SOLUTIONS OF INITIAL AND BOUNDARY VALUE PROBLEMS FOR INHOMOGENEOUS BURGERS EQUATIONS WITH TIME-VARIABLE COEFFICIENTS

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ABSTRACT

SOLUTIONS OF INITIAL AND BOUNDARY VALUE PROBLEMS FOR INHOMOGENEOUS BURGERS EQUATIONS WITH TIME-VARIABLE COEFFICIENTS

In this thesis, we have investigated initial-boundary value problems on semi-infinite line for inhomogeneous Burgers equation with time-variable coefficients. We have formulated the solutions for the cases with Dirichlet and Neumann boundary conditions. We showed that the Dirichlet problem for the variable parametric Burgers equation is solvable in terms of a linear ordinary differential equation and a linear second kind singular Volterra integral equation. Then, for particular models with special initial and Dirichlet boundary conditions we found a class of exact solutions. Next, we considered the Neumann problem and showed that it reduces to a second order linear ordinary differential equation and the standard heat equation with initial and nonlinear boundary conditions. Finally, we formulated the Cauchy problem for the variable parametric Burgers equation on the non-characteristic line, and obtained its solution in terms of a linear ODE and the series solution of the corresponding Cauchy problem for the heat equation. We gave examples to illustrate how some well known solutions of the Burgers equation can be recovered by solving a corresponding Cauchy problem.

ÖZET

KATSAYILARI ZAMANA BAĞLI HOMOJEN OLMAYAN BURGERS DENKLEMLERİ İÇİN BAŞLANGIÇ VE SINIR DEĞER PROBLEMLERİNİN ÇÖZÜMLERİ

Bu tezde zamana bağlı değişken katsayılı, homojen olmayan Burger denklemi için yarı sonsuz aralıkta başlangıç-sınır değer problemlerini araştırdık. Dirichlet ve Neumann sınır koşulları durumlarında çözümler için formülasyonlar elde ettik. Zamana bağlı değişken katsayılı Burger denkleminin bir lineer adi diferansiyel denklem ve bir lineer ikinci çeşit tekil Volterra integral denklemi cinsinden çözülebilir olduğunu gösterdik. Ardından, özel başlangıç ve Dirichlet sınır değer koşullu özel modeller için kesin çözüm sınıfları bulduk. Neumann problemini göz önüne aldık ve bu problemin ikinci mertebeden lineer adi diferansiyel denklem ile başlangıç ve nonlineer sınır koşullarına sahip standart ısı denklemine indirgendiğini gösterdik. Son olarak karakteristik olmayan doğru üzerinde değişken katsayılı Burger denklemi için Cauchy problemini formüle ettik ve bu problemin çözümünü lineer adi diferansiyel denklem ile ısı denklemi için Cauchy problemine karşılık gelen seri çözümü türünden elde ettik. Burger denkleminin bazı iyi bilinen çözümlerinin, ilgili Caucy problemini çözerek nasıl elde edilebileceğini göstermek için örnekler verdik.

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CHAPTER 1

INTRODUCTION

Burgers equation is a nonlinear model which appears in the study of many physical phenomena such as diffusion, acoustics, fluid dynamics, formation and development of shocks. Basically, it describes the balance which occurs between the nonlinear convection and the linear diffusion processes. Mathematically, Burgers equation is one of the simplest nonlinear models, since by the Cole-Hopf transformation it can be directly linearized to a heat equation. Then, solutions and many properties of Burgers equation can be investigated using the corresponding linear model. Precisely, the initial value problem for Burgers equation is defined as

$$\begin{cases} V_\tau + VV_\eta = \nu V_{\eta\eta}, & -\infty < \eta < \infty, \quad \tau > 0, \\ V(\eta, 0) = F(\eta), & -\infty < \eta < \infty, \end{cases} \tag{1.1}$$

where the subscripts denote partial derivatives, V mostly represents a velocity field, τ is the time variable, η is the space variable, $\nu > 0$, and $F(\eta)$ is a given initial data. Then, by the Cole-Hopf transformation

$$V(\eta, \tau) = -2\nu \frac{\varphi_\eta(\eta, \tau)}{\varphi(\eta, \tau)}, \tag{1.2}$$

this problem reduces to the Heat IVP. Indeed, letting $V = \Phi_\eta$ in Burgers equation (1.1) and then integrating with respect to η, we get the potential Burgers equation $\Phi_\tau + (1/2)((\Phi_\eta)^2) = \nu\Phi_{\eta\eta}$. Then, using $\Phi = -2\nu \ln \varphi$ gives the heat equation $\varphi_\tau = \nu\varphi_{\eta\eta}$, up to an additional term $a(\tau)\varphi$ which can be neglected. The initial condition (IC) for Burgers equation is also easily transformed to IC for the Heat equation by solving $V(\eta, 0) = -2\nu\varphi_\eta(\eta, 0)/\varphi(\eta, 0)$ for $\varphi(\eta, 0)$. As a result, we have the IVP for the heat equation

$$\begin{cases} \varphi_\tau = \nu\varphi_{\eta\eta} & -\infty < \eta < \infty, \quad \tau > 0, \\ \varphi(\eta, 0) = f(\eta), \quad \eta > 0, \end{cases} \tag{1.3}$$

where $f(\eta) = \exp(-\frac{1}{2\nu}\int^{\eta} V(\eta', 0)d\eta')$. Using the Fourier transform $\tilde{\varphi}(s, \tau) = \int_{-\infty}^{\infty} \varphi(\eta, \tau)e^{-is\eta}d\eta$, solution of heat problem (1.3) is obtained in the form

$$\varphi(\eta, \tau) = \int_{-\infty}^{\infty} K(\eta - \xi, \tau)f(\xi)d\xi = K(\eta, \tau) * f(\eta), \tag{1.4}$$

where $K(\eta, \tau) = \frac{e^{-\frac{\eta^2}{4\nu\tau}}}{\sqrt{4\nu\pi\tau}}$ is the Heat Kernel and it is the fundamental solution corresponding to initial condition $f(\eta) = \delta(\eta)$, where $\delta(\eta)$ is the Dirac-delta distribution, briefly defined by $\int_{-\infty}^{\infty} \delta(\eta - \xi)f(\xi)d\xi = f(\eta)$. Then, IVP for the Burgers equation has general solution

$$V(\eta, \tau) = -2\nu \frac{\int_{-\infty}^{\infty} K_\eta(\eta - \xi, \tau)e^{-\frac{1}{2\nu}\int^{\xi} V(\eta',0)d\eta'}d\xi}{\int_{-\infty}^{\infty} K(\eta - \xi, \tau)e^{-\frac{1}{2\nu}\int^{\xi} V(\eta',0)d\eta'}d\xi}.$$

Clearly, depending on the initial data $F(\eta)$, solution of the IVP for Burgers equation can not be always obtained explicitly. However, knowing explicit solutions are always of considerable interest. As known the IVP for Burgers equation (1.1) have many physically interesting exact solutions in explicit form, such as traveling shock and multi-shock waves, diffusive waves (triangular and N-waves) and rational type solutions, see [Whitham, Debnath, Büyükaşık].

In this thesis, we consider initial-boundary value problems for the standard Burgers equation and an inhomogeneous Burgers equation with variable coefficients on the semi-infinite line. As we have seen, the initial condition for the Burgers equation can easily be transformed to IC for the Heat equation. However, the same is not always true for the boundary conditions. For example, the Dirichlet boundary condition at $\eta = 0$ for the Burgers equation on the semi-infinite line $0 < \eta < \infty$ transforms to the Robin boundary condition at $\eta = 0$ for the heat equation on $0 < \eta < \infty$. Similarly, the Neumann boundary condition for the Burgers equation transforms to nonlinear boundary condition for the heat equation. Therefore, to investigate the initial-boundary value problems for the Burgers models, the thesis is organized as follows.

In Chapter 2 we review the well known solutions of IBVP's for the standard Heat equation on the semi-infinite line. First we consider the IBVP with Dirichlet boundary condition and obtain solution by the Fourier Sine transform. We also demonstrate the use of the reflection principle, by extending the initial condition as an odd function, when boundary condition is homogeneous. Then, we consider the IBVP with Neumann boundary condition and obtain its solution by the Fourier Cosine transform. The reflection principle is used to the initial condition by extending it as an even function, when Neumann boundary condition is homogeneous. To solve the IBVP with the Robin boundary conditions, we use alternative approaches, such as assuming that the Dirichlet condition is known, or assuming that the Neumann BC is known. In both cases, the problem reduces to solving a linear integral equation of Volterra type with weakly singular kernel. After this, we briefly discuss the IBVP on semi-infinite line for a heat equation with time-variable coefficients.

In Chapter 3, we study the IBVP's for standard Burgers equation on semi-infinite line. First, we consider the IBVP with the Dirichlet BC and give two ways for solving it. One way is by using the Cole-Hopf transform and the other way is by using generalized Cole-Hopf transform **[4]**. Second, we consider the IBVP with the Neumann BC and again we show two ways for finding the solution **[?]**. At the end, we compare the different approaches. Finally, we investigate the IBVP's with special nonlinear boundary conditions **[1]**.

In Chapter 4, we study the IBVP's for inhomogeneous Burgers equation with variable coefficients on semi-infinite line. In the first section, we consider the IBVP with Dirichlet boundary condition and obtain general solution leads to the corresponding standard models discussed in Chapter 2 and Chapter 3. Some exactly solvable models **[3]** are discussed in details to obtain explicit results for the Dirichlet problem for inhomogeneous BE with variable coefficients. In the second section, we investigate the IBVP with Neumann boundary condition. The general solution is obtained by transforming the model firstly to the IBVP with Neumann BC for the standard Burgers equation and secondly to the IBVP with the nonlinear boundary condition for the standard Heat equation. At the end, we show that solving IBVP's for inhomogeneous Burgers equation with variable coefficients corresponds to solving either Volterra type integral equation or nonlinear integro-differential equation.

In Chapter 5, we review the solution of Cauchy problem for the Heat and Burgers equation, **[7]**, **[8]**. Then we show how some special solutions of Burgers equation can be obtained as solutions of a Cauchy problem. Finally, we investigate the Cauchy problem for the inhomogeneous Burgers equation with time-variable coefficients.

CHAPTER 2

THE HEAT PROBLEMS ON SEMI-INFINITE LINE

In this chapter, we review solution of the Heat equation on semi-infinite line with initial condition at time $\tau = 0$ and the Dirichlet and Neumann boundary conditions at $\eta = 0$. The similarity solutions of the Heat equation are discussed in the third section. In the other two sections, we write solution of IBVP's with the Robin boundary condition and special boundary condition for the Heat equation. Then, we consider the Dirichlet and Neumann problems for the variable parametric parabolic equation on semi-infinite line and we obtain general solutions of the IBVP's. Finally, we investigate the IBVP with the Robin boundary condition for variable parametric parabolic equation on semi-infinite line.

2.1. The Dirichlet Problem on Semi-infinite Line

The Heat equation on semi-infinite line with initial condition at time $\tau = 0$ and the Dirichlet boundary condition at $\eta = 0$ is given as follows

$$\begin{cases} \varphi_\tau = \frac{1}{2}\varphi_{\eta\eta}, & 0 < \eta < \infty, \quad \tau > 0, \\ \varphi(\eta, 0) = A(\eta), & 0 < \eta < \infty, \\ \varphi(0, \tau) = H(\tau), & \tau > 0, \end{cases} \tag{2.1}$$

where $A(\eta)$ and $H(\tau)$ are given functions of η and τ respectively and one assumes $\varphi(\eta, \tau) \to 0$ and $\varphi_\eta(\eta, \tau) \to 0$ as $\eta \to \infty$. Also, we assume that $A(\eta)$ has sufficient smoothness and decays as $\eta \to \infty$, and $H(\tau)$ is sufficiently smooth. To solve the IBVP (2.1), we use linearity of the Heat equation. So that the problem (2.1) can be replaced by two problems **[5]** as follows, The first one is IBVP with homogeneous Dirichlet BC,

$$\begin{cases} \varphi_\tau = \frac{1}{2}\varphi_{\eta\eta}, \quad 0 < \eta < \infty, \quad \tau > 0, \\ \varphi(\eta, 0) = A(\eta), \quad 0 < \eta < \infty, \\ \varphi(0, \tau) = 0, \quad \tau > 0. \end{cases} \tag{2.2}$$

The second IBVP is with homogeneous initial condition and inhomogeneous Dirichlet boundary condition,

$$\begin{cases} \varphi_\tau = \frac{1}{2}\varphi_{\eta\eta}, \quad 0 < \eta < \infty, \quad \tau > 0, \\ \varphi(\eta, 0) = 0, \quad 0 < \eta < \infty, \\ \varphi(0, \tau) = H(\tau), \quad \tau > 0. \end{cases} \tag{2.3}$$

For solution of the IBVP (2.2), firstly we apply the Fourier sine transform, $F_s[\varphi] = \frac{2}{\pi}\int_0^\infty \varphi \sin(\eta y) d\eta$, and using $\varphi(\eta, \tau) \to 0$ and $\varphi_\eta(\eta, \tau) \to 0$ as $\eta \to \infty$, we get

$$\begin{aligned} F_s[\varphi_\tau] &= \frac{2}{\pi}\int_0^\infty \frac{\partial \varphi}{\partial \tau} \sin(\eta y) d\eta = \frac{\partial}{\partial \tau}\hat{\varphi}(y, \tau), \\ F_s[\varphi_{\eta\eta}] &= \frac{2}{\pi}\int_0^\infty \frac{\partial^2 \varphi}{\partial \eta^2} \sin(\eta y) d\eta. \end{aligned}$$

Applying integration by parts to the last integral i.e. $\frac{\partial^2\varphi}{\partial\eta^2} = dv \Rightarrow \frac{\partial\varphi}{\partial\eta} = v$ and $\sin(\eta y) = u \Rightarrow y\cos(\eta y)d\eta = du$, we have

$$F_s[\varphi_{\eta\eta}] = \frac{2}{\pi}\Big(\underbrace{\frac{\partial \varphi}{\partial \eta}\sin(\eta y)|_0^\infty}_{=0} - y\int_0^\infty \frac{\partial \varphi}{\partial \eta}\cos(\eta y) d\eta\Big). \tag{2.4}$$

Then, again integrating by parts in the second integral in (2.4), i.e. $\frac{\partial\varphi}{\partial\eta} = dv \Rightarrow \varphi = v$ and $\cos(\eta y) = u \Rightarrow -y\sin(\eta y)d\eta = du$

$$F_s[\varphi_{\eta\eta}] = -\frac{2}{\pi}y\Big(\underbrace{\varphi(\eta, \tau)\cos(\eta y)|_0^\infty}_{=0} + y\int_0^\infty \varphi(\eta, \tau)\sin(\eta y) d\eta\Big),$$

thus we get

$$F_s[\varphi_{\eta\eta}] \quad = \quad -y^2\hat{\varphi}. \tag{2.5}$$

For the Fourier sine transform of the initial condition we have

$$F_s[\varphi(\eta,0)] = \frac{2}{\pi}\int_0^\infty A(\eta)\sin(\eta y)d\eta = \hat{\varphi}(y,0) = \hat{A}(y).$$

Then we obtain the ordinary differential equation with the initial condition as follows

$$\begin{cases} \frac{\partial\hat{\varphi}}{\partial\tau} = -\frac{1}{2}y^2\hat{\varphi}, \\ \hat{\varphi}(y,0) = \hat{A}(y). \end{cases} \tag{2.6}$$

The solution to IVP (2.6) is given as

$$\hat{\varphi}(y,\tau) = \hat{A}(y)e^{-\frac{y^2}{2}\tau} = \frac{2}{\pi}\Big(\int_0^\infty A(\eta)\sin(\eta y)d\eta\Big)e^{-\frac{y^2}{2}\tau}. \tag{2.7}$$

Applying the inverse sine transform

$$\begin{aligned} \varphi(\eta,\tau) &= \int_0^\infty \hat{\varphi}(y,\tau)\sin(y\eta)dy, && (2.8)\\ &= \frac{2}{\pi}\int_0^\infty \Big[\int_0^\infty e^{-\frac{y^2\tau}{2}}A(\xi)\sin(y\xi)d\xi\Big]\sin(\eta y)dy, && (2.9)\\ &= \frac{2}{\pi}\int_0^\infty A(\xi)\Big[\int_0^\infty e^{-\frac{y^2}{2}\tau}\sin(\xi y)\sin(\eta y)dy\Big]d\xi, && (2.10) \end{aligned}$$

and using relation $\sin(y\xi)\sin(\eta y) = [\cos(\eta-\xi)y - \cos(\eta+\xi)y]/2$, we get

$$\varphi(\eta,\tau) \quad = \quad \frac{1}{\pi}\int_0^\infty A(\xi)\Big[\underbrace{\int_0^\infty e^{-\frac{y^2}{2}\tau}\cos(\eta-\xi)ydy}_{I} - \underbrace{\int_0^\infty e^{-\frac{y^2}{2}\tau}\cos(\eta+\xi)ydy}_{II}\Big]. \tag{2.11}$$

For the integral (I) in above, using relation $\cos(\eta - \xi)y = (e^{i(\eta-\xi)y} + e^{-i(\eta-\xi)y})/2$ and by completing exponential functions to the squares, we obtain

$$I = \sqrt{\frac{\pi}{2\tau}} e^{-\frac{(\eta-\xi)^2}{2\tau}}. \tag{2.12}$$

For the second integral (II) in (2.11), using relation $\cos(\eta+\xi)y = (e^{i(\eta+\xi)y} + e^{-i(\eta+\xi)y})/2$ and by completing exponential functions to the squares, we get

$$II = \sqrt{\frac{\pi}{2\tau}} e^{-\frac{(\eta+\xi)^2}{2\tau}}. \tag{2.13}$$

Substituting (I) and (II) into (2.11), we obtain the solution of IBVP (2.2) in the following form

$$\varphi(\eta, \tau) = \frac{1}{\sqrt{2\pi\tau}} \int_0^\infty \left(e^{-\frac{(\eta-\xi)^2}{2\tau}} - e^{-\frac{(\eta+\xi)^2}{2\tau}}\right) A(\xi) d\xi. \tag{2.14}$$

Note that, when the Dirichlet BC is homogeneous, $H(\tau) = 0$, as an alternative approach we can use the reflection principle, by extending initial condition $A(\eta)$ as an odd function,

$$A_0(\eta) = \begin{cases} A(\eta), & \eta > 0, \\ -A(-\eta), & \eta < 0, \\ 0, & \eta = 0. \end{cases} \tag{2.15}$$

Here, our aim is reduce problem (2.2) to the IVP on the whole line, for which the solution is known. This is achieved by extending the initial data $A(\eta)$ to the whole line, so that the boundary condition (2.2) is automaticallly satisfied. It is the following IVP on the whole line

$$\begin{cases} \varphi_\tau = \frac{1}{2}\varphi_{\eta\eta}, & -\infty < \eta < \infty, \quad \tau > 0, \\ \varphi(\eta, 0) = A_0(\eta), & -\infty < \eta < \infty. \end{cases} \tag{2.16}$$

Then, it satisfies the BC $\varphi(0,\tau) = 0$, since if the initial condition is odd, the solution $\varphi(\eta,\tau)$ is also odd w.r.t η. It's well known that the problem (2.16) has solution

$$\varphi(\eta,\tau) = \int_{-\infty}^{\infty} K(\eta-\xi,\tau)A_0(\xi)d\xi. \tag{2.17}$$

Substituting $A_0(\eta)$ from (2.15) into (2.17), we get

$$\begin{aligned}
\varphi(\eta,\tau) &= \int_{-\infty}^{0} K(\eta-\xi,\tau)A_0(\xi)d\xi - \int_{0}^{\infty} K(\eta-\xi,\tau)A_0(\xi)d\xi, \\
&= \int_{-\infty}^{0} K(\eta-\xi,\tau)A(\xi)d\xi - \int_{0}^{\infty} K(\eta-\xi,\tau)A(-\xi)d\xi,
\end{aligned}$$

and by the change of variable $\xi \to -\xi$ in the second integral, finally we have

$$\varphi(\eta,\tau) = \int_{0}^{\infty} K(\eta-\xi,\tau)A(\xi)d\xi - \int_{0}^{\infty} K(\eta+\xi,\tau)A(\xi)d\xi.$$

By using definition of the Heat kernel, the solution of the IBVP (2.2) in explicit form is

$$\varphi(\eta,\tau) = \int_{0}^{\infty} \Big(\frac{e^{-\frac{(\eta-\xi)^2}{2\tau}} - e^{-\frac{(\eta+\xi)^2}{2\tau}}}{\sqrt{2\pi\tau}}\Big)A(\xi)d\xi. \tag{2.18}$$

It coincides with the solution (2.14) obtained previously.

To solve IBVP (2.3)

$$\begin{cases}
\varphi_\tau = \frac{1}{2}\varphi_{\eta\eta}, \quad 0<\eta<\infty, \quad \tau>0, \\
\varphi(\eta,0) = 0, \quad 0<\eta<\infty, \\
\varphi(0,\tau) = H(\tau), \quad \tau>0,
\end{cases} \tag{2.19}$$

we again apply the Fourier sine transform to equation (2.19)

$$\begin{aligned} F_s(\varphi_\tau) &= \frac{2}{\pi}\int_0^\infty \frac{\partial\varphi}{\partial\tau}\sin(\eta y)d\eta = \frac{\partial}{\partial\tau}\hat{\varphi}(y,\tau), \\ F_s(\varphi_{\eta\eta}) &= \frac{2}{\pi}\int_0^\infty \frac{\partial^2\varphi}{\partial\eta^2}\sin(\eta y)d\eta, \end{aligned}$$

and integration by parts, i.e $\frac{\partial^2\varphi}{\partial\eta^2} = dv \Rightarrow \frac{\partial\varphi}{\partial\eta} = v$ and $\sin(\eta y) = u \Rightarrow y\cos(\eta y)d\eta = du$,

$$F_s[\varphi_{\eta\eta}] = \frac{2}{\pi}\Big(\underbrace{\frac{\partial\varphi}{\partial\eta}\sin(\eta y)|_0^\infty}_{=0} - y\int_0^\infty \frac{\partial\varphi}{\partial\eta}\cos(\eta y)d\eta\Big). \tag{2.20}$$

Then, integration by parts in the second integral (2.20), $\frac{\partial\varphi}{\partial\eta} = dv \Rightarrow \varphi = v$ and $\cos(\eta y) = u \Rightarrow -y\sin(\eta y)d\eta = du$,

$$F_s[\varphi_{\eta\eta}] = -\frac{2}{\pi}y\Big(\underbrace{\varphi(\eta,\tau)\cos(\eta y)|_0^\infty}_{=-H(\tau)} + y\int_0^\infty \varphi(\eta,\tau)\sin(\eta y)d\eta\Big).$$

Thus we obtain

$$F_s[\varphi_{\eta\eta}] = \frac{2}{\pi}yH(\tau) - y^2\hat{\varphi}. \tag{2.21}$$

By the Fourier sine transform of the initial condition

$$F_s(\varphi(\eta,0)) \quad = \hat{\varphi}(y,0) = 0,$$

we obtain the following inhomogeneous IVP for the $\hat{\varphi}$,

$$\begin{cases} \frac{\partial\hat{\varphi}}{\partial\tau} + \frac{y^2}{2}\hat{\varphi} = \frac{1}{\pi}yH(\tau), \\ \hat{\varphi}(y,0) = 0. \end{cases} \tag{2.22}$$

The solution of IVP (2.22) is

$$\hat{\varphi}(y,\tau) \quad = \tfrac{y}{\pi} e^{-\frac{y^2}{2}\tau} \int_0^\tau H(\tau')e^{\frac{y^2}{2}\tau'} d\tau'.$$

By the inverse fourier sine transform

$$\begin{aligned}
\varphi(\eta,\tau) &= \int_0^\infty \hat{\varphi}(\xi,\tau)\sin(\xi\eta)d\xi, &(2.23)\\
&= \int_0^\infty \Big[\frac{\xi}{\pi}e^{-\frac{\xi^2}{2}\tau}\int_0^\tau H(\tau')e^{\frac{\xi^2}{2}\tau'}d\tau'\Big]\sin(\eta\xi)d\xi, &(2.24)\\
&= \frac{1}{\pi}\int_0^\tau H(\tau')\Big[\int_0^\infty \xi e^{-\frac{\xi^2}{2}(\tau-\tau')}\sin(\eta\xi)d\xi\Big]d\tau'. &(2.25)
\end{aligned}$$

By using relation $\sin(\eta\xi) = (e^{i\eta\xi} - e^{-i\eta\xi})/2$ and completing the exponential functions to the squares we obtain solution of IBVP (2.22) as follows

$$\varphi(\eta,\tau) = \int_0^\tau \Big(\frac{\eta}{\tau-\tau'}\Big)\frac{e^{-\frac{\eta^2}{2(\tau-\tau')}}}{\sqrt{2\pi(\tau-\tau')}}H(\tau')d\tau'. \qquad (2.26)$$

By superposition of (2.14) and (2.26), we have solution of IBVP (2.1) in the form

$$\varphi(\eta,\tau) = \int_0^\infty \underbrace{\Bigg(\frac{e^{-\frac{(\eta-\xi)^2}{2\tau}} - e^{-\frac{(\eta+\xi)^2}{2\tau}}}{\sqrt{2\pi\tau}}\Bigg)}_{Dirichlet\ heat\ kernel\ G(\eta,\xi,\tau)} A(\xi)d\xi + \int_0^\tau \underbrace{\Big(\frac{\eta}{\tau-\tau'}\Big)\frac{e^{-\frac{\eta^2}{2(\tau-\tau')}}}{\sqrt{2\pi(\tau-\tau')}}}_{-K_\eta:\ derivative\ of\ Heat\ Kernel} H(\tau')d\tau'.$$

Equivalently, we can write it in closed form

$$\varphi(\eta,\tau) = \int_0^\infty G(\eta,\xi,\tau)A(\xi)d\xi - \int_0^\tau K_\eta(\eta,\tau-\tau')H(\tau')d\tau',$$

where we have used notation $G(\eta,\xi,\tau) = K(\eta-\xi,\tau) - K(\eta+\xi,\tau)$.

