discretization equation for the equation (4.13)

$$\frac{y_{k+1} - y_k}{\left(\frac{1-e^{-nh}}{n}\right)} = y_k(1 - y_k^n) \tag{4.30}$$

where

$$R_1 = 1, \qquad R_2 = R_3 = ... = R_n = (-1)^n n, \qquad R^* = n. \tag{4.31}$$

Next, we rewrite the nonlinear terms of the equation (4.13) by applying the non-standard finite difference rules introduced in chapter 2. The nonstandard modeling rules require that nonlinear terms can be rewritten nonlocally on the computational grid as follows:

For n=1, the discrete scheme, with a nonlocal nonlinear term, is

$$\frac{y_{k+1} - y_k}{1 - e^{-h}} = y_k(1 - y_{k+1}). \tag{4.32}$$

This difference equation can be solved exactly by using the transformation

$$y_k = \frac{1}{w_k}. \tag{4.33}$$

This gives

$$w_{k+1} - (\frac{1}{2 - e^{-h}})w_k = \frac{1 - e^{-h}}{2 - e^{-h}}, \tag{4.34}$$

whose exact solution is

$$w_k = 1 + A(2 - e^{-h})^{-k}, \tag{4.35}$$

where A is an arbitrary constant. Imposing the initial condition, $y(0) = y_0$, we get the following equation

$$y_k = \frac{y_0}{y_0 + (1 - y_0)(2 - e^{-h})^{-k}}. \tag{4.36}$$

Note that

$$1 < 2 - e^{-h} < 2, \qquad h > 0 \tag{4.37}$$

consequently,

$$g_k = (2 - e^{-h})^{-k} \tag{4.38}$$

is an exponentially decreasing function of k. Examination of equation (4.36) shows that all the solutions of equation (4.32) have the same qualitative properties as the solutions to the logistic differential equation for all step-sizes, $h > 0$.

For n=2, a discrete model for the equation (4.21) with a nonlocal nonlinear term, is

$$\frac{y_{k+1} - y_k}{\frac{1-e^{-2h}}{2}} = y_k(1 - y_{k+1}y_k) \tag{4.39}$$

In the same manner, the nonstandard finite difference scheme for any $n > 0$ can be written as follows,

$$\frac{y_{k+1} - y_k}{\frac{1-e^{-nh}}{n}} = y_k(1 - y_{k+1}y_k^{n-1}). \tag{4.40}$$

We use this discreatization equation for our computation.

To illustrate the construction of discrete finite difference models of differential equations, we begin with the scalar ordinary equation

$$\frac{dy}{dt} = f(y) \tag{4.41}$$

where $f(y)$ is, in general, a nonlinear function of y. For a uniform lattice, with step-size, $\Delta t = h$, we replace the independent variable t by

$$t \rightarrow t_k = hk, \tag{4.42}$$

where k is an integer, i.e.,

$$t \in \{..., -2, -1, 0, 1, 2, 3, ...\}. \tag{4.43}$$

The dependent variable $y(t)$ is replaced by

$$y(t) \rightarrow y_k, \tag{4.44}$$

where y_k is approximation of $y(t_k)$. Likewise, the function $f(y)$ is replaced by

$$f(y) \rightarrow f_k, \tag{4.45}$$

where f_k is the approximation to $f[y(t_k]$. The simplest possibility for f_k is

$$f_k = f(y_k). \tag{4.46}$$

4.3 Numerical Verifications

In this section, we present some numerical simulations by using standard, non-standard discretization equation for the nonlinear dynamic equation given in equation (4.13) for various n and h. Then, we compare standard and nonstandard solutions of the equation (4.13).

In Figure 4.1, we compare the standard, nonstandard finite difference solutions and exact solution for n=1 and h=0.1. As it can be seen in Figure 4.1, both methods work and converge to the exact solution. However, from the Figure 4.2 shows that nonstandard finite difference method converges better than the standard finite difference method to the exact solution of the equation (4.13).

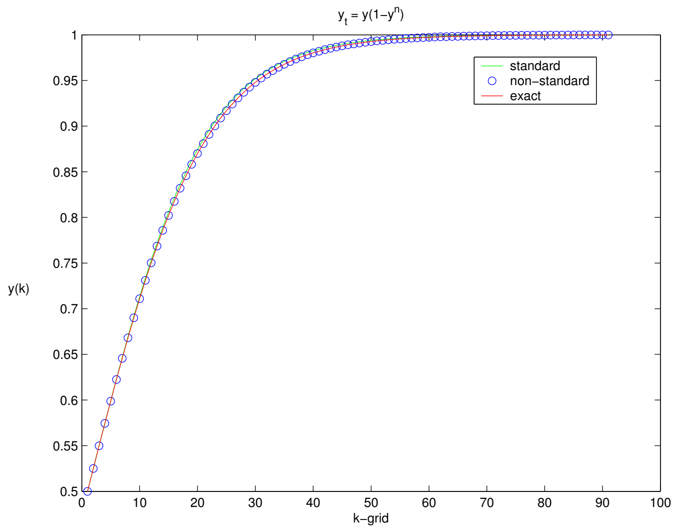

Figure 4.1. Comparison of standard, nonstandard finite difference methods and exact solution for n=1, h=0.1

Next, we fix h=0.1 as before, but the degree of the nonlinearity is increased to n=20. Although, for such big n, the numerical instability occurs when standard finite difference method is applied to the same equation, the nonstandard discretization for this equation still gives the numerical stability solution.

In Figure 4.4, we compare the standard and nonstandard finite difference solutions of the equation (4.13) for n=2 and h=0.1. These two discretization forms give the numerical stability solutions.

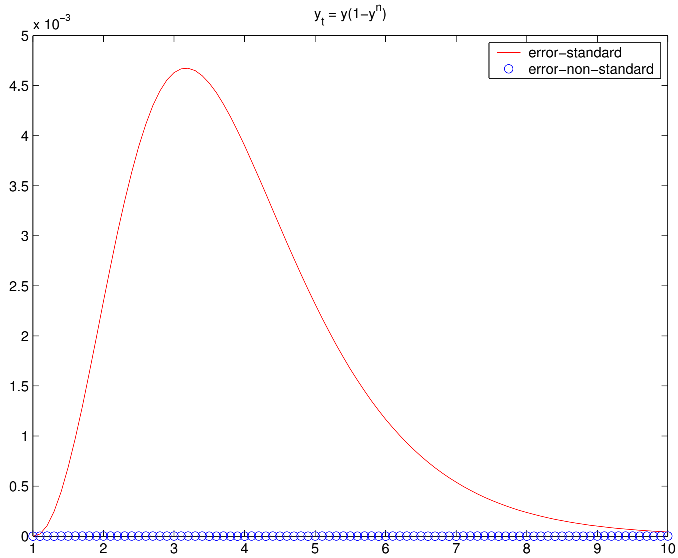

Figure 4.2. The error plot for the standard and nonstandard finite difference methods for n=1, h=0.1

