# EXACTLY SOLVABLE GENERALIZED QUANTUM HARMONIC OSCILLATORS RELATED WITH THE CLASSICAL ORTHOGONAL POLYNOMIALS

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# ABSTRACT

## EXACTLY SOLVABLE GENERALIZED QUANTUM HARMONIC OSCILLATORS RELATED WITH THE CLASSICAL ORTHOGONAL POLYNOMIALS

In this thesis, we study exactly solvable generalized parametric oscillators related with the classical orthogonal polynomials of Hermite, Laguerre and Jacobi type. These quantum models with specific damping term, frequency and external forces are solved using Wei-Norman Lie algebraic approach. The exact form of the evolution operator is explicitly obtained in terms of two linearly independent homogeneous solutions and a particular solution of the corresponding classical equation of motion. Then, time evolution of wave functions and Glauber coherent states are constructed. Probability densities, expectation values and uncertainty relations are found and their properties are investigated according to the influence of the external forces. Besides, some examples with explicit solutions are given and their plots are constructed for the probability densities and uncertainty relations.

# ÖZET

## KLASİK ORTOGONAL POLİNOMLARLA İLGİLİ TAM ÇÖZÜLEBİLEN GENELLEŞTİRİLMİŞ KUANTUM HARMONİK OSİLATÖRLER

Bu tezde Hermite, Laguerre ve Jacobi tipi klasik ortogonal polinomlarla ilişkili tam çözülebilen genelleştirilmiş parametrik osilatörler çalışılmıştır. Bu özel sönümleyici terimli, frekanslı ve dış kuvvetli kuantum modeller Wei-Norman Lie cebri yaklaşımı kullanılarak çözülmüştür. Evrim operatörünün tam formu buna karşılık gelen hareket denkleminin homojen iki lineer bağımsız ve bir özel çözümü cinsinden açıkça elde edilmiştir. Daha sonra, dalga fonksiyonlarının ve Glauber eş uyumlu durumlarının zamanla evrimi inşa edilmiştir. Olasılık yoğunlukları, beklenen değerler ve belirsizlik ilişkileri bulunmuş ve bunların özellikleri dış kuvvetlerin etkisine göre incelenmiştir. Bunun yanı sıra, açık çözümlü bazı örnekler verilmiş ve bunların grafikleri olasılık yoğunlukları ve eş uyumlu durumları için oluşturulmuştur.

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# CHAPTER 1

# INTRODUCTION

The standard harmonic oscillator is one of the most important physical models. In this model, a particle is subject to the restoring force proportional to the distance. For more realistic situation, one can consider the damped quantum oscillator with an addition of a frictional forcing term. It can be formulated as a quantum model with time dependent mass (Caldirola, 1941), (Kanai, 1948). Time evolution of quantum systems with time-variable parameters has been studied in many works (Lewis, & Riesenfeld, 1969), (Khandekar, & Lawande, 1979), (Pedrosa, 1997), (Dantas, & Pedrosa, & Baseia, 1992), (Fernandez, 1989), (Yeon, & Pandey, 1997), (Song, 1999).

The quantum harmonic oscillator with explicitly time-dependent Hamiltonian is one of the most fundamental models. It has applications in various areas of physics such as quantum optics (Colegrave & Abdalla, 1981), (Pedrosa, & Rosas, & Guedes), quantum fluid dynamics (Nassar, 1984), ion-traps (Malkin, & Manko, &Trifonov, 1979), and cosmology (Sakharov, 1965). It is a useful model also in quantum information (Schleich, & Walther, 2007) and quantum computation (Sarandy, & Duzzioni, & Serra, 2011). As known, the Caldirola-Kanai quantum oscillator has exact solutions and is used to study dissipation in quantum mechanics (Dekker, 1981). To obtain quantum states of a time-dependent oscillator, several methods have been developed, such as path integral method (Feynman, 1951), the Lewis-Riesenfeld time invariant method (Lewis, & Riesenfeld, 1969), the Wei-Norman dynamical symmetry method (Wei, & Norman, 1963), etc.