2.2. The Neumann Problem on Semi-infinite Line

The IBVP for the Heat equation with initial condition at time $\tau = 0$ and the Neumann boundary condition at $\eta = 0$ is given by

$$\begin{cases} \varphi_\tau = \frac{1}{2}\varphi_{\eta\eta}, \quad 0 < \eta < \infty, \quad \tau > 0, \\ \varphi(\eta, 0) = A(\eta), \quad 0 < \eta < \infty, \\ \varphi_\eta(0, \tau) = Q(\tau), \quad \tau > 0, \end{cases} \tag{2.27}$$

where $A(\eta)$ and $Q(\tau)$ are given functions. To solve IBVP (2.27), as we did in Sec.2.1, we use linearity to replace the problem (2.27) by two problems **[5]** as follows,
The first one is IBVP with homogeneous Neumann BC,

$$\begin{cases} \varphi_\tau = \frac{1}{2}\varphi_{\eta\eta}, \quad 0 < \eta < \infty, \quad \tau > 0, \\ \varphi(\eta, 0) = A(\eta), \quad \eta > 0, \\ \varphi_\eta(0, \tau) = 0, \quad \tau > 0. \end{cases} \tag{2.28}$$

The second IBVP is inhomogeneous Neumann BC with homogeneous IC

$$\begin{cases} \varphi_\tau = \frac{1}{2}\varphi_{\eta\eta}, \quad 0 < \eta < \infty, \quad \tau > 0, \\ \varphi(\eta, 0) = 0, \quad 0 < \eta < \infty, \\ \varphi_\eta(0, \tau) = Q(\tau), \quad \tau > 0. \end{cases} \tag{2.29}$$

Applying Fourier cosine transform to the problem (2.28), i.e

$$\begin{aligned} F_c(\varphi_\tau) &= \frac{2}{\pi}\int_0^\infty \frac{\partial \varphi}{\partial \tau}\cos(\eta y)d\eta = \frac{\partial}{\partial \tau}\hat{\varphi}(y, \tau), \\ F_c(\varphi_{\eta\eta}) &= \frac{2}{\pi}\int_0^\infty \frac{\partial^2 \varphi}{\partial \eta^2}\cos(\eta y)d\eta = -y^2\hat{\varphi}, \end{aligned}$$

and to the initial condition

$$F_c(\varphi(\eta, 0)) = \frac{2}{\pi}\int_0^\infty A(\eta)\cos(\eta y)d\eta = \hat{\varphi}(y, 0) = S(y),$$

we obtain the following ordinary differential equation with initial condition

$$\begin{cases} \frac{\partial \hat{\varphi}}{\partial \tau} = -\frac{1}{2} y^2 \hat{\varphi}, \\ \hat{\varphi}(y, 0) = S(y). \end{cases} \tag{2.30}$$

The solution of IVP (2.30) is given by

$$\hat{\varphi}(y, \tau) = S(y) e^{-\frac{y^2}{2}\tau} = \Big(\frac{2}{\pi} \int_0^{\infty} A(\eta) \cos(\eta y) d\eta\Big) e^{-\frac{y^2}{2}\tau}.$$

Then, applying the inverse Fourier cosine transform, i.e

$$\begin{aligned} \varphi(\eta, \tau) &= \int_0^{\infty} \hat{\varphi}(\xi, \tau) \cos(\xi \eta) d\xi, && (2.31) \\ &= \frac{2}{\pi} \int_0^{\infty} \Big[\int_0^{\infty} A(\xi) \cos(\xi y) d\xi \Big] e^{-\frac{y^2}{2}\tau} \cos(\eta y) dy, && (2.32) \\ &= \frac{2}{\pi} \int_0^{\infty} A(\xi) \Big[\int_0^{\infty} e^{-\frac{y^2}{2}\tau} \cos(\xi y) \cos(\eta y) dy \Big] d\xi, && (2.33) \end{aligned}$$

and using relation $\cos(y\xi) \cos \eta y = [\cos(\eta - \xi) y + \cos(\eta + \xi) y]/2$, we get

$$\varphi(\eta, \tau) = \frac{1}{\pi} \int_0^{\infty} A(\xi) \Big[\underbrace{\int_0^{\infty} e^{-\frac{y^2}{2}\tau} \cos(\eta - \xi) y dy}_{I} + \underbrace{\int_0^{\infty} e^{-\frac{y^2}{2}\tau} \cos(\eta + \xi) y dy}_{II} \Big]. \tag{2.34}$$

For integral (I) in above, using relation $\cos(\eta - \xi) y = (e^{i(\eta-\xi)y} + e^{-i(\eta-\xi)y})/2$ and completing exponential functions to the squares, we obtain

$$I = \sqrt{\frac{\pi}{2\tau}} e^{-\frac{(\eta-\xi)^2}{2\tau}}, \tag{2.35}$$

and for the second integral (II) in (2.34), using relation $\cos(\eta + \xi) y = (e^{i(\eta+\xi)y} + e^{-i(\eta+\xi)y})/2$ and completing exponential functions to the squares, we get

$$II = \sqrt{\frac{\pi}{2\tau}} e^{-\frac{(\eta+\xi)^2}{2\tau}}. \tag{2.36}$$

Substituting (I) and (II) into (2.34), we obtain solution of IBVP (2.28) in the form

$$\varphi(\eta,\tau) = \frac{1}{\sqrt{2\pi\tau}} \int_0^\infty \left(e^{-\frac{(\eta-\xi)^2}{2\tau}} + e^{-\frac{(\eta+\xi)^2}{2\tau}}\right) A(\xi) d\xi. \tag{2.37}$$

Note that, the Neumann BC is homogeneous, that is $\varphi_\eta(0,\tau) = 0$, we can use the reflection principle. We seek to reduce the IBVP (2.27) to an IVP on whole line by extending the initial data $A(\eta)$ as an even function, in such a way that boundary condition is automatically satisfied. The even extension of $A(\eta)$ is

$$A_e(\eta) = \begin{cases} A(\eta), & \eta \geq 0, \\ A(-\eta), & \eta \leq 0, \end{cases}$$

and we consider the following IVP

$$\begin{cases} \varphi_\tau = \frac{1}{2}\varphi_{\eta\eta}, & -\infty < \eta < \infty, \quad \tau > 0, \\ \varphi(\eta,0) = A_e(\eta), & -\infty < \eta < \infty. \end{cases} \tag{2.38}$$

It's well-known that problem (2.38) has solution

$$\varphi(\eta,\tau) = \int_{-\infty}^\infty K(\eta-\xi,\tau) A_e(\xi) d\xi.$$

Substituting the expressions for $A_e(\eta)$ and changing variable $\xi \to -\xi$ in the second integral, we obtain

$$\varphi(\eta,\tau) = \int_0^\infty K(\eta-\xi,\tau) A(\xi) d\xi + \int_0^\infty K(\eta+\xi,\tau) A(\xi) d\xi,$$

By using definition of the Heat kernel, we have in explicit form

$$\varphi(\eta,\tau) = \int_0^\infty \left(\frac{e^{-\frac{(\eta-\xi)^2}{2\tau}} + e^{-\frac{(\eta+\xi)^2}{2\tau}}}{\sqrt{2\pi\tau}}\right) A(\xi) d\xi. \tag{2.39}$$

To solve the IBVP (2.29), we use the Fourier cosine transform

$$
\begin{aligned}
F_c(\varphi_\tau) &= \frac{2}{\pi}\int_0^\infty \frac{\partial \varphi}{\partial \tau}\cos(\eta y)d\eta = \frac{\partial}{\partial \tau}\hat{\varphi}(y,\tau), \\
F_c(\varphi_{\eta\eta}) &= \frac{2}{\pi}\int_0^\infty \frac{\partial^2 \varphi}{\partial \eta^2}\cos(\eta y)d\eta, && (2.40)\\
F_c(\varphi(\eta,0)) &= \hat{\varphi}(y,0) = 0. && (2.41)
\end{aligned}
$$

Thus, we have the following IVP for the first order linear ODE in τ variable

$$
\begin{cases}
\frac{\partial \hat{\varphi}}{\partial \tau} + \frac{y^2}{2}\hat{\varphi} = -\frac{1}{2}Q(\tau), \\
\hat{\varphi}(y,0) = 0,
\end{cases} \tag{2.42}
$$

which is

$$
\hat{\varphi}(y,\tau) = -\frac{1}{2}e^{-\frac{y^2}{2}\tau}\int_0^\tau Q(\tau')e^{\frac{y^2}{2}\tau'}d\tau'.
$$

By inverse cosine transform

$$
\begin{aligned}
\varphi(\eta,\tau) &= \frac{2}{\pi}\int_0^\infty \hat{\varphi}(\xi,\tau)\cos(\eta\xi)d\xi, \\
&= -\frac{2}{\pi}\int_0^\infty \left[\frac{1}{2}e^{-\frac{\xi^2}{2}\tau}\int_0^\tau Q(\tau')e^{\frac{\xi^2}{2}\tau'}d\tau'\right]\cos(\eta\xi)d\xi, \\
&= -\frac{1}{2\pi}\int_0^\tau Q(\tau)\Big[\int_0^\infty (e^{-\frac{\xi^2}{2}(\tau-\tau')+i\eta\xi} + e^{-\frac{\xi^2}{2}(\tau-\tau')-i\eta\xi})\Big].
\end{aligned}
$$

Thus, we have solution of IVP (2.42) as

$$
\varphi(\eta,\tau) = -\int_0^\tau \frac{e^{-\frac{\eta^2}{2(\tau-\tau')}}}{\sqrt{2\pi(\tau-\tau')}}Q(\tau')d\tau'. \tag{2.43}
$$

By superposition of (2.39) and (2.43), finally we have solution of the IBVP (2.27) as follows

$$\varphi(\eta,\tau) = \int_0^\infty \underbrace{\left(\frac{e^{-\frac{(\eta-\xi)^2}{2\tau}} + e^{-\frac{(\eta+\xi)^2}{2\tau}}}{\sqrt{2\pi\tau}}\right)}_{Neumann\ \ heat\ \ kernel\ \ N(\eta,\xi,\tau)} A(\xi)d\xi - \int_0^\tau \underbrace{\frac{e^{-\frac{\eta^2}{2(\tau-\tau')}}}{\sqrt{2\pi(\tau-\tau')}}}_{K(\eta,\tau-\tau')\ :\ Heat\ \ Kernel} Q(\tau')d\tau'. \quad (2.44)$$

Equivalently, in closed form we have

$$\varphi(\eta,\tau) = \int_0^\infty N(\eta,\xi,\tau)A(\xi)d\xi - \int_0^\tau K(\eta,\tau-\tau')Q(\tau')d\tau',$$

where $N(\eta,\xi,\tau) = K(\eta-\xi,\tau) + K(\eta+\xi,\tau)$.

2.3. Similarity Solutions of Heat Equation

If $\varphi(\eta,\tau)$ is a solution of the Heat equation $\varphi_\tau = (1/2)\varphi_{\eta\eta}$ $(*)$, then $\varphi_\lambda(\eta,\tau) = \lambda^{-c}\varphi(\lambda\eta,\lambda^2\tau)$ is also a rescaled solution of the Heat equation. We look for solutions satisfying

$$\varphi(\eta,\tau) = \lambda^{-c}\varphi(\lambda\eta,\lambda^2\tau), \quad -\infty < \eta < \infty, \quad \tau > 0, \ \ and \ \ c,\lambda \in \mathbb{R}, \quad (2.45)$$

which are known as similarity solutions (or homogeneous solutions **[10]**). For $\lambda > 0$ and $\tau > 0$, let $\lambda = 1/\sqrt{2\tau}$, we have

$$\varphi(\eta,\tau) = \left(\frac{1}{\sqrt{2\tau}}\right)^{-c} \varphi\left(\frac{\eta}{\sqrt{2\tau}}, \frac{1}{2}\right). \quad (2.46)$$

For $z = \eta/\sqrt{2\tau}$ define $f(z) = \varphi(\frac{\eta}{\sqrt{2\tau}}, \frac{1}{2})$, then

$$\varphi(\eta,\tau) \ = \ \left(\frac{1}{\sqrt{2\tau}}\right)^{-c} f(z). \quad (2.47)$$

Substituting (2.47) into Heat equation (∗), then we obtain ODE $f'' + 2zf' - 2cf = 0$. For the special case $c = n$, this equation becomes

$$f'' + 2zf' - 2nf = 0, \quad -\infty < z < \infty. \tag{2.48}$$

It's easily showed that the following functions are solutions of (2.48).

$$Hh_n^-(z) = \int_0^\infty e^{-(z-y)^2} y^n dy, \quad -\infty < z < \infty, \tag{2.49}$$

$$Hh_n^+(z) = \int_0^\infty e^{-(z+y)^2} y^n dy, \quad -\infty < z < \infty, \tag{2.50}$$

$$H_n^k(z) = \int_{-\infty}^\infty e^{-(z-y)^2} y^n dy. \quad -\infty < z < \infty. \tag{2.51}$$

Using similarity variable $z = \eta/\sqrt{2\tau}$, we can write also

$$Hh_n^-\left(\frac{\eta}{\sqrt{2\tau}}\right) = \int_0^\infty e^{-\frac{(\eta-\sqrt{2\tau}y)^2}{2\tau}} y^n dy, \quad -\infty < \eta < \infty, \quad \tau > 0, \tag{2.52}$$

$$Hh_n^+\left(\frac{\eta}{\sqrt{2\tau}}\right) = \int_0^\infty e^{-\frac{(\eta+\sqrt{2\tau}y)^2}{2\tau}} y^n dy, \quad -\infty < \eta < \infty, \quad \tau > 0, \tag{2.53}$$

$$H_n^k\left(\frac{\eta}{\sqrt{2\tau}}\right) = \int_{-\infty}^\infty e^{-\frac{(\eta-\sqrt{2\tau}y)^2}{2\tau}} y^n dy, \quad -\infty < \eta < \infty, \quad \tau > 0, \tag{2.54}$$

and by changing variable $\sqrt{2\tau}y \to \xi$, we have

$$Hh_n^-\left(\frac{\eta}{\sqrt{2\tau}}\right) = \int_0^\infty \frac{e^{-\frac{(\eta-\xi)^2}{2\tau}}}{\sqrt{2\tau}} \left(\frac{\xi}{\sqrt{2\tau}}\right)^n d\xi, \tag{2.55}$$

$$Hh_n^+\left(\frac{\eta}{\sqrt{2\tau}}\right) = \int_0^\infty \frac{e^{-\frac{(\eta+\xi)^2}{2\tau}}}{\sqrt{2\tau}} \left(\frac{\xi}{\sqrt{2\tau}}\right)^n d\xi, \tag{2.56}$$

$$H_n^k\left(\frac{\eta}{\sqrt{2\tau}}\right) = \int_{-\infty}^\infty \frac{e^{-\frac{(\eta-\xi)^2}{2\tau}}}{\sqrt{2\tau}} \left(\frac{\xi}{\sqrt{2\tau}}\right)^n d\xi, \tag{2.57}$$

and we define

$$
\begin{aligned}
h_n^-(\eta,\tau) &= \frac{1}{\sqrt{\pi}}(\sqrt{2\tau})^n Hh_n^-(\frac{\eta}{\sqrt{2\tau}}) = \int_0^\infty \frac{e^{-\frac{(\eta-\xi)^2}{2\tau}}}{\sqrt{2\pi\tau}}\xi^n d\xi, &(2.58)\\
h_n^+(\eta,\tau) &= \frac{1}{\sqrt{\pi}}(\sqrt{2\tau})^n Hh_n^+(\frac{\eta}{\sqrt{2\tau}}) = \int_0^\infty \frac{e^{-\frac{(\eta+\xi)^2}{2\tau}}}{\sqrt{2\pi\tau}}\xi^n d\xi, &(2.59)\\
H_n^k(\eta,\tau/2) &= \frac{1}{\sqrt{\pi}}(\sqrt{2\tau})^n H_n^k(\frac{\eta}{\sqrt{2\tau}}) = \int_{-\infty}^\infty \frac{e^{-\frac{(\eta-\xi)^2}{2\tau}}}{\sqrt{2\pi\tau}}\xi^n d\xi, &(2.60)
\end{aligned}
$$

where function (2.60) is Kampe de Feriet polynomials, defined by

$$
H_n^k(\eta,\tau/2) = n!\sum_{m=0}^{[n/2]} \frac{(\tau/2)^m}{m!(n-2m)!}\eta^{n-2m}. \tag{2.61}
$$

Clearly, these functions are similarity solutions which satisfy (2.46). In function (2.59), replacing $\xi \to -\xi$, we get

$$
h_n^+(\eta,\tau) = (-1)^n \int_{-\infty}^0 \frac{e^{-\frac{(\eta-\xi)^2}{2\tau}}}{\sqrt{2\pi\tau}}\xi^n d\xi.
$$

Then it's easily seen that for even n, i.e $n = 2p$ for $p = 0, 1, 2, \cdots$, we have even KFP in terms of h_n^+ and h_n^-,

$$
H_{2p}^k(\eta,\tau) = h_{2p}^-(\eta,\tau) + h_{2p}^+(\eta,\tau), \quad p = 0, 1, 2, \cdots,
$$

and for odd n, say $n = 2p + 1$, we have odd KFP

$$
H_{2p+1}^k(\eta,\tau) = h_{2p+1}^-(\eta,\tau) - h_{2p+1}^+(\eta,\tau), \quad p = 0, 1, 2, \cdots.
$$

For fixed $\tau > 0$ and at $\eta = 0$, we have

$$\begin{aligned}
h_n^-(0,\tau) &= h_n^+(0,\tau) = \int_0^\infty \frac{e^{-\frac{\xi^2}{2\tau}}}{\sqrt{2\pi\tau}}\xi^n d\xi = \frac{2^{\frac{n-1}{2}}\tau^{\frac{n+1}{2}}\Gamma[\frac{n+1}{2}]}{\sqrt{2\pi\tau}},\\
H_{2p}^k(0,\tau) &= h_{2p}^-(0,\tau) + h_{2p}^+(0,\tau) = 2\int_0^\infty \frac{e^{-\frac{\xi^2}{2\tau}}}{\sqrt{2\pi\tau}}\xi^{2p} d\xi,\\
H_{2p+1}^k(0,\tau) &= h_{2p+1}^-(0,\tau) - h_{2p+1}^+(0,\tau) = 0.
\end{aligned}$$

For fixed $\eta \in (-\infty,\infty)$ and as $\tau \to 0$,

$$\begin{aligned}
h_n^-(\eta,0) &= \eta^n,\\
h_n^+(\eta,0) &= 0,\\
H_n^k(\eta,0) &= \eta^n.
\end{aligned}$$

Using the above solutions of the Heat equation, we can obtain solutions of the Dirichlet and Neumann IBVP as in the following.

Example 2.1 *We consider the following IBVP for the Heat equation with homogeneous Dirichlet BC*

$$\begin{cases}
\varphi_\tau = \frac{1}{2}\varphi_{\eta\eta}, & 0<\eta<\infty,\ 0<\tau<\infty,\\
\varphi(\eta,0) = \eta^n, & n = 0,1,2,\cdots,\ 0<\eta<\infty,\\
\varphi(0,\tau) = 0, & 0<\tau<\infty,
\end{cases} \tag{2.62}$$

which has solution

$$\varphi(\eta,\tau) = \int_0^\infty \Big(\frac{e^{-\frac{(\eta-\xi)^2}{2\tau}} - e^{-\frac{(\eta+\xi)^2}{2\tau}}}{\sqrt{2\pi\tau}}\Big)\xi^n d\xi. \tag{2.63}$$

If n is odd, then solutions of problem (2.62) are odd Kampe de Feriet Polynomials, i.e

$$\varphi_{2p+1}(\eta,\tau) = h_{2p+1}^-(\eta,\tau) - h_{2p+1}^+(\eta,\tau) = H_{2p+1}^k(\eta,\tau).$$

However, if n is even, then solutions of Heat problem are no longer even KFP, since even KFP does not satisfy the Dirichlet BC $\varphi(0,\tau) = 0$*. Then, solution in that case can be written in terms of functions (2.58) and (2.59) as*

$$\varphi_{2p}(\eta,\tau) \quad = \quad h^{-}_{2p}(\eta,\tau) - h^{+}_{2p}(\eta,\tau). \qquad (2.64)$$

Example 2.2 *Now consider the IBVP with homogeneous Neumann BC*

$$\begin{cases} \varphi_\tau = \frac{1}{2}\varphi_{\eta\eta}, \quad 0 < \eta < \infty, \ 0 < \tau < \infty, \\ \varphi(\eta,0) = \eta^n, \quad n = 0,1,2,\cdots, \ 0 < \eta < \infty, \\ \varphi_\eta(0,\tau) = 0, \quad 0 < \tau < \infty, \end{cases} \qquad (2.65)$$

which has solution

$$\varphi(\eta,\tau) = \int_0^\infty \Big(\frac{e^{-\frac{(\eta-\xi)^2}{2\tau}} + e^{-\frac{(\eta+\xi)^2}{2\tau}}}{\sqrt{2\pi\tau}}\Big)\xi^n d\xi. \qquad (2.66)$$

If n is even, then solutions of problem (2.65) are even Kampe de Feriet Polynomials, i.e

$$\varphi_{2p}(\eta,\tau) = h^{-}_{2p}(\eta,\tau) + h^{+}_{2p}(\eta,\tau) = H^{k}_{2p}(\eta,\tau).$$

However, if n is odd, then solutions of Heat problem are no longer odd KFP, since odd KFP does not satisfy the Neumann BC $\varphi_\eta(0,\tau) = 0$*. Then solution in that case can be written in terms of functions (2.58) and (2.59), that's*

$$\varphi_{2p+1}(\eta,\tau) \quad = \quad h^{-}_{2p+1}(\eta,\tau) + h^{+}_{2p+1}(\eta,\tau). \qquad (2.67)$$

2.4. Heat Equation with Robin Boundary Condition on Semi-infinite Line

Now, we consider the IBVP given by

$$\begin{cases} \varphi_\tau = \frac{1}{2}\varphi_{\eta\eta}, \quad 0 < \eta < \infty, \quad \tau > 0, \\ \varphi(\eta, 0) = A(\eta), \quad 0 < \eta < \infty, \\ \alpha(\tau)\varphi_\eta(0,\tau) + \beta(\tau)\varphi(0,\tau) = g(\tau), \quad \tau > 0. \quad (Robin\ \ BC) \end{cases} \tag{2.68}$$

where $A(\eta)$, $\alpha(\tau)$, $\beta(\tau)$, $g(\tau)$ are given functions. To solve the IBVP (2.68), we apply two approaches :

1) Dirichlet Approach : If $\beta(\tau)$ is not identically zero for $\tau > 0$, assume temporary we know $\varphi(0,\tau) = H(\tau)$, then we have the following IBVP with Dirichlet BC

$$\begin{cases} \varphi_\tau = \frac{1}{2}\varphi_{\eta\eta}, \quad 0 < \eta < \infty, \quad \tau > 0, \\ \varphi(\eta, 0) = A(\eta), \quad 0 < \eta < \infty, \\ \varphi(0,\tau) = H(\tau), \quad \tau > 0. \end{cases} \tag{2.69}$$

From previous section, we know the solution of IBVP (2.69) in the form

$$\varphi(\eta,\tau) = \int_0^\infty \Big(\frac{e^{-\frac{(\eta-\xi)^2}{2\tau}} - e^{-\frac{(\eta+\xi)^2}{2\tau}}}{\sqrt{2\pi\tau}}\Big)\varphi(\xi,0)d\xi + \int_0^\tau \Big(\frac{\eta}{\tau-\tau'}\Big)\frac{e^{-\frac{\eta^2}{2(\tau-\tau')}}}{\sqrt{2\pi(\tau-\tau')}}H(\tau')d\tau', \tag{2.70}$$

where in closed form

$$\varphi(\eta,\tau) = \int_0^\infty G(\eta,\xi,\tau)\varphi(\xi,0)d\xi - \int_0^\tau K_\eta(\eta,\tau-\tau')H(\tau')d\tau'.$$

But this solution contains unknown function $H(\tau)$. To fix this function, we have to solve the Robin BC. By taking derivative of (2.70) with respect to η,

$$
\begin{aligned}
\varphi_\eta(\eta,\tau) &= \int_0^\infty G_\eta(\eta,\xi,\tau)\varphi(\xi,0)d\xi - \int_0^\tau \underbrace{K_{\eta\eta}(\eta,\tau-\tau')}_{-2K_{\tau'}\ :\ from\ Heat\ eq.} H(\tau')d\tau', \\
\varphi_\eta(0,\tau) &= \int_0^\infty G_\eta(0,\xi,\tau)\varphi(\xi,0)d\xi + 2\int_0^\tau K_{\tau'}(0,\tau-\tau')H(\tau')d\tau',
\end{aligned}
$$

or explicitly

$$
\varphi_\eta(0,\tau) = 2\int_0^\infty \left(\frac{\xi}{\tau}\right)\frac{e^{-\frac{\xi^2}{2\tau}}}{\sqrt{2\pi\tau}}A(\xi)d\xi + \int_0^\tau \frac{H(\tau')}{\sqrt{2\pi(\tau-\tau')^3}}d\tau', \tag{2.71}
$$

and substituting $\varphi_\eta(0,\tau)$ with $\varphi(0,\tau) = H(\tau)$ into Robin BC (2.68), we obtain the following integral equation **[5]** for the unknown function $H(\tau)$,

$$
\beta(\tau)H(\tau) + \alpha(\tau)\left(2\int_0^\infty \left(\frac{\xi}{\tau}\right)\frac{e^{-\frac{\xi^2}{2\tau}}}{\sqrt{2\pi\tau}}A(\xi)d\xi + \int_0^\tau \frac{H(\tau')}{\sqrt{2\pi(\tau-\tau')^3}}d\tau'\right) = g(\tau),
$$

or equivalently we have,

$$
H(\tau) = F_D(\tau) - \frac{\alpha(\tau)}{\beta(\tau)}\left(\int_0^\tau \frac{H(\tau')}{\sqrt{2\pi(\tau-\tau')^3}}d\tau'\right), \tag{2.72}
$$

where

$$
F_D(\tau) = \frac{g(\tau)}{\beta(\tau)} - \frac{2\alpha(\tau)}{\beta(\tau)}\left(\int_0^\infty \left(\frac{\xi}{\tau}\right)\frac{e^{-\frac{\xi^2}{2\tau}}}{\sqrt{2\pi\tau}}A(\xi)d\xi\right).
$$

The function $F_D(\tau)$ can be obtained directly, since we know $A(\eta)$, $g(\tau)$, $\alpha(\tau)$ and $\beta(\tau)$. Equation (2.72) is an inhomogeneous linear integral equation of Volterra type. In general, the solution of integral equation (2.72) can be obtained numerically. If we can solve it explicitly and find the unknown $H(\tau)$, then the solution of Heat IBVP with Robin BC is (2.70).

Thus, the problem of solving the Heat IBVP (2.68) with Robin BC is reduced to solving integral equation (2.72) for the unknown function $H(\tau)$.

2) Neumann Approach : Now, we consider again the IBVP (2.68) with Robin BC. If function $\alpha(\tau)$ is not identically zero for $\tau > 0$, assume temporary we know $\varphi_\eta(0,\tau) = Q(\tau)$. Then we have the following IBVP with Neumann BC

$$\begin{cases} \varphi_\tau = \frac{1}{2}\varphi_{\eta\eta}, \quad 0 < \eta < \infty, \quad \tau > 0, \\ \varphi(\eta, 0) = A(\eta), \quad 0 < \eta < \infty, \\ \varphi_\eta(0,\tau) = Q(\tau), \quad \tau > 0. \end{cases} \tag{2.73}$$

From previous part, we know solution of the IBVP (2.73) in the form

$$\varphi(\eta,\tau) = \int_0^\infty \Big(\frac{e^{-\frac{(\eta-\xi)^2}{2\tau}} + e^{-\frac{(\eta+\xi)^2}{2\tau}}}{\sqrt{2\pi\tau}} \Big) A(\xi)d\xi - \int_0^\tau \frac{e^{-\frac{\eta^2}{2(\tau-\tau')}}}{\sqrt{2\pi(\tau-\tau')}} \underbrace{Q(\tau')}_{\varphi_\eta(0,\tau')} d\tau', \tag{2.74}$$

and contains the unknown function $Q(\tau)$. To fix this function we need to solve the Robin BC. From solution (2.74), we obtain $\varphi(0,\tau)$ as follows

$$\varphi(0,\tau) = 2\int_0^\infty \Big(\frac{e^{-\frac{\xi^2}{2\tau}}}{\sqrt{2\pi\tau}} \Big) A(\xi)d\xi - \int_0^\tau \frac{Q(\tau')}{\sqrt{2\pi(\tau-\tau')}} d\tau'. \tag{2.75}$$

Substituting $\varphi(0,\tau)$ and $\varphi_\eta(0,\tau) = Q(\tau)$ into Robin BC (2.68), we get

$$\alpha(\tau)Q(\tau) + \beta(\tau)\Big(2\int_0^\infty \Big(\frac{e^{-\frac{\xi^2}{2\tau}}}{\sqrt{2\pi\tau}} \Big) A(\xi)d\xi - \int_0^\tau \frac{Q(\tau')}{\sqrt{2\pi(\tau-\tau')}} d\tau' \Big) = g(\tau),$$

or equivalently

$$Q(\tau) = F_N(\tau) + \frac{\beta(\tau)}{\alpha(\tau)} \Big(\int_0^\tau \frac{Q(\tau')}{\sqrt{2\pi(\tau-\tau')}} d\tau' \Big), \tag{2.76}$$

where

$$F_N(\tau) = \frac{g(\tau)}{\alpha(\tau)} - 2\frac{\beta(\tau)}{\alpha(\tau)} \Big(\int_0^\infty \Big(\frac{e^{-\frac{\xi^2}{2\tau}}}{\sqrt{2\pi\tau}} \Big) A(\xi)d\xi \Big).$$

The function $F_N(\tau)$ can be obtained directly since $A(\xi)$, $g(\tau)$, $\alpha(\tau)$ and $\beta(\tau)$ are known, but $Q(\tau)$ is unknown function. Thus, equation (2.76) is an inhomogeneous integral equation of Volterra type for the unknown function $Q(\tau)$. If we can solve it explicitly, we will fix $\varphi_\eta(0,\tau) = Q(\tau)$, so that the solution of Heat IBVP (2.68) can be found. If $\beta(\tau)/\alpha(\tau)$ is constant, then it's known Abel's integral equation. In the next section, we give solution of Abel's integral equation.

2.5. Abel's Integral Equation and Solutions of Heat Problem with Robin Boundary Condition

The integral equation

$$f(\tau) = \int_0^\tau \frac{1}{\sqrt{\tau-\tau'}} u(\tau')d\tau', \tag{2.77}$$

where $f(\tau)$ is given and $u(\tau)$ is unknown, is called a first-kind Abel's integral equation. This equation is a special Volterra type integral equation with weakly singular kernel $K(\tau,\tau') = 1/\sqrt{\tau-\tau'}$, where $K(\tau,\tau') \to \infty$ as $\tau' \to \tau$.

This equation can be solved by applying the Laplace transform and then by inverse Laplace transform so that we have

$$u(\tau) = \frac{1}{\pi}\frac{d}{d\tau}\int_0^\tau \frac{f(\tau')}{\sqrt{\tau-\tau'}}d\tau'. \tag{2.78}$$

Clearly, the formula (2.78) will be used for solving Abel's integral equation (2.77). It's known that for some special functions $f(\tau)$, the solution (2.78) can be obtained explicitly. The followings are some examples.