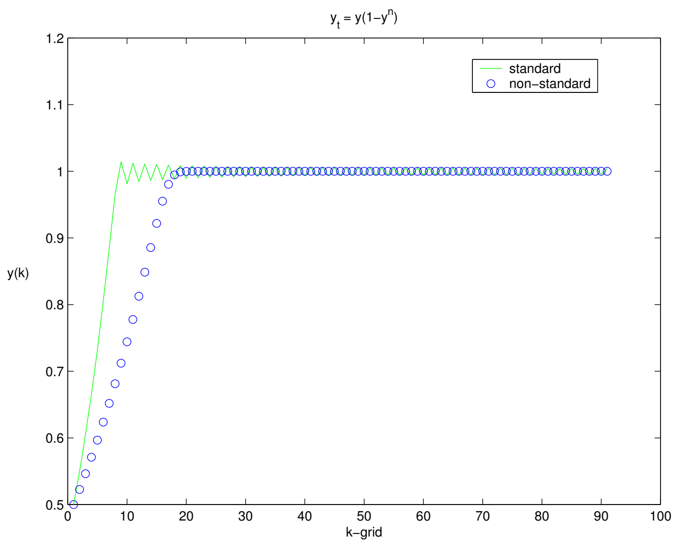

Figure 4.3. Comparison of standard and nonstandard finite difference methods for n=20, h=0.1

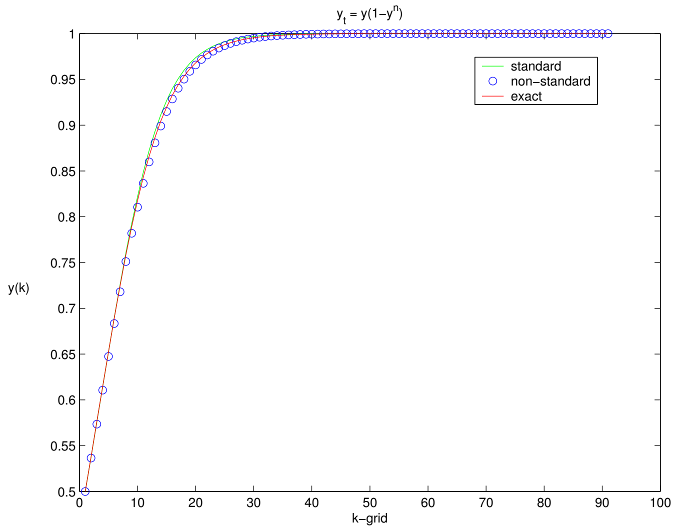

Figure 4.4. Comparison of standard, nonstandard finite difference methods and exact solution for n=2, h=0.1

In Figure 4.5, we increase the step-size as h=1.5 for n=2. The nonstandard finite difference method works, however standard finite difference method does not, as we expected.

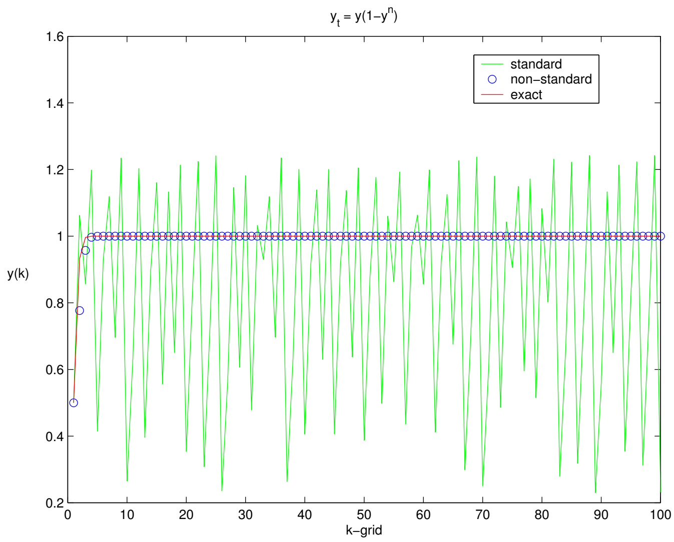

Figure 4.5. Comparison of standard, nonstandard finite difference methods and exact solution for n=2, h=1.5

4.4 Discussion

In this chapter, we study the equation (4.13) for a variable n. For our numerical simulations, we use the nonstandard discrete form given in equation (4.40). Although for $h < 1$, both methods are in a good agreement for $n < 15$, for $h > 1$, standard finite difference method exhibits the numerical instability for all n. In addition, when n is increased, nonstandard discrete models do not exhibit numerical instabilities for all h.

CHAPTER 5

SECOND ORDER ORDINARY DIFFERENTIAL EQUATION

In this chapter, we construct the standard and nonstandard finite difference schemes for the singularly perturbed convection-diffusion problem. We analyze both methods for this problem. Then we simulate some numerical results to compare both methods for various perturbation parameter ϵ.

5.1 The Problem Statement

In this section, we consider the convection-diffusion problem. Convection-diffusion problems have many applications in flows, water quality problems, convective heat transfer problems. Also this equation arise, from the linearization of the Navier-Stokes equation and the drift-diffusion equation of semiconductor device modelling. Consequently it is especially important to devise effective numerical methods for their approximate solution.

We now consider the following convection-diffusion problem

$$\epsilon y^{''} + y^{'} = -1 \quad on \quad [0,1] \tag{5.1}$$

$$y(0) = 0$$
$$y(1) = 0$$

which we can solve exactly:

$$y(x) = \frac{1 - exp(-\frac{x}{\epsilon})}{1 - exp(-\frac{1}{\epsilon})} - x. \tag{5.2}$$

If ϵ is big enough, the solution will be smooth and standard finite difference methods will give good results. However, as ϵ tends to zero, there is a boundary layer around the x=0, then we will show that nonstandard finite difference methods will give better results.

5.2 Implementation of Nonstandard Finite Difference Method for Convection-Diffusion Problem

We construct the nonstandard finite difference scheme for the equation (5.1). Approximating diffusion term by second special order central difference approximation and convective term by backward difference approximation, we obtain the following discrete equation for the equation (5.1).

$$\epsilon\frac{y_{k+1}-2y_k+y_{k-1}}{\phi(h)}+(\frac{y_k-y_{k-1}}{h})=-1 \tag{5.3}$$

where

$$\phi(h)=[\frac{exp(-\frac{h}{\epsilon})-1}{-\frac{1}{\epsilon}}]h.$$

After some algebraic manipulation, we can obtain the following implicit discrete equation for the equation (5.1).

$$y_{k+1}-[1+exp(-\frac{h}{\epsilon})]y_k+exp(-\frac{h}{\epsilon})y_{k-1}=h[exp(-\frac{h}{\epsilon})-1]. \tag{5.4}$$

We will use the equation (5.4) to simulate the solution of the convection-diffusion problem by nonstandard finite difference approximation.

5.3 Implementation of Standard Finite Difference Method for Convection-Diffusion Problem

We present and analyze standard finite difference approximation for equation (5.1). Our discrete model is constructed by using a central difference scheme for the second derivative and a forward difference scheme for the first derivative.

$$\epsilon(\frac{y_{k+1}-2y_k+y_{k-1}}{h^2})+(\frac{y_{k+1}-y_k}{h})=-1. \tag{5.5}$$

Then we have

$$(1+\frac{h}{\epsilon})y_{k+1}-(2+\frac{h}{\epsilon})y_k+y_{k-1}=-\frac{h^2}{\epsilon}. \tag{5.6}$$

We will use the equation (5.6) to simulate the solution of the convection-diffusion problem by standard finite difference approximation.

