CO-COATOMICALLY SUPPLEMENTED MODULES

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ABSTRACT

CO-COATOMICALLY SUPPLEMENTED MODULES

The purpose of this study to define co-coatomically supplemented modules, ⊕-co-coatomically supplemented modules, co-coatomically weak supplemented modules and co-coatomically amply supplemented modules and examine them over arbitrary rings and over commutative Noetherian rings, in particular over Dedekind domains. Motivated by cofinite submodule which is defined by R. Alizade, G. Bilhan and P. F. Smith, we define co-coatomic submodule. A proper submodule is called co-coatomic if the factor module by this submodule is coatomic. Then we define co-coatomically supplemented module. A module is called co-coatomically supplemented if every co-coatomic submodule has a supplement in this module. Over a discrete valuation ring, a module is co-coatomically supplemented if and only if the basic submodule of this module is coatomic. Over a non-local Dedekind domain, if a reduced module is co-coatomically amply supplemented then the factor module of this module by its torsion part is divisible and P-primary components of this module are bounded for each maximal ideal P. Conversely, over a non-local Dedekind domain, if the factor module of a reduced module by its torsion part is divisible and P-primary components of this module are bounded for each maximal ideal P, then this module is co-coatomically supplemented. A ring R is left perfect if and only if any direct sum of copies of the ring is ⊕-co-coatomically supplemented left R-module. Over a discrete valuation ring, co-coatomically weak supplemented and co-coatomically supplemented modules coincide. Over a Dedekind domain, if the torsion part of a module has a weak supplement in this module, then the module is co-coatomically weak supplemented if and only if the torsion part is co-coatomically weak supplemented and the factor module of the module by its torsion part is co-coatomically weak supplemented. Every left R-module is co-coatomically weak supplemented if and only if the ring R is left perfect.

ÖZET

EŞ EŞATOMİK TÜMLENEN MODÜLLER

Bu çalışmada eş eşatomik tümlenen, ⊕-eş eşatomik tümlenen, eş eşatomik zayıf tümlenen ve eş eşatomik bol tümlenen modüllerin tanımlanması ve bu modüllerin herhangi bir halka üzerinde ve değişmeli Noether halkaları özellikle Dedekind bölgeleri üzerinde incelenmesi amaçlanmıştır. R. Alizade, G. Bilhan and P. F. Smith'in tanımladığı eş sonlu alt modül tanımından hareketle eş eşatomik alt modülü tanımladık. Bir modülün öz alt modülüne göre bölüm modülü eşatomik oluyorsa o alt modüle eş eşatomik denir. Daha sonra eş eşatomik tümlenen modülü tanımladık. Tüm eş eşatomik alt modüllerinin tümleyeni bulunan modüle eş eşatomik tümlenen modül denir. Bir ayrık değerleme halkası üzerinde bir modülün eş eşatomik tümlenen olması için gerek ve yeter koşul modülün temel alt modülünün eşatomik olmasıdır. Yerel olmayan bir Dedekind bölgesi üzerinde, indirgenmiş bir modül eş eşatomik bol tümlenen ise modülün burulma alt modülüne göre bölüm modülü bölünebilirdir ve her maksimal ideal P için modülün P-bileşenleri sınırlıdır. Tersine, yerel olmayan bir Dedekind bölgesi üzerinde, indirgenmiş bir modülün burulma alt modülüne göre bölüm modülü bölünebilir ve her maksimal ideal P için modülün P-bileşenleri sınırlı ise modül eş eşatomik tümlenendir. Bir R halkasının sol mükemmel olması için gerek ve yeter koşul halkanın her dik toplamının ⊕-eş eşatomik tümlenen sol R-modül olmasıdır. Bir ayrık değerleme halkası üzerinde eş eşatomik tümlenen ve eş eşatomik zayıf tümlenen modüller çakışır. Her sol modülün eş eşatomik zayıf tümlenen olması için gerek ve yeter koşul halkanın sol mükemmel halka olmasıdır.

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LIST OF ABBREVIATIONS

R	an associative ring with unit unless otherwise stated
$\mathbb{N}$	the set of all positive integers
$\mathbb{Z}$	the ring of integers
$\mathbb{Q}$	the field of rational numbers
${}_RR$	left R-module R
${}_RM$	left R-module M
$End_R(M)$	a ring of homomorphisms from R-module M to M
$Hom_R(M,N)$	all R-module homomorphisms from M to N
$\mathrm{Ext}_R(C,A)$	the set of all equivalence classes of short exact sequences starting with the R-module A and ending with the R-module C
$\mathrm{Ker}\, f$	the kernel of the map f
$\mathrm{Im}\, f$	the image of the map f
$T(M)$	the torsion submodule of the R-module M for an integral domain R: $T(M) = \{m \in M \mid rm = 0 \text{ for some } 0 \neq r \in R\}$
$E(M)$	the injective envelope (hull) of a module M
$\mathrm{Soc}(M)$	the socle of the R-module M
$\mathrm{Soc}^{\oplus}(M)$	the sum of simple submodules of M which are direct summands of M
$\mathrm{Rad}(M)$	the radical of the R-module M
$\mathrm{Jac}(R)$	the Jacobson radical of the ring R
$\mathrm{Loc}(M)$	the sum of all local submodules of M
$\mathrm{Loc}^{\oplus}(M)$	the sum of all local submodules of M which are direct summand of M
$\mathrm{Art}(M)$	the sum of all artinian submodules of M
$\mathrm{Cof}(M)$	the sum of all cofinitely supplemented submodules of M
$\mathrm{cws}(M)$	the sum of all submodules K of M such that K is a weak supplement of a maximal submodule of M
$M^{(\mathbb{N})}$	the direct sum of M for index set $\mathbb{N}$
$\mathcal{P}$	the set of all maximal ideals for commutative rings
$\leq$	submodule
$\nleq$	not submodule

$\lneqq$ proper submodule

$\ll$ small(=superfluous) submodule

$\trianglelefteq$ essential submodule

$\cong$ isomorphic

CHAPTER 1

INTRODUCTION

Supplement submodules, supplemented modules (the modules every submodules of which have supplements) and some generalizations were intensively investigated in 1970's mainly by H. Zöschinger. The main results on this topic are published in monographs (Wisbauer, 1991) and (Clark et al., 2006).

The modules M, cofinitely supplemented, for which every submodule N with M/N finitely generated has supplements were studied in (Alizade et al., 2001). Afterwards $\oplus$-cofinitely supplemented, cofinitely semiperfect and cofinitely weak supplemented modules were studied intensively in the last ten years (see, e.g. (Çalışıcı and Pancar, 2004), (Çalışıcı and Pancar, 2005) and (Alizade and Büyükaşık, 2003)).

Coatomic modules are thoroughly investigated in 1980's by H. Zöschinger. H. Zöschinger gave characterizations of coatomic modules both over local and non-local rings in (Zöschinger, 1980). Then G. Güngöroğlu and A. Harmancı investigated coatomic modules (see (Güngöroğlu, 1998) and (Güngöroğlu and Harmancı, 1999)).

Every finitely generated module is coatomic, so we decided to generalize the results about cofinite submodules (i.e. submodules N of M with M/N finitely generated) to co-coatomic submodules (i.e. submodules N of M with M/N coatomic). In this thesis we obtain some results about co-coatomically supplemented, $\oplus$-co-coatomically supplemented, co-coatomically semiperfect, co-coatomically weak supplemented and co-coatomically amply supplemented modules.

Throughout this thesis R denotes associative ring with unity and M a left R-module unless otherwise stated.

In Chapter 2 we introduce some basic terminology for rings and modules and the fundamental results about modules and rings to be used in this study. There are also some information about supplements, supplemented modules, weakly supplemented modules, $\oplus$-supplemented modules, $\oplus$-cofinitely supplemented modules and cofinitely weak supplemented modules in this chapter. If every submodule U of M has a supplement V, i.e. V is minimal with respect to $M = U + V$, then M is said to be supplemented. A module M is said to be cofinitely supplemented if every cofinite submodule has a supplement. A module M is cofinitely supplemented if and only if every maximal submodule of M has a supplement over an arbitrary ring (see ((Alizade et al., 2001), Theorem 2.8)). Any sum of cofinitely supplemented modules is cofinitely supplemented (see (Alizade et al., 2001), Lemma 2.3). A module M is called $\oplus$-supplemented if every submodule has a supplement that is a direct summand of M. $\oplus$-supplemented modules are widely investigated in (Harmancı et al., 1999). If every cofinite submodule of a module M has a supplement that is a direct summand of M, then M is said to be $\oplus$-cofinitely supplemented module. A submodule N of M has a weak supplement in M if $M = N + K$ and $N \cap K \ll M$ for some submodule K of M. M is called cofinitely weak supplemented module if every cofinite submodule has a weak supplement in M. M is cofinitely weak supplemented if and only if every maximal submodule has a weak supplement (see (Alizade and Büyükaşık, 2003)).

In Chapter 3 we define co-coatomic submodule. A proper submodule N of M is called co-coatomic if M/N is coatomic, i.e. every proper submodule of M/N is contained in a maximal submodule of M/N. We generalize Lemma 2.7 in (Alizade et al., 2001) to co-coatomic submodules. A module M is said to be co-coatomically supplemented if every co-coatomic submodule has a supplement. For a co-coatomically supplemented submodule N of a module M, if M/N has no maximal submodule, then M is co-coatomically supplemented. Supplemented modules are co-coatomically supplemented, but the converse does not hold by Example 3.2. A finite direct sum of co-coatomically supplemented modules is co-coatomically supplemented, but an infinite direct sum of co-coatomically supplemented modules need not be co-coatomically supplemented by Examples 3.3 and 3.4. A co-coatomically supplemented module is cofinitely supplemented, but the converse is not true even over Dedekind domains, in particular over discrete valuation rings (DVR) and over semiperfect rings again by Examples 3.3 and 3.4. A ring R is called a left V-ring if every simple R-module is injective. Over a left V-ring, a module M is co-coatomically supplemented if and only if M is semisimple. Moreover, any direct sum of co-coatomically supplemented modules is co-coatomically supplemented over a left V-ring. A ring R is called left perfect if every left R-module has a projective cover. Every left R-module is co-coatomically supplemented if and only if R is left perfect ring. We examine co-coatomically supplemented modules over discrete valuation rings. An R-module M is called radical-supplemented if $\operatorname{Rad}(M)$ has a supplement in M (see (Zöschinger, 1974b)). Over a DVR, co-coatomically supplemented modules and radical-supplemented modules coincide. Therefore we obtain the following results by this equivalency. For a module M over a DVR, M is co-coatomically supplemented if and only if the basic submodule of M is coatomic. We deduce the structure of co-coatomically supplemented modules over a DVR: M is co-coatomically supplemented if and only if $M = T(M) \oplus X$ where the reduced part of $T(M)$ is bounded and $X/\operatorname{Rad}(X)$ is finitely generated. Then we examine co-coatomically supplemented modules over Dedekind domains. Over a Dedekind domain, M is a module whose co-coatomic submodules are direct summands if and only if the torsion part $T(M) = M_1 \oplus M_2$ where M_1 is semisimple, M_2 is divisible and $M/T(M)$ is divisible. Over a Dedekind domain, a torsion module M is co-coatomically supplemented if and only if M is co-coatomically weak supplemented. Over an integral domain R, the submodule $\{m \in M \mid P^n m = 0$ for some integer $n \geq 1\}$ is said to be the P-primary component of M for some prime ideal of R, and it is denoted by $T_P(M)$. Over a non-local Dedekind domain R, if the torsion part of a reduced module M has a weak supplement, then M is co-coatomically supplemented if and only if $M/T(M)$ is divisible and $T_P(M)$ is bounded for each maximal ideal P of R.

In Chapter 4 we deal with $\oplus$-co-coatomically supplemented modules. A module M is called $\oplus$-co-coatomically supplemented module if every co-coatomic submodule has a supplement that is a direct summand in M. An $\oplus$-Supplemented module is an $\oplus$-co-coatomically supplemented module, but the converse does not hold by Example 4.1. $\oplus$-Co-coatomically supplemented modules are $\oplus$-cofinitely supplemented, but the converse need not be true by Example 4.2. In contrast to co-coatomically supplemented modules, a direct summand of an $\oplus$-co-coatomically supplemented module need not be $\oplus$-co-coatomically supplemented by Example 4.3. A submodule U of M is called fully invariant if $f(U) \leq U$ for each $f \in End_R(M)$. If a module M is $\oplus$-co-coatomically supplemented, then M/U is $\oplus$-co-coatomically supplemented for a fully invariant submodule U of M. Furthermore, if U is a co-coatomic direct summand of M, then U is also $\oplus$-co-coatomically supplemented. Since $\operatorname{Rad}(M)$ and $\operatorname{Soc}(M)$ are fully invariant, if M is $\oplus$-co-coatomically supplemented, then $M/\operatorname{Rad}(M)$ and $M/\operatorname{Soc}(M)$ are $\oplus$-co-coatomically supplemented. Similar to co-coatomically supplemented modules, a finite direct sum of $\oplus$-co-coatomically supplemented modules is $\oplus$-co-coatomically supplemented. Property (D3) for an R-module M is the following: If M_1 and M_2 are direct summands of M with $M = M_1 + M_2$, then $M_1 \cap M_2$ is also a direct summand of M. If $\oplus$-co-coatomically supplemented module M has the property (D3), then a direct summand of M is $\oplus$-co-coatomically supplemented. An indecomposable module M such that $\operatorname{Rad}(M) \neq M$ is $\oplus$-co-coatomically supplemented if and only if M is local. A ring R is left perfect if and only if $R^{(\mathbb{N})}$ is a $\oplus$-co-coatomically supplemented R-module, where $\mathbb{N}$ is the set of natural numbers. An R-module M is called a multiplication module if every submodule of M is of the form IM for some ideal I of R. If a multiplication module M with $\operatorname{Rad}(M) \ll M$ is $\oplus$-co-coatomically supplemented, then M can be written as an irredundant sum of local direct summands of M. In Section 4.1 we define co-coatomically semiperfect modules.

A module M is called co-coatomically semiperfect if every coatomic factor module of M has a projective cover. A projective module M is a co-coatomically semiperfect module if and only if M is a $\oplus$-co-coatomically supplemented module. Every factor module of a co-coatomically semiperfect module is co-coatomically semiperfect. Therefore, every factor module of a projective $\oplus$-co-coatomically supplemented module is $\oplus$-co-coatomically supplemented. Let M be a module and N be a submodule of M. N is said to lie above a direct summand of M if there is a decomposition $M = K \oplus K'$ such that $K \leq N$ and $K' \cap N \ll K'$. For a projective module M, each co-coatomic submodule of M lies above a direct summand of M if and only if M is co-coatomically semiperfect. A small cover of a co-coatomically semiperfect module is co-coatomically semiperfect. A ring R is semiperfect if and only every finitely generated free R-module is semiperfect if and only if every finitely generated free R-module is co-coatomically semiperfect.

In Chapter 5 we define co-coatomically weak supplemented modules. A module M is called co-coatomically weak supplemented if every co-coatomic submodule has a weak supplement in M. A small cover of a co-coatomically weak supplemented module is a co-coatomically weak supplemented module. It follows that for an R-module M with small $\operatorname{Rad}(M)$, if $M/\operatorname{Rad}(M)$ is a co-coatomically weak supplemented module, then M is a co-coatomically weak supplemented module. A finite sum of co-coatomically weak supplemented modules is co-coatomically weak supplemented, but this is not valid for arbitrary sum of co-coatomically weak supplemented modules (see Example 5.4). A co-coatomically weak supplemented module is cofinitely weak supplemented, but the inverse is not true (see Example 5.4). A supplement in a co-coatomically weak supplemented module is co-coatomically weak supplemented. Over Dedekind domains, every weak supplement of a co-coatomic submodule is coatomic. For a short exact sequence $0 \to L \to M \to N \to 0$, if L and N are co-coatomically weak supplemented and L has a weak supplement, then M is co-coatomically weak supplemented. Every left R-module is co-coatomically weak supplemented if and only if R is a left perfect ring. A module M is co-coatomically weak supplemented if and only if $M/\bigoplus_{i=1}^{n} L_i$ is co-coatomically weak supplemented where each L_i is a local submodule of M. Over a discrete valuation ring, co-coatomically weak and co-coatomically supplemented modules coincide. Over a Dedekind domain, if $T(M)$ has a weak supplement in M, then M is co-coatomically weak supplemented if and only if $T(M)$ and $M/T(M)$ are co-coatomically weak supplemented.

In Chapter 6 we study co-coatomically amply supplemented modules and show some basic properties of co-coatomically amply supplemented modules. A submodule U of an R-module M has ample supplements in M if, for every submodule V of M with

$U + V = M$, there exists a supplement V' of U such that $V' \leq V$. A module is called co-coatomically amply supplemented if every submodule has an ample supplement. If M is co-coatomically amply supplemented module, then every supplement of a co-coatomic submodule is co-coatomically amply supplemented. Factor modules and direct summands of co-coatomically amply supplemented modules are also co-coatomically amply supplemented. If every submodule U of M is of the form $U = X+Y$, with X co-coatomically supplemented and $Y \ll M$, then M is co-coatomically amply supplemented. A finite direct sum of co-coatomically amply supplemented module need not be co-coatomically amply supplemented by Example 6.1. But by the condition as stated in the following, a finite direct sum of co-coatomically amply supplemented modules is co-coatomically amply supplemented. Let an R-module $M = M_1 \oplus \ldots \oplus M_n$ be a finite direct sum of co-coatomically amply supplemented submodules $M_i\,(1 \leq i \leq n)$ for some positive integer $n \geq 2$ such that $R = \operatorname{ann}(M_i) + \operatorname{ann}(M_j)$ for all $1 \leq i < j \leq n$. Then M is co-coatomically amply supplemented. An R-module M is called totally co-coatomically supplemented module if every submodule of M is co-coatomically supplemented. Every totally co-coatomically supplemented module is co-coatomically amply supplemented. A co-coatomically amply supplemented module M with coatomic factor module $M/\operatorname{Rad}(M)$ can be written as an irredundant sum of local modules and $\operatorname{Rad}(M)$. Therefore, over a discrete valuation ring, a co-coatomically amply supplemented module can be written as an irredundant sum of local modules and $\operatorname{Rad}(M)$. Over a non-local Dedekind domain, if a reduced module M is co-coatomically amply supplemented, then $M/T(M)$ is divisible and $T_P(M)$ is bounded for every maximal ideal P of R. Conversely, over a non-local Dedekind domain, if $M/T(M)$ is divisible and $T_P(M)$ is bounded for each maximal ideal P, then M is co-coatomically supplemented.

In Chapter 7 we define coatomically supplemented modules and coatomically $\oplus$-supplemented modules. A module is said to be coatomically supplemented if every coatomic submodule has a supplement in M. For a coatomically supplemented module, its factor module by a coatomic submodule is also coatomically supplemented. The sum $M = M_1 + M_2$ of two coatomically supplemented and coatomic modules M_1 and M_2 is also coatomically supplemented if every intersection of two coatomic submodules of M is coatomic. The module $M = M_1 \oplus M_2$ where M_1, M_2 are coatomic and coatomically supplemented is coatomically supplemented if M is quasi-projective. A module M is called coatomically H-supplemented if for every coatomic submodule N of M, there exists a direct summand L of M such that $M = N + X$ holds if and only if $M = L + X$ for some submodule X of M. M is called coatomically $\oplus$-supplemented if every coatomic submodule of M has a supplement that is a direct summand of M. Every coatomically H-supplemented module is coatomically $\oplus$-supplemented. For a coatomically supplemented module, if every maximal submodule of M is a direct summand of M, then M is coatomically $\oplus$-supplemented. For a coatomically supplemented module M with coatomic radical, every coatomic submodule of $M/\operatorname{Rad}(M)$ is a direct summand. For a module M with $\operatorname{Rad}(M) \ll M$ if every coatomic submodule of $M/\operatorname{Rad}(M)$ is a direct summand and every coatomic direct summand of $M/\operatorname{Rad}(M)$ lifts to a direct summand of M, then M is coatomically $\oplus$-supplemented. If a module M with property (D3) is coatomically $\oplus$-supplemented, then every direct summand of M is coatomically $\oplus$-supplemented. The direct sum $M = M_1 \oplus M_2$ of two coatomically $\oplus$-supplemented and coatomic modules M_1 and M_2 is also coatomically $\oplus$-supplemented if every intersection of two coatomic submodules of M is coatomic. The module $M = M_1 \oplus M_2$ where M_1, M_2 are coatomic and coatomically $\oplus$-supplemented is coatomically $\oplus$-supplemented if M is quasi-projective.

CHAPTER 2

PRELIMINARIES

Throughout this chapter R will be an associative ring with unity unless otherwise stated. Basic information about modules can be found in related references (Kasch, 1982), (Wisbauer, 1991) and (Anderson and Fuller, 1992). During this study we will use the following well known results.

2.1. Nilpotent Element, Nil and t-Nilpotent Ideals

An element a of a ring R is called nilpotent if there exists a positive integer n such that $a^n = 0$. An ideal is called a nil-ideal if all its elements are nilpotent .

A subset I of a ring R is called right (resp. left) t-nilpotent if for every sequence $a_1, a_2, \ldots, a_n, \ldots$ of elements $a_i \in I$ there exists a positive integer k such that $a_k a_{k-1} \ldots a_1 = 0$ (resp. $a_1 a_2 \ldots a_k = 0$). If I is a right (resp. left) ideal, then it is called right (left) t-nilpotent ideal.

Remark 2.1 *Clearly a left or right t-nilpotent ideal is a nil-ideal .*

2.2. Socle and Radical of A Module

Let M be an R-module. A submodule K of M is small (superfluous) in M, abbreviated $K \ll M$, if for every submodule $L \leq M$, $K + L = M$ implies $L = M$.

An epimorphism $g : M \to N$ is called small in case $\ker g \ll M$; such a module M with small epimorphism $g : M \to N$ is called small cover of N.

Lemma 2.1 *((Wisbauer, 1991), 19.2) An epimorphism $f : M \to N$ is small if and only if every homomorphism $h : L \to M$ with epimorphism fh is epimorphism.*

Lemma 2.2 *((Wisbauer, 1991), 19.3) Let M, N, K and L be R-modules. Then the following hold:*

1. *If $f : M \to N$ and $g : N \to L$ are two epimorphisms, then $g \circ f$ is small if and only if f and g are small.*

2. *If $K \leq L \leq M$, then $L \ll M$ if and only if $K \ll M$ and $L/K \ll M/K$.*

3. $L_1 + L_2 + \cdots + L_n \ll M$ *if and only if* $L_i \ll M$ $(1 \leq i \leq n)$.

4. *If $L \ll M$ and $\varphi : M \longrightarrow N$, then $\varphi(L) \ll N$.*

5. *If $K \leq L \leq M$ and L is a direct summand of M, then $K \ll L$ if and only if $K \ll M$.*

Definition 2.1 *Let M be an R-module. If every proper submodule of M is small in M, then M is called a hollow module.*

Definition 2.2 *Let M be an R-module. A submodule K of M is said to be large or essential if $K \cap L \neq 0$ for every non-zero submodule $L \leq M$, and this is denoted by $K \trianglelefteq M$.*

Definition 2.3 *An R-module M is called uniform if every non-zero submodule of M is essential in M.*

A submodule $A \leq M$ is called a simple submodule of M if $A \neq 0$ and for all $B \leq M$, $B \lneqq A$ then $B = 0$.

Definition 2.4 *Let $(S_\alpha)_{\alpha \in A}$ be an indexed set of simple submodules of M. If M is a direct sum of this set, then $M = \oplus_A S_\alpha$ is a semisimple decomposition of M. A module M is called semisimple if it has a semisimple decomposition.*

A sequence of R-modules and homomorphisms

$$\cdots \longrightarrow M_2 \longrightarrow M_1 \xrightarrow{\sigma_1} M_0 \xrightarrow{\sigma_0} M_{-1} \xrightarrow{\sigma_{-1}} M_{-2} \longrightarrow \cdots \tag{2.1}$$

is said to be exact at M_i if $\operatorname{Im} \sigma_{i+1} = \operatorname{Ker} \sigma_i$. The sequence is said to be exact if it is exact at each M_i. A sequence

$$0 \longrightarrow A \xrightarrow{f} B \tag{2.2}$$

of R-modules is exact if and only if f is one-to-one, and a sequence

$$B \xrightarrow{g} C \longrightarrow 0 \tag{2.3}$$

is exact if and only if g is onto. An exact sequence of the form

$$0 \longrightarrow K \xrightarrow{f} M \xrightarrow{g} N \longrightarrow 0 \tag{2.4}$$

is said to be a short exact sequence. In this case, $M/\operatorname{Im} f \cong N$. Furthermore, if $\operatorname{Im} f$ is a direct summand of M, then the short exact sequence is said to be split exact (or it is pronounced as the sequence splits).