For $f(\tau) = \tau^{n+1/2}$, n is a positive integer, we have solutions for $u(\tau)$ as follows

$$\begin{aligned}
&n = 1, \quad f(\tau) = \tfrac{4}{3}\tau^{3/2} \;\Rightarrow\; u(\tau) = \tau,\\
&n = 2, \quad f(\tau) = \tfrac{16}{15}\tau^{5/2} \;\Rightarrow\; u(\tau) = \tau^2,\\
&n = 3, \quad f(\tau) = \tfrac{32}{35}\tau^{7/2} \;\Rightarrow\; u(\tau) = \tau^3,\\
&\vdots
\end{aligned} \tag{2.79}$$

In general, $n = 1, 2, 3, \cdots$,

$$f(\tau) = \frac{2^{n+1}\Gamma(n+1)}{1.3.5\cdots(2n+1)}\tau^{n+1/2} \;\Rightarrow\; u(\tau) = \tau^n. \tag{2.80}$$

For $f(\tau) = \tau^n$, n is a positive integer, we have solutions for $u(\tau)$ as follows

$$\begin{aligned} &n = 1, \quad f(\tau) = \tfrac{1}{2}\pi\tau, \;\Rightarrow\; u(\tau) = \tau^{1/2}, \\ &n = 2, \quad f(\tau) = \tfrac{3}{8}\pi\tau^2 \;\Rightarrow\; u(\tau) = \tau^{3/2}, \\ &n = 3, \quad f(\tau) = \tfrac{5}{16}\pi\tau^3 \;\Rightarrow\; u(\tau) = \tau^{5/2}, \\ &\vdots \end{aligned} \tag{2.81}$$

In general, $n = 1, 2, 3, \cdots$,

$$f(\tau) = \frac{\Gamma(n+1/2)}{\Gamma(n+1}\sqrt{\pi}\tau^n \;\Rightarrow\; u(\tau) = \tau^{n-1/2}. \tag{2.82}$$

The weakly-singular Abel's integral equations of the second kind are given by

$$Q(\tau) = F(\tau) + \int_0^\tau \frac{\beta}{\sqrt{\tau - \tau'}} Q(\tau')d\tau', \quad \tau \in [0, T], \tag{2.83}$$

where $F(\tau)$ is known and $Q(\tau)$ is unknown with constant β. To solve this type of integral equation, one can use again Laplace transform or method of successive approximations. As we have seen in previous chapter, the Heat equation with Robin boundary condition reduces to Volterra type integral equation. If this integral equation is of the form (2.83), then we obtain explicit result for the solution of IBVP with Robin boundary condition for the Heat equation.

⋆ **First type initial condition :** The following IBVP

$$\begin{cases} \varphi_\tau = \frac{1}{2}\varphi_{\eta\eta}, \quad 0 < \eta < \infty, \quad \tau > 0, \\ \varphi(\eta, 0) = \left(\frac{\sqrt{2\pi}}{2^n\Gamma[n+\frac{1}{2}]}\xi^{2n} + \frac{\sqrt{2\pi}}{1.3.5\cdots(2n+1)}\xi^{2n+1}\right), \quad 0 < \eta < \infty, \\ \varphi(0, \tau) - \varphi_\eta(0, \tau) = 0, \quad \tau > 0, \end{cases} \tag{2.84}$$

has solution

$$\varphi(\eta,\tau) = \int_0^\infty N(\eta,\xi,\tau)\varphi(\xi,0)d\xi - \int_0^\tau K(\eta,\tau-\tau')\varphi_\eta(0,\tau')d\tau',$$

which is solvable. Indeed,

For example if n=1 : $A(\eta) = \sqrt{2\pi}\eta^2 + \frac{\sqrt{2\pi}}{3}\eta^3$, the following IBVP with Robin BC and initial data $A(\eta)$,

$$\begin{cases} \varphi_\tau = \frac{1}{2}\varphi_{\eta\eta}, \quad 0 < \eta < \infty, \quad \tau > 0, \\ \varphi(\eta,0) = \sqrt{2\pi}\eta^2 + \frac{\sqrt{2\pi}}{3}\eta^3, \quad 0 < \eta < \infty, \\ \varphi(0,\tau) - \varphi_\eta(0,\tau) = 0, \quad \tau > 0, \end{cases} \tag{2.85}$$

reduces to solving integral equation

$$Q(\tau) = 2\int_0^\infty \left(\frac{e^{-\frac{\xi^2}{2\tau}}}{\sqrt{2\pi\tau}}\right)\left[\sqrt{2\pi}\xi^2 + \frac{\sqrt{2\pi}}{3}\xi^3\right]d\xi - \int_0^\tau \frac{Q(\tau')}{\sqrt{2\pi(\tau-\tau')}}d\tau', \tag{2.86}$$

$$Q(\tau) = \sqrt{2\pi}\tau + \frac{4}{3}\tau^{3/2} - \int_0^\tau \frac{Q(\tau')}{\sqrt{2\pi(\tau-\tau')}}d\tau'. \tag{2.87}$$

By Laplace transform, we obtain solution $Q(\tau) = \sqrt{2\pi}\tau$ with $f(\tau) = \frac{4}{3}\tau^{3/2}$. Thus, the IBVP for the Heat equation with Robin BC is exactly solvable by special initial condition $A(\eta)$. And we have solution for the problem (2.85) as follows

$$\varphi(\eta,\tau) = \int_0^\infty \left(\frac{e^{-\frac{(\eta-\xi)^2}{2\tau}} + e^{-\frac{(\eta+\xi)^2}{2\tau}}}{\sqrt{2\pi\tau}}\right)A(\xi)d\xi - \int_0^\tau \frac{e^{-\frac{\eta^2}{2(\tau-\tau')}}}{\sqrt{2\pi(\tau-\tau')}}Q(\tau')d\tau', \tag{2.88}$$

$$\varphi(\eta,\tau) = \underbrace{\int_0^\infty \left(\frac{e^{-\frac{(\eta-\xi)^2}{2\tau}} + e^{-\frac{(\eta+\xi)^2}{2\tau}}}{\sqrt{2\pi\tau}}\right)\left[\sqrt{2\pi}\xi^2 + \frac{\sqrt{2\pi}}{3}\xi^3\right]d\xi}_{I} - \underbrace{\int_0^\tau \frac{e^{-\frac{\eta^2}{2(\tau-\tau')}}}{\sqrt{\tau-\tau'}}\tau' d\tau'}_{II}, \tag{2.89}$$

where

$$I = \frac{2}{3}e^{-\frac{\eta^2}{2\tau}}\left(\eta^2\sqrt{\tau}+2\tau^{3/2}\right)+\sqrt{2\pi}\left(\eta^2+\tau+\frac{[\eta^3+3\eta\tau]}{3}Erf\left[\frac{\eta}{\sqrt{2\tau}}\right]\right), \quad (2.90)$$

$$II = \frac{2}{3}e^{-\frac{\eta^2}{2\tau}}\left(2\tau^{3/2}+\eta^2\sqrt{\tau}\right)-\sqrt{2\pi}\left(\eta\tau+\frac{\eta^3}{3}-\frac{[\eta^3+3\eta\tau]}{3}Erf\left[\frac{\eta}{\sqrt{2\tau}}\right]\right), \quad (2.91)$$

then $\varphi(\eta,\tau)$= I- II gives the exact solution to the IBVP (2.85) with Robin BC as follows

$$\varphi(\eta,\tau) = \sqrt{2\pi}\left[\frac{\eta^3}{3}+\eta^2+\eta\tau+\tau\right]. \quad (2.92)$$

For example if n=2 : $A(\eta) = \frac{\sqrt{2\pi}}{3}\eta^4+\frac{\sqrt{2\pi}}{15}\eta^5$, the following IBVP with Robin BC and initial data $A(\eta)$,

$$\begin{cases} \varphi_\tau = \frac{1}{2}\varphi_{\eta\eta}, \quad 0<\eta<\infty, \quad \tau>0, \\ \varphi(\eta,0) = \frac{\sqrt{2\pi}}{3}\eta^4+\frac{\sqrt{2\pi}}{15}\eta^5, \quad 0<\eta<\infty, \\ \varphi(0,\tau)-\varphi_\eta(0,\tau)=0, \quad \tau>0, \end{cases} \quad (2.93)$$

we have the following integral equation for $Q(\tau)$

$$Q(\tau) = 2\int_0^\infty\left(\frac{e^{-\frac{\xi^2}{2\tau}}}{\sqrt{2\pi\tau}}\right)\left[\frac{\sqrt{2\pi}}{3}\xi^4+\frac{\sqrt{2\pi}}{15}\xi^5\right]d\xi-\int_0^\tau\frac{Q(\tau')}{\sqrt{2\pi(\tau-\tau')}}d\tau', \quad (2.94)$$

$$Q(\tau) = \sqrt{2\pi}\tau^2+\frac{16}{15}\tau^{5/2}-\int_0^\tau\frac{Q(\tau')}{\sqrt{2\pi(\tau-\tau')}}d\tau'. \quad (2.95)$$

By solving the above integral equation motivated by the first kind Abels integral equation, we obtain $Q(\tau) = \sqrt{2\pi}\tau^2$ with $f(\tau)=\frac{16}{15}\tau^{5/2}$. Thus the IBVP for the Heat equation with Robin BC is exactly solvable by special initial condition $A(\eta)$. And we have solution for the problem (2.101) as follows

$$\varphi(\eta,\tau) = \underbrace{\int_0^\infty\left(\frac{e^{-\frac{(\eta-\xi)^2}{2\tau}}+e^{-\frac{(\eta+\xi)^2}{2\tau}}}{\sqrt{2\pi\tau}}\right)\left[\frac{\sqrt{2\pi}}{3}\xi^4+\frac{\sqrt{2\pi}}{15}\xi^5\right]d\xi}_{I}-\underbrace{\int_0^\tau\frac{e^{-\frac{\eta^2}{2(\tau-\tau')}}}{\sqrt{\tau-\tau'}}(\tau')^2d\tau'}_{II}, \quad (2.96)$$

where

$$
\begin{aligned}
I &= \frac{1}{15}\left(e^{-\frac{\eta^2}{2\tau}}(2\eta^2\sqrt{\tau}+2\tau^{3/2})(\eta^2+8\tau)+\sqrt{2\pi}(5\eta^4+30\eta^2\tau+15\tau^2)\right)\\
&+ \frac{1}{15}\left(\sqrt{2\pi}(\eta^5+10\eta^3\tau+15\eta\tau^2)Erf\left[\frac{\eta}{\sqrt{2\tau}}\right]\right), \qquad (2.97)\\
II &= \frac{1}{15}\left(2e^{-\frac{\eta^2}{2\tau}}\sqrt{\tau}(\eta^2+\tau)(\eta^2+8\tau)-\sqrt{2\pi}(15\eta\tau^2+10\eta^3\tau+\eta^5)\right)\\
&+ \frac{1}{15}\left(\sqrt{2\pi}(15\eta\tau^2+10\eta^3\tau+\eta^5)Erf\left[\frac{\eta}{\sqrt{2\tau}}\right]\right), \qquad (2.98)
\end{aligned}
$$

then $\varphi(\eta,\tau)$= I- II gives the exact solution to the IBVP (2.93) with Robin BC as follows

$$
\varphi(\eta,\tau) = \sqrt{2\pi}\left[\frac{\eta^5}{15}+\frac{\eta^4}{3}+\frac{5}{3}\eta^3\tau+2\eta^2\tau+\eta\tau^2+\tau^2\right]. \qquad (2.99)
$$

⋆ **Second type initial condition :** For the following IBVP

$$
\begin{cases}
\varphi_\tau = \frac{1}{2}\varphi_{\eta\eta}, \quad 0<\eta<\infty, \quad \tau>0,\\
\varphi(\eta,0) = \left(\frac{\sqrt{2\pi}}{2^n\Gamma(n)}\xi^{2n-1}+\frac{\pi}{2^n\Gamma(n+1)}\xi^{2n}\right),\ 0<\eta<\infty,\\
\varphi(0,\tau)-\varphi_\eta(0,\tau)=0 \quad \tau>0,
\end{cases} \qquad (2.100)
$$

we also obtain explicit solution. Indeed,
For example if n=1 : $A(\eta) = \sqrt{\frac{\pi}{2}}\eta+\frac{\pi}{2}\eta^2$, the following IBVP with Robin BC with initial data $A(\eta)$,

$$
\begin{cases}
\varphi_\tau = \frac{1}{2}\varphi_{\eta\eta}, \quad 0<\eta<\infty, \quad \tau>0,\\
\varphi(\eta,0) = \sqrt{\frac{\pi}{2}}\eta+\frac{\pi}{2}\eta^2, \quad 0<\eta<\infty,\\
\varphi(0,\tau)-\varphi_\eta(0,\tau)=0, \quad \tau>0,
\end{cases} \qquad (2.101)
$$

reduces to the the following integral equation for $Q(\tau)$

$$
\begin{aligned}
Q(\tau) &= 2\int_0^\infty\left(\frac{e^{-\frac{\xi^2}{2\tau}}}{\sqrt{2\pi\tau}}\right)\left[\sqrt{\frac{\pi}{2}}\xi+\frac{\pi}{2}\xi^2\right]d\xi-\int_0^\tau\frac{Q(\tau')}{\sqrt{2\pi(\tau-\tau')}}d\tau', \qquad (2.102)\\
Q(\tau) &= \sqrt{\tau}+\frac{\pi}{2}\tau-\int_0^\tau\frac{Q(\tau')}{\sqrt{2\pi(\tau-\tau')}}d\tau'. \qquad (2.103)
\end{aligned}
$$

By solving above integral equation motivated by the first kind Abels integral equation, we obtain $Q(\tau) = \tau^{1/2}$ with $f(\tau) = \frac{\pi}{2}\tau$. Thus the IBVP for Heat equation with Robin BC is exactly solvable by special initial condition $A(\eta)$. And we have solution for the problem (2.101) as follows

$$\varphi(\eta,\tau) = \int_0^\infty \Big(\frac{e^{-\frac{(\eta-\xi)^2}{2\tau}} + e^{-\frac{(\eta+\xi)^2}{2\tau}}}{\sqrt{2\pi\tau}}\Big)A(\xi)d\xi - \int_0^\tau \frac{e^{-\frac{\eta^2}{2(\tau-\tau')}}}{\sqrt{2\pi(\tau-\tau')}}Q(\tau')d\tau', \quad (2.104)$$

$$\varphi(\eta,\tau) = \underbrace{\int_0^\infty \Big(\frac{e^{-\frac{(\eta-\xi)^2}{2\tau}} + e^{-\frac{(\eta+\xi)^2}{2\tau}}}{\sqrt{2\pi\tau}}\Big)\Big[\sqrt{\frac{\pi}{2}}\xi + \frac{\pi}{2}\xi^2\Big]d\xi}_{I} \quad (2.105)$$

$$- \underbrace{\int_0^\tau \frac{e^{-\frac{\eta^2}{2(\tau-\tau')}}}{\sqrt{2\pi(\tau-\tau')}}(\tau')^{1/2}d\tau'}_{II},$$

where

$$I = \sqrt{\tau}e^{-\frac{\eta^2}{2\tau}} + \frac{\pi}{2}(\tau+\eta^2) + \sqrt{\frac{\pi}{2}}\eta Erf\left[\frac{\eta}{\sqrt{2\tau}}\right], \quad (2.106)$$

$$II = -\frac{1}{2}\eta\sqrt{\tau}e^{-\frac{\eta^2}{2\tau}} + \frac{\sqrt{\pi}}{2\sqrt{2}}(\eta^2+\tau)Erfc\left[\frac{\eta}{\sqrt{2\tau}}\right], \quad (2.107)$$

then $\varphi(\eta,\tau)$= I- II gives the exact solution to the IBVP (2.101) with Robin BC as follows

$$\varphi(\eta,\tau) = \sqrt{\tau}e^{-\frac{\eta^2}{2\tau}}\left(1+\frac{\eta}{2}\right) + \frac{\pi}{2}\left(\tau+\eta^2\right) + \sqrt{\frac{\pi}{2}}\left(\eta Erf\left[\frac{\eta}{\sqrt{2\tau}}\right] - \frac{(\eta^2+\tau)}{2}Erfc\left[\frac{\eta}{\sqrt{2\tau}}\right]\right).$$

2.6. Heat Equation with Special Boundary Condition on Semi-infinite Line

Now, we consider the IBVP given by

$$\begin{cases} \varphi_\tau = \frac{1}{2}\varphi_{\eta\eta}, \quad 0<\eta<\infty, \quad \tau>0, \\ \varphi(\eta,0) = A(\eta), \quad 0<\eta<\infty, \\ \varphi_\tau(0,\tau) + \alpha(\tau)\varphi_\eta(0,\tau) + \beta(\tau)\varphi(0,\tau) = g(\tau), \quad \tau>0, \end{cases} \quad (2.108)$$

where $A(\eta)$, $\alpha(\tau)$, $\beta(\tau)$, $g(\tau)$ are given functions. Assume temporary we know $\varphi(0,\tau) = H(\tau)$. We know that the IBVP with Dirichlet BC has solution

$$\varphi(\eta,\tau) = \int_0^\infty \Big(\frac{e^{-\frac{(\eta-\xi)^2}{2\tau}} - e^{-\frac{(\eta+\xi)^2}{2\tau}}}{\sqrt{2\pi\tau}}\Big)\varphi(\xi,0)d\xi + \int_0^\tau \Big(\frac{\eta}{\tau-\tau'}\Big)\frac{e^{-\frac{\eta^2}{2(\tau-\tau')}}}{\sqrt{2\pi(\tau-\tau')}}H(\tau')d\tau'. \quad (2.109)$$

But we see that the unknown function $H(\tau)$ is in the solution. We can fix this function from the Robin BC. By taking derivative of (2.109) with respect to η,

$$\varphi_\eta(0,\tau) = 2\int_0^\infty \Big(\frac{\xi}{\tau}\Big)\frac{e^{-\frac{\xi^2}{2\tau}}}{\sqrt{2\pi\tau}}A(\xi)d\xi + \int_0^\tau \frac{H(\tau')}{\sqrt{2\pi(\tau-\tau')^3}}d\tau', \quad (2.110)$$

and substituting $\varphi_\eta(0,\tau)$, $\varphi(0,\tau) = H(\tau)$ and $\varphi_\tau(0,\tau) = \dot{H}(\tau)$ into Robin BC (2.108), we obtain the following integral equation for the unknown function $H(\tau)$,

$$\dot{H}(\tau) + \beta(\tau)H(\tau) + \alpha(\tau)\Big(2\int_0^\infty \Big(\frac{\xi}{\tau}\Big)\frac{e^{-\frac{\xi^2}{2\tau}}}{\sqrt{2\pi\tau}}A(\xi)d\xi + \int_0^\tau \frac{H(\tau')}{\sqrt{2\pi(\tau-\tau')^3}}d\tau'\Big) = g(\tau),$$

or equivalently we have,

$$\dot{H}(\tau) = F_D(\tau) - \alpha(\tau)\Big(\int_0^\tau \frac{H(\tau')}{\sqrt{2\pi(\tau-\tau')^3}}d\tau'\Big) - \beta(\tau)H(\tau), \quad (2.111)$$

where

$$F_D(\tau) = g(\tau) - 2\alpha(\tau)\Big(\int_0^\infty \Big(\frac{\xi}{\tau}\Big)\frac{e^{-\frac{\xi^2}{2\tau}}}{\sqrt{2\pi\tau}}A(\xi)d\xi\Big).$$

The function $F_D(\tau)$ can be obtained directly, since we know $A(\eta)$, $g(\tau)$, $\alpha(\tau)$ and $\beta(\tau)$. The equation (2.111) is an inhomogeneous linear integral equation. Thus, the problem of solving the Heat IBVP (2.108) with Robin BC is reduced to solving integral equation (2.111) for the unknown function $H(\tau)$.

2.7. The Dirichlet Problem for Variable Parametric Parabolic Equation on Semi-infinite Line

In this section, we consider IBVP for a linear parabolic equation with variable coefficients. We show that it can be transformed to IBVP for standard Heat equation discussed in previous section.

Proposition 2.1 *The IBVP for variable parametric linear parabolic equation on semi-infinite line given as*

$$\begin{cases} \Phi_t = \frac{1}{2\mu(t)}\Phi_{xx} + \frac{\mu(t)\omega^2(t)}{2}x^2\Phi, \quad x > 0, \quad t_0 < t < T, \\ \Phi(x, t_0) = A(x), \quad x > 0, \\ \Phi(0, t) = B_D(t), \quad t_0 < t < T, \end{cases} \tag{2.112}$$

where $A(x)$ and $B_D(t)$ are known functions of x and t respectively and $\mu(t) > 0$ and continuously differentible and $\omega(t)$ are given smooth continuous functions of t, has solution of the form

$$\Phi(x, t) = \sqrt{\frac{r_0(t)}{r(t)}}\exp\Big[-\frac{\mu(t)\dot{r}(t)}{2r(t)}x^2\Big]\varphi(\eta(x, t), \tau(t)),$$

if $r(t)$ is strictly positive and solution of the ordinary differential equation with initial conditions

$$\begin{aligned} &\ddot{r} + \frac{\dot{\mu}(t)}{\mu(t)}\dot{r} + \omega^2(t)r = 0, \\ &r(t_0) = r_0 \neq 0, \quad \dot{r}(t_0) = 0, \end{aligned} \tag{2.113}$$

with

$$\eta(x, t) = \frac{r_0}{r(t)}x, \quad \tau(t) = r_0^2\int_{t_0}^{t}\frac{d\xi}{\mu(\xi)r^2(\xi)},$$

and $\varphi(\eta,\tau)$ satisfies the IBVP for the standard Heat equation,

$$\begin{cases} \varphi_\tau = \frac{1}{2}\varphi_{\eta\eta}, \quad \eta > 0, \quad 0 < \tau < \tau(T), \\ \varphi(\eta, 0) = \Phi(\eta, t_0), \quad \eta > 0, \\ \varphi(0,\tau) = \sqrt{\frac{r(t(\tau))}{r_0}}\Phi(0, t(\tau)), \quad 0 < \tau < \tau(T). \end{cases} \tag{2.114}$$

where $\tau(t) = \tau \Leftrightarrow t = t(\tau)$.

Proof:

Using the ansatz, **[2]**

$$\Phi(x,t) = e^{\frac{g(t)-\rho(t)x^2}{2}}\varphi(e^{g(t)}x, \tau(t)),$$

we can show that, if the auxiliary functions $\rho(t), \tau(t)$ and $g(t)$ satisfy the nonlinear system of ordinary differential equations

$$\dot{\rho} + \frac{\rho^2}{\mu(t)} + \mu(t)\omega^2(t) = 0, \quad \rho(t_0) = 0,$$

$$\dot{\tau} - \frac{e^{2g(t)}}{\mu(t)} = 0, \quad \tau(t_0) = 0, \tag{2.115}$$

$$\dot{g} + \frac{\rho(t)}{\mu(t)} = 0, \quad g(t_0) = 0, \tag{2.116}$$

and $\varphi(\eta,\tau)$ satisfies the standard HE (2.114), then $\Phi(x,t) = \exp[\frac{g(t)-\rho(t)x^2}{2}]\varphi(e^{g(t)}x, \tau(t))$ satisfies the variable parametric Heat equation (2.112). Noticing that equation (2.115) is a nonlinear Riccati equation which can be linearized by using $\rho(t) = \mu(t)\dot{r}(t)/r(t)$. Then the system is easily solved and we obtain the following auxiliary functions in terms of solution $r(t)$ to the IVP (2.113) as follows

$$\rho(t) = \mu(t)\frac{\dot{r}(t)}{r(t)},$$

$$\tau(t) = r_0^2 \int^t \frac{d\xi}{\mu(\xi) r^2(\xi)}, \quad \tau(t_0) = 0, \tag{2.117}$$

$$g(t) = ln\left(\frac{r_0}{r(t)}\right), \tag{2.118}$$

then, by back substitution these functions in the ansatz, we get $\Phi(x,t)$ in the form

$$\Phi(x,t) = \sqrt{\frac{r_0}{r(t)}} \exp\left[-\frac{\mu(t)\dot{r}(t)}{2r(t)}x^2\right]\varphi(\eta(x,t),\tau(t)).$$

Using ansatz, the initial condition $\Phi(x,t)\mid_{t=t_0} = \Phi(x,t_0)$ is easily transformed to the initial condition $\varphi(\eta,0) = \Phi(\eta,t_0)$ for standard HE. And Dirichlet BC for variable parametric parabolic equation $\Phi(0,t) = B_D(t)$ is transformed to Dirichlet BC for standard HE as follows

$$\varphi(0,\tau) = \sqrt{\frac{r(t(\tau))}{r_0}}\Phi(0,t(\tau)),$$

where we use that $\mu(t) > 0$ and $r^2(t) > 0$, so that $\tau(t) = r_0^2\int^t \frac{d\xi}{\mu(\xi)r^2(\xi)}$, $\tau(t_0) = 0$ is strictly increasing and thus its inverse $t(\tau)$ exists. Thus, IBVP (2.112) is transformed to the IBVP (2.114). This shows that solution of variable parametric parabolic equation is explicitly obtained in terms of solution of $\varphi(\eta,\tau)$ to the standard HE (2.114) and solution $r(t)$ of the IVP for the linear ODE (2.113). ■

Now, we give some basic examples to apply the above proposition.

Example 2.3 *For the constant coefficient parabolic equation where* $\mu(t) = 1$*,* $\omega^2(t) = -\omega_0^2$*,* $\omega_0 > 0$*, we have*

$$\Phi_t = \frac{1}{2}\Phi_{xx} + \frac{\omega_0^2}{2}x^2\Phi,$$

where ω_0*:constant and the related IVP as follows*

$$\begin{aligned} &\ddot{r} - {\omega_0}^2 r = 0, \\ &r(t_0) = r_0 \neq 0 \quad \dot{r}(t_0) = 0, \end{aligned} \tag{2.119}$$

where it has solution $r(t) = r_0\cosh(\omega_0)$*. Now, for example taking initial condition* $\Phi(x,0) = 0$ *and BC* $\Phi(0,t) = c_0\sqrt{sech(\omega_0 t)}$ *where* c_0*:constant, we obtain the following IBVP for constant coefficient variable parabolic equation,*

$$\begin{cases} \Phi_t = \frac{1}{2}\Phi_{xx} + \frac{\omega_0^2}{2}x^2\Phi, \quad x > 0, \quad t > 0, \\ \Phi(x,0) = 0, \quad x > 0, \\ \Phi(0,t) = c_0\sqrt{sech(\omega_0 t)}, \quad t > 0, \end{cases} \tag{2.120}$$

from Proposition-I, the corresponding functions as follows

$$\begin{aligned} \eta(x,t) &= sech(\omega_0 t)x, \\ \tau(t) &= \tanh(\omega_0 t)/\omega_0, \\ g(t) &= ln(sech(\omega_0 t)), \\ \rho(t) &= \omega_0 \tanh(\omega_0 t). \end{aligned} \tag{2.121}$$

Then we have solution in following form

$$\Phi(x,t) = \sqrt{sech(\omega_0 t)} e^{-\frac{1}{2}\omega_0 \tanh(\omega_0 t)x^2} \varphi(\eta(x,t), \tau(t)),$$

where $\varphi(x,t)$ satisfies the IBVP for standard Heat equation,

$$\begin{cases} \varphi_\tau = \frac{1}{2}\varphi_{\eta\eta}, \quad \eta > 0, \quad \tau > 0, \\ \varphi(\eta, 0) = 0, \quad \eta > 0, \\ \varphi(0,\tau) = c_0, \quad \tau > 0, \ c_0 : constant. \end{cases} \tag{2.122}$$

Applying Fourier sine and inverse Fourier sine transform, we get solution as follows

$$\varphi(\eta,\tau) = c_0 \Big(\underbrace{1 - erf(\frac{\eta}{\sqrt{2\tau}})}_{Erfc(\frac{\eta}{\sqrt{2\tau}})} \Big).$$

Thus, the corresponding solution of variable parametric parabolic equation is given by

$$\Phi(x,t) = c_0 \sqrt{sech(\omega_0 t)} e^{-\frac{1}{2}\omega_0 \tanh(\omega_0 t)x^2} Erfc\Big(\sqrt{\omega_0 cosech(2\omega_0 t)} x \Big).$$

Example 2.4 *The IBVP*

$$\begin{cases} \Phi_t = \frac{1}{2}\Phi_{xx} + \frac{\omega_0^2}{2}x^2\Phi, \quad x > 0, \quad t > 0, \\ \Phi(x,0) = 0, \quad x > 0, \\ \Phi(0,t) = \frac{\tanh(\omega_0 t)}{\omega_0} \sqrt{sech(\omega_0 t)} \quad t > 0 \end{cases} \tag{2.123}$$

which is transformed to the following IBVP for the standard HE

$$\begin{cases} \varphi_\tau = \frac{1}{2}\varphi_{\eta\eta}, & \eta > 0, \quad \tau > 0, \\ \varphi(\eta, 0) = 0, & \eta > 0, \\ \varphi(0, \tau) = \tau, & \tau > 0. \end{cases} \tag{2.124}$$

We obtain the solution as follows

$$\varphi(\eta, \tau) = (\eta^2 + \tau)Erfc\Big[\frac{\eta}{\sqrt{2\tau}}\Big] - \sqrt{\frac{2}{\pi}} e^{-\frac{\eta^2}{2\tau}} \sqrt{\tau}\eta. \tag{2.125}$$

And the corresponding solution for the parabolic problem is given by

$$\Phi(x, t) = \sqrt{sech(\omega_0 t)} e^{-\frac{\omega_0 \tanh(\omega_0 t)}{2} x^2} \Big([\eta^2(x, t) + \tau(t)]Erfc\Big[\frac{\eta(x, t)}{\sqrt{2\tau(t)}}\Big] - \sqrt{\frac{2}{\pi}} e^{-\frac{\eta^2(x,t)}{2\tau(t)}} \sqrt{\tau(t)}\eta(x, t)\Big),$$

where $\eta(x, t) = sech(\omega_0 t)x$ *and* $\tau(t) = \frac{\tanh(\omega_0 t)}{\omega_0}$.