5.4 Analysis of Nonstandard Finite Difference Approximation

We consider the equation (5.4) and establish the solution of the difference equation. We consider homogeneous case of equation. The characteristic equation for equation (5.4) is

$$r^2 - (1 + exp(-\frac{h}{\epsilon}))r + exp(-\frac{h}{\epsilon}) = 0 \tag{5.7}$$

$$\Rightarrow r_{1,2} = \frac{1 + exp(-\frac{h}{\epsilon}) \mp \sqrt{(1 + exp(-\frac{h}{\epsilon}))^2 - 4exp(-\frac{h}{\epsilon})}}{2} \tag{5.8}$$

$$\Rightarrow r_{1,2} = \frac{1 + exp(-\frac{h}{\epsilon}) \mp \sqrt{(1 - exp(-\frac{h}{\epsilon}))^2}}{2} \tag{5.9}$$

Then we get

$$r_1 = 1 \quad \text{and} \quad r_2 = \frac{1}{exp(\frac{h}{\epsilon})}. \tag{5.10}$$

Since both characteristic roots less and equal to 1, the stability of the solution exists for all $h > 0$ and $\epsilon > 0$.

5.5 Analysis of Standard Finite Difference Approximation

We can go back to the equation (5.6) and establish the solution of the difference equation. First, we consider homogeneous case. The characteristic equation for equation (5.6) is

$$(1 + \frac{h}{\epsilon})r^2 - (2 + \frac{h}{\epsilon})r + 1 = 0 \tag{5.11}$$

$$\Rightarrow r_{1,2} = \frac{2 + \frac{h}{\epsilon} \mp \sqrt{(2 + \frac{h}{\epsilon})^2 - 4(1 + \frac{h}{\epsilon})}}{2(1 + \frac{h}{\epsilon})} \tag{5.12}$$

Then we obtain

$$r_1 = \frac{2(1 + \frac{h}{\epsilon})}{2(1 + \frac{h}{\epsilon})} = 1 \quad \text{and} \quad r_2 = \frac{1}{1 + \frac{h}{\epsilon}}. \tag{5.13}$$

Since both characteristic roots less and equal to 1, the stability of the solution exists for all $h > 0$ and $\epsilon > 0$.

5.6 Numerical Verifications

In this section, we solve our convection-diffusion problem by using discretization form of nonstandard finite difference approximation given in equation (5.4) and by standard finite difference approximation given in equation (5.6). We compare standard and nonstandard finite difference methods to the exact solution of the problem.

In Figure 5.1, we exhibit nonstandard finite difference, standard finite difference and exact solutions of the equation for ϵ=1. The step-size is taken as h=0.02. Although three curves are in a good agreement, the error shown in Figure 5.2 reveals that there is a slight deviation for standard finite difference method from the exact solution. Thus, nonstandard finite difference method still works better than the standard finite difference method for this choice of ϵ. In next figure, we decrease the perturbation

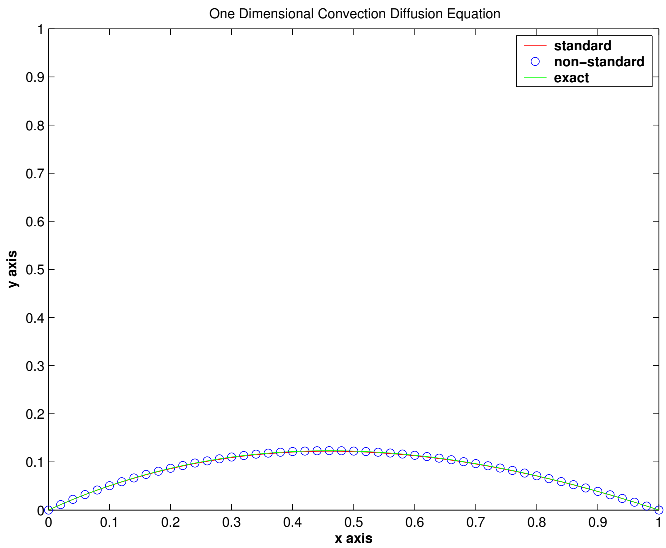

Figure 5.1. Comparison of the nonstandard, standard finite difference methods and exact solution for n=50, ϵ=1

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parameter ϵ as $\epsilon = 0.1$. In this case, nonstandard finite difference method works better than the standard finite difference method. There is a slight deviation from the exact solution for the standard finite difference method. However, from the error shown in Figure 5.4 nonstandard finite difference method fits the exact solution very well.

In Figure 5.5, we compare the nonstandard finite difference, standard finite

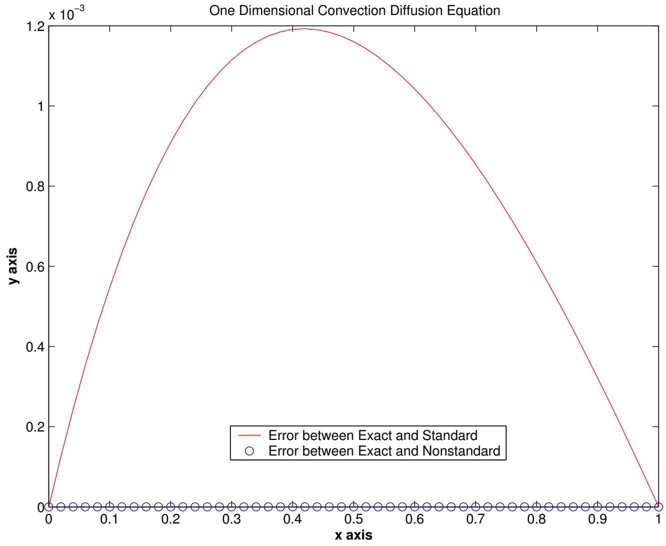

Figure 5.2. The error for the nonstandard and standard finite difference methods with ϵ=1.

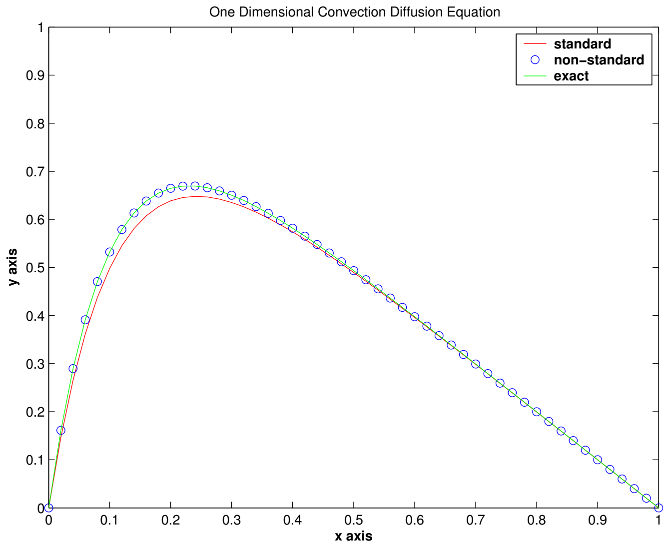

Figure 5.3. Comparison of the nonstandard, standard finite difference methods and exact solution for n=50, ϵ=0.1

