SPACETIME COMPACTIFICATION INDUCED BY SCALAR FIELDS

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ABSTRACT

SPACETIME COMPACTIFICATION INDUCED BY SCALAR FIELDS

This thesis work is devoted to a discussion of spacetime compactification via scalar fields. We first provide an introduction to basic concepts and mechanisms, and review existing compactification methods. We then review and discuss spacetime compactification triggered by non-linear sigma model fields.

We study spacetime compactification via a single scalar field by requiring scalar field in higher dimensions to gravitate only in a subset of spacetime dimensions. For this purpose we first review fully non-gravitating scalar field configurations and then determine conditions and mechanisms for obtaining a partially gravitating scalar field. In each case Ricci and hence energy-momentum tensor of the scalar field vanishes completely or partially though this does not imply or require scalar field itself to vanish.

By making use of the partially-gravitating scalar fields, we discuss how spacetime dimensions get compactified if the scalar field gravitates in those dimensions, only. We illustrate how this mechanism works in special cases, like generating a constant-curvature manifold of extra dimensions.

ÖZET

SKALER ALANLAR YARDIMIYLA UZAYZAMAN KOMPAKTİFİKASYONU

Bu tez çalışması, skaler alanlar yolu ile uzayzaman kompaktifikasyonunun bir tartışması olarak hazırlanmıştır.Öncelikle, temel içerikler ve mekanizmalar için bir giriş yaptık ve varolan kompaktifikasyon metotlarını gözden geçirdik. Daha sonra, lineer olmayan sigma model alanlarını gözden geçirip, tartıştık.

Bir skaler alan yolu ile uzayzaman kompaktifikayonunu çalıştık öyle ki bunun için uzayzamanın sadece bir alt kümesinde çekim alanına katkıda bulunan yüksek boyutlardaki skaler alana gerek vardır. Bunun için, ilk olarak çekim alanına tamamen katkıda bulunmayan skaler alan biçimlenimlerini gözden geçirdik ve daha sonra çekim alanına kısmen katkıda bulunan bir skaler alan elde etmek için koşullar ve mekanizmalar belirledik. Herbir durumda, Ricci tensörü ve dolayısıyla skaler alanın enerji-momentum tensörü tamamen ya da kısmen yok olur fakat bu skaler alanın kendisinin yokolması anlamına gelmez ya da bunu gerektirmez.

Çekim alanına kısmen katkıda bulunan skaler alanları kullanarak, eger skaler alan sadece kompaktife olan boyutlarda çekime katkıda bulunuyorsa, uzayzaman boyutları nasıl kompaktife olur durumunu tartıştık. Bu mekanizmanın özel durumlarda, ekstra boyutların sabit egrilikli bir manifoldunu oluşturarak, nasıl çalıştıgını gösterdik.

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CHAPTER 1

INTRODUCTION

A true understanding of Nature will be achieved when ideas such as matter, space, time and the number of dimensions we live in are explained. Paul Ehrenfest in (1917) pointed out that the equations which describe the motion of electrons bound to nuclei in atoms and the planets around the sun had stable solutions only in three space dimensions. So it can be inferred from this result that the solar system and atoms can conserve their structure only if we live in three space dimensions.

A German mathematician Theodor Kaluza pointed out () that if Einstein's theory of general relativity is extended to a five-dimensional (5D) spacetime, the equations can be separated out into ordinary 4D gravitation plus an extra set, which is equivalent to Maxwell's equations for the electromagnetic field, plus an extra field known as dilaton. So in his programme electromagnetism is explained as a manifestation of curvature in an extra dimension of physical space, in the same way that gravitation is explained in the theory of general relativity as a manifestation of curvature in the first three dimensions. There is only one force, gravity, in the theory and since it is a universal force it effects everything. The component of the gravitational field in the direction of the fifth (extra) dimension obeyed the same equation as the electromagnetic field, that is to say, the gravitational force in the fifth direction was the electromagnetism that we knew. Then a Swedish physicist Oskar Klein proposed (Klein 1926) that the reason the extra spatial dimension goes unseen is that it is compact (curled up) like a ball with a fantastically small radius. He assumed the extra dimension to be compact instead of assuming total independence from it. So, the extra dimension would have the topology of a circle, with a radius of the order of the Plank length. The 5D spacetime has the topology $\mathrm{M}^4 \times \mathrm{S}^1$ where M^4 is the usual Minkowski spacetime, and S^1 is a circle on which the extra coordinate takes values.

Since then a wealth of higher-dimensional theories have been motivated by Kaluza-Klein programme (Overduin and Wesson 1997). The extra dimensions are assumed to roll up to form a sufficiently small and compact space in the theories that we mentioned above and in many other ones. The present experiments tell us that the characteristic size of the extra dimensions can vary from Planck length up to a few *mm* (Long *et. al.*1999).

It is important to have an understanding of how or why extra dimensions differed so markedly in size and topology from the ordinary four dimensions. In other words, it is necessary to find the dynamical mechanism that leads to compactification of the extra dimensions, that is, the higher dimensional space assumes the topology of ordinary four-dimensional spacetime of macroscopic dimensions times a compact manifold of extra dimensions. Namely, the configuration $\mathrm{M}^4 \times \mathrm{E}^D$ should be an energetically-preferred solution of the higher-dimensional Einstein equations. This programme has been studied by utilizing higher-curvature gravity (Wetterich 1982, Mueller-Hoissen 1985) and by coupling Einstein gravity to matter in a judicious way. The latter leads to spontaneous compactification of extra dimensions was pointed out in (Cremmer and Scherk 1976, Cremmer and Scherk 1977). Spontaneous compactification has been realized with Yang-Mills fields (Randjbar-Daemi and Percacci 1982), antisymmetric tensor fields (Freund and Rubin 1980), sigma model fields (Gell-Mann and Zwiebach 1984) and conformally-coupled scalars (Gerard *et. al.* 1984). In each case, components of the Ricci tensor are balanced by those of the stress tensor, and depending on the structure of the latter a subset of dimensions are found to get compactified.

In this thesis work, we are interested in the dynamical compactification induced by scalar fields. The role of scalars in dynamical compactification process was first analyzed in (Omero and Percacci 1980, Gell-Mann and Zwiebach 1984, Gell-Mann and Zwiebach 1985) where a D–dimensional minimally-coupled non-linear sigma model with metric $h_{ij}(\phi)$ $(i, j = 1, \ldots, D)$ was shown to lead to a dynamical compactification of D extra dimensions provided that sigma model metric is in the Riemannian metric form. In other words, equations of motion for the metric field requires the Ricci tensor $\mathcal{R}_{ij}$ to be proportional to $h_{ij}(\phi)$, and thus, D-dimensional extra space relaxes to the geometry of the sigma model. The remaining dimensions x^μ $(\mu = 0, 1, \ldots)$ span a strictly flat Minkowski space. That this set-up compactifies the extra dimensions y^i becomes especially clear with the ansatz $\phi^i = y^i$ or any function of y^i is used (Gell-Mann and Zwiebach 1984).

Obviously, the space of extra dimensions may (Cremmer and Scherk 1976, Cremmer and Scherk 1977, Randjbar-Daemi and Percacci 1982, Omero and Percacci 1980, Gell-Mann and Zwiebach 1984, Gell-Mann and Zwiebach 1985) or may not (Gell-Mann and Zwiebach 1984, Gell-Mann and Zwiebach 1985, Randjbar-Daemi and Wetterich 1986) form a compact space. The extra dimensions can possess negative curvature yet they can still be compact (Demir and Shifman 2002). Moreover, the geometry does not need to be factorizable (Randall and Sundrum 1999). In general, shape and topology of the extra space are entirely determined by the mechanism of dynamical compactification.

In this thesis work we discuss yet another compactification mechanism (Demir and Pulice 2006) induced by scalar fields. We will show that a single scalar field living in a higher dimensional spacetime can lead to dynamical compactification of the extra dimensions without inducing a classical cosmological constant when it gravitates only in those dimensions which are to be compactified. In what follows we will first review the Kaluza-Klein programme and the method of compactification induced by scalar fields (Gell-Mann and Zwiebach 1984) in the next chapter.

For a coherent discussion of the compactification mechanism by (Demir and Pulice 2006), it is necessary to show first that a strictly flat spacetime supports non-trivial scalar field configurations (Ayon-Beato *et. al.* 2005). This we will do in Chapter 3. The next step is to show the compactification of the extra dimensions into a D–dimensional manifold, and this we will show in Chapter 4. Following that, we will see the details of the compactification mechanism in chapter 5. We will conclude in Chapter 6. There are some reviews on curvature tensor ('t Hooft 2002, Landau and Lifshitz 1975) and energy-momentum tensor (Wienberg 1971) which are reviewed for completeness in Appendix A and B.

CHAPTER 2

THE KALUZA-KLEIN PROGRAMME

In the early 20th century the theory of electromagnetism was complete and well understood, and the theory of gravitation had just been completed by Einstein. Einstein used Riemannian differential geometry and formulated his theory in four dimensions by admitting generally relativistic transformations among spacetime coordinates. Then it was natural to try to unify electromagnetism and gravity inspired by Maxwell's unification of electricity and magnetism as well as Einstein's unification of time and space. Several people like Nordstr*ö*m in 1914, Weyl in 1918 and Kaluza in 1921 attempted to accomplish this. As he was working before general relativity, Nordstr*ö*m assumed a scalar gravitational potential while Weyl and Kaluza used Einstein's tensor potential.

Kaluza realized (1919) that Einstein's four-dimensional theory of gravity and Maxwell's theory of electromagnetism can be unified into five-dimensional general relativity. This theory was entirely a classical theory with the only goal of deriving both Maxwell's equations for electromagnetism and Einstein's general relativity from $5D$ general relativity. He further assumed that, because no empirical evidence so far pointed towards a fifth dimension, the field variable would not depend on the fifth coordinate, but only on the four coordinates of the ordinary space-time continuum. He achieved this by having some sort of axis of symmetry around which the fifth coordinate is measured, that is to say, making the metric cylindrical. However, after that Klein provided an important contribution (1926) where he accomplished compactification of the fifth dimension. He assumed that the extra dimension was microscopically small.The idea of Klein has motivated field theorists to unify the long-range and short-range interactions of physics in higher dimensions. So a wealth of higher-dimensional theories (such as eleven-dimensional supergravity theories of 1980s and ten-dimensional superstrings) has been motivated by this idea.

The Kaluza-Klein programme is essentially general relativity in five dimensions with imposition certain constraints so as to take into account the facts that extra dimensions differ markedly in size and topology from the ordinary four dimensions, and we seem to perceive only a four-dimensional continuum. More explicitly, the fundamental assumptions included:

a) *Cylinder condition*: The first condition which consists in setting all partial derivatives with respect to the fifth coordinate (x^4) to zero was introduced by Kaluza. He assumed this condition because no empirical evidence (even today) was found in favor of a fifth dimension; the field variable would not depend on the fifth coordinate, but only on the four coordinates of an ordinary space-time continuum. In other words, physics was to take place in a four-dimensional hypersurface in a five-dimensional universe. This condition reduced the algebraic complexity of the programme to a manageable level.

b) *Compactification Condition*: This condition, brought about by Klein, has rectified the cylinder condition above. Indeed, instead of full independence of fields from the extra dimension, Klein has made use of a compact topology (for only one extra dimension it is a circle) so that fields admitted a zero-mode (a mode which is completely independent of the extra coordinate) followed by higher harmonics (which may not be accessible at ordinary energies) was introduced by Klein. The Klein's idea was convincing in that extra dimensions do not appear in physics in four dimensions because they are compactified and unobservable in experimentally accessible energy scales (even today). Given absence of deviations from Newtonian gravitational attraction within present-day experimental precision, the characteristic size of the extra dimensions can vary from Planck length up to a few *mm* (Long 1999).

2.1. Compactification *á la* Kaluza

The field equations of both electromagnetism and gravity were obtained from a single five-dimensional theory with the assumption of *cylinder condition*. Kaluza demonstrated that five-dimensional general relativity in vacuum (*i.e.*, $\hat{G}_{AB} = 0, A, B = 0, 1, 2, 3, 4$) contained four-dimensional general relativity plus electromagnetic field (*i.e.*, $G_{\alpha\beta} = T^{EM}_{\alpha\beta}, \alpha, \beta = 0, 1, 2, 3$). The vacuum Einstein equations in five dimensions are given by:

$$\hat{G}_{AB} = 0 \tag{2.1}$$

or equivalently

$$\hat{R}_{AB} = 0 \tag{2.2}$$

where $\hat{G}_{AB} \equiv \hat{R}_{AB} - \frac{1}{2}\hat{R}\hat{g}_{AB}$ is the Einstein tensor, $\hat{R}_{AB}$ and $\hat{R} = \hat{g}_{AB}\hat{R}^{AB}$ are the five-dimensional Ricci tensor and Ricci scalar, respectively, and $\hat{g}_{AB}$ is the five-dimensional metric tensor, (A,B=0,1,2,3,4 and the quantities with hat are five-dimensional). The absence of matter sources in the field equations shows one of Kaluza's assumptions which was inspired by Einstein: *the universe in higher dimensions is empty.* This idea explains matter (in four dimensions) as a manifestation of pure geometry (in higher dimensions), that is to say, four-dimensional matter arises purely from the geometry of empty five-dimensional spacetime. Though emptiness of the higher dimensions sounds appealing, there is no solid ground to prove this. Therefore, a more general setting that replaces (2.1) would be to use $\hat{G}_{AB} = k\hat{T}_{AB}$ where k is a constant and $\hat{T}_{AB}$ is a five-dimensional energy-momentum tensor (of matter living in five dimensions).

The field equations can be derived from a five-dimensional version of the usual Einstein-Hilbert action by taking the variation of it with respect to the five-dimensional metric. The action is

$$S = \frac{1}{16\pi\hat{G}} \int \hat{R}\sqrt{-\hat{g}}d^4xdy \tag{2.3}$$

where $y = x^4$ is the fifth coordinate and $\hat{G}$ is five-dimensional gravitational constant.

The five-dimensional Ricci tensor and Christoffel symbols are defined in terms of the metric as in general number of dimensions:

$$\begin{aligned} \hat{R}_{AB} &= \partial_C\hat{\Gamma}^C_{AB} - \partial_B\hat{\Gamma}^C_{AC} + \hat{\Gamma}^C_{AB}\hat{\Gamma}^D_{CD} - \hat{\Gamma}^C_{AD}\hat{\Gamma}^D_{BC} \\ \hat{\Gamma}^C_{AB} &= \frac{1}{2}\hat{g}^{CD}(\partial_A\hat{g}_{DB} + \partial_B\hat{g}_{DA} - \partial_D\hat{g}_{AB}) \end{aligned} \tag{2.4}$$

It is now important to determine the form of the five-dimensional metric. There are 15 relations that serve to determine the 15 $\hat{g}_{AB}$ but this is only possible by making some educated guess about $\hat{g}_{AB}$. Kaluza was interested in Maxwell's theory and realized that $\hat{g}_{AB}$ could be expressed in a way involving the four-dimensional vector potential A_α which is the one in Maxwell's theory.He adopted the cylinder condition and took the 44-component of the metric constant, g_{44} =constant.