2.8. The Neumann Problem for Varible Parametric Parabolic Equation on Semi-infinite Line

In this section we consider that IBVP for variable parametric parabolic equation with Neumann BC.

$$\begin{cases} \Phi_t = \frac{1}{2\mu(t)}\Phi_{xx} + \frac{\mu(t)\omega^2(t)}{2} x^2 \Phi, & x > 0, \quad t_0 < t < T, \\ \Phi(x, t_0) = A(x), & x > 0, \\ \Phi_x(0, t) = B_N(t), & t_0 < t < T, \end{cases} \tag{2.126}$$

where $A(x)$ and $B_N(t)$ are given functions. Then IBVP (2.126) has solution of the form

$$\Phi(x, t) = \sqrt{\frac{r_0(t)}{r(t)}} \exp\Big[- \frac{\mu(t)\dot{r}(t)}{2r(t)} x^2 \Big] \varphi(\eta(x, t), \tau(t)), \tag{2.127}$$

where $r(t)$ is the solution of IVP $\ddot{r} + \frac{\dot{\mu}(t)}{\mu(t)}\dot{r} + \omega^2(t)r = 0, \quad r(t_0) = r_0 \neq 0, \quad \dot{r}(t_0) = 0,$ with $\eta(x,t) = \frac{r(t_0)}{r(t)}x, \quad \tau(t) = r_0^2 \int_{t_0}^{t} \frac{d\xi}{\mu(\xi)r^2(\xi)}$, and $\varphi(\eta,\tau)$ satisfies the following IBVP for the HE,

$$\begin{cases} \varphi_\tau = \frac{1}{2}\varphi_{\eta\eta}, \quad \eta > 0, \quad 0 < \tau < \tau(T), \\ \varphi(\eta, 0) = \Phi(\eta, t_0), \quad \eta > 0, \\ \varphi_\eta(0,\tau) = \sqrt{\left(\frac{r(t(\tau))}{r_0}\right)^3}\Phi_x(0, t(\tau)), \quad 0 < \tau < \tau(T). \end{cases} \tag{2.128}$$

Proof:

Using the ansatz

$$\Phi(x,t) = e^{\frac{g(t)-\rho(t)x^2}{2}}\varphi(e^{g(t)}x, \tau(t)),$$

we can show that initial condition $\Phi(x,t)\mid_{t=t_0} = \Phi(x,t_0)$ is easily transformed to the initial condition $\varphi(\eta,0) = \Phi(\eta,t_0)$. And Neumann BC for (2.126) is transformed to Neumann BC for (2.128).

$$\varphi_\eta(0,\tau) = \exp\left[-\frac{3}{2}g(t(\tau))\right]\Phi_x(0, t(\tau)),$$

where we use that $\mu(t) > 0$ and $r(t) > 0$, so that $\tau(t) = r_0^2 \int_{t_0}^{t} \frac{d\xi}{\mu(\xi)r^2(\xi)}$ is strictly increasing and thus its inverse $t(\tau)$ exists. Thus, IBVP (2.126) for the variable parametric Heat equation is transformed to the IBVP (2.128) for the standard HE. Thus, solution of the problem (2.126) is explicitly obtained in terms of solution of $\varphi(\eta,\tau)$ to the standard HE (2.128) and solution $r(t)$ of the IVP for the linear ODE (2.113). ■

2.9. Robin Boundary Condition for Variable Parametric Parabolic Equation on Semi-infinite Line

In this section we consider that IBVP for variable parametric parabolic equation with Robin BC.

$$\begin{cases} \Phi_t = \frac{1}{2\mu(t)}\Phi_{xx} + \frac{\mu(t)\omega^2(t)}{2}x^2\Phi, \quad x > 0, \quad t_0 < t < T, \\ \Phi(x,t_0) = A(x), \quad x > 0, \\ \alpha(t)\Phi_x(0,t) + \beta(t)\Phi(0,t) = h(t), \quad t_0 < t < T, \end{cases} \tag{2.129}$$

where $A(x)$, $\alpha(t)$, $\beta(t)$ and $h(t)$ are given functions, $\alpha(t)$ and $\beta(t)$ are not zero simultaneously. Then the IBVP (2.129) is transformed to the following IBVP for standard Heat equation by ansatz which we define in previous sections,

$$\begin{cases} \varphi_\tau = \frac{1}{2}\varphi_{\eta\eta}, \quad 0 < \eta < \infty, \quad 0 < \tau < \tau(T), \\ \varphi(\eta, 0) = \Phi(\eta, t_0), \quad 0 < \eta < \infty, \\ \alpha(t(\tau)) \sqrt{\left(\frac{r_0}{r(t)}\right)^3} \varphi_\eta(0, \tau) + \beta(t(\tau)) \sqrt{\frac{r_0}{r(t)}} \varphi(0, \tau) = h(t(\tau)), \quad 0 < \tau < \tau(T), \end{cases} \tag{2.130}$$

where we use

$$\varphi(0, \tau) = \exp\left[-\frac{1}{2} g(t(\tau)) \right] \Phi(0, t(\tau)), \tag{2.131}$$

$$\varphi_\eta(0, \tau) = \exp\left[-\frac{3}{2} g(t(\tau)) \right] \Phi_x(0, t(\tau)). \tag{2.132}$$

We see that variable parametric parabolic problem with Robin BC is reduced to the Heat problem with Robin BC easily. In previous section, we have shown how to find the solution of Heat problem with Robin BC. Thus, solution of the problem (2.129) is explicitly obtained in terms of solution of $\varphi(\eta, \tau)$ to HE (2.130) and solution $r(t)$ of the IVP for the linear ODE as

$$\Phi(x, t) = \sqrt{\frac{r_0(t)}{r(t)}} \exp\left[-\frac{\mu(t)\dot{r}(t)}{2r(t)} x^2 \right] \varphi(\eta(x, t), \tau(t)).$$

CHAPTER 3

THE IBVP FOR BURGERS EQUATION ON SEMI-INFINITE LINE

In this chapter, we consider Burgers equation with three types of boundary conditions. Firstly, we obtain solution of the problem with Dirichlet boundary condition on semi-infinite line. Then, we investigate the IBVP with Neumann boundary condition and finally we consider the problem with special nonlinear boundary condition for Burgers equation on semi-infinite line.

3.1. The IBVP with Dirichlet Boundary Condition

Consider the IBVP for Burgers equation on semi-infinite line

$$\begin{cases} V_\tau + VV_\eta = \frac{1}{2}V_{\eta\eta}, & 0 < \eta < \infty, \quad \tau > 0, \\ V(\eta, 0) = f(\eta), & 0 < \eta < \infty \\ V(0, \tau) = \beta(\tau), & \tau > 0 \qquad (Dirichlet\ \ condition), \end{cases} \tag{3.1}$$

where $f(\eta)$ and $g(\tau)$ are given functions. We will use two ways for solving this IBVP.

•First Way -Direct Cole-Hopf :
Applying directly the Cole-Hopf transform $V(\eta, \tau) = -\varphi_\eta(\eta, \tau)/\varphi(\eta, \tau)$ to problem (3.1), we obtain the corresponding IBVP for the Heat equation with Robin boundary condition,

$$\begin{cases} \varphi_\tau = \frac{1}{2}\varphi_{\eta\eta}, & 0 < \eta < \infty, \quad \tau > 0, \\ \varphi(\eta, 0) = \exp\left[-\int^\eta V(\eta', 0)d\eta'\right], & 0 < \eta < \infty, \\ \varphi_\eta(0, \tau) + \beta(\tau)\varphi(0, \tau) = 0, & \tau > 0, \quad (Robin\ BC), \end{cases} \tag{3.2}$$

where $B(\tau)$ is not identically zero. The boundary condition is directly obtained using $V(0, \tau) \equiv -\varphi_\eta(0, \tau)/\varphi(0, \tau) = \beta(\tau)$. From Chapter 2, we know how to solve Heat problem with Robin BC using two approaches.

(i) Dirichlet approach : Assume we know $\varphi(0,\tau) = H(\tau)$, then solution for the problem (3.2) is given by

$$\varphi(\eta,\tau) = \int_0^\infty G(\eta,\xi,\tau)e^{-\int_0^\xi V(\eta',0)d\eta'}d\xi - \int_0^\tau K_\eta(\eta,\tau-\tau')H(\tau')d\tau', \tag{3.3}$$

where $H(\tau)$ is obtained by solving the integral equation

$$H(\tau) = -\frac{2}{\beta(\tau)}\Big[\int_0^\infty \Big(\frac{\xi}{\tau}\Big)\frac{e^{-\frac{\xi^2}{2\tau}}}{\sqrt{2\pi\tau}}e^{-\int^\xi V(\eta',0)d\eta'}d\xi\Big] + \frac{1}{\beta(\tau)}\Big[\int_0^\tau \frac{H(\tau')}{\sqrt{2\pi(\tau-\tau')^3}}d\tau'\Big]. \tag{3.4}$$

Then, solution of Burgers problem (3.1) becomes

$$V(\eta,\tau) = -\frac{\int_0^\infty G_\eta(\eta,\xi,\tau)e^{-\int^\xi V(\eta',0)d\eta'}d\xi - \int_0^\tau K_{\eta\eta}(\eta,\tau-\tau')H(\tau')d\tau'}{\int_0^\infty G(\eta,\xi,\tau)e^{-\int^\xi V(\eta',0)d\eta'}d\xi - \int_0^\tau K_\eta(\eta,\tau-\tau')H(\tau')d\tau'}, \tag{3.5}$$

where $K(\eta,\tau)$ is the Heat kernel, $G(\eta,\xi,\tau) = K(\eta-\xi,\tau) - K(\eta+\xi,\tau)$ is the Dirichlet heat kernel. Then explicitly the solution takes the form

$$V(\eta,\tau) = \frac{\int_0^\infty \Big(\frac{(\frac{\eta-\xi}{\tau})e^{-\frac{(\eta-\xi)^2}{2\tau}} - (\frac{\eta+\xi}{\tau})e^{-\frac{(\eta+\xi)^2}{2\tau}}}{\sqrt{2\pi\tau}}\Big)e^{-\int^\xi V(\eta',0)d\eta'}d\xi - \int_0^\tau \Big[\frac{e^{-\frac{\eta^2}{2(\tau-\tau')}}}{\sqrt{2\pi(\tau-\tau')^3}} - \eta^2\frac{e^{-\frac{\eta^2}{2(\tau-\tau')}}}{\sqrt{2\pi(\tau-\tau')^5}}\Big]H(\tau')d\tau'}{\int_0^\infty \Big(\frac{e^{-\frac{(\eta-\xi)^2}{2\tau}} - e^{-\frac{(\eta+\xi)^2}{2\tau}}}{\sqrt{2\pi\tau}}\Big)e^{-\int^\xi V(\eta',0)d\eta'}d\xi + \int_0^\tau \Big(\frac{\eta}{\tau-\tau'}\Big)\frac{e^{-\frac{\eta^2}{2(\tau-\tau')}}}{\sqrt{2\pi(\tau-\tau')}}H(\tau')d\tau'} \tag{3.6}$$

If we find a solution of integral equation (3.4) for $H(\tau)$, then we obtain solution (3.6) of IBVP (3.1) for standard Burgers equation.

(ii) Neumann approach : Assume we know $\varphi_\eta(0,\tau) = Q(\tau)$, then the solution is

$$\varphi(\eta,\tau) = \int_0^\infty N(\eta,\xi,\tau)\varphi(\xi,0)d\xi - \int_0^\tau K(\eta,\tau-\tau')\varphi_\eta(0,\tau')d\tau', \tag{3.7}$$

where $Q(\tau)$ is determined by solving the following integral equation

$$Q(\tau) = -2\beta(\tau)\Big[\int_0^\infty \frac{e^{-\frac{\xi^2}{2\tau}}}{\sqrt{2\pi\tau}} e^{-\int^\xi V(\eta',0)d\eta'} d\xi\Big] + \beta(\tau)\Big[\int_0^\tau \frac{Q(\tau')}{\sqrt{2\pi(\tau-\tau')}} d\tau'\Big]. \tag{3.8}$$

Then, the corresponding solution of the IBVP (3.1) is of the form

$$V(\eta,\tau) = -\frac{\int_0^\infty N_\eta(\eta,\xi,\tau)\varphi(\xi,0)d\xi - \int_0^\tau K_\eta(\eta,\tau-\tau')Q(\tau')d\tau'}{\int_0^\infty N(\eta,\xi,\tau)\varphi(\xi,0)d\xi - \int_0^\tau K(\eta,\tau-\tau')Q(\tau')d\tau'}, \tag{3.9}$$

where $K(\eta,\tau)$ is the Heat kernel, and $N(\eta,\xi,\tau) = K(\eta-\xi,\tau) + K(\eta+\xi,\tau)$ is the Neumann heat kernel. Then explicitly the solution is given as follows

$$V(\eta,\tau) = \frac{\int_0^\infty \Big(\frac{(\frac{\eta-\xi}{\tau})e^{-\frac{(\eta-\xi)^2}{2\tau}} + (\frac{\eta+\xi}{\tau})e^{-\frac{(\eta+\xi)^2}{2\tau}}}{\sqrt{2\pi\tau}}\Big) e^{-\int^\xi V(\eta',0)d\eta'} d\xi - \int_0^\tau (\frac{\eta}{\tau-\tau'}) \frac{e^{-\frac{\eta^2}{2(\tau-\tau')}}}{\sqrt{2\pi(\tau-\tau')}} Q(\tau')d\tau'}{\int_0^\infty \Big(\frac{e^{-\frac{(\eta-\xi)^2}{2\tau}} + e^{-\frac{(\eta+\xi)^2}{2\tau}}}{\sqrt{2\pi\tau}}\Big) e^{-\int^\xi V(\eta',0)d\eta'} d\xi - \int_0^\tau \frac{e^{-\frac{\eta^2}{2(\tau-\tau')}}}{\sqrt{2\pi(\tau-\tau')}} Q(\tau')d\tau'}.$$

•**Second way - Generalized Cole-Hopf :**

Now we briefly outline the way of solving the IBVP (3.1) by generalized Cole-Hopf transform, which was used by **[4]**. The generalized Cole-Hopf transformation is given by

$$\begin{aligned}
&V(\eta,\tau) = -\frac{\Psi(\eta,\tau)}{\Big(C(\tau) + \int_0^\eta \Psi(\eta',\tau)d\eta'\Big)} = -\frac{\partial}{\partial\eta}\Big[\ln[C(\tau) + \int_0^\eta \Psi(\eta',\tau)d\eta']\Big],\\
&\textit{or equivalently}\\
&\Psi(\eta,\tau) = -C(\tau)V(\eta,\tau)\exp[-\int_0^\eta V(\eta',\tau)d\eta'], \qquad (3.10)\\
&\textit{with}\\
&C(0) = 1,\\
&\dot{C}(\tau) = \tfrac{1}{2}\Psi_\eta(0,\tau).
\end{aligned}$$

Under the generalized Cole-Hopf, the IBVP (3.1) for BE transforms to Heat problem

$$\begin{cases} \Psi_\tau = \frac{1}{2}\Psi_{\eta\eta}, \quad 0 < \eta < \infty, \quad \tau > 0, \\ \Psi(\eta, 0) = -V(\eta, 0)e^{-\int^{\eta} V(\eta',0)d\eta'}, \qquad 0 < \eta < \infty, \\ \Psi(0, \tau) = -C(\tau)V(0, \tau), \qquad \tau > 0, \\ \textit{and} \\ \dot{C}(\tau) = \frac{1}{2}\Psi_\eta(0, \tau). \end{cases} \tag{3.11}$$

where $C(\tau)$ is unknown. Following the work of **[4]**, assume temporary we know $C(\tau)$. Then we have IBVP with the Dirichlet BC for the Heat equation. The solution to this Dirichlet Heat problem is given by

$$\Psi(\eta, \tau) = \int_0^\infty G(\eta, \xi, \tau)\Psi(\xi, 0)d\xi - \int_0^\tau K_\eta(\eta, \tau - \tau')\Psi(0, \tau')d\tau'. \tag{3.12}$$

From solution (3.12) we can obtain

$$\begin{aligned} \Psi_\eta(\eta, \tau) &= \int_0^\infty G_\eta(\eta, \xi, \tau)\Psi(\xi, 0)d\xi + 2\int_0^\tau K_{\tau'}(\eta, \tau - \tau')\Psi(0, \tau')d\tau', \\ \Psi_\eta(0, \tau) &= \int_0^\infty G_\eta(0, \xi, \tau)\Psi(\xi, 0)d\xi + 2\int_0^\tau K_{\tau'}(0, \tau - \tau')\Psi(0, \tau')d\tau', \\ &= 2\int_0^\infty \left(\frac{\xi}{\tau}\right)\frac{e^{-\frac{\xi^2}{2\tau}}}{\sqrt{2\pi\tau}}\Psi(\xi, 0) + \int_0^\tau \frac{\Psi(0, \tau')}{\sqrt{2\pi(\tau - \tau')^3}}d\tau'. \end{aligned}$$

By using $\Psi(0, \tau) = -C(\tau)V(0, \tau)$ and $\Psi(\xi, 0) = -V(\eta, 0)e^{-\int^{\xi} V(\eta',0)d\eta'}$ and the relation $\Psi_\eta(0, \tau) = 2\dot{C}(\tau)$, we have the following integro-differential equation for the unknown function of $C(\tau)$ i.e.

$$\dot{C}(\tau) = -\int_0^\infty \left(\frac{\xi}{\tau}\right)\frac{e^{-\frac{\xi^2}{2\tau}}}{\sqrt{2\pi\tau}}V(\xi, 0)e^{-\int^{\xi} V(\eta',0)d\eta'}d\xi - \frac{1}{2}\int_0^\tau \frac{C(\tau')V(0, \tau')}{\sqrt{2\pi(\tau - \tau')^3}}d\tau'.$$

This integro-differential equation can be transformed to integral equation for $C(\tau)$. Thus, we see that solving the IBVP for the Burgers equation with Dirichlet BC (3.1) again reduces to the problem of solving a linear integral equation where solution of the problem is given

$$V(\eta,\tau) = -\frac{\Psi(\eta,\tau)}{C(\tau) + \int_0^\eta \Psi(\eta',\tau)d\eta'}.$$

Comparison: The relation between both approaches can be easily established if we let $\Psi(\eta,\tau) = \varphi_\eta(\eta,\tau)$. Using directly the Cole-Hopf transform, we have seen that the Dirichlet BC for Burgers equation transforms to Robin BC for the corresponding heat problem. On the other hand, using the generalized Cole-Hopf the Dirichlet BC for the Burgers problem was transformed again to Dirichlet BC for the heat equation. However, at the end, both approaches lead to solving integral equations. Indeed, it is not difficult to see that the heat IBVP's (3.2) and (3.11) are related by $\Psi = \varphi_\eta$. Then, we have $\Psi_\eta = \varphi_{\eta\eta} = 2\varphi_\tau$, which implies $\dot{C}(\tau) = (1/2)\Psi_\eta(0,\tau) = \varphi_\tau(0,\tau)$. Then, $C(\tau) = c\varphi(0,\tau)$, where c is constant and $\varphi_\eta(0,\tau) = -C(\tau)V(0,\tau)$ implies $V(0,\tau)\varphi(0,\tau) + \varphi_\eta(0,\tau) = 0$ which is the Robin BC for (3.2).

3.2. The IBVP with Neumann Boundary Condition

Consider the IBVP

$$\begin{cases} V_\tau + VV_\eta = \frac{1}{2}V_{\eta\eta}, & 0 < \eta < \infty, \quad \tau > 0, \\ V(\eta,0) = f(\eta), & 0 < \eta < \infty, \\ V_\eta(0,\tau) = h(\tau), & \tau > 0, \end{cases} \tag{3.13}$$

where $f(\eta)$ and $h(\tau)$ are given functions. Again we shall give two ways to solve the above problem.

• **First way - Direct Cole-Hopf :**

By Cole-Hopf, the IBVP (3.13) reduces to Heat problem with nonlinear BC

$$\begin{cases} \varphi_\tau = \frac{1}{2}\varphi_{\eta\eta}, \quad 0<\eta<\infty, \quad \tau>0, \\ \varphi(\eta,0) = \exp\left[-\int^{\eta} V(\eta',0)d\eta'\right], \quad 0<\eta<\infty, \\ \varphi_\eta^2(0,\tau) - 2\varphi_\tau(0,\tau)\varphi(0,\tau) - h(\tau)\varphi^2(0,\tau) = 0, \quad \tau>0. \end{cases} \tag{3.14}$$

To transform the Neumann BC in (3.13), we use

$$V(\eta,\tau) = -\frac{\varphi_\eta(\eta,\tau)}{\varphi(\eta,\tau)}, \tag{3.15}$$

$$V_\eta(\eta,\tau) = -\frac{\varphi_{\eta\eta}(\eta,\tau)}{\varphi(\eta,\tau)} + \left(\frac{\varphi_\eta(\eta,\tau)}{\varphi(\eta,\tau)}\right)^2, \tag{3.16}$$

$$V_\eta(0,\tau) = -\frac{\varphi_{\eta\eta}(0,\tau)}{\varphi(0,\tau)} + \left(\frac{\varphi_\eta(0,\tau)}{\varphi(0,\tau)}\right)^2 = -2\frac{\varphi_\tau}{\varphi} + \left(\frac{\varphi_\eta}{\varphi}\right)^2, \tag{3.17}$$

$$\tag{3.18}$$

which implies the nonlinear BC in (3.14). Here $\varphi(0,\tau)$, $\varphi_\eta(0,\tau)$ and $\varphi_\tau(0,\tau)$ are unknown functions, but they are related with the nonlinear BC. To solve the Heat problem (3.14) by Dirichlet Approach, assume temporary we know $\varphi(0,\tau)$=$H(\tau)$. Then solution of (3.14) is given by

$$\begin{aligned} \varphi(\eta,\tau) = &\int_0^\infty \left(\frac{e^{-\frac{(\eta-\xi)^2}{2\tau}} - e^{-\frac{(\eta+\xi)^2}{2\tau}}}{\sqrt{2\pi\tau}}\right) e^{-\int^{\xi} V(\eta',0)d\eta'} d\xi \\ &+ \int_0^\tau \left(\frac{\eta}{\tau-\tau'}\right) \frac{e^{-\frac{\eta^2}{2(\tau-\tau')}}}{\sqrt{2\pi(\tau-\tau')}} H(\tau')d\tau'. \end{aligned} \tag{3.19}$$

In fact, we need to solve Heat problem with the nonlinear boundary condition (3.14). Thus we obtain $\varphi_\eta(0,\tau)$ from solution (3.19),

$$\varphi_\eta(\eta,\tau) = \int_0^\infty \frac{(\frac{\eta+\xi}{\tau})e^{-\frac{(\eta+\xi)^2}{2\tau}} - (\frac{\eta-\xi}{\tau})e^{-\frac{(\eta-\xi)^2}{2\tau}}}{\sqrt{2\pi\tau}} e^{-\int^{\xi} V(\eta',0)d\eta'} d\xi - 2\int_0^\tau K_{\tau'}(\eta,\tau-\tau')H(\tau')d\tau',$$

$$\varphi_\eta(0,\tau) = 2\int_0^\infty \left(\frac{\xi}{\tau}\right)\left(\frac{e^{-\frac{\xi^2}{2\tau}}}{\sqrt{2\pi\tau}}\right) e^{-\int^{\xi} V(\eta',0)d\eta'} d\xi - \int_0^\tau \frac{H(\tau')}{\sqrt{2\pi(\tau-\tau')^3}} d\tau'. \tag{3.20}$$

Substituting $\varphi_\eta(0,\tau)$, $\varphi(0,\tau) = H(\tau)$ and $\varphi_\tau(0,\tau) = \dot{H}(\tau)$ into nonlinear boundary condition (3.14), we obtain the following

$$2\dot{H}(\tau)H(\tau) + h(\tau)H^2(\tau) - \left(F(\tau) - \int_0^\tau \frac{H(\tau')}{\sqrt{2\pi(\tau-\tau')^3}}d\tau'\right)^2 = 0, \tag{3.21}$$

where

$$\begin{aligned} F(\tau) &= 2\int_0^\infty \left(\frac{\xi}{\tau}\right)\frac{e^{-\frac{\xi^2}{2\tau}}e^{-\int^\xi V(\eta',0)d\eta'}}{\sqrt{2\pi\tau}}d\xi, \\ 2\dot{H}(\tau)H(\tau) &= \frac{d}{d\tau}[H^2(\tau)]. \end{aligned}$$

Then we have

$$\frac{d}{d\tau}[H^2(\tau)] + h(\tau)H^2(\tau) - \left(F(\tau) - \int_0^\tau \frac{H(\tau')}{\sqrt{2\pi(\tau-\tau')^3}}d\tau'\right)^2 = 0, \tag{3.22}$$

which is a nonlinear integro-differential equation for the unknown function $H(\tau)$. Thus, solving Heat problem (3.14) is equivalent to solving the nonlinear integral equation (3.21). Then the corresponding Burgers solution of the problem (3.13) is given by

$$V(\eta,\tau) = \frac{\int_0^\infty \left(\frac{(\frac{\eta-\xi}{\tau})e^{-\frac{(\eta-\xi)^2}{2\tau}} - (\frac{\eta+\xi}{\tau})e^{-\frac{(\eta+\xi)^2}{2\tau}}}{\sqrt{2\pi\tau}}\right)e^{-\int^\xi V(\eta',0)d\eta'}d\xi + \int_0^\tau \left[\frac{e^{-\frac{\eta^2}{2(\tau-\tau')}}}{\sqrt{2\pi(\tau-\tau')^3}} - \eta^2\frac{e^{-\frac{\eta^2}{2(\tau-\tau')}}}{\sqrt{2\pi(\tau-\tau')^5}}\right]H(\tau')d\tau'}{\int_0^\infty \left(\frac{e^{-\frac{(\eta-\xi)^2}{2\tau}} - e^{-\frac{(\eta+\xi)^2}{2\tau}}}{\sqrt{2\pi\tau}}\right)e^{-\int^\xi V(\eta',0)d\eta'}d\xi + \int_0^\tau \left(\frac{\eta}{\tau-\tau'}\right)\frac{e^{-\frac{\eta^2}{2(\tau-\tau')}}}{\sqrt{2\pi(\tau-\tau')}}H(\tau')d\tau'}.$$

⋆ **Special Case:** If we have homogeneous Neumann BC in (3.13) , that's $V_\eta(0,\tau) = h(\tau) = 0$, then the BC of the corresponding Heat problem (3.14) becomes

$$\varphi_\eta^2(0,\tau) - \underbrace{2\varphi_\tau(0,\tau)\varphi(0,\tau)}_{(\varphi^2(0,\tau))_\tau} = 0. \tag{3.23}$$

Therefore, the solution of the Heat problem is given by (3.19), where $H(\tau)$ is determined by solving the nonlinear integro-differential equation of the form

$$\frac{d}{dt}[H^2(\tau)] - \left(F(\tau) - \int_0^\tau \frac{H(\tau')}{\sqrt{2\pi(\tau-\tau')^3}}d\tau'\right)^2 = 0, \tag{3.24}$$

where $F(\tau)$ is known.