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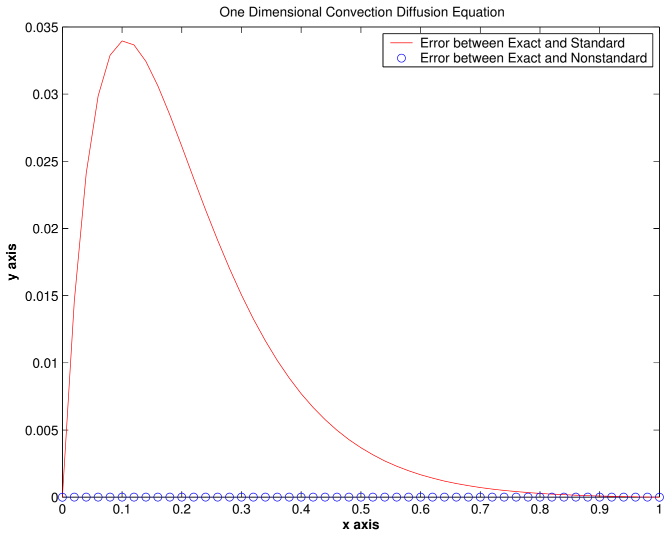

Figure 5.4. The error for the nonstandard and standard finite difference methods with ϵ=0.1.

difference and exact solutions of the equation (5.1) for $\epsilon = 0.01$. We have seen that nonstandard finite difference method works better than standard finite difference method. We also see that as ϵ gets smaller, both techniques are in a good agreement away from the boundary layer. The error shown in Figure 5.6 supports this result.

Finally, we solve the problem for $\epsilon = 0.001$. In this case, nonstandard finite difference method works better than the standard finite difference method. The error shown in Figure 5.8 exhibits that two methods work very well away from the boundary layer.

5.7 Discussion

In this chapter, we considered the singularly perturbed convection-diffusion problem. First, we find the new discretization for the convection-diffusion equation we considered. In this discretization, the characteristic root for the nonstandard finite difference method is $r_2 = \frac{1}{exp(\frac{h}{\epsilon})}$. On the other hand, the characteristic root for the standard finite difference method is $r_2 = \frac{1}{1+\frac{h}{\epsilon}}$. Therefore, the characteristic root for the nonstandard finite difference method decays faster than the root for the standard finite difference method. By using this discretization equation, we show that nonstandard finite difference method

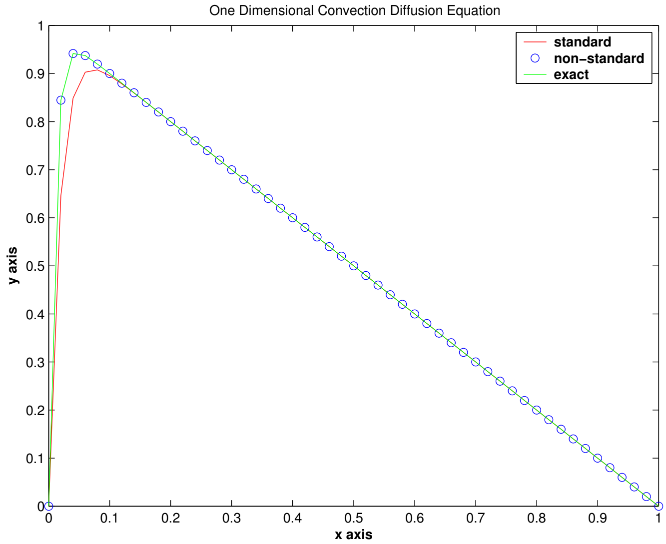

Figure 5.5. Comparison of the nonstandard, standard finite difference methods and exact solution for n=50, ϵ=0.01

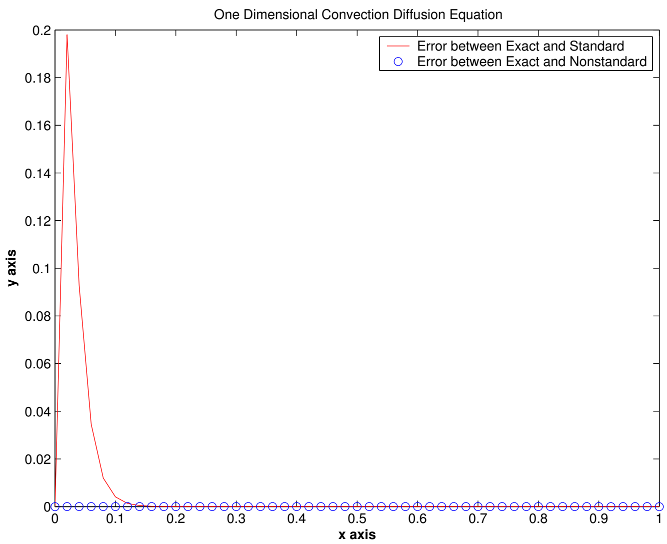

Figure 5.6. The error for the nonstandard and standard finite difference methods with ϵ=0.01

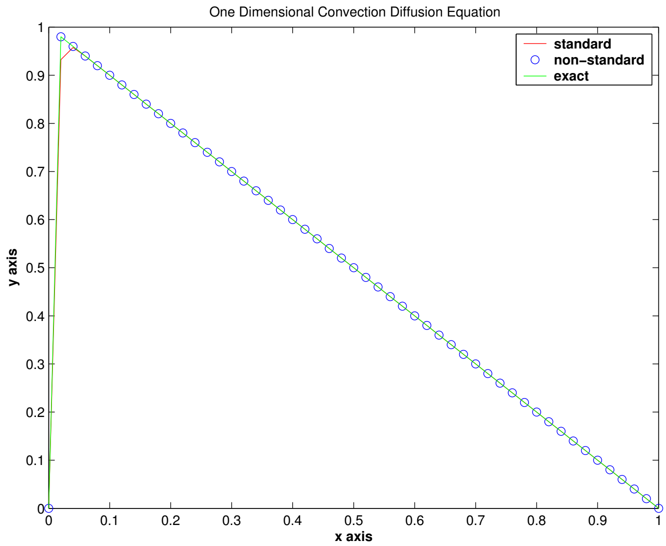

Figure 5.7. The nonstandard and standard finite difference methods with n=50, ϵ=0.001

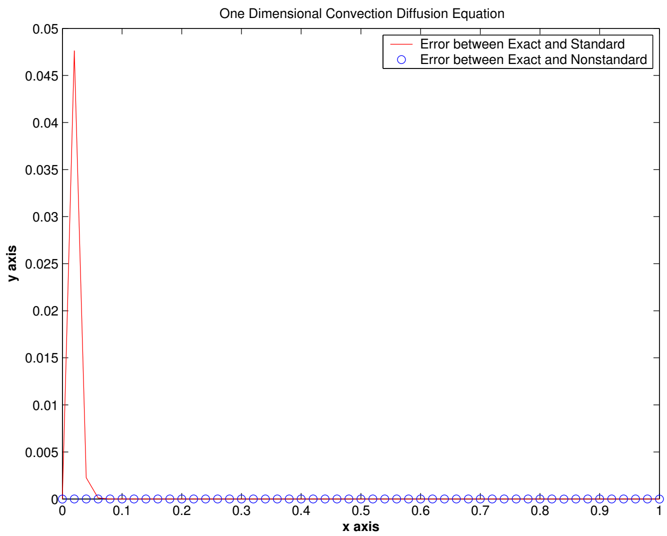

Figure 5.8. The error at the layer for the nonstandard and standard finite difference methods with ϵ=0.001

works better than the standard finite difference method for all choice of h and ϵ and as perturbation parameter ϵ gets smaller, nonstandard finite difference method works better than the standard finite difference method near the boundary layer. In addition, both methods are in a good agreement away from the boundary layer.