In general, the $\alpha\beta$-part of $\hat{g}_{AB}$ is described by $g_{\alpha\beta}$ (the four dimensional metric tensor), the $\alpha 4$-part is defined with A_α (the electromagnetic potential) and the 44-part is defined with Φ (a scalar field). So the five-dimensional metric tensor can be decomposed as:

$$\hat{g}_{AB} = \begin{bmatrix} g_{\alpha\beta} - \kappa^2\phi^2 A_\alpha A_\beta & -\kappa\phi^2 A_\alpha \\ -\kappa\phi^2 A_\beta & -\phi^2 \end{bmatrix} \tag{2.5}$$

where κ is a coupling constant and Greek indices $\alpha\beta$ run over 0,1,2,3, 0 designating the temporal component. If the cylinder condition is used, that is to say, if all derivatives with respect to the fifth coordinate drop out, the five-dimensional field equations (2.2) reduce to the equations below:

$$G_{\alpha\beta} = \frac{\kappa^2\phi^2}{2} T^{EM}_{\alpha\beta} - \frac{1}{\phi}[\nabla_\alpha\nabla_\beta\phi - g_{\alpha\beta}g^{\mu\nu}\nabla_\mu\nabla_\nu\phi] \tag{2.6}$$

$$\nabla^\alpha F_{\alpha\beta} = -3\frac{\nabla^\alpha\phi}{\phi}F_{\alpha\beta} \tag{2.7}$$

$$g^{\alpha\beta}\nabla_\alpha\nabla_\beta\phi = -\frac{\kappa^2\phi^3}{4}F_{\alpha\beta}F^{\alpha\beta} \tag{2.8}$$

where

$$G_{\alpha\beta} \equiv R_{\alpha\beta} - \frac{1}{2}Rg_{\alpha\beta} \tag{2.9}$$

is the Einstein tensor,

$$T^{EM}_{\alpha\beta} \equiv \frac{1}{4}g_{\alpha\beta}F_{\gamma\delta}F^{\gamma\delta} - F^\gamma_\alpha F_{\beta\gamma} \tag{2.10}$$

is the electromagnetic energy-momentum tensor,and

$$F_{\alpha\beta} \equiv \partial_\alpha A_\beta - \partial_\beta A_\alpha \tag{2.11}$$

is the field strength tensor of electromagnetism.

We can see that Kaluza succeeded in unifying electromagnetism and gravity, we recover not only the Einstein equations but also the Maxwell equations as well as a Klein-Gordon equation for the massless scalar field ϕ. He showed that five dimensional general relativity in vacuum contains both four-dimensional general relativity in the presence of an electromagnetic field as well as Maxwell's laws for electromagnetism. In short, Kaluza-Klein theory is in general a unified account of gravity, electromagnetism and a scalar field.

However the presence of the scalar field influences the theory. If it is set to being a constant throughout spacetime (Kaluza set $\phi = 1$) then the first two field equations (2.6) and (2.7) reduce to Einstein and Maxwell equations:

$$G_{\alpha\beta} = 8\pi G T^{EM}_{\alpha\beta} \tag{2.12}$$

$$\nabla^{\alpha} F_{\alpha\beta} = 0 \tag{2.13}$$

where the scaling parameter κ is identified in terms of the four-dimensional gravitational constant G as $\kappa \equiv (16\pi G)$. This is the result that was obtained by Kaluza and Klein by setting $\phi = 1$ as we mentioned before. However, it was first pointed out by Jordan and Thiry that the condition $\phi =$ constant is consistent with the third of the field equations (2.8) if one sets $F_{\alpha\beta}F^{\alpha\beta} = 0$.

By using the metric decomposition in (2.5) and definitions of curvature tensor in five dimensions (2.4), one can see that (2.3) contains three terms. By pulling $\int dy$ out of the action while making use of the cylinder condition (dropping derivatives with respect to y) one finds:

$$S = \int d^4x \sqrt{-g}\phi\Big(\frac{R}{16\pi G} + \frac{1}{4}\phi^2 F_{\mu\nu}F^{\mu\nu} + \frac{2}{3\kappa^2}\frac{\partial^{\mu}\phi\partial_{\mu}\phi}{\phi^2}\Big) \tag{2.14}$$

where the five-dimensional gravitational constant $\hat{G}$ is defined in terms of the four-dimensional one G by:

$$\hat{G} \equiv G \int dy \tag{2.15}$$

which expresses that the fundamental scale of gravity in four-dimensions ($M^2_{Pl} = 8\pi G$) equals the fundamental scale of gravity in five dimensions $M^3_5 = 8\pi\hat{G}$ times the volume of the extra space $V_{extra} = \int dy$. More explicitly, $M^2_{Pl} \equiv M^3_5 V_{extra}$.

The action (2.14) is seen to consist of Einstein and Maxwell actions which couple to a scalar field ϕ coming from the 44 component of the metric. The ϕ field in (2.14) is seen to possess a non-minimal kinetic term, and this makes it a perfect candidate from Brans-Dicke (Brans 1961) type scalar field. The Brans-Dicke models provide a general parametrization of gravity plus scalar field theories *i.e.* scalar-tensor theories of gravity. To exemplify such theories more clearly, set $A_\mu = 0$ in (2.14) then (2.5) reduces to

$$\hat{g}_{AB} = \begin{bmatrix} g_{\mu\nu} & 0 \\ 0 & \phi^2 \end{bmatrix} \tag{2.16}$$

which exhibits a completely decoupled structure between the four-dimensional sector and the fifth dimension. For (2.16), the action (2.3) takes the form

$$S = \frac{1}{16\pi G} \int d^4x \sqrt{-g} R\phi \tag{2.17}$$

which is nothing but the special case of the Brans-Dicke action with vanishing Brans-Dicke constant ($\omega = 0$)

$$S_{BD} = \int d^4x \sqrt{-g} \Big(\frac{R\phi}{16\pi G} + \omega \frac{\partial^\mu \phi \partial_\mu \phi}{\phi} \Big) + S_m \tag{2.18}$$

where S_m stands for matter fields that may be coupled to the metric and scalar field. We here note that the Brans-Dicke constant ω is forced to take very large values by present-day experiments on the moon.

2.2. Compactification *á la* Klein

The Swedish theoretical physicist Oscar Klein made a major contribution to Kaluza's theory in 1926 by making a physical explanation for Kaluza's cylinder condition. In a sense, Klein has rectified the trivial-looking cylinder condition of Kaluza. Klein assumed that the fifth coordinate has:

1. a small size

2. a circular topology(S^1)

which hand-shake in shaping the experimental and phenomenological studies of extra dimensions. Under these properties, in particular, the second one any field existing in 5-dimensional bulk possesses is periodic in the fifth dimension. In other words,

$$f(x^\mu, y) = f(x^\mu, y + 2\pi r) \tag{2.19}$$

where r is the radius of S^1, and $\mu = 0, 1, 2, 3$ as usual. Here f is a generic field in 5-dimensional space. This periodicity condition is satisfied if f admits a Fourier expansion

$$f(x^\mu, y) = \sum_{n=-\infty}^{n=+\infty} f_n(x^\mu) e^{in\frac{y}{r}} \tag{2.20}$$

from which we infer that:

1. The lowest mode $n = 0$ (the so-called zero-mode) is independent of y. This stems directly from the compact nature of the extra space *i.e.* S^1 on which the extra dimension y extends.

2. Higher modes $n \neq 0$ (the so-called higher harmonics) depend explicitly on y with a wavelength (or equivalently inverse-mass, in natural units) $\lambda_n = r/n$ for n-th harmonic.

3. The x^μ-dependence of $f(x^\mu, y)$ does not exhibit any periodicity at all because these macroscopic dimensions are not compact; they extend to infinity in both directions. Thus, the energy spectrum of $f(x^\mu, y)$ in four-dimensional space-time can be extracted via Fourier integral rather than Fourier series.

It is clear that Item 1 above, the one about the zero-mode, comprises Kaluza's cylinder condition. The Item 2 tells us that higher harmonics can be hidden from present-day experiments as they may not have reached yet the energies $\sim n/r$ which is the main reason behind assuming 'r' small.

The Fourier expansion (2.20) is valid for any bulk field such as the components of the five-dimensional metric tensor (2.5):

$$g_{\mu\nu}(x, y) \quad = \quad \sum_{n=-\infty}^{n=\infty} g_{\mu\nu}^{(n)}(x) e^{\frac{iny}{r}}$$

$$
\begin{aligned}
A_\mu(x,y) &= \sum_{n=-\infty}^{n=\infty} A_\mu^{(n)}(x) e^{\frac{iny}{r}} \\
\phi(x,y) &= \sum_{n=-\infty}^{n=\infty} \phi^{(n)}(x) e^{\frac{iny}{r}}
\end{aligned}
\tag{2.21}
$$

where each Kaluza-Klein mode $g_{\mu\nu}^{(n)}(x)$ or $A_\mu^{(n)}(x)$ or $\phi^{(n)}(x)$ carries a momentum $p_n = n/r$ along the extra dimension. The zero-mode has, of course, no momentum into the extra space.

The zero-modes of $g_{\mu\nu}(x,y)$, $A_\mu(x,y)$ and $\phi(x,y)$ are nothing but the fields which have already been established by experiments *i.e.* photon, quarks, leptons *etc.*, that is, they are the fields which are strictly bound to live in M^4. On the other hand, their higher harmonics do have a sinusoidal extension into the extra space with a wavelength decreasing with increasing Kaluza-Klein index, n. For instance, to be able to disentangle effects of $g_{\mu\nu}^{(9)}(x)$ on a physical process it is necessary to have a collider with a characteristic energy $\sim 9/r$ apart from additional effects that might come from strength of its coupling to colliding matter species.

In general, taking $\ell_{Pl} \sim 10^{-35}$ m to be the smallest scale above which one can have a sensible notion of field theories, one can take size of the extra dimensions, r, to lie from ℓ_{Pl} up to a few mm where the latter follows from experimental studies on deviations from Newton's law of attraction (Long 1999).

The higher-dimensional unification, in the sense of metric decomposition (2.5) and action (2.14) there are three key features:

1. The electromagnetic and gravitational fields are contained in the higher dimensional Eintein tensor ${}^{(4+D)}G_{AB}$, that is, in the metric and its derivatives. Therefore, there is no need to have an explicit source of energy and momentum ${}^{4+D}T_{AB}$. In this sense, matter species in four-dimensions, follow from pure geometry.

2. The mathematical structure of higher dimensional theories are the same as Einstein gravity in four dimensions. The only change is that tensor indices run over 0 to $(3+D)$ instead of 0 to 3.

3. The simple S^1 compactification illustrated above or its direct extension to

more extra dimensions assume that extra dimensions are curled up to form a compact manifold, in other words there is a cylindrical structure with respect to the extra dimensions. However, this procedure does not explain or attempt at explaining why and how such a geometrical structure has emerged dynamically.

The third feature above has always been regarded as some problematic aspect of the existing compactification schemes, and several approaches have been devised to put dynamical origin of geometrical structure into a solid ground. In other words, it is necessary to find the dynamical mechanism that leads to compactification of the extra dimensions, that is, the macroscopic four-dimensional spacetime times the compact manifold of extra dimensions should be an energetically-preferred solution of the higher-dimensional Einstein equations. As we mentioned in the introduction part, there are several methods for compactification of extra dimensions. Out of all those methods, in the next section, we will discuss spontaneous spacetime compactification method of Gell-Mann and Zwiebach in order to study roles of scalar fields in compactification. This section will prove useful for analyses in the following chapters.

2.3. Compactification *á la* Gell-Mann and Zwiebach

This method shows that scalar fields of a nonlinear sigma model coupled to gravity can trigger spontaneous compactification of spacetime if the scalar manifold has an Einstein metric and the scalar self-coupling constant takes a specific value. It is then possible to obtain a flat (actually, Ricci-flat) four-dimensional spacetime times a curved manifold of extra dimensions as the geometrical structure.

The action density consists of Einstein gravity coupled to a non-linear sigma model in $(4 + D)$ dimensions:

$$S = \frac{1}{2}\int d^{4+D}x\sqrt{-g}\left\{-\frac{1}{2}M_\star^{D+2}R + \frac{g^{\mu\nu}}{\lambda^2}h_{ij}(\phi)\partial_\mu\phi^i\partial_\nu\phi^j\right\} \tag{2.22}$$

where $M_\star$ is the fundamental scale of gravity in $(4 + D)$ dimensions, and metric signature is taken to be $(-, +, +, ..., +)$. Here scalar fields ϕ^i $(i = 1, 2, \ldots, D)$ are regarded as coordinates of a D–dimensional Riemannian manifold E^D with metric $h_{ij}(\phi)$. It is clear that number of scalar fields ϕ^i equals the number of extra dimensions to be compactified.

The equations of motion for metric and scalar fields, as follows form (2.22), read as:

$$R_{\mu\nu} = \frac{2}{\lambda^2} h_{ij}(\phi)\partial_\mu \phi^i \partial_\nu \phi^j \tag{2.23}$$

$$\frac{2}{\sqrt{-g}}\partial_\mu(\sqrt{-g} g^{\mu\nu} h_{ij}\partial_\nu \phi^j) = g^{\mu\nu}\frac{\partial h_{pq}}{\partial \phi^i}\partial_\mu \phi^p \partial_\nu \phi^q \tag{2.24}$$

where the Greek indices $\alpha, \beta, \gamma, \ldots$ with values running from 0 to 3 denote curved vector indices in the physical 4-dimensional space, and indices $i, j, k, \ldots$ with values running from 1 to D denote curved vector indices in the extra space.

The compactification method of Gell-Mann and Zwiebach (Gell-Mann and Zwiebach 1984) is based on establishing a one-to-one relation between the scalar fields ϕ^i and extra dimensions x^i. In fact, they take

$$\phi^i(x) = x^i \tag{2.25}$$

where right-hand side may be replaced by any function $f(x^i)$ of the extra dimensions. This correspondence between scalar fields and extra dimensions enforces the system of equations (2.23,2.24) to have a specific solution enabling one to trade, effectively, scalars ϕ^i for extra dimensions and vice versa.

Under the ansatze (2.25), it is clear that (2.23) give rise to the solutions

$$R_{\alpha\beta} = 0 \tag{2.26}$$

$$R_{ij}(\bar{g}) = \frac{2}{\lambda^2 M_\star^{D+2}} h_{ij}(\phi) \tag{2.27}$$

which explicitly leads to a background geometry with (metric $\bar{g}$) a Ricci-flat M^4 and curved (in a way similar to the manifold of scalar fields themselves) extra space E^D.

It is clear that (2.27) is easily solved if $h_{ij}(\phi)$ is a positive-signature Riemannian metric for a positively curved manifold E^D, that is, if there is a real constant α such that

$$R_{ij}(h(\phi)) = \alpha^2 h_{ij}(\phi) \tag{2.28}$$

so that it suffices to take $\bar{g}_{ij} = -a^2 h_{ij}$ (with real a) to find that $\lambda^2 = 2M_\star^{D+2}/\alpha^2$ to satisfy (2.27). This relation determines the self-coupling in kinetic term of the scalar fields in terms of the curvature of the internal manifold.

From this analysis one concludes that extra dimensions roll up to form a manifold having the same shape as the manifold of sigma model fields ϕ^i. The metric of the manifold $\bar{g}_{ij}$ is determined in terms of the Riemannian metric of scalars h_{ij} up to a constant $-a^2$ which cannot be determined by the classical equations of motion.

One notes that the solution above does not admit existence of a scalar self-interaction potential $V(\phi)$. One is also reminded of the fact that a compact manifold cannot, in general, be covered by a single coordinate patch. For this reason, the scalar fields are set equal to the extra coordinates over patches.

For scalar fields living on a 2–sphere, the parameter λ^2 is calculated to be 1/2. This number agrees with results of (Witten and Bagger 1982) which discusses consistency of S^2 sigma model with N=1 supergravity in four dimensions. This entails the conclusion that (Witten and Bagger 1982) realizes Gell-Mann–Zwiebach model in $(2+2)$ dimensions.

Our goal is to realize compactification via a single scalar field living in the bulk, and requisite preparatory work will be given in Chapter 3, below.

CHAPTER 3

FIELDS WITH VANISHING ENERGY-MOMENTUM TENSORS

In general, vector fields and fermions are to vanish for them to have vanishing energy-momentum tensor. Naively, one expects this to hold also for scalar fields. However, as we will study in detail in this chapter, the scalar fields can have non-trivial configurations with vanishing energy-momentum tensors.