Let A be an R-module, B a submodule of A, $\alpha : B \to A$ the inclusion map, i.e. $\alpha(b) = b$ for all $b \in B$, and $\beta : A \to A/B$ be the natural projection (or canonical epimorphism), i.e. $\beta(a) = a + B$, $a \in A$. Then α is a monomorphism, β is an epimorphism and $\operatorname{Im}\alpha = B = \ker\beta$. Thus the sequence

$$0 \longrightarrow B \xrightarrow{\alpha} A \xrightarrow{\beta} A/B \longrightarrow 0 \tag{2.5}$$

is exact.

Let $\mathcal{U}$ be a class of modules. A module M is (finitely) generated by $\mathcal{U}$ if there is a (finite) indexed set $(U_\alpha)_{\alpha\in A}$ in $\mathcal{U}$ and an epimorphism

$$\bigoplus_A U_\alpha \longrightarrow M \longrightarrow 0\ . \tag{2.6}$$

Let $\mathcal{U}$ be a class of modules. A module M is (finitely) cogenerated by $\mathcal{U}$ if there is a (finite) indexed set $(U_\alpha)_{\alpha\in A}$ in $\mathcal{U}$ and a monomorphism

$$0 \longrightarrow M \longrightarrow \prod_A U_\alpha \tag{2.7}$$

Theorem 2.1 *((Anderson and Fuller, 1992), Theorem 9.6) For an R-module M, the following are equivalent:*

1. *M is semisimple ;*

2. *M is generated by simple modules;*

3. *M is the sum of some set of simple submodules;*

4. *M is the sum of its simple submodules;*

5. *Every submodule of M is a direct summand;*

6. *Every short exact sequence*

$$0 \longrightarrow L \longrightarrow M \longrightarrow N \longrightarrow 0 \tag{2.8}$$

of R-modules splits.

Definition 2.5 *Let M be an R-module. The socle of M, denoted by* $\mathrm{Soc}(M)$*, is the sum of all simple submodules of M, equivalently the intersection of all essential submodules of M.*

Remark 2.2 *Note that* $\mathrm{Soc}(M)$ *is the largest semisimple submodule, and M is semisimple if and only if $M = \mathrm{Soc}(M)$.*

A submodule $A \leq M$ is called a maximal submodule of M if $A \neq M$ and for all $B \leq M$, $A \lneqq B$ then $B = M$.

Definition 2.6 *Let M be an R-module. The radical of M is the sum of all small submodules of M, equivalently the intersection of all maximal submodules of M. The radical of M is denoted by* $\mathrm{Rad}(M)$*. The radical of ${}_RR$ is said to be Jacobson radical of R and it is denoted by* $\mathrm{Jac}(R)$*, i.e* $\mathrm{Jac}(R) = \mathrm{Rad}({}_RR)$.

Definition 2.7 *A module M is called radical if* $\mathrm{Rad}(M) = M$.

Let $P(M) = \sum\{N \leq M | \mathrm{Rad}(N) = N\}$ for an R-module M. The module M is said to be reduced if $P(M) = 0$.

2.3. Projective and Injective Modules

Let P be an R-module. If M is an R-module, then P is called M-projective (or projective relative to M) if for each epimorphism $f : M \to N$ and each homomorphism

$g : P \to N$ there is an R-homomorphism $h : P \to M$ such that the following diagram

$$\begin{array}{ccccc} & & P & & \\ & \overset{h}{\swarrow} & \big\downarrow g & & \\ M & \underset{f}{\longrightarrow} & N & \longrightarrow & 0 \end{array} \qquad (2.9)$$

commutes. If P is P-projective, then P is also called self-projective (or quasi-projective).

An R-module P is called projective if it is projective relative to every module ${}_RM$.

Definition 2.8 *Let M be an R-module. A pair (P, p) is a projective cover of M if P is a projective R-module and*

$$P \xrightarrow{p} M \longrightarrow 0 \qquad (2.10)$$

is a small epimorphism (Ker $p \ll P$).

Let I be an R-module. If ${}_RM$ is a module, then I is called M-injective (injective relative to M) if for each monomorphism $f : N \to M$ and each homomorphism $g : N \to I$ there is an R-homomorphism $h : M \to I$ such that the diagram

$$\begin{array}{ccccc} 0 & \longrightarrow & N & \overset{f}{\longrightarrow} & M \\ & & g\big\downarrow & \overset{h}{\swarrow} & \\ & & I & & \end{array} \qquad (2.11)$$

commutes.

A module ${}_RI$ is injective if it is injective relative to every module ${}_RM$.

Definition 2.9 *Let M be an R-module. A pair (I, ε) is an injective hull of M in case I is an injective R-module and*

$$0 \longrightarrow M \xrightarrow{\varepsilon} I \qquad (2.12)$$

is an essential monomorphism ($\mathrm{Im}\,\varepsilon \trianglelefteq I$).

2.4. Noetherian and Artinian Modules

An R-module M is called Noetherian if every non-empty set of submodules of M has a maximal element.

Theorem 2.2 *((Kasch, 1982), Theorem 6.1.2) Let M be an R-module and L a submodule of M. The following properties are equivalent:*

1. *M is Noetherian.*
2. *L and M/L are Noetherian.*
3. *Every ascending chain $L_1 \leq L_2 \leq L_3 \leq \cdots$ of submodules of M is stationary.*
4. *Every submodule of M is finitely generated.*
5. *In every set $\{L_i \mid i \in I\} \neq \emptyset$ of submodules $L_i \leq M$ there is a finite subset $\{L_i \mid i \in I_0\}$ (i.e. finite $I_0 \subset I$) with*

$$\sum_{i \in I} L_i = \sum_{i \in I_0} L_i. \tag{2.13}$$

An R-module M is Artinian if every non-empty set of submodules has a minimal element.

Theorem 2.3 *((Kasch, 1982), Theorem 6.1.2) Let M be an R-module and L a submodule of M. The following properties are equivalent:*

1. *M is Artinian.*
2. *L and M/L are Artinian.*
3. *Every descending chain $L_1 \geq L_2 \geq L_3 \geq \cdots$ of submodules of M is stationary.*
4. *Every factor module of M is finitely cogenerated.*
5. *In every set $\{L_i \mid i \in I\} \neq \emptyset$ of submodules $L_i \leq M$ there is a finite subset $\{L_i \mid i \in I_0\}$ (i.e. finite $I_0 \subset I$) with*

$$\bigcap_{i \in I} L_i = \bigcap_{i \in I_0} L_i. \tag{2.14}$$

A ring R is left Noetherian, respectively Artinian if $_RR$ is Noetherian, respectively Artinian.

2.5. Local and Semilocal Modules

A module M is called local if M has a largest submodule, i.e. a proper submodule which contains all other proper submodules.

Proposition 2.1 *((Wisbauer, 1991), 41.4(2)) For an R-module M, the following assertions are equivalent:*

1. *M is hollow and* $\mathrm{Rad}(M) \neq M$,
2. *M is hollow and cyclic (or finitely generated),*
3. *M is local.*

Note that a module is local if and only if it is non-zero, cyclic, and has a unique maximal proper submodule.

An R-module M is called semilocal if $M/\mathrm{Rad}(M)$ is semisimple.

2.6. Local and Semilocal Rings

A ring R is called a local ring in case the set of non-invertible elements of R is closed under addition.

Definition 2.10 *Let R be a ring. If every non-zero element of R is invertible, then R is called a division ring. A commutative division ring is a field.*

Proposition 2.2 *((Anderson and Fuller, 1992), Proposition 15.15) The following are equivalent for a ring R with* $\mathrm{Jac}(R)$*:*

1. *R is local;*
2. *R has a unique maximal left ideal;*
3. $\mathrm{Jac}(R)$ *is a maximal left ideal;*
4. *The set of elements of R without left inverses is closed under addition;*

5. $\operatorname{Jac}(R) = \{x \in R \mid Rx \neq R\}$;

6. $R/\operatorname{Jac}(R)$ *is a division ring;*

7. $\operatorname{Jac}(R) = \{x \in R \mid x$ *is not invertible* $\}$;

8. *If* $x \in R$*, then either* x *or* $1 - x$ *is invertible.*

   A commutative ring is called a local ring if it has a unique maximal ideal.

Definition 2.11 *A ring* R *is called semilocal if* ${}_RR$ *is semilocal, i.e.* $R/\operatorname{Jac}(R)$ *is left semisimple.*

Proposition 2.3 *((Lam, 2001), Proposition 20.2) For a ring* R*, consider the following two conditions:*

1. *R is semilocal,*

2. *R has finitely many maximal left ideals.*

In general, (2) $\Rightarrow$ (1). The converse holds if $R/\operatorname{Jac}(R)$ is commutative.

2.7. Dedekind Domains

An integral domain is a commutative ring without zero-divisors. Let R be an integral domain and M an R-module. The submodule $T(M) = \{m \in M \mid rm = 0$ for some $0 \neq r \in R\}$ of M is said to be the torsion submodule of M. If $T(M) = M$, then M is called a torsion module; if $T(M) = 0$, then M is called a torsion-free module. Let P be a prime ideal of R. The submodule $\{m \in M \mid P^n m = 0$ for some $n \geq 1\}$ is called the P-primary part of M. This submodule is denoted by $T_P(M)$.

By R_S we denote the localization of R at the multiplicatively closed subset S of R (see (Sharp, 2000)).

Let M be a torsion-free R-module. The maximum number of linearly independent elements of M is called rank of M.

A commutative ring R is a valuation ring in case its ideals are totally ordered by inclusion. Additionally, if R is an integral domain, it is called a valuation domain. A Noetherian valuation domain is said to be a discrete valuation ring (in brief DVR). If R is a DVR that is not a field, then all of its non-zero ideals are: $R > Rp > \cdots > Rp^n > \cdots$ where $p \in R$ is the unique prime element (see ((Fuchs and Salce, 1985), Proposition 1.7)).

Let R be an integral domain and Q its field of fractions. An element of Q is said to be integral over R if it is a root of a monic polynomial in $R[X]$ (the ring of polynomials in X with coefficients in R). A commutative domain R is integrally closed if the elements of Q which are integral over R are just the elements of R.

Definition 2.12 *An integral domain R is a Dedekind domain if the following hold:*

1. *R is a Noetherian ring.*

2. *R is integrally closed in its field of fractions Q.*

3. *All non-zero prime ideals of R are maximal.*

For commutative rings, $\mathcal{P}$ denotes the set of all maximal ideals.

Remark 2.3 *Every local Dedekind domain is a DVR.*

Let R be a commutative ring; for $a \in R$ the ideal $aR := \{ar : r \in R\}$ of R is called principal ideal of R generated by a. An integral domain R is called a principal ideal domain if every ideal of R is principal. A principal ideal domain is a Dedekind domain.

Theorem 2.4 *((Cohn, 2002), Proposition 10.6.9) Over a Dedekind domain R, any torsion module M is a direct sum of its primary parts in a unique way:*

$$M = \bigoplus_{P \in \mathcal{P}} T_P(M) \tag{2.15}$$

and when M is finitely generated, only finitely many terms on the right side are different from zero.

Over an integral domain, a module M is divisible if $M = cM$ for every $0 \neq c \in R$.

Every injective module is divisible. Over a Dedekind domain, every divisible module is injective ((Sharpe and Vámos, 1972), Proposition 2.10). Therefore the following lemma is a direct result of Lemma 4.4 in (Alizade et al., 2001).

Lemma 2.3 *((Büyükaşık, 2005), Lemma 1.7.2) Let R be a Dedekind domain and M an R-module. The following statements are equivalent:*

1. *M is injective.*

2. *M is divisible.*

3. *$M = PM$ for every maximal ideal P of R.*

4. *M does not contain a maximal submodule.*

2.8. Coatomic Modules

This section includes some results about coatomic modules. More information can be found in (Zöschinger, 1980) and (Güngöroğlu, 1998).

An R-module M is called coatomic if every proper submodule is contained in a maximal submodule of M (Zöschinger, 1980).

Proposition 2.4 *(Zöschinger, 1974a) Let M be an R-module. Then M is coatomic if and only if for every submodule N of M,* $\mathrm{Rad}(M/N) = M/N$ *implies* $M/N = 0$.

Proof Clear by definition of coatomic module. □

Theorem 2.5 *((Kasch, 1982), Theorem 2.3.11) If an R-module M is finitely generated, then every proper submodule of M is contained in a maximal submodule of M.*

Corollary 2.1 *Every finitely generated nonzero module is coatomic.*

Example 2.1 *A semisimple module M is coatomic since every submodule of M is a direct summand, and therefore every submodule is contained in a maximal submodule of M (see (Anderson and Fuller, 1992), Lemma 9.2).*

The following proposition gives a general property of coatomic modules.

Proposition 2.5 *For a coatomic module M,* $\mathrm{Rad}(M) \ll M$.

Proof Let M be a coatomic R-module. Suppose $M = \mathrm{Rad}(M) + N$ for some proper submodule N of M. Since M is coatomic, N is contained in a maximal submodule of M, say K. It follows from the definition of radical that $M = \mathrm{Rad}(M) + N \leq K$, so $M = K$, contradicting to the maximality of K. Thus for every proper submodule N of M, $\mathrm{Rad}(M) + N \neq M$, i.e. $\mathrm{Rad}(M) \ll M$. □

As the following example shows a submodule of a coatomic module need not be coatomic in general.

Example 2.2 *Let p be a prime number. The abelian group $\mathbb{Z}_{(p^\infty)}$ (p-component of $\mathbb{Q}/\mathbb{Z}$) is a $\mathbb{Z}$-module. Now consider the ring:*

$$R = \begin{pmatrix} \mathbb{Z} & 0 \\ \mathbb{Z}_{(p^\infty)} & \mathbb{Z} \end{pmatrix} \tag{2.16}$$

${}_RR$ is coatomic since it is finitely generated. Consider the submodule

$$I = \begin{pmatrix} 0 & 0 \\ \mathbb{Z}_{(p^\infty)} & 0 \end{pmatrix} \tag{2.17}$$

of ${}_RR$. The left R-module I has the same structure with the left $\mathbb{Z}$-module $\mathbb{Z}_{(p^\infty)}$. ${}_{\mathbb{Z}}\mathbb{Z}_{(p^\infty)}$ is not coatomic since it has no maximal submodule. Therefore I is not coatomic since ${}_{\mathbb{Z}}\mathbb{Z}_{(p^\infty)}$ is not coatomic.

Theorem 2.6 *((Zöschinger, 1980), Lemma 1.1) Let M be a coatomic module over a commutative Noetherian ring. Then every submodule of M is coatomic.*

Proof Let R be a commutative ring and M a coatomic module. Let U be a submodule of M. Suppose for the contrast, U has a nonzero radical factor module, that is, there exists $N \lneqq U$,

$$\mathrm{Rad}(U/N) = U/N \neq 0 \tag{2.18}$$

Since $U/N \neq 0$, there exists a nonzero homomorphism $h : U/N \to E$ where E is the injective hull of a simple module by ((Sharpe and Vámos, 1972), Proposition 2.24). Let f be the composition $f = h \circ \sigma$, where

$$U \xrightarrow{\sigma} U/N \xrightarrow{h} E, \tag{2.19}$$

σ is the canonical epimorphism. Then

$$\operatorname{Im} f = \operatorname{Im} h = h(U/N) = h(\operatorname{Rad}(U/N)) \leq \operatorname{Rad}(\operatorname{Im} h) \leq \operatorname{Im} h \tag{2.20}$$

and so

$$\operatorname{Rad}(\operatorname{Im} h) = \operatorname{Im} h. \tag{2.21}$$

Therefore $\operatorname{Im} f$ is radical module, that is

$$\operatorname{Rad}(\operatorname{Im} f) = \operatorname{Im} f, \tag{2.22}$$

and $\operatorname{Im} f \neq 0$. By ((Matlis, 1960), Proposition 3), E is Artinian. Let T be a coatomic submodule of E. Since E is Artinian, T is also Artinian. $T/\operatorname{Rad}(T)$ is also Artinian and $\operatorname{Rad}(T/\operatorname{Rad}(T)) = 0$. By ((Anderson and Fuller, 1992), Proposition 10.15), the module $T/\operatorname{Rad}(T)$ is finitely generated and semisimple. Since T is coatomic, $\operatorname{Rad}(T) \ll T$. Since $T/\operatorname{Rad}(T)$ is finitely generated and $\operatorname{Rad}(T) \ll T$, T is finitely generated. As a result, every coatomic submodule of E is finitely generated. Since E is injective, there exists an extension $g : M \to E$ of $f : U \to E$. Therefore $\operatorname{Im} f \leq \operatorname{Im} g$. Since M is coatomic, so is any homomorphic image of it by ((Zöschinger, 1974a), Lemma 1.5). Hence, $\operatorname{Im} g$ is a coatomic submodule of E, and so it is finitely generated. Thus $\operatorname{Im} g$ is a Noetherian R-module since R is Noetherian ring and $\operatorname{Im} g$ is finitely generated. Then $\operatorname{Im} f$ is finitely generated. But $\operatorname{Rad}(\operatorname{Im} f) = \operatorname{Im} f \neq 0$, so $\operatorname{Im} f$ can not be finitely generated. This contradiction shows that U must be coatomic. □

Lemma 2.4 *((Zöschinger, 1974a), Lemma 1.13(1) and (Güngöroğlu, 1998), Lemma 3) Let*

$$0 \to L \to M \to K \to 0 \tag{2.23}$$

be an exact sequence of R-modules. Then

1. *If M is a coatomic module, then K is a coatomic module.*

2. *If K and L are coatomic modules, then M is a coatomic module.*

3. *If R is a Discrete Valuation Ring and M is a coatomic module, then L is a coatomic module.*

Proof

1. We can assume that $L \leq M$ and $K = M/L$. Let L'/L be a submodule of M/L such that $\text{Rad}(M/L') = M/L'$. Since M is coatomic, $M = L'$, i.e. $M/L = L'/L$. Thus M/L is coatomic.

2. Let N be a proper submodule of M.
Case 1: Assume $L + N = M$. Let P be a maximal submodule of L containing $L \cap N$. Then

$$\begin{aligned} M/(N+P) &= (L+N)/(N+P) \\ &= (L+N+P)/(N+P) \\ &\cong L/(L \cap (P+N)) \\ &= L/P \end{aligned} \tag{2.24}$$

is simple. Therefore $N + P$ is maximal submodule in M. Thus N is contained in a maximal submodule of M.
Case 2: Now assume $N + L \neq M$. Then $(L + N)/L \neq M/L$. Since $K = M/L$ is coatomic, there exists a maximal submodule N'/L of M/L such that $(L + N)/L \leq N'/L$. Then N' becomes maximal submodule of M that contains N. Hence M is coatomic.

3. Clear by Theorem 2.6.

□

Corollary 2.2 *((Güngöroğlu, 1998), Lemma 4) Let $M = M_1 \oplus M_2$. Then M is coatomic if and only if M_1 and M_2 are coatomic.*

Corollary 2.3 *((Güngöroğlu, 1998), Corollary 5) Let $M = \bigoplus_{i=1}^{n} M_i$ be a finite direct sum of submodules M_i of M for $(i = 1, 2, \ldots, n)$. Then M is coatomic if and only if each M_i is coatomic, $(i = 1, 2, \ldots, n)$.*

Note that by Corollary 2.3, any direct summand of a coatomic module is coatomic.

Lemma 2.5 *Let M be an R-module. For every small submodule N of M, if M/N is coatomic, then M is coatomic.*

Proof Suppose M/N is coatomic, where N is a small submodule of M. Let K be a proper submodule of M.
Case 1: Assume $N \leq K$. Since K is a proper submodule of M, K/N is a proper submodule of M/N. Since M/N is coatomic, there exists a maximal submodule of M/N that contains K/N. Thus K is contained in a maximal submodule of M.
Case 2: Assume $N \nleq K$. Consider the submodule $(K+N)/N$ of M/N which is proper. Since M/N is coatomic, $(K+N)/N$ is contained in a maximal submodule of M/N. Hence K is contained in a maximal submodule of M. □

Lemma 2.6 *((Zöschinger, 1974a), Lemma 1.13(2) and (Güngöroğlu, 1998), Lemma 8) Let M be a module and U, V submodules of M such that V is a supplement of U (i.e. $U + V = M$ and $U \cap V \ll V$). Then V is coatomic if and only if M/U is coatomic.*

Proof (⇒) Since V is a supplement of the submodule U of M, $M = U + V$ and $V \cap U \ll V$. Then

$$M/U = (U + V)/U \cong V/(V \cap U). \tag{2.25}$$

By hypothesis V is coatomic, and so $V/(V \cap U)$ is coatomic. Therefore M/U is coatomic.
(⇐) Let K be a proper submodule of V. Since $V \cap U$ is small in V, $K + (V \cap U) \neq V$. Therefore

$$(K + (V \cap U))/(V \cap U) \neq V/(V \cap U). \tag{2.26}$$

Since $V/(V \cap U)$ is coatomic, there exists a maximal submodule N of V that contains $K + (V \cap U)$. Thus V is coatomic. □

Lemma 2.7 *((Güngöroğlu and Harmancı, 1999), Lemma 2.1) Let R be a Dedekind domain and M an R-module. Then the following hold:*

1. *If N is a small submodule in M, then N is coatomic.*

2. $\mathrm{Rad}(M)$ *is small in M if and only if* $\mathrm{Rad}(M)$ *is coatomic.*

3. *If M is a divisible R-module then M is not coatomic.*

2.9. Supplemented Modules

This section contains definitions, some results about supplement and supplemented modules. For more information see ((Wisbauer, 1991), §41)

Definition 2.13 *A module M is called supplemented if every submodule U of M has a supplement V, i.e. V is minimal in the collection of submodules L of M such that $M = U + L$.*

Lemma 2.8 *(Zöschinger, 1974a) V is a supplement of U in M if and only if $U + V = M$ and $U \cap V \ll V$.*

The following proposition gives some properties of supplement.

Lemma 2.9 *((Zöschinger, 1974a), Lemma 1.2) Let U and V be submodules of M such that V is a supplement of U in M. Then*

1. *For $X \lneqq V$, V/X is not small in M/X.*
2. *For every $X \leq V$, if $X \ll M$ then $X \ll V$.*
3. $\mathrm{Rad}(V) = V \cap \mathrm{Rad}(M)$.
4. $\mathrm{Rad}(M/U) = (\mathrm{Rad}(M) + U)/U$.
5. $\mathrm{Rad}(M) = (V + \mathrm{Rad}(M)) \cap (U + \mathrm{Rad}(M)) = (V \cap \mathrm{Rad}(M)) + (U \cap \mathrm{Rad}(M))$.

Lemma 2.10 *((Wisbauer, 1991), 41.1) Let $U, V \leq M$ such that V is a supplement of U in M.*

1. *If U is a maximal submodule of M, then V is cyclic and $U \cap V = \mathrm{Rad}(V)$ is a (the unique) maximal submodule of V.*
2. *For a submodule $L \leq U$, $(V + L)/L$ is a supplement of U/L in M/L.*
3. *If M is finitely generated, then V is also finitely generated.*

Note that a hollow module is supplemented since every proper submodule is small. Since a local module is hollow, a local module is supplemented (see Proposition 2.1).

Definition 2.14 *Let M be an R-module. Then an R-module N is called (finitely) M-generated if it is a homomorphic image of a (finite) direct sum of copies of M.*

The following proposition gives some basic and important properties of supplemented modules.