Quantum systems with the generalized quadratic Hamiltonian and time-variable parameters can be solved using the several approaches given above and formal solutions are obtained. But exact solutions are investigated mostly for the driven Caldirola-Kanai oscillator. In (Büyükaşık & Pashaev& Ulaş-Tigrak, 2009) exactly solvable models with specific damping and frequency are introduced in terms of the quantum Sturm-Liouville problems for the classical orthogonal polynomials. The aim of this thesis is to provide exact explicit solutions of the general quadratic oscillator models related with the classical orthogonal polynomials. For this, we use Wei-Norman algebraic approach (Wei, & Norman, 1963), also known as evolution operator approach. This technique is useful for solving evolution problems whose Hamiltonian is a linear combination of generators of a finite dimensional Lie group, so that the evolution operator can be represented as a product of exponential operators. The thesis is organized in the following way:

In Chapter 2 we give the definitions of some basic tools and their properties which are useful for our further studies. Also, we mention about the fundamental postulates of quantum mechanics and some of their consequences.

In Chapter 3 we introduce the IVP for the time-dependent Schrödinger equation related with the standard Hamiltonian $\hat{H}_0 = \hat{p}^2/2m + (m\omega_0^2/2)\hat{q}^2$. To solve this problem, we first find the eigenstates of $\hat{H}_0$. Then we give the solution of the IVP in terms of these states. Also, we recall the definitions of coherent states for standard harmonic oscillator and review some of their properties.

In Chapter 4 we introduce the quantum evolution problem related with the general quadratic Hamiltonian with time-dependent parameters. Using Wei-Norman algebraic approach, we obtain the evolution operator explicitly, in terms of two linearly independent homogeneous solutions and a particular solution of the corresponding classical equation of motion. Then, using the exact evolution operator we find the corresponding wave functions and probability densities. Also, we obtain time evolution of Glauber coherent states under the generalized evolution operator. We show that time-evolved coherent states of the generalized harmonic oscillator are the eigenstates of time-dependent annihilation operator. Furthermore, we obtain the position and momentum operators in Heisenberg picture, and find the expectation values and uncertainty relation in wave functions and coherent states.

In Chapter 5 we introduce the generalized oscillator models related with the classical orthogonal polynomials. The original frequency $\omega^2(t)$ of these models depends on the eigenvalues of the related Sturm-Liouville problem. Moreover, the mixed term parameter $B(t)$ modifies the original frequency and by the special choice of this parameter, we obtain the modified frequency $\Omega^2(t)$ in terms of different eigenvalues of the same Sturm-Liouville problem. Then, we define the Hermite type generalized quantum oscillator including also the linear external terms. We find the solution of the corresponding classical equation of motion with specific initial conditions as a linear combination of a Hermite polynomial and a confluent hypergeometric function of the first-kind. We also give examples with explicit solutions of Hermite type general oscillator with and without linear external terms. And we discuss their properties according to the influence of the external terms. Illustrative plots are constructed for the probability densities and uncertainty relations.

In Chapter 6 we formulate the Laguerre type generalized quantum oscillator. By the special choice of the mixed term parameter $B(t)$, the corresponding classical equation of motion is a forced Laguerre differential equation. Assuming the total force of this equation is continuous for $t > 0$, solution of the quantum oscillator is written in terms of two linearly independent homogeneous solutions and a particular solution satisfying some initial conditions. Furthermore, we give some examples of Laguerre type generalized oscillator with exact solutions and their plots are analyzed according to the influence of the linear external terms.

In Chapter 7 we define Jacobi type generalized quantum oscillator. Here, we treat explicitly two special cases, the Legendre generalized oscillator and the first-kind Chebyshev oscillator. For the Legendre type oscillator, we obtain the homogeneous solution of the classical equation of motion, which is a forced Legendre differential equation, as a linear combination of a Legendre polynomial and a Legendre function of the second kind. For the fist-kind Chebyshev oscillator, we introduce the homogenous solution of the classical oscillator as a linear combination of Chebyshev polynomials of the first and second kind. And for each oscillator type, we give examples with exact explicit solutions and discuss their properties.

In Conclusion we summarize the main results obtained in this thesis.

# CHAPTER 2

# PRELIMINARIES

This chapter consists of basic concepts about Hilbert spaces, linear operators and quantum mechanics which are used in the next chapters.