Let us consider a real scalar field ϕ living in a $(4+D)$–dimensional spacetime with coordinates $z^A = (x^\mu, y^i)$ where $\mu = 0, \dots, 3$ and $i = 1, \dots, D$. Keeping the gravitational sector minimal, the most general action integral takes the form

$$S = \int d^{4+D}z\, \sqrt{-g} \left\{ \frac{1}{2} M_\star^{D+2} \mathcal{R} - \frac{1}{2} g^{AB} \partial_A \phi\, \partial_B \phi - \frac{1}{2} \zeta \mathcal{R} \phi^2 - V(\phi) \right\} \tag{3.1}$$

where we have adopted $(-1, +1, \dots, +1)$ metric signature, and denoted the curvature scalar by $\mathcal{R}$ and fundamental scale of gravity by $M_\star$. There is no symmetry principle for avoiding direct coupling of ϕ to the curvature scalar, namely, a scalar field should always exhibit $\zeta \mathcal{R} \phi^2$ type interaction with Ricci scalar. The main exception here is Goldstone bosons. Indeed, Goldstone bosons of spontaneously broken continuous symmetries do not couple to curvature scalar directly.

Note that the scalar field theory in (3.1) exhibits conformal invariance when $V(\phi) \propto \phi^{4+D}$ and $\zeta = \zeta_{4+D}$, where

$$\zeta_{4+D} = \frac{D+2}{4\,(D+3)} \tag{3.2}$$

which equals $1/6$ for $D = 0$ and $1/4$ for $D = \infty$. The conformal invariance implies invariance of system under resizings.

The field equations, as usual, follow from (3.1) by the variational principle. We start analysis by computing the first variation of the action against variations in the metric field:

$$\delta S \;=\; \int d^{4+D}z \Bigg\{ \delta\sqrt{-g} \left(\frac{1}{2} M_\star^{D+2} \mathcal{R} - \frac{1}{2} g^{AB} \partial_A \phi \partial_B \phi - \frac{1}{2} \zeta \mathcal{R} \phi^2 - V(\phi) \right)$$

$$+ \quad \sqrt{-g}\left(\frac{1}{2}M_\star^{D+2}\delta\mathcal{R} - \frac{1}{2}\partial_A\phi\partial_B\phi\delta g^{AB} - \frac{1}{2}\zeta\phi^2\delta\mathcal{R}\right)\Bigg\} \tag{3.3}$$

where we took self-interaction potential $V(\phi)$ to be independent of g_{AB} which is always the case. By using the equalities

$$\delta\sqrt{-g} = \frac{-1}{2}\sqrt{-g}g_{CD}\delta g^{CD} \tag{3.4}$$

$$\delta\mathcal{R} = \delta(\mathcal{R}_{AB}g^{AB}) = \delta\mathcal{R}_{AB}g^{AB} + \mathcal{R}_{AB}\delta g^{AB} \tag{3.5}$$

$$\delta\mathcal{R}_{AB} = \nabla_C\delta\Gamma^C_{AB} - \nabla_B\delta\Gamma^C_{AC} \tag{3.6}$$

in (3.3) we get:

$$\begin{aligned}
\delta S \;=\; & \int d^{4+D}z\sqrt{-g}\Bigg\{ g^{CD}(-\frac{1}{4}M_\star^{D+2}\mathcal{R} + \frac{1}{4}g^{AB}\partial_A\phi\partial_B\phi + \frac{1}{4}\zeta\mathcal{R}\phi^2 + \frac{1}{2}V(\phi))\delta g^{CD} \\
+ \; & \frac{1}{2}(M_\star^{D+2} - \zeta\phi^2)\mathcal{R}_{AB}\delta g^{AB} - \frac{1}{2}\partial_A\phi\partial_B\phi\delta g^{AB} - \delta V(\phi) \\
+ \; & \frac{1}{2}(M_\star^{D+2} - \zeta\phi^2)(\nabla_C\delta\Gamma^C_{AB} - \nabla_B\delta\Gamma^C_{AC})\Bigg\}
\end{aligned} \tag{3.7}$$

whose last line can be further simplified into

$$\delta I \;=\; \int d^{4+D}z\sqrt{-g}\frac{1}{2}\zeta\left(\partial_A\partial_B\phi^2\delta g^{AB} - g^{AB}\partial_A\partial_B\phi^2 g_{CD}\delta g^{CD}\right) \tag{3.8}$$

which stems solely from the direct coupling between ϕ^2 and curvature scalar $\mathcal{R}$.

Substitution of (3.8) into (3.7) and use of the action principle *i.e.* the principle that the first variation of the action, δS, must vanish for classical g_{AB} configuration, lead us to

$$\begin{aligned}
\frac{\delta S}{\delta g^{EF}(x)} \;=\; & 0 \\
=\; & \int d^{4+D}z\sqrt{-g}\Bigg\{\left(\frac{-1}{4}M_\star^{D+2}\mathcal{R} + \frac{1}{4}g^{CD}\partial_C\phi\partial_D\phi + \frac{1}{4}\zeta\mathcal{R}\phi^2 + \frac{1}{2}V(\phi)\right)g_{AB} \\
+\; & \frac{1}{2}(M_\star^{D+2} - \zeta\phi^2)\mathcal{R}_{AB} - \frac{1}{2}\partial_A\phi\partial_B\phi \\
+\; & \frac{1}{2}\zeta\partial_A\partial_B\phi^2 - \frac{1}{2}\zeta g^{CD}\partial_C\partial_D\phi^2 g_{AB}\Bigg\}\delta^{4+D}(x-z)
\end{aligned} \tag{3.9}$$

which implies

$$\mathcal{G}_{AB} = \mathcal{R}_{AB} - \frac{1}{2}\mathcal{R}g_{AB} = \frac{T_{AB}}{M_\star^{D+2} - \zeta\phi^2} \tag{3.10}$$

as the equation of motion for metric field components g_{AB}. These are nothing but gravitational field equations (the Einstein equations) in $(4+D)$ dimensions with a variable Newton constant: $M_{\star}^{D+2} - \zeta\phi^2$ instead of $M_{\star}^{D+2}$ alone.

T_{AB} at the right-hand side designates energy-momentum distribution of the scalar field. In fact, from (3.9) it follows that

$$\begin{aligned} T_{AB} &= \partial_A\phi\partial_B\phi - g_{AB}\left(\frac{1}{2}g^{CD}\partial_C\phi\partial_D\phi + V(\phi)\right) \\ &+ \zeta(g_{AB}\partial_\mu\partial^\mu - \nabla_A\nabla_B)\phi^2 \,. \end{aligned} \tag{3.11}$$

whose last term, which involves second derivatives of ϕ^2, follow from direct coupling between curvature scalar and ϕ in the original action (3.1).

As usual, one can rewrite (3.10) for Ricci tensor $\mathcal{R}_{AB}$ by first determining the curvature scalar from the trace of (3.10):

$$\mathcal{R} = -\frac{2}{D+2}\frac{T}{M_{\star}^{D+2} - \zeta\phi^2} \tag{3.12}$$

where $T = g^{AB}T_{AB}$ is the trace of the energy-momentum tensor. Plugging (3.12) into (3.10) above gives us

$$\mathcal{R}_{AB} = \frac{1}{M_{\star}^{D+2} - \zeta\phi^2}(T_{AB} - \frac{1}{D+2}T g_{AB}) \tag{3.13}$$

which is a dynamical equation for Ricci tensor rather than Einstein tensor. The source term at right-hand side is given by

$$\begin{aligned} \mathcal{T}_{AB}(\phi) &= T_{AB} - \frac{1}{D+2}g_{AB}g^{CD}T_{CD} \\ &= \partial_A\phi\partial_B\phi - \zeta\nabla_A\nabla_B\phi^2 \\ &+ \frac{1}{D+2}\left(2V(\phi) - \zeta\partial_C\partial^C\phi^2\right)g_{AB} \end{aligned} \tag{3.14}$$

which will be utilized throughout the chapters to come.

Having formed equations of motion for g_{AB} we turn to that of the scalar field ϕ. This requires computing the first variation of (3.1) with respect to variations in ϕ. More explicitly we have:

$$\delta S = \int d^{4+D}z\sqrt{-g}\left\{ - g^{AB}\partial_A\phi\partial_B(\delta\phi) - \zeta\mathcal{R}\delta(\phi) - \delta V(\phi)\right\} \tag{3.15}$$

where we keep in mind that derivative ∂ and variation δ operations commute since the latter is related to deviations of ϕ from a fixed configuration at each spacetime point. This fact allows us to rewrite (3.15) in the form

$$\frac{\delta S}{\delta \phi(x)} = \int d^{4+D}z\sqrt{-g}\Big\{g^{AB}\partial_A\phi\partial_B\phi - \zeta R\phi - V'(\phi)\Big\}\delta^{4+D}(x-z) \tag{3.16}$$

which has to vanish according to action principle if ϕ is to represent the true configuration of the system.

As a result, the correct configurations of g_{AB} and ϕ fields, equivalently, the field configurations that extremize the action functional (3.1) obey the system of partial differential equations

$$\mathcal{R}_{AB} = \frac{\mathcal{T}_{AB}(\phi)}{M_\star^{D+2} - \zeta\phi^2} \tag{3.17}$$

$$\nabla_A\nabla^A\phi = \zeta R\phi + V'(\phi) \tag{3.18}$$

where prime denotes differentiation with respect to ϕ.

In the next two sections we will be exclusively dealing with self-consistent solutions of (3.17) and (3.18). As will be detailed below, our primary inquiry will be to nullify $\mathcal{T}_{AB}$ fully or partially depending on what we want.

3.1. Non-gravitating Scalar Field

We start our analysis by considering first a completely non-gravitating scalar *i.e.* we impose $\mathcal{T}_{AB} = 0$ for all $A = (\mu, i)$ and $B = (\nu, j)$. This implies that $\mathcal{R}_{AB}$ vanishes for all A, B so that metric tensor may be assumed to take the form $\eta_{AB} = (-1, 1, \ldots, 1)$, as mentioned before. In other words, we take spacetime having a vanishing Ricci tensor to be 'flat'. This is actually a critical assumption since there are curved spacetimes having vanishing Ricci tensor. In this sense, our concern in this thesis work is Ricci-flat spacetimes or spacetimes which are taken to be flat once the corresponding Ricci tensor vanishes.

The non-gravitating scalar fields have originally been analyzed in (Ayon-Beato *et. al.* 2005). Here we essentially repeat their analysis as a preparatory work for partially-gravitating scalars to be discussed in the next section.

It is convenient to nullify first $\mathcal{T}_{AB}$ for $A \neq B$.

$$\begin{aligned} \mathcal{T}_{AB}^{A\neq B} &= \partial_A\phi\partial_B\phi - \zeta\partial_A\partial_B\phi^2 \\ &= 0 \end{aligned} \tag{3.19}$$

These equations receive contributions from the first line of the second equality in (3.14) only, and the form that they enforce ϕ to have can be guessed to be

$$\phi = \psi^{\alpha}, \alpha : constant \tag{3.20}$$

where derivatives of ϕ can be related to those of ψ via

$$\begin{aligned} \partial_B\phi &= \alpha\psi^{\alpha-1}\partial_B\psi \\ \partial_B\phi^2 &= 2\alpha\psi^{2\alpha-1}\partial_B\psi \\ \partial_A\partial_B\phi^2 &= 2\alpha(2\alpha-1)\psi^{2\alpha-2}\partial_A\psi\partial_B\psi + 2\alpha\psi^{2\alpha-1}\partial_A\partial_B\psi \end{aligned} \tag{3.21}$$

as needed for analyzing (3.19). Indeed, by substituting these equalities into (3.19) one arrives at

$$\begin{aligned} \mathcal{T}_{AB}^{(A\neq B)} &= \alpha^2\psi^{2\alpha-2}\partial_A\psi\partial_B\psi - \zeta 2\alpha(2\alpha-1)\psi^{2\alpha-2}\partial_A\psi\partial_B\psi - \zeta 2\psi^{2\alpha-1}\partial_A\partial_B\psi \\ &= 0 \end{aligned} \tag{3.22}$$

from which it is easy to obtain

$$\alpha - 2\zeta(2\alpha-1) = 0 \tag{3.23}$$

or equivalently

$$\alpha = -\frac{2\zeta}{1-4\zeta} \tag{3.24}$$

so that ϕ is determined to have the form

$$\phi = \psi^{-\frac{2\zeta}{1-4\zeta}} . \tag{3.25}$$

Here ψ is another scalar field introduced as a result of the transformation (3.20).

The vanishing of the diagonal entries of $\mathcal{T}_{AB}$ further determines ψ to be a second order polynomial in z^A, and $V(\phi)$ to be a function of ϕ only. This can be seen by an explicit calculation. First, one notes that both $\mathcal{T}_{00}$ and $\mathcal{T}_{ii}$ (no sum over i) must vanish:

$$\mathcal{T}_{00} = 0 = (\partial_0\phi)^2 - \frac{2}{D+2}V(\phi) - \frac{\zeta}{D+2}(-\partial_\mu\partial^\mu + (D+2)\partial_0^2)\phi^2 \tag{3.26}$$

$$\mathcal{T}_{ii} = 0 = (\partial_i\phi)^2 + \frac{2}{D+2}V(\phi) - \frac{\zeta}{D+2}(\partial_\mu\partial^\mu + (D+2)\partial_i^2)\phi^2 \tag{3.27}$$

and hence their sum

$$\begin{aligned}\mathcal{T}_{00} + \mathcal{T}_{ii} &= (\partial_0\phi)^2 + (\partial_i\phi)^2 - \zeta(\partial_0^2 + \partial_i^2)\phi^2 \\ &= 0\,. \end{aligned} \tag{3.28}$$

By using (3.25) in (3.28) we get

$$\begin{aligned}\mathcal{T}_{00} + \mathcal{T}_{ii} &= \alpha^2\psi^{2\alpha-2}\Bigg[(\partial_0\psi)^2 + (\partial_i\psi)^2\Bigg] \\ &- \zeta\Bigg[2\alpha(2\alpha-1)\psi^{2\alpha-2}(\partial_0\psi)^2 + 2\alpha\psi^{2\alpha-1}\partial_0^2\psi \\ &+ 2\alpha(2\alpha-1)\psi^{2\alpha-2}(\partial_i\psi)^2 + 2\alpha\psi^{2\alpha-1}\partial_i^2\psi\Bigg] \\ &= -2\zeta\alpha\psi^{2\alpha-1}(\partial_0^2\psi + \partial_i^2\psi) \\ &= 0\,. \end{aligned} \tag{3.29}$$