Proposition 2.6 *((Wisbauer, 1991), 41.2) For an R-module M, the following properties hold:*

1. *Let U and V be submodules of M such that U is supplemented and $U + V$ has a supplement in M. Then V has a supplement in M.*
2. *If $M = M_1 + M_2$ with M_1 and M_2 are supplemented modules, then M is also supplemented.*
3. *If M is supplemented , then:*
   (a) *Every finitely M-generated module is supplemented .*
   (b) *$M/\operatorname{Rad}(M)$ is semisimple .*

Example 2.3 *Semisimple and Artinian modules are supplemented.*

If $M = \sum_{\Lambda} M_\lambda$, then the sum is called irredundant if, for every $\lambda_0 \in \Lambda$, $\sum_{\lambda \neq \lambda_0} M_\lambda \neq M$.

Proposition 2.7 *((Inoue, 1983), Proposition 9) If M is a supplemented module and* $\operatorname{Rad}(M)$ *is small in M, then M is written as an irredundant sum of local modules.*

2.9.1. Supplemented Modules Over Dedekind Domains

An R module F with a linearly independent spanning set $(x_\alpha)_{(\alpha \in A)}$ is called a free R-module (of rank card A) with basis $(x_\alpha)_{\alpha \in A}$. A module ${}_RF$ is free if it is isomorphic to a direct sum of copies of ${}_RR$.

A module B is called bounded if $rB = 0$ for some non-zero $r \in R$.

Lemma 2.11 *((Zöschinger, 1974a), Lemma 2.1) Let R be a DVR. For an R-module M, the following are equivalent:*

1. *M has a small radical.*
2. *M is coatomic.*
3. *M is a direct sum of a finitely generated free submodule and a bounded submodule.*
4. *M is reduced and supplemented .*

Corollary 2.4 *((Zöschinger, 1974a), 2nd Folgerung p.48) Let R be a DVR and M an R module. If M is torsion, reduced and radical of M has a supplement in M, then M is bounded.*

Theorem 2.7 *((Zöschinger, 1974a), Theorem 3.1) Let R be a non-local Dedekind domain. An R-module M is supplemented if and only if it is torsion and every primary component is supplemented.*

2.10. Weakly Supplemented Modules

Let M be an R-module and U a submodule of M. A submodule V of M is called weak supplement of U if $M = U + V$ and $U \cap V \ll M$. M is called weakly supplemented module if every submodule of M has a weak supplement in M.

A submodule N of a module M is said to be closed if N has no proper essential extension in M, i.e if $N \trianglelefteq L$ for some submodule L of M then $L = N$.

Definition 2.15 *A submodule L of M is called coclosed in M if L has no proper submodule K for which* $L/K \ll M/K$.

Lemma 2.12 *((Clark et al., 2006), 20.2) For a submodule N of M, the following are equivalent:*

1. *N is a supplement in M;*
2. *N is a weak supplement in M that is coclosed in M;*
3. *N is a weak supplement in M and, whenever* $K \leq N$ *and* $K \ll M$*, then* $K \ll N$.

Proposition 2.8 *((Lomp, 1999), Proposition 2.2) Assume that M a is weakly supplemented module. Then the following hold:*

1. *M is semilocal;*
2. $M = M_1 + M_2$ *with* M_1 *semisimple,* M_2 *semilocal and* $\mathrm{Rad}(M) \trianglelefteq M_2$*;*
3. *Every factor module of M is weakly supplemented;*
4. *Every small cover of M is weakly supplemented;*
5. *Every supplement in M and every direct summand of M is weakly supplemented.*

2.11. Perfect and Semiperfect Rings

This section contains definitions and some results about semiperfect rings and perfect rings. One can find more information in ((Wisbauer, 1991), §42 and §43).

A left R-module is called a semiperfect module if it is projective and every homomorphic image of it has projective cover.

Every non-zero module M has at least two direct summands, namely 0 and M. A non-zero module M is indecomposable if 0 and M are its only direct summands.

A submodule U of an R-module M has ample supplements in M if, for every submodule V of M with $U + V = M$, there exists a supplement V' of U such that $V' \leq V$. A module M is called amply supplemented if every submodule of M has an ample supplement.

Proposition 2.9 *((Wisbauer, 1991), 42.6) The following statements are equivalent for a ring R:*

1. ${}_RR$ *is semiperfect;*
2. ${}_RR$ *is supplemented ;*
3. *Every finitely generated R-module is semiperfect;*
4. *Every finitely generated R-module has projective cover;*
5. *Every finitely generated R-module is (amply) supplemented ;*
6. $R/\operatorname{Jac}(R)$ *is left semisimple and idempotents in* $R/\operatorname{Jac}(R)$ *can be lifted to R;*
7. *Every simple R-module has a projective cover;*
8. *Every maximal left ideal has a supplement in R;*
9. ${}_RR$ *is a direct sum of local (projective covers of simple) modules;*
10. $R = Re_1 \oplus \ldots \oplus Re_k$ *for local orthogonal idempotents* e_i*;*
11. R_R *is semiperfect.*

If R satisfies these conditions, then R is said to be a semiperfect ring.

The following proposition gives the relation between a perfect ring and its t-nilpotent Jacobson radical. ${}_RR^{(\mathbb{N})}$ denotes the direct sum of R-module R by index set $\mathbb{N}$. Note that $\mathbb{N}$ denotes the set of all positive integers.

Proposition 2.10 *((Wisbauer, 1991), 43.9) For a ring R, the following assertions are equivalent:*

1. *$_RR$ is perfect;*
2. *Every (indecomposable) flat left R-module is semiperfect;*
3. *Every left R-module (or only $R^{(\mathbb{N})}$) is semiperfect;*
4. *Every left R-module has a projective cover;*
5. *Every left R-module is (amply) supplemented ;*
6. *$R/\operatorname{Jac}(R)$ is left semisimple and $\operatorname{Rad}({}_RR^{(\mathbb{N})}) \ll_R R^{(\mathbb{N})}$;*
7. *The ascending chain condition for cyclic left R-modules holds;*
8. *$End_R(R^{(\mathbb{N})})$ is f-semiperfect;*
9. *$R/\operatorname{Jac}(R)$ is left semisimple and $\operatorname{Jac}(R)$ is right t-nilpotent ;*
10. *R satisfies the descending chain condition for cyclic right ideals;*
11. *R contains no infinite set of orthogonal idempotents and every non-zero right R-module has non-zero socle.*

A ring which satisfies one of these equivalent properties is called left perfect.

Lemma 2.13 *((Smith, 2000), Lemma 1.6) For a ring R, the following are equivalent:*

1. *R is a perfect ring;*
2. *Every left R-module is supplemented ;*
3. *Every left R-module is amply supplemented ;*
4. *The left R-module $R^{(\mathbb{N})}$ is supplemented .*

The following theorem gives a convenient characterization of perfect rings.

Theorem 2.8 *((Büyükaşık and Lomp, 2009), Theorem 1) The following statements are equivalent for a ring R:*

1. *Every left R-module is weakly supplemented;*
2. *$R^{(\mathbb{N})}$ is weakly supplemented as a left R-module;*

3. *R is semilocal and* $\mathrm{Rad}({}_RR^{(\mathbb{N})})$ *has a weak supplement in* ${}_RR^{(\mathbb{N})}$.

4. *R is left perfect.*

2.12. ⊕-Supplemented Modules

Let R be an arbitrary ring and M an R-module. M is called ⊕-supplemented module if every submodule of M has a supplement that is a direct summand of M.

Note that hollow modules are ⊕-supplemented.

Clearly ⊕-supplemented modules are supplemented, but the converse is not correct in general (see (Mohamed and Müller, 1990), Lemma A.4(2)).

The following lemma gives the equivalency of supplemented and ⊕-supplemented module if the module is projective.

Lemma 2.14 *((Harmancı et al., 1999), Lemma 1.2) Let M be a projective module. Then the following statements are equivalent.*

1. *M is semiperfect.*

2. *M is supplemented .*

3. *M is ⊕-supplemented.*

Let R be a ring and M an R-module. Property (D3) for a module M is the following:

(D3) If M_1 and M_2 are direct summands of M with $M = M_1 + M_2$, then $M_1 \cap M_2$ is also a direct summand of M (see (Mohamed and Müller, 1990), p.57).

Definition 2.16 *An R-module M is called completely ⊕-supplemented if every direct summand of M is ⊕-supplemented.*

Proposition 2.11 *((Harmancı et al., 1999), Proposition 2.3) Let M be an ⊕-supplemented module with (D3). Then M is completely ⊕-supplemented .*

2.13. Cofinitely Supplemented Modules

This section involves definition and some results about cofinitely supplemented modules. For more information see (Alizade et al., 2001).

Definition 2.17 *(Alizade et al., 2001) A submodule N of an R-module M is called cofinite if M/N is finitely generated.*

Lemma 2.15 *((Alizade et al., 2001), Lemma 2.7) Let R be any ring. For an R-module M, the following statements are equivalent:*

1. *Every cofinite submodule of M is a direct summand of M.*
2. *Every maximal submodule of M is a direct summand of M.*
3. *$M/\operatorname{Soc}(M)$ does not contain a maximal submodule.*

Definition 2.18 *(Alizade et al., 2001) An R-module M is called cofinitely supplemented if every cofinite submodule of M has a supplement in M.*

Lemma 2.16 *((Alizade et al., 2001), Lemma 2.3) Let N_i, $(i \in I)$, I an index set, be any collection of cofinitely supplemented submodules of a module M. Then $\sum_{i\in I} N_i$ is a cofinitely supplemented submodule of M.*

Corollary 2.5 *((Alizade et al., 2001), Corollary 2.4) Any direct sum of cofinitely supplemented modules is cofinitely supplemented .*

$\operatorname{Loc}(M)$ denotes the sum of all local submodules of M. $\operatorname{Cof}(M)$ denotes the sum of all cofinitely supplemented submodules of M (Alizade et al., 2001).

Theorem 2.9 *((Alizade et al., 2001), Theorem 2.8) Let R be a ring. For an R-module M, the following statements are equivalent:*

1. *M is cofinitely supplemented .*
2. *Every maximal submodule of M has a supplement in M.*
3. *The module $M/\operatorname{Loc}(M)$ does not contain a maximal submodule.*
4. *The module $M/\operatorname{Cof}(M)$ does not contain a maximal submodule.*

A module M is called amply cofinitely supplemented if every cofinite submodule of M has an ample supplement.

Theorem 2.10 *((Alizade et al., 2001), Theorem 2.13) For any ring R, the following statements are equivalent:*

1. *R is semiperfect.*

2. *Every right R-module is amply cofinitely supplemented.*

3. *Every right R-module is cofinitely supplemented.*

4. *Every left R-module is amply cofinitely supplemented.*

5. *Every left R-module is cofinitely supplemented.*

Lemma 2.17 *((Alizade et al., 2001), Corollary 4.2) Let M be a module over a non-local commutative domain R. If M is cofinitely supplemented, then $M/T(M)$ does not contain a maximal submodule.*

2.14. ⊕-Cofinitely Supplemented Modules

A module M is called ⊕-cofinitely supplemented if every cofinite submodule of M has a supplement that is a direct summand in M (Çalışıcı and Pancar, 2004).

Proposition 2.12 *((Çalışıcı and Pancar, 2004), Proposition 2.4) Let M be an ⊕-cofinitely supplemented module with (D3). Then every cofinite direct summand of M is ⊕-cofinitely supplemented .*

Let $\{L_\lambda\}_{\lambda\in\Lambda}$, where Λ is an index set, be the family of local submodules of M such that each of them is a direct summand of M. $\mathrm{Loc}^{\oplus}(M)$ will denote the sum of L_λs for all $\lambda \in \Lambda$, that is $\mathrm{Loc}^{\oplus}(M) = \sum_{\lambda\in\Lambda} L_\lambda$ (Çalışıcı and Pancar, 2004). Clearly $\mathrm{Loc}^{\oplus}(M) \leq \mathrm{Loc}(M)$.

Theorem 2.11 *((Çalışıcı and Pancar, 2004), Theorem 2.6) For any ring R, arbitrary direct sum of ⊕-cofinitely supplemented R-modules is ⊕-cofinitely supplemented .*

2.15. Cofinitely Weak Supplemented Modules

A module M is called cofinitely weak supplemented (cws-module in short) if every cofinite submodule has a weak supplement in M (Alizade and Büyükaşık, 2003).

For a module M, let Γ denote the set of all submodules K such that K is a weak supplement for some maximal submodule of M and $\mathrm{cws}(M)$ the sum of all submodules from Γ. $\mathrm{cws}(M) = 0$ if $\Gamma = \emptyset$ (see (Alizade and Büyükaşık, 2003)).

Theorem 2.12 *((Alizade and Büyükaşık, 2003), Theorem 2.16) For a module M, the following are equivalent:*

1. *M is a* cws*-module.*
2. *Every maximal submodule of M has a weak supplement.*
3. *$M/\operatorname{cws}(M)$ has no maximal submodule.*

Lemma 2.18 *((Smith, 2000), by Lemma 2.4) Let M be a finitely generated module with zero radical and N a non-finitely generated submodule of M. Then N does not have any weak supplement in M.*

Lemma 2.19 *((Alizade and Büyükaşık, 2003), Lemma 2.4) Let M and N be modules. If $f : M \to N$ is a homomorphism and a submodule L including $\operatorname{Ker} f$ is a weak supplement in M, then $f(L)$ is a weak supplement in $f(M)$.*

Lemma 2.20 *((Alizade and Büyükaşık, 2003), Lemma 2.8) Let M and N be modules. If $f : M \to N$ is a small epimorphism, then a submodule L of M is a weak supplement in M if and only if $f(L)$ is a weak supplement in N.*

Lemma 2.21 *((Alizade and Büyükaşık, 2003), Corollary 2.22) Let R be a ring. Then R is semilocal if and only if every left R-module a* cws*-module.*

Proposition 2.13 *((Alizade and Büyükaşık, 2003), Proposition 2.12) An arbitrary sum of cws-modules is a cws-module.*

2.16. Extensions as Short Exact Sequences

This section consists of definition of pull back, push out and information about the group extensions by short exact sequences. For more information see (Wisbauer, 1991), (Fuchs, 1970) and (Mac Lane, 1963).

Definition 2.19 *Let $f_1 : M_1 \to M$, $f_2 : M_2 \to M$ be two homomorphisms. A commutative diagram*

$$\begin{array}{ccc} P & \xrightarrow{p_2} & M_2 \\ {\scriptstyle p_1}\downarrow & & \downarrow{\scriptstyle f_2} \\ M_1 & \xrightarrow{f_1} & M \end{array} \tag{2.27}$$

is called the pullback for the pair (f_1, f_2) *if, for every pair of homomorphisms*

$$g_1 : X \to M_1,\ g_2 : X \to M_2 \tag{2.28}$$

with $f_1 g_1 = f_2 g_2$*, there is a unique homomorphism* $g : X \to P$ *with* $p_1 g = g_1$ *and* $p_2 g = g_2$.

If M_1 and M_2 are submodules of M and $M_i \to M$ the natural embeddings (i.e. injection), then the following diagram exists as pullback

$$\begin{array}{ccc} M_1 \cap M_2 & \longrightarrow & M_1 \\ \downarrow & & \downarrow \\ M_2 & \longrightarrow & M \end{array} \tag{2.29}$$

Theorem 2.13 *((Wisbauer, 1991), Noether Isomorphism Theorem) For two submodules* M_1, M_2 *of an R-module M, the commutative diagram with exact rows exists:*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & (M_1 \cap M_2) & \longrightarrow & M_2 & \longrightarrow & M_2/(M_1 \cap M_2) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow \cong & & \\ 0 & \longrightarrow & M_1 & \longrightarrow & M_1 + M_2 & \longrightarrow & (M_1 + M_2)/M_1 & \longrightarrow & 0 \end{array} \tag{2.30}$$

Lemma 2.22 *((Wisbauer, 1991), 10.3) If the diagram*

$$\begin{array}{ccc} P & \xrightarrow{h_2} & M_2 \\ {\scriptstyle h_1}\downarrow & & \downarrow{\scriptstyle f_2} \\ M_1 & \xrightarrow{f_1} & M \end{array} \tag{2.31}$$

is pullback, then the following diagram with exact rows exists:

$$\begin{array}{ccccccc} 0 & \longrightarrow & K & \longrightarrow & P & \xrightarrow{h_2} & M_2 \\ & & \| & & \downarrow{\scriptstyle h_1} & & \downarrow{\scriptstyle f_2} \\ 0 & \longrightarrow & K & \longrightarrow & M_1 & \xrightarrow{f_1} & M \end{array} \tag{2.32}$$

Definition 2.20 *Let* $g_1 : N \to N_1$, $g_2 : N \to N_2$ *be two homomorphisms. A commutative* *diagram*

$$\begin{array}{ccc} N & \xrightarrow{g_2} & N_2 \\ {\scriptstyle g_1}\downarrow & & \downarrow{\scriptstyle q_2} \\ N_1 & \xrightarrow{q_1} & Q \end{array} \tag{2.33}$$

is called the pushout for the pair (g_1, g_2) *if, for every pair of homomorphisms*

$$h_1 : N_1 \to Y, h_2 : N_2 \to Y \tag{2.34}$$

with $h_1 g_1 = h_2 g_2$*, there is a unique morphism* $h : Q \to Y$ *with* $hq_1 = h_1$, $hq_2 = h_2$*.*

Lemma 2.23 *((Wisbauer, 1991), 10.6) If the diagram*

$$\begin{array}{ccc} N & \xrightarrow{f_2} & N_2 \\ {\scriptstyle f_1}\downarrow & & \downarrow{\scriptstyle g_2} \\ N_1 & \xrightarrow{g_1} & Q \end{array} \tag{2.35}$$

is pushout, then the following diagram with exact rows exists:

$$\begin{array}{ccccccc} N & \xrightarrow{f_2} & N_2 & \longrightarrow & C & \longrightarrow & 0 \\ {\scriptstyle f_1}\downarrow & & \downarrow{\scriptstyle g_2} & & \| & & \\ N_1 & \xrightarrow{g_1} & Q & \longrightarrow & C & \longrightarrow & 0 \end{array} \tag{2.36}$$

Definition 2.21 *A category* $\mathcal{C}$ *is given by:*

1. *A class of objects,* $Obj(\mathcal{C})$*,*
2. *For every ordered pair* (A, B) *of objects in* $\mathcal{C}$ *there exists a set* $Mor_{\mathcal{C}}(A, B)$*, the morphism of A to B, such that* $Mor_{\mathcal{C}}(A, B) \cap Mor_{\mathcal{C}}(A', B') = \emptyset$ *for* $(A, B) \neq (A', B')$*.*
3. *A composition of morphisms, i.e. a map* $Mor_{\mathcal{C}}(A, B) \times Mor_{\mathcal{C}}(B, C) \to Mor_{\mathcal{C}}(A, C)$*,* $(f, g) \mapsto gf$*, for every triple* (A, B, C) *of objects in* $\mathcal{C}$*, with the properties:*
4. *For every* A, B, C, D *in* $Obj(\mathcal{C})$ *and* $f \in Mor_{\mathcal{C}}(A, B)$*,* $g \in Mor_{\mathcal{C}}(B, C)$*,* $h \in Mor_{\mathcal{C}}(C, D)$*,* $h(gf) = (hg)f$*;*

5. *For every object A in $\mathcal{C}$ there is a morphism $id_A \in Mor_{\mathcal{C}}(A, A)$, the identity of A, with $f id_A = id_B f = f$ for every $f \in Mor_{\mathcal{C}}(A, B)$, $B \in Obj(\mathcal{C})$.*

Let A and C be left R-modules. An extension of A by C means an exact sequence

$$0 \longrightarrow A \xrightarrow{\mu} B \xrightarrow{\nu} C \longrightarrow 0\,, \tag{2.37}$$

where μ is a monomorphism and ν is an epimorphism with kernel $\mu(A)$. Then the objects short exact sequences constitute a category $\mathscr{E}$ and a morphism between two exact sequences E and E' is a triple (α, β, γ) of module homomorphisms such that the diagram

$$\begin{array}{rccccccccc} E: & 0 & \longrightarrow & A & \xrightarrow{\mu} & B & \xrightarrow{\nu} & C & \longrightarrow & 0 \\ & & & \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\gamma} & & \\ E': & 0 & \longrightarrow & A' & \xrightarrow{\mu'} & B' & \xrightarrow{\nu'} & C' & \longrightarrow & 0 \end{array} \tag{2.38}$$

has commutative squares.

The extensions E and E' with $A = A'$, $C = C'$ are called equivalent if there is a morphism $(1_A, \beta, 1_C)$ with $\beta : B \to B'$ an isomorphism and denoted by $E \equiv E'$.

Let A be a fixed R-module. If $\gamma : C' \to C$ is any homomorphism, then to the extension E in the diagram (2.38) , there is a pullback square

$$\begin{array}{ccccccccc} & & & & B' & \xrightarrow{\nu'} & C' & & \\ & & & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\gamma} & & \\ 0 & \longrightarrow & A & \xrightarrow{\mu} & B & \xrightarrow{\nu} & C & \longrightarrow & 0 \end{array} \tag{2.39}$$

for some B', β and ν'. By properties of pullback diagram, ν' is an epimorphism (since ν is an epimorphism), and $\operatorname{Ker}\nu' \cong \operatorname{Ker}\nu \cong A$, therefore there is a monomorphism $\mu' : A \to B'$ (namely, $\mu' a = (\mu a, 0) \in B'$ if B' is a submodule of $B \oplus C'$) such that the diagram

$$\begin{array}{rccccccccc} E\gamma: & 0 & \longrightarrow & A & \xrightarrow{\mu'} & B' & \xrightarrow{\nu'} & C' & \longrightarrow & 0 \\ & & & \| & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\gamma} & & \\ E: & 0 & \longrightarrow & A & \xrightarrow{\mu} & B & \xrightarrow{\nu} & C & \longrightarrow & 0 \end{array} \tag{2.40}$$

with exact rows and pullback right square is commutative. The row which is an extension of A by C' in diagram (2.40) is denoted by $E\gamma$. Note that $\gamma^* = (1_A, \beta, \gamma)$ is a morphism $E\gamma \to E$ in $\mathscr{E}$.

Now let C be fixed and A vary. For a given $\alpha : A \to A'$, let B' be defined by the pushout square

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \xrightarrow{\mu} & B & \xrightarrow{\nu} & C & \longrightarrow & 0\,. \\ & & \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta} & & & & \\ & & A' & \xrightarrow{\mu'} & B' & & & & \end{array} \tag{2.41}$$

Here μ' is a monomorphism since μ is a monomorphism by the properties of pushout. Furthermore, if B' is defined as the quotient module $(B \oplus A')/H$, where H is the submodule of $B \oplus A'$ including elements of the form $(\mu(a), -\alpha(a))$ for $a \in A$, then $\nu' : B' \to C$ defined by $\nu'((b, a') + H) = \nu(b)$ for $(b, a') \in B \oplus A'$, makes the diagram with exact rows

$$\begin{array}{cccccccccc} E: & 0 & \longrightarrow & A & \xrightarrow{\mu} & B & \xrightarrow{\nu} & C & \longrightarrow & 0 \\ & & & {\scriptstyle\alpha}\downarrow & & \downarrow{\scriptstyle\beta} & & \Vert & & \\ \alpha E: & 0 & \longrightarrow & A' & \xrightarrow{\mu'} & B' & \xrightarrow{\nu'} & C & \longrightarrow & 0 \end{array} \tag{2.42}$$

commutative. The bottom row which is denoted by αE is an extension of A' by C. Here $\alpha_* = (\alpha, \beta, 1_C)$ is a morphism $E \to \alpha E$ in $\mathscr{E}$.

With $\alpha : A \to A'$ and $\gamma : C' \to C$, there exists the important associative law:

$$\alpha(E\gamma) \equiv (\alpha E)\gamma. \tag{2.43}$$

To describe the group operation in the language of short exact sequences, the diagonal map $\Delta_G : g \mapsto (g, g)$ and the codiagonal map $\nabla_G : (g_1, g_2) \mapsto g_1 + g_2$ of a module G are used. If it is understood by the direct sum of two extensions

$$E_i : 0 \longrightarrow A_i \xrightarrow{\mu_i} B_i \xrightarrow{\nu_i} C_i \longrightarrow 0 \qquad (i = 1, 2) \tag{2.44}$$

the extension

$$E_1 \oplus E_2 : 0 \longrightarrow A_1 \oplus A_2 \xrightarrow{\mu_1 \oplus \mu_2} B_1 \oplus B_2 \xrightarrow{\nu_1 \oplus \nu_2} C_1 \oplus C_2 \longrightarrow 0\,, \tag{2.45}$$

then the sum of two extensions is given in the following theorem.