## 2.1. Hilbert Space

**Definition 2.1** *Let $X$ be a vector space over the field $\mathbb{F}$.*

**(a)** *A mapping $\langle .|. \rangle : X \times X \to \mathbb{F}$ is called an **inner product** in $X$ if $\forall x, y, z \in X$ and $\forall \alpha \in \mathbb{F}$ the following conditions are satisfied:*

**(i)** $\langle x|x \rangle \geq 0$ *and* $\langle x|x \rangle = 0$ *if and only if* $x = 0$;

**(ii)** $\langle \alpha x + y|z \rangle = \alpha \langle x|z \rangle + \langle y|z \rangle$;

**(iii)** $\langle x|y \rangle = \langle y|x \rangle^*$ *(* denotes the complex conjugate).*

**(b)** *A vector space with an inner product defined on it is called an **inner product space**.*

**(c)** *A complete inner product space is called a **Hilbert space**.*

**Proposition 2.1** *An inner product space is a normed space with a norm defined by*

$$||x|| = \sqrt{\langle x|x \rangle}.$$

**Theorem 2.1** ***(Cauchy-Schwarz inequality)*** *For any two elements $x$ and $y$ of an inner product space $X$ over $\mathbb{F}$, we have*

$$|\langle x|y \rangle| \leq ||x|| \, ||y||,$$

*where the equality $|\langle x|y \rangle| = ||x|| \, ||y||$ holds if and only if $x$ and $y$ are linearly dependent.*

**Definition 2.2** *Let $X$ be an inner product space. Then*

**(a)** *the vectors x and y in X are called* ***orthogonal vectors*** *if* $\langle x|y\rangle = 0$,

**(b)** *the sequence* $\{x_n\}_{n=1}^{\infty}$ *in X is called* ***orthonormal sequence*** *if*

$$\langle x_i|x_j\rangle = \begin{cases} 0 & \text{if } i \neq j, \\ 1 & \text{if } i = j. \end{cases}$$

**Definition 2.3** *An orthonormal sequence* $\{x_n\}_{n=1}^{\infty}$ *in a Hilbert space* $\mathcal{H}$ *is called an* ***orthonormal basis*** *if* $\langle x|x_n\rangle = 0$ *for all* $n \in \mathbb{N}$ *implies* $x = 0$.

**Definition 2.4** *An orthonormal sequence* $\{x_n\}_{n=1}^{\infty}$ *in a Hilbert space* $\mathcal{H}$ *is called* ***complete (total) sequence*** *if the set*

$$span\,\{x_n\}_{n=1}^{\infty} = \left\{ \sum_{k=1}^{n} \alpha_k x_k \,\middle|\, n \in \mathbb{N}, \alpha_k \in \mathbb{C} \right\}$$

*is dense in* $\mathcal{H}$, *that is if* $\overline{span\,\{x_n\}} = \mathcal{H}$.

**Proposition 2.2** *Let* $\{x_n\}_{n=1}^{\infty}$ *be an orthonormal sequence in a Hilbert space* $\mathcal{H}$. *Then* $\{x_n\}_{n=1}^{\infty}$ *is an orthonormal basis for* $\mathcal{H}$ *if and only if* $\{x_n\}_{n=1}^{\infty}$ *is a complete sequence for* $\mathcal{H}$.

**Theorem 2.2** *Let* $\{x_n\}_{n=1}^{\infty}$ *be an orthonormal sequence in a Hilbert space* $\mathcal{H}$. *Then the following statements are equivalent:*

**(a)** $\{x_n\}_{n=1}^{\infty}$ *is an orthonormal basis for* $\mathcal{H}$;

**(b)** *for every* $x \in \mathcal{H}$, *one has the unique representation*

$$x = \sum_{n=1}^{\infty} \langle x|x_n\rangle x_n;$$

**(c)** $\sum_{n=1}^{\infty} |\langle x|x_n\rangle|^2 = \|x\|^2$ *(Parseval's Identity).*

**Definition 2.5** *The space* $L^2(\mathbb{R})$ *is the space of all complex-valued functions* $f(x)$ *for which*

$$\|f\|_2 := \left( \int |f(x)|^2 dx \right)^{1/2} < \infty.$$

*Then the function $f(x)$ is said to be* ***square integrable****. For any two functions $f(x), g(x)$ in $L^2(\mathbb{R})$ the inner product is defined by*

$$\langle f|g\rangle := \int_{-\infty}^{\infty} \overline{f(x)} g(x) dx.$$

*The space $L^2(\mathbb{R})$ is a Hilbert space, that is, it is complete inner product space.*