Consequently, $\phi(z)$ must have the form

$$\phi(z) \equiv \phi_0(z) = \left(\frac{\tilde{a}}{2}\eta^{AB}z_A z_B + \eta^{AB}z_A\tilde{p}_B + \tilde{b}\right)^{-\frac{2\zeta}{1-4\zeta}} \tag{3.30}$$

where $\tilde{a}$, $\tilde{b}$ and $\tilde{p}_A$ are constants of integration. For $\phi(z)$ to take this rather specific form its self-interaction potential must take a special form which can be found by using (3.26):

$$\begin{aligned}\mathcal{T}_{00} &= \alpha^2\psi^{2\alpha-2}(\partial\psi)^2 - \frac{2}{D+2}V(\phi) \\ &- \frac{\zeta}{D+2}\Bigg\{-2\alpha(2\alpha-1)\psi^{2\alpha-2}\eta^{AB}\partial_A\psi\partial_B\psi - 2\alpha\psi^{2\alpha-1}\eta^{AB}\partial_A\partial_B\psi \\ &+ (D+2)\Bigg(2\alpha(2\alpha-1)\psi^{2\alpha-2}(\partial_0\psi)^2 + 2\alpha\psi^{2\alpha-1}\partial_0^2\psi\Bigg)\Bigg\} \\ &= \alpha^2\psi^{2\alpha-2}(\tilde{a}z_0 + \tilde{p}_0)^2 - \frac{2}{D+2}V(\phi)\end{aligned}$$

$$
\begin{aligned}
&- \frac{\zeta}{D+2}\Bigg\{ - 2\alpha(2\alpha-1)\psi^{2\alpha-2}\Bigg(-(\tilde{a}z_0+\tilde{p}_0)^2 + \sum_i (\tilde{a}z_i+\tilde{p}_i)^2 \Bigg) \\
&- 2\alpha\psi^{2\alpha-1}\tilde{a}(D+4) \\
&+ (D+2)\Bigg(2\alpha(2\alpha-1)\psi^{2\alpha-2}(\tilde{a}z_0+\tilde{p}_0)^2 \quad - \quad 2\alpha\psi^{2\alpha-1}\tilde{a} \Bigg)\Bigg\}
\end{aligned}
\tag{3.32}
$$

whose algebraic rearrangements gives

$$
\begin{aligned}
\mathcal{T}_{00} &= \psi^{2\alpha-2}(\alpha^2 - 2\zeta\alpha(2\alpha-1)) - \frac{2}{D+2}V(\phi) + \frac{4\zeta\alpha\tilde{a}(D+3)}{D+2}\psi^{2\alpha-1} \\
&+ \frac{2\zeta\alpha(2\alpha-1)}{D+2}\psi^{2\alpha-2}\Bigg(-(\tilde{a}x_0+\tilde{p}_0)^2 + \sum_i (\tilde{a}x_i+p_i)^2 \Bigg).
\end{aligned}
\tag{3.33}
$$

The first term at the right-hand side vanishes due to (3.23) so that

$$
\begin{aligned}
\mathcal{T}_{00} &= \frac{2\zeta\alpha(2\alpha-1)}{D+2}\psi^{2\alpha-2}\Bigg(-(\tilde{a}z_0+\tilde{p}_0)^2 + \sum_i (\tilde{a}z_i+\tilde{p}_i)^2 \Bigg) \\
&+ \frac{4\zeta\tilde{a}(D+3)}{D+2}\psi^{2\alpha-1} - \frac{2}{D+2}V(\phi)\,.
\end{aligned}
\tag{3.34}
$$

Now, by using the functional dependence of ϕ on z in (3.30) we can rearrange the last term in (3.34) as

$$
\begin{aligned}
\eta^{AB}(\tilde{a}z_A+\tilde{p}_A)(\tilde{a}z_B+\tilde{p}_B) &= \eta^{AB}\tilde{a}^2 z_A z_B + \eta^{AB}2\tilde{a}\tilde{p}_A z_B + 2\tilde{a}\tilde{b} + \eta^{AB}\tilde{p}_A\tilde{p}_B - 2\tilde{a}\tilde{b} \\
&= 2\tilde{a}\psi + \eta^{AB}\tilde{p}_A\tilde{p}_B - 2\tilde{a}\tilde{b}
\end{aligned}
\tag{3.35}
$$

whose replacement into (3.34) gives

$$
\begin{aligned}
\mathcal{T}_{00} &= \frac{4\zeta\alpha(2\alpha-1)\tilde{a}}{D+2}\psi^{2\alpha-1} + \frac{4\zeta\alpha(D+3)\tilde{a}\psi^{2\alpha-1}}{D+2} \\
&+ \frac{2\zeta\alpha(2\alpha-1)}{D+2}(\tilde{p}^2 - 2\tilde{a}\tilde{b})\psi^{2\alpha-2} - \frac{2}{D+2}V(\phi) \\
&= \frac{32\tilde{a}(D+3)}{(D+2)}\frac{\zeta^2}{(1-4\zeta)^2}(\zeta-\zeta_{D+4})\psi^{2\alpha-1} \\
&+ 4\frac{(\eta^{AB}\tilde{p}_A\tilde{p}_B - 2\tilde{a}\tilde{b})}{(D+2)}\frac{\zeta^2}{(1-4\zeta)^2}\psi^{2\alpha-2} - \frac{2}{D+2}V(\phi) \\
&= 0\,.
\end{aligned}
\tag{3.36}
$$

This then enforces a specific form for the self-interaction potential of ϕ:

$$
\begin{aligned}
V(\phi_0) &= 16\tilde{a}(D+3)\frac{\zeta^2}{(1-4\zeta)^2}\left(\zeta-\zeta_{4+D}\right)\phi_0^{\frac{1}{2\zeta}} \\
&+ 2\left(\eta^{AB}\tilde{p}_A\tilde{p}_B - 2\tilde{a}\tilde{b}\right)\frac{\zeta^2}{(1-4\zeta)^2}\phi_0^{\frac{1-2\zeta}{\zeta}}
\end{aligned}
\tag{3.37}
$$

which explicitly depends on the parameters of (3.30). Consequently, for $\mathcal{T}_{AB}$ to vanish the scalar field itself does not need to vanish; all that is required is to devise a self-interaction potential (3.37) on the specific solution (3.30) for $\phi(z)$. One notices that this non-gravitating nontrivial field configuration arises thanks to the ζ dependent terms in $\mathcal{T}_{AB}$ or equivalently the non-minimal coupling of ϕ to the curvature scalar. Indeed, when $\zeta \to 0$ the scalar field reduces to a constant and $V(\phi) \to 0$, which is the well-known trivial configuration.

It is not hard to see that (3.30) and (3.37) also nullify T_{AB}, the true energy-momentum tensor of ϕ in (3.11). Actually, this coincidence is expected since the Einstein tensor vanishes whenever the Ricci tensor vanishes. The fact that a non-minimally coupled scalar field possesses a non-trivial configuration despite its vanishing T_{AB} has recently been discussed in (Ayon-Beato *et. al.* 2005), and field and potential solutions in (3.30) and (3.37) have already been obtained therein. The wave front is spherical for $\tilde{p}_A = 0$ and planar for $\widetilde{a} = 0$. When $\zeta = \zeta_{4+D}$ the first term in potential drops out, and the second term becomes proportional to $\phi_0^{-(D+4)}$, which is precisely what is required by conformal invariance (Demir 2004).

An interesting property of the potential function (3.37) is that its minimum varies with ζ.

$$V(\phi) = A\phi^{\frac{1}{2\zeta}} + B\phi^{\frac{1-2\zeta}{\zeta}} \tag{3.38}$$

where

$$\begin{aligned} A &= 16\tilde{a}(D+3)\frac{\zeta^2}{(1-4\zeta)^2}\left(\zeta - \zeta_{4+D}\right) \\ B &= 2\left(\eta^{AB}\tilde{p}_A\tilde{p}_B - 2\tilde{a}\tilde{b}\right)\frac{\zeta^2}{(1-4\zeta)^2} \end{aligned} \tag{3.39}$$

If we differentiate (3.38) we get

$$\begin{aligned} V' &= \frac{A}{2\zeta} + B\phi^{\frac{1-2\zeta}{2\zeta}} + B\left(\frac{1-2\zeta}{\zeta}\right)\phi^{\frac{1-2\zeta}{\zeta}} \\ &= \phi^{\frac{1-2\zeta}{2\zeta}}\left\{\frac{A}{2\zeta} + B\left(\frac{1-2\zeta}{\zeta}\right)\phi^{\frac{1-4\zeta}{2\zeta}}\right\} \\ &= 0 \end{aligned} \tag{3.40}$$

One can see that for $\zeta > \zeta_{4+D}$ it is minimized at $\phi = 0$ whereas its minimum occurs at

$$\overline{\phi} = \left(\frac{(D+3)(\zeta - \zeta_{4+D})}{2\zeta - 1} \frac{4\tilde{a}}{\eta^{AB}\tilde{p}_A\tilde{p}_B - 2\tilde{a}\tilde{b}} \right)^{\frac{2\zeta}{1-4\zeta}} \tag{3.41}$$

when $\zeta < \zeta_{4+D}$ and $\eta^{AB}\tilde{p}_A\tilde{p}_B - 2\tilde{a}\tilde{b} > 0$. In this sense the conformal value of ζ represents a threshold level below and above which the lowest energy configuration for $V(\phi_0)$ drastically changes.

So far we have discussed only the solution of $\mathcal{T}_{AB} = 0$ with no mention of the equation of motion of ϕ in (3.18). Actually, the field configuration (3.30) with $V(\phi)$ given in (3.37) automatically satisfies (3.18). This observation is correct for all parameter ranges; in particular, at the two possible minima of the potential: $\phi = 0$ and $\phi = \overline{\phi}$.

3.2. Partially-gravitating Scalar Field

In this section we discuss cases where ϕ gravitates only in a subset of dimensions. The construction of completely non-gravitating scalar above will serve as a guide for our analysis. We will look for metric and scalar field configurations in agreement with the following $\mathcal{T}_{AB}$ texture:

$$\mathcal{T}_{\mu\nu}(\phi) = 0 \tag{3.42}$$

$$\mathcal{T}_{\mu j}(\phi) = \mathcal{T}_{i\nu}(\phi) = 0 \tag{3.43}$$

$$\mathcal{T}_{ij}(\phi) \neq 0 \tag{3.44}$$

where $\mathcal{T}_{ij}(\phi)$ determines topology and shape of the extra space via (3.17). As mentioned before, when $\mathcal{T}_{AB}$ vanishes for a certain range of indices so does the Ricci tensor. This, however, is not a trivial condition when ϕ gravitates in a subset of dimensions only. To clarify this point consider, for instance, the constraint (3.42) above. It guarantees that $\mathcal{R}_{\mu\nu} = 0$; however, it cannot guarantee, even with $g_{\mu\nu} = \eta_{\mu\nu}$, that the quartet (x_0, x_1, x_2, x_3) forms a flat space. The reason is that $\nabla_\mu\nabla_\nu\phi^2 = \partial_\mu\partial_\nu\phi^2$ if and only if the connection coefficients, Γ^A_{BC}, satisfy $\Gamma^A_{\mu\nu} = 0$ for all (A, μ, ν). This is guaranteed if $g_{\mu j}$ and $g_{i\nu}$ depend only on the extra dimensions. On the other hand, considering $T_{\mu j}$ and $T_{i\nu}$, one finds that $\nabla_\mu\nabla_i = \partial_\mu\partial_i$ if $g_{\mu j}$ and $g_{i\nu}$ both are constants with respect to all coordinates x^A, and if g_{ij} depends only on the extra dimensions. These flatness conditions on different groups of coordinates implies that the metric tensor g_{AB} must conform to structure of $\mathcal{T}_{AB}$ in (3.42-3.44):

$$
\begin{aligned}
g_{\mu\nu} &= \eta_{\mu\nu} && (3.45)\\
g_{\mu j} &= g_{i\nu} = 0 && (3.46)\\
g_{ij} &= g_{ij}(\vec{y}) && (3.47)
\end{aligned}
$$

which exhibits a block-diagonal structure as it should for extra coordinates $\{y^i\}$ to be compactified *i.e.* decoupled from the usual non-compact dimensions. With this structure for the metric tensor, the source term of the Ricci tensor $\mathcal{R}_{\mu\nu}$ in four dimensions can be found as

$$
\begin{aligned}
\mathcal{T}_{\mu\nu} &= \partial_\mu\phi\partial_\nu\phi - \zeta\partial_\mu\partial_\nu\phi^2 + \frac{1}{D+2}(2V(\phi)\\
&- \zeta\eta^{\alpha\beta}\partial_\alpha\partial_\beta\phi^2 - \zeta g^{ij}\partial_i\partial_j\phi^2)\eta_{\mu\nu} \qquad (3.48)
\end{aligned}
$$

which can be put into a more familiar form

$$
\begin{aligned}
\mathcal{T}_{\mu\nu}(\phi) &= \partial_\mu\phi\partial_\nu\phi - \zeta\partial_\mu\partial_\nu\phi^2\\
&+ \frac{1}{D+2}\left(2V_{\text{new}}(\phi) - \zeta\eta^{\alpha\beta}\partial_\alpha\partial_\beta\phi^2\right)\eta_{\mu\nu} \qquad (3.49)
\end{aligned}
$$

as follows from (3.14). This expression is similar to the modified energy-momentum tensor of a scalar field living in four dimensions except for two key features: (i) instead of 1/2 factor in four dimensions we have $1/(D+2)$, and (ii) the scalar field lives in the entire $(4+D)$–dimensional spacetime *i.e.* it depends also on the extra dimensions.

From (3.49) it is seen that, as seen from four dimensions, the self-interaction potential of ϕ is not the original one $V(\phi)$, but

$$V_{\text{new}}(\phi) = V(\phi) - \frac{1}{2}\zeta g^{ij}\nabla_i\nabla_j\phi^2 \qquad (3.50)$$

which involves derivatives of ϕ^2 with respect to extra coordinates $\{y^i\}$.

For $\mathcal{T}_{\mu\nu}(\phi)$ to vanish, first of all, the scalar field must have the special form

$$\phi(z) \equiv \phi_0(z) = \left(\frac{a}{2}\eta^{\mu\nu}x_\mu x_\nu + \eta^{\mu\nu}x_\mu p_\nu + b\right)^{-\frac{2\zeta}{1-4\zeta}} \qquad (3.51)$$

in analogy with (3.30) derived in Sec. 3.1 above. Here, in principle, all the parameters a, b and p_μ are functions of the extra coordinates $\{y^i\}$, and their mass dimensions are $2-(1-4\zeta)(D+2)/4\zeta$, $-(1-4\zeta)(D+2)/4\zeta$ and $1-(1-4\zeta)(D+2)/4\zeta$, respectively. The scalar field configuration (3.51) describes a shock wave propagation in four dimensions at each point $\{y^i\}$ of the extra space.