Theorem 2.14 *((Mac Lane, 1963), Ch. III, Theorem 2.1) For given R-modules A and C, the set* $\mathrm{Ext}_R(C, A)$ *of all congruence classes of extensions of A by C is an abelian group under the binary operation which assigns to the congruence classes of extensions E_1 and E_2, the congruence class of the extension*

$$E_1 + E_2 = \nabla_A(E_1 \oplus E_2)\Delta_C. \tag{2.46}$$

The class of the split extension $0 \longrightarrow A \longrightarrow A \oplus C \longrightarrow C \longrightarrow 0$ *is the zero element of this group, while the inverse of any E is the extension* $(-1_A)E$*. For homomorphisms* $\alpha : A \longrightarrow A'$ *and* $\gamma : C' \longrightarrow C$*, one has*

$$\alpha(E_1 + E_2) \equiv \alpha E_1 + \alpha E_2, \qquad (E_1 + E_2)\gamma \equiv E_1\gamma + E_2\gamma, \tag{2.47}$$

$$(\alpha_1 + \alpha_2)E \equiv \alpha_1 E + \alpha_2 E, \qquad E(\gamma_1 + \gamma_2) \equiv E\gamma_1 + E\gamma_2. \tag{2.48}$$

The equivalences in (2.47) and (2.48) show that $\alpha_* : E \mapsto \alpha E$ and $\gamma^* : E \mapsto E\gamma$ are group homomorphisms

$$\alpha_* : \mathrm{Ext}_R(C, A) \to \mathrm{Ext}_R(C, A'), \qquad \gamma^* : \mathrm{Ext}_R(C, A) \to \mathrm{Ext}_R(C', A), \tag{2.49}$$

and that $(\alpha_1 + \alpha_2)_* = (\alpha_1)_* + (\alpha_2)_*$ and $(\gamma_1 + \gamma_2)^* = (\gamma_1)^* + (\gamma_2)^*$ for module homomorphisms $\alpha_1, \alpha_2 : A \longrightarrow A'$, $\gamma_1, \gamma_2 : C' \longrightarrow C$.

Let M and N be R-modules. The set $Hom_R(M, N)$ of all R-module homomorphisms f of M into N is an abelian group, under the addition defined by $(f + g)m = fm + gm$ for $f, g : M \to N$.

Let $\alpha : A' \to A$ and $\gamma : C \to C'$ be fixed R-module homomorphisms. Every $\eta \in Hom_R(A, C)$ results in a homomorphism $A' \to C'$ which is the composite

$$A' \xrightarrow{\alpha} A \xrightarrow{\eta} C \xrightarrow{\gamma} C' . \tag{2.50}$$

The correspondence $\eta \mapsto \gamma\eta\alpha$ is group homomorphism of $Hom_R(A, C)$ into $Hom_R(A', C')$ which is denoted as

$$Hom_R(\alpha, \gamma) : Hom_R(A, C) \to Hom_R(A', C') \tag{2.51}$$

and is said to be the induced homomorphism by η. Fore more information about induced homomorphism see ((Mac Lane, 1963), Ch. I).

For a given extension

$$0 \longrightarrow A \xrightarrow{\alpha} B \xrightarrow{\beta} C \longrightarrow 0 , \tag{2.52}$$

representing an element of $Ext_R(C, A)$, and a homomorphism $\eta : A \to G$, it is known that ηE is an extension of G by C, i.e., ηE represents an element of $Ext_R(C, G)$. Therefore a map

$$E^* : Hom_R(A, G) \to Ext_R(C, G) \tag{2.53}$$

defined as

$$E^* : \eta \mapsto \eta E. \tag{2.54}$$

Similarly, a homomorphism $\xi : G \to C$ produces an extension $E\xi$ of A by G from E, and

$$E_* : Hom_R(G, C) \to Ext_R(G, A) \tag{2.55}$$

is a map acting as follows:

$$E_* : \xi \mapsto E\xi. \tag{2.56}$$

By Theorem 2.14, E^* and E_* are group homomorphisms. If $\phi : G \to H$ is any homomorphism for some module H, then since $(\phi\eta)E \equiv \phi(\eta E)$ and $E(\xi\phi) \equiv (E\xi)\phi$ the following diagrams

$$\begin{array}{ccc} Hom_R(A,G) & \longrightarrow & Ext_R(C,G) \\ \downarrow & & \downarrow \\ Hom_R(A,H) & \longrightarrow & Ext_R(C,H) \end{array} \qquad \begin{array}{ccc} Hom_R(H,C) & \longrightarrow & Ext_R(H,A) \\ \downarrow & & \downarrow \\ Hom_R(G,C) & \longrightarrow & Ext_R(G,A) \end{array} \tag{2.57}$$

with the obvious maps are commutative. E^* and E_* are said to be the connecting homomorphisms for the short exact sequence (2.52).

Theorem 2.15 *((Mac Lane, 1963), Theorem 3.2 and Theorem 3.4) If*

$$0 \longrightarrow A \xrightarrow{\alpha} B \xrightarrow{\beta} C \longrightarrow 0 \tag{2.58}$$

is a short exact sequence of R-modules, then the sequences

1. $0 \longrightarrow Hom_R(C,\ G) \longrightarrow Hom_R(B,\ G) \longrightarrow Hom_R(A,\ G)$

   $\xrightarrow{E^*} Ext_R(C,\ G) \xrightarrow{\beta^*} Ext_R(B,\ G) \xrightarrow{\alpha^*} Ext_R(A,\ G)$

2. $0 \longrightarrow Hom_R(G,\ A) \longrightarrow Hom_R(G,\ B) \longrightarrow Hom_R(G,\ C)$

   $\xrightarrow{E_*} Ext_R(G,\ A) \xrightarrow{\alpha_*} Ext_R(G,\ B) \xrightarrow{\beta_*} Ext_R(G,\ C)$

are exact for every R-module G.

A submodule ${}_RU \leq_R M$ is pure in M if $IU = U \cap IM$ for each right ideal I in R_R.

Over a discrete valuation ring R with maximal ideal Rp where $p \in R$ is the unique prime element, a submodule S of a module M is called basic submodule if S is pure in M (i.e. $p^nS = S \cap p^nM$ for all positive integer n), S is a direct sum of cyclic modules and M/S is divisible. Any R-module has a basic submodule and two basic submodules are isomorphic (see (Kaplansky, 1969), Lemma 21).

A short exact sequence $0 \longrightarrow A \xrightarrow{\alpha} B \xrightarrow{\beta} C \longrightarrow 0$ is called pure-exact if $\operatorname{Im}\alpha$ is a pure subgroup (i.e. $n\operatorname{Im}\alpha = \operatorname{Im}\alpha \cap nB$, for all integer n) of B.

CHAPTER 3

CO-COATOMICALLY SUPPLEMENTED MODULES

In this chapter, we define co-coatomic submodules and co-coatomically supplemented modules and give some results about co-coatomically supplemented modules. Throughout this chapter, R will be an arbitrary ring unless otherwise stated.

Definition 3.1 *A proper submodule N of M is called co-coatomic in M if M/N is coatomic.*

Example 3.1 *Every submodule of a coatomic module is co-coatomic, in particular every submodule of semisimple and finitely generated modules is co-coatomic.*

Let $\{M_\lambda\}_{\lambda\in\Lambda}$, where Λ is an index set, be the family of simple submodules of M such that each of them is a direct summand of M. $\operatorname{Soc}^{\oplus}(M)$ will be the sum of M_λs for all $\lambda \in \Lambda$, that is, $\operatorname{Soc}^{\oplus}(M) = \sum_{\lambda\in\Lambda} M_\lambda$. Clearly $\operatorname{Soc}^{\oplus}(M) \leq \operatorname{Soc}(M)$.

Theorem 3.1 *Let R be a ring. The following are equivalent for an R-module M.*

1. *Every co-coatomic submodule of M is a direct summand of M.*
2. *Every cofinite submodule of M is a direct summand of M.*
3. *Every maximal submodule of M is a direct summand of M.*
4. *$M/\operatorname{Soc}^{\oplus} M$ does not contain a maximal submodule.*
5. *$M/\operatorname{Soc}(M)$ does not contain a maximal submodule.*

Proof (1) $\Rightarrow$ (2) Let N be a cofinite submodule of M. Since every cofinite submodule is co-coatomic, N is a direct summand of M.
(2) $\Rightarrow$ (3) It follows easily from the fact that every maximal submodule is cofinite.
(3) $\Rightarrow$ (4) Suppose $M/\operatorname{Soc}^{\oplus} M$ contains a maximal submodule, say $K/\operatorname{Soc}^{\oplus} M$. Then K is a maximal submodule of M. By hypothesis, $M = K \oplus K'$, and K' is simple. $K' \leq \operatorname{Soc}^{\oplus} M \leq K$. Contradiction.
(4) $\Rightarrow$ (5) Since $\operatorname{Soc}^{\oplus} M \leq \operatorname{Soc}(M)$, it is clear.
(5) $\Rightarrow$ (1) Let N be a co-coatomic submodule of M. Then M/N is coatomic. Since $M/(N + \operatorname{Soc}(M)) \cong (M/N)/((N + \operatorname{Soc}(M))/N)$, $M/(N + \operatorname{Soc}(M))$ is also coatomic. Thus

$M = N + \mathrm{Soc}(M)$ by (5). It follows that $M = N \oplus N'$ for any submodule N' such that $\mathrm{Soc}(M) = (N \cap \mathrm{Soc}(M)) \oplus N'$. The proof is completed, as desired. □

Definition 3.2 *An R-module M is called co-coatomically supplemented if every co-coatomic submodule of M has a supplement in M.*

A supplemented module is co-coatomically supplemented but a co-coatomically supplemented module need not be supplemented by the following example.

Example 3.2 *The $\mathbb{Z}$-module $\mathbb{Q}$ is co-coatomically supplemented since the only co-coatomic submodule is $\mathbb{Q}$ itself. But $\mathbb{Z}$-module $\mathbb{Q}$ is not supplemented since $\mathbb{Q}$ is not torsion (see Theorem 2.7).*

Proposition 3.1 *Let M be a semilocal module with small radical* $\mathrm{Rad}(M)$. *Then M is co-coatomically supplemented if and only if M is supplemented.*

Proof Let N be a submodule of M. Since M is semilocal, $M/\mathrm{Rad}(M)$ is semisimple, i.e. coatomic. Consider the following statement:

$$M/(N + \mathrm{Rad}(M)) \cong (M/\mathrm{Rad}(M))/((N + \mathrm{Rad}(M))/\mathrm{Rad}(M)). \tag{3.1}$$

Since $M/\mathrm{Rad}(M)$ is coatomic, $M/(N + \mathrm{Rad}(M))$ is coatomic. Therefore $N + \mathrm{Rad}(M)$ has a supplement in M, say K. Then $M = N + \mathrm{Rad}(M) + K$ and $(N + \mathrm{Rad}(M)) \cap K \ll K$. Since $\mathrm{Rad}(M) \ll M$, it follows that $M = N + K$ and $N \cap K \leq (N + \mathrm{Rad}(M)) \cap K \ll K$. Thus M is supplemented. □

If a module M is coatomic and co-coatomically supplemented, then it is, clearly, supplemented. In fact the module M in Proposition 3.1 is coatomic (see Lemma 2.5). Therefore the result is clear but we proved the proposition without using the coatomic property of that module.

Proposition 3.2 *Co-coatomically supplemented modules are closed under homomorphic image.*

Proof Let M be a co-coatomically supplemented R-module and L a submodule of M. Then any co-coatomic submodule of M/N is of the form L/N such that L is co-coatomic submodule of M and $N \leq L$. Since M is co-coatomically supplemented, $L + K = M$ and $L \cap K \ll K$ for some submodule K of M. It follows that $M/N = (L + K)/N = (L/N) + ((K + N)/N)$ and $(L/N) \cap ((K + N)/N) = ((L \cap K) + N)/N \ll (K + N)/N$ by

Lemma 2.2(4). $(K + N)/N$ is a supplement of L/N in M/N by Lemma 2.8. Thus M/N is co-coatomically supplemented. □

Proposition 3.3 *Let M be a co-coatomically supplemented R-module. Then every co-coatomic submodule of the module $M/Rad(M)$ is a direct summand.*

Proof Let $N/\operatorname{Rad}(M)$ be a co-coatomic submodule of $M/\operatorname{Rad}(M)$. Then N is also co-coatomic submodule of M such that $\operatorname{Rad}(M) \leq N$. Since M is co-coatomically supplemented, there exists a submodule K of M such that $M = N + K$ and $N \cap K \ll K$. It follows that $N \cap K \leq \operatorname{Rad}(M)$. Thus $M/\operatorname{Rad}(M) = (N + K)/\operatorname{Rad}(M) = (N/\operatorname{Rad}(M)) + (K + \operatorname{Rad}(M))/\operatorname{Rad}(M)$ and

$$\begin{aligned} N/\operatorname{Rad}(M) \cap (K + \operatorname{Rad}(M))/\operatorname{Rad}(M) &= N \cap (K + \operatorname{Rad}(M))/\operatorname{Rad}(M) \\ &= ((N \cap K) + \operatorname{Rad}(M))/\operatorname{Rad}(M) \\ &= 0. \end{aligned} \tag{3.2}$$

Hence $M/\operatorname{Rad}(M) = (N/\operatorname{Rad}(M)) \oplus (K + \operatorname{Rad}(M))/\operatorname{Rad}(M)$. □

Lemma 3.1 *Let M be an R-module, N and U submodules of M such that N is co-coatomically supplemented, U is co-coatomic and $N + U$ has a supplement A in M. Then $N \cap (U + A)$ has a supplement B in N, and $A + B$ is a supplement of U in M.*

Proof Since A is a supplement of $(N + U)$ in M, then

$$M = N + U + A \text{ and } (N + U) \cap A \ll A. \tag{3.3}$$

Since M/U is coatomic,

$$\begin{aligned} N/(N \cap (U + A)) &\cong (N + U + A)/(U + A) \\ &= M/(U + A) \\ &\cong (M/U)/((U + A)/U) \end{aligned} \tag{3.4}$$

is coatomic. Therefore $N \cap (U + A)$ is a co-coatomic submodule of N. Since N is co-coatomically supplemented, $N \cap (U + A)$ has a supplement B in N, i.e. $N \cap (U + A) + B = N$

and $B \cap (U + A) \ll B$. Then

$$M = N + U + A = U + A + B. \tag{3.5}$$

Furthermore

$$\begin{aligned} U \cap (A + B) &\leq (A \cap (U + B)) + (B \cap (U + A)) \\ &\leq (A \cap (U + N)) + (B \cap (U + A)) \\ &\ll A + B \end{aligned} \tag{3.6}$$

by Lemma 2.2. Hence $A + B$ is a supplement of U in M. □

Corollary 3.1 *Let N and L be submodules of an R-module M such that N is co-coatomic, L is co-coatomically supplemented, and $N + L$ has a supplement in M. Then N has a supplement in M.*

Proposition 3.4 *A finite sum of co-coatomically supplemented modules is co-coatomically supplemented .*

Proof Let M_1 and M_2 be co-coatomically supplemented modules. It is adequate to show that $M = M_1 + M_2$ is co-coatomically supplemented. Let U be a co-coatomic submodule of M. Then $M = M_1 + M_2 + U$. Since $M_2 + U$ is co-coatomic submodule of M, M_1 is co-coatomically supplemented and 0 is trivial supplement of M, $M_2 + U$ has a supplement in M by Corollary 3.1. Since M_2 is co-coatomically supplemented and U is co-coatomic, then again by Corollary 3.1, U has a supplement in M. Thus M is co-coatomically supplemented. □

Corollary 3.2 *A finite direct sum of co-coatomically supplemented modules is co-coatomically supplemented .*

Proof It is clear by Proposition 3.4. □

Proposition 3.5 *If M is co-coatomically supplemented R-module, then every finitely M-generated module is co-coatomically supplemented .*

Proof Let N be a finitely M-generated module and F a finite index set. Then there is an epimorphism f

$$\bigoplus_{F} M \stackrel{f}{\longrightarrow} N \longrightarrow 0 \tag{3.7}$$

from a finite direct sum of copies of M to N. Since M is co-coatomically supplemented, $\bigoplus_{F} M$ is also co-coatomically supplemented module by Corollary 3.2. Thus N is co-coatomically supplemented by Proposition 3.2. □

Proposition 3.6 *If N is a co-coatomically supplemented submodule of an R-module M such that M/N has no maximal submodule, then M is co-coatomically supplemented .*

Proof Let L be a submodule of M such that M/L is coatomic. Clearly $M/(N + L)$ is also coatomic. Since M/N has no maximal submodule, $M = N + L$. By Corollary 3.1, L has a supplement. Thus M is co-coatomically supplemented. □

Corollary 3.3 *Let M be a module and $M/\operatorname{Soc}(M)$ have no maximal. Then M is co-coatomically supplemented.*

Proof Since $\operatorname{Soc}(M)$ is semisimple, it is supplemented. The proof follows by Proposition 3.6. □

Proposition 3.7 *Let M be a co-coatomically supplemented R-module. If M contains a maximal submodule, then M contains a local submodule.*

Proof Let L be a maximal submodule of M. Since M is a co-coatomically supplemented module, there exists a submodule K of M such that K is a supplement of L in M, i.e. $M = K + L$ and $K \cap L \ll K$. Now consider a proper submodule X of K. Suppose X is not contained in L. Since L is a maximal submodule, $M = X + L$. By the minimality of K, $X = K$. This contradicts with the assumption. Thus X is contained in L. Therefore $X \leq K \cap L \ll K$. Hence K is hollow, and so local. □

Definition 3.3 *A module M is called linearly compact if for every family of cosets $\{x_i + M_i\}_{\triangle}$, $x_i \in M$ and submodules $M_i \leq M$ (with M/M_i finitely cogenerated) such that the intersection of any finitely many of these cosets is not empty, the intersection is also not empty.*

Lemma 3.2 *((Smith, 2000), Corollary 2.7) Let K be a supplemented submodule of a module M such that for every submodule H of M with $K \leq H$, K has a supplement in H. Let*

N be a submodule of M such that $(N+K)/K$ *has supplement in* M/K*. Then N has a supplement in M.*

The following proposition gives a characterization of co-coatomically supplemented module by a linearly compact submodule.

Proposition 3.8 *Let K be a linearly compact submodule of an R-module M. Then M is co-coatomically supplemented if and only if* M/K *is co-coatomically supplemented .*

Proof (⇒) By Proposition 3.2.
(⇐) Let N be a co-coatomic submodule of M. Then $(N+K)/K$ is also a co-coatomic submodule of M/K since $N+K$ is co-coatomic submodule of M. Since M/K is co-coatomically supplemented, $(N+K)/K$ has a supplement in M/K. Note that K has a supplement in every submodule L of M with $K \leq L$ since K is linearly compact (see (Smith, 2000), Lemma 2.3). Furthermore K is supplemented since every submodule of K is linearly compact (see (Wisbauer, 1991), 29.8(2)). By Lemma 3.2, N has a supplement in M. Thus M is co-coatomically supplemented. □

Proposition 3.9 *A co-coatomically supplemented module is cofinitely supplemented .*

Proof Let M be a co-coatomically supplemented R-module and N a cofinite submodule of M. Then M/N is finitely generated, so is coatomic. It follows that N is a co-coatomic submodule of M. Since M is co-coatomically supplemented module, N has a supplement. Thus M is cofinitely supplemented. □

On the other hand, a cofinitely supplemented module need not be co-catomically supplemented by the following examples.

Example 3.3 *((Alizade and Büyükaşık, 2003), Example 2.14) Let p be a prime integer and M the* $\mathbb{Z}$*-module*

$$M = \bigoplus_{i=1}^{\infty} < a_i > \tag{3.8}$$

which is the direct sum of cyclic subgroups $< a_i >$ *of order* p^i*. Then M is cofinitely supplemented module, but not co-coatomically supplemented module.*

Proof Since each $< a_i >$ is local, it is supplemented, so it is both co-coatomically supplemented and cofinitely supplemented. M is a cofinitely supplemented module as it is the direct sum of cofinitely supplemented modules (see Corollary 2.5). Rad(M) of M is

$$\begin{aligned}\operatorname{Rad}(M) &= \operatorname{Rad}\left(\bigoplus_{i=1}^{\infty} < a_i >\right) \\ &= \bigoplus_{i=1}^{\infty} \operatorname{Rad}(< a_i >) \\ &= \bigoplus_{i=1}^{\infty} p < a_i > .\end{aligned} \tag{3.9}$$

It follows that

$$\begin{aligned}M/\operatorname{Rad}(M) &= \left(\bigoplus_{i=1}^{\infty} < a_i >\right) / \left(\bigoplus_{i=1}^{\infty} p < a_i >\right) \\ &\cong \bigoplus_{i=1}^{\infty} (< a_i > / p < a_i >)\end{aligned} \tag{3.10}$$

is semisimple so coatomic, i.e. Rad(M) is co-coatomic submodule of M and Rad(M) = pM. Say $T = pM$. Suppose that T has a weak supplement L in M, i.e.

$$M = T + L \text{ and } N = T \cap L \ll M. \tag{3.11}$$

Then $N \ll E(M)$, where $E(M)$ is an injective hull of M. Since injective hull $E(N)$ of N is direct summand of $E(M)$, $N \ll E(N)$. It follows from ((Leonard, 1966), Theorem 4) that if a torsion abelian group is small in its injective hull, then it is bounded, i.e. $p^n N = 0$ for some positive integer n. Then, since $pL \leq L \cap pM = L \cap T = N$,

$$p^{n+1}M = p^{n+1}T + p^n(pL) \leq p^{n+1}T + p^n N = p^{n+1}T. \tag{3.12}$$

Therefore $p^{n+1}a_{n+2} = p^{n+1}b$ for some $b \in T = pM$. Since $b = pc$ for some $c = (m_i a_i)_{i=1}^{\infty} \in M$, we have

$$0 \neq p^{n+1}a_{n+2} = p^{n+1}(pm_{n+2}a_{n+2}) = m_{n+2}p^{n+2}a_{n+2} = 0 \tag{3.13}$$

This contradiction implies that Rad(M) does not have a weak supplement, therefore does not have a supplement. Thus M is not co-coatomically supplemented. □

By Example 3.3, it is seen that the direct sum of infinitely many co-coatomically supplemented modules need not be co-coatomically supplemented. Example 3.3 also shows that over Dedekind domains, co-coatomically supplemented modules and cofinitely supplemented modules need not coincide.

Example 3.4 *Let p be a prime integer and consider the following ring*

$$R = \mathbb{Z}_{(p)} = \left\{ \frac{a}{b} \mid a,\, b \in \mathbb{Z},\, b \neq 0,\, (b, p) = 1 \right\} \tag{3.14}$$

which is localization of $\mathbb{Z}$ at (p). Then the left R-module $R^{(\mathbb{N})}$ is cofinitely supplemented, but not co-coatomically supplemented .

Proof Since R is local, ${}_RR$ is supplemented. Therefore R is a semiperfect ring by Proposition 2.9. R is not a perfect ring since its jacobson radical is not t-nilpotent (see Proposition 2.10). Since R-module R is supplemented, it is cofinitely supplemented. Therefore R-module $R^{(\mathbb{N})}$ is a cofinitely supplemented as it is the direct sum of cofinitely supplemented modules (see Corollary 2.5).
The radical of ${}_RR^{(\mathbb{N})}$

$$\mathrm{Rad}({}_RR^{(\mathbb{N})}) = (\mathrm{Jac}(R))^{(\mathbb{N})} = \bigoplus p\mathbb{Z}_{(p)}. \tag{3.15}$$

It follows that

$${}_RR^{(\mathbb{N})} / \mathrm{Rad}({}_RR^{(\mathbb{N})}) = \bigoplus \mathbb{Z}_{(p)} / \bigoplus p\mathbb{Z}_{(p)} \tag{3.16}$$

is semisimple, so it is coatomic, i.e. Rad(${}_RR^{(\mathbb{N})}$) is co-coatomic submodule of ${}_RR^{(\mathbb{N})}$. Since R is not a perfect ring and R is local, Rad(${}_RR^{(\mathbb{N})}$) does not have a weak supplement by Theorem 2.8. Therefore Rad(${}_RR^{(\mathbb{N})}$) does not have a supplement in ${}_RR^{(\mathbb{N})}$. Hence R-module $R^{(\mathbb{N})}$ is not co-coatomically supplemented. □

In fact the module M in the Example 3.3 considered as an $\mathbb{Z}_{(p)}$-module is also cofinitely supplemented, but not co-coatomically supplemented.