## 2.2. Linear Operators

**Definition 2.6** *Let X and Y be two normed spaces. An operator $\hat{T} : X \to Y$ is said to be* ***linear*** *if for all $x_1, x_2 \in X$ and $\alpha, \beta \in \mathbb{F}$,*

$$\hat{T}(\alpha x_1 + \beta x_2) = \alpha \hat{T}(x_1) + \beta \hat{T}(x_2).$$

**Definition 2.7** *Let X and Y be normed spaces and $\hat{T} : X \to Y$ be a linear operator. Then, $\hat{T}$ is* ***a bounded operator*** *if there exists a real number $c > 0$ such that*

$$||\hat{T}x|| \leq c||x|| \quad \text{for all } x \in \mathcal{D}(\hat{T}).$$

**Theorem 2.3** *Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be two Hilbert spaces over the field $\mathbb{F}$. Then for every linear bounded operator $\hat{T} : \mathcal{H}_1 \to \mathcal{H}_2$ there exists a unique linear bounded operator $\hat{T}^\dagger : \mathcal{H}_2 \to \mathcal{H}_1$ such that*

$$\langle \hat{T}x|y\rangle = \langle x|\hat{T}^\dagger y\rangle \quad \forall x \in \mathcal{H}_1, \ \forall y \in \mathcal{H}_2 \ \text{ and } \ ||\hat{T}^\dagger|| = ||\hat{T}||.$$

**Definition 2.8** *Let $\hat{T}$ be a bounded linear operator on a Hilbert space $\mathcal{H}$. The operator $\hat{T}^\dagger : \mathcal{H} \to \mathcal{H}$ defined by*

$$\langle \hat{T}x|y\rangle = \langle x|\hat{T}^\dagger y\rangle \quad \forall x, y \in \mathcal{H}$$

*is called* ***the adjoint operator*** *of $\hat{T}$ and if $\hat{T} = \hat{T}^\dagger$, then $\hat{T}$ is called self-adjoint.*

**Theorem 2.4** *Let $\hat{T}$ be a bounded self-adjoint operator on a Hilbert space $\mathcal{H}$. Then,*

**(a)** *all eigenvalues of $\hat{T}$ are real,*

**(b)** *eigenvectors corresponding to distinct eigenvalues are orthogonal.*

**Definition 2.9** *A bounded linear operator $\hat{U} : \mathcal{H} \to \mathcal{H}$ on a Hilbert space $\mathcal{H}$ is said to be* ***unitary*** *if $\hat{U}$ is one-to-one, onto and $\hat{U}^\dagger = \hat{U}^{-1}$, or equivalently*

$$\hat{U}^\dagger \hat{U} = \hat{U}\hat{U}^\dagger = \hat{I}.$$

**Definition 2.10** *An operator $\hat{T} : X \to Y$, where $X$ and $Y$ are normed spaces, is called* ***an unbounded operator*** *if there exists a sequence $\{x_n\} \in \mathcal{D}(\hat{T})$ with $\|x_n\| = 1$ for all $n \in \mathbb{N}$ which implies $\|\hat{T}x_n\| \to \infty$.*

**Definition 2.11** *An operator $\hat{T}$ defined in a Hilbert space $\mathcal{H}$ is called* ***densely defined*** *if its domain is a dense subset of $\mathcal{H}$, that is, $\overline{\mathcal{D}(\hat{T})} = \mathcal{H}$.*