•Second way - Generalized Cole-Hopf :

$$\begin{cases} V_\tau + VV_\eta = \frac{1}{2}V_{\eta\eta}, & 0<\eta<\infty, \quad \tau>0, \\ V(\eta,0) = f(\eta), & 0<\eta<\infty, \\ V_\eta(0,\tau) = h(\tau), & \tau>0. \end{cases} \tag{3.25}$$

By Generalized Cole-Hopf, the IBVP (3.25) transforms to the IBVP for the Heat equation **[6]**, i.e

$$\begin{cases} \Psi_\tau = \frac{1}{2}\Psi_{\eta\eta}, \quad 0<\eta<\infty, \quad \tau>0, \\ \Psi(\eta,0) = -V(\eta,0)\exp\left[-\int^\eta V(\eta',0)d\eta'\right], \quad 0<\eta<\infty, \\ \Psi^2(0,\tau) - C(\tau)\Psi_\eta(0,\tau) - C^2(\tau)V_\eta(0,\tau) = 0, \quad \tau>0, \\ with \\ C(0) = 1, \\ \dot{C}(\tau) = \frac{1}{2}\Psi_\eta(0,\tau), \end{cases} \tag{3.26}$$

or equivalently using the relation $\Psi_\eta(0,\tau) = 2\dot{C}(\tau)$, the boundary condition (3.26) becomes $\Psi^2(0,\tau) - 2\dot{C}(\tau)C(\tau) - C^2(\tau)h(\tau) = 0$. We now assume temporarily that $C(\tau)$ is known. Then we have following IBVP

$$\begin{cases} \Psi_\tau = \frac{1}{2}\Psi_{\eta\eta}, \quad 0<\eta<\infty, \quad \tau>0, \\ \Psi(\eta,0) = -V(\eta,0)\exp\left[-\int^\eta V(\eta',0)d\eta'\right], \quad 0<\eta<\infty, \\ \Psi_\eta(0,\tau) = 2\dot{C}(\tau), \quad \tau>0. \end{cases} \tag{3.27}$$

Then the solution to the problem (3.27) with Neumann BC is given

$$\Psi(\eta,\tau) = \int_0^\infty N(\eta,\xi,\tau)\Psi(\xi,0)d\xi - 2\int_0^\tau K(\eta,\tau-\tau')\dot{C}(\tau')d\tau'. \tag{3.28}$$

From solution (3.28) we can obtain $\Psi(0,\tau)$ i.e

$$\Psi(0,\tau) = \int_0^\infty N(0,\xi,\tau)\Psi(\xi,0)d\xi - 2\int_0^\tau K(0,\tau-\tau')\dot{C}(\tau')d\tau', \tag{3.29}$$

or explicitly

$$\Psi(0,\tau) = 2\int_0^\infty \frac{e^{-\frac{\xi^2}{2\tau}}}{\sqrt{2\pi\tau}}\Psi(\xi,0)d\xi - 2\int_0^\tau \frac{\dot{C}(\tau')}{\sqrt{2\pi(\tau-\tau')}}d\tau'. \tag{3.30}$$

Substituting (3.30) and $\Psi_\eta(0,\tau) = 2\dot{C}(\tau)$ into nonlinear BC of (3.26), we obtain

$$2C(\tau)\dot{C}(\tau) + C^2(\tau)h(\tau) - \left(2\int_0^\infty \frac{e^{\frac{\xi^2}{2\tau}}}{\sqrt{2\pi\tau}}\Psi(\xi,0)d\xi - 2\int_0^\tau \frac{\dot{C}(\tau')}{\sqrt{2\pi(\tau-\tau')}}d\tau'\right)^2 = 0. \tag{3.31}$$

Notice that, again the IBVP is reduced to solving the nonlinear integral equation (3.31) which determines $C(\tau)$ together with $C(0) = 1$. Thus explicitly, the solution to the problem (3.25) is given

$$V(\eta,\tau) = -\frac{\Psi(\eta,\tau)}{C(\tau) + \int_0^\eta \Psi(\eta',\tau)d\eta'}.$$

Comparison: Using both, the direct Cole-Hopf transform and generalized Cole-Hopf, we have seen that the Neumann BC for Burgers equation transform to nonlinear BC for the corresponding Heat problem. At the end, both approaches lead to solving the same nonlinear integral equation.

3.3. Burgers Equation with Special Nonlinear Boundary Condition-I

We consider IBVP for Burgers equation with special nonlinear BC defined by

$$\begin{cases} V_\tau + VV_\eta = \frac{1}{2}V_{\eta\eta}, \quad 0 < \eta < \infty, \quad \tau > 0, \\ V(\eta, 0) = f(\eta), \;\; 0 < \eta < \infty, \\ V^2(0,\tau) - V_\eta(0,\tau) = g(\tau), \qquad \tau > 0, \end{cases} \tag{3.32}$$

which is linearized to the following IBVP

$$\begin{cases} \varphi_\tau = \frac{1}{2}\varphi_{\eta\eta}, \quad 0 < \eta < \infty, \quad \tau > 0, \\ \varphi(\eta, 0) = \exp\left[-\int^\eta V(\eta', 0)d\eta'\right], \quad 0 < \eta < \infty, \\ g(\tau)\varphi(0,\tau) - 2\varphi_\tau(0,\tau) = 0, \quad \tau > 0. \end{cases} \tag{3.33}$$

Again we solve the problem (3.33) by two ways.

- **First way - Direct Cole-Hopf :**

By Cole-Hopf, the IBVP (3.32) reduces to the Heat problem with Dirichlet BC

$$\begin{cases} \varphi_\tau = \frac{1}{2}\varphi_{\eta\eta}, \quad 0 < \eta < \infty, \quad \tau > 0, \\ \varphi(\eta, 0) = \exp\left[-\int^\eta V(\eta', 0)d\eta'\right], \quad 0 < \eta < \infty, \\ \varphi(0,\tau) = \exp\left[\frac{1}{2}\int_0^\tau g(\tau')d\tau'\right], \quad \tau > 0. \end{cases} \tag{3.34}$$

Note that $V^2(0,\tau) - V_\eta(0,\tau) = -(\frac{\varphi_\eta(0,\tau)}{\varphi(0,\tau)})^2 - [-\frac{\varphi_{\eta\eta}(0,\tau)}{\varphi(0,\tau)} + (\frac{\varphi_\eta(0,\tau)}{\varphi}(0,\tau))^2] = g(\tau)$ which implies $2\varphi_\tau/\varphi = g(\tau)$. Integrating the last equation w.r.t τ, we get BC (3.34). Then the solution of this Dirichlet Heat problem is given by

$$\varphi(\eta,\tau) = \int_0^\infty G(\eta,\xi,\tau)\varphi(\xi,0)d\xi - \int_0^\tau K_\eta(\eta,\tau-\tau')e^{\frac{1}{2}\int_0^{\tau'} g(t)dt}d\tau'.$$

And explicitly the corresponding solution for Burgers problem is the following

$$V(\eta,\tau)=\frac{\int_0^\infty\Big(\frac{(\eta-\xi)e^{-\frac{(\eta-\xi)^2}{2\tau}}-(\eta+\xi)e^{-\frac{(\eta+\xi)^2}{2\tau}}}{\sqrt{2\pi\tau^3}}\Big)e^{-\int^\xi f(\eta')d\eta'}d\xi+2\int_0^\tau\left[\frac{e^{-\frac{\eta^2}{2(\tau-\tau')}}}{\sqrt{2\pi(\tau-\tau')^3}}-\eta^2\frac{e^{-\frac{\eta^2}{2(\tau-\tau')}}}{\sqrt{2\pi(\tau-\tau')^5}}\right]e^{\frac{1}{2}\int_0^{\tau'}g(t)dt}d\tau'}{\int_0^\infty\Big(\frac{e^{-\frac{(\eta-\xi)^2}{2\tau}}-e^{-\frac{(\eta+\xi)^2}{2\tau}}}{\sqrt{2\pi\tau}}\Big)e^{-\int^\xi f(\eta')d\eta'}d\xi+\int_0^\tau\eta\frac{e^{-\frac{\eta^2}{2(\tau-\tau')}}}{\sqrt{2\pi(\tau-\tau')^3}}e^{\frac{1}{2}\int_0^{\tau'}g(t)dt}d\tau'}.$$

•Second way - Generalized Cole-Hopf :

By Generalized Cole-Hopf, the IBVP (3.32) transforms to the IBVP for the HE **[1]**

$$\begin{cases}\Psi_\tau=\frac{1}{2}\Psi_{\eta\eta}, \quad 0<\eta<\infty, \quad \tau>0,\\ \Psi(\eta,0)=-f(\eta)\exp\left[-\int^\eta V(\eta',0)d\eta'\right], \quad 0<\eta<\infty,\\ \Psi_\eta(0,\tau)=-C(\tau)\Big(V_\eta(0,\tau)-V^2(0,\tau)\Big)\equiv C(\tau)g(\tau), \quad \tau>0.\\ with\\ C(0)=1,\\ \dot{C}(\tau)=\frac{1}{2}\Psi_\eta(0,\tau).\end{cases} \tag{3.35}$$

Notice that the relation $\dot{C}(\tau)=\frac{1}{2}\Psi_\eta(0,\tau)$ together with BC implies $C(\tau)=e^{\frac{1}{2}\int^\tau g(\tau')d\tau'}$, which is the same as $\varphi(0,\tau)$ found in the first way.

Comparison: By Cole-Hopf, we have seen that the Nonlinear BC for Burgers equation transforms to Dirichlet BC for the corresponding heat problem. On the other hand, using the generalized Cole-Hopf the nonlinear BC for the Burgers problem was transformed to Neumann BC for the Heat equation.

3.4. Burgers Equation with Special Nonlinear Boundary Condition-II

In this section, we consider IBVP for Burgers equation with other special nonlinear BC on semi-infinite line defined by

$$\begin{cases} V_\tau + VV_\eta = \frac{1}{2}V_{\eta\eta}, & 0 < \eta < \infty, \quad \tau > 0, \\ V(\eta, 0) = f(\eta), & 0 < \eta < \infty, \\ V^2(0,\tau) - \alpha(\tau)V(0,\tau) - V_\eta(0,\tau) = 0, & \tau > 0, \end{cases} \tag{3.36}$$

where $\alpha(\tau)$ and $f(\tau)$ are given functions. Then, problem (3.36) reduces to

$$\begin{cases} \varphi_\tau = \frac{1}{2}\varphi_{\eta\eta}, \quad 0 < \eta < \infty, \quad \tau > 0, \\ \varphi(\eta, 0) = \exp\left[-\int^\eta V(\eta', 0)d\eta'\right], \quad 0 < \eta < \infty, \\ \alpha(\tau)\varphi_\eta(0,\tau) - 2\varphi_\tau(0,\tau) = 0, \quad \tau > 0. \end{cases} \tag{3.37}$$

Assume temporary we know $\varphi(0,\tau) = H(\tau)$. Then we have Heat problem with Dirichlet BC which has solution

$$\varphi(\eta,\tau) = \int_0^\infty \Big(\frac{e^{-\frac{(\eta-\xi)^2}{2\tau}} - e^{-\frac{(\eta+\xi)^2}{2\tau}}}{\sqrt{2\pi\tau}}\Big)\varphi(\xi,0)d\xi + \int_0^\tau \Big(\frac{\eta}{\tau-\tau'}\Big)\frac{e^{-\frac{\eta^2}{2(\tau-\tau')}}}{\sqrt{2\pi(\tau-\tau')}}H(\tau')d\tau'. \tag{3.38}$$

We can obtain $\varphi_\eta(0,\tau)$ from above solution, that's

$$\varphi_\eta(0,\tau) = 2\int_0^\infty \Big(\frac{\xi}{\tau}\Big)\frac{e^{-\frac{\xi^2}{2\tau}}}{\sqrt{2\pi\tau}}\varphi(\xi,0)d\xi + \int_0^\tau \frac{H(\tau')}{\sqrt{2\pi(\tau-\tau')^3}}d\tau'. \tag{3.39}$$

Then substituting (3.39) and $\varphi_\tau = \dot{H}(\tau)$ into Robin BC in (3.37), we obtain

$$\alpha(\tau)\Big(2\int_0^\infty \Big(\frac{\xi}{\tau}\Big)\frac{e^{-\frac{\xi^2}{2\tau}}}{\sqrt{2\pi\tau}}\varphi(\xi,0)d\xi + \int_0^\tau \frac{H(\tau')}{\sqrt{2\pi(\tau-\tau')^3}}d\tau'\Big) - 2\dot{H}(\tau) = 0,$$

equivalently

$$\dot{H}(\tau) = \alpha(\tau)\Big(\int_0^\infty \left(\frac{\xi}{\tau}\right) \frac{e^{-\frac{\xi^2}{2\tau}}}{\sqrt{2\pi\tau}} \varphi(\xi, 0) d\xi + \frac{1}{2} \int_0^\tau \frac{H(\tau')}{\sqrt{2\pi(\tau - \tau')^3}} d\tau' \Big).$$

CHAPTER 4

THE IBVP FOR INHOMOGENEOUS BURGERS EQUATION WITH TIME-VARIABLE COEFFICIENTS ON SEMI-INFINITE LINE

In this chapter, firstly we investigate the Dirichlet problem for inhomogeneous Burgers equation with time-variable coefficients on semi-infinite line. We show that solution of the Dirichlet problem for variable Burgers equation corresponds to either solution of the problem with Dirichlet boundary condition for standard Burgers equation or solution of the problem with Robin boundary condition for standard Heat equation. Some exactly solvable different Burgers models **[3]** are investigated for Dirichlet problem. Finally, we consider the Neumann problem for variable Burgers equation on semi-infinite line.

4.1. Dirichlet Problem for Inhomogeneous Burgers Equation with Time-variable Coefficients on Semi-infinite Line

In this section, we consider the IBVP for inhomogeneous Burgers equation with time-variable coefficients given by

$$\begin{cases} U_t + \frac{\dot{\mu}(t)}{\mu(t)}U + UU_x = \frac{1}{2\mu(t)}U_{xx} - \omega^2(t)x, \quad 0 < x < \infty, \ \ t_0 < t < T, \\ U(x, t_0) = F(x), \quad 0 < x < \infty, \\ U(0, t) = D(t), \quad t_0 < t < T, \end{cases} \tag{4.1}$$

where $\mu(t) > 0$ is continuously differentiable, $\omega^2(t)$ is a real-valued continuous function on $[t_0, T)$. Assume $D(t)$, $F(x)$ are sufficiently smooth and $F(x)$ is not increasing too fast as $x \to \infty$.

Proposition 4.1 *If for $t_0 \le t < T$ the function $r(t)$ is strictly positive (or strictly negative) solution of the IVP for the second order linear ODE*

$$\ddot{r} + \frac{\dot{\mu}(t)}{\mu(t)}\dot{r} + \omega^2(t) r = 0, \tag{4.2}$$

$$r(t_0) = r_0 \neq 0 \quad \dot{r}(t_0) = 0, \tag{4.3}$$

and

$$\eta(x,t) = \frac{r(t_0)}{r(t)} x, \quad \tau(t) = r_0^2 \int_{t_0}^{t} \frac{d\xi}{\mu(\xi) r^2(\xi)}, \tag{4.4}$$

then the IBVP (4.1) has solution in the following forms:

a)

$$U(x,t) = \frac{\dot{r}(t)}{r(t)} x + \frac{r(t_0)}{\mu(t) r(t)} V(\eta(x,t), \tau(t)), \tag{4.5}$$

where $V(\eta, \tau)$ satisfies the IBVP for the standard Burgers equation with Dirichlet BC

$$\begin{cases} V_\tau + V V_\eta = \frac{1}{2} V_{\eta\eta}, \quad 0 < \eta < \infty, \quad 0 < \tau < \tau(T), \\ V(\eta, 0) = \mu_0 U(\eta, t_0), \quad 0 < \eta < \infty, \\ V(0,\tau) = \left[\mu(t(\tau)) r(t(\tau))/r_0\right] U(0, t(\tau)), \quad 0 < \tau < \tau(T), \end{cases} \tag{4.6}$$

and $\tau = \tau(t) \Leftrightarrow t = t(\tau)$, $\mu_0 = \mu(t_0)$, $r_0 = r(t_0)$.

b)

$$U(x,t) = \frac{\dot{r}(t)}{r(t)} x - \frac{r_0}{\mu(t) r(t)} \frac{\varphi_\eta(\eta(x,t), \tau(t))}{\varphi(\eta(x,t), \tau(t))}, \tag{4.7}$$

where $\varphi(\eta, \tau)$ satisfies the IBVP for the Heat equation with Robin BC

$$\begin{cases} \varphi_\tau = \frac{1}{2} \varphi_{\eta\eta}, \quad 0 < \eta < \infty, \ 0 < \tau < \tau(T), \\ \varphi(\eta, 0) = \exp\left[-\mu_0 \int^{\eta} U(\eta', t_0) d\eta'\right], \quad 0 < \eta < \infty, \\ \left[r(t(\tau)) \mu(t(\tau)) U(0, t(\tau))\right] \varphi(0,\tau) + r_0 \varphi_\eta(0,\tau) = 0, \quad 0 < \tau < \tau(T). \end{cases} \tag{4.8}$$

Proof:

a) If the functions $\rho(t)$, $\tau(t)$ and $s(t)$ satisfy the nonlinear system of ordinary differential equations

$$\dot{\rho} + \frac{\rho^2}{\mu(t)} + \mu(t)\omega^2(t) = 0, \quad \rho(t_0) = 0, \tag{4.9}$$

$$\dot{\tau} - \frac{s^2}{\mu(t)} = 0, \quad \tau(t_0) = 0, \tag{4.10}$$

$$\dot{s} + \frac{\rho(t)}{\mu(t)s} = 0, \quad s(t_0) = 1, \tag{4.11}$$

and $V(\eta, \tau)$ satisfies the standard Burgers equation in (4.6), then

$$U(x,t) = \frac{\rho(t)x + s(t)V\big(s(t)x, \tau(t)\big)}{\mu(t)}, \tag{4.12}$$

satisfies the Burgers equation in (4.1), **[3]**. Notice that equation (4.9) is a nonlinear Riccati equation and substitution $\rho(t) = \mu(t)\dot{r}(t)/r(t)$ gives

$$\ddot{r}(t) + \frac{\dot{\mu}(t)}{\mu(t)}\dot{r}(t) + \omega^2(t)r(t) = 0, \tag{4.13}$$

then the system is easily solved as follows

$$\rho(t) = \mu(t)\frac{\dot{r}(t)}{r(t)}, \tag{4.14}$$

$$\tau(t) = r_0^2 \int_{t_0}^{t} \frac{d\xi}{\mu(\xi)r^2(\xi)}, \tag{4.15}$$

$$s(t) = \frac{r_0}{r(t)}, \tag{4.16}$$

where $r(t)$ is the solution IVP (4.2) and substituting back above functions into (4.12) gives solution in the form (4.5), **[3]**. Then, initial condition $U(x,t_0) = F(x)$ easily transforms to the initial condition $V(\eta, 0) = \mu_0 F(\eta)$ of for the standard BE. And Dirichlet boundary condition for the inhomogeneous BE $U(0,t) = D(t)$ transforms to Dirichlet boundary condition for standard BE

$$V(0,\tau) = [\mu(t(\tau))r(t(\tau))/r_0]U(0,t(\tau)) = [\mu(t(\tau))r(t(\tau))/r_0]D(t(\tau)), \quad 0 < \tau < \tau(T),$$

where we used that $\mu(t) > 0$ and $r^2(t) > 0$, so that $\tau(t) = r_0^2 \int_{t_0}^{t} \frac{d\xi}{\mu(\xi)r^2(\xi)}$ is strictly increasing continuous function on $[t_0, T)$ and thus its inverse $t(\tau)$ exists for $\tau \in [0, \tau(T))$. Thus, IBVP (4.1) for inhomogeneous BE transforms to the IBVP (4.6) for the standard BE, and solution $U(x,t)$ of the variable BE is explicitly obtained in terms of solution of $V(\eta,\tau)$ to the standard BE (4.6) and solution $r(t)$ of the IVP for the linear ODE (4.2).

Part **(b)** of the proposition follows directly from the Cole-Hopf transformation $V = -\varphi_\eta/\varphi$. Again, initial condition $V(\eta,0) = \mu_0 U(\eta,t_0)$ transform directly to initial condition for HE

$$\varphi(\eta,0) = \exp\left[-\mu_0 \int^{\eta} U(\eta',t_0)d\eta'\right].$$

However, by Cole-Hopf transformation $V = -\varphi_\eta/\varphi$, the Dirichlet BC for BE

$$V(0,\tau) = [\mu(t(\tau))r(t(\tau))/r_0]\, U(0,t(\tau)),$$

transforms to Robin boundary condition for HE

$$[r(t(\tau))\mu(t(\tau))U(0,t(\tau))]\,\varphi(0,\tau) + r_0\varphi_\eta(0,\tau) = 0.$$

Then the IBVP (4.1) for the BE transforms to the IBVP for the HE as (4.8). ■

Therefore, we see that solving the IBVP for inhomogeneous Burgers equation with Dirichlet BC is reduced to the problem of solving IBVP for Heat equation with Robin BC. To solve this Heat problem one can use different approaches. We write two of them.

• **First Approach (Dirichlet):**

Let $U(0,t) = D(t)$ be not identically zero. Assume temporary we know $\varphi(0,\tau) = H(\tau)$. Then from Chapter 3, we know that the Dirichlet IBVP

$$\begin{cases} \varphi_\tau = \frac{1}{2}\varphi_{\eta\eta}, \quad 0 < \eta < \infty, \quad 0 < \tau < \tau(T), \\ \varphi(\eta,0) = \exp\left[-\int^{\eta} \mu_0 U(\eta', t_0) d\eta'\right], \quad 0 < \eta < \infty, \\ \varphi(0,\tau) = H(\tau), \quad 0 < \tau < \tau(T), \end{cases} \tag{4.17}$$

has solution of the form

$$\varphi(\eta,\tau) = \int_0^\infty \underbrace{\left(\frac{e^{-\frac{(\eta-\xi)^2}{2\tau}} - e^{-\frac{(\eta+\xi)^2}{2\tau}}}{\sqrt{2\pi\tau}}\right)}_{G(\eta,\xi,\tau)} \underbrace{e^{-\int^{\xi} \mu_0 U(\eta',t_0)d\eta'}}_{\varphi(\xi,0)} d\xi + \int_0^\tau \underbrace{\left(\frac{\eta}{\tau-\tau'}\right)\frac{e^{-\frac{\eta^2}{2(\tau-\tau')}}}{\sqrt{2\pi(\tau-\tau')}}}_{-K_\eta} H(\tau')d\tau'.$$

From this solution we can obtain $\varphi_\eta(0,\tau)$ as follows

$$\varphi_\eta(0,\tau) = \int_0^\infty G_\eta(0,\xi,\tau)e^{-\int^{\xi} \mu_0 U(\eta',t_0)d\eta'} d\xi + 2\int_0^\tau K_{\tau'}(0,\tau-\tau')H(\tau')d\tau'. \tag{4.18}$$

Substituting (4.18) and $\varphi(0,\tau) = H(\tau)$ into Robin BC (4.8) we obtain

$$\begin{aligned} H(\tau) \quad = \quad & -\frac{r_0}{r(t(\tau))\mu(t(\tau))U(0,t(\tau))}\left(\int_0^\tau \frac{H(\tau')}{\sqrt{2\pi(\tau-\tau')^3}}d\tau'\right) \\ & - \frac{2r_0}{r(t(\tau))\mu(t(\tau))U(0,t(\tau))}\left(\int_0^\infty \left(\frac{\xi}{\tau}\right)\frac{e^{-\frac{\xi^2}{2\tau}}}{\sqrt{2\pi\tau}}e^{-\int^{\xi} \mu_0 U(\eta',t_0)d\eta'} d\xi\right). \end{aligned} \tag{4.19}$$

The equation (4.19) is an integral equation for the unknown function $H(\tau)$. Thus, solving Heat problem is equivalent to solving the integral equation. The corresponding standard Burgers solution is given by in closed form

$$V(\eta,\tau) = -\frac{\int_0^\infty G_\eta(\eta,\xi,\tau)e^{-\int^{\xi} \mu_0 U(\eta',t_0)d\eta'} d\xi + 2\int_0^\tau K_{\tau'}(\eta,\tau-\tau')H(\tau')d\tau'}{\int_0^\infty G(\eta,\xi,\tau)e^{-\int^{\xi} \mu_0 U(\eta',t_0)d\eta'} d\xi - \int_0^\tau K_\eta(\eta,\tau-\tau')H(\tau')d\tau'},$$

or explicitly

$$V(\eta,\tau)=\frac{\int_0^{\infty}(\frac{(\frac{\eta-\xi}{\tau})e^{-\frac{(\eta-\xi)^2}{2\tau}}-(\frac{\eta+\xi}{\tau})e^{-\frac{(\eta+\xi)^2}{2\tau}}}{\sqrt{2\pi\tau}})e^{-\int^{\xi}V(\eta',0)d\eta'}d\xi-\int_0^{\tau}\left(\frac{e^{-\frac{\eta^2}{2(\tau-\tau')}}}{\sqrt{2\pi(\tau-\tau')^3}}-\frac{\eta^2e^{-\frac{\eta^2}{2(\tau-\tau')}}}{\sqrt{2\pi(\tau-\tau')^5}}\right)H(\tau')d\tau'}{\int_0^{\infty}(\frac{e^{-\frac{(\eta-\xi)^2}{2\tau}}-e^{-\frac{(\eta+\xi)^2}{2\tau}}}{\sqrt{2\pi\tau}})e^{-\int^{\xi}V(\eta',0)d\eta'}d\xi+\int_0^{\tau}\left(\frac{\eta}{\tau-\tau'}\right)\frac{e^{-\frac{\eta^2}{2(\tau-\tau')}}}{\sqrt{2\pi(\tau-\tau')}}H(\tau')d\tau'}.$$

Therefore, using the above proposition, we can obtain formal solution of the IBVP (4.1) for inhomogeneous BE in terms of $r(t)$ of the linear ODE (4.9) and $\varphi(\eta,\tau)$

$$\begin{aligned}
U(x,t) &= \frac{\dot r(t)}{r(t)}x\\
&- \frac{r_0}{\mu(t)r(t)}\frac{\int_0^{\infty}G_{\eta}(\eta(x,t),\xi,\tau(t))e^{-\int^{\xi}\mu_0U(\eta',t_0)d\eta'}d\xi}{\int_0^{\infty}G(\eta(x,t),\xi,\tau(t))e^{-\int^{\xi}\mu_0U(\eta',t_0)d\eta'}d\xi-\int_0^{\tau(t)}K_{\eta}(\eta(x,t),\tau(t)-\tau')H(\tau')d\tau'}\\
&- \frac{r_0}{\mu(t)r(t)}\frac{2\int_0^{\tau(t)}K_{\tau'}(\eta(x,t),\tau(t)-\tau')H(\tau')d\tau'}{\int_0^{\infty}G(\eta(x,t),\xi,\tau(t))e^{-\int^{\xi}\mu_0U(\eta',t_0)d\eta'}d\xi-\int_0^{\tau(t)}K_{\eta}(\eta(x,t),\tau(t)-\tau')H(\tau')d\tau'},
\end{aligned}$$

or explicitly

$$\begin{aligned}
U(x,t) &= \frac{\dot r(t)}{r(t)} \qquad\qquad (4.20)\\
&+ \frac{r_0}{\mu(t)r(t)}\frac{\int_0^{\infty}\left(\frac{(\frac{\eta(x,t)-\xi}{\tau(t)})e^{-\frac{(\eta(x,t)-\xi)^2}{2\tau(t)}}-(\frac{\eta(x,t)+\xi}{\tau})e^{-\frac{(\eta(x,t)+\xi)^2}{2\tau(t)}}}{\sqrt{2\pi\tau(t)}}\right)e^{-\int^{\xi}\mu_0U(\eta',t_0)d\eta'}d\xi}{\int_0^{\infty}(\frac{e^{-\frac{(\eta(x,t)-\xi)^2}{2\tau(t)}}-e^{-\frac{(\eta(x,t)+\xi)^2}{2\tau(t)}}}{\sqrt{2\pi\tau(t)}})e^{-\int^{\xi}\mu_0U(\eta',t_0)d\eta'}d\xi+\int_0^{\tau(t)}\left(\frac{\eta(x,t)}{\tau(t)-\tau'}\right)\frac{e^{-\frac{\eta^2(x,t)}{2(\tau(t)-\tau')}}}{\sqrt{2\pi(\tau(t)-\tau')}}H(\tau')d\tau'}\\
&- \frac{r_0}{\mu(t)r(t)}\frac{\int_0^{\tau(t)}\left(\frac{e^{-\frac{\eta^2(x,t)}{2(\tau(t)-\tau')}}}{\sqrt{2\pi(\tau(t)-\tau')^3}}-\frac{\eta^2(x,t)e^{-\frac{\eta^2(x,t)}{2(\tau(t)-\tau')}}}{\sqrt{2\pi(\tau(t)-\tau')^5}}\right)H(\tau')d\tau'}{\int_0^{\infty}(\frac{e^{-\frac{(\eta(x,t)-\xi)^2}{2\tau(t)}}-e^{-\frac{(\eta(x,t)+\xi)^2}{2\tau(t)}}}{\sqrt{2\pi\tau(t)}})e^{-\int^{\xi}\mu_0U(\eta',t_0)d\eta'}d\xi+\int_0^{\tau(t)}\left(\frac{\eta(x,t)}{\tau(t)-\tau'}\right)\frac{e^{-\frac{\eta^2(x,t)}{2(\tau(t)-\tau')}}}{\sqrt{2\pi(\tau(t)-\tau')}}H(\tau')d\tau'},
\end{aligned}$$

where $\tau(t)$ and $\eta(x,t)$ are defined before and the time interval on which the solution exists depends on the properties of the auxiliary functions. Thus, if the integral equation (4.19) for $H(\tau)$ is solved, then we obtain the solution (4.20) for IBVP (4.1).