Having $\phi(z)$ obeying to (3.51) is not sufficient for nullifying all components of $\mathcal{T}_{\mu\nu}$, however. Indeed, for $\mathcal{T}_{\mu\nu}$ to vanish completely we have to find the special form of the self-interaction potential which would yield $\phi_0(z)$ as a solution. Let us consider

$$\mathcal{T}_{00} = (\partial_0\phi)^2 - \zeta\partial_0^2\phi^2 - \frac{1}{D+2}(2\tilde{V}(\phi) - \zeta\eta^{\alpha\beta}\partial_\alpha\partial_\beta\phi^2) \tag{3.52}$$

which can be rearranged to give

$$\mathcal{T}_{00} = (\partial_0\phi)^2 - \zeta\left(\frac{D+3}{D+2}\right)\partial_0^2\phi^2 - \frac{2}{D+2}\tilde{V}(\phi) + \frac{\zeta}{D+2}\sum_j^2 \phi^2\,. \tag{3.53}$$

As in Sec.3.1 above, it is convenient to perform a transformation of ϕ to another scalar field ε similar to (3.20):

$$\phi(z) = \varepsilon^{-\frac{2}{1-4\zeta}} \tag{3.54}$$

so that various derivatives take the form

$$\begin{aligned}
\partial_0\phi &= \left(-\frac{2\zeta}{1-4\zeta}\right)\varepsilon^{\frac{2\zeta-1}{1-4\zeta}}\partial_0\varepsilon \\
\partial_0^2\phi^2 &= \frac{4\zeta}{(1-4\zeta)^2}\varepsilon^{\frac{4\zeta-2}{1-4\zeta}}(\partial_0\varepsilon)^2 - \frac{4\zeta}{1-4\zeta}\varepsilon^{-\frac{1}{1-4\zeta}}\partial_0^2\varepsilon \\
\partial_i^2\phi^2 &= \frac{4\zeta}{(1-4\zeta)^2}\varepsilon^{\frac{4\zeta-2}{1-4\zeta}}(\partial_i\varepsilon)^2 - \frac{4\zeta}{1-4\zeta}\varepsilon^{-\frac{1}{1-4\zeta}}\partial_i^2\varepsilon
\end{aligned} \tag{3.55}$$

and their substitution in in (3.52) gives

$$\begin{aligned}
\mathcal{T}_{00} &= -\frac{2}{D+2}\tilde{V}(\phi) - \frac{4\zeta^2}{1-4\zeta}\frac{1}{D+2}\frac{\phi^2}{\varepsilon}\left(-\frac{1}{1-4\zeta}\frac{\partial_\mu\varepsilon\partial^\mu\varepsilon}{\varepsilon}\right. \\
&- \left.(D+3)\partial_0^2\varepsilon + \sum_j \partial_j^2\varepsilon\right)\,.
\end{aligned} \tag{3.56}$$

Furthermore, by using the identities

$$\begin{aligned}
\frac{\partial_\mu\varepsilon\partial^\mu\varepsilon}{\varepsilon} &= \frac{1}{\varepsilon}\eta^{\mu\nu}(ax_\mu+p_\mu)(ax_\nu+p_\nu) \\
&= \frac{1}{\varepsilon}(a^2\eta^{\mu\nu}x_\mu x_\nu + 2a\eta^{\mu\nu}x_\mu p_\nu + p^2) \\
&= \frac{1}{\varepsilon}\left(2a(\frac{a}{2}\eta^{\mu\nu}x_\mu x_\nu + \eta^{\mu\nu}x_\mu p_\nu + b) + p^2 - 2ab\right) \\
&= 2a + \frac{p^2-2ab}{\varepsilon}
\end{aligned} \tag{3.57}$$

and

$$\begin{aligned}\sum_j \partial_j^2 \varepsilon &= 3a \\ \partial_0^2 \varepsilon &= -a\end{aligned} \tag{3.58}$$

(3.56) takes the form

$$\begin{aligned}\mathcal{T}_{00} &= -\frac{2}{D+2}\tilde{V}(\phi) - \frac{4\zeta^2}{(1-4\zeta)^2}\frac{a[(D+6)(1-4\zeta)-2]}{D+2}\phi^{\frac{1}{2\zeta}} \\ &+ \frac{4\zeta^2}{(1-4\zeta)^2}\frac{1}{D+2}(p^2-2ab)\phi^{\frac{1-2\zeta}{\zeta}}\end{aligned} \tag{3.59}$$

whose vanishing yields a solution for $\tilde{V}(\phi)$

$$\begin{aligned}\tilde{V}(\phi) &= -\frac{2\zeta^2 a}{(1-4\zeta)^2}\Bigg((D+6)(1-4\zeta)-2\Bigg)\phi^{\frac{1}{2\zeta}} \\ &+ \frac{2\zeta^2}{(1-4\zeta)^2}(p^2-2ab)\phi^{\frac{1-2\zeta}{\zeta}}\,.\end{aligned} \tag{3.60}$$

This expression for self-interaction potential can be put into a more suggestive form by defining

$$\begin{aligned}(D+6)(1-4\zeta)-2 &= -4(D+6)\left(\zeta-\frac{D+4}{4(D+6)}\right) \\ &= -4(D+6)(\zeta-\zeta_{crit})\end{aligned} \tag{3.61}$$

which finally yields

$$\begin{aligned}\widetilde{V}(\phi_0) &= 8a(D+6)\frac{\zeta^2}{(1-4\zeta)^2}(\zeta-\zeta_{crit})\,\phi_0^{\frac{1}{2\zeta}} \\ &+ 2\left(\eta^{\mu\nu}\tilde{p}_\mu\tilde{p}_\nu-2ab\right)\frac{\zeta^2}{(1-4\zeta)^2}\phi_0^{\frac{1-2\zeta}{\zeta}}\end{aligned} \tag{3.62}$$

which is to be contrasted with the potential function (3.37) of purely non-gravitating scalar field discussed in Sec. 3.1 above. The most important difference between the two potentials comes from replacement of ζ_{4+D} in (3.37) by

$$\zeta_{crit} = \frac{(D+4)}{4(D+6)} \tag{3.63}$$

which ranges from 1/6 at $D=0$ to 1/4 at $D=\infty$. These two critical ζ values, ζ_{crit} and ζ_{4+D}, agree at $D=0$ and $D=\infty$, but behave differently in between. Clearly,

ζ_{crit} arises from $1/(D+2)$ factor in (3.49), and the two potentials (3.37) and (3.62) coincide when $D = 0$. In other words, (3.49) is not the true stress tensor of a scalar field living in four-dimensions; it is just projection of the stress tensor of a scalar field living in $(4 + D)$ dimensions upon four-dimensional subspace. It is with the special solution (3.62) *i.e.* it is with

$$V_{\rm new}(\phi_0) = \widetilde{V}(\phi_0) \qquad (3.64)$$

which holds on $\phi(z) = \phi_0(z)$ that all ten components of $\mathcal{T}_{\mu\nu}$ and hence those of $\mathcal{R}_{\mu\nu}$ vanish with a strictly flat metric $\eta_{\mu\nu}$.

Having determined under what conditions $\mathcal{T}_{\mu\nu}$ vanishes, we now look for implications of (3.43). Obviously, vanishing of $\mathcal{T}_{\mu j}$ and $\mathcal{T}_{i\nu}$ is guaranteed if $\phi_0(z)$ in (3.51) does not involve mixed terms of x^μ and y^i. In other words, the parameters a, ζ and p_μ must be global constants yet $b = b(\vec{y})$. The dependence of b on extra dimensions is rather general; all that is needed is to satisfy equations of motion self-consistently. For future reference, the two values of ϕ which makes the potential minimum are found as

$$\tilde{V}(\phi) = A\phi^{\frac{1}{2\zeta}} + B\phi^{\frac{1-2\zeta}{\zeta}} \qquad (3.65)$$

where

$$\begin{aligned} A &= 8a(D+6)\frac{\zeta^2}{(1-4\zeta)^2}(\zeta - \zeta_{crit}) \\ B &= \frac{2\zeta^2}{(1-4\zeta)^2}(\eta^{\mu\nu}p_\mu p_\nu - 2ab)\,. \end{aligned} \qquad (3.66)$$

First, by requiring (3.65) to have a vanishing derivative we get

$$\begin{aligned} \tilde{V}'(\phi) &= \frac{A}{2\zeta}\phi^{\frac{1-2\zeta}{2\zeta}} + B\left(\frac{1-2\zeta}{\zeta}\right)\phi^{\frac{1-3\zeta}{\zeta}} \\ &= \phi^{\frac{1-2\zeta}{2\zeta}}\left\{\frac{A}{2\zeta} + B\left(\frac{1-2\zeta}{\zeta}\right)\phi^{\frac{1-4\zeta}{2\zeta}}\right\} \\ &= 0 \end{aligned} \qquad (3.67)$$

from which it follows that (by taking $a > 0$ and $p_\mu p^\mu - 2ab(\vec{y}) > 0$) $\widetilde{V}(\phi_0)$ is minimized at $\phi_0 = 0$ for $1/4 > \zeta > \zeta_{crit}$, and at $\phi_0 = \overline{\phi}$ with

$$\overline{\phi} = \left(\frac{(D+6)(\zeta - \zeta_{crit})}{2\zeta - 1}\frac{2a}{\eta^{\mu\nu}p_\mu p_\nu - 2ab(\vec{y})}\right)^{\frac{2\zeta}{1-4\zeta}} \qquad (3.68)$$

for $\zeta < \zeta_{crit}$. Clearly, unless the shock wave propagation in four dimensions is a spherical one, $a \neq 0$, this very minimum of $\widetilde{V}(\phi_0)$ is neither possible nor meaningful.

Finally, we analyze implications of a finite $\mathcal{T}_{ij}$. By construction, $\mathcal{T}_{ij}$ does not vanish and hence extra space experiences a nontrivial curving. On the field configuration (3.51) for which $\mathcal{T}_{\mu\nu}$, $\mathcal{T}_{i\nu}$ and $\mathcal{T}_{\mu j}$ vanish identically, equations of motion for the metric tensor and ϕ_0 take the form

$$\begin{aligned} \mathcal{R}_{ij} &= \frac{\mathcal{T}_{ij}(\phi_0)}{M_\star^{D+2} - \zeta\phi_0^2} && (3.69)\\ g^{ij}\nabla_i\nabla_j\phi_0 &= \zeta\mathcal{R}\phi_0 + V'(\phi_0) - \widetilde{V}'(\phi_0) - aD\frac{\zeta}{1-4\zeta}\phi_0^{\frac{1-2\zeta}{2\zeta}} && (3.70)\end{aligned}$$

where the source of Ricci tensor is found from (3.14). Explicitly,

$$\begin{aligned} \mathcal{T}_{ij}(\phi_0) &= \partial_i\phi_0\partial_j\phi_0 - \zeta\nabla_i\nabla_j\phi_0^2 \\ &+ \frac{1}{D+2}\left(2V(\phi_0) - \zeta\eta^{\mu\nu}\partial_\mu\partial_\nu\phi_0^2 - \zeta g^{ij}\nabla_i\nabla_j\phi_0^2\right)g_{ij} \\ &= \partial_i\phi_0\partial_j\phi_0 - \zeta\nabla_i\nabla_j\phi_0^2 + \frac{1}{D+2}\left(2\tilde{V}(\phi_0) - \zeta\eta^{\mu\nu}\partial_\mu\partial_\nu\phi_0^2\right) && (3.71)\end{aligned}$$

where the terms in the parenthesis can be computed by using (3.62) and the identity

$$\partial_\mu\partial_\nu\phi_0^2 = \frac{4\zeta}{(1-4\zeta)^2}\varepsilon^{\frac{4\zeta-2}{1-4\zeta}}\partial_\mu\varepsilon\partial_\nu\varepsilon - \frac{4\zeta}{1-4\zeta}\varepsilon^{-\frac{1}{1-4\zeta}}\partial_\mu\partial_\nu\varepsilon \qquad (3.72)$$

so that

$$\begin{aligned} 2\tilde{V}(\phi_0) - \zeta\eta^{\mu\nu}\phi_0^2 &= 16a(D+6)\frac{\zeta^2}{(1-4\zeta)^2}\left(\zeta - \frac{D+4}{4(D+6)}\right)\phi_0^{\frac{1}{2\zeta}} \\ &+ \frac{4\zeta^2}{(1-4\zeta)^2}(p^2-2ab)\phi_0^{\frac{1-2\zeta}{\zeta}} \\ &- \frac{4\zeta^2}{(1-4\zeta)^2}\varepsilon^{\frac{4\zeta-2}{1-4\zeta}}\partial_\mu\varepsilon\partial^\mu\varepsilon + \frac{4\zeta^2}{1-4\zeta}\varepsilon^{-\frac{1}{1-4\zeta}}\partial_\mu\partial_\nu\varepsilon \\ &= 4a\frac{\zeta^2}{(1-4\zeta)^2}\left(4(D+6)\zeta - (D+4)\right)\phi_0^{\frac{1}{2\zeta}} \\ &+ \frac{4\zeta^2}{(1-4\zeta)^2}(p^2-2ab)\phi_0^{\frac{1-2\zeta}{\zeta}} \\ &- \frac{4\zeta^2}{(1-4\zeta)^2}\varepsilon^{\frac{4\zeta-2}{1-4\zeta}}(2a\sigma + p^2 - 2ab) \\ &+ \frac{4\zeta^2}{1-4\zeta}\varepsilon^{-\frac{1}{1-4\zeta}}4a && (3.73)\end{aligned}$$

where use has been made of

$$
\begin{aligned}
-\frac{\partial^2 \varepsilon}{\partial x_0^2} = \frac{\partial^2 \varepsilon}{\partial x_1^2} &= \frac{\partial^2 \varepsilon}{\partial x_2^2} = \frac{\partial^2 \varepsilon}{\partial x_3^2} = a \\
\eta^{\mu\nu}\partial_\mu \partial_\nu \varepsilon &= -\partial_0^2 \varepsilon + \partial_i^2 \varepsilon = 4a
\end{aligned} \tag{3.74}
$$

Then by rewriting ε in terms of ϕ one finds

$$
2\tilde{V}(\phi_0) - \zeta \eta^{\mu\nu}\partial_\mu \partial_\nu \phi_0^2 = -\frac{4a\zeta^2}{1-4\zeta}(D+2) \tag{3.75}
$$

so that the source term for Ricci tensor takes the form:

$$
\mathcal{T}_{ij}(\phi_0) = \partial_i \phi_0 \partial_j \phi_0 - \zeta \nabla_i \nabla_j \phi_0^2 - \frac{4a\zeta^2}{1-4\zeta}\phi_0^{\frac{1}{2\zeta}} g_{ij} \tag{3.76}
$$

which requires ϕ to possess the specific solution ϕ_0 given in (3.51).

A simultaneous solution of (3.69) and (3.70) completely determines the curvature scalar. To see this, first let us take trace of (3.69). It gives

$$
\mathcal{R} = \frac{\mathcal{T}}{M_\star^{D+2} - \zeta \phi^2} \tag{3.77}
$$

whose right-hand side requires trace of (3.76):

$$
\begin{aligned}
\mathcal{T}(\phi_0) &= g^{ij}\mathcal{T}_{ij}(\phi_0) \\
&= \partial_i \phi_0 \partial^i \phi_0 - \zeta \nabla_i \nabla^i \phi_0^2 - \frac{4aD\zeta^2}{1-4\zeta}\phi_0^{\frac{1}{2\zeta}} .
\end{aligned} \tag{3.78}
$$

Then by using (3.50) we find

$$
-\zeta g^{ij}\nabla_i \nabla_j \phi_0^2 = 2\left(\tilde{V}(\phi)_0 - V(\phi_0)\right) \tag{3.79}
$$

so that (3.78) takes the form

$$
\mathcal{T}(\phi_0) = \partial_i \phi_0 \partial^i \phi_0 + 2\left(\tilde{V}(\phi_0) - V(\phi_0)\right) - \frac{4aD\zeta^2}{1-4\zeta}\phi_0^{\frac{1}{2\zeta}} . \tag{3.80}
$$

We can further iterate the first term of this expression. Consider first

$$
g^{ij}\nabla_i \nabla_j \phi_0^2 = 2g^{ij}\nabla_i \phi_0 \nabla_j \phi_0 + 2g^{ij}\phi_0 \nabla_i \nabla_j \phi_0 \tag{3.81}
$$

where the second term at right-hand side follows from (3.70) to be

$$
\phi_0 g^{ij}\nabla_i \nabla_j \phi_0 = \zeta \mathcal{R}\phi_0^2 + \phi_0\left(V'(\phi_0) - \tilde{V}'(\phi_0)\right) - \frac{aD\zeta}{1-4\zeta}\phi_0^{\frac{1}{2\zeta}} \tag{3.82}
$$

whose substitution back in (3.81) gives

$$g^{ij}\partial_i\phi_0\partial_j\phi_0 = -\frac{1}{\zeta}\Big(\tilde{V}(\phi_0) - V(\phi_0)\Big) - \zeta\mathcal{R}\phi_0^2 - \phi_0\left(V'(\phi_0) - \tilde{V}'(\phi_0)\right) + \frac{aD\zeta}{1-4\zeta}\phi_0^{\frac{1}{2\zeta}} \quad (3.83)$$

This last expression when plugged (3.80) yields

$$\begin{aligned}
\mathcal{T}(\phi_0) &= -\frac{1}{\zeta}\Big(\tilde{V}(\phi_0) - V(\phi_0)\Big) - \zeta\mathcal{R}\phi_0^2 - \phi_0\left(V'(\phi_0) - \tilde{V}'(\phi_0)\right) + \frac{aD\zeta}{1-4\zeta}\phi_0^{\frac{1}{2\zeta}} \\
&+ 2\left(\tilde{V}(\phi_0) - V(\phi_0)\right) - \frac{4aD\zeta^2}{1-4\zeta}\phi_0^{\frac{1}{2\zeta}} \\
&= \mathcal{R}M_\star^{D+2} - \zeta\mathcal{R}\phi_0^2
\end{aligned} \quad (3.84)$$

from which the scalar curvature follows:

$$\begin{aligned}
\mathcal{R} &= \frac{1}{M_\star^{D+2}}\Bigg\{\left(2-\frac{1}{\zeta}\right)\left(\widetilde{V}(\phi_0) - V(\phi_0)\right) \\
&+ \phi_0\left(\widetilde{V}'(\phi_0) - V'(\phi_0)\right) + aD\zeta\phi_0^{\frac{1}{2\zeta}}\Bigg\}
\end{aligned} \quad (3.85)$$

which is a measure of the degree to which the extra space is curved.