**Definition 2.12** ***(Adjoint of a densely defined operator)***
*Let $\hat{T}$ be a densely defined operator in a Hilbert space $\mathcal{H}$. The adjoint $\hat{T}^\dagger$ of $T$ is the operator defined on the set of all $y \in \mathcal{H}$ for which $\langle \hat{T}x|y\rangle$ is a continuous functional on $\mathcal{D}(\hat{T})$ and such that*

$$\langle \hat{T}x|y\rangle = \langle x|\hat{T}^\dagger y\rangle \quad \text{for all } \; x \in \mathcal{D}(\hat{T}) \text{ and } y \in \mathcal{D}(\hat{T}^\dagger).$$

**Definition 2.13** *Let $\hat{T}$ be a densely defined linear operator in a Hilbert space $\mathcal{H}$. Then,*

**(a)** *$\hat{T}$ is called* ***symmetric (Hermitian)*** *if*

$$\langle \hat{T}x|y\rangle = \langle x|\hat{T}y\rangle \quad \forall x, y \in \mathcal{D}(\hat{T}).$$

*In other words, $\hat{T}$ is symmetric if $\hat{T} \subset \hat{T}^\dagger$, which means $\hat{T} = \hat{T}^\dagger$ on $\mathcal{D}(\hat{T})$, and $\mathcal{D}(\hat{T}) \subset \mathcal{D}(\hat{T}^\dagger)$,*

**(b)** *$\hat{T}$ is called* ***self-adjoint*** *if $\hat{T} = \hat{T}^\dagger$, which means $\langle \hat{T}x|y\rangle = \langle x|\hat{T}y\rangle \quad \forall x \in \mathcal{D}(\hat{T}), \forall y \in \mathcal{D}(\hat{T}^\dagger)$ and $\mathcal{D}(\hat{T}) = \mathcal{D}(\hat{T}^\dagger)$.*

## 2.3. The Fundamental Postulates of Quantum Mechanics

In this section, we present the basic principles of quantum mechanics as postulates and some of their consequences.

**Postulate 1** *The state of a quantum mechanical system is completely specified by a wave function* $\Psi(x,t)$ *in a Hilbert space.*

According to the probabilistic interpretation of the wave function, the probability that a particle lies in the volume element $d\sigma$ located at $x$ at time $t$ is

$$\rho(x,t)d\sigma = \Psi^*(x,t)\Psi(x,t)d\sigma = |\Psi(x,t)|^2 d\sigma,$$

and since the probability of a particle being somewhere in space is one, we have

$$\int_{-\infty}^{\infty} |\Psi(x,t)|^2 d\sigma = 1,$$

which in fact is the normalization of $\Psi(x,t)$, that is, $||\Psi(x,t)||^2 = 1$.

The function $\rho(x,t) = |\Psi(x,t)|^2$ is called ***the probability density function***.

**Postulate 2** *To every physical observable in quantum mechanics, there corresponds a linear Hermitian operator* $\hat{A}$ *in the Hilbert space. Conversely, to each such operator in the Hilbert space there corresponds some physical observable.*

As a consequence of this postulate, a quantum observable is mathematically represented by a linear Hermitian operator on a Hilbert space and there is one such operator for each quantum observable such as the position, the momentum, the energy, and so on. Some examples of Hermitian operators are:

| Observable | Classical quantities | Quantum operators |
|---|---|---|
| Position | x | $\hat{x}$ (multiplication by $x$) |
| Momentum | $p = mv$ | $\hat{p} = -i\hbar\frac{\partial}{\partial x}$ |
| Potential Energy | $V(x)$ | $\hat{V}(x)$ (multiplication by $V(x)$) |
| Kinetic Energy | $T = \frac{1}{2}mv^2$ | $\hat{T} = \frac{\hat{p}^2}{2m} = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}$ |
| Hamiltonian | $H(x,p) = \frac{p^2}{2m} + V(x)$ | $\hat{H}(x,p) = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + \hat{V}(x)$ |

Table 2.1. Physical quantities in classical mechanics and the corresponding quantum mechanical operators.

**Postulate 3** *In any measurement of an observable associated with the operator $\hat{A}$, the only values that will ever be observed are the eigenvalues $\lambda_n$, that satisfy the eigenvalue equation*

$$\hat{A}\psi_n = \lambda_n\psi_n, \ \ n = 0, 1, 2, \ldots.$$

This postulate asserts that if the system is in an eigenstate $\psi_n$ of $\hat{A}$ with eigenvalue $\lambda_n$, then any measurement of the observable $A$ will always yield the value $\lambda_n$.