CHAPTER 6

PARTIAL DIFFERENTIAL EQUATIONS

In this chapter, we construct the nonstandard finite difference scheme for nonlinear reaction-diffusion equation. We state the lemma related the positivity and boundedness condition for discretization equation. We simulate some numerical results to check the correctness of the lemma.

6.1 The Problem Statement

Partial differential equations provide valuable mathematical models for dynamical systems that involve both space and time variables. We study the partial differential equation first order in the time derivative and second order in the space derivative. This equation includes various one space dimension modifications of wave, diffusion and Burgers' partial differential equations. The nonlinearity considered is third order polynomial with three distinct roots. One can find the exact solution by using the Hirota method. These special solution can then be used in the construction of nonstandard discrete models. However, it should be noted that exact-finite difference schemes are not expected to exist for partial differential equations. For the partial differential equation considered, a comparison will be made to the standard finite-difference schemes and how the solutions of the various nonstandard and standard discrete models differ from each other.

6.2 Nonstandard Finite Difference Scheme for a Nonlinear PDE

We consider the following reaction-diffusion equation which has a nonlinear cubic source term

$$u_t = u_{xx} - (u - a_1)(u - a_2)(u - a_3) \tag{6.1}$$

where $a_1 = -1, a_2 = 0, a_3 = 1$. For this choice of parameters, the equation (6.1) can be written as follows:

$$u_t = u_{xx} - u^3 + u. \tag{6.2}$$

Before proceeding with the construction of the non-standard numerical scheme for equation (6.2), a brief summary of its significant mathematical properties will be given. The reason why this is being done is to make sure that the non-standard finite difference scheme to be derived has these properties, otherwise, numerical instabilities will occur. First note that equation (6.2) has three fixed-points or constant solutions,

$$\bar{u}^{(1)} = -1 \quad \bar{u}^{(2)} = 0 \quad \bar{u}^{(3)} = 1. \tag{6.3}$$

The first and third fixed-points are linearly stable, while the second is linearly unstable. We use these stable fixed-points to check the boundedness condition for the discretization solution of the discrete equation, i.e.,

$$-1 \leq u_m^n \leq 1 \Rightarrow -1 \leq u_{m+1}^n \leq 1, \quad t > 0, \quad \text{fixed} \quad \text{n} \quad \text{all} \quad \text{m}.$$

6.3 Implementation of Standard Finite Difference Method for Reaction-Diffusion Equation

We present standard finite difference scheme for equation (6.2). Our discrete model is constructed by using a forward difference scheme for the first derivative and a central difference scheme for the second derivative.

$$\frac{u_m^{n+1} - u_m^n}{\Delta t} = \frac{u_{m+1}^n - 2u_m^n + u_{m-1}^n}{(\Delta x)^2} - (u_m^n)^3 + (u_m^n). \tag{6.4}$$

Then we have

$$u_m^{n+1} = u_m^n + \frac{\Delta t}{(\Delta x)^2}(u_{m+1}^n - 2u_m^n + u_{m-1}^n) - \Delta t(u_m^n)^3 + \Delta t(u_m^n). \tag{6.5}$$

We will use the equation (6.5) to simulate the solution of the reaction-diffusion equation by standard finite difference approximation.

6.4 Implementation of Nonstandard Finite Difference Method for Reaction-Diffusion Equation

Based on the previous works on non-standard finite difference schemes and the enforcement of a positivity condition, the following discrete model is selected for equation (6.2)

$$\frac{u_m^{n+1} - u_m^n}{\Delta t} = \frac{u_{m+1}^n - 2u_m^n + u_{m-1}^n}{(\Delta x)^2} - (\frac{3u_m^{n+1} - u_{m-1}^n}{2})(u_{m-1}^n)^2 + u_{m-1}^n. \tag{6.6}$$

We can write the denominator functions in a more complicated form then we have

$$\frac{u_m^{n+1}-u_m^n}{\frac{1-e^{-2\Delta t}}{2}} = \frac{u_{m+1}^n - 2u_m^n + u_{m-1}^n}{4\sin^2(\frac{\Delta x}{2})} - (\frac{3u_m^{n+1}-u_{m-1}^n}{2})(u_{m-1}^n)^2 + u_{m-1}^n. \tag{6.7}$$

Note that this scheme has the following features:

(i) The first order time derivative is replaced by a forward-Euler form.

(ii) A central difference scheme replaces the second order space derivative.

(iii) The non-linear u^3 term is modelled non-locally, i.e.,

$$u^3 \rightarrow (\frac{3u_m^{n+1}-u_{m-1}^n}{2})(u_{m-1}^n)^2. \tag{6.8}$$

(iv) The linear u term is modelled non-locally, i.e.,

$$u \rightarrow u_{m-1}^n. \tag{6.9}$$

Inspection of equation (6.6) shows that it is linear in u_m^{n+1} and solving for it gives the expression;

$$[1+\frac{3\Delta t}{2}(u_{m-1}^n)^2]u_m^{n+1} = (1-2\beta)u_m^n + \beta u_{m+1}^n + [\beta + \Delta t + \frac{\Delta t}{2}(u_{m-1}^n)^2](u_{m-1}^n)$$

where β is defined as

$$\beta = \frac{\Delta t}{(\Delta x)^2}. \tag{6.10}$$

After some algebraic manipulation, we can obtain the explicit discrete equation

$$u_m^{n+1} = \frac{(1-2\beta)u_m^n + \beta u_{m+1}^n + [\beta + \Delta t + \frac{\Delta t}{2}(u_{m-1}^n)^2](u_{m-1}^n)}{[1+\frac{3\Delta t}{2}(u_{m-1}^n)^2]}. \tag{6.11}$$

In the next section, we analyze this discrete equation.

6.5 Analysis of Nonstandard Finite Difference Approximation

According to the nonstandard finite difference rules, the equation (6.11) has to satisfy two criteria; positivity and boundedness conditions:

The discrete version of the positivity condition is

$$0 \le u_m^n \quad \Rightarrow \quad 0 \le u_m^{n+1} \quad \text{fixed n all } m, \quad \text{and} \tag{6.12}$$

that of the boundedness condition is

$$-1 \le u_m^n \le 1 \quad \Rightarrow \quad -1 \le u_m^{n+1} \le 1. \tag{6.13}$$

The positivity condition given in equation (6.12) is satisfied if

$$1 - 2\beta \geq 0 \quad \Rightarrow \quad \frac{\Delta t}{(\Delta x)^2} \leq \frac{1}{2}. \tag{6.14}$$

This can be easily seen from the discretization equation given in equation (6.11) due to each coefficient of the discrete solution is positive except the first one in the numerator.