Having worked out the question of under what conditions a bulk scalar in $4+D$ dimensions gravitates only in a subgroup of dimensions, we now turn to a discussion of the role and nature of the self-interaction potential $V_{\rm new}(\phi)$ of $\phi(z)$. First of all, $V_{\rm new}(\phi)$ is the scalar potential felt by a generic scalar field when the higher dimensional metric obtains the block diagonal structure in (3.45-3.47). In other words, it refers to part of the action density when all derivatives with respect to x_μ are dropped out. In fact, it is not more than a rearrangement of the terms involving derivatives with respect to extra dimensions so that action density looks like a four-dimensional one to facilitate analysis of $\mathcal{T}_{\mu\nu} = 0$. In particular, $V_{\rm new}(\phi)$ has nothing to do with the effective potential one would obtain by integrating out degrees of freedom associated with extra dimensions. It is neither a four-dimensional effective potential in the common sense of the word nor a $(4+D)$–dimensional effective potential; it is a local function of coordinates, and by taking the specific form $\widetilde{V}(\phi)$, it directly participates in flattening of the four-dimensional spacetime and in curving of the extra space via the equations of motion (3.69) and (3.70). To stress again, $\widetilde{V}(\phi)$ is just an analog of (3.37), and mathematically it is highly useful since its extrema in (3.68) will feature in the next section when we discuss compactification of the extra dimensions.

In summary, the entire dynamical problem has thus reduced to a self-consistent solution of (3.69) and (3.70). The unknowns of the problem are the metric tensor $g_{ij}(\vec{y})$ and $b(\vec{y})$. Once these two parameters are fixed one obtains a precise description of the geometry and shape of the extra space. The terms involving derivatives with respect to x^μ in the original equations of motion (3.17) and (3.18) have been eliminated by using the explicit expression of ϕ in (3.51). It is easy to see that, when $b(\vec{y}) = \frac{a}{2}\eta^{ij}y_i y_j + \eta^{ij}y_i p_j + b_0$, b_0 being a constant, all components of $\mathcal{T}_{ij}$ vanish and entire $(4+D)$–dimensional spacetime becomes flat, as discussed in detail in Sec.3.1 above. All other forms of $b(\vec{y})$ lead to a nontrivial curving of the extra space. In the next section we will analyze (3.69) and (3.70), and discuss their implications for compactification of the extra dimensions.

In the next chapter we will discuss role of partially gravitating scalars in compactification of the extra dimensions.

CHAPTER 4

SPACETIME COMPACTIFICATION

Spontaneous compactification of $(4+D)$–dimensional spacetime M^{4+D} into a four-dimensional flat spacetime M^4 spanned by the four macroscopic dimensions times a D-dimensional manifold E^D means that $\mathrm{M}^4 \otimes \mathrm{E}^D$ is an energetically preferred solution compared to M^{4+D} (Omero and Percacci 1980, Forste *et. al.* 2000). The analysis in Chapter 3 should have made it clear that flatness of M^4 is governed by $\widetilde{V}(\phi)$ not by $V(\phi)$. Indeed, $V(\phi)$ is the self-interaction potential of ϕ in $(4+D)$ dimensions whereas $V_{\mathrm{new}}(\phi)$ is the potential of the same scalar as seen from a four-dimensional perspective (see (3.49) which has to vanish for flattening the four-dimensional subspace). In this sense, higher-dimensional spacetime configuration consisting of a strictly flat four-dimensional geometry times an extra curved manifold becomes energetically preferable only at those ϕ_0 values for which $\widetilde{V}(\phi_0)$ is a minimum.

As follows from Chapter 3, by taking $a > 0$ and $\eta^{\mu\nu}p_\mu p_\nu - 2ab(\vec{y}) > 0$ for definiteness, the scalar potential $\widetilde{V}(\phi_0)$ is found to possess two minima: $\phi_0 = 0$ (for $\zeta > \zeta_{crit}$) and $\phi_0 = \overline{\phi}$ (for $\zeta < \zeta_{crit}$) given in (3.68). In the minimum of $\widetilde{V}(\phi_0)$ at $\phi_0 = 0$, the scalar field equation (3.70) is consistently solved if $V(\phi_0) = \widetilde{V}(\phi_0)$ *i.e.* $V(0) = 0$. This, in fact, follows from (3.50) which implies that $V(\phi_0)$ must be equal to $\widetilde{V}(\phi_0)$ for any $\vec{y}$ independent ϕ_0 configuration. With $\phi_0 = 0$ and $V(0) = 0$, Ricci tensor and curvature scalar are found to vanish identically, as follows from (3.69) and (3.85). It is clear that the whole picture is consistent since a vanishing ϕ possesses a vanishing $\mathcal{T}_{AB}$ if its potential does also vanish at the field configuration under concern. Consequently, the minimum of $\widetilde{V}(\phi_0)$ at $\phi_0 = 0$ represents a Ricci-flat manifold. This, as mentioned at the beginning of Sec.2.1, may be taken to indicate a strictly flat space *i.e.* $g_{ij} = \eta_{ij}$. One thus arrives at the conclusion that if $\widetilde{V}(\phi_0)$ is minimized at $\phi_0 = 0$ and if $V(0) = 0$ then the resulting spacetime is a $(4+D)$ dimensional Minkowski spacetime M^{4+D} *i.e.* there is no compactification effect at all. The extra space is a strictly flat manifold as the four-dimensional subspace itself.

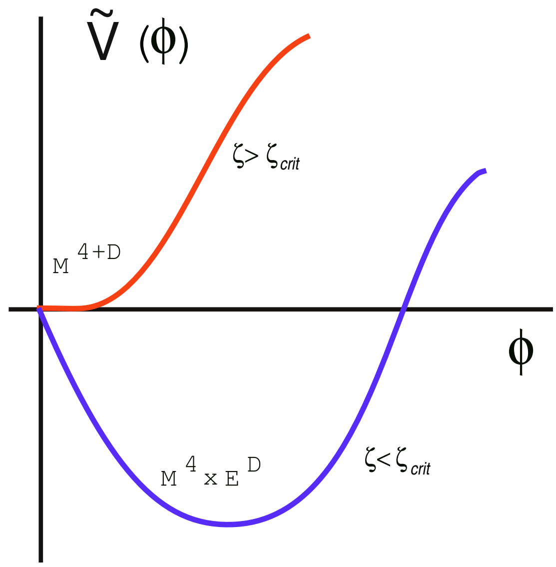

Figure 4.1: The two minima of $\widetilde{V}(\phi)$ and the corresponding spacetime structures.

For $\zeta < \zeta_{crit}$, the potential $\widetilde{V}(\phi_0)$ is minimized at a nonzero ϕ_0 value given in (3.68). The dynamical equations governing the compactification process are (3.69) and (3.70) where now ϕ_0 is replaced by $\overline{\phi}$. All one is to do is to solve dynamical equations for determining $g_{ij}(\vec{y})$ (with $D(D+1)/2$ independent components) and $b(\vec{y})$ in a self-consistent fashion. These two must give a complete description of the shape and topology of the extra space.

We schematically illustrate the two minima and corresponding spacetime structures of $\widetilde{V}(\phi)$ in Fig. 4. The overall picture is that as ζ makes a transition from $\zeta > \zeta_{crit}$ regime to $\zeta < \zeta_{crit}$ regime the spacetime structure changes from M^{4+D} to $\mathrm{M}^4 \otimes \mathrm{E}^D$ spontaneously. The topology and shape of the extra space are determined by simultaneous solutions of (3.69) and (3.70) for $\phi_0 = \overline{\phi}$, defined in (3.68).

An analytic solution of the topology and shape of the extra space is quite difficult to implement since (3.69) and (3.70) exhibit a functional dependence on $b(\vec{y})$ and $b(\vec{y})$ itself depends on $g_{ij}(\vec{y})$ via contraction of the extra coordinates. Therefore, one may eventually need to resort numerical techniques to determine the structure of the extra space. Despite these difficulties in establishing an analytic solution, it may be instructive to analyze certain simple cases by explicit examples:

Constant Curvature Space: The simplest $\overline{\phi}$ configuration which admits an analytic solution of (3.69) and (3.70) is provided by the ansatze $b(\vec{y}) = b_0$, a completely $\vec{y}$ independent configuration. The equation of motion for $\overline{\phi}$ (3.70) is satisfied with $V(\overline{\phi}) = \widetilde{V}(\overline{\phi})$ as expected from (3.50). A self-consistent solution of (3.69), (3.70) and (3.85) gives $\mathcal{R}$ after the following steps

$$\begin{aligned}
\phi_0 &= \overline{\phi} \\
b(\vec{y}) &= b_0 \\
&\Rightarrow \nabla_i \nabla_j \overline{\phi} = 0 \\
&\Rightarrow \nabla_i \nabla_j \overline{\phi}^2 = 0 \\
\tilde{V}(\overline{\phi}) - V(\overline{\phi}) &= 0 \\
\tilde{V}'(\overline{\phi}) - V'(\overline{\phi}) &= 0
\end{aligned} \tag{4.1}$$

By replacing these equalities in (3.70) we get the curvature scalar as

$$\mathcal{R}_{ij} = \frac{\mathcal{R}}{D} g_{ij} \quad \text{with} \quad \mathcal{R} = \frac{aD}{1-4\zeta}\overline{\phi}^{-\frac{1-4\zeta}{2\zeta}} \tag{4.2}$$

where vacuum expectation value of the scalar field is fixed via the consistency condition which is found as below

From (3.85) and (4.1) one can see

$$\mathcal{R} = \frac{aD\zeta}{M_\star^{D+2}}\overline{\phi}^{\frac{1}{2\zeta}} \tag{4.3}$$

Then by using the consistence between (4.2) and (4.3) we get

$$M_\star^{D+2} = \frac{\zeta}{1-4\zeta}\overline{\phi}^2 \tag{4.4}$$

In other words, the fundamental scale of gravity in $(4+D)$ dimensions, $M_\star$, fixes the vacuum expectation value of the bulk scalar ϕ_0 which is already designed not to gravitate in the four-dimensional subspace. The integration constants a, b and p_μ in (3.51) are naturally $\mathcal{O}(M_\star)$ – the only mass scale in the bulk. In fact, by taking $a = \lambda M_\star^{2-\frac{1-4\zeta}{4\zeta}(D+2)}$ with λ being a dimensionless constant, one finds curvature scalar in several steps

Firstly, let us replace value of a in (4.3)

$$\mathcal{R} = \frac{\lambda D\zeta}{M_\star^{\frac{D+2}{4\zeta}}}\overline{\phi}^{\frac{1}{2\zeta}} \tag{4.5}$$

Then replacing the value of $M_\star$ given in (4.4) gives

$$\mathcal{R} = \lambda D \zeta^{\frac{4\zeta-1}{4\zeta}} (1-4\zeta)^{-\frac{1}{4\zeta}} M_\star^2 \tag{4.6}$$

which is completely determined by ζ, D, λ and $M_\star$. The resulting spacetime topology is obviously $\mathrm{M}^4 \otimes \mathrm{E}^D$ with E^D being a D dimensional manifold with positive constant curvature. The coordinates $\{y_i\}$ may or may not be compact. The constant $b(\vec{y})$ case under discussion offers an elegant way of solving (3.69) and (3.70) and it results in an intuitively simple interpretation of the manifold formed by extra dimensions. Indeed, the self-interaction potential $V(\phi)$, on the partially-gravitating configuration ϕ_0 in (3.51), gets converted into $\widetilde{V}(\phi_0)$ whose minimum at $\phi_0 = \overline{\phi}$ results in a non-trivial constant-curvature space. In essence, the would-be cosmological term, $V(\overline{\phi})$, as seen from a four-dimensional Poincare-invariant perspective via (3.49) is off-loaded and utilized in curving the extra space (in similarity with the mechanism advocated in (Arkani-Hamed *et. al.* 2000) for solving the cosmological constant problem).

More General Cases: Some further properties of (3.69) and (3.70) can be revealed by using an appropriate coordinate system. A suitable setting for such an analysis is provided by the Riemann normal coordinates which are defined by a locally-flat space attached to a point N of the manifold of extra dimensions. The local flatness of the space at (not in any neighborhood of) the point N implies that $\partial_i g_{jk} \equiv 0$ for all $i, j, k = 1, \dots, D$ at N *i.e.* all components of the connection coefficients Γ^i_{jk} vanish at N. Clearly, curvature tensors do not need to vanish at N since they involve not only Γ^i_{jk} but also their first derivatives. Consequently, one finds

$$\mathcal{T}^{(N)}_{ij}(\overline{\phi}) \;=\; \frac{4\zeta^2}{1-4\zeta}\overline{\phi}^{\frac{1}{2\zeta}}\left(\frac{1-2\zeta}{(D+6)(\zeta-\zeta_{crit})}\partial_i\partial_j b - a g_{ij}\right) \tag{4.7}$$

so that $\mathcal{R}_{ij}$, unlike (4.2) where it is strictly proportional to g_{ij}, now picks up novel structures generated by $\partial_i\partial_j b$. In other words, it is the $\vec{y}$ dependence of $b(\vec{y})$ that enables $\mathcal{R}_{ij}$ to develop new components not necessarily related to those of the metric field.

Having replaced covariant derivatives with ordinary ones in this particular coordinate system, it is now possible to examine implications of different $\vec{y}$ dependencies of $b(\vec{y})$. If $b(\vec{y})$ exhibits a linear dependence, $b(\vec{y}) = g^{ij} p'_i y_j$, then the Ricci tensor turns out to depend on ${p'}^k x^l \partial_i \partial_j g_{kl}$ which involves curvature tensors rather than the metric tensor itself. When $b(\vec{y})$ is quadratic in $\vec{y}$, $b(\vec{y}) = (a'/2) y_i y^i$, the Ricci tensor now involves $2a' g_{ij} + a' x^k x^l \partial_i \partial_j g_{kl}$ which again depends on curvature tensors computed at the point N. Consequently, when $b(\vec{y})$ exhibits an explicit $\vec{y}$ dependence the Ricci tensor involves not only the metric tensor itself (as in (4.2) holding for constant-curvature spaces) but also double derivatives of the metric tensor *i.e.* the curvature tensors. More general dependencies are expected to yield more general structures for the geometry and topology of the extra space.