• **Second approach (Neumann):**

Assume temporary we know $\varphi_\eta(0,\tau) = Q(\tau)$. Then we have the following IBVP

$$\begin{cases} \varphi_\tau = \frac{1}{2}\varphi_{\eta\eta}, \quad 0 < \eta < \infty, \quad 0 < \tau < \tau(T), \\ \varphi(\eta,0) = \exp\left[-\int^\eta \mu_0 U(\eta',t_0)d\eta'\right], \quad 0 < \eta < \infty, \\ \varphi_\eta(0,\tau) = Q(\tau), \quad 0 < \tau < \tau(T). \end{cases} \tag{4.21}$$

Solution to IBVP (4.21) is

$$\varphi(\eta,\tau) = \int_0^\infty N(\eta,\xi,\tau)e^{-\int^\xi \mu_0 U(\eta',t_0)d\eta'}d\xi - \int_0^\tau K(\eta,\tau-\tau')Q(\tau')d\tau',$$

or explicitly

$$\varphi(\eta,\tau) = \int_0^\infty \Big(\frac{e^{-\frac{(\eta-\xi)^2}{2\tau}} + e^{-\frac{(\eta+\xi)^2}{2\tau}}}{\sqrt{2\pi\tau}}\Big)e^{-\int^\xi \mu_0 U(\eta',t_0)d\eta'}d\xi - \int_0^\tau \frac{e^{-\frac{\eta^2}{2(\tau-\tau')}}}{\sqrt{2\pi(\tau-\tau')}}\underbrace{Q(\tau')}_{\varphi_\eta(0,\tau')}d\tau'. \tag{4.22}$$

From solution (4.22), we can obtain $\varphi(0,\tau)$ as follows

$$\varphi(0,\tau) = 2\int_0^\infty \Big(\frac{e^{-\frac{\xi^2}{2\tau}}}{\sqrt{2\pi\tau}}\Big)e^{-\int^\xi \mu_0 U(\eta',t_0)d\eta'}d\xi - \int_0^\tau \frac{Q(\tau')}{\sqrt{2\pi(\tau-\tau')}}d\tau'.$$

Substituting $\varphi(0,\tau)$ and $\varphi_\eta(0,\tau)$ into Robin BC (4.8), we have

$$Q(\tau) = \frac{r(t(\tau))\mu(t(\tau))U(0,t(\tau))}{r_0}\Big(\int_0^\tau \frac{Q(\tau')}{\sqrt{2\pi(\tau-\tau')}}d\tau' - 2\int_0^\infty \Big(\frac{e^{-\frac{\xi^2}{2\tau}}}{\sqrt{2\pi\tau}}\Big)e^{-\int^\xi \mu_0 U(\eta',t_0)d\eta'}d\xi\Big),$$

which is an integral equation of Volterra type for the unknown function $Q(\tau)$. Again we see that solving Heat problem is equivalent to solving integral equation. By Cole-Hopf, the solution of the IBVP (4.41) for the standard BE

$$V(\eta,\tau) = -\frac{\int_0^\infty N_\eta(\eta,\xi,\tau)e^{-\int^\xi V(\eta',0)d\eta'}d\xi - \int_0^\tau K_\eta(\eta,\tau-\tau')Q(\tau')d\tau'}{\int_0^\infty N(\eta,\xi,\tau)e^{-\int^\xi V(\eta',0)d\eta'}d\xi - \int_0^\tau K(\eta,\tau-\tau')Q(\tau')d\tau'},$$

or explicitly

$$V(\eta,\tau) = \frac{\int_0^\infty \left(\frac{(\frac{\eta-\xi}{\tau})e^{-\frac{(\eta-\xi)^2}{2\tau}}+(\frac{\eta+\xi}{\tau})e^{-\frac{(\eta+\xi)^2}{2\tau}}}{\sqrt{2\pi\tau}}\right)e^{-\int^\xi V(\eta',0)d\eta'}d\xi}{\int_0^\infty \left(\frac{e^{-\frac{(\eta-\xi)^2}{2\tau}}+e^{-\frac{(\eta+\xi)^2}{2\tau}}}{\sqrt{2\pi\tau}}\right)e^{-\int^\xi V(\eta',0)d\eta'}d\xi - \int_0^\tau \frac{e^{-\frac{\eta^2}{2(\tau-\tau')}}}{\sqrt{2\pi(\tau-\tau')}}Q(\tau')d\tau'}$$

$$-\frac{\int_0^\tau \left(\frac{\eta}{\tau-\tau'}\right)\frac{e^{-\frac{\eta^2}{2(\tau-\tau')}}}{\sqrt{2\pi(\tau-\tau')}}Q(\tau')d\tau'}{\int_0^\infty \left(\frac{e^{-\frac{(\eta-\xi)^2}{2\tau}}+e^{-\frac{(\eta+\xi)^2}{2\tau}}}{\sqrt{2\pi\tau}}\right)e^{-\int^\xi V(\eta',0)d\eta'}d\xi - \int_0^\tau \frac{e^{-\frac{\eta^2}{2(\tau-\tau')}}}{\sqrt{2\pi(\tau-\tau')}}Q(\tau')d\tau'}.$$

And the solution of the IBVP (4.1) is obtained in terms of $r(t)$ of the linear ODE (4.9) and $\varphi(\eta,\tau)$, which is given in closed and explicit form respectively as follows

$$U(x,t) = \frac{\dot{r}(t)}{r(t)}x - \frac{r_0}{\mu(t)r(t)}\frac{\int_0^\infty N_\eta(\eta,\xi,\tau)e^{-\int^\xi \mu_0 U(\eta',t_0)d\eta'}d\xi - \int_0^\tau K_\eta(\eta,\tau-\tau')Q(\tau')d\tau'}{\int_0^\infty N(\eta,\xi,\tau)e^{-\int^\xi \mu_0 U(\eta',t_0)d\eta'}d\xi - \int_0^\tau K(\eta,\tau-\tau')Q(\tau')d\tau'},$$

or explicitly

$$\begin{aligned}
U(x,t) &= \frac{\dot{r}(t)}{r(t)}x \\
&- \frac{r_0}{\mu(t)r(t)}\frac{\int_0^\infty \left(\frac{(\frac{\eta(x,t)-\xi}{\tau(t)})e^{-\frac{(\eta(x,t)-\xi)^2}{2\tau(t)}}+(\frac{\eta(x,t)+\xi}{\tau(t)})e^{-\frac{(\eta(x,t)+\xi)^2}{2\tau(t)}}}{\sqrt{2\pi\tau(t)}}\right)e^{-\int^\xi \mu_0 U(\eta',t_0)d\eta'}d\xi}{\int_0^\infty \left(\frac{e^{-\frac{(\eta(x,t)-\xi)^2}{2\tau(t)}}+e^{-\frac{(\eta(x,t)+\xi)^2}{2\tau(t)}}}{\sqrt{2\pi\tau(t)}}\right)e^{-\int^\xi \mu_0 U(\eta',t_0)d\eta'}d\xi - \int_0^{\tau(t)} \frac{e^{-\frac{\eta^2(x,t)}{2(\tau(t)-\tau')}}}{\sqrt{2\pi(\tau(t)-\tau')}}Q(\tau')d\tau'} \\
&- \frac{r_0}{\mu(t)r(t)}\frac{\int_0^{\tau(t)} \left(\frac{\eta(x,t)}{\tau(t)-\tau'}\right)\frac{e^{-\frac{\eta^2(x,t)}{2(\tau(t)-\tau')}}}{\sqrt{2\pi(\tau(t)-\tau')}}Q(\tau')d\tau'}{\int_0^\infty \left(\frac{e^{-\frac{(\eta(x,t)-\xi)^2}{2\tau(t)}}+e^{-\frac{(\eta(x,t)+\xi)^2}{2\tau(t)}}}{\sqrt{2\pi\tau(t)}}\right)e^{-\int^\xi \mu_0 U(\eta',t_0)d\eta'}d\xi - \int_0^{\tau(t)} \frac{e^{-\frac{\eta^2(x,t)}{2(\tau(t)-\tau')}}}{\sqrt{2\pi(\tau(t)-\tau')}}Q(\tau')d\tau'},
\end{aligned}$$

where $\tau(t)$ and $\eta(x,t)$ are defined before and the time interval on which the solution exists depends on the properties of the auxiliary functions.

4.1.1. Exactly Solvable Models

In this section we shall give examples to show the application of the general results given in Propositon-I. Also, we will investigate exact solutions of three different Burgers models **[3]** on the semi-infinite line. Precisely, we consider the IBVP with Dirichlet BC for the following Burgers equations :

(A) Forced Burgers equation with constant coefficients:

$$U_t + UU_x = \frac{1}{2}U_{xx} + {\omega_0}^2 x, \quad 0 < x < \infty, \;\; 0 < t < \infty, \quad \omega_0 > 0.$$

(B) Forced Burgers equation-Critical damping case:

$$U_t + \gamma U + UU_x = \frac{1}{2}e^{-\gamma t}U_{xx} - \omega_0^2 x, \quad 0 < x < \infty, \;\; 0 < t < \infty, \qquad \omega_0^2 - (\gamma^2/4) = 0.$$

(C) Forced Burgers equation -Over damping case:

$$U_t + \gamma U + UU_x = \frac{1}{2}e^{-\gamma t}U_{xx} - \omega_0^2 x, \quad 0 < x < \infty, \;\; 0 < t < T, \qquad \omega_0^2 - (\gamma^2/4) < 0.$$

We choose these models since the corresponding ODE with the given IC's **[3]** is exactly solvable and its solution $r(t)$ is positive for $t \geq 0$, so that $\tau(t)$ is positive and invertible as required for application of the Proposition.

(A) Forced Burgers equation with constant coefficients

Consider the following IBVP for the forced Burgers equation with constant coefficients defined by

$$\begin{cases} U_t + UU_x = \frac{1}{2}U_{xx} + {\omega_0}^2 x, \quad 0 < x < \infty, \;\; 0 < t < \infty, \\ U(x,t)\mid_{t=0} = U(x,0), \quad 0 < x < \infty, \\ U(0,t) = D(t), \quad 0 < t < \infty, \end{cases} \tag{4.23}$$

with $\mu(t) = 1$, $\omega_0 > 0$. The corresponding ODE has solution $r(t) = r_0 \cosh(\omega_0 t)$, which is positive for $r_0 > 0$ and $0 \le t < T$, and $\eta(x,t) = sech(\omega_0 t)x$, $\tau(t) = \tanh(\omega_0 t)/\omega_0$. Then, solution of (4.23) is found in the form

$$U(x,t) = \omega_0 \tanh(\omega_0 t)x + sech(\omega_0 t)V(\eta(x,t), \tau(t)), \tag{4.24}$$

where $V(\eta, \tau)$ satisfies the IBVP for the standard BE with the Dirichlet BC

$$\begin{cases} V_\tau + VV_\eta = \frac{1}{2}V_{\eta\eta}, & 0 < \eta < \infty, \;\; 0 < \tau < 1/\omega_0, \\ V(\eta, 0) = U(\eta, 0), & 0 < \eta < \infty, \\ V(0,\tau) = \left(\frac{1+(\omega_0\tau)^2}{1-(\omega_0\tau)^2}\right) U(0, t(\tau)), & 0 < \tau < 1/\omega_0, \end{cases} \tag{4.25}$$

with $t(\tau) = \tanh^{-1}(\omega_0\tau)/\omega_0 = 1/(2\omega_0)\,[\ln[(1+\omega_0\tau)/(1-\omega_0\tau)]]$,
$r(t(\tau)) = r_0 \cosh(\tanh^{-1}(\omega_0\tau)) = r_0[1 + (\omega_0\tau)^2)]/[1 - (\omega_0\tau)^2)]$. Also, solution of (4.23) is of the form

$$U(x,t) = \omega_0 \tanh(\omega_0 t)x - sech(\omega_0 t)\frac{\varphi_\eta(\eta(x,t), \tau(t))}{\varphi(\eta(x,t), \tau(t))},$$

if $\varphi(\eta, \tau)$ satisfies the IBVP for the Heat equation with the Robin BC

$$\begin{cases} \varphi_\tau = \frac{1}{2}\varphi_{\eta\eta}, & 0 < \eta < \infty, \;\; 0 < \tau < 1/\omega_0, \\ \varphi(\eta, 0) = \exp\left[-\int^\eta U(\eta', 0)d\eta'\right], & 0 < \eta < \infty, \\ \left[1 + (\omega_0\tau)^2\right] U(0, t(\tau))\varphi(0,\tau) + \left[1 - (\omega_0\tau)^2\right]\varphi_\eta(0,\tau) = 0, & 0 < \tau < 1/\omega_0. \end{cases} \tag{4.26}$$

From previous chapter we know that by Neumann approach the solution of this Heat problem with Robin BC is of the form

$$\varphi(\eta, \tau) = \int_0^\infty \left(\frac{e^{-\frac{(\eta-\xi)^2}{2\tau}} + e^{-\frac{(\eta+\xi)^2}{2\tau}}}{\sqrt{2\pi\tau}}\right) e^{-\int^\xi U(\eta', 0)d\eta'} d\xi - \int_0^\tau \frac{e^{-\frac{\eta^2}{2(\tau-\tau')}}}{\sqrt{2\pi(\tau-\tau')}} \underbrace{Q(\tau')}_{\varphi_\eta(0,\tau')} d\tau',$$

where $Q(\tau)$ is found by solving the integral equation

$$Q(\tau) = \left[\frac{1 + (\omega_0\tau)^2}{1 - (\omega_0\tau)^2}\right] U(0, t(\tau))\Big(\int_0^\tau \frac{Q(\tau')}{\sqrt{2\pi(\tau - \tau')}} d\tau' - 2\int_0^\infty \Big(\frac{e^{-\frac{\xi^2}{2\tau}}}{\sqrt{2\pi\tau}}\Big) e^{-\int^\xi U(\eta',0)d\eta'} d\xi\Big).$$

In general, this integral equation requires numerical methods and can be solved only approximately. The simplest case is when $U(0, t(\tau)) = 0$, so that the BC of the heat problem becomes of Neumann type. Another special case is when the BC is chosen to be $U(0, t(\tau)) = D_0(1 - (\omega_0\tau)^2)/(1 + (\omega_0\tau)^2)$, where D_0 is constant, so that above integral equation becomes of the form

$$Q(\tau) = F(\tau) + D_0 \int_0^\tau \frac{Q(\tau')}{\sqrt{2\pi(\tau - \tau')}} d\tau',$$

which is the well known second kind Abel's integral equation for the unknown $Q(\tau)$ and known $F(\tau)$, and can be solved by Laplace transform.

1) Problems with Homogeneous Boundary Condition $U(0, t) = 0$

Example 4.1 *The IBVP with homogeneous Dirichlet BC*

$$\begin{cases} U_t + UU_x = \frac{1}{2}U_{xx} + {\omega_0}^2 x, & 0 < x < \infty, \ 0 < t < \infty, \\ U(x, 0) = 1, & 0 < x < \infty, \\ U(0, t) = 0, & 0 < t < \infty, \end{cases} \tag{4.27}$$

reduces to IBVP for the Heat equation with homogeneous Neumann BC

$$\begin{cases} \varphi_\tau = \frac{1}{2}\varphi_{\eta\eta}, & 0 < \eta < \infty, \quad 0 < \tau < 1/\omega_0, \\ \varphi(\eta, 0) = e^{-\eta}, & 0 < \eta < \infty, \\ \varphi_\eta(0, \tau) = 0, & 0 < \tau < 1/\omega_0. \end{cases} \tag{4.28}$$

Solution to this Heat problem is

$$\begin{aligned}\varphi(\eta,\tau) &= \int_0^\infty \left(\frac{e^{-\frac{(\eta-\xi)^2}{2\tau}} + e^{-\frac{(\eta+\xi)^2}{2\tau}}}{\sqrt{2\pi\tau}}\right) e^{-\xi} d\xi, \\ &= \frac{e^{\tau/2}}{2}\left[e^{\eta} Erfc\left(\frac{\tau+\eta}{\sqrt{2\tau}}\right) + e^{-\eta} Erfc\left(\frac{\tau-\eta}{\sqrt{2\tau}}\right)\right],\end{aligned}$$

so that solution of the Burgers IBVP (4.27) becomes

$$\begin{aligned}U(x,t) &= \omega_0 \tanh(\omega_0 t)x \\ &+ sech(\omega_0 t)\frac{e^{-\eta(x,t)}\left(Erfc\left(\frac{\tau(t)-\eta(x,t)}{\sqrt{2\tau(t)}}\right) - \frac{2}{\sqrt{\pi}} e^{-\frac{(\tau(t)-\eta(x,t))^2}{2\tau(t)}}\right)}{e^{\eta(x,t)} Erfc\left(\frac{\tau(t)+\eta(x,t)}{\sqrt{2\tau(t)}}\right) + e^{-\eta(x,t)} Erfc\left(\frac{\tau(t)-\eta(x,t)}{\sqrt{2\tau(t)}}\right)} \\ &- sech(\omega_0 t)\frac{e^{\eta(x,t)}\left(Erfc\left(\frac{\tau(t)+\eta(x,t)}{\sqrt{2\tau(t)}}\right) - \frac{2}{\sqrt{\pi}} e^{-\frac{(\eta(x,t)+\tau(t))^2}{2\tau(t)}}\right)}{e^{\eta(x,t)} Erfc\left(\frac{\tau(t)+\eta(x,t)}{\sqrt{2\tau(t)}}\right) + e^{-\eta(x,t)} Erfc\left(\frac{\tau(t)-\eta(x,t)}{\sqrt{2\tau(t)}}\right)},\end{aligned}$$

where $\eta(x,t) = sech(\omega_0 t)x$, $\tau(t) = \tanh(\omega_0 t)/\omega_0$.

Example 4.2

$$\begin{cases} U_t + UU_x = \frac{1}{2}U_{xx} + {\omega_0}^2 x, \quad 0 < x < \infty, \ 0 < t < \infty, \\ U(x,0) = -A\tanh(Ax), \quad 0 < x < \infty, \\ U(0,t) = 0, \quad 0 < t < \infty. \end{cases} \tag{4.29}$$

$\mu(t) = 1$, $\omega_0 > 0$. *The corresponding Heat problem is*

$$\begin{cases} \varphi_\tau = \frac{1}{2}\varphi_{\eta\eta}, \quad 0 < \eta < \infty, \ 0 < \tau < 1/\omega_0, \\ \varphi(\eta,0) = \cosh(A\eta), \quad 0 < \eta < \infty, \\ \varphi_\eta(0,\tau) = 0, \quad 0 < \tau < 1/\omega_0, \end{cases} \tag{4.30}$$

which has solution

$$\varphi(\eta,\tau) = \int_0^{\infty} \frac{e^{-\frac{(\eta-\xi)^2}{2\tau}} + e^{-\frac{(\eta+\xi)^2}{2\tau}}}{\sqrt{2\pi\tau}} \cosh(A\xi)d\xi = \int_{-\infty}^{\infty} \frac{e^{-\frac{(\eta-\xi)^2}{2\tau}}}{\sqrt{2\pi\tau}} \cosh(A\xi)d\xi = e^{\frac{A^2}{2}\tau}\cosh(A\eta),$$

and solution of the Burgers problem is therefore

$$U(x,t) = \omega_0 \tanh(\omega_0 t)x - Asech(\omega_0 t)\tanh(Asech(\omega_0 t)x).$$

Example 4.3

$$\begin{cases} U_t + UU_x = \frac{1}{2}U_{xx} + {\omega_0}^2 x, & 0 < x < \infty,\ 0 < t < \infty, \\ U(x,0) = -\frac{m}{x}, \quad m = 0,1,2,\cdots, & 0 < x < \infty, \\ U(0,t) = 0, & 0 < t < \infty. \end{cases} \tag{4.31}$$

The solution of (4.31) is of the form

$$U(x,t) = \omega_0 \tanh(\omega_0 t)x - sech(\omega_0 t)\frac{\varphi_\eta(\eta(x,t),\tau(t))}{\varphi(\eta(x,t),\tau(t))},$$

where $\eta(x,t) = xsech(\omega_0 t)$*,* $\tau(t) = \tanh(\omega_0 t)/\omega_0$ *and* $\varphi(\eta,\tau)$ *satisfies the IBVP for HE*

$$\begin{cases} \varphi_\tau = \frac{1}{2}\varphi_{\eta\eta}, & 0 < \eta < \infty,\ 0 < \tau < 1/\omega_0, \\ \varphi(\eta,0) = \eta^m, \quad m = 0,1,2,\cdots, & 0 < \eta < \infty, \\ \varphi_\eta(0,\tau) = 0, & 0 < \tau < 1/\omega_0. \end{cases} \tag{4.32}$$

Using the functions (2.58) and (2.59) which are defined in Chapter 2, the solution of IBVP (4.32) can be found as follows:
If m is even, then solutions of problem (4.32) are even Kampe de Feriet Polynomials,

$$\varphi_{2p}(\eta,\tau) = H^k_{2p}(\eta,\tau) = (2p)!\sum_{n=0}^{p}\frac{\eta^{2p-2n}}{n!(2p-2n)!}\tau^n,$$

and the corresponding inhomogeneous BE solution is given by

$$U_{2p}(x,t) = \omega_0 \tanh(\omega_0 t)x - 2p\,sech(\omega_0 t)\frac{H^k_{2p-1}\Big(sech(\omega_0 t)x, \tanh(\omega_0 t)/\omega_0\Big)}{H^k_{2p}\Big(sech(\omega_0 t)x, \tanh(\omega_0 t)/\omega_0\Big)}.$$

⋆ ***For example if m=2***, *then solution of problem (4.32) is*

$$\varphi_2(\eta,\tau) = H^k_2(\eta,\tau) = \eta^2 + \tau,$$

and the corresponding inhomogeneous Burgers solution

$$U_2(x,t) = \omega_0 \tanh(\omega_0 t)x - (2\omega_0 x)\,\frac{sech^2(\omega_0 t)}{\omega_0 sech^2(\omega_0 t)x^2 + \tanh(\omega_0 t)}.$$

However, if m is odd, then solutions of the Heat problem are no longer odd KFP, since odd KFP does not satisfy the Neumann BC $\varphi_\eta(0,\tau) = 0$. *Then solution in that case can be written in terms of functions (2.58) and (2.59), that's*

$$\varphi_{2p+1}(\eta,\tau) = \underbrace{\int_0^\infty \frac{e^{-\frac{(\eta-\xi)^2}{2\tau}}}{\sqrt{2\pi\tau}}\xi^{2p+1}d\xi}_{h^-_{2p+1}(\eta,\tau)} + \underbrace{\int_0^\infty \frac{e^{-\frac{(\eta+\xi)^2}{2\tau}}}{\sqrt{2\pi\tau}}\xi^{2p+1}d\xi}_{h^+_{2p+1}(\eta,\tau)}, \tag{4.33}$$

and the corresponding solution of the problem (4.31) is

$$U_{2p+1}(x,t) = \omega_0 \tanh(\omega_0 t)x - (2p+1)sech(\omega_0 t)\left[\frac{h^-_{2p}(\eta(x,t),\tau(t)) - h^+_{2p}(\eta(x,t),\tau(t))}{h^-_{2p+1}(\eta(x,t),\tau(t)) + h^+_{2p+1}(\eta(x,t),\tau(t))}\right],$$

where $\eta(x,t) = sech(\omega_0 t)x$ *and* $\tau(t) = \tanh(\omega_0 t)\omega_0$.

⋆ ***For example if m=3***, *then solution of problem (4.32) is*

$$\varphi_3(\eta,\tau) = h^-_3(\eta,\tau) + h^+_3(\eta,\tau) = \sqrt{\frac{2}{\pi}}e^{-\frac{\eta^2}{2\tau}}\sqrt{\tau}(\eta^2 + 2\tau) + (\eta^3 + 3\eta\tau)Erf[\frac{\eta}{\sqrt{2\tau}}],$$

and the corresponding inhomogeneous Burgers solution

$$
\begin{aligned}
U_3(x,t) &= \omega_0 \tanh(\omega_0 t)x \\
&- sech(\omega_0 t)\frac{3\left(\sqrt{\frac{2}{\pi}}\eta(x,t)\tau(t)e^{-\frac{\eta^2(x,t)}{2\tau(t)}} + (\eta^2(x,t)+\tau(t))Erf[\frac{\eta(x,t)}{\sqrt{2\tau(t)}}]\right)}{\sqrt{\frac{2}{\pi}}e^{-\frac{\eta^2(x,t)}{2\tau(t)}}\sqrt{\tau(t)}(\eta^2(x,t)+2\tau(t)) + \left(\eta^3(x,t)+3\eta(x,t)\tau(t)\right)Erf\left[\frac{\eta(x,t)}{\sqrt{2\tau(t)}}\right]},
\end{aligned}
$$

where $\eta(x,t) = sech(\omega_0 t)x$ *and* $\tau(t) = \tanh(\omega_0 t)/\omega_0$.

2) Problems with Nonhomogeneous Boundary Condition :

$U(0,t(\tau)) = D_0(1-(\omega_0\tau)^2)/(1+(\omega_0\tau)^2)$. For simplicty, we take $D_0 = -1$.

Example 4.4 *The IBVP with nonhomogeneous Dirichlet BC*

$$
\begin{cases}
U_t + UU_x = \frac{1}{2}U_{xx} + {\omega_0}^2 x, & 0 < x < \infty, \ 0 < t < \infty, \\
U(x,0) = -\frac{3x^2+6x}{x^3+3x^2}, & 0 < x < \infty, \\
U(0,t) = -sech(\omega_0 t), & 0 < t < \infty,
\end{cases}
\tag{4.34}
$$

reduces to IBVP for the Heat equation with special Robin BC

$$
\begin{cases}
\varphi_\tau = \frac{1}{2}\varphi_{\eta\eta}, & 0 < \eta < \infty, \quad 0 < \tau < 1/\omega_0, \\
\varphi(\eta,0) = \sqrt{2\pi}\eta^2 + \frac{\sqrt{2\pi}}{3}\eta^3, & 0 < \eta < \infty, \\
\varphi_\eta(0,\tau) - \varphi(0,\tau) = 0, & 0 < \tau < 1/\omega_0.
\end{cases}
\tag{4.35}
$$

In Chapter 2, we obtained the solution to the problem (4.35) as

$$
\varphi(\eta,\tau) = \sqrt{2\pi}\left[\frac{\eta^3}{3} + \eta^2 + \eta\tau + \tau\right]. \tag{4.36}
$$

And the corresponding solution to the problem (4.34)

$$U(x,t) = \omega_0 \tanh(\omega_0 t)x - \frac{sech^3(\omega_0 t)x^2 + 2sech^2(\omega_0 t)x + \frac{sech(\omega_0 t)\tanh(\omega_0 t)}{\omega_0}}{\frac{sech^3(\omega_0 t)x^3}{3} + sech^2(\omega_0 t)x^2 + sech(\omega_0 t)x\frac{\tanh(\omega_0 t)}{\omega_0} + \frac{\tanh(\omega_0 t)}{\omega_0}}.$$

(B) Forced Burgers equation with constant damping and exponentially decaying diffusion coefficient-Critical damping case:

We consider the IBVP for Burgers equation

$$\begin{cases} U_t + \gamma U + UU_x = \frac{1}{2}e^{-\gamma t}U_{xx} - \omega_0^2 x, \qquad 0 < x < \infty, \quad 0 < t < \infty, \\ U(x,t)\mid_{t=0} = U(x,0), \quad 0 < x < \infty, \\ U(0,t) = D(t), \quad 0 < t < \infty, \end{cases} \tag{4.37}$$

with constant damping $\Gamma(t) = \gamma > 0$, $\mu(t) = e^{\gamma t}$ and $\omega_0^2 - (\gamma^2/4) = 0$. The corresponding IVP for the linear ODE is then

$$\ddot{r} + \gamma\dot{r} + \omega_0^2 r = 0, \quad r(0) = r_0 \neq 0, \;\; \dot{r}(0) = 0, \tag{4.38}$$

which has solution

$$r(t) = r_0 e^{-\frac{\gamma t}{2}}\left(1 + \frac{\gamma}{2}t\right), \tag{4.39}$$

and thus $\eta(x,t) = (e^{\gamma t/2}x)/(1 + \frac{\gamma}{2}t)$, $\tau(t) = t/(1 + \gamma t/2)$. Therefore, the BE (4.37) has solution of the form

$$U(x,t) = -\left(\frac{\gamma}{2}\right)^2\left(\frac{t}{1+\frac{\gamma}{2}t}x\right) + \left(\frac{e^{-\gamma t/2}}{1+\frac{\gamma}{2}t}\right)V\left(\frac{e^{\gamma t/2}x}{1+\frac{\gamma}{2}t}, \frac{t}{1+\frac{\gamma}{2}t}\right), \tag{4.40}$$

where $V(\eta,\tau)$ satisfies the IBVP for the standard BE with the Dirichlet BC

$$\begin{cases} V_\tau + VV_\eta = \frac{1}{2}V_{\eta\eta}, \quad 0 < \eta < \infty, \quad 0 < \tau < 2/\gamma, \\ V(\eta,0) = U(\eta,t_0), \quad 0 < \eta < \infty, \\ V(0,\tau) = \left[e^{-\frac{\gamma\tau}{2-\gamma\tau}}\left(\frac{2}{2-\gamma\tau}\right)\right]U(0,t(\tau)), \quad 0 < \tau < 2/\gamma, \end{cases} \tag{4.41}$$

with $t(\tau) = \tau/(1-\gamma\tau/2)$, $r(t(\tau)) = r_0 e^{-\frac{\gamma\tau}{2-\gamma\tau}}(1+\frac{\gamma\tau}{2-\gamma\tau})$. The solution can be found also in the form

$$U(x,t) = -\left(\frac{\gamma}{2}\right)^2\left(\frac{t}{1+\frac{\gamma}{2}t}x\right) - \left(\frac{e^{-\gamma t/2}}{1+\frac{\gamma}{2}t}\right)\frac{\varphi_\eta(\eta(x,t),\tau(t))}{\varphi(\eta(x,t),\tau(t))},$$

where $\varphi(\eta,\tau)$ satisfies the IBVP for the Heat equation with the Robin BC

$$\begin{cases} \varphi_\tau = \frac{1}{2}\varphi_{\eta\eta}, \quad 0<\eta<\infty,\ 0<\tau<2/\gamma, \\ \varphi(\eta,0) = \exp\left[-\int_0^\eta \mu_0 U(\eta',0)d\eta'\right], \quad 0<\eta<\infty, \\ \left[e^{+\frac{\gamma\tau}{2-\gamma\tau}}(\frac{2}{2-\gamma\tau})U(0,t(\tau))\right]\varphi(0,\tau) + \varphi_\eta(0,\tau) = 0, \quad 0<\tau<2/\gamma. \end{cases} \tag{4.42}$$

As we found in previous Chapter, solution of this Heat IBVP is formally

$$\varphi(\eta,\tau) = \int_0^\infty \Big(\frac{e^{-\frac{(\eta-\xi)^2}{2\tau}} + e^{-\frac{(\eta+\xi)^2}{2\tau}}}{\sqrt{2\pi\tau}}\Big)e^{-\int^\xi U(\eta',0)d\eta'}d\xi - \int_0^\tau \frac{e^{-\frac{\eta^2}{2(\tau-\tau')}}}{\sqrt{2\pi(\tau-\tau')}}\underbrace{Q(\tau')}_{\varphi_\eta(0,\tau')}d\tau',$$

where $Q(\tau)$ is found by solving the integral equation

$$Q(\tau) = \left[\frac{2e^{\frac{\gamma\tau}{2-\gamma\tau}}}{(2-\gamma\tau)}U(0,t(\tau))\right]\Big(\int_0^\tau \frac{Q(\tau')}{\sqrt{2\pi(\tau-\tau')}}d\tau' - 2\int_0^\infty \frac{e^{-\frac{\xi^2}{2\tau}}}{\sqrt{2\pi\tau}}e^{-\int^\xi U(\eta',0)d\eta'}d\xi\Big).$$

Clearly, when $U(0,t(\tau)) = 0$, the BC of the Heat problem becomes of Neumann type. Another special case is when the BC is chosen to be $U(0,t(\tau)) = D_0(2-\gamma\tau)/(2e^{\frac{\gamma\tau}{2-\gamma\tau}})$, where D_0 is constant, so that the above integral equation becomes of the form

$$Q(\tau) = F(\tau) + D_0\int_0^\tau \frac{Q(\tau')}{\sqrt{2\pi(\tau-\tau')}}d\tau',$$

which is the well known inhomogeneous Abel's integral equation for the unknown $Q(\tau)$ and known $F(\tau)$, and can be solved by Laplace transform.