Example 3.4 shows that over semiperfect rings, cofinitely supplemented and co-coatomically supplemented modules need not coincide. This example also shows that an infinite direct sum of co-coatomically supplemented modules need not be co-coatomically supplemented .

Definition 3.4 *A ring is called a left V-ring if every simple left R-module is injective.*

Theorem 3.2 *((Lam, 1999), Theorem 3.75) For any ring R, the following are equivalent:*

1. *R is a left V-ring;*
2. *Any left ideal A of R is an intersection of maximal left ideals;*
3. *For any left R-module M,* $\mathrm{Rad}(M) = 0$

Let R be a ring. An element $a \in R$ is called von Neumann regular if $a \in aRa$. If every $a \in R$ is von Neumann regular, then R is said to be von Neumann regular ring. A commutative ring R is V-ring if and only if R is von Neumann regular ring (see (Wisbauer, 1991), 23.5).

Proposition 3.10 *((Alizade et al., 2001), Proposition 3.6) Let R be a left V-ring . Then* $\mathrm{Soc}(M) = \mathrm{Art}(M) = \mathrm{Loc}(M)$ *for any R-module M.*

Proposition 3.11 *Over a left V-ring R, a left R-module M is co-coatomically supplemented if and only if M is semisimple.*

Proof ($\Leftarrow$) Clear.
($\Rightarrow$) Since M is co-coatomically supplemented, M is cofinitely supplemented. Therefore $M/\mathrm{Soc}(M)$ has no maximal submodule by Theorem 2.9 and Proposition 3.10. Since R is a left V-ring, $M/\mathrm{Soc}(M) = \mathrm{Rad}(M/\mathrm{Soc}(M)) = 0$ by Theorem 3.2. Thus M is semisimple. □

Corollary 3.4 *Over a left V-ring, any direct sum of co-coatomically supplemented modules is co-coatomically supplemented .*

Proof Since over a left V-ring co-coatomically supplemented and cofinitely supplemented modules coincide, it is clear. □

Definition 3.5 *(Zöschinger, 1974b) A module M is called Σ-self-projective if for each index set* $\mathbb{N}$*, the module* $M^{(\mathbb{N})}$ *is self-projective.*

Proposition 3.12 *((Zöschinger, 1974b), Satz 4.1) Let R be a ring and M an R-module. If M is Σ-self-projective and* $U \leq \text{Rad}(M)$*, then the following holds: U has a supplement in M, so U is small in M.*

Remark 3.1 *By using Proposition 3.12, the following deduction can be obtained:* $_RR$ *is projective, so* $_RR^{(\mathbb{N})}$ *is projective. Thus* $_RR^{(\mathbb{N})}$ *is self-projective. Hence* $_RR^{(\mathbb{N})}$ *is Σ-self-projective. Since* $\text{Rad}(_RR^{(\mathbb{N})}) \leq \text{Rad}(_RR^{(\mathbb{N})})$*, if* $\text{Rad}(_RR^{(\mathbb{N})})$ *has a supplement in* $_RR^{(\mathbb{N})}$ *then* $\text{Rad}(_RR^{(\mathbb{N})}) \ll {}_RR^{(\mathbb{N})}$.

Theorem 3.3 *Every left R-module is co-coatomically supplemented if and only if R is left perfect ring.*

Proof (⇐) Clear.
(⇒) By hypothesis, every left R-module is co-coatomically supplemented, so every left R-module is cofinitely supplemented. Then R is semiperfect by Theorem 2.10. Therefore $R/\operatorname{Jac}(R)$ is left semisimple by Proposition 2.9. It follows that $_RR^{(\mathbb{N})}/\operatorname{Rad}(_RR^{(\mathbb{N})})$ is semisimple. Thus $\operatorname{Rad}(_RR^{(\mathbb{N})})$ is co-coatomic submodule of $_RR^{(\mathbb{N})}$. By hypothesis, $_RR^{(\mathbb{N})}$ is co-coatomically supplemented so $\operatorname{Rad}(_RR^{(\mathbb{N})})$ has a supplement in $_RR^{(\mathbb{N})}$. By Remark 3.1 and Proposition 3.12, $\operatorname{Rad}(_RR^{(\mathbb{N})}) \ll {}_RR^{(\mathbb{N})}$. Therefore since $R/\operatorname{Jac}(R)$ is left semisimple, $_RR$ is perfect by Proposition 2.10. Thus R is a left perfect ring. □

3.1. Co-coatomically Supplemented Modules Over Discrete Valuation Rings

In this section, we investigate co-coatomically supplemented modules over discrete valuation ring (DVR). Throughout this section, R is a DVR. We use the results about radical-supplemented modules over DVR in (Zöschinger, 1974b) to obtain the results on co-coatomically supplemented modules.

Definition 3.6 *(Zöschinger, 1974b) A module M is called radical-supplemented if* $\text{Rad}(M)$ *has a supplement in M.*

Proposition 3.13 *((Zöschinger, 1974b), Satz 3.1) Let R be a DVR. For an R-module M, the following are equivalent:*

1. *M is radical-supplemented.*

2. $\operatorname{Rad}^n(M)/\operatorname{Rad}^{n+1}(M)$ *is finitely generated for some* $n \geq 0$.

3. *"The"basic-submodule of M is coatomic.*

4. $M = T(M) \oplus X$, *where the reduced part of* $T(M)$ *is bounded and* $X/\operatorname{Rad}(X)$ *is finitely generated.*

Proposition 3.14 *Let R be a DVR and M an R-module. Then M is co-coatomically supplemented module if and only if the basic submodule of M is coatomic.*

Proof ($\Rightarrow$) Over DVR, $M/\operatorname{Rad}(M) = M/pM$ is semisimple, i.e. coatomic. Since M is co-coatomically supplemented module, pM has a supplement. Thus M is radical-supplemented module. It follows that the basic submodule of M is coatomic by Proposition 3.13.
($\Leftarrow$) Let X be a submodule of M such that M/X is coatomic and B the basic submodule of M. Then $M/(X+B)$ is also coatomic. Furthermore, $M/(X+B)$ is reduced by Lemma 2.11. On the other hand, $M/(X+B)$ is divisible since M/B is divisible. In this case $M/(X+B)$ is zero. Thus $M = X + B$. By hypothesis, B is coatomic, so supplemented by Lemma 2.11. Therefore X has a supplement by Corollary 3.1. Hence M is co-coatomically supplemented module. □

Corollary 3.5 *Over a DVR, co-coatomically supplemented modules and radical-supplemented modules coincide.*

Corollary 3.6 *Over a discrete valuation ring, M is co-coatomically supplemented module if and only if* $M = T(M) \oplus X$, *where the reduced part of* $T(M)$ *is bounded and* $X/\operatorname{Rad}(X)$ *is finitely generated.*

The following properties are given for radical-supplemented modules over a DVR in ((Zöschinger, 1974b), Lemma 3.2). Since co-coatomically supplemented modules and radical-supplemented modules coincide, clearly they hold also for co-coatomically supplemented modules.

Proposition 3.15 *Let R be a DVR. Then for an R-module M, the following hold:*

1. *The class of co-coatomically supplemented modules is closed under pure submodules and extensions.*

2. *If M is co-coatomically supplemented and M/U is reduced, then U is also co-coatomically supplemented.*

3. *Every submodule of M is co-coatomically supplemented if and only if $T(M)$ is supplemented and $M/T(M)$ has finite rank.*

3.2. Co-coatomically Supplemented Modules Over Dedekind Domains

Throughout this section R is a Dedekind domain unless otherwise stated. A module M is called md-module if every maximal submodule of M is a direct summand. M is md-module if and only if $T(M) = M_1 \oplus M_2$ where M_1 is semisimple, M_2 and $M/T(M)$ are divisible over a Dedekind domain ((Büyükaşık and Pusat-Yılmaz, 2010), Theorem 6.11).

Recall that every maximal submodule is a direct summand if and only if every co-coatomic submodule is a direct summand (see Theorem 3.1). Hence the characterization for modules whose co-coatomic submodules are direct summands can be given by the following theorem.

Theorem 3.4 *Let R be a Dedekind domain and M an R-module. M is a module whose co-coatomic submodules are direct summands if and only if*

1. *$T(M) = M_1 \oplus M_2$, where M_1 is semisimple and M_2 is divisible,*
2. *$M/T(M)$ is divisible.*

Proof Clear by Theorem 3.1 and ((Büyükaşık and Pusat-Yılmaz, 2010), Theorem 6.11). □

Lemma 3.3 *((Lam, 1999), Example 6.34) Let R be a domain and M an R-module. Then the torsion submodule $T(M)$ is a closed submodule of M.*

Lemma 3.4 *((Zöschinger, 1974a), Lemma 3.3) Let R be a Dedekind domain, M an R-module and N a submodule of M. Then N is closed if and only if N is coclosed.*

A module M is called co-coatomically weak supplemented if every co-coatomic submodule of M has a weak supplement in M. A small cover of a co-coatomically weak supplemented module is co-coatomically weak supplemented (see Proposition 5.3). Details of these modules are provided in Chapter 5.

Proposition 3.16 *Let R be a Dedekind domain and M a torsion R-module. Then M is co-coatomically weak supplemented if and only if it is co-coatomically supplemented.*

Proof Clearly co-coatomically supplemented module is co-coatomically weak supplemented. To prove converse, let K be a submodule of M such that M/K is coatomic. Since M is co-coatomically weak supplemented, K has a weak supplement, say N. Then $M = K + N$ and $K \cap N \ll M$. Since M is torsion, N is also torsion, so it is coclosed by Lemma 3.3 and Lemma 3.4. Therefore $K \cap N \ll N$ by Lemma 2.12. Hence M is co-coatomically supplemented. □

Theorem 3.5 *Let R be a non-local Dedekind domain and M a reduced R-module. If $T(M)$ has a weak supplement then M is co-coatomically supplemented if and only if $M/T(M)$ is divisible and $T_P(M)$ is bounded for each $P \in \mathcal{P}$.*

Proof ($\Rightarrow$) Let R be a non-local Dedekind domain and M a co-coatomically supplemented reduced R-module. Then the module $M/T(M)$ is radical: Let K be a maximal submodule of M with $T(M) \leq K$. Since M is co-coatomically supplemented, K has a supplement, say V. Since K is maximal, V is local, therefore V is cyclic, i.e. $V \cong R/I$. On the other hand, R is non-local, so $I \neq 0$, i.e. V is torsion so $V \leq T(M)$, contradiction. Hence $M/T(M)$ has no maximal submodule. By Lemma 2.3, $M/T(M)$ is divisible. $T(M)$ is closed by Lemma 3.3, i.e. it is coclosed by Lemma 3.4. Since $T(M)$ has (is) a weak supplement, it is a supplement by Lemma 2.12. Therefore there is a submodule N in M such that $T(M) + N = M$ and $T(M) \cap N \ll T(M)$. Then $T(M)/(T(M) \cap N) \cong (T(M) + N)/N = M/N$. Since M is co-coatomically supplemented, it is co-coatomically weak supplemented, so $T(M)/(T(M) \cap N)$ is co-coatomically weak supplemented. Small cover of a co-coatomically weak supplemented module is co-coatomically weak supplemented, therefore $T(M)$ is co-coatomically weak supplemented (see Proposition 5.3). $T_P(M)$ is also co-coatomically weak supplemented for each $P \in \mathcal{P}$ as it is a direct summand of $T(M)$ (see Theorem 2.4 and Proposition 5.1). By Proposition 3.16, $T_P(M)$ is co-coatomically supplemented module. By Corollary 2.4, $T_P(M)$ is bounded for each $P \in \mathcal{P}$.

($\Leftarrow$) Each $T_P(M)$ is bounded so it is supplemented by Lemma 2.11. Therefore $T(M)$ is supplemented by Theorem 2.7. Now let K be a submodule of M such that M/K is coatomic. Then $M/(K + T(M))$ is also coatomic. By hypothesis, $M/T(M)$ is divisible, i.e. it has no maximal submodule (see Lemma 2.3). Therefore $M = K + T(M)$. By Corollary 3.1, K has a supplement. Hence M is co-coatomically supplemented. □

Lemma 3.5 *((Büyükaşık, 2005), Corollary 4.1.2) Let R be a Dedekind domain and M a torsion module, then $M/\operatorname{Rad}(M)$ is semisimple.*

Corollary 3.7 *Let R be a non-local Dedekind domain and M a reduced R-module. If*

$\text{Rad}(T(M)) \ll T(M)$, *then* M *is co-coatomically supplemented if and only if* $M/T(M)$ *is divisible.*

Proof (⇒) Let M be a co-coatomically supplemented module. By the proof of Theorem 3.5, $M/T(M)$ is divisible.
(⇐) By Lemma 3.5, $T(M)/\text{Rad}(T(M))$ is semisimple, so it is co-coatomically weak supplemented. Then $T(M)$ is co-coatomically weak supplemented since $\text{Rad}(T(M)) \ll T(M)$ (see Proposition 5.3). Therefore $T(M)$ is co-coatomically supplemented by Proposition 3.16. Since $M/T(M)$ is divisible, $M/T(M)$ has no maximal submodule. Therefore M is co-coatomically supplemented by Proposition 3.6. □

CHAPTER 4

⊕-CO-COATOMICALLY SUPPLEMENTED MODULES

In this chapter we define ⊕-co-coatomically supplemented module and give some results about ⊕-co-coatomically supplemented modules. Most of the results we give in this chapter are the generalizations of the results about ⊕-cofinitely supplemented modules to ⊕-co-coatomically supplemented modules.

Throughout this chapter R will be an associative ring with unity unless otherwise stated .

Definition 4.1 *A module M is called an ⊕-co-coatomically supplemented module if every co-coatomic submodule of M has a supplement that is a direct summand of M.*

If every maximal submodule of a module M is a direct summand of M, then M is an ⊕-co-coatomically supplemented module by Theorem 3.1.

Obviously an ⊕-co-coatomically supplemented module is co-coatomically supplemented.

An ⊕-supplemented module is ⊕-co-coatomically supplemented module. But an ⊕-co-coatomically supplemented module need not be ⊕-supplemented in general by the following example.

Example 4.1 *The $\mathbb{Z}$-module $\mathbb{Q}$ does not have any proper co-coatomic submodule. Thus $\mathbb{Q}$ is ⊕-co-coatomically supplemented . But $\mathbb{Z}$-module $\mathbb{Q}$ is not supplemented, so it is not ⊕-supplemented (see Theorem 2.7).*

Proposition 4.1 *An ⊕-co-coatomically supplemented module is an ⊕-cofinitely supplemented module.*

Proof Let M be an ⊕-co-coatomically supplemented R-module and N a cofinite submodule of M. Since every cofinite submodule is co-coatomic and M is ⊕-co-coatomically supplemented module, N has a supplement that is a direct summand of M. Thus M is ⊕-cofinitely supplemented. □

By the following example, it is seen that an $\oplus$-cofinitely supplemented module need not be $\oplus$-co-coatomically supplemented module.

Example 4.2 *((Kasch, 1982), 11.3) Let R denote the ring $K[[x]]$ of all power series $\sum_{i=0}^{\infty} k_i x^i$ with coefficients from a field K in an indeterminate x which is a local ring. Then R-module $R^{(\mathbb{N})}$ is $\oplus$-cofinitely supplemented module but not $\oplus$-co-coatomically supplemented module.*

Proof If we consider R as R-module, R is supplemented, and hence R is semiperfect by Proposition 2.9. Note that the Jacobson radical of R

$$\mathrm{Jac}(R) = \left\{ \sum_{i=1}^{\infty} k_i x^i \mid k_i \in K \right\} = Rx \tag{4.1}$$

is not t-nilpotent. Thus R is not perfect by Proposition 2.10. Since R is semiperfect, $R/\,\mathrm{Jac}(R)$ is semisimple by Proposition 2.9. Since R is local, R-module R is $\oplus$-supplemented, therefore R-module R is both $\oplus$-co-coatomically supplemented and $\oplus$-cofinitely supplemented. By Theorem 2.11, any direct sum of ${}_RR$, i.e. left R-module $R^{(\mathbb{N})}$ is $\oplus$-cofinitely supplemented. Consider

$$\begin{aligned} {}_RR^{(\mathbb{N})}/\,\mathrm{Rad}({}_RR^{(\mathbb{N})}) &= \Big(\bigoplus R\Big)/\Big(\bigoplus \mathrm{Jac}(R)\Big) \\ &\cong \bigoplus (R/\,\mathrm{Jac}(R)) \end{aligned} \tag{4.2}$$

which is semisimple. Since semisimple modules are coatomic, $\mathrm{Rad}({}_RR^{(\mathbb{N})})$ is a co-coatomic submodule of ${}_RR^{(\mathbb{N})}$. By Theorem 2.8, $\mathrm{Rad}({}_RR^{(\mathbb{N})})$ does not have a supplement. Thus ${}_RR^{(\mathbb{N})}$ is not co-coatomically supplemented, so it is not $\oplus$-co-coatomically supplemented module. □

By the example above, it is seen that arbitrary direct sum of $\oplus$-co-coatomically supplemented module need not be $\oplus$-co-coatomically supplemented .

Proposition 4.2 *A finite direct sum of $\oplus$-co-coatomically supplemented R-modules is $\oplus$-co-coatomically supplemented.*

Proof Let n be a positive integer and M_i an $\oplus$-co-coatomically supplemented R-module for each $1 \leq i \leq n$. Let $M = M_1 \oplus \cdots \oplus M_n$. To prove that M is $\oplus$-co-coatomically supplemented, it is sufficient to prove the case for $n = 2$. Thus suppose $n = 2$. Let L be any co-coatomic submodule of M. Then $M = M_1 + M_2 + L$, so that $M_1 + M_2 + L$ has a supplement 0 in M. Consider the submodule $M_2 \cap (M_1 + L)$ of M_2. It follows that

$$\begin{aligned} M_2/(M_2 \cap (M_1 + L)) &\cong (M_1 + M_2 + L)/(M_1 + L) \\ &= M/(M_1 + L). \end{aligned} \tag{4.3}$$

Since $M_1 + L$ is co-coatomic submodule of M, $M_2 \cap (M_1 + L)$ is co-coatomic submodule of M_2. Let H be a supplement of $M_2 \cap (M_1 + L)$ in M_2 such that H is a direct summand of M_2. H is a supplement of $M_1 + L$ in M by Lemma 3.1. Now consider the submodule $M_1 \cap (L + H)$ of M_1. Then

$$\begin{aligned} M_1/(M_1 \cap (L + H)) &\cong (M_1 + L + H)/(L + H) \\ &= M/(L + H). \end{aligned} \tag{4.4}$$

Since $L + H$ is co-coatomic submodule of M, $M_1 \cap (L + H)$ is co-coatomic submodule of M_1. Let K be a supplement of $M_1 \cap (L+H)$ in M_1 such that K is a direct summand of M_1. Again applying Lemma 3.1, we get $H + K$ is supplement of L in M. Since H is a direct summand of M_2 and K is a direct summand of M_1, it follows that $H + K = H \oplus K$ is a direct summand of M. Thus $M = M_1 \oplus M_2$ is $\oplus$-co-coatomically supplemented. □

A finitely generated R-module M is said to be finitely presented in case in every exact sequence $0 \longrightarrow K \longrightarrow F \longrightarrow M \longrightarrow 0$ with F finitely generated and free and the kernel K is also finitely generated.

Example 4.3 *Let R be a commutative local ring which is not a valuation ring and let $n \geq 2$. By ((Warfield Jr., 1970), Theorem 2), there exists a finitely presented indecomposable module $M = R^{(n)}/K$ which can not be generated by fewer than n elements. By ((Idelhadj and Tribak, 2000), Corollary 1), $R^{(n)}$ is $\oplus$-supplemented, and therefore $\oplus$-co-coatomically supplemented. However, M is not $\oplus$-cofinitely supplemented, so it is not $\oplus$-co-coatomically supplemented (see (Idelhadj and Tribak, 2000), Proposition 2 and (Wang and Sun, 2007), Example 2.1).*

Example 4.3 is an example for that factor module of an $\oplus$-co-coatomically supplemented module need not be $\oplus$-co-coatomically supplemented. But under some conditions this is possible as in the following results.

Let M be a nonzero R-module and let U be a fully invariant submodule of M, i.e. $f(U) \leq U$ for each $f \in End_R(M)$ (endomorphism ring of ${}_RM$). If $M = M_1 \oplus M_2$, then

$U = (U \cap M_1) \oplus (U \cap M_2)$ for a fully invariant submodule U of M (see ((Fuchs, 1970), Lemma 9.3) for abelian groups).

Proposition 4.3 *Let M be a nonzero module and U a fully invariant submodule of M. If M is $\oplus$-co-coatomically supplemented module, then M/U is $\oplus$-co-coatomically supplemented. Furthermore, if U is a co-coatomic direct summand of M, then U is also $\oplus$-co-coatomically supplemented.*

Proof Suppose that M is $\oplus$-co-coatomically supplemented module, and L/U is co-coatomic submodule of M/U. Therefore $M/L \cong (M/U)/(L/U)$ is coatomic. Since M is $\oplus$-co-coatomically supplemented, there exist submodules N and N' of M such that $M = N \oplus N'$, $M = N + L$ and $N \cap L \ll N$. Then $(N + U)/U$ is a supplement of L/U in M/U. By hypothesis, U is fully invariant, therefore $U = (U \cap N) \oplus (U \cap N')$ (see (Fuchs, 1970), Lemma 9.3). Thus $U = (N+U) \cap (N'+U)$ and $M/U = ((N+U)/U) \oplus ((N'+U)/U)$. Hence M/U is $\oplus$-co-coatomically supplemented.

Now suppose that U is a co-coatomic direct summand of M. Then there exist a submodule U' of M such that $M = U \oplus U'$ and U' is coatomic. Let V be a co-coatomic submodule of U. Therefore $M/V = (U \oplus U')/V \cong (U/V) \oplus U'$ is coatomic as it is direct sum of two coatomic modules. Since M is $\oplus$-co-coatomically supplemented, there exist submodules K and K' of M such that $M = K \oplus K'$, $M = V + K$ and $V \cap K \ll K$. Thus $U = V + (U \cap K)$. Since U is fully invariant, $U = (U \cap K) \oplus (U \cap K')$, and so $U \cap K$ is direct summand of U. Furthermore, $V \cap (U \cap K) = V \cap K \ll K$. Then $V \cap (U \cap K) \ll U \cap K$ (see Lemma 2.2(5)). Therefore $U \cap K$ is a supplement of V in U, and it is a direct summand of U. Hence U is $\oplus$-co-coatomically supplemented. □

Corollary 4.1 *Let M be a $\oplus$-co-coatomically supplemented module. Then $M/\operatorname{Rad}(M)$ and $M/\operatorname{Soc}(M)$ are also $\oplus$-co-coatomically supplemented modules.*

For an $\oplus$-cofinitely supplemented module with property (D3), every cofinite direct summand of M is $\oplus$-cofinitely supplemented (see Proposition 2.12). The following proposition is an analoguous result for $\oplus$-co-coatomically supplemented modules.

Proposition 4.4 *Let M be a $\oplus$-co-coatomically supplemented module with property (D3). Then every co-coatomic direct summand of M is $\oplus$-co-coatomically supplemented .*

Proof Let N be a co-coatomic direct summand of M. Then there exists a submodule N' of M such that $M = N \oplus N'$ and N' is coatomic. Let U be a co-coatomic submodule of N. Note that $M/U = (N \oplus N')/U \cong (N/U) \oplus N'$ is coatomic as it is a finite direct sum of coatomic modules. Since M is $\oplus$-co-coatomically supplemented module, there exists a direct summand V of M such that $M = U + V$ and $U \cap V \ll V$. Hence $N = N \cap M = N \cap (U + V) = U + (N \cap V)$. Since M has property (D3), $N \cap V$ is a direct summand of M. Moreover, $N \cap V$ is a direct summand of N since N is a direct summand of M. Then $U \cap (N \cap V) = U \cap V$ is small in $N \cap V$ by Lemma 2.2(5). Hence N is $\oplus$-co-coatomically supplemented. □

Proposition 4.5 *Over a left V-ring R, a left R-module M is $\oplus$-co-coatomically supplemented if and only if M is semisimple.*

Proof ($\Leftarrow$) Clear.