In general, irrespective of what coordinate system is chosen $b(\vec{y})$ is a bounded quantity. Therefore, it forces extra dimensions to take values within a hyperboloid. Indeed, a quadratic polynomial dependence for $b(\vec{y})$, for instance, results in

$$\frac{a'}{2} y_i y^i + p'_i y^i + b_0 < \frac{p_\mu p^\mu}{2a} \tag{4.8}$$

so that extra dimensions are bounded to have a finite size. For a purely quadratic dependence one finds $y_i y^i < p_\mu p^\mu / a a'$ which gives an idea on the maximal size a given dimension y^i can have. However, for more general, in particular, non-polynomial $\vec{y}$ dependencies of $b(\vec{y})$ its bounded nature may not imply any size restriction on the extra space at all. One keeps in mind that all model parameters must eventually return the correct value of Newton's constant in four dimensions: $\int d^D y \sqrt{-g} = 8\pi G_N M_\star^{D+2}$. This constraint requires the extra space to be of finite volume irrespective of the nature of the manifold (Cremmer and Scherk 1976, Cremmer and Scherk 1977, Randjbar-Daemi and Percacci 1982, Omero and Percacci 1980, Gell-Mann and Zwiebach 1984, Gell-Mann and Zweibach 1985, Gerard *et. al.* 1984, Randjbar-Daemi and Wetterich 1986)

CHAPTER 5

CONCLUSION

In this thesis work we have introduced a new method of spontaneous compactification triggered by a partially gravitating bulk scalar field (Demir and Pulice 2006). We have systematically constructed first a completely non-gravitating scalar field and then a partially gravitating one. We have examined scalar field configurations and minimum energy configurations in each case. Finally, we have discussed implications of a partially gravitating scalar for spacetime compactification. Our analysis here serves as an existence proof of a novel scalar-induced compactification. In particular, existence of a constant-curvature manifold for extra dimensions, and other novel properties observed in the frame of Riemann normal coordinates are particularly encouraging indications for the fact that a single scalar field, non-minimally coupled to the curvature scalar, can indeed lead to spontaneous compactification of the extra dimensions.

It is necessary to determine a simultaneous solution of (3.69) and (3.70) for having a precise knowledge of the shape and topology of the aimed-at manifold. In particular, these equations cannot be guaranteed to be free of singularities in the extra space. A detailed analysis is expected to shed light on nature of such singularities (see, for instance, (Forste *et. al.* 2000) for an analysis of the singularities in braneworld scenarios with a self-tuning cosmological term). Moreover, a full account of the spontaneous compactification might require a numerical determination of variables for sample values of the parameters. It will be after such an analysis that one will have detailed information on under what conditions the extra space takes a given shape and topology.

Another important issue is the determination of excitation spectrum about the background geometry we have determined. In other words, it is necessary to determine the gravi-particle spectra corresponding to normal modes generated by small oscillations about the background (see (Gell-Mann and Zwiebach 1984), for instance). This involves shifts $\eta_{\mu\nu} \rightarrow \eta_{\mu\nu}+h_{\mu\nu}$, $g_{ij} \rightarrow g_{ij}+h_{ij}$, $\phi(x^{\mu}, y^{i}) \rightarrow \phi(x^{\mu}, y^{i})+$

$\delta(x^\mu, y^i)$ as well as small but finite values of $g_{\mu j}$ and $g_{i\nu}$. In doing the spectrum analysis, particular care should be payed to the fact that the partially gravitating scalar field configuration in (3.51) depends explicitly on the metric tensor, and thus, its variation stems from both $\delta(x^\mu, y^i)$ and variations of the metric components.

One final remark concerns the use of higher curvature gravity. Indeed, higher-curvature gravity theories which generalize Einstein-Hilbert action to a function $f(\mathcal{R}, \mathcal{R})$ of the curvature scalar can be mapped, via conformal transformations, into Einstein-Hilbert action plus a scalar field theory (Maeda 1989, Demir and Tanyildizi 2006). In this context, the scalar field theory which facilitates the compactification may be interpreted to have a purely gravitational origin, and this may entail possibility of spontaneous compactification via higher curvature gravity.

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APPENDIX A

AFFINE CONNECTION AND CURVATURE TENSOR

The curvature of a Riemannian manifold (or more generally, any manifold with affine connection) is completely described by the Riemann tensor. In this Appendix we will give some basics about curvature tensor and dynamical equations thereof.

Consider a contravariant vector field $\zeta^{\mu}(x)$ and the spacetime trajectory $x^{\mu}(\tau)$ on a curve S of an observer. Let us assume that the observer can determine whether $\zeta^{\mu}(x)$ is constant or varies as its eigentime τ goes by. If we show the observed time derivative by a dot, we can write the time derivative of the contravariant vector field as

$$\dot{\zeta}^{\mu} = \frac{d}{d\tau}\zeta(x(\tau)) \tag{A.1}$$

Let us try to write this equation in some other coordinate frame v instead of coordinate frame x:

$$\zeta^{\mu}(x) = \frac{\partial x^{\mu}}{\partial v^{\nu}}\tilde{\zeta}^{\nu}(v(x)) \tag{A.2}$$

$$\begin{aligned}
\frac{\partial x^{\mu}}{\partial v^{\nu}}\dot{\tilde{\zeta}}^{\nu}(v(x)) &= \frac{d}{d\tau}\zeta^{\mu}(x(\tau)) \\
&= \frac{d}{d\tau}\Big(\frac{\partial x^{\mu}}{\partial v^{\nu}}\tilde{\zeta}^{\nu}(v(x))\Big) \\
&= \frac{\partial x^{\mu}}{\partial v^{\nu}}\frac{d}{d\tau}\tilde{\zeta}^{\nu}(v(x(\tau))) + \frac{dv^{\lambda}}{d\tau}\frac{\partial}{\partial v^{\lambda}}\frac{\partial x^{\mu}}{\partial v^{\nu}}\tilde{\zeta}^{\nu}(v) \\
&= \frac{\partial x^{\mu}}{\partial v^{\nu}}\frac{d}{d\tau}\tilde{\zeta}^{\nu}(v(x(\tau))) + \frac{\partial^2 x^{\mu}}{\partial v^{\nu}\partial v^{\lambda}}\frac{dv^{\lambda}}{d\tau}\tilde{\zeta}^{\nu}(v)
\end{aligned} \tag{A.3}$$

where v^{μ} and x^{μ} obey a rather general relationship. Apart from invertibility of their functional interdependence, they do not need to satisfy any specific condition at all.

It can be inferred from this last equation that the transformation of a contravariant vector field in a general coordinate frame is written as

$$\frac{\partial v^{\nu}}{\partial x^{\mu}}\frac{\partial x^{\mu}}{\partial v^{\nu}}\dot{\tilde{\zeta}}^{\nu}(v(x)) = \frac{\partial v^{\nu}}{\partial x^{\mu}}\frac{\partial x^{\mu}}{\partial v^{\nu}}\frac{d}{d\tau}\tilde{\zeta}(v(x(\tau))) + \frac{\partial v^{\nu}}{\partial x^{\mu}}\frac{\partial^2 x^{\mu}}{\partial v^{\nu}\partial v^{\lambda}}\frac{dv^{\lambda}}{d\tau}\tilde{\zeta}^{\nu}(v) \tag{A.4}$$

$$\dot{\zeta}^{\nu}(v(\tau)) \;\;=\;\; \frac{d}{d\tau}\zeta^{\nu}(v(\tau)) + \Gamma^{\nu}_{\kappa\lambda}\frac{dv^{\lambda}}{d\tau}\zeta^{\kappa}(v(\tau))$$

This is the equation that defines the parallel displacement of a contravariant vector along a curve S, and $\Gamma^{\nu}_{\lambda\mu}$ is a new field which is called "affine connection". It is not a tensor field as we can see this from its general transformation rule

$$\tilde{\Gamma}^{\nu}_{\kappa\lambda}(v(x)) = \frac{\partial v^{\nu}}{\partial x^{\mu}}\frac{\partial x^{\alpha}}{\partial v^{\kappa}}\frac{\partial x^{\beta}}{\partial v^{\lambda}}\Gamma^{\mu}_{\alpha\beta}(x) + \frac{\partial v^{\nu}}{\partial x^{\mu}}\frac{\partial^2 x^{\mu}}{\partial v^{\kappa}\partial v^{\lambda}} \tag{A.5}$$

where the second term at right-hand side indeed shows that it is not a tensor. That it is not a tensor is important in that it can be generated by a coordinate transformation even if it vanishes in the original frame.

A preferred coordinate frame x can be used near the point v by the local observer such that

$$\tilde{\Gamma}^{\nu}_{\kappa\lambda} = \frac{\partial v^{\nu}}{\partial x^{\mu}}\frac{\partial^2 x^{\mu}}{\partial v^{\kappa}\partial v^{\lambda}} \tag{A.6}$$

so that Γ vanishes in the preferred coordinate frame of the observer (only on the observer's curve S) but, in general, it does not vanish everywhere.

One can observe that (A.6) implies

$$\Gamma^{\nu}_{\lambda\kappa} = \Gamma^{\nu}_{\kappa\lambda} \tag{A.7}$$

and this symmetry will hold in any other coordinate frame since

$$\frac{\partial^2 x^{\mu}}{\partial v^{\kappa}\partial v^{\lambda}} = \frac{\partial^2 x^{\mu}}{\partial v^{\lambda}\partial v^{\kappa}} \tag{A.8}$$

From (A.7) one arrives at Newton's second law equation. Indeed, a curve $x^{\mu}(\delta)$ is a geodesic curve if it obeys

$$\frac{d^2 x^{\mu}(\delta)}{d\delta^2} + \Gamma^{\mu}_{\kappa\lambda}\frac{dx^{\kappa}}{d\delta}\frac{dx^{\lambda}}{d\delta} = 0 \tag{A.9}$$

where one notices that this is the particular case of (A.4) for a contravariant vector $\zeta^{\nu} = \frac{dx^{\lambda}}{d\delta}$.

One notes that the second term at the left-hand side of (A.9) is effectively the force acting on a particle and balanced by particle's acceleration (the first term at left-hand side). The fact that $\Gamma^{\mu}_{\kappa\lambda}$ is not a tensor tells us that this force term can be nullified or modified, in general, by going to different frames of reference. This is familiar, actually, even from the Newtonian mechanics where one can always go to a frame where gravitational field is eliminated locally.

Spacetime trajectories of particles that are moving in a gravitational field are described by the curves which obey the geodesic equation above. There exists a coordinate frame for every point x in which Γ vanishes so in that coordinate frame, the frame of the freely falling elevator, the trajectories are straight but they are curved in an accelerated elevator. This curvature can be attributed to a gravitational field by an observer inside the elevator.

Since the partial derivative is not a good tensor operator we now need to define a covariant derivative which reduces to the partial derivative in flat space with Cartesian coordinates, and as an operator transforms as a tensor on an arbitrary manifold. Partial derivative operator ∂_μ is a map that transforms (k, l) tensor fields to $(k, l+1)$ tensor fields and acts linearly on its arguments and obeys the Leibniz rule on tensor products. The map, which the partial derivative provides, depends on the coordinate system used since for every local point coordinate system changes. Then, we need to try to define a covariant derivative of a covariant vector field V_μ which means that, for each direction μ, the covariant derivative ∇_μ will be given by the partial derivative ∂_μ plus a correction specified by a matrix $\Gamma^\nu_{\mu\lambda}$ which is an $n \times n$ matrix (where n is the dimensionality of the manifold, for each μ) and which will perform the functions of the partial derivative, but not depending on coordinates. There are two properties that the covariant derivative ∇ should obey:

1. Linearity:

$$\nabla(A+V) = \nabla A + \nabla V \tag{A.10}$$

2. Leibniz rule:

$$\nabla(A \otimes V) = (\nabla A) \otimes V + A \otimes (\nabla V) \tag{A.11}$$

The covariant derivative can be written as the partial derivative plus some linear transformation that makes the result covariant when the first property is obeyed. It can be defined for a co-vector as

$$\nabla_\alpha V_\mu = \partial_\alpha V_\mu - \Gamma^\nu_{\alpha\mu} V_\nu \tag{A.12}$$

The transformation rule for this quantity is

$$\nabla_\alpha \tilde{V}_\nu(u) = \frac{\partial x^\mu}{\partial u^\nu}\frac{\partial x^\beta}{\partial u^\alpha}\nabla_\beta V_\mu(x) \tag{A.13}$$

which is the same for a tensor. The covariant derivative of a contravariant vector field can be defined similarly:

$$\nabla_\alpha V^\mu = \partial_\alpha V^\mu + \Gamma^\mu_{\alpha\beta}V^\beta \tag{A.14}$$

It is now not difficult to take the covariant derivative of a tensor of arbitrary rank by introducing a term with single $+\Gamma$ for each upper index and a term with a single $-\Gamma$ for each lower index:

$$\begin{aligned}
\nabla_\alpha T^{\mu\nu\ldots}_{\kappa\lambda\ldots} &= \partial_\alpha T^{\mu\nu\ldots}_{\kappa\lambda\ldots} \\
&+ \Gamma^\mu_{\alpha\beta}T^{\beta\nu\ldots}_{\kappa\lambda\ldots} + \Gamma^\nu_{\alpha\beta}T^{\mu\beta\ldots}_{\kappa\lambda\ldots}\ldots \\
&- \Gamma^\beta_{\kappa\alpha}T^{\mu\nu\ldots}_{\beta\lambda\ldots} - \Gamma^\beta_{\lambda\alpha}T^{\mu\nu\ldots}_{\kappa\beta\ldots}\ldots
\end{aligned} \tag{A.15}$$

The covariant derivative of a scalar field ϕ is the ordinary derivative:

$$\nabla_\alpha\phi = \partial_\alpha\phi \tag{A.16}$$

Let us now discuss curvature property after discussing parallel transport and covariant derivative. One conventional way to introduce the Riemann tensor is to consider parallel transport around an infinitesimal loop. Let us consider a curved two-dimensional space,that is to say, any curved surface and try to find the change of this vector after parallel displacement. The change of a vector after parallel displacement $\triangle A_\mu$ around an infinitesimal closed contour can be written in the form $\oint \delta A_\mu$ (where the integral is taken over the loop) and it can be expressed as

$$\delta A_\alpha = \Gamma^\gamma_{\alpha\beta}A_\gamma dx^\beta \tag{A.17}$$

Then, by replacing this last expression into the integral, we get:

$$\triangle A_\alpha = \oint \Gamma^\gamma_{\alpha\beta}A_\gamma dx^\beta \tag{A.18}$$

There are some important points to notice at this stage. The values of the vector A_α depend on the path, that is to say, they are not unique at points inside the contour.

This is related to the second order terms discussed below. Therefore, we have the components of the vector A_α as being uniquely determined by their values on the contour itself by the formulas (A.17), in another way as below

$$\frac{\partial A_\alpha}{\partial x^\beta} = \Gamma^\gamma_{\alpha\beta} A_\gamma \tag{A.19}$$

The Stokes' theorem states that

$$\oint A_\mu dx^\mu = \int df^{\mu\nu} \frac{\partial A_\nu}{\partial x^\mu} = \frac{1}{2} \int df^{\mu\nu} \Big(\frac{\partial A_\mu}{\partial x^\nu} - \frac{\partial A_\nu}{\partial x^\mu} \Big) \tag{A.20}$$

where $df^{\mu\nu}$ stands for a differential surface element. Consequently, by applying Stokes' theorem to the integral (A.18) we get:

$$\begin{aligned} \Delta A_\alpha &= \frac{1}{2} \left[\frac{\partial \left(\Gamma^\gamma_{\alpha\beta} A_\gamma \right)}{\partial x^\rho} - \frac{\partial \left(\Gamma^\gamma_{\alpha\rho} A_\gamma \right)}{\partial x^\beta} \right] \Delta f^{\rho\beta} \\ &= \frac{1}{2} \left[\frac{\partial \Gamma^\gamma_{\alpha\beta}}{\partial x^\rho} A_\gamma - \frac{\partial \Gamma^\gamma_{\alpha\rho}}{\partial x^\beta} A_\gamma + \Gamma^\gamma_{\alpha\rho} \frac{\partial A_\gamma}{\partial x^\rho} - \Gamma^\gamma_{\alpha\rho} \frac{\partial A_\gamma}{\partial x^\beta} \right] \Delta f^{\rho\beta} \end{aligned} \tag{A.21}$$

where we considered that the area enclosed by the contour has the infinitesimal value $\triangle A^{\rho\beta}$. Now, by replacing the derivatives (A.19) into this equation we get

$$\begin{aligned} \frac{\partial A_\nu}{\partial x^\lambda} &= \Gamma^\gamma_{\nu\lambda} A_\gamma \\ \frac{\partial A_\nu}{\partial x^\mu} &= \Gamma^\gamma_{\nu\mu} A_\gamma \\ \Delta A_\alpha &= \frac{1}{2} \Big[\frac{\partial \Gamma^\gamma_{\alpha\mu}}{\partial x^\lambda} A_\gamma - \frac{\partial \Gamma^\gamma_{\alpha\lambda}}{\partial x^\mu} A_\gamma + \Gamma^\nu_{\alpha\mu} \Gamma^\gamma_{\nu\lambda} A_\gamma - \Gamma^\nu_{\alpha\lambda} \Gamma^\gamma_{\nu\mu} A_\gamma \Big] \Delta f^{\lambda\mu} \end{aligned} \tag{A.22}$$

and finally we have

$$\Delta A_\alpha = \frac{1}{2} R^\gamma_{\alpha\lambda\mu} A_\gamma \Delta f^\mu \tag{A.23}$$

where $R^\gamma_{\alpha\lambda\mu}$ is the curvature tensor, a (1,3) tensor, commonly known as the Riemann tensor. From the expressions above, one finds its explicit form to be

$$R^\gamma_{\kappa\lambda\mu} = \frac{\partial \Gamma^\gamma_{\kappa\mu}}{\partial x^\lambda} - \frac{\partial \Gamma^\gamma_{\kappa\lambda}}{\partial x^\mu} + \Gamma^\nu_{\kappa\mu} \Gamma^\gamma_{\nu\lambda} - \Gamma^\nu_{\kappa\lambda} \Gamma^\gamma_{\nu\mu} \tag{A.24}$$

which involves terms which are either linear in the derivatives of connection coefficients or quadratic in connection coefficients themselves. If we restrict ourselves to at most first derivative of the connection coefficients then $R^\gamma_{\kappa\lambda\mu}$ is unique. It completely specifies the curvature of the manifold under concern.