Although measurement will always yield an eigenvalue $\lambda_n$, the initial state does not have to be an eigenstate of $\hat{A}$, so we do not know which eigenvalue it is. What we can predict is the expectation value of $\hat{A}$, which is defined as follows:

**Definition 2.14** ***The expectation value*** *$\langle\hat{A}\rangle$ of an observable operator $\hat{A}$ in the state $\psi$ of a physical system is defined by*

$$\langle\hat{A}\rangle_\psi = \langle\psi|\hat{A}|\psi\rangle.$$

Note that, since expectation value must be real, it shows why $\hat{A}$ must be Hermitian.

As a consequence of the third postulate, if $\{\psi_n\}$ is a complete set of eigenfunctions corresponding to $\hat{A}$ such that $\langle\psi_n|\psi_m\rangle = \delta_{nm}$ and $||\psi_n(x)|| = 1$ for all $n = 0, 1, 2, \ldots,$ then $\{\psi_n\}$ forms an orthonormal basis in the associated Hilbert space. So any arbitrary state $\psi(x,t)$ can be expanded in terms of eigenvectors $\{\psi_n\}$ as $\psi(x,t) = \sum_{n=1}^{\infty} c_n\psi_n(x) = \sum_{n=1}^{\infty}\langle\psi|\psi_n\rangle\psi_n$. Then from Parseval's identity, we have $\sum_{n=1}^{\infty}|c_n|^2 = \sum_{n=1}^{\infty}|\langle\psi|\psi_n\rangle|^2 = ||\psi||^2 = 1$. In that case;

$$\langle\hat{A}\rangle_\psi = \left\langle\psi\,\middle|\,\hat{A}\,\middle|\,\psi\right\rangle = \left\langle\sum_{n=1}^{\infty} c_n\psi_m \,\middle|\, \sum_{n=1}^{\infty} c_n\hat{A}\psi_n\right\rangle = \left\langle\sum_{n=1}^{\infty} c_n\psi_m \,\middle|\, \sum_{n=1}^{\infty} c_n\lambda_n\psi_n\right\rangle = \sum_{n=1}^{\infty}\lambda_n|c_n|^2.$$

The probability of observing the eigenvalue $\lambda_n$ is $\rho_n = |c_n|^2 = |\langle\psi|\psi_n\rangle|^2$, if $\hat{A}$ is in the state $\psi$.

This postulate also implies that, after the measurement of $\hat{A}$ the wave function $\psi$ collapses into the eigenstate $\{\psi_n\}$ corresponding to $\lambda_n$. Thus the act of measurement affects the state of the system.

In terms of the expectation value of $\hat{A}$, we define the mean square deviation, $(\Delta\hat{A})_\psi$, which measures the dispersion around the mean value $\langle\hat{A}\rangle_\psi$.

**Definition 2.15** ***The mean square deviation (or uncertainty)*** $(\Delta\hat{A})_\psi$ *is defined by the square root of the expectation (mean) value of* $(\hat{A} - \langle\hat{A}\rangle_\psi)^2$ *in the state* $\psi$ *in which* $\langle\hat{A}\rangle_\psi$ *is computed.*

**Theorem 2.5** *For any Hermitian operator* $\hat{A}$ *and any normalized state* $\psi$*, we have*

**(a)** $(\Delta\hat{A})^2_\psi = \langle\hat{A}^2\rangle_\psi - \langle\hat{A}\rangle^2_\psi$,

**(b)** $\langle\hat{A}^2\rangle_\psi = ||\hat{A}\psi||^2$.

**Proof**

**(a)** By using the fact that the states $\psi$ are normalized, we obtain

$$\begin{aligned}(\Delta\hat{A})^2_\psi &= \langle(\hat{A} - \langle\hat{A}\rangle_\psi)^2\rangle_\psi = \langle\psi|(\hat{A} - \langle\hat{A}\rangle_\psi)^2\psi\rangle = \langle\psi|(\hat{A}^2 - 2\hat{A}\langle\hat{A}\rangle_\psi + \langle\hat{A}\rangle^2_\psi)\psi\rangle \\ &= \langle\psi|\hat{A}^2\psi\rangle - 2\langle\hat{A}\rangle_\psi\langle\psi|\hat{A}\psi\rangle + \langle\hat{A}\rangle^2\langle\psi|\psi\rangle = \langle\hat{A}^2\rangle_\psi - \langle\hat{A}\rangle^2_\psi.\end{aligned}$$