($\Rightarrow$) Since M is an $\oplus$-co-coatomically supplemented, M is $\oplus$-cofinitely supplemented, and so cofinitely supplemented. Therefore $M/\operatorname{Soc}(M)$ has no maximal submodule by Theorem 2.9 and Proposition 3.10. Since R is a left V-ring, $M/\operatorname{Soc}(M) = \operatorname{Rad}(M/\operatorname{Soc}(M)) = 0$ by Theorem 3.2. Thus M is semisimple. □

Proposition 4.6 *Let M be an indecomposable R-module. The following are equivalent:*

1. *Every co-coatomic submodule of M has a supplement that is a direct summand.*
2. *Every maximal submodule of M has a supplement that is a direct summand.*
3. *M is radical or M is local.*

Proof (1) $\Rightarrow$ (2) Clear since every maximal submodule is co-coatomic.

(2) $\Rightarrow$ (3) By ((Tribak, 2008), Proposition 2.7).

(3) $\Rightarrow$ (1) Let M be radical module, i.e. M has no maximal submodule. Therefore the only co-coatomic submodule of M itself. Thus M has supplement 0, and since M is indecomposable 0 is direct summand. Now let M be local module, and so hollow. Therefore M is $\oplus$-supplemented. Thus every co-coatomic submodule has a supplement that is a direct summand in M. □

Corollary 4.2 *Let M be an indecomposable R-module such that* $\operatorname{Rad}(M) \neq M$. *$M$ is $\oplus$-co-coatomically supplemented if and only if M is local.*

Theorem 4.1 *A ring R is left perfect if and only if $R^{(\mathbb{N})}$ is an $\oplus$-co-coatomically supplemented left R-module.*

Proof ($\Rightarrow$) By ((Mohamed and Müller, 1990), Theorem 4.41 and Proposition 4.8), ${}_RR^{(\mathbb{N})}$ is $\oplus$-supplemented, and so $\oplus$-co-coatomically supplemented.

($\Leftarrow$) Since ${}_RR^{(\mathbb{N})}$ is $\oplus$-co-coatomically supplemented module, it is co-coatomically supplemented. Therefore ${}_RR$ is co-coatomically supplemented since it is direct summand of

${}_RR^{(\mathbb{N})}$. Since ${}_RR$ is finitely generated, and so coatomic, it is supplemented. Thus $R/\operatorname{Jac}(R)$ is left semisimple by Proposition 2.9. It follows that ${}_RR^{(\mathbb{N})}/\operatorname{Rad}({}_RR^{(\mathbb{N})})$ is semisimple, i.e. $\operatorname{Rad}({}_RR^{(\mathbb{N})})$ is co-coatomic submodule of ${}_RR^{(\mathbb{N})}$. By hypothesis, ${}_RR^{(\mathbb{N})}$ is co-coatomically supplemented, so $\operatorname{Rad}({}_RR^{(\mathbb{N})})$ has a supplement in ${}_RR^{(\mathbb{N})}$. By Remark 3.1 and Proposition 3.12, $\operatorname{Rad}({}_RR^{(\mathbb{N})}) \ll {}_RR^{(\mathbb{N})}$. Therefore, since $R/\operatorname{Jac}(R)$ is left semisimple, R is left perfect by Proposition 2.10. □

Corollary 4.3 *The following are equivalent for a ring R:*

1. *R is left perfect.*
2. *The R-module $R^{(\mathbb{N})}$ is $\oplus$-supplemented.*
3. *The R-module $R^{(\mathbb{N})}$ is $\oplus$-co-coatomically supplemented.*

Proof (1) $\Leftrightarrow$ (2) By ((Keskin et al., 1999), Theorem 2.10).
(1) $\Leftrightarrow$ (3) By Theorem 4.1. □

An R-module M is called a multiplication module if every submodule of M is of the form IM for some ideal I of R.

Let M be a multiplication R-module. If M is $\oplus$-cofinitely supplemented module with $\operatorname{Rad}(M) \ll M$, then M can be written as an irredundant sum of local direct summands of M (see (Wang and Sun, 2007), Theorem 2.7).

Proposition 4.7 *Let M be a $\oplus$-co-coatomically supplemented multiplication module with* $\operatorname{Rad}(M) \ll M$. *Then M can be written as an irredundant sum of local direct summands of M.*

Proof Since every $\oplus$-co-coatomically supplemented module is $\oplus$-cofinitely supplemented, the proof is clear by ((Wang and Sun, 2007), Theorem 2.7). □

4.1. Co-coatomically Semiperfect Modules

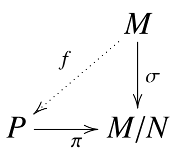
A projective module M is $\oplus$-supplemented if and only if M is semiperfect (see (Azumaya, 1991), Proposition 1.4). An R-module M is cofinitely semiperfect if every finitely generated factor module of M has a projective cover. Semiperfect modules are cofinitely semiperfect, and finitely generated cofinitely semiperfect modules are semiperfect (Çalışıcı and Pancar, 2005). A projective module M is cofinitely semiperfect if and only if M is $\oplus$-cofinitely supplemented (see (Çalışıcı and Pancar, 2005), Theorem 2.1). Therefore we give the following analogous definition.

Definition 4.2 *Let M be an R-module. M is called co-coatomically semiperfect module if every coatomic factor module of M has a projective cover.*

The following result gives a characterization of a projective ⊕-co-coatomically supplemented module.

Proposition 4.8 *Let M be a projective R-module. Then M is co-coatomically semiperfect module if and only if M is ⊕-co-coatomically supplemented module.*

Proof (⇒) Let N be a co-coatomic submodule of M. Then M/N is coatomic. By hypothesis, there exists a projective cover $\pi : P \to M/N$. Let $\sigma : M \to M/N$ be the canonical epimorphism. Since M is projective, there exists a homomorphism $f : M \to P$ such that the diagram



(4.5)

is commutative, i.e. $\pi \circ f = \sigma$. Since π is a small epimorphism, f is epic by Lemma 2.1. Since P is projective, f splits, i.e. there exists a homomorphism $g : P \to M$ such that $f \circ g = 1_P$ by ((Kasch, 1982), 3.9.3). Thus $\pi = \pi \circ f \circ g = \sigma \circ g$. It follows that $M = \ker f \oplus g(P)$ and $\ker f \leq N$, so $M = N + g(P)$. Let $\mu = \sigma |_{g(P)}: g(P) \to M/N$. Then $\pi = \mu \circ g$, and therefore μ is epic since π is epimorphism. Furthermore, since π is a small epimorphism, μ is also a small epimorphism by Lemma 2.2. Therefore $\ker \mu = N \cap g(P) \ll g(P)$. Thus $g(P)$ is a supplement of N.

(⇐) Let M/N be a coatomic factor module of M. Since M is ⊕-co-coatomically supplemented, there exist submodules K and K' such that $M = K \oplus K'$, $M = N + K$ and $N \cap K \ll K$. Since M is projective, K is projective. For the inclusion homomorphism $i : K \to M$ and the canonical epimorphism $\sigma : M \to M/N$, $\sigma \circ i : K \to M/N$ is an epimorphism and $\ker \sigma \circ i = N \cap K \ll K$. □

A co-coatomically semiperfect module is cofinitely semiperfect, but inverse need not be true by Example 4.2 since the projective R-module $R^{(\mathbb{N})}$ in that example is ⊕-cofinitely supplemented module, but not ⊕-co-coatomically supplemented module.

Let M be an R-module and N a submodule of M. N is said to lie above a direct summand of M if there is a decomposition $M = K \oplus K'$ such that $K \leq N$ and $K' \cap N \ll K'$.

A module M is called co-coatomically amply supplemented module if every co-coatomic submodule of M has an ample supplement. Details of these modules are provided in Chapter 6.

Proposition 4.9 *Let M be a projective module. Then the following are equivalent:*

1. *M is co-coatomically semiperfect.*
2. *M is $\oplus$-co-coatomically supplemented.*
3. *Every co-coatomic submodule of M lies above a direct summand of M.*
4. *M is co-coatomically amply supplemented by supplements which have projective covers.*
5. *M is co-coatomically supplemented by supplements which have projective covers.*

Proof (1) $\Leftrightarrow$ (2) By Proposition 4.8.
(2) $\Rightarrow$ (3) Let N be a co-coatomic submodule of M. By hypothesis, there exist submodules K and K' of M such that $M = N + K$, $N \cap K \ll K$ and $M = K \oplus K'$. Since M is projective, there exists a submodule $K'' \leq N$ such that $M = K'' \oplus K$ (see (Wisbauer, 1991), 41.14).
(3) $\Rightarrow$ (2) Clear.
(1) $\Rightarrow$ (4) Let N be a co-coatomic submodule of M such that $M = N + L$ for some submodule L of M. Let (P, f) be a projective cover of M/N. Since P is projective and $M/N \cong L/N \cap L$, there exists a homomorphism $g : P \to L$. Since $\operatorname{Ker} f \ll P$ and $g(\operatorname{Ker} f) = \operatorname{Im} g \cap N \cap L = \operatorname{Im} g \cap N$, $\operatorname{Im} g \cap N \cap L \ll \operatorname{Im} g$. $\operatorname{Im} g + (N \cap L) = L$ since f is an epimorphism. Therefore $\operatorname{Im} g$ is a supplement of $N \cap L$ in L. $M = N + L = N + \operatorname{Im} g + N \cap L = \operatorname{Im} g + N$ and $\operatorname{Im} g \cap N \ll \operatorname{Im} g$, i.e. $\operatorname{Im} g$ is a supplement of N in M and $\operatorname{Im} g$ is contained in L. Since $\operatorname{Ker} g \leq \operatorname{Ker} f$ and $\operatorname{Ker} f \ll P$, P is a projective cover of $\operatorname{Im} g$.
(4) $\Rightarrow$ (5) Clear.
(5) $\Rightarrow$ (1) Let N be a co-coatomic submodule of M and L a supplement of N in M. Therefore L is a small cover of $L/(N \cap L)$. By hypothesis, every projective cover L is also projective cover of $L/(N \cap L)$. Since $M/N \cong L/(N \cap L)$, M/N has a projective cover. Thus M is co-coatomically semiperfect. □

Proposition 4.10 *Every homomorphic image of a co-coatomically semiperfect module is co-coatomically semiperfect.*

Proof Let $f : M \to N$ be a homomorphism and let M be a co-coatomically semiperfect module. Let $f(M)/U$ be a coatomic factor module of $f(M)$. There is an epimorphism

$\sigma : M \to f(M)/U$, $m \mapsto f(m) + U$. Since M is co-coatomically semiperfect,

$$M/f^{-1}(U) \cong f(M)/U \tag{4.6}$$

that $f(M)/U$ has a projective cover. Thus $f(M)$ is co-coatomically semiperfect. □

Corollary 4.4 *Every factor module of a co-coatomically semiperfect module is co-coatomically semiperfect.*

Corollary 4.5 *Let M be a projective module. If M is $\oplus$-co-coatomically supplemented, then every factor module of $\oplus$-co-coatomically supplemented module is also $\oplus$-co-coatomically supplemented.*

Proof By Corollary 4.4. □

Proposition 4.11 *Every small cover of a co-coatomically semiperfect module is co-coatomically semiperfect .*

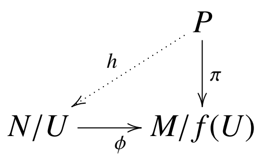
Proof Let N be a small cover of M and $f : N \to M$ a small epimorphism. For a coatomic factor module N/U of N, the homomorphism

$$\phi : N/U \to M/f(U),\ n + U \mapsto f(n) + f(U) \tag{4.7}$$

is epic and $\ker \phi \ll N/U$ since $\ker f \ll N$. Thus

$$M/f(U) = \phi(N/U) \cong (N/U)/((U + \ker f)/U) \tag{4.8}$$

so that $M/f(U)$ is coatomic. Since M is co-coatomically semiperfect, $M/f(U)$ has a projective cover $\pi : P \to M/f(U)$. Since P is projective, there is a homomorphism $h : P \to N/U$ such that the diagram



(4.9)

commutes, i.e. $\pi = \phi \circ h$. Thus h is epimorphism by Lemma 2.1 and since π is small, h is small by Lemma 2.2. Hence P is a projective cover of the module N/U. □

Corollary 4.6 *If $K \ll M$ and M/K is co-coatomically semiperfect module, then M is co-coatomically semiperfect.*

Corollary 4.7 *Let $\pi : P \to M$ be a projective cover of module M. Then the following statements are equivalent:*

1. *M is co-coatomically semiperfect.*

2. *P is co-coatomically semiperfect.*

3. *P is $\oplus$-co-coatomically supplemented.*

Proof (1) ⇒ (2) By Proposition 4.11.
(2) ⇒ (1) By Proposition 4.10.
(2) ⇔ (3) By Proposition 4.8. □

Lemma 4.1 *Let M be a projective module. If M is semiperfect, then every finitely M-generated module is co-coatomically semiperfect. The converse holds if M is finitely generated.*

Proof Let N be a finitely M-generated module. Since M is semiperfect, M is $\oplus$-supplemented. Therefore M is $\oplus$-co-coatomically supplemented. By Proposition 4.2, a finite direct sum of M, i.e. for a finite set Λ, $M^{(\Lambda)}$ is also $\oplus$-co-coatomically supplemented. Therefore $M^{(\Lambda)}$ is co-coatomically semiperfect by Proposition 4.8. By Corollary 4.4, N is co-coatomically semiperfect. Conversely, suppose that M is finitely generated, and so it is coatomic. By hypothesis, M is co-coatomically semiperfect. Therefore M is semiperfect. □

Proposition 4.12 *For a ring R, the following statements are equivalent:*

1. *R is semiperfect.*

2. *Every finitely generated free R-module is semiperfect.*

3. *Every finitely generated free R-module is co-coatomically semiperfect.*

Proof (1) ⇔ (2) By Lemma 2.14 and ((Keskin et al., 1999), Theorem 2.1).
(1) ⇔ (3) By Lemma 4.1. □

CHAPTER 5

CO-COATOMICALLY WEAK SUPPLEMENTED MODULES

Throughout this chapter R is an arbitrary ring unless otherwise stated. Let M be an R-module. If every co-coatomic submodule of M has a weak supplement, then M is called a co-coatomically weak supplemented module.

Example 5.1 *The $\mathbb{Z}$-module $\mathbb{Q}$ is co-coatomically weak supplemented.*

A co-coatomically supplemented module is co-coatomically weak supplemented, but the converse is not correct by the following example.

Example 5.2 *Consider the ring*

$$R = \mathbb{Z}_{p,q} = \left\{ \frac{a}{b} \mid a,\ b \in \mathbb{Z},\ b \neq 0,\ (b, p) = 1,\ (b, q) = 1 \right\}. \tag{5.1}$$

${}_RR$ is (co-coatomically weak) weakly supplemented, but it is not (co-coatomically) supplemented (see ((Lomp, 1999), Remark 3.3)).

A ring R is semilocal if and only if ${}_RR$ is weakly supplemented ((Lomp, 1999), Corollary 3.2). Since ${}_RR$ is finitely generated, so coatomic, ${}_RR$ is co-coatomically weak supplemented if and only if R is semilocal.

Proposition 5.1 *A homomorphic image of a co-coatomically weak supplemented module is a co-coatomically weak supplemented module.*

Proof Let $f : M \to N$ be a homomorphism and M a co-coatomically weak supplemented module. Suppose that K is a co-coatomic submodule of $f(M)$, then

$$M/f^{-1}(K) \cong ((M/\operatorname{Ker} f)/(f^{-1}(K)/\operatorname{Ker} f)) \cong f(M)/K. \tag{5.2}$$

Therefore the submodule $f^{-1}(K)$ of M is co-coatomic. Since M is co-coatomically weak supplemented module, $f^{-1}(K)$ is a weak supplement in M. $K = f(f^{-1}(K))$ is a weak supplement in $f(M)$ by Lemma 2.19. Hence K has a weak supplement in $f(M)$. □

Proposition 5.2 *Let N be a co-coatomically weak supplemented submodule of an R-module M such that M/N has no maximal submodule. Then M is a co-coatomically weak supplemented module.*

Proof See the proof of Proposition 3.2. □

Proposition 5.3 *A small cover of a co-coatomically weak supplemented module is a co-coatomically weak supplemented module.*

Proof Let N be a co-coatomically weak supplemented module, $f : M \to N$ a small epimorphism and L a co-coatomic submodule of M. Then the module $N/f(L)$ is an epimorphic image of M/L under the epimorphism $\overline{f} : M/L \to N/f(L)$ defined by $\overline{f}(m + L) = f(m) + f(L)$. Therefore $f(L)$ is a co-coatomic submodule of N. Since N is a co-coatomically weak supplemented module, $f(L)$ is a weak supplement. By Lemma 2.20, L is (has) a weak supplement in M. □

Corollary 5.1 *Let M be an R-module with $\text{Rad}(M) \ll M$. If the factor module $M/\text{Rad}(M)$ is co-coatomically weak supplemented module, then M is a co-coatomically weak supplemented module.*

Proposition 5.4 *Every supplement in a co-coatomically weak supplemented module is co-coatomically weak supplemented .*

Proof Let M be a co-coatomically weak supplemented module. If N is a supplement in M, then $N + K = M$ and $N \cap K \ll N$ for some submodule K of M. Therefore $M/K = (N + K)/K \cong N/(N \cap K)$. Since M/K is co-coatomically weak supplemented, $N/(N \cap K)$ is co-coatomically weak supplemented. Thus N is co-coatomically weak supplemented by Proposition 5.3. □

Over arbitrary rings, a supplement of a co-coatomic submodule is coatomic. In the following proposition we prove that over Dedekind domains, a weak supplement of a co-coatomic submodule is coatomic.

Proposition 5.5 *Let R be a Dedekind domain and M an R-module, K and L submodules of M. If L is a weak supplement of K, then L is coatomic if and only if M/K is coatomic.*

Proof Let M be an R-module, K a co-coatomic submodule of M and L a weak supplement of K in M. Then $M = L + K$ and $L \cap K \ll M$. It follows that $M/K = (L + K)/K \cong$

$L/(L \cap K)$. By Lemma 2.7(1), $L \cap K$ is coatomic. Since $L \cap K$ and $L/(L \cap K)$ are coatomic, L is also coatomic by Lemma 2.4(2). Conversely, since $M/K \cong L/(K \cap L)$ and L is coatomic M/K is coatomic. □

Lemma 5.1 *((Alizade and Büyükaşık, 2003), Lemma 2.2) The $\mathbb{Z}$-submodule*

$$M = \sum_{q prime} \mathbb{Z}.\frac{1}{q}, \tag{5.3}$$

consisting of all rational numbers with square-free denominators, is a small submodule of the $\mathbb{Z}$-module $\mathbb{Q}$ of all rational numbers, i.e. ${}_{\mathbb{Z}}M \ll_{\mathbb{Z}} \mathbb{Q}$.

Example 5.3 *Consider $\mathbb{Q} \oplus \mathbb{Z}_p$ as a $\mathbb{Z}$-module, where p is a prime. Then $\mathbb{Q} \oplus 0$ is a maximal submodule of $\mathbb{Q} \oplus \mathbb{Z}_p$, therefore it is co-coatomic. Let M be as in Lemma 5.1.*

$$(M \oplus \mathbb{Z}_p) \cap (\mathbb{Q} \oplus 0) = M \oplus 0 \ll \mathbb{Q} \oplus 0 \leq \mathbb{Q} \oplus \mathbb{Z}_p \tag{5.4}$$

by Lemma 5.1. Therefore $M \oplus \mathbb{Z}_p$ is a weak supplement of $\mathbb{Q} \oplus 0$. Hence $M \oplus \mathbb{Z}_p$ is a coatomic submodule of $\mathbb{Q} \oplus \mathbb{Z}_p$ by Proposition 5.5.

Remark 5.1 *In the example above, M is coatomic since it is a direct summand of a coatomic module. $M/\mathbb{Z}$ is a direct sum of the cyclic groups $< \frac{1}{q} + \mathbb{Z} >$, q prime, is a coatomic module, but not a finitely generated module, and therefore M is not a finitely generated module. Furthermore, M is not semisimple. Therefore, M is an example of a coatomic module that is neither finitely generated nor semisimple.*

Lemma 5.2 *Let N and U be submodules of M with co-coatomically weak supplemented N and co-coatomic U. If $N + U$ has a weak supplement in M, then U has a weak supplement in M.*

Proof Let X be a weak supplement of $N + U$ in M. Then

$$\begin{aligned} N/(N \cap (U + X)) &\cong (N + U + X)/(U + X) \\ &= M/(U + X) \\ &\cong (M/U)/((U + X)/U). \end{aligned} \tag{5.5}$$

Since M/U is coatomic, $N/(N \cap (U + X))$ is also a coatomic module. It follows that $N \cap (U+X)$ has a weak supplement Y in N, i.e. $Y+(N \cap (U+X)) = N$ and $Y \cap N \cap (U+X) = Y \cap (U + X) \ll N \leq M$. Therefore

$$\begin{aligned} M &= U + X + N \\ &= U + X + Y + (N \cap (U + X)) \\ &= U + X + Y, \end{aligned} \tag{5.6}$$

and

$$\begin{aligned} U \cap (X + Y) &\leq (X \cap (U + Y)) + (Y \cap (U + X) \\ &\leq (X \cap (U + N)) + (Y \cap (U + X)) \\ &\ll M. \end{aligned} \tag{5.7}$$

Hence $X + Y$ is a weak supplement of U in M. □

A finite sum of co-coatomically weak supplemented modules is co-coatomically weak supplemented by the following proposition, but the direct sum of infinitely many co-coatomically weak supplemented modules need not be co-coatomically weak supplemented by Example 3.3.

Proposition 5.6 *A finite sum of co-coatomically weak supplemented modules is co-coatomically weak supplemented .*

Proof Let $M = M_1 + M_2$ be a module, where M_1 and M_2 are co-coatomically weak supplemented modules. Let U be a co-coatomic submodule of M. Thus $M = M_1 + M_2 + U$ has a weak supplement which is 0. Since U is a co-coatomic submodule of M, $M_2 + U$ is also a co-coatomic submodule of M. In this case, $M_2 + U$ has a weak supplement in M by Lemma 5.2. Again by Lemma 5.2, U has a weak supplement in M. □

Corollary 5.2 *A finite direct sum of co-coatomically weak supplemented modules is also co-coatomically weak supplemented.*

Corollary 5.3 *If M is co-coatomically weak supplemented module, then every finitely M-generated module is co-coatomically weak supplemented .*

A co-coatomically weak supplemented module is a cofinitely weak supplemented module, but the converse is not correct by the Example 3.3. Example 3.3 also shows that over Dedekind domains, co-coatomically weak supplemented modules and cofinitely weak supplemented modules need not coincide.

Example 5.4 *Let p be a prime integer and consider the following ring which is semiperfect but not perfect*

$$R = \mathbb{Z}_{(p)} = \left\{ \frac{a}{b} \mid a,\, b \in \mathbb{Z},\, b \neq 0,\, (b, p) = 1 \right\}. \tag{5.8}$$

Since R-module R is local, ${}_RR$ is supplemented. Therefore R-module R is both co-coatomically weak supplemented and cofinitely weak supplemented. R-module $R^{(\mathbb{N})}$ is a cofinitely weak supplemented module as it is the direct sum of cofinitely weak supplemented modules (see Proposition 2.13). $\mathrm{Rad}({}_RR^{(\mathbb{N})})$ *is co-coatomic submodule of R-module $R^{(\mathbb{N})}$, but it does not have a weak supplement in ${}_RR^{(\mathbb{N})}$ by Theorem 2.8 (see Example 3.4). R-module R is co-coatomically weak supplemented, but R-module $R^{(\mathbb{N})}$ is not co-coatomically weak supplemented.*

Example 5.4 shows that over semiperfect rings cofinitely weak supplemented and co-coatomically weak supplemented modules need not coincide. This example also shows that infinite direct sum of co-coatomically weak supplemented modules need not be co-coatomically weak supplemented. By Example 5.4, it is seen that over semiperfect rings every module need not be co-coatomically weak supplemented, but it is proved in the following theorem that over perfect rings every module is co-coatomically weak supplemented.