Next, we show that boundedness condition for the discretization equation (6.11) is satisfied under some restrictions. For $\beta \leq \frac{1}{2}$, equation (6.11) can be written as follows:

$$u_m^{n+1} \leq \frac{\frac{1}{2}u_{m+1}^n + [\frac{1}{2} + \Delta t + \frac{\Delta t}{2}(u_{m-1}^n)^2](u_{m-1}^n)}{[1 + \frac{3\Delta t}{2}(u_{m-1}^n)^2]}. \tag{6.15}$$

Since, it is first assumed that $-1 \leq u_m^n \leq 0$ for n-fixed and all m, it follows that

$$\frac{1}{2}u_{m+1}^n \leq \frac{1}{2} \tag{6.16}$$

$$\frac{1}{2}u_{m-1}^n \leq \frac{1}{2} \tag{6.17}$$

$$\Delta t(u_{m-1}^n) \leq \Delta t(u_{m-1}^n)^2 \tag{6.18}$$

$$\frac{\Delta t}{2}(u_{m-1}^n)^3 \leq \frac{\Delta t}{2}(u_{m-1}^n)^2. \tag{6.19}$$

We add the equations (6.16)-(6.19), to obtain the following equation.

$$\frac{1}{2}u_m^{n+1} + \frac{1}{2}u_{m-1}^n + \Delta t(u_{m-1}^n + \frac{\Delta t}{2}(u_{m-1}^n)^3) \leq 1 + \frac{3\Delta t}{2}(u_{m-1}^n)^2. \tag{6.20}$$

Dividing equation (6.20) by the expression on its right side gives the following equation:

$$\frac{\frac{1}{2}u_m^{n+1} + \frac{1}{2}u_{m-1}^n + \Delta t(u_{m-1}^n + \frac{\Delta t}{2}(u_{m-1}^n)^3)}{1 + \frac{3\Delta t}{2}(u_{m-1}^n)^2} \leq 1. \tag{6.21}$$

However, the left-side of equation (6.20) is just u_m^{n+1}. Therefore, the result in equation (6.13) is shown to be true if $\beta \leq \frac{1}{2}$.

Next, we will show that $0 \leq u_m^n \leq 1$, defining Φ as follows:

$$u_m^n = u_{m-1}^n = u_{m+1}^n = \phi.$$

We use the equation (6.11) to obtain the following inequality:

$$u_m^{n+1} \leq \frac{\Phi - 2\beta\Phi + \beta\Phi + \beta\Phi + \Delta t\Phi + \frac{\Delta t}{2}\Phi^3}{1 + \frac{3\Delta t}{2}\Phi^2}. \tag{6.22}$$

The requirement for boundedness condition, i.e., $u_m^{n+1} \leq 1$, implies the following inequality:

$$\frac{\Phi + \Delta t \Phi + \frac{\Delta t}{2}\Phi^3}{1 + \frac{3\Delta t}{2}\Phi^2} \leq 1. \tag{6.23}$$

From this inequality, we find the restriction for the time step-size as follows:

$$\Delta t \leq \frac{1 - \Phi}{\Phi + \frac{\Phi^3}{2} - \frac{3}{2}\Phi^2}. \tag{6.24}$$

Thus the above analysis can be given as the following lemma:

Lemma 6.4.1. If under the following conditions are satisfied:

a)$-1 \leq u_m^n \leq 1$

b)$\frac{\Delta t}{(\Delta x)^2} \leq \frac{1}{2}$.

c)$\Delta t \leq \alpha$, where α= { Φ: sup($\frac{1-\Phi}{\Phi + \frac{\Phi^3}{2} - \frac{3}{2}\Phi^2}$), where $\Phi = u_m^n$, for all m, fixed n}

Then $-1 \leq u_m^{n+1} \leq 1$.

In the next section, we exhibit some numerical verifications for the equation (6.2) by using both standard and nonstandard finite difference methods.

6.6 Numerical Verifications

In this section, we present some numerical simulation by using nonstandard finite difference discretization for the equation (6.2). We compare these numerical solutions with the standard finite difference discretization for the same equation for the various β. Finally, we exhibit nonstandard, standard and exact solution for this equation, for fixed β.

In Figure 6.1, Figure 6.2 and Figure 6.3 we exhibit the solution of our problem for β=0.1, β=0.5, β=0.6 respectively and different values for t by using nonstandard difference discretization form given in equation (6.11). It can be seen in these figures that discretization form works for these choice of $\beta \leq \frac{1}{2}$. For $\beta \geq \frac{1}{2}$, as we claimed in Lemma 6.4.1, the nonstandard finite difference method does not work.

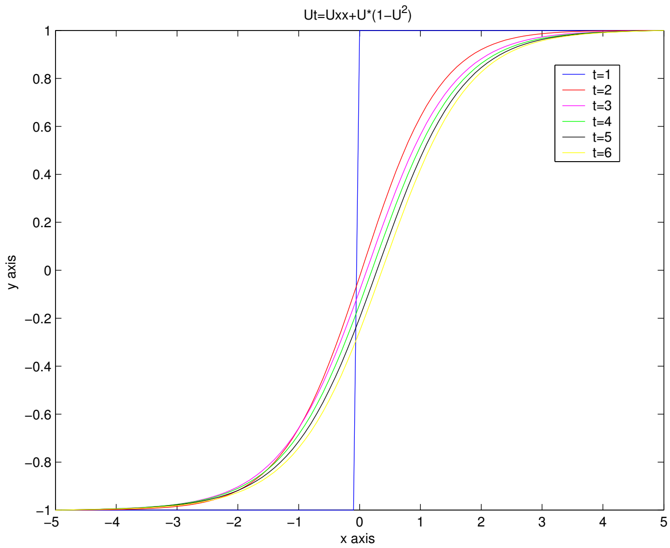

Figure 6.1. The nonstandard finite difference scheme with β=0.1 and different values of t.

Next, we compare the standard and nonstandard finite difference method for $\beta = \frac{1}{2}$ in Figure 6.4 for fixed t. Although, our new discretization form of the equation (6.2) work, standard finite difference discretization for the same equation doesn't work. Finally, for fixed β and for fixed time nonstandard, standard and exact solution of this equation are compared in Figure 6.5. We claim that nonstandard finite difference discretization of this problem converges better than the standard finite difference discretization for the equation (6.2).

6.7 Discussion

A new nonstandard finite difference scheme was constructed for the nonlinear reaction-diffusion equation. This new scheme has the correct fixed-points, satisfies both the positivity and boundedness conditions of equation (6.2), and easy to implement for obtaining numerical solutions since the scheme is effectively explicit. The validity of the scheme depends on the inequalities stated in equation (6.14), i.e., once Δx is selected, then Δt must satisfy equation (6.14). Numerical studies indicate that the derived nonstandard scheme provides excellent numerical solutions.

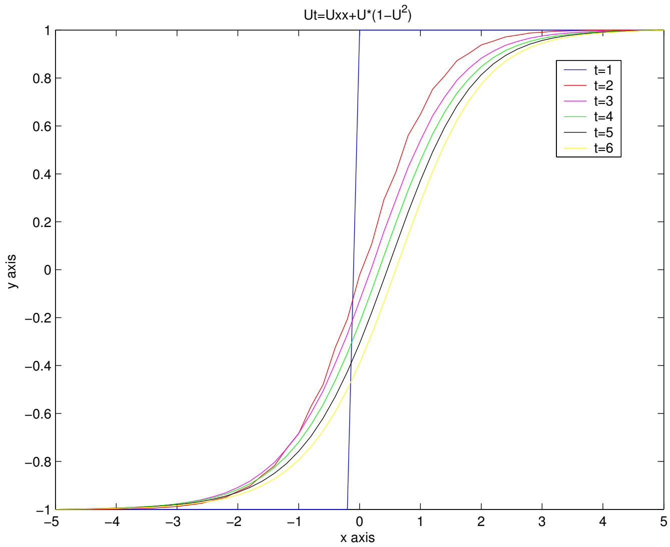

Figure 6.2. The nonstandard finite difference scheme with β=0.5 and different values of t.