Curvature tensor can be written in another form,

$$g_{\lambda\rho}R^{\lambda}_{\sigma\mu\nu} = R_{\rho\sigma\mu\nu} \tag{A.25}$$

It is invariant under interchange of the first pair of indices with the second:

$$R_{\rho\sigma\mu\nu} = R_{\mu\nu\rho\sigma} \tag{A.26}$$

and it is antisymmetric in its first two indices and in its last two indices separately:

$$R_{\rho\sigma\mu\nu} = -R_{\sigma\rho\mu\nu} \tag{A.27}$$

$$R_{\rho\sigma\mu\nu} = -R_{\rho\sigma\nu\mu} \tag{A.28}$$

The sum of cyclic permutations of the last three indices vanishes:

$$R_{\rho\sigma\mu\nu} + R_{\rho\mu\nu\sigma} + R_{\rho\nu\sigma\mu} = 0 \tag{A.29}$$

or equivalently

$$R_{\rho[\sigma\mu\nu]} = 0 \tag{A.30}$$

These last properties will be true in any coordinates since they are all tensor equations.

Let us now see how many independent quantities remain after these relations between the different components of the Riemann tensor.We know curvature tensor $R_{\rho\sigma\mu\nu}$ is antisymmetric in the first two indices, antisymmetric in the last two indices and symmetric under interchange of these two pairs so it can be thought as a symmetric matrix $R_{[\rho\sigma][\mu\nu]}$ where the pairs $\rho\sigma$ and $\mu\nu$ may be thought as individual indices.An $n \times n$ antisymmetric matrix has $n(n-1)/2$ independent components, while an $m \times m$ symmetric matrix has $m(m+1)/2$ independent components.Then the number of independent components is found as

$$\frac{1}{2}[\frac{1}{2}n(n-1)][\frac{1}{2}n(n-1)+1] = \frac{1}{8}(n^4 - 2n^3 + 3n^2 - 2n) \tag{A.31}$$

There is still the identity (A.29) to deal with.The totally antisymmetric part of the Riemann tensor vanishes as a consequence of this identity:

$$R_{[\rho\sigma\mu\nu]} = 0 \tag{A.32}$$

Let us consider decomposing

$$R_{\rho\sigma\mu\nu} = X_{\rho\sigma\mu\nu} + R_{[\rho\sigma\mu\nu]} \tag{A.33}$$

Any totally antisymmetric 4-index tensor is automatically antisymmetric in its first and last indices, and symmetric under interchange of the two pairs.A totally antisymmetric 4-index tensor has $n(n-1)(n-2)(n-3)/4!$ terms, and therefore (A.32) reduces the number of independent components by this amount.So the number of independent components of the Riemann tensor is

$$\frac{1}{8}(n^4 - 2n^3 + 3n^2 - 2n) - \frac{1}{24}n(n-1)(n-2)(n-3) = \frac{1}{12}n^2(n^2-1) \tag{A.34}$$

For instance, in four dimensions the Riemann tensor has 20 independent components and in one dimension it has no components.

We have seen the algebraic symmetries of the Riemann tensor that constrain the number of independent components at any point.There is an additional differential identity that the Riemann tensor obeys.So let us consider the covariant derivative of the Riemann tensor which is evaluated in Riemann normal coordinates:

$$\begin{aligned} \nabla_\lambda R_{\rho\sigma\mu\nu} &= \partial_\lambda R_{\rho\sigma\mu\nu} \\ &= \frac{1}{2}\partial_\lambda(\partial_\mu\partial_\sigma g_{\rho\nu} - \partial_\mu\partial_\rho g_{\nu\sigma} - \partial_\nu\partial_\sigma g_{\rho\mu} + \partial_\nu\partial_\rho g_{\mu\sigma}) \end{aligned} \tag{A.35}$$

When we consider the sum of cyclic permutation of the first three indices we get:

$$\begin{aligned} \nabla_\lambda R_{\rho\sigma\mu\nu} &+ \nabla_\rho R_{\sigma\lambda\mu\nu} + \nabla_\sigma R_{\lambda\rho\mu\nu} \\ &= \frac{1}{2}(\partial_\lambda\partial_\mu\partial_\sigma g_{\rho\nu} - \partial_\lambda\partial_\mu\partial_\rho g_{\nu\sigma} - \partial_\lambda\partial_\nu\partial_\sigma g_{\rho\mu} + \partial_\lambda\partial_\nu\partial_\rho g_{\mu\sigma} \\ &= \partial_\rho\partial_\mu\partial_\lambda g_{\sigma\nu} - \partial_\rho\partial_\mu\partial_\sigma g_{\nu\lambda} - \partial_\rho\partial_\nu\partial_\lambda g_{\sigma\mu} + \partial_\rho\partial_\nu\partial_\sigma g_{\mu\lambda} \\ &= \partial_\sigma\partial_\mu\partial_\rho g_{\lambda\nu} - \partial_\sigma\partial_\mu\partial_\lambda g_{\nu\rho} - \partial_\sigma\partial_\nu\partial_\rho g_{\nu\mu} + \partial_\sigma\partial_\nu\partial_\lambda g_{\mu\rho}) \\ &= 0 \end{aligned} \tag{A.36}$$

It can be recognized that the antisymmetry property (A.29) allows us to write the last result as

$$\nabla_{[\lambda} R_{\rho\sigma]\mu\nu} = 0 \tag{A.37}$$

which is called as the *Bianchi identity*.

One can directly obtain a symmetric second rank tensor from the Riemann tensor by contracting γ and λ:

$$R_{\kappa\mu} \equiv R^{\gamma}_{\kappa\gamma\mu} \tag{A.38}$$

which is known as the Ricci tensor.

So far, the manifold we have worked on has been required to have only geodesics, that is, trajectories of material points in a curved spacetime. Neither the Riemann nor the Ricci tensor need any structure on the manifold other than the connection coefficients. However, if the manifold is equipped with a metric field, that is, if the observers on the manifold have *meter sticks* and *clocks* to measure distances between the events then we obtain a more structured manifold. In fact, if a metric field is given then one can contract Ricci tensor to obtain a scalar, that is, the Ricci scalar:

$$R = g^{\kappa\mu} R_{\kappa\mu}\,. \tag{A.39}$$

In general, the connection coefficients are unknown non-tensorial structures. They are contained in the covariant derivative of all objects on the manifold, and they make up the curvature tensor of the manifold. However, *metric compatibility*, that is, the fact that metric tensor is a *covariantly constant tensor* determines the connection coefficients uniquely. Indeed, from

$$\nabla_{\mu} g_{\alpha\beta} = 0 \tag{A.40}$$

we can solve for the connection coefficients to obtain

$$\Gamma^{\gamma}_{\alpha\beta} = \frac{1}{2} g^{\gamma\rho} \left(\partial_{\alpha} g_{\beta\rho} + \partial_{\beta} g_{\rho\alpha} - \partial_{\rho} g_{\alpha\beta}\right) \tag{A.41}$$

which is known as 'metric connection'. Since difference between any two connections is a tensor, any other connection differs from (A.41) by a tensorial structure.

The Riemann tensor (A.24) and its contractions Ricci tensor and Ricci scalar, computed for the metric connection (A.41), satisfies the gravitational field equations

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = \frac{1}{M_{\star}^{D+2}} T_{\mu\nu} \tag{A.42}$$

in a $(4+D)$–dimensional spacetime populated by some general form of matter and energy described by the energy-momentum tensor $T_{\mu\nu}$.

APPENDIX B

STRESS-ENERGY-MOMENTUM TENSOR

The stress-energy tensor (sometimes it is also called energy-momentum tensor) is a tensor quantity that describes the density and current of the energy-momentum four-vector p^α. In other words, energy-momentum four-vector p^α is the conserved charge of the energy-momentum flow described by $T_{\alpha\beta}$. We here discuss these quantities in detail by considering a system of particles labeled by n with energy-momentum four-vectors $p_n^\alpha(t)$. We can define the density of p^α as

$$T^{\alpha 0}(\vec{x}, t) \equiv \sum_n p_n^\alpha(t)\delta^3(\vec{x} - \vec{x}_n(t)) \tag{B.1}$$

the current of it is defined as

$$T^{\alpha i}(\vec{x}, t) \equiv \sum_n p_n^\alpha(t)\frac{dx_n^i(t)}{dt}\delta^3(\vec{x} - \vec{x}_n(t))\,. \tag{B.2}$$

These two quantities are indeed densities of energy and momentum since the right-hand side involve Dirac δ-functions over the space.

The two definitions (B.1) and (B.2) can be unified into a single quantity:

$$T^{\alpha\beta}(x) = \sum_n p_n^\alpha \frac{dx_n^\beta(t)}{dt}\delta^3(x - x_n(t)) \tag{B.3}$$

where $\delta x^0/dt = 1$ by definition. In general, for a relativistic particles obey the relation

$$\frac{\vec{p}}{E} = \vec{v} \tag{B.4}$$

where $\vec{p}$ is the momentum, E is the energy and $\vec{v}$ is the velocity of the particle. We can write this formula for our system of particles as

$$p_n^\beta = E_n \frac{dx_n^\beta}{dt} \tag{B.5}$$

relating thus four-momentum of a particle to four-velocity.

If we substitute (B.5) into (B.3), we get:

$$T^{\alpha\beta}(x) = \sum_n \frac{p_n^\alpha p_n^\beta}{E_n}\delta^3(\vec{x} - \vec{x}_n(t)) \tag{B.6}$$

which is clearly a symmetric second rank tensor.

It could be instructive to check check if $T^{\alpha\beta}(x)$ is indeed conserved. This is a important issue because if it is not conserved then momentum four-vector of the particle cannot be a conserved charge.

By utilizing (B.1) and (B.2):

$$\begin{aligned}\frac{\partial}{\partial x^i}T^{\alpha i}(\vec{x},t) &= -\sum_n p_n^\alpha(t)\frac{dx_n^i(t)}{dt}\frac{\partial}{\partial x_n^i}\delta^3(\vec{x}-\vec{x}_n(t)) \\ &= -\sum_n p_n^\alpha(t)\frac{\partial}{\partial t}\delta^3(\vec{x}-\vec{x}_n(t)) \\ &= -\frac{\partial}{\partial t}T^{\alpha 0}(\vec{x},t)+\sum_n\frac{dp_n^\alpha(t)}{dt}\delta^3(\vec{x}-\vec{x}_n(t)) \end{aligned} \tag{B.7}$$

so that we can write

$$\frac{\partial}{\partial x^\beta}T^{\alpha\beta} = G^\alpha \tag{B.8}$$

where G^α is the density of force defined as

$$G^\alpha(\vec{x},t) \equiv \sum_n \delta^3(\vec{x}-\vec{x}_n(t))\frac{dp_n^\alpha(t)}{dt} \tag{B.9}$$

It is clear that if the momentum four-vectors p_n^α of particles are constant, that is to say, if the particles are free (from any external force) then the energy-momentum tensor is conserved:

$$\frac{\partial}{\partial x^\beta}T^{\alpha\beta}(x) = 0 \tag{B.10}$$

establishing thus momentum four-vector of particles as conserved charges.

So far we have discussed only a set of particles to establish some basic features of the energy-momentum tensor. The concepts above can be put in a more general setting by considering the action functional. In field theories, all interactions and dynamics are contained in the action

$$S[\psi] = \int d^4x\,\sqrt{-g}\mathcal{L}(\psi,\nabla\psi) \tag{B.11}$$

where $g_{\alpha\beta}$ is the metric tensor and ψ are some generic matter fields encoded in the lagrangian density $\mathcal{L}$. The energy-momentum tensor of the matter fields ψ is defined via the relation

$$T^{\alpha\beta}(x) = \frac{2}{\sqrt{-g}} \frac{\delta S}{\delta g_{\alpha\beta}(x)} = 2\frac{\delta \mathcal{L}}{\delta g_{\alpha\beta}(x)} - \mathcal{L} g^{\alpha\beta} \tag{B.12}$$

By using this expression one can compute energy-momentum tensor of any field theory. Here we list down energy-momentum tensors of some basic fields:

The energy-momentum tensor of a massive vector field A_μ:

$$T_{\mu\nu}^{(J=1)} = \eta_{\mu\nu}\left(\frac{1}{4}F^{\lambda\rho}F_{\lambda\rho} - \frac{1}{2}M_A^2 A_\lambda A^\lambda\right) - \left(F_\mu^\rho F_{\nu\rho} - M_A^2 A_\mu A_\nu\right) \tag{B.13}$$

The energy-momentum tensor of a massive fermion field ψ:

$$\begin{aligned} T_{\mu\nu}^{(J=1/2)} &= -\eta_{\mu\nu}\left(\overline{\psi} i\partial\psi - m_\psi \overline{\psi}\psi\right) + \frac{i}{2}\overline{\psi}\left(\gamma_\mu\partial_\nu + \gamma_\nu\partial_\mu\right)\psi \\ &+ \frac{1}{4}\left[2\eta_{\mu\nu}\partial^\lambda\left(\overline{\psi} i\gamma_\lambda\psi\right) - \partial_\mu\left(\overline{\psi} i\gamma_\nu\psi\right) - \partial_\nu\left(\overline{\psi} i\gamma_\mu\psi\right)\right] \end{aligned} \tag{B.14}$$

The energy-momentum tensor of a massive complex scalar field Φ:

$$\begin{aligned} T_{\mu\nu}^{(J=0)} &= -\eta_{\mu\nu}\left[\partial^\rho\Phi^\dagger\partial_\rho\Phi - M_\Phi^2\Phi^\dagger\Phi - \lambda\left(\Phi^\dagger\Phi\right)^2\right] + \partial_\mu\Phi^\dagger\partial_\nu\Phi + \partial_\nu\Phi^\dagger\partial_\mu\Phi \\ &+ 2\zeta\left(\eta_{\mu\nu} - \partial_\mu\partial_\nu\right)\Phi^\dagger\Phi \end{aligned} \tag{B.15}$$