Theorem 5.1 *The following are equivalent:*

1. *Every left R-module is co-coatomically weak supplemented.*
2. *${}_RR^{(\mathbb{N})}$ is co-coatomically weak supplemented.*
3. *R is semilocal and* $\mathrm{Rad}({}_RR^{(\mathbb{N})})$ *has a weak supplement in ${}_RR^{(\mathbb{N})}$.*
4. *R is left perfect ring.*

Proof (4) ⇒ (1) ⇒ (2) Clear.
(2) ⇒ (3) Since ${}_RR^{(\mathbb{N})}$ is co-coatomically weak supplemented, ${}_RR$ is co-coatomically weak supplemented. Since ${}_RR$ is finitely generated, i.e. coatomic, ${}_RR$ is weakly supplemented. Therefore ${}_RR$ is semilocal (see Proposition 2.8). By definition of semilocal ring, $R/\operatorname{Jac} R$ is left semisimple. It follows that ${}_RR^{(\mathbb{N})}/\mathrm{Rad}({}_RR^{(\mathbb{N})})$ is semisimple left R-module. Thus

$\mathrm{Rad}({}_RR^{(\mathbb{N})})$ is co-coatomic submodule in ${}_RR^{(\mathbb{N})}$. Hence $\mathrm{Rad}({}_RR^{(\mathbb{N})})$ has a weak supplement in ${}_RR^{(\mathbb{N})}$.
(3) $\Rightarrow$ (4) By Theorem 2.8. □

Proposition 5.7 *Let*

$$0 \to L \to M \to N \to 0 \tag{5.9}$$

be a short exact sequence. If L and N are co-coatomically weak supplemented modules and L has a weak supplement in M, then M is a co-coatomically weak supplemented module.

If L is coclosed and co-coatomic, then the converse holds, that is if M is co-coatomically weak supplemented, then L and N are co-coatomically weak supplemented.

Proof Let S be a weak supplement of L in M i.e. $L + S = M$ and $L \cap S \ll M$. Then

$$M/L \cap S \cong (L/L \cap S) \oplus (S/L \cap S). \tag{5.10}$$

$L/L \cap S$ is co-coatomically weak supplemented module since L is co-coatomically weak supplemented module. On the other hand,

$$S/L \cap S \cong M/L \cong N \tag{5.11}$$

is a co-coatomically weak supplemented module. Then $M/L \cap S$ is a co-coatomically weak supplemented module as a sum of co-coatomically weak supplemented modules. Hence M is a co-coatomically weak supplemented module since M is a small cover of $M/L \cap S$ (see Proposition 5.3). If L is co-coatomic submodule of M, then it has a weak supplement in M, say S. Since L is coclosed $L \cap S \ll L$, i.e. L is a supplement of S in M. By Proposition 5.4, L is co-coatomically weak supplemented and by Proposition 5.1, N is co-coatomically weak supplemented. □

In Proposition 5.7, if L has no weak supplement in M then the result may not be correct. The following example explains this situation.

Over a von Neumann regular ring, a module need not be co-coatomically weak supplemented module by the Example 5.5.

Example 5.5 *((Smith, 2000), Example 2.5) Let F be a field and S the direct product $\prod_{n\in\mathbb{N}} F_n$, where $F_n = F\ (n \geq 1)$. Then the elements of S are the sequences $\{a_n\}$, where $a_n \in F\ (n \in \mathbb{N})$. Let R be the subring of S consisting of all sequences $\{a_n\}$ such that there exists $a \in F$, $k \in \mathbb{N}$ with $a_n = a$ for all $n \geq k$. Then R is a von Neumann regular ring, so that the R-module R has zero radical. The $\mathrm{Soc}({}_RR)$ of the R-module R consists of all sequences $\{a_n\}$ in R such that $a_n = 0$ for all $n \geq k$ for some $k \in \mathbb{N}$. Hence $\mathrm{Soc}({}_RR)$ is not finitely generated, and $\mathrm{Soc}({}_RR)$ does not have any weak supplement. On the other hand, $\mathrm{Soc}_R R$ is a maximal submodule of ${}_RR$. Therefore ${}_RR$ is not co-coatomically weak supplemented.*

Proof Let $(a_1, \ldots, a_{k-1}, a, a, \ldots)$ be an element of R. If each $a_i\ (1 \leq i \leq k-1)$ and a are non-zero elements of F, they have inverses. Hence

$$\begin{aligned}(a_1, \ldots, a_{k-1}, a, a, \ldots)(a_1^{-1}, \ldots, a_{k-1}^{-1}, a^{-1}, a^{-1}, \ldots)(a_1, \ldots, a_{k-1}, a, a, \ldots)\\ = (a_1, \ldots, a_{k-1}, a, a, \ldots).\end{aligned} \tag{5.12}$$

Thus R is a von Neumann regular ring. Clearly $\mathrm{Jac}(R) = 0$. Now let T be a simple submodule of R-module R and $0 \neq \overline{a} = (a_1, \ldots, a_{k-1}, a, a, \ldots) \in T \leq R$. For an element $r = (a_1^{-1}, 0, 0, \ldots, 0, \ldots) \in R$, $r\overline{a} = (1, 0, 0, \ldots, 0, \ldots) \in T$. Then

$$A_1 =< (1, 0, 0, \ldots, 0, \ldots) > \tag{5.13}$$

is a submodule of R-module R which is generated by $r\overline{a} \in T$. Hence $A_1 \leq T$. Since T is simple and A_1 is different from zero, $A_1 = T$. Hence $\mathrm{Soc}({}_RR)$ is not finitely generated, and by Lemma 2.18, $\mathrm{Soc}({}_RR)$ does not have any weak supplement.

Now let M denote ${}_RR$ and $\mathrm{Soc}(M) \lneqq A \leq M$ for some submodule A of M. Since $\mathrm{Soc}(M) \lneqq A$, there exists an element x of $A \setminus \mathrm{Soc}(M)$ such that $x = (a_1, \ldots, a_n, a, a, \ldots)$. For an element $r = (0, \ldots, 0, a^{-1}, a^{-1}, \ldots)$ of R, $rx = (0, \ldots, 0, 1, 1, \ldots) \in A$. Now let $\overline{m} = (m_1, \ldots, m_k, m, m, \ldots) \in M$. Then $\overline{m}$ can be represented in the following way:

$$\overline{m} = (m_1, \ldots, m_k, 0, 0, \ldots) + (0, \ldots, 0, m, m, \ldots) \tag{5.14}$$

where $(m_1, \ldots, m_k, 0, 0, \ldots) \in \mathrm{Soc}(M) \lneqq A$ and $(0, \ldots, 0, m, m, \ldots) \in A$, so $\overline{m} \in A$ implies $M = A$. Hence $\mathrm{Soc}(M)$ is a maximal submodule of M and $M/\mathrm{Soc}(M)$ is simple. Thus, since Soc(M) is a co-coatomic submodule of M but it does not have a weak supplement in M, M is not co-coatomically weak supplemented. □

Theorem 5.2 *Let M be an R-module. M is co-coatomically weak supplemented if and only if $M/(\bigoplus_{i=1}^{n} L_i)$ is co-coatomically weak supplemented, where each L_i is a local submodule of M.*

Proof (⇒) is clear.

(⇐) Suppose $n = 1$ and M/L is co-coatomically weak supplemented. Consider the following exact sequence:

$$0 \to L \to M \to M/L \to 0 \tag{5.15}$$

Case 1: If L is small in M, then M is co-coatomically weak supplemented since it is small cover of M/L (see Proposition 5.3).

Case 2: If L is not small M, then $M = L + K$ for some proper submodule K of M. Since L is local, and so hollow, therefore $L \cap K \ll L \leq M$. Hence K is a weak supplement of L in M. Since L and M/L are co-coatomically weak supplemented and L has a weak supplement, M is co-coatomically weak supplemented by Theorem 5.7.

Now suppose that the claim holds when $i < n$. Let $M/\left(\bigoplus_{i=1}^{n} L_i\right)$ be co-coatomically weak supplemented. We obtain the following exact sequence:

$$0 \to \left(\bigoplus_{i=1}^{n} L_i\right)/\left(\bigoplus_{i=1}^{n-1} L_i\right) \to M/\left(\bigoplus_{i=1}^{n-1} L_i\right) \to M/\left(\bigoplus_{i=1}^{n} L_i\right) \to 0 \tag{5.16}$$

Since

$$\left(\bigoplus_{i=1}^{n} L_i\right)/\left(\bigoplus_{i=1}^{n-1} L_i\right) \cong L_n \tag{5.17}$$

is a local submodule of $M/\left(\bigoplus_{i=1}^{n-1} L_i\right)$ and $M/\left(\bigoplus_{i=1}^{n} L_i\right)$ is co-coatomically weak supplemented, $M/\left(\bigoplus_{i=1}^{n-1} L_i\right)$ is co-coatomically weak supplemented. Hence M is co-coatomically weak supplemented by induction. □

Proposition 5.8 *Let R be a Dedekind domain and M a torsion module over R. M is co-coatomically supplemented if and only if* $M/\left(\bigoplus_{i=1}^{n} L_i\right)$ *is co-coatomically supplemented for some local submodules L_i of M.*

Proof Since over a Dedekind domain, co-coatomically supplemented torsion module is co-coatomically supplemented the proof is the same with the Theorem 5.2. □

To recall $\mathrm{cws}(M)$ for a module M, see section 2.15.

Lemma 5.3 *Over a left V-ring* , $\mathrm{Soc}(M) = \mathrm{cws}(M)$.

Proof ($\subseteq$): Let S be a simple submodule of an R-module M. Since R is a left V-ring, S is injective, and so it is a direct summand in M, i.e. $M = S \oplus K$ for some submodule K of M. Therefore

$$M/K = S \oplus K/K \cong S \tag{5.18}$$

and so K is maximal submodule in M. Since $M = S \oplus K$, S is weak a supplement of a maximal submodule K of M. In this case, $S \in \Gamma$. Thus $\mathrm{Soc}(M) \subseteq \mathrm{cws}(M)$.
($\supseteq$): Now let $T \in \Gamma$. Therefore T is a weak supplement of a maximal submodule K of M, i.e. $M = T + K$ and $T \cap K \ll M$. Since R is a V-ring, $\mathrm{Rad}(M) = 0$ by Theorem 3.2. Hence $T \cap K = 0$, and so $M = T \oplus K$. It follows that

$$M/K = T \oplus K/K \cong T. \tag{5.19}$$

Since M/K is simple, T is simple. Thus $\mathrm{cws}(M) \subseteq \mathrm{Soc}(M)$. □

Proposition 5.9 *Over a left V-ring, a left R-module M is co-coatomically weak supplemented if and only if M is semisimple.*

Proof ($\Leftarrow$) is clear.
($\Rightarrow$) Since M is co-coatomically weak supplemented, M is cofinitely weak supplemented. Then $M/cws(M)$ has no maximal submodule by Theorem 2.12. Therefore $M/\mathrm{Soc}(M)$ has no maximal submodule by Lemma 5.3. Since R is a left V-ring, $M/\mathrm{Soc}(M) = \mathrm{Rad}(M/\mathrm{Soc}(M)) = 0$ by Theorem 3.2. Hence M is semisimple. □

5.1. Co-coatomically Weak Supplemented Modules Over Discrete Valuation Rings

In this section we use equivalency of co-coatomically supplemented module and radical supplemented module over a DVR.

Theorem 5.3 *Over a DVR, a module M is co-coatomically weak supplemented if and only if co-coatomically supplemented.*

Proof ($\Leftarrow$) is clear.
($\Rightarrow$) Over DVR, $M/\operatorname{Rad}(M)$ is semisimple, i.e. coatomic. Since M is co-coatomically weak supplemented, $\operatorname{Rad}(M)$ has a weak supplement, say K. Therefore $M = \operatorname{Rad}(M) + K$. K is coatomic by Proposition 5.5. Over a DVR, a coatomic module is supplemented (see Lemma 2.11). Thus $\operatorname{Rad}(M)$ has a supplement by Proposition 2.6. Therefore M is radical supplemented, and so M is co-coatomically supplemented (see Corollary 3.5). □

Corollary 5.4 *Over a DVR, M is co-coatomically weak supplemented if and only if $M = T(M) \oplus X$ where the reduced part of $T(M)$ is bounded and $X/\operatorname{Rad}(X)$ is finitely generated.*

Proof Since co-coatomically supplemented and co-coatomically weak supplemented modules coincide over a DVR, it is clear by Corollary 3.6. □

5.2. Co-coatomically Weak Supplemented Modules Over Dedekind Domains

In Chapter 3 it is proved that over Dedekind domains, a torsion module M is both co-coatomically supplemented and co-coatomically weak supplemented.

Proposition 5.10 *Let R be a Dedekind domain and M an R-module. If $T(M)$ has a weak supplement in M, then M is co-coatomically weak supplemented if and only if $T(M)$ and $M/T(M)$ are co-coatomically weak supplemented.*

Proof ($\Leftarrow$) By Theorem 5.7.
($\Rightarrow$) Clearly $M/T(M)$ is co-coatomically weak supplemented. $T(M)$ is closed by Lemma 3.3, i.e. it is coclosed by Lemma 3.4. Since $T(M)$ has a weak supplement, it is a supplement by Lemma 2.12. Therefore $T(M)$ is co-coatomically weak supplemented by Proposition 5.4. □

CHAPTER 6

CO-COATOMICALLY AMPLY SUPPLEMENTED MODULES

Throughout this chapter R will be an arbitrary ring unless otherwise stated. Recall that a submodule U of an R-module M has ample supplements in M if, for every submodule V of M with $U + V = M$, there exists a supplement V' of U such that $V' \leq V$.

Definition 6.1 *Let R be a ring and M an R-module. M is called co-coatomically amply supplemented module if every co-coatomic submodule of M has an ample supplement.*

Proposition 6.1 *Let R be an arbitrary ring and M a co-caotomically amply supplemented R-module. Then:*

1. *Every supplement of a co-coatomic submodule is also co-coatomically amply supplemented.*
2. *Factor modules and direct summands of M are co-coatomically amply supplemented.*

Proof

1. Let U be a co-coatomic submodule of M and V a supplement of U. Then $M = U + V$. Let X be a co-coatomic submodule of V and $V = X + Y$ for some submodule Y of V. Then $M = U + V = U + X + Y$. Since $U + X$ is a co-coatomic submodule of M and M is co-coatomically amply supplemented module, there is an ample supplement Y' of $U + X$ such that $Y' < Y$. It follows that

$$X \cap Y' \leq (U + X) \cap Y' \ll Y' \text{ and } M = U + X + Y'. \tag{6.1}$$

   Since V is a supplement of U, $V = X + Y'$ and $Y' \leq Y$. Thus V is co-coatomically amply supplemented.

2. Let U/K be a co-coatomic submodule of M/K and $M/K = (U/K) + (N/K)$ for some submodule N of M including K. Then U is also a co-coatomic submodule

of M and $M = U + N$. Since M is co-coatomically amply supplemented module, U has a supplement N' such that $N' \leq N$, that is $M = U + N'$ and $U \cap N' \ll N'$. Then $M/K = (U/K) + (N' + K)/K$ and $(N' + K)/K$ is supplement of U/K. Also $(N' + K)/K \leq N/K$. Thus M/K is a co-coatomically amply supplemented module. Since a factor module of a co-coatomically amply supplemented module is co-coatomically amply supplemented, every direct summand of a co-coatomically amply supplemented module is also co-coatomically amply supplemented.

□

Proposition 6.2 *Let R be an arbitrary ring and M an R-module. Consider the following statements.*

1. *Every submodule U of M is of the form $U = X + Y$ with X co-coatomically supplemented and $Y \ll M$.*
2. *For every submodule U of M, there is a co-coatomically supplemented submodule X of U with $U/X \ll M/X$.*
3. *M is co-coatomically amply supplemented.*

Then $(1) \Rightarrow (2) \Rightarrow (3)$. *Furthermore,* $(3) \Rightarrow (1)$ *if M is coatomic.*

Proof $(1) \Rightarrow (2)$ If $U = X + Y$ with X is co-coatomically supplemented and $Y \ll M$, then $Y/X \cap Y \cong U/X \ll M/X$.

$(2) \Rightarrow (3)$ Let U be a co-coatomic submodule of M and $M = U + V$ for some submodule V of M. By (2), there exists a co-coatomically supplemented submodule X of V such that $V/X \ll M/X$. Then $M = U + X$. Since $U \cap X$ is a co-coatomic submodule of X, there is a supplement V' of $U \cap X$ in X. Therefore, $M = U + (U \cap X) + V' = U + V'$ and $U \cap V' = (U \cap X) \cap V' \ll V'$. Thus V' is supplement of U in M and $V' \leq V$.

$(3) \Rightarrow (1)$ Since M is coatomic, it is clear by ((Wisbauer, 1991), 41.9). □

A module M is called amply cofinitely supplemented if every cofinite submodule of M has an ample supplement in M (see (Alizade et al., 2001)). Clearly a co-coatomically amply supplemented module M is amply cofinitely supplemented since every cofinite submodule is co-coatomic.

A co-coatomically amply supplemented module is co-coatomically supplemented, but converse does not hold by the following example.

Example 6.1 *((Alizade et al., 2001), Corollary 4.9) Let R be a non-local commutative domain, Q the field of fractions of R and X a non-zero semisimple R-module. Then* *the R-module $Q \oplus X$ is co-coatomically supplemented as it is the direct sum of two co-coatomically supplemented modules, but it is not co-coatomically amply supplemented since it is not amply cofinitely supplemented (see the proof of ((Alizade et al., 2001), Corollary 4.9)).*

An R-module M is called totally co-coatomically supplemented module if every submodule of M is co-coatomically supplemented.

Proposition 6.3 *Every totally co-coatomically supplemented module is co-coatomically amply supplemented module.*

Proof Let M be a totally co-coatomically supplemented R-module. Then every submodule N of M is co-coatomically supplemented, $N = N + 0$, i.e. it is the sum of a co-coatomically supplemented submodule and a small submodule of M. Therefore M is co-coatomically amply supplemented module by Proposition 6.2. □

Every co-coatomically amply supplemented module is amply cofinitely supplemented, but an amply cofinitely supplemented module need not be co-coatomically amply supplemented by the following example.

Example 6.2 *Let p be a prime integer and consider the following ring which is semiperfect, but not perfect*

$$R = \mathbb{Z}_{(p)} = \left\{ \frac{a}{b} \mid a,\, b \in \mathbb{Z},\, b \neq 0,\, (b, p) = 1 \right\}. \tag{6.2}$$

Since R is semiperfect, R-module $R^{(\mathbb{N})}$ is an amply cofinitely supplemented module by Theorem 2.10. $\mathrm{Rad}({}_R R^{(\mathbb{N})})$ *is a co-coatomic submodule of the R-module $R^{(\mathbb{N})}$, but it does not have a supplement in $R^{(\mathbb{N})}$. Therefore R-module $R^{(\mathbb{N})}$ is not co-coatomically supplemented. Hence R-module $R^{(\mathbb{N})}$ is not co-coatomically amply supplemented.*

Let X be a submodule of M. The left annihilator of X in R is $\mathrm{ann}(X) = \{r \in R | rx = 0,\ x \in X\}$ and $\mathrm{ann}(X)$ is an ideal of R.

Lemma 6.1 *((Smith, 2000), Lemma 4.1) Let a module $M = M_1 \oplus \ldots \oplus M_n$ be a finite direct sum of submodules M_i $(1 \leq i \leq n)$, for some $n \geq 2$, such that $R = \mathrm{ann}(M_i) + \mathrm{ann}(M_j)$ for all $1 \leq i < j \leq n$. Then $N = (N \cap M_1) \oplus \ldots \oplus (N \cap M_n)$ for every submodule N of M.*

In Example 6.1, Q and X are co-coatomically amply supplemented, but $M = Q \oplus X$ is not co-coatomically amply supplemented. Therefore Example 6.1 shows that a finite direct sum co-coatomically amply supplemented modules need not be co-coatomically amply supplemented. However, under some conditions, a finite direct sum of co-coatomically amply supplemented modules is co-coatomically amply supplemented as it is shown in the following proposition.

Proposition 6.4 *Let a module $M = M_1 \oplus \ldots \oplus M_n$ be a finite direct sum of co-coatomically amply supplemented submodules M_i $(1 \leq i \leq n)$, for some positive integer $n \geq 2$, and $R = \operatorname{ann}(M_i) + \operatorname{ann}(M_j)$ for all $1 \leq i < j \leq n$. Then M is co-coatomically amply supplemented.*

Proof Suppose that M_i is a co-coatomically amply supplemented module for all $1 \leq i \leq n$. Let N be a co-coatomic submodule of M and $M = N + K$ for some submodule K of M. By Lemma 6.1, $N = (N \cap M_1) \oplus \ldots \oplus (N \cap M_n)$ and $K = (K \cap M_1) \oplus \ldots \oplus (K \cap M_n)$. Then $M_i = (N \cap M_i) + (K \cap M_i)$ for each $1 \leq i \leq n$. Since

$$\begin{aligned} M/N &= M_1 \oplus \ldots \oplus M_n/(N \cap M_1) \oplus \ldots \oplus (N \cap M_n) \\ &\cong M_1/(N \cap M_1) \oplus \ldots M_n/(N \cap M_n) \end{aligned} \tag{6.3}$$

is coatomic, each $M_i/(N \cap M_i)$ is coatomic. Since each M_i is co-coatomically amply supplemented, there exists a submodule L_i of $K \cap M_i$ such that $M_i = (N \cap M_i) + L_i$ and $N \cap L_i \ll L_i$. Let $L = L_1 \oplus \ldots \oplus L_n$. Then $L \leq K$, $M = N + L$ and $N \cap L = (N \cap L_1) \oplus \ldots \oplus (N \cap L_n) \ll L$ by Lemma 2.2. Therefore L is a supplement of N in M. Thus M is co-coatomically amply supplemented. □

Proposition 6.5 *Let M be a reduced module with $M/\operatorname{Rad}(M)$ a coatomic R-module. If M is co-coatomically amply supplemented, then M can be written as an irredundant sum*

$$M = \sum L_k + \operatorname{Rad}(M) \tag{6.4}$$

with L_k local modules where $k \in \Lambda$, Λ an index set.

Proof Since M is co-coatomically amply supplemented module and $M/\operatorname{Rad}(M)$ is coatomic, there is a supplement K of $\operatorname{Rad}(M)$, i.e. $M = \operatorname{Rad}(M) + K$. K is co-coatomically amply supplemented by Proposition 6.1, and so co-coatomically supplemented. Also, $\operatorname{Rad}(K) = K \cap \operatorname{Rad}(M) \ll K$ since K is a supplement (see Lemma 2.9). Furthermore, K is coatomic by Lemma 2.6. Since K is coatomic and co-coatomically supplemented, it is supplemented. Since K is supplemented and $\operatorname{Rad}(K) \ll K$, by Proposition 2.7, K is written as an irredundant sum of local modules, i.e. $K = \sum L_k$ where each $L_k(k \in \Lambda)$ is local. Hence $M = \sum L_k + \mathrm{Rad}(M)$ and the sum is irredundant. $\square$

Corollary 6.1 *Over a DVR, a reduced co-coatomically amply supplemented module M is written as an irredundant sum $M = \sum L_k + \mathrm{Rad}(M)$ with L_k local modules where $k \in \Lambda$.*

Corollary 6.2 *Let M be a reduced and co-coatomically amply supplemented R-module. If $M/\mathrm{Rad}(M)$ is finitely generated, then the sum is finite in Proposition 6.5.*

Corollary 6.3 *Let M be a reduced and coatomic. Then a co-cotomically amply supplemented module is written as an irredundant sum $M = \sum L_k$ with L_k local modules, where $k \in \Lambda$.*

Theorem 6.1 *Let R be a non-local Dedekind domain and M a reduced R-module. If M is co-coatomically amply supplemented, then $M/T(M)$ is divisible and $T_P(M)$ is bounded for each $P \in \mathcal{P}$.*

Conversely, if $M/T(M)$ is divisible and $T_P(M)$ is bounded for each maximal ideal P of R, then M is co-coatomically supplemented.