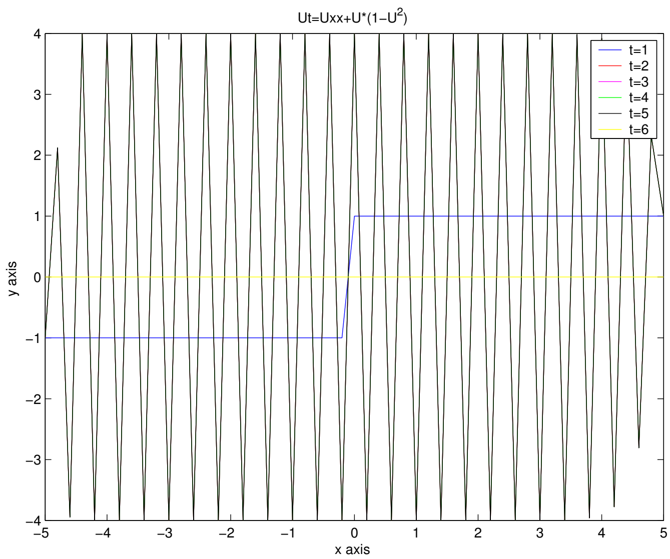

Figure 6.3. The nonstandard finite difference scheme with β=0.6 and different values of t.

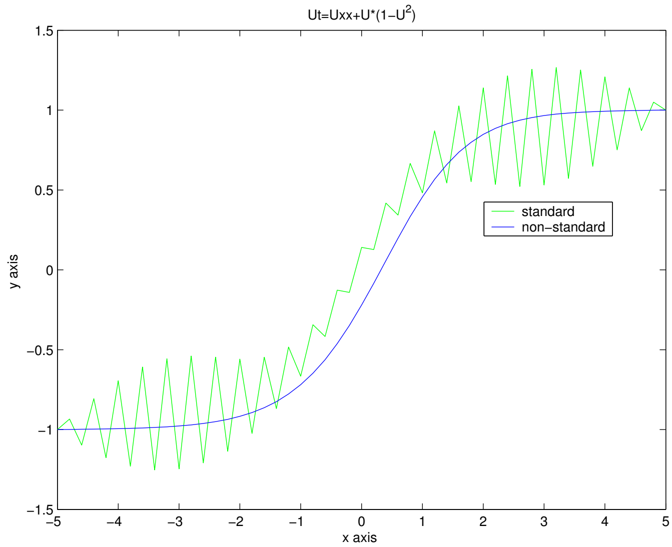

Figure 6.4. The standard and nonstandard finite difference scheme with β=0.5 and t=4.

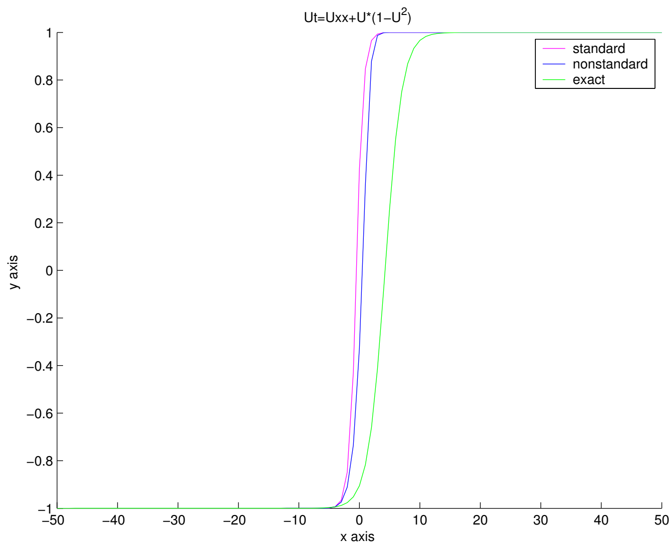

Figure 6.5. The standard, nonstandard and exact solution with β=0.1 and t=2.

CHAPTER 7

CONCLUSION

In Chapter 1, we reviewed the standard finite difference scheme and connected this to the nonstandard finite difference scheme. After, we shortly mentioned about non-standard finite difference rules and explained the necessary of the nonstandard finite difference models.

In Chapter 2, we explained why the numerical instabilities occur in the solution of the differential equations.

Chapter 3 introduced the notion of an exact finite difference scheme. It was shown, by means of a theorem, that, in general, ordinary differential equations have exact finite-difference equation representations. This theorem was then used to construct exact discrete models for several differential equations. A study of these exact schemes then led to the formulation of a set of nonstandard modelling rules.

Chapter 4 dealt with the construction of discrete representations for a single scalar ordinary differential equation, such that the linear stability properties of the fixed-points of the finite difference scheme were exactly the same as the corresponding fixed-points of the differential equation for all values of the step-size. This result eliminated all the elementary numerical instabilities was based on the idea of using a renormalized denominator function.

In Chapter 5, we constructed nonstandard finite difference scheme for the convection-diffusion problem. We both analyzed standard and nonstandard finite difference approximations. We have observed that nonstandard finite difference method works better than the standard finite difference method for all choice of h and ϵ. Although, both methods have been in a good agreement away from the boundary layer, nonstandard finite difference method has done better job near the boundary layer.

In Chapter 6, we constructed the nonstandard finite difference scheme for nonlinear reaction-diffusion partial differential equation. This new scheme had the correct fixed-points, satisfies both the positivity and boundedness conditions and easy to implement for obtaining numerical solutions since the scheme is effectively explicit. Numerical studies indicated that the derived nonstandard scheme converges to the exact solution better than the standard finite difference discretization of the equation.

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APPENDIX A

COMPUTER SOFTWARE

Here, the Matlab codes we wrote to solve the dynamic equation is shown.

```
clear all n=input('enter n= '); h=input('enter h= ');

% INITIAL VALUES

y(1)=0.5;

z(1)=0.5;

% DENOMINATOR FUNCTION

a(n)=(1-exp(-n*h))/n;

% THE CASE OF STEP SIZES

if h==0.01
   w2=10;

else if h==0.1
        w2=10;

else if h==0.2
        w2=19;

else if h==0.5 | h==0.7
            w2=26;

else if h==0.65 | h==0.8
```

```
            w2=26;
else if h==1 | h==1.2
            w2=26;

else if h==1.5 | h==2
            w2=46;

else if h==2.5
            w2=51;

else break

        end
           end
              end
                 end
                    end
                       end
                          end
                             end
w1=1;

m=((w2-w1)/h);

x=w1:h:w2;

% EXACT EQUATION

z1=(1+(2.^n-1)*exp(n*(1-x))).^(-1/n);

for k=1:m
```

```
% STANDARD EQUATION

y(k+1)=y(k)*(1+h*(1-y(k).^n));

% NON-STANDARD EQUATION
z(k+1)=((1+a(n))*z(k))/(1+a(n)*(z(k).^n));
end
er1=abs(z1-y);
er2=abs(z1-z);
%  PLOTS
figure(1)
plot(y,'g')
hold on
plot(z,'bo')
hold on
plot(z1,'r')
legend('standard','non-standard')
title('{y}_t = y(1-y^n)')
x label('k-grid');
y label('y(k) ');
figure(2)
plot(x,er1,'r',x,er2)
legend('error-standard','error-non-standard')
title('{y}_t =y(1-y^n)')
```