**(b)** Since $\hat{A}$ is Hermitian,

$$\langle\hat{A}^2\rangle_\psi = \langle\psi|\hat{A}^2\psi\rangle = \langle\hat{A}\psi|\hat{A}\psi\rangle = ||\hat{A}\psi||^2.$$

□

**Definition 2.16** *Let* $\hat{A}$ *and* $\hat{B}$ *be two linear operators on a Hilbert space* $H$*, then* ***the commutator*** *of these operators is defined by*

$$[\hat{A}, \hat{B}] = \hat{A}\hat{B} - \hat{B}\hat{A}.$$

From the definition, we have $[\hat{x}, \hat{p}] = i\hbar$, where $\hat{x}$ is the position operator, $\hat{p} = -i\hbar(\partial/\partial x)$ is the momentum operator and $\hbar$ is the Planck constant.

## 2.4. The Heisenberg Uncertainty Principle

**Theorem 2.6** ***(Uncertainty Principle)*** *Let $\hat{A}$ and $\hat{B}$ be two Hermitian operators on a Hilbert space $H$, then for any state vector $\psi$*

$$(\Delta\hat{A})_\psi(\Delta\hat{B})_\psi \geq \frac{1}{2}|\langle[\hat{A}, \hat{B}]\rangle|.$$

**Proof** We have $(\Delta\hat{A})_\psi = ||(\hat{A} - \langle\hat{A}\rangle)\psi||$, and $(\Delta\hat{B})_\psi = ||(\hat{B} - \langle\hat{B}\rangle)\psi||$. So using these we get

$$\begin{aligned}
(\Delta\hat{A})_\psi(\Delta\hat{B})_\psi &= ||(\hat{A} - \langle\hat{A}\rangle)\psi||||(\hat{B} - \langle\hat{B}\rangle)\psi|| \\
&\geq |\langle(\hat{A} - \langle\hat{A}\rangle)\psi|(\hat{B} - \langle\hat{B}\rangle)\psi\rangle|, \quad \text{by Cauchy-Schwarz inequality,} \\
&= |\langle\psi|(\hat{A} - \langle\hat{A}\rangle)(\hat{B} - \langle\hat{B}\rangle)|\psi\rangle| \\
&\geq |\mathrm{Im}\,\langle\psi|(\hat{A} - \langle\hat{A}\rangle)(\hat{B} - \langle\hat{B}\rangle)|\psi\rangle| \\
&= \frac{1}{2}|\langle\psi|(\hat{A} - \langle\hat{A}\rangle)(\hat{B} - \langle\hat{B}\rangle) - (\hat{B} - \langle\hat{B}\rangle)(\hat{A} - \langle\hat{A}\rangle)|\psi\rangle| \\
&= \frac{1}{2}|\langle\psi|[\hat{A}, \hat{B}]|\psi\rangle| = \frac{1}{2}|\langle[\hat{A}, \hat{B}]\rangle|.
\end{aligned}$$

□

**Corollary 2.1** *For any state vector $\psi$,* ***the Heisenberg uncertainty principle*** *states that*

$$(\Delta\hat{x})_\psi(\Delta\hat{p})_\psi \geq \frac{\hbar}{2}.$$

**Proof** Since $\hat{x}$ and $\hat{p}$ are Hermitian operators and since $[\hat{x}, \hat{p}] = i\hbar$, we can apply the Uncertainty principle to these operators and obtain

$$(\Delta\hat{x})_\psi(\Delta\hat{p})_\psi \geq \frac{1}{2}|\langle[\hat{x}, \hat{p}]\rangle| = \frac{\hbar}{2}.$$

□

**Definition 2.17** *States which the Heisenberg uncertainty principle holds with equality are called* ***the minimum uncertainty states****.*

**Postulate 4** *If a physical system is not disturbed by any experiment, the Hamiltonian operator $\hat{H}$ determines the time evolution of the state vector of the system $\Psi(x,t)$ through the partial differential equation*

$$i\hbar\frac{\partial\Psi}{\partial t} = \hat{H}\Psi(x,t).$$

This is called the time-dependent Schrödinger equation, and represents the fundamental equation of motion in quantum mechanics first discovered by Erwin Schrödinger (1887-1961).