1) Problems with BC U(0,t)=0

Example 4.5

$$\begin{cases} U_t + \gamma U + UU_x = \frac{1}{2}e^{-\gamma t}U_{xx} - \omega_0^2 x, & 0 < x < \infty, \ 0 < t < \infty, \\ U(x, t_0) = -A\tanh(Ax), & 0 < x < \infty, \\ U(0, t) = 0, & 0 < t < \infty. \end{cases} \tag{4.43}$$

with the corresponding IBVP for the HE

$$\begin{cases} \varphi_\tau = \frac{1}{2}\varphi_{\eta\eta}, & 0 < \eta < \infty, \ 0 < \tau < 2/\gamma, \\ \varphi(\eta, 0) = \cosh(A\eta), & 0 < \eta < \infty, \\ \varphi_\eta(0, \tau) = 0, & 0 < \tau < 2/\gamma. \end{cases} \tag{4.44}$$

The solution of (4.44) is given by

$$\varphi(\eta, \tau) = \int_{-\infty}^{\infty} \frac{e^{-\frac{(\eta-\xi)^2}{2\tau}}}{\sqrt{2\pi\tau}} \cosh(A\xi)d\xi = e^{\frac{A^2}{2}\tau}\cosh(A\eta).$$

And the corresponding solution of the problem (4.43) is

$$U(x, t) = -\left(\frac{\gamma}{2}\right)^2 \left(\frac{t}{1 + \frac{\gamma}{2}t}x\right) - A\left(\frac{e^{-\gamma t/2}}{1 + \frac{\gamma}{2}t}\right)\tanh\left(A\frac{e^{\gamma t/2}}{1 + \gamma t/2}x\right),$$

which is a shock-wave solution.

Example 4.6 *The IBVP (4.43) with IC $U(x, 0) = 1$ and BC $U(0, t) = 0$, then the IC for the Heat equation is $\varphi(\eta, 0) = e^{-\eta}$ with BC $\varphi_\eta(0, \eta) = 0$, we have solution of the Heat problem as*

$$\begin{aligned} \varphi(\eta, \tau) &= \int_0^{\infty} \Big(\frac{e^{-\frac{(\eta-\xi)^2}{2\tau}} + e^{-\frac{(\eta+\xi)^2}{2\tau}}}{\sqrt{2\pi\tau}}\Big)e^{-\xi}d\xi, \\ &= \frac{e^{\tau/2}}{2}\Big(e^{\eta}Erfc\left[\frac{\tau + \eta}{\sqrt{2\tau}}\right] + e^{-\eta}Erfc\left[\frac{\tau - \eta}{\sqrt{2\tau}}\right]\Big). \end{aligned}$$

And solution of the Burgers problem is

$$
\begin{aligned}
U(x,t) &= -\left(\frac{\gamma}{2}\right)^2\left(\frac{t}{1+\frac{\gamma}{2}t}x\right)\\
&- \left[\frac{e^{-\gamma t/2}}{1+\gamma t/2}\right]\frac{e^{-\eta(x,t)}Erfc(\frac{\tau(t)-\eta(x,t)}{\sqrt{2\tau(t)}}) - \frac{2}{\sqrt{\pi}}e^{-\eta(x,t)}e^{-\frac{(\tau(t)-\eta(x,t))^2}{2\tau(t)}}}{e^{\eta(x,t)}Erfc\left(\frac{\tau(t)+\eta(x,t)}{\sqrt{2\tau(t)}}\right)+e^{-\eta(x,t)}Erfc\left(\frac{\tau(t)-\eta(x,t)}{\sqrt{2\tau(t)}}\right)}\\
&+ \left[\frac{e^{-\gamma t/2}}{1+\gamma t/2}\right]\frac{e^{\eta(x,t)}Erfc\left(\frac{\tau(t)+\eta(x,t)}{\sqrt{2\tau(t)}}\right) - \frac{2}{\sqrt{\pi}}e^{\eta(x,t)}e^{-\frac{(\eta(x,t)+\tau(t))^2}{2\tau(t)}}}{e^{\eta(x,t)}Erfc\left(\frac{\tau(t)+\eta(x,t)}{\sqrt{2\tau(t)}}\right)+e^{-\eta(x,t)}Erfc\left(\frac{\tau(t)-\eta(x,t)}{\sqrt{2\tau(t)}}\right)}.
\end{aligned}
$$

Example 4.7 *The IBVP (4.43) with IC* $U(x,t_0) = \frac{-m}{x}$ *is transformed to* $\varphi(\eta,0) = \eta^m$ *and BC* $U(0,t) = 0$ *is transformed to* $\varphi_\eta(0,\tau) = 0$*. Clearly, if m is even positive integer i.e* $m = 2p$*, then solutions of the Heat problem are even Kampe de Feriet Polynomials i.e*

$$\varphi_{2p}(\eta,\tau) = H_{2p}(\eta,\tau) = (2p)!\sum_{k=0}^{p}\frac{\eta^{2p-2k}}{k!(2p-2k)!}\tau^k.$$

The corresponding solution for the inhomogeneous Burger problem is

$$U_{2p}(x,t) = -\left(\frac{\gamma}{2}\right)^2\left(\frac{t}{1+\frac{\gamma}{2}t}x\right) - 2p\left[\frac{e^{-\gamma t/2}}{1+\gamma t/2}\right]\frac{H_{2p-1}\left(\frac{e^{\gamma t/2}}{1+\gamma t/2}x, \frac{t}{1+\frac{\gamma}{2}t}\right)}{H_{2p}\left(\frac{e^{\gamma t/2}}{1+\gamma t/2}x, \frac{t}{1+\frac{\gamma}{2}t}\right)}.$$

If m is odd, $m = 2p+1$*, then solution of the Heat problem are no longer odd Kampe de Feriet polynomials.*

2) Problem with Nonhomogeneous BC : $U(0,t) = D_0e^{\gamma t/2}/(1+\gamma t/2)$

Example 4.8 *Taking $D_0 = -1$, the IBVP with the nonhomogeneous Dirichlet BC*

$$\begin{cases} U_t + \gamma U + UU_x = \frac{1}{2}e^{-\gamma t}U_{xx} - \omega_0^2 x, \ 0 < x < \infty, \ 0 < t < \infty, \\ U(x,0) = -\frac{3x^2+6x}{x^3+3x^2}, \quad 0 < x < \infty, \\ U(0,t) = -\frac{e^{-\gamma t/2}}{1+\gamma t/2}, \quad 0 < t < \infty, \end{cases} \tag{4.45}$$

reduces to IBVP for the Heat equation with the special Robin BC

$$\begin{cases} \varphi_\tau = \frac{1}{2}\varphi_{\eta\eta}, \quad 0 < \eta < \infty, \quad 0 < \tau < 2/\gamma, \\ \varphi(\eta,0) = \sqrt{2\pi}\eta^2 + \frac{\sqrt{2\pi}}{3}\eta^3, \quad 0 < \eta < \infty, \\ \varphi_\eta(0,\tau) - \varphi(0,\tau) = 0, \quad 0 < \tau < 2/\gamma. \end{cases} \tag{4.46}$$

In Chapter 2, we obtained solution of the problem (4.46) as

$$\varphi(\eta,\tau) = \sqrt{2\pi}\left[\frac{\eta^3}{3} + \eta^2 + \eta\tau + \tau\right]. \tag{4.47}$$

And the corresponding solution of the problem (4.45)

$$U(x,t) = -\left(\frac{\gamma}{2}\right)^2\left(\frac{t}{1+\frac{\gamma}{2}t}x\right) - \left[\frac{e^{-\gamma t/2}}{1+\gamma t/2}\right]\frac{\eta^2(x,t) + 2\eta(x,t) + \tau(t)}{\frac{\eta^3(x,t)}{3} + \eta^2(x,t) + \eta(x,t)\tau(t) + \tau(t)}. \tag{4.48}$$

(C) Forced Burgers equation with constant damping and exponentially decaying diffusion coefficient-Over damping case:

We consider IBVP for Burgers equation

$$\begin{cases} U_t + \gamma U + UU_x = \frac{1}{2}e^{-\gamma t}U_{xx} - \omega_0^2 x, \quad 0 < x < \infty, \quad 0 < t < \infty, \\ U(x,t)\,|_{t=t_0} = U(x,t_0), \quad 0 < x < \infty, \\ U(0,t) = D(t), \quad 0 < t < \infty, \end{cases} \tag{4.49}$$

with constant damping $\Gamma(t) = \gamma > 0$, $\mu(t) = e^{\gamma t}$, and $\omega_0^2 - (\gamma^2/4) < 0$. The corresponding IVP (4.38) for the ODE has solution

$$r(t) = r_0 \frac{\omega_0}{\Omega'} e^{-\frac{\gamma t}{2}} \sinh[\Omega' t + \beta], \tag{4.50}$$

where $\Omega' = \sqrt{|\omega_0^2 - (\gamma^2/4)|}$, and $\beta = \coth^{-1}(\frac{\gamma}{2\Omega'})$ **[3]**. Then, Burgers problem has solutions of the form

$$U(x,t) = \Big(-\frac{\gamma}{2} + \Omega' \coth[\Omega' t + \beta]\Big)x - \Big(\frac{\Omega' e^{-\gamma t/2}}{\omega_0 \sinh[\Omega' t + \beta]}\Big)\frac{\varphi_\eta(\eta(x,t),\tau(t))}{\varphi(\eta(x,t),\tau(t))},$$

where

$$\eta(x,t) = \frac{\Omega' e^{\gamma t/2} x}{\omega_0 \sinh[\Omega' t + \beta]}, \quad \tau(t) = -\frac{\Omega'}{\omega_0^2}\Big(\coth[\Omega' t + \beta] - \frac{\gamma}{2\Omega'}\Big),$$

and $\varphi(\eta,\tau)$ satisfies the IBVP for the Heat equation with the Robin BC

$$\begin{cases} \varphi_\tau = \frac{1}{2}\varphi_{\eta\eta}, \quad 0 < \eta < \infty, \quad 0 < \tau < \infty, \\ \varphi(\eta,0) = \exp\left[-\int^\eta U(\eta',0)d\eta'\right], \quad 0 < \eta < \infty, \\ \left[r(t(\tau))\mu(t(\tau))U(0,t(\tau))\right]\varphi(0,\tau) + \varphi_\eta(0,\tau) = 0, \quad 0 < \tau < \infty. \end{cases} \tag{4.51}$$

As we found in previous Chapter, solution of this Heat IBVP is formally

$$\varphi(\eta,\tau) = \int_0^\infty \Big(\frac{e^{-\frac{(\eta-\xi)^2}{2\tau}} + e^{-\frac{(\eta+\xi)^2}{2\tau}}}{\sqrt{2\pi\tau}}\Big) e^{-\int^\xi U(\eta',0)d\eta'} d\xi - \int_0^\tau \frac{e^{-\frac{\eta^2}{2(\tau-\tau')}}}{\sqrt{2\pi(\tau-\tau')}} \underbrace{Q(\tau')}_{\varphi_\eta(0,\tau')} d\tau',$$

where $Q(\tau)$ is found by solving the integral equation

$$Q(\tau) = \big[r(t(\tau))\mu(t(\tau))U(0,t(\tau))\big]\Big(\int_0^\tau \frac{Q(\tau')}{\sqrt{2\pi(\tau-\tau')}}d\tau' - 2\int_0^\infty \frac{e^{-\frac{\xi^2}{2\tau}}}{\sqrt{2\pi\tau}} e^{-\int^\xi U(\eta',0)d\eta'} d\xi\Big).$$

4.2. The Neumann Problem for Inhomogeneous Burgers Equation with Time-variable Coefficients on Semi-infinite Line

Proposition 4.2 *The IBVP for the variable parametric BE*

$$\begin{cases} U_t + \frac{\dot\mu(t)}{\mu(t)}U + UU_x = \frac{1}{2\mu(t)}U_{xx} - \omega^2(t)x, \quad 0 < x < \infty, \quad t_0 < t < T, \\ U(x,t)\,|_{t=t_0} = U(x,t_0), \quad 0 < x < \infty, \\ U_x(0,t) = \Theta(t), \quad t_0 < t < T, \end{cases} \tag{4.52}$$

where $\mu(t) > 0$ is continuously differentiable, $\omega^2(t)$, $\Theta(t)$ are real-valued continuous for $t \geqslant t_0$, has solution in the following forms:

a)

$$U(x,t) = \frac{\dot r(t)}{r(t)}x + \frac{r_0}{\mu(t)r(t)}V(\eta(x,t),\tau(t)),$$

if for $t \geq t_0$ $r(t)$ is a positive solution of the IVP for the following linear ODE

$$\ddot r + \frac{\dot\mu(t)}{\mu(t)}r + \omega^2(t) = 0, \tag{4.53}$$

$$r(t_0) = r_0 \neq 0, \quad \dot r(t_0) = 0, \tag{4.54}$$

with functions $\eta(x,t) = \frac{r_0}{r(t)}x,\ \ \tau(t) = r_0^2\int_{t_0}^{t}\frac{d\xi}{\mu(\xi)r^2(\xi)}$ *and* $V(\eta,\tau)$ *satisfies the IBVP for BE*

$$\begin{cases} V_\tau + VV_\eta = \frac{1}{2}V_{\eta\eta}, \quad 0<\eta<\infty, \quad 0<\tau<\tau(T), \\ V(\eta,0) = \mu_0 U(\eta,t_0), \quad 0<\eta<\infty, \\ V_\eta(0,\tau) = r_0^{-2}[U_x(0,t(\tau))\mu(t(\tau))r^2(t(\tau)) - \mu(t(\tau))\dot{r}(t(\tau))r(t(\tau))], \quad 0<\tau<\tau(T). \end{cases} \tag{4.55}$$

b)

$$U(x,t) = \frac{\dot{r}(t)}{r(t)}x - \frac{r_0}{\mu(t)r(t)}\frac{\varphi_\eta(\eta(x,t),\tau(t))}{\varphi(\eta(x,t),\tau(t))},$$

where η*,* τ *and* r *defined in* a) *and* $\varphi(\eta,\tau)$ *satisfies the IBVP for the HE*

$$\begin{cases} \varphi_\tau = \frac{1}{2}\varphi_{\eta\eta}, \quad 0<\eta<\infty, \quad 0<\tau<\tau(T), \\ \varphi(\eta,0) = \exp\left[-\int^{\eta}\mu_0 U(\xi,t_0)d\xi\right], \quad 0<\eta<\infty, \\ \varphi_\eta^2(0,\tau) - 2\varphi_\tau(0,\tau)\varphi(0,\tau) - \varphi^2(0,\tau)\Big[r_0^{-2}[U_x(0,t(\tau))\mu(t(\tau))r^2(t(\tau)) - \mu(t(\tau))\dot{r}(t(\tau))r(t(\tau))]\Big] = 0. \end{cases}$$

Proof:

a) If the functions $\rho(t)$, $\tau(t)$ and $s(t)$ satisfy the nonlinear system of ordinary differential equations

$$\dot{\rho} + \frac{\rho^2}{\mu(t)} + \mu(t)\omega^2(t) = 0, \quad \rho(t_0) = 0, \tag{4.56}$$

$$\dot{\tau} - \frac{s^2}{\mu(t)} = 0, \quad \tau(t_0) = 0, \tag{4.57}$$

$$\dot{s} + \frac{\rho(t)}{\mu(t)s} = 0, \quad s(t_0) = 1, \tag{4.58}$$

and $V(\eta,\tau)$ satisfies the standard BE (4.55), then

$$U(x,t) = \frac{\rho(t)x + s(t)V\big(s(t)x,\tau(t)\big)}{\mu(t)}, \tag{4.59}$$

satisfies the inhomogeneous BE (4.52). Notice that, equation (4.56) is a nonlinear Riccati equation and substitution $\rho(t) = \dot{r}(t)/r(t)$ gives $\ddot{r} + \frac{\dot{\mu}(t)}{\mu(t)}r + \omega^2(t) = 0$, then system is easily solved and we obtain $\rho(t) = \mu(t)\frac{\dot{r}(t)}{r(t)}$, $\tau(t) = r_0^2\int_{t_0}^{t}\frac{d\xi}{\mu(\xi)r^2(\xi)}$, $s(t) = \frac{r_0}{r(t)}$ where $r(t)$ is the solution of linear ODE and by substituting in (4.59) gives solution in the form

$$U(x,t) = \frac{\dot{r}(t)}{r(t)}x + \frac{r_0}{\mu(t)r(t)}V(\eta(x,t),\tau(t)).$$

Then, IC $U(x,t)\mid_{t=t_0} = U(x,t_0)$ easily transforms to IC $V(\eta,0) = \mu_0 U(\eta,t_0)$. But the Neumann BC for the inhomogeneous BE $U_x(0,t) = \Theta(t)$ transforms to the Neumann BC for the standard BE. Thus, solution of the inhomogeneous Burgers problem is explicitly obtained in terms of solution of $V(\eta,\tau)$ (4.55) and solution $r(t)$ of the IVP for the linear ODE (4.53).

Part **b)** of the proposition follows directly from the Cole-Hopf transformation $V = -\varphi_\eta/\varphi$. The Neumann BC for BE $r_0^{-2}[U_x(0,t(\tau))\mu(t)r^2(t) - \mu(t)\dot{r}(t)r(t)]$ transforms to nonlinear BC for HE by Cole-Hopf and using the relation $V_\eta = \frac{2\varphi_\eta}{\varphi} - \frac{\varphi_\eta^2}{\varphi^2}$ as follows

$$\varphi_\eta^2(0,\tau) - 2\varphi_\tau(0,\tau)\varphi(0,\tau) - \varphi^2(0,\tau)\Big[r_0^{-2}[U_x(0,t(\tau))\mu(t)r^2(t) - \mu(t)\dot{r}(t)r(t)]\Big] = 0,$$

and IC $V(\eta,0) = \mu_0 U(\eta,t_0)$ for the standard BE transforms to IC for the HE $\varphi(\eta,0) = \exp\left[-\int^{\eta}\mu_0 U(\xi,t_0)d\xi\right]$. Then the IBVP (4.55) for the standard BE is reduced to the IBVP (4.56) for the HE. ∎

CHAPTER 5

THE CAUCHY PROBLEM

In this Chapter, firstly we review the closed form solution of Cauchy problem for the Heat equation, **[10]**. Then, motivated by the works of **[7]** and **[8]**, we consider Cauchy problem for standard Burgers equation. And we will show that some special well-known solutions of Burgers equation can be obtained as solution of Cauchy problem for Burgers equation. Finally, we investigate the Cauchy problem for the variable Burgers equation.

5.1. The Cauchy Problem for Heat Equation

We consider Cauchy problem for the HE on the non-characteristic line $\eta = 0$

$$\begin{cases} \varphi_\tau = \frac{1}{2}\varphi_{\eta\eta}, \quad -\infty < \eta < \infty \quad \tau > 0, \\ \varphi(0,\tau) = F(\tau), \qquad \tau > 0, \\ \varphi_\eta(0,\tau) = G(\tau), \qquad \tau > 0, \end{cases} \tag{5.1}$$

where $F(\tau)$ and $G(\tau)$ are analytic functions. Assuming a power series solution of the form,

$$\varphi(\eta,\tau) = \sum_{n=0}^{\infty} a_n(\tau)\eta^n,$$

one can easily determine the functions $a_n(\tau)$ in terms of $F(\tau)$ and $G(\tau)$ for all $n = 0, 1, 2, \cdots$. Indeed, by substituting φ into Heat equation (5.1), we have

$$\varphi_\tau - \frac{1}{2}\varphi_{\eta\eta} = \sum_{n=0}^{\infty}\left(a_n'(\tau) - \frac{(n+2)(n+1)}{2}a_{n+2}(\tau)\right)\eta^n = 0,$$

which requires that

$$a_{n+2}(\tau) \quad = \quad 2\frac{a_n'(\tau)}{(n+2)(n+1)}, \qquad n = 0, 1, 2, \cdots .$$

From this recursion relation, we see that

$$\begin{aligned} a_{2n}(\tau) &= 2^n \frac{a_0^{(n)}(\tau)}{(2n)!}, \qquad n = 0, 1, 2, \cdots , \\ a_{2n+1}(\tau) &= 2^n \frac{a_1^{(n)}(\tau)}{(2n+1)!}. \qquad n = 0, 1, 2, \cdots . \end{aligned}$$

Note that $a_0(\tau) = F(\tau)$ and $a_1(\tau) = G(\tau)$. Thus, solution of the problem (5.1) is obtained in the form, **[5]**

$$\varphi(\eta, \tau) = \sum_{n=0}^{\infty} \left(F^{(n)}(\tau) \frac{\eta^{2n}}{(2n)!} 2^n + G^{(n)}(\tau) \frac{\eta^{2n+1}}{(2n+1)!} 2^n \right). \tag{5.2}$$

When $F(\tau)$ and $G(\tau)$ are analytic functions for $\tau > 0$, then solution (5.2) is unique on $-\infty < \eta < \infty$, and also it is analytic for $\tau > 0$ and fixed η, and is entire in η for fixed τ, **[10]**.

5.2. The Cauchy Problem for Burgers equation

In the works **[9]** and **[7]**, the Cauchy problem for the Burgers equation was formulated and its solution was obtained in terms of the series solution of the corresponding Cauchy problem for the Heat equation, **[10]**. Then, these results were used by Rodin to solve some concrete problems and show that some well-known solutions of Burgers equation can be recovered as solutions of a Cauchy problem. The following Proposition 1 is a basic result and consequence of the Cole-Hopf transform, **[9]**.

Proposition 5.1 *If $f(\tau)$ and $g(\tau)$ are analytic functions for $\tau > 0$, then the Cauchy problem for the Burgers equation defined by*

$$\begin{cases} V_\tau + VV_\eta = \frac{1}{2}V\eta\eta, & -\infty < \eta < \infty, \quad \tau > 0, \\ V(0,\tau) = f(\tau), & \tau > 0, \\ V_\eta(0,\tau) = g(\tau), & \tau > 0, \end{cases} \tag{5.3}$$

has a solution of the form

$$V(\eta,\tau) = -\frac{\sum_{n=0}^{\infty}\left(F^{(n+1)}(\tau)\frac{\eta^{2n+1}}{(2n+1)!}2^{n+1} + G^{(n)}(\tau)\frac{\eta^{2n}}{(2n)!}2^n\right)}{\sum_{n=0}^{\infty}\left(2^nF^{(n)}(\tau)\frac{\eta^{2n}}{(2n)!} + 2^nG^{(n)}(\tau)\frac{\eta^{2n+1}}{(2n+1)!}\right)},$$

where

$$F(\tau) = e^{-\frac{1}{2}\int_0^\tau (g(\tau')-f^2(\tau'))d\tau'}, \quad \tau > 0, \tag{5.4}$$

$$G(\tau) = -f(\tau)e^{-\frac{1}{2}\int_0^\tau (g(\tau')-f^2(\tau'))d\tau'}, \quad \tau > 0. \tag{5.5}$$

Proof:

It is enough to show that the Cauchy problem for Burgers equation (5.3) is reducible to the Cauchy problem for heat equation

$$\begin{cases} \varphi_\tau = \frac{1}{2}\varphi_{\eta\eta}, & -\infty < \eta < \infty, \quad \tau > 0, \\ \varphi(0,\tau) = c_0F(\tau), & \tau > 0, \\ \varphi_\eta(0,\tau) = c_0G(\tau), & \tau > 0, \end{cases} \tag{5.6}$$

where $F(\tau)$ and $G(\tau)$ are as defined in (5.4) and $c_0 = \varphi(0,0^+)$. Indeed, the Cauchy data can be transformed using that

$$V(\eta,\tau) = -\frac{\varphi_\eta(\eta,\tau)}{\varphi(\eta,\tau)}, \qquad V_\eta(\eta,\tau) = -2\frac{\varphi_\tau(\eta,\tau)}{\varphi(\eta,\tau)} + \left(\frac{\varphi_\eta(\eta,\tau)}{\varphi(\eta,\tau)}\right)^2. \tag{5.7}$$

This implies

$$f(\tau) = -\frac{\varphi_\eta(0,\tau)}{\varphi(0,\tau)}, \qquad g(\tau) = -2\frac{\varphi_\tau(0,\tau)}{\varphi(0,\tau)} + \left(\frac{\varphi_\eta(0,\tau)}{\varphi(0,\tau)}\right)^2, \tag{5.8}$$

so that

$$g(\tau) = -2\frac{\varphi_\tau(0,\tau)}{\varphi(0,\tau)} + f^2(\tau). \tag{5.9}$$

Then, integrating (5.9) with respect to τ we obtain

$$\varphi(0,\tau) \quad = \quad c_0 e^{-\frac{1}{2}\int_0^\tau (g(\tau')-f^2(\tau'))d\tau'}, \tag{5.10}$$

and this immediately gives

$$\varphi_\eta(0,\tau) \quad = \quad -c_0 f(\tau) e^{-\frac{1}{2}\int_0^\tau (g(\tau')-f^2(\tau'))d\tau'}. \tag{5.11}$$

The rest of the proposition follows from the solution (5.2) of the Cauchy problem for the heat equation. ■

Given some special Cauchy data for the Burgers equation, we write the Cauchy data for the corresponding heat equation as follows.

$$\begin{cases} V(0,\tau) = f(\tau) = 0 \quad \Longrightarrow \quad \varphi(0,\tau) = constant, \\ V_\eta(0,\tau) = g(\tau) = 0 \quad \Longrightarrow \quad \varphi_\eta(0,\tau) = 0. \end{cases}$$

$$\begin{cases} V(0,\tau) = f(\tau) = 0 \quad \Longrightarrow \quad \varphi(0,\tau) = e^{-\frac{1}{2}\int_0^\tau g(\tau')d\tau'}, \\ V_\eta(0,\tau) = g(\tau) \neq 0 \quad \Longrightarrow \quad \varphi_\eta(0,\tau) = 0. \end{cases}$$

$$\begin{cases} V(0,\tau) = f(\tau) \neq 0 \quad \Longrightarrow \quad \varphi(0,\tau) = e^{\frac{1}{2}\int_0^\tau f^2(\tau')d\tau'}, \\ V_\eta(0,\tau) = g(\tau) = 0 \quad \Longrightarrow \quad \varphi_\eta(0,\tau) = -f(\tau)e^{\frac{1}{2}\int_0^\tau f^2(\tau')d\tau'}. \end{cases}$$

In particular, if the Burgers conditions are of the form

$$V(0,\tau) = -\frac{G(\tau)}{F(\tau)}, \qquad V_\eta(0,\tau) = \frac{G^2(\tau)}{F^2(\tau)} - 2\frac{F'(\tau)}{F(\tau)}, \tag{5.12}$$

then the corresponding heat conditions become

$$\varphi(0,\tau) = F(\tau), \qquad \varphi_\eta(0,\tau) = G(\tau). \tag{5.13}$$

Using the above results, Rodin **[7]** obtained solution of the problem related with moving piston, and showed that the Fay's solution and Benton's solutions can be found also as solutions of a Cauchy problem.

Here, we give different examples. We show how the shock-wave, triangular wave and N-wave solutions of the Burgers equation can be obtained as solutions of a Cauchy problem by the approach described in this section.