Here, the Matlab codes we wrote to solve the convection-diffusion equation is shown.

```
% eps.y''+y'=-1
%  y(0)=0;
%  y(1)=0;
```

```
clear all format long n=50; L=1; h=L/n; e=1; h=L/n; w=-(1/e);
b=-2-(h/e); c=1; a=1+(h/e); w1=-(1/e); b1=-1-exp(w*h);
c1=exp(w*h);

% -a*y(k+1)+b*y(k)+c*y(k-1);
%solution of linear equation as AU=F,SET MATRIX A1
for i=1:n-1;
for j=1:n-1;
if i==j;
     A(i,j)=b;
else if i==j+1;
     A(i,j)=c;
else if i==j-1;
     A(i,j)=a;
else parity=0;
end
   end
      end
         end
            end
      % set matrix F
for i=1:n-1;
for j=1:1;
F(i,j)=(-h*h)/e;
     end
        end
for i=1:n-1;
for j=1:n-1;
if i==j;
     B(i,j)=b1;
else if i==j+1;
     B(i,j)=c1;
```

```
else if i==j-1;
     B(i,j)=1;
else parity=0;
end
   end
      end
         end
            end
      % set matrix F
for i=1:n-1;
for j=1:1;
F1(i,j)=(exp(w*h)-1)*h;
     end
        end
u11=A/F;
u21=B/F;
U11(1)=0;
U11(n+1)=0;
U21(1)=0;
U21(n+1)=0;
for i=2:n;

      U11(i)=u11(i-1);
      U21(i)=u21(i-1);

end

 x=[0:h:1];
 %exact
 z1=(-x)+(1-exp(-x/e))/(1-exp(-1/e));

  x1=[0:h:1];
```

```
%error analysis
er11=abs(z1-U11);
er21=abs(z1-U21);figure(1)
plot(x1,U11,'r',x1,U21,'bo',x,z1,'g');
title('One Dimensional Convection Diffusion Equation')
legend('\bf{standard}','\bf{non-standard}','\bf{exact}');
x label('\bf{x axis}');
y label('\bf{y axis}');
axis([0 1 0 1 ])
figure(2)
plot(x1,er11,'r',x1,er21,'bo');
title('One Dimensional Convection Diffusion Equation')
legend('Error between Exact and Standard','Error between

Exact and Nonstandard');
x label('\bf{x axis}');
y label('\bf{y axis}');
```

Here, the Fortran codes we wrote to solve the reaction-diffusion equation is shown.

```
*      u(t)=u(xx)-(u-A1)(u-A2)(u-A3)
*      STANDARD
*      A1=-1  A2=0  A3=1
*      DX=0.1
*      DT=0.001
*      H=0.1


       IMPLICIT DOUBLE PRECISION(A-H,O-Z)
       DOUBLE PRECISION X(1500),T(6500),S(1500,6500)
       DOUBLE PRECISION E(1500,6500),U(1500,6500),F(1500,6500)
       DOUBLE PRECISION G(1500,6500)
```

```
       OPEN(11,FILE='data1.txt',STATUS='UNKNOWN')
       OPEN(12,FILE='data2.txt',STATUS='UNKNOWN')
       OPEN(13,FILE='data3.txt',STATUS='UNKNOWN')
       OPEN(14,FILE='data4.txt',STATUS='UNKNOWN')
       OPEN(15,FILE='data5.txt',STATUS='UNKNOWN')
       OPEN(16,FILE='data6.txt',STATUS='UNKNOWN')

      A1=-1.0D0
      A2=0.0D0
      A3=1.0D0
      L=5.0D0
      M=100.0D0

      DX=(2.0D0*L) / DFLOAT(M)
      DT=0.0010D0
      T(1)=0.0D0
      R=((6-T(1))/DT)+1
*************** T(N)LERI BULMA ******************
      DO 30 N=2,R
      T(N)=T(1)+(N-1)*DT

   30 CONTINUE
*************     X(I) LARI BULMA       ***************
      X(1)=-L
      DO 5 I=2,M+1
      X(I)=X(1)+(I-1)*DX

    5 CONTINUE
*     BOUNDARY CONDITION

      DO 10 N=1,R
```

```
      S(1,N)=-1.0D0
      S(M+1,N)=1.0D0
      U(1,N)=-1.0D0
      U(M+1,N)=1.0D0
      E(M+1,N)=1.0D0
      E(1,N)=-1.0D0
   10 CONTINUE
      C=(L/DX)+1
      X(C)= 0.0D0
*     INITIAL CONDITION
      DO 20 I=1,M+1
      IF (X(I).LT.0.0D0) THEN
      S(I,1)=-1.0D0
      U(I,1)=-1.0D0
      E(I,1)=-1.0D0
      ELSE
      S(I,1)=1.0D0
      U(I,1)=1.0D0
      E(I,1)=1.0D0
      END IF
  20  CONTINUE
      DO 36 N=1,R-1
      DO 42 I=2,M
*      STANDART EQUATION(1)
       S(I,N+1)=S(I,N)+((DT)/(DX*DX))*(S(I+1,N)
     --2.0D0*S(I,N)+S(I-1,N))
     --(DT)*(S(I,N)-A1)*(S(I,N)-A2)*(S(I,N)-A3)
*****************************************************
*       NON-STANDARD EQUATION(1)
        U(I,N+1)=(U(I,N)+((1.0D0-EXP(-2.0D0*DT))
     //(8.0D0*SIN(DX*0.50D0)*SIN(DX*0.50D0)))*(U(I+1,N)
```

```
--2.0D0*U(I,N)+U(I-1,N))
++(0.250D0*(1.0D0-EXP(-2.0D0*DT))*U(I-1,N)*U(I-1,N)*U(I-1,N))
--(0.250D0*(1.0D0-EXP(-2.0D0*DT))*U(I-1,N)*U(I-1,N)*(A1+A2+A3))
--(0.50D0*(1.0D0-EXP(-2.0D0*DT))*U(I-1,N)*((A1*A2)+(A1*A3)+(A2*A3)))
++(1.0D0-EXP(-2.0D0*DT))*(A1*A2*A3)))
//(1.0D0+(0.750D0*(1.0D0-EXP(-2.0D0*DT))*U(I-1,N)*U(I-1,N))
-- (0.750D0*(1.0D0-EXP(-2.0D0*DT))*U(I-1,N)*(A1+A2+A3)))

****************************************************************************
42 CONTINUE

36

CONTINUE

   DO 60 I=1,M+1
   WRITE(11,*) X(I),S(I,1),U(I,1)
   WRITE(12,*) X(I),S(I,1201),U(I,1201)
   WRITE(13,*) X(I),S(I,2401),U(I,2401)
   WRITE(14,*) X(I),S(I,3601),U(I,3601)
   WRITE(15,*) X(I),S(I,4801),U(I,4801)
   WRITE(16,*) X(I),S(I,6001),U(I,6001)
60 CONTINUE
   WRITE(*,*) 'BITTI'
   STOP
   END
```