Proof ($\Rightarrow$) Let R be a non-local Dedekind domain and M a co-coatomically amply supplemented reduced R-module. By the proof of Theorem 3.5, $M/T(M)$ is divisible. Now suppose that $T_P(M)$ is not bounded for some $P \in \mathcal{P}$. If the basic submodule $B_p(M)$ is bounded, then by ((Kaplansky, 1952), Theorem 5), $T_P(M) = B_P(M) \oplus D$, where D is divisible. Therefore M is not reduced. Contradiction. Therefore $B_p(M)$ is not bounded. We will prove that $B_P(M)$ is co-coatomically supplemented. Let K be a co-coatomic submodule of $B_P(M)$, i.e. $B_P(M)/K$ is coatomic. Therefore $B_P(M)/K$ is bounded by Corollary 2.4. We have the following commutative diagram (pushout) with exact rows and columns

$$\begin{array}{cccccccccc} & & & K & & & & & & \\ & & & \downarrow{\scriptstyle i} & & & & & & \\ E: & 0 & \longrightarrow & B_P(M) & \xrightarrow{pure} & M & \longrightarrow & X & \longrightarrow & 0 \\ & & & \downarrow{\scriptstyle \sigma} & & \downarrow & & \| & & \\ E': & 0 & \longrightarrow & B_P(M)/K & \xrightarrow{pure} & M' & \longrightarrow & X & \longrightarrow & 0 \end{array} \tag{6.5}$$

Since E is pure E' is also pure (see (Fuchs, 1970), Lemma 26.1). Hence E' is splitting since $B_P(M)/K$ is bounded (see (Kaplansky, 1952), Theorem 5). By applying *Ext*, we obtain exact sequence

$$\to Ext_R(X, K) \overset{i_*}{\to} Ext_R(X, B_P(M)) \overset{\sigma_*}{\to} Ext_R(X, B_P(M)/K) \tag{6.6}$$

by Theorem 2.15. Since $Ext_R(X, B_P(M)/K) = 0$, $\sigma_*(E) = 0$. Therefore

$$E \in \operatorname{Ker} \sigma_* = \operatorname{Im} i_*. \tag{6.7}$$

Thus there is a short exact sequence

$$E'' : 0 \to K \to N \to X \to 0 \tag{6.8}$$

such that $i_*(E'') = E$. Therefore we obtain the following diagram:

$$\begin{array}{ccccccccc} & & 0 & & 0 & & & & \\ & & \downarrow & & \downarrow & & & & \\ 0 & \longrightarrow & K & \longrightarrow & N & \longrightarrow & X & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \| & & \\ 0 & \longrightarrow & B_P(M) & \longrightarrow & M & \longrightarrow & X & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & & & \\ 0 & \longrightarrow & B_P(M)/K & = & B_P(M)/K & & & & \\ & & \downarrow & & \downarrow & & & & \\ & & 0 & & 0 & & & & \end{array} \tag{6.9}$$

Without loss of generality, we can assume that K, $B_P(M)$ and N are submodules of M. In diagram (6.9) $B_P(M) \cap N = K$, $B_P(M) + N = M$ (see Theorem 2.13). Moreover M/N is coatomic. Since M is co-coatomically amply supplemented, there exists a submodule L of $B_P(M)$ such that $N + L = M$ and $N \cap L \ll L$. Therefore

$$\begin{aligned} B_P(M) &= B_P(M) \cap (N + L) \\ &= L + (B_P(M) \cap N) \\ &= L + K \end{aligned} \tag{6.10}$$

and $L \cap K \leq L \cap N \ll L$. Thus K has a supplement in $B_P(M)$, and so $B_P(M)$ is cocoatomically supplemented. Therefore $B_P(M)$ is bounded by Corollary 2.4. Contradiction. Thus $T_P(M)$ is bounded for each $P \in \mathcal{P}$.
Converse is clear by Theorem 3.5. □

CHAPTER 7

COATOMICALLY SUPPLEMENTED AND COATOMICALLY ⊕-SUPPLEMENTED MODULES

The results in this chapter are the generalizations of the results about finitely supplemented modules and finitely ⊕-supplemented modules to coatomically supplemented and coatomically ⊕-supplemented modules. Throughout this chapter R is an arbitrary ring unless otherwise stated.

Definition 7.1 *Let M be an R-module. If every coatomic submodule of M has a supplement in M, then M is called a coatomically supplemented module.*

Lemma 7.1 *Let M be a coatomically supplemented module. For a coatomic submodule N of M, M/N is also coatomically supplemented.*

Proof Let K/N be a coatomic submodule of M/N. Since N and K/N are coatomic, K is coatomic by Lemma 2.4. By hypothesis, K has a supplement in M, say L. It follows that $L + N/N$ is a supplement of K/N. Thus M/N is coatomically supplemented. □

Lemma 7.2 *Let M be an R-module, N a coatomic submodule and L a coatomically supplemented submodule of M. If $N + L$ has a supplement X in M such that $(N + X) \cap L$ is coatomic in L, then N has a supplement in M.*

Proof Let X be a supplement of $N+L$ in M. Then $X+N+L = M$ and $X \cap (N+L) \ll X$. Since $(N + X) \cap L$ is coatomic submodule of L and L is coatomically supplemented, $(N + X) \cap L$ has a supplement Y in L, i.e. $(N + X) \cap L + Y = L$ and $(N + X) \cap Y = (N + X) \cap Y \cap L \ll Y$. Then

$$\begin{aligned} M &= X + N + L \\ &= X + N + ((X + N) \cap K) + Y \qquad (7.1) \\ &= N + X + Y \end{aligned}$$

and

$$\begin{aligned} N \cap (X + Y) &\leq (X \cap (N + Y)) + (Y \cap (N + X)) \\ &\leq (X \cap (N + L)) + (Y \cap (N + X)) \\ &\ll X + Y. \end{aligned} \tag{7.2}$$

Thus $X + Y$ is a supplement of N in M. □

Remark 7.1 *Let M be an R-module and let N and K be coatomic submodules of M. By Corollary 2.2, $N \oplus K$ is coatomic. Consider the exact sequence*

$$0 \to N \cap K \to N \oplus K \to N + K \to 0. \tag{7.3}$$

By Lemma 2.4, $N + K$ is coatomic.

Theorem 7.1 *Let M be an R-module and $M = M_1+M_2$ such that M_1 and M_2 are coatomic and coatomically supplemented. Assume that intersection of two coatomic submodules of M is coatomic. Then M is coatomically supplemented.*

Proof Let U be a coatomic submodule of M. It is clear that 0 is a trivial supplement of $M = M_1+M_2+U$ in M. Since M_1+U and M_2 are coatomic, $M_2 \cap (M_1+U+0)$ is coatomic as an intersection of coatomic submodules by assumption. Since M_2 is coatomically supplemented, $M_2 \cap (M_1 + U + 0)$ has a supplement X in M_2. By Lemma 7.2, X is a supplement of $M_1 + U$ in M, i.e. $M = M_1 + U + X$ and $(M_1 + U) \cap X \ll X$.

$$\begin{aligned} M/(M_1 + U) &= (M_1 + U + X)/(M_1 + U) \\ &\cong X/((M_1 + U) \cap X) \end{aligned} \tag{7.4}$$

is coatomic, so X is coatomic by Lemma 2.5. Hence $M_1 \cap (X + U)$ is coatomic by assumption. Since M_1 is coatomically supplemented, $M_1 \cap (X + U)$ has a supplement Y in M_1. Thus $X + Y$ is a supplement of U in M by Lemma 7.2. □

Theorem 7.2 *Let M be an R-module and $M = M_1 \oplus M_2$ such that M_1, M_2 are coatomic and coatomically supplemented. Assume that M is quasi-projective. Then M is coatomically supplemented.*

Proof Let M be a quasi-projective module and U a coatomic submodule of M. $M = M_1 + M_2 + U$ has the trivial supplement 0 in M.

$$\begin{aligned}(M_1 + U)/(M_2 \cap (M_1 + U)) &\cong (M_1 + M_2 + U)/M_2 \\ &= M/M_2 \\ &\cong M_1\end{aligned} \tag{7.5}$$

is M-projective by ((Wisbauer, 1991), 18.1). Furthermore, it is $(M_1 + U)$-projective (see (Anderson and Fuller, 1992), 16.12). Therefore $M_2 \cap (M_1 + U)$ is a direct summand of $M_1 + U$. Since $M_1 + U$ is coatomic, $M_2 \cap (M_1 + U)$ is also coatomic. Since M_2 is coatomically supplemented, $M_2 \cap (M_1 + U)$ has a supplement X in M_2. By Lemma 7.2, X is a supplement of $M_1 + U$ in M, that is $M = M_1 + U + X$ and $(M_1 + U) \cap X \ll X$. Since

$$\begin{aligned}M/(M_1 + U) &= (M_1 + U + X)/(M_1 + U) \\ &\cong X/((M_1 + U) \cap X)\end{aligned} \tag{7.6}$$

is coatomic and $(M_1 + U) \cap X \ll X$, X is also coatomic by Lemma 2.5.

$$\begin{aligned}(X + U)/(M_1 \cap (X + U)) &\cong (M_1 + X + U)/M_1 \\ &= M/M_1 \\ &\cong M_2\end{aligned} \tag{7.7}$$

is M-projective by ((Wisbauer, 1991), 18.1). Furthermore, it is $(X + U)$-projective by ((Anderson and Fuller, 1992), 16.12). Therefore $M_1 \cap (X + U)$ is a direct summand of $X + U$, and so $M_1 \cap (X + U)$ is coatomic. Since M_1 is coatomically supplemented, $M_1 \cap (X + U)$ has a supplement Y in M_1. Thus $X + Y$ is a supplement of U in M. □

Definition 7.2 *Let M be an R-module. M is called coatomically H-supplemented module if for every coatomic submodule N of M, there exists a direct summand L of M such that $M = N + X$ holds if and only if $M = L + X$ for some submodule X of M.*

Definition 7.3 *Let M be an R-module. M is called coatomically $\oplus$-supplemented module if every coatomic submodule of M has a supplement that is a direct summand of M.*

Proposition 7.1 *Every coatomically H-supplemented module is coatomically $\oplus$-supplemented.*

Proof Let M be a coatomically H-supplemented module and N a coatomic submodule of M. By hypothesis, there exists a direct summand L of M such that $M = N + K$ if and only if $M = L + K$ and $M = L \oplus L'$ for some submodule K and a direct summand L'. By hypothesis, $M = N + L'$. Suppose $L' = N \cap L' + U$ for some submodule U of L'. Then

$$N + L' = N + N \cap L' + U = N + U = L + U = M = L \oplus L' \tag{7.8}$$

It follows that $M = L + U$ and $L' = L' \cap M = L' \cap (L + U) = U$. Therefore $N \cap L' \ll L'$. Thus M is coatomically $\oplus$-supplemented. □

Proposition 7.2 *Let M be a coatomically supplemented module. If every maximal submodule of M is a direct summand of M, then M is coatomically $\oplus$-supplemented.*

Proof Let N be a coatomic submodule of M. Since M is coatomically supplemented, N has a supplement in M, say K, i.e. $M = N + K$ and $N \cap K \ll K$. It follows that $M/K = (N+K)/K \cong N/(N \cap K)$ is coatomic. Therefore K is a co-coatomic submodule of M. By Theorem 3.1, K is a direct summand. Thus M is coatomically $\oplus$-supplemented. □

Lemma 7.3 *Let M be a coatomically supplemented module such that* $\operatorname{Rad}(M)$ *is coatomic. Then every coatomic submodule of* $M/\operatorname{Rad}(M)$ *is a direct summand.*

Proof Let $N/\operatorname{Rad}(M)$ be a coatomic submodule of $M/\operatorname{Rad}(M)$. Since $\operatorname{Rad}(M)$ and $N/\operatorname{Rad}(M)$ are coatomic, N is coatomic by Lemma 2.4. Since M is a coatomically supplemented module, N has a supplement, say K. Therefore $M = N + K$ and $N \cap K \ll K$. It follows that

$$\begin{aligned} M/\operatorname{Rad}(M) &= (N + K)/\operatorname{Rad}(M) \\ &= (N/\operatorname{Rad}(M)) + ((K + \operatorname{Rad}(M))/\operatorname{Rad}(M)) \end{aligned} \tag{7.9}$$

and

$$\begin{aligned} (N/\operatorname{Rad}(M)) \cap ((K + \operatorname{Rad}(M))/\operatorname{Rad}(M)) &= ((N \cap K) + \operatorname{Rad}(M))/\operatorname{Rad}(M) \\ &= 0. \end{aligned} \tag{7.10}$$

Thus $N/\operatorname{Rad}(M)$ is a direct summand of $M/\operatorname{Rad}(M)$. □

The following corollary is a direct result of Lemma 7.3.

Corollary 7.1 *If M is coatomically supplemented module such that* $\operatorname{Rad}(M)$ *is coatomic, then $M/\operatorname{Rad}(M)$ is coatomically $\oplus$-supplemented.*

Proposition 7.3 *Let M be a module with* $\operatorname{Rad}(M) \ll M$. *If every coatomic submodule of $M/\operatorname{Rad}(M)$ is a direct summand and every coatomic direct summand of $M/\operatorname{Rad}(M)$ lifts to a direct summand of M, then M is coatomically $\oplus$-supplemented.*

Proof Let N be a coatomic submodule of M. Then $(N+\operatorname{Rad}(M))/\operatorname{Rad}(M)$ is coatomic. By hypothesis, $(N+\operatorname{Rad}(M))/\operatorname{Rad}(M)$ is a direct summand of $M/\operatorname{Rad}(M)$ with

$$M/\operatorname{Rad}(M) = ((N+\operatorname{Rad}(M))/\operatorname{Rad}(M)) \oplus (K/\operatorname{Rad}(M)) \tag{7.11}$$

for some submodule $K/\operatorname{Rad}(M)$ of $M/\operatorname{Rad}(M)$ and by hypothesis, there exists a direct summand L of M such that

$$(L+\operatorname{Rad}(M))/\operatorname{Rad}(M) = (N+\operatorname{Rad}(M))/\operatorname{Rad}(M). \tag{7.12}$$

It follows that $M = N + \operatorname{Rad}(M) + K$. Since $\operatorname{Rad}(M) \ll M$, $M = N + K$. By (7.12), $M = L+K+\operatorname{Rad}(M)$. Therefore $M = L+K$ since $\operatorname{Rad}(M) \ll M$. Hence M is coatomically H-supplemented. By Proposition 7.1, M is coatomically $\oplus$-supplemented. □

Proposition 7.4 *Let M be a coatomically $\oplus$-supplemented with (D3). Then every direct summand of M is coatomically $\oplus$-supplemented.*

Proof Let N be a direct summand of M and K a coatomic submodule of N. By hypothesis, there exists a direct summand L of M such that $M = K + L$ and $K \cap L \ll L$. It follows that $N = N \cap (K + L) = K + (N \cap L)$ and $M = N + L$ Since M has property (D3), $N \cap L$ is also a direct summand of M. Since N is a direct summand of M, $N \cap L$ is a direct summand of N. Since $K \cap (L \cap N) = K \cap L \leq N \cap L$, $K \cap (L \cap N) = K \cap L$ is small in $N \cap L$ by ((Wisbauer, 1991), 19.3). Thus N is coatomically $\oplus$-supplemented. □

Corollary 7.2 *Let M be a self-projective module. Then M is coatomically $\oplus$-supplemented if and only if every direct summand of M is coatomically $\oplus$-supplemented.*

Proof (⇐) Clear.

(⇒) Let ${}_RM$ be self-projective. Then by ((Mohamed and Müller, 1990), Lemma 4.6 and Proposition 4.38), M has property (D3). Thus M is coatomically ⊕-supplemented by Proposition 7.4. □

Proposition 7.5 *Let M be an R-module and $M = M_1 \oplus M_2$ such that M_1 and M_2 are coatomic and coatomically ⊕-supplemented. Assume that the intersection of two coatomic submodules of M is coatomic. Then M is coatomically ⊕-supplemented.*

Proof Let U be a coatomic submodule of M. It is clear that 0 is a trivial supplement of $M = M_1 + M_2 + U$ in M. Since $M_1 + U$ and M_2 are coatomic, $M_2 \cap (M_1 + U + 0)$ is coatomic as an intersection of coatomic submodules by assumption. Since M_2 is coatomically ⊕-supplemented, $M_2 \cap (M_1 + U + 0)$ has a supplement X in M_2 such that X is a direct summand of M_2. By Lemma 7.2, X is a supplement of $M_1 + U$ in M, i.e. $M = M_1 + U + X$ and $(M_1 + U) \cap X \ll X$.

$$\begin{aligned} M/(M_1 + U) &= (M_1 + U + X)/(M_1 + U) \\ &\cong X/((M_1 + U) \cap X) \end{aligned} \tag{7.13}$$

is coatomic, so X is coatomic by Lemma 2.5. Hence $M_1 \cap (X + U)$ is coatomic by assumption. Since M_1 is coatomically ⊕-supplemented, $M_1 \cap (X + U)$ has a supplement Y in M_1 such that Y is a direct summand of M_1. Thus $X + Y$ is a supplement of U in M by Lemma 7.2. Since M_1 and M_2 are direct summands of M, it follows that $X + Y = X \oplus Y$ is a direct summand of M. □

Proposition 7.6 *Let M be an R-module and $M = M_1 \oplus M_2$ such that M_1, M_2 are coatomic and coatomically ⊕-supplemented. Assume that M is quasi-projective. Then M is coatomically ⊕-supplemented.*

Proof Let M be a quasi-projective module and U be a coatomic submodule of M. $M = M_1 + M_2 + U$ has the trivial supplement 0 in M.

$$\begin{aligned} (M_1 + U)/(M_2 \cap (M_1 + U)) &\cong (M_1 + M_2 + U)/M_2 \\ &= M/M_2 \\ &\cong M_1 \end{aligned} \tag{7.14}$$

is M-projective by ((Wisbauer, 1991), 18.1). Furthermore it is $(M_1 + U)$-projective (see (Anderson and Fuller, 1992), 16.12). Therefore $M_2 \cap (M_1 + U)$ is a direct summand of $M_1 + U$. Since $M_1 + U$ is coatomic, $M_2 \cap (M_1 + U)$ is also coatomic. Since M_2 is coatomically $\oplus$-supplemented, $M_2 \cap (M_1 + U)$ has a supplement X in M_2 such that X is a direct summand of M_2. By Lemma 7.2, X is a supplement of $M_1 + U$ in M that is $M = M_1 + U + X$ and $(M_1 + U) \cap X \ll X$. Since

$$\begin{aligned} M/(M_1 + U) &= (M_1 + U + X)/(M_1 + U) \\ &\cong X/((M_1 + U) \cap X) \end{aligned} \tag{7.15}$$

is coatomic and $(M_1 + U) \cap X \ll X$, X is also coatomic by Lemma 2.5.

$$\begin{aligned} (X + U)/(M_1 \cap (X + U)) &\cong (M_1 + X + U)/M_1 \\ &= M/M_1 \\ &\cong M_2 \end{aligned} \tag{7.16}$$

is M-projective by ((Wisbauer, 1991), 18.1). Furthermore it is $(X + U)$-projective by ((Anderson and Fuller, 1992), 16.12). Therefore $M_1 \cap (X + U)$ is a direct summand of $X + U$ and so $M_1 \cap (X + U)$ is coatomic. Since M_1 is coatomically $\oplus$-supplemented, $M_1 \cap (X + U)$ has a supplement Y in M_1 such that Y is a direct summand of M_1. Thus $X + Y$ is a supplement of U in M. Since M_1 and M_2 are direct summands of M, $X + Y = X \oplus Y$ is a direct summand of M. □

Definition 7.4 *Let M be an R-module. M is called a coatomically semiperfect module if for every coatomic submodule N of M, M/N has a projective cover.*

Proposition 7.7 *Let M be a projective module. Then M is coatomically semiperfect module if and only if M is coatomically ⊕-supplemented.*

Proof The proof is similar to that of Proposition 4.8. □

Proposition 7.8 *Let M be a projective module. Then the following are equivalent:*

1. *M is coatomically semiperfect.*
2. *M is coatomically ⊕-supplemented.*
3. *Each coatomic submodule of M lies above a direct summand of M.*

Proof (1) $\Leftrightarrow$ (2) By Proposition 7.7.

(2) $\Rightarrow$ (3) By the proof of Proposition 4.9.

(3) $\Rightarrow$ (2) Clear.

□

Proposition 7.9 *Every factor module of a coatomically semiperfect module by a coatomic submodule is coatomically semiperfect.*

Proof Let M be a coatomically semiperfect module and L a coatomic submodule of M. Assume that N/L is a coatomic submodule of M/L. Therefore $(M/L)/(N/L) \cong M/N$. Since L and N/L are coatomic, N is coatomic. Since M is coatomically semiperfect, M/N has a projective cover. Thus M/L is a coatomically semiperfect module. □

Corollary 7.3 *For a projective module M, if M is coatomically $\oplus$-supplemented then M/L is coatomically $\oplus$-supplemented, where L is coatomic submodule of M.*

Proposition 7.10 *Every small cover of a coatomically semiperfect module is coatomically semiperfect.*

Proof The proof is similar to the proof of Proposition 4.11. □

Corollary 7.4 *If $K \ll M$ and M/K is coatomically semiperfect, then M is coatomically semiperfect.*

CHAPTER 8

CONCLUSIONS

In this thesis we have defined co-coatomic submodules and co-coatomically supplemented, $\oplus$-co-coatomically supplemented, co-coatomically weak supplemented, co-coatomically amply supplemented, coatomically supplemented and $\oplus$-coatomically supplemented modules, and obtained some results about these modules.

Co-coatomically supplemented modules have a place between supplemented modules and cofinitely supplemented modules, i.e. a supplemented module is co-coatomically supplemented and a co-coatomically supplemented module is cofinitely supplemented. The basic properties of co-coatomically supplemented modules are similar to the properties of supplemented modules, e.g. a finite sum of co-coatomically supplemented modules is co-coatomically supplemented like supplemented modules. Every left R-module is co-coatomically supplemented if and only if R is left perfect if and only if every left R-module is supplemented. We have given the following characterizations of co-coatomically supplemented modules: Over a V-ring, a module M is co-coatomically supplemented if and only if M is semisimple. Over a DVR, a module M is co-coatomically supplemented if and only the basic submodule of M is coatomic. Over a non-local Dedekind domain, if $T(M)$ has a weak supplement, then M is co-coatomically supplemented if and only if $M/T(M)$ is divisible and $T_P(M)$ is bounded for each maximal ideal P. Although we have given some characterizations of co-coatomically supplemented modules over a V-ring, DVR and non-local Dedekind domain, we could not give a characterization of co-coatomically supplemented modules by its submodules over arbitrary rings; it is still a problem.

Results about $\oplus$-co-coatomically supplemented modules we have given are the generalizations of $\oplus$-cofinitely supplemented modules.

For co-coatomically weak supplemented modules, we have obtained: Every left R-module is co-coatomically weak supplemented if and only if R is semilocal and $\mathrm{Rad}({}_RR^{(\mathbb{N})})$ has a weak supplement in ${}_RR^{(\mathbb{N})}$ if and only if every left R-module is weakly supplemented if and only if R is left perfect. Over a DVR, co-coatomically weak supplemented modules and co-coatomically supplemented modules coincide. Over a Dedekind domain, if $T(M)$ has a weak supplement in M, then M is co-coatomically weak supplemented if and only if $T(M)$ and $M/T(M)$ are co-coatomically weak supplemented.

We have examined properties of co-coatomically amply supplemented modules, and we have seen that in contrast to co-coatomically supplemented, ⊕-co-coatomically supplemented and co-coatomically weak supplemented modules, even a finite sum of co-coatomically amply supplemented modules need not be co-coatomically amply supplemented. A reduced co-coatomically amply supplemented module with a coatomic factor module $M/\operatorname{Rad}(M)$ can be written as an irredundant sum of local modules and $\operatorname{Rad}(M)$.

Results about coatomically supplemented and coatomically ⊕-supplemented modules are generalizations of finitely supplemented and finitely ⊕-supplemented modules.

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