Example 5.1 *: **Shock-Wave Solution***
We show that the well-known solution for Burgers can be obtained by solving the following BVP for Burgers

$$\begin{cases} V_\eta + VV_\eta = \frac{1}{2}V\eta\eta, \quad -\infty < \eta < \infty \quad \tau > 0, \\ V(0,\tau) = c - A\tanh(-A(c\tau - c_0)), \quad A,\ c,\ c_0 : constant, \quad \tau > 0, \\ V_\eta(0,\tau) = A^2\tanh^2(-A(c\tau - c_0)) - A^2 \qquad \tau > 0. \end{cases} \tag{5.14}$$

The corresponding BC's $F(\tau)$ and $G(\tau)$ for Heat are found as follows

$$V(0,\tau) = c - A\tanh(-A(c\tau - c_0)) = -\frac{G(\tau)}{F(\tau)}, \tag{5.15}$$

$$V_\eta(0,\tau) = A^2\tanh^2(-A(c\tau - c_0)) - A^2 = \frac{G^2(\tau)}{F^2(\tau)} - 2\frac{F'(\tau)}{F(\tau)}. \tag{5.16}$$

Taking square of the first BC (5.15) and substituting into second one, we get

$$c^2 - 2Ac\tanh(-A(c\tau - c_0)) + A^2\tanh^2(-A(c\tau - c_0)) - 2\frac{F'}{F} = A^2(\tanh^2(-A(c\tau - c_0)) - 1),$$

where we have

$$\begin{aligned}\frac{F'}{F} &= \frac{c^2+A^2}{2} - Ac\tanh(-A(c\tau - c_0)),\\ F(\tau) &= \cosh(A(c_0 - c\tau))e^{\frac{c^2+A^2}{2}\tau} = \frac{1}{2}\Big(e^{-Ac_0}e^{\frac{(c+A)^2}{2}\tau} + e^{Ac_0}e^{\frac{(c-A)^2}{2}\tau}\Big).\end{aligned}$$

The derivatives of $F(\tau)$ are given as follows

$$\begin{aligned}&F'(\tau) = \tfrac{e^{-Ac_0}}{2}\tfrac{(c+A)^2}{2}e^{(\frac{(c+A)^2}{2})\tau} + \tfrac{e^{Ac_0}}{2}\tfrac{(c-A)^2}{2}e^{\frac{(c-A)^2}{2}\tau},\\ &\vdots\\ &F^{(n)}(\tau) = \tfrac{e^{-Ac_0}}{2}\tfrac{(c+A)^2 n}{2}e^{(\frac{(c+A)^2}{2})\tau} + \tfrac{e^{Ac_0}}{2}\tfrac{(c-A)^2 n}{2}e^{\frac{(c-A)^2}{2}\tau}.\end{aligned} \tag{5.17}$$

From condition $V(0,\tau)$, we have $G(\tau) = -F(\tau)(c - A\tanh(Ac_0 - Ac\tau))$, then substituting $F(\tau)$ we get

$$\begin{aligned}G(\tau) &= -c\cosh(A(c_0 - c\tau))e^{\frac{c^2+A^2}{2}\tau} + A\sinh(A(c_0 - c\tau))e^{\frac{c^2+A^2}{2}\tau}, && (5.18)\\ &= -\frac{e^{-Ac_0}}{2}(c+A)e^{\frac{(c+A)^2}{2}\tau} - \frac{e^{Ac_0}}{2}(c-A)e^{\frac{(c-A)^2}{2}\tau}. && (5.19)\end{aligned}$$

The derivatives of $G(\tau)$ are given by

$$\begin{aligned}&G'(\tau) = -\tfrac{e^{-Ac_0}}{2}(c+A)\tfrac{(c+A)^2}{2}e^{\frac{(c+A)^2}{2}\tau} - \tfrac{e^{Ac_0}}{2}(c-A)e^{\frac{(c-A)^2}{2}\tau}\tfrac{(c-A)^2}{2},\\ &\vdots\\ &G^{(n)}(\tau) = -\tfrac{e^{-Ac_0}}{2}\tfrac{(c+A)^{2n+1}}{2^n}e^{\frac{(c+A)^2}{2}\tau} - \tfrac{e^{Ac_0}}{2}\tfrac{(c-A)^{2n+1}}{2^n}e^{\frac{(c-A)^2}{2}\tau}.\end{aligned} \tag{5.20}$$

Substituting $F(\tau)$ and $G(\tau)$ into closed form solution of Burgers (5.24), we obtain

$$V(\eta,\tau) = \frac{(c+A)e^{\frac{(c+A)^2}{2}\tau - Ac_0 - \eta(c+A)} + (c-A)e^{\frac{(c-A)^2}{2}\tau + Ac_0 - \eta(c-A)}}{e^{\frac{(c+A)^2}{2}\tau - Ac_0 - \eta(c+A)} + e^{\frac{(c-A)^2}{2}\tau + Ac_0 - \eta(c-A)}},$$

or equivalently the shock-wave solution is

$$V(\eta,\tau) = c - A\tanh(A(\eta - c\tau + c_0)).$$

Example 5.2 *: **Triangular Wave Solution***

Here, we show that the well-known Triangular-Wave can be obtained as solution of a Cauchy problem for standard Burgers,

$$\begin{cases} V_\eta + VV_\eta = \frac{1}{2}V\eta\eta, \quad -\infty < \eta < \infty \quad \tau > 0, \\ V(0,\tau) = \frac{2}{\sqrt{2\pi\tau}}(\frac{e^A-1}{e^A+1}), \quad \tau > 0, \\ V_\eta(0,\tau) = \frac{2}{\pi\tau}\left(\frac{e^A-1}{e^A+1}\right)^2, \quad \tau > 0. \end{cases} \tag{5.21}$$

The corresponding BC's $F(\tau)$ and $G(\tau)$ for Heat are found as follows

$$\begin{aligned} V(0,\tau) &= \tfrac{2}{\sqrt{2\pi\tau}}(\tfrac{e^A-1}{e^A+1}) = -\tfrac{G(\tau)}{F(\tau)}, \\ V_\eta(0,\tau) &= \tfrac{2}{\pi\tau}\left(\tfrac{e^A-1}{e^A+1}\right)^2 = \tfrac{G^2(\tau)}{F^2(\tau)} - 2\tfrac{F'(\tau)}{F(\tau)}. \end{aligned} \tag{5.22}$$

Taking square of first BC in (5.22) and substituting into second one

$$\begin{aligned} \frac{4}{2\pi\tau}\left(\frac{e^A-1}{e^A+1}\right)^2 - 2\frac{F'(\tau)}{F(\tau)} &= \frac{2}{\pi\tau}\left(\frac{e^A-1}{e^A+1}\right)^2 \\ -2\frac{F'(\tau)}{F(\tau)} &= 0 \ => \ F(\tau) = c : constant, \\ G(\tau) = -F(\tau)\frac{2}{\sqrt{2\pi\tau}}\left(\frac{e^A-1}{e^A+1}\right) &= -\frac{2c}{\sqrt{2\pi\tau}}\left(\frac{e^A-1}{e^A+1}\right). \end{aligned}$$

Since $F(\tau)$ is constant, $F^{(n)}(\tau) = 0$ for $n = 1, 2, 3, \cdots$.
$G(\tau) = -\frac{2c}{\sqrt{2\pi}}(\frac{e^A-1}{e^A+1})\frac{1}{\sqrt{\tau}}$ and define $-\frac{2c}{\sqrt{2\pi}}\left(\frac{e^A-1}{e^A+1}\right) = K$: constant, then the derivatives of $G(\tau)$ are in the following form

$$\begin{aligned} G(\tau) &= K\tau^{-1/2}, \\ G'(\tau) &= K\tfrac{-1}{2}\tau^{-3/2}, \\ G''(\tau) &= K\tfrac{(-1)^2}{2^2}1.3.\tau^{-5/2}, \\ G'''(\tau) &= K\tfrac{(-1)^3}{2^3}1.3.5.\tau^{-7/2}, \\ &\vdots \\ G^{(n)}(\tau) &= K\tfrac{(-1)^n}{2^n}\tfrac{(2n)!}{2^n n!}.\tau^{-n-1/2}. \end{aligned} \tag{5.23}$$

Previously, we get general closed form solution of the standard Burgers i.e,

$$V(\eta,\tau) = -\frac{\sum_{n=0}^{\infty}\left(F^{(n+1)}(\tau)\frac{\eta^{2n+1}}{(2n+1)!}2^{n+1} + G^{(n)}(\tau)\frac{\eta^{2n}}{(2n)!}2^{n}\right)}{\sum_{n=0}^{\infty}\left(F^{(n)}(\tau)\frac{\eta^{2n}}{(2n)!}2^{n} + G^{(n)}(\tau)\frac{\eta^{2n+1}}{(2n+1)!}2^{n}\right)}. \tag{5.24}$$

Substituting $F(\tau)$ and $G(\tau)$ into (5.24),

$$\begin{aligned} V(\eta,\tau) &= -\frac{K\sum_{n=0}^{\infty}\frac{(-1)^n}{2^n}\frac{(2n)!}{2^n.n!}.\tau^{-n-1/2}.\frac{\eta^{2n}}{(2n)!}.2^n}{c + K\sum_{n=0}^{\infty}\frac{(-1)^n}{2^n}\frac{(2n)!}{2^n.n!}.\tau^{-n-1/2}.\frac{\eta^{2n+1}}{(2n+1)!}.2^n}, \\ &= -\frac{\frac{K}{\sqrt{\tau}}\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}.(\frac{\eta^2}{2\tau})^n}{c + \frac{\eta K}{\sqrt{\tau}}\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}(\frac{\eta^2}{2\tau})^n.\frac{1}{2n+1}}. \end{aligned} \tag{5.25}$$

Using $e^{-\frac{\eta^2}{2\tau}} = \sum_{n=0}^{\infty}\frac{1}{n!}(-\frac{\eta^2}{2\tau})^n$ and substituting $K = -\frac{2c}{\sqrt{2\pi}}(\frac{e^A-1}{e^A+1})$ into (5.25), we obtain

$$V(\eta,\tau) = \frac{\frac{2c}{\sqrt{2\pi}}(\frac{e^A-1}{e^A+1}).e^{-\frac{\eta^2}{2\tau}}}{c - \frac{2\eta c}{\sqrt{2\pi\tau}}(\frac{e^A-1}{e^A+1})\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}(\frac{\eta^2}{2\tau})^n.\frac{1}{2n+1}}. \tag{5.26}$$

We know $Erf(x) = \frac{2}{\sqrt{\tau}}\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{(2n+1)n!}$. By changing variable $x = \frac{\eta}{\sqrt{2\tau}}$, we have

$$Erf\left(\frac{\eta}{\sqrt{2\tau}}\right) = \frac{2}{\sqrt{\tau}}\sum_{n=0}^{\infty}(-1)^n\frac{\eta^{2n+1}}{(\sqrt{2\tau})^{2n+1}}\frac{1}{(2n+1)n!}. \tag{5.27}$$

Then equation (5.26) is equivalent to the following

$$V(\eta,\tau) = \frac{\frac{2}{\sqrt{2\pi\tau}}(\frac{e^A-1}{e^A+1})e^{-\frac{\eta^2}{2\tau}}}{1 - (\frac{e^A-1}{e^A+1})\underbrace{\frac{2}{\sqrt{\pi}}\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{(2n+1)n!}}_{erf(\frac{\eta}{\sqrt{2\tau}})}} \tag{5.28}$$

$$= \frac{2}{\sqrt{2\pi\tau}}\left(\frac{(\frac{e^A-1}{e^A+1})e^{-\frac{\eta^2}{2\tau}}}{\frac{2}{e^A+1} + (\frac{e^A-1}{e^A+1})Erfc\left[\frac{\eta}{\sqrt{2\tau}}\right]}\right).$$

If we multiply and divide right hand side of (5.28) by $(e^A + 1)/2$*, then we obtain the solution of (5.21) for Burgers equation as follows*

$$V(\eta,\tau) \quad = \quad \frac{1}{\sqrt{2\pi\tau}}\left(\frac{(e^A - 1)e^{-\frac{\eta^2}{2\tau}}}{1 + \frac{1}{2}(e^A - 1)Erfc\left[\frac{\eta}{\sqrt{2\tau}}\right]}\right).$$

Example 5.3 *: **N-Wave Solution***

$$\begin{cases} V_\eta + VV_\eta = \frac{1}{2}V\eta\eta, \quad -\infty < \eta < \infty \quad \tau > 0, \\ V(0,\tau) = 0, \qquad \tau > 0, \\ V_\eta(0,\tau) = \frac{1}{\tau}\left(\frac{\sqrt{\frac{a}{\tau}}}{1+\sqrt{\frac{a}{\tau}}}\right), \qquad \tau > 0. \end{cases} \tag{5.29}$$

The BC's for the Heat are found as follows,

$$\begin{aligned} &V(0,\tau) = 0 = -\frac{G(\tau)}{F(\tau)}, \\ &V_\eta(0,\tau) = \frac{1}{\tau}\frac{\sqrt{\frac{a}{\tau}}}{(1+\sqrt{\frac{a}{\tau}})} = \frac{G^2(\tau)}{F^2(\tau)} - 2\frac{F'(\tau)}{F(\tau)}. \end{aligned} \tag{5.30}$$

Taking square of first BC and substituting into second one we have

$$-2\frac{F'(\tau)}{F(\tau)} = \frac{1}{\tau}\frac{\sqrt{\frac{a}{\tau}}}{(1 + \sqrt{\frac{a}{\tau}})}, \tag{5.31}$$

and integrating both sides of (5.31), we get

$$F(\tau) = \left(1 + \sqrt{\frac{a}{\tau}}\right) = 1 + \sqrt{a}\tau^{-1/2}, \tag{5.32}$$

where the derivatives of $F(\tau)$ as follows

$$\begin{aligned}
F'(\tau) &= \sqrt{a}.\tfrac{(-1)}{2}\tau^{-3/2},\\
F''(\tau) &= \sqrt{a}.\tfrac{(-1)}{2}.\tfrac{(-3)}{2}\tau^{-5/2},\\
&\vdots\\
F^{(n+1)}(\tau) &= \sqrt{a}.\tfrac{(-1)^{n+1}}{2^{n+1}}.\tfrac{(2n+2)!}{2^{n+1}n!}\tau^{-(2n+3)/2}.
\end{aligned} \tag{5.33}$$

Since $G(\tau) = 0$, we have closed form solution for the standard Burgers as follows

$$\begin{aligned}
V(\eta,\tau) &= -\frac{\sqrt{a}\sum_{n=0}^{\infty}\frac{(-1)^{n+1}}{2^{n+2}}.\frac{(2n+2)!}{(n+1)!}\frac{\eta^{2n+1}}{(2n+1)!}2^{n+1}\tau^{(-n-3/2)}}{1+\sqrt{\frac{a}{\tau}}+\sqrt{a}\sum_{n=1}^{\infty}\frac{(-1)^n}{2^n}.\frac{(2n)!}{2^n(n)!}\frac{\eta^{2n}}{(2n)!}2^n\tau^{(-n-1/2)}},\\
&= \frac{\sqrt{a}\frac{\eta}{\sqrt{\tau}}\frac{1}{\tau}\sum_{n=0}^{\infty}(-1)^n(\frac{\eta^2}{2\tau})^n\frac{1}{n!}}{1+\sqrt{\frac{a}{\tau}}+\sqrt{\frac{a}{\tau}}\sum_{n=1}^{\infty}\frac{(-1)^n}{n!}.(\frac{\eta^2}{2\tau})^n},
\end{aligned}$$

or equivalently we obtain the N-wave solution

$$V(\eta,\tau) = \frac{\eta}{\tau}\left(\frac{\sqrt{\frac{a}{\tau}}e^{-\frac{\eta^2}{2\tau}}}{1+\sqrt{\frac{a}{\tau}}e^{-\frac{\eta^2}{2\tau}}}\right).$$

5.3. The Cauchy Problem for the Inhomogeneous Burgers Equation with Time-variable Coefficients

Proposition 5.2 *Let the Cauchy problem for inhomogeneous Burgers equation with variable coefficients be given as*

$$\begin{cases}
U_t + \frac{\dot{\mu}(t)}{\mu(t)}U + UU_x = \frac{1}{2\mu(t)}U_{xx} - \omega^2(t)x, \quad -\infty < x < \infty, \quad t_0 < t < T,\\
U(0,t) = A(t), \quad t_0 < t < T,\\
U_x(0,t) = B(t), \quad t_0 < t < T,
\end{cases} \tag{5.34}$$

where $\mu(t) > 0$, $A(t)$ and $B(t)$ are analytic functions for $t \in (t_0, T)$. If for $t_0 \le t < T$ the function $r(t)$ is strictly positive analytic solution of the IVP for the second order linear

ODE

$$\ddot{r} + \frac{\dot{\mu}(t)}{\mu(t)}\dot{r} + \omega^2(t) r = 0, \tag{5.35}$$

$$r(t_0) = r_0 \neq 0 \quad \dot{r}(t_0) = 0, \tag{5.36}$$

and

$$\eta(x,t) = \frac{r(t_0)}{r(t)} x, \quad \tau(t) = r^2(t_0) \int_{t_0}^{t} \frac{d\xi}{\mu(\xi) r^2(\xi)}, \tag{5.37}$$

then the problem (5.34) has solution of the form

$$U(x,t) = \frac{\dot{r}(t)}{r(t)} x - \frac{r(t_0)}{\mu(t) r(t)} \frac{\sum_{n=0}^{\infty} \left(F^{(n+1)}(\tau(t)) \frac{[\eta(x,t)]^{2n+1}}{(2n+1)!} 2^{n+1} + G^{(n)}(\tau(t)) \frac{[\eta(x,t)]^{2n}}{(2n)!} 2^n \right)}{\sum_{n=0}^{\infty} \left(2^n F^{(n)}(\tau(t)) \frac{[\eta(x,t)]^{2n}}{(2n)!} + 2^n G^{(n)}(\tau(t)) \frac{[\eta(x,t)]^{2n+1}}{(2n+1)!} \right)},$$

where

$$F(\tau) = \exp\left(-\frac{1}{2} \int_0^{\tau} \frac{\mu(t(\tau')) r^2(t(\tau'))}{r_0^2} \left(B(t(\tau')) - A^2(t(\tau')) - \frac{\dot{r}(t(\tau'))}{r(t(\tau'))} \right) d\tau' \right), \tag{5.38}$$

$$G(\tau) = -\frac{\mu(t(\tau) r(t(\tau))}{r_0} A(t(\tau)) F(\tau). \tag{5.39}$$

Proof:

If $V(\eta,\tau)$ satisfies the standard Burgers equation $V_\tau + V V_\eta = \frac{1}{2} V_{\eta\eta}$, then

$$U(x,t) = \frac{\dot{r}(t)}{r(t)} x + \frac{r_0}{\mu(t) r(t)} V(\eta(x,t), \tau(t)),$$

satisfies the Burgers equation of problem (5.34). Then the Cauchy problem (5.34) reduces to the Cauchy problem for the standard Burgers equation

$$\begin{cases} V_\tau + V V_\eta = \frac{1}{2} V_{\eta\eta}, \quad -\infty < \eta < \infty, \quad 0 < \tau < \tau(T), \\ V(0,\tau) = \frac{\mu(t(\tau)) r(t(\tau))}{r_0} A(t(\tau)), \quad 0 < \tau < \tau(T), \\ V_\eta(0,\tau) = \frac{\mu(t(\tau)) r^2(t(\tau))}{r_0^2} \left(B(t(\tau)) - \frac{\dot{r}(t(\tau))}{r(t(\tau))} \right), \quad 0 < \tau < \tau(T). \end{cases} \tag{5.40}$$

Using the results obtained in previous section, problem (5.40) reduces to the following heat problem

$$\begin{cases} \varphi_\eta = \frac{1}{2}\varphi_{\eta\eta}, & -\infty < \eta < \infty, \quad 0 < \tau < \tau(T), \\ \varphi(0,\tau) = c_0 F(\tau), & 0 < \tau < \tau(T), \\ \varphi_\eta(0,\tau) = c_0 G(\tau), & 0 < \tau < \tau(T), \end{cases} \tag{5.41}$$

where $F(\tau)$ and $G(\tau)$ are given by (5.38), (5.39). ■

CHAPTER 6

CONCLUSION

In this thesis, we have investigated initial-boundary value problems (IBVP's) on semi-infinite line $0 < x < \infty$ for inhomogeneous Burgers equations with time-variable coefficients (VBE). We have formulated the solutions for the cases with Dirichlet BC and Neumann BC imposed at $x = 0$.

First, we showed that the Dirichlet problem for the VBE reduces to solving a linear ODE and a second kind singular Volterra integral equation. Therefore, solutions in general can be obtained using approximate and numerical techniques. However, for particular VBE models with special initial and Dirichlet BCs we found some class of exact solutions. Next, we worked on the Neumann problem for the VBE and we found that it reduces to a second order linear ODE and an IBVP for the standard heat equation with nonlinear boundary conditions.

Finally, we recalled the Cauchy problem for the heat equation on the non-characteristic line, and derived its well-known solution as an infinite power series, whose coefficients are obtained in terms of the Cauchy data **[10]**. This result was used in **[9]** to solve the Cauchy problem for the standard Burgers equation. We gave examples to illustrate how some well known solutions of the Burgers equation can be recovered by solving a corresponding Cauchy problem. Then we formulated the Cauchy problem for the VBE, and obtained its solution in terms of a linear ODE and the series solution of the corresponding Cauchy problem for the heat equation.

In this work, we were able to derive mostly the general form of the solutions for the given problems. Our research on exact solutions for the Burgers problems with variable coefficients is not complete and we plan to extend the list of the exactly solvable models and study their properties in details.

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APPENDIX A

SOME BASIC PROPERTIES AND DEFINITIONS

•Properties of the fundamental solution $K(\eta, \tau)$:

1. $K(\eta, \tau) > 0$ for $\tau > 0$.
2. For fixed $\tau > 0$, K and its derivatives tends to zero exponentially fast as $|\eta|$ tends to infinity.
3. For any fixed $\delta > 0$, $\lim_{\tau->0} K(\eta, \tau) = 0$ uniformly for all $|\eta| \geqslant \delta$.
4. For all $\tau > 0$, $\int_{-\infty}^{\infty} K(\eta, \tau) d\eta = 1$.
5. For $\tau > 0$, K is an analytic function of η and τ.
6. $\lim_{t->\tau} \frac{\partial K}{\partial \eta}(\eta, \tau - t) = 0$ for $\eta > 0$ and $\tau > 0$.
7. $\lim_{\tau->0} \int_0^{\tau} \frac{\partial K}{\partial \eta}(\eta, t - \tau) dt = 0$ for $\eta > 0$.
8. $\lim_{\eta->0} \int_0^{\tau} \frac{\partial K}{\partial \eta}(\eta, t - \tau) dt = 1$ for $\tau > 0$.

•Some integrals for the special initial data $A(\eta) = \eta^n$:

$$
\begin{aligned}
\int_0^{\infty} \frac{e^{-\frac{\xi^2}{2\tau}}}{\sqrt{2\pi\tau}} A(\xi) d\xi \implies \int_0^{\infty} \frac{e^{-\frac{\xi^2}{2\tau}}}{\sqrt{2\pi\tau}} d\xi &= \frac{1}{2}, \\
\int_0^{\infty} \frac{e^{-\frac{\xi^2}{2\tau}}}{\sqrt{2\pi\tau}} \xi d\xi &= \frac{1}{\sqrt{2\pi}} \tau^{\frac{1}{2}}, \\
\int_0^{\infty} \frac{e^{-\frac{\xi^2}{2\tau}}}{\sqrt{2\pi\tau}} \xi^2 d\xi &= \frac{1}{2} \tau, \\
\int_0^{\infty} \frac{e^{-\frac{\xi^2}{2\tau}}}{\sqrt{2\pi\tau}} \xi^3 d\xi &= \sqrt{\frac{2}{\pi}} \tau^{\frac{3}{2}}, \\
\int_0^{\infty} \frac{e^{-\frac{\xi^2}{2\tau}}}{\sqrt{2\pi\tau}} \xi^4 d\xi &= \frac{3}{2} \tau^2, \\
&\vdots \\
\int_0^{\infty} \frac{e^{-\frac{\xi^2}{2\tau}}}{\sqrt{2\pi\tau}} \xi^n d\xi &= \frac{2^{-\frac{1}{2}(1-n)} \tau^{\frac{n+1}{2}} \Gamma(\frac{n+1}{2})}{\sqrt{2\pi\tau}}.
\end{aligned}
$$

•Dirac Delta function

The Dirac delta can be loosely thought of as a function on the real line which is zero everywhere except at $x = x_0$, where it is infinite,

$$\delta(x - x_0) = \begin{cases} +\infty & , \; x = x_0 \,, \\ 0, & x \neq x_0, \end{cases}$$

and which is also constrained to satisfy the identity

$$\int_{-\infty}^{\infty} \delta(x - x_0) = 1,$$

with the convolution property

$$\delta(x) * f(x) = \int_{-\infty}^{\infty} \delta(x - x_0) f(x) dx = f(x_0),$$

for any $f : \mathbb{R} \to \mathbb{R}$ continuous around $x = x_0$. And as a special case

$$\int_{-\infty}^{\infty} \delta(x - y)\delta(y - \xi) dy = \delta(x - \xi).$$

•Leibnitz's Rule

Suppose that $F = F(x, t)$ is defined on $[a, b]x[\alpha, \beta]$ such that, for each $t \in [\alpha, \beta]$, $F(x, t)$ is an integrable function of x and that for each x, $(\partial F/\partial t)(x, t)$ exists and is continuous. Suppose that for all $t \in [\alpha, \beta]$,

$$\left| \frac{\partial F}{\partial t}(x, t) \right| \leqslant g(x),$$

for some nonnegative integrable function g. Then, $G(t) = \int_a^b F(x, t) dx$ is differentible and

$$G'(t) = \int_a^b \frac{\partial F}{\partial t}(x, t) dx,$$

for $t \in [\alpha, \beta]$.

•Error Function

The error function is defined as

$$Erf(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt.$$

The complementary error function, denoted erfc, is defined

$$\begin{aligned} Erfc(x) &= 1 - Erf(x) \\ &= \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt. \\ &= e^{x^2} Erfcx(x) \end{aligned}$$

The Taylor series of error function

$$Erf(z) = \frac{2}{\sqrt{\pi}} \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+1}}{n!(2n+1)}.$$

The derivative of the error function follows immediately from its definition:

$$\frac{d}{dz} Erf(z) = \frac{2}{\sqrt{\pi}} e^{-z^2}.$$

•Analytic Function

Let f be a real-valued function defined on an open set S in the xy-plane, and let $(x_0, y_0) \in S$. Then f is analytic at point (x_0, y_0) if f has continuous partial derivatives of all orders w.r.t x and y, and the Taylor's series of f about $P_0 = (x_0, y_0)$

$$f(x_0, y_0) + \frac{\partial f}{\partial x}|_{P_0}(x - x_0) + \frac{\partial f}{\partial y}|_{P_0}(y - y_0)$$
$$+\frac{1}{2}\left\{\left[\frac{\partial^2 f}{\partial x^2}\right]_{P_0}(x - x_0)^2 + 2\frac{\partial^2 f}{\partial x \partial y}|_{P_0}(x - x_0)(y - y_0) + \frac{\partial^2 f}{\partial y^2}|_{P_0}(y - y_0)^2 + \cdots\right\},$$

converges to $f(x, y)$ for points (x, y) in some neighbourhood of (x_0, y_0).

$\star$ f is analytic on S, if f is analytic at each point of S.

Properties :

1) The sums, products and composiions of analytic functions are analytic.

2) The inverse of an analytic function whose derivative is nowhere zero is also analytic.

If a complex function is analytic on a region $\mathbb{R}$, it is infinitely differentiable in $\mathbb{R}$. A complex function may fail to be analytic at one or more points through the presence of singularities, or along lines or line segments through the presence of branch cuts. The situation is quite different when one considers complex analytic functions and complex derivatives. It can be proved that any complex function differentiable (in the complex sense) in an open set is analytic. Consequently, in complex analysis, the term analytic function is synonymous with holomorphic function. A complex function that is analytic at all finite points of the complex plane is said to be entire.

•Convolution Properties

$$f(t) * g(t) = \int_{-\infty}^{\infty} f(\tau)g(t-\tau)d\tau, \tag{A.1}$$

1. If $f(t) = \delta(t)$, then $\delta(t) * g(t) = g(t)$ and also $g(t) * \delta(t+y) = g(t+y)$.
2. $f * (g * h) = (f * g) * h$.
3. $f * (g + h) = (f * g) + (f * h)$.
4. $a(f * g) = (af) * g$ where a is scalar.
5. For the derivative case $f'(t) * g(t) = f(t) * g'(t)$.
6. Fourier transform $F(f * g) = F(f).F(g)$.

•Fubini's Theorem

Let $f = f(x,y)$ denote an integral function on the rectangle $D = \{(x,y) : a \leqslant x \leqslant b,\ c \leqslant y \leqslant d\}$. If one of the following integrals exists, then the other two exists and

$$\int_D \int f dxdy = \int_c^d \Big\{ \int_a^b f(x,y)dx \Big\} dy = \int_a^b \Big\{ \int_c^d f(x,y)dy \Big\} dx.$$