## 2.5. The Evolution Operator

If we consider the Schrödinger equation

$$\begin{cases} i\hbar\frac{\partial}{\partial t}\Psi(x,t) = \hat{H}(t)\Psi(x,t), \\ \Psi(x,t_0) = \Psi_0(x), \end{cases} \tag{2.1}$$

the wave function solution $\Psi(x,t)$ of this problem can be obtained by applying an evolution operator $\hat{U}(t,t_0)$ to the initial state $\Psi(x,t_0)$, that is

$$\Psi(x,t) = \hat{U}(t,t_0)\Psi(x,t_0). \tag{2.2}$$

As we know, a state must be normalized, that is $||\Psi(x,t)|| = 1$ for all $t$. Therefore, normalization does not depend on time and this implies that

$$1 = \langle\Psi(x,t)|\Psi(x,t)\rangle = \langle\Psi(x,t_0)|\hat{U}^\dagger\hat{U}|\Psi(x,t_0)\rangle = \langle\Psi(x,t_0)|\Psi(x,t_0)\rangle\,,$$

so that $\hat{U}^\dagger\hat{U} = \hat{I}$. Thus, we conclude that $\hat{U}(t,t_0)$ is a unitary operator. Also, the operator $\hat{U}(t,t_0)$ does not depend on $\Psi(x,t_0)$. Consequently,

$$\Psi(x,t_2) = \hat{U}(t_2,t_1)\Psi(x,t_1) = \hat{U}(t_2,t_1)\hat{U}(t_1,t_0)\Psi(x,t_0) = \hat{U}(t_2,t_0)\Psi(x,t_0).$$

Therefore, the evolution operator has the property

$$\hat{U}(t_2, t_0) = \hat{U}(t_2, t_1)\hat{U}(t_1, t_0). \tag{2.3}$$

From the equation (2.2), we have that $\hat{U}(t,t) = \hat{I}$. So, $\hat{U}(t, t_0)\hat{U}(t_0, t) = \hat{U}(t_0, t)\hat{U}(t, t_0) = \hat{I}$, which means $\hat{U}^{-1}(t, t_0) = \hat{U}(t_0, t)$.

Replacing $t_2$ by $t$ and $t_1$ by $t - \delta t$, where $\delta t$ is infinitesimal in (2.3), we obtain

$$\hat{U}(t, t_0) = \hat{U}(t, t - \delta t)\hat{U}(t - \delta t, t_0). \tag{2.4}$$

Now, $\hat{U}(t, t - \delta t)$ is an infinitesimal unitary operator, which may be written in the form

$$\hat{U}(t, t - \delta t) = 1 - \frac{i}{\hbar}\delta t\hat{H}(t), \tag{2.5}$$

Substituting (2.5) into (2.4), we get

$$\hat{U}(t, t_0) = \hat{U}(t - \delta t, t_0) - \frac{i}{\hbar}\delta t\hat{H}(t)\hat{U}(t - \delta t, t_0)$$

or

$$\lim_{\delta t \to 0} \frac{1}{\delta t}[\hat{U}(t, t_0) - \hat{U}(t - \delta t, t_0)] = \frac{1}{i\hbar}\hat{H}(t)\hat{U}(t, t_0).$$

Thus, we obtain the corresponding operator equation

$$\begin{cases} i\hbar\frac{\partial}{\partial t}\hat{U}(t, t_0) = \hat{H}(t)\hat{U}(t, t_0), \\ \hat{U}(t_0, t_0) = \hat{I}. \end{cases} \tag{2.6}$$

We note that, if the Hamiltonian is time-independent, then $\hat{U}(t, t_0) = e^{-\frac{i}{\hbar}(t-t_0)\hat{H}}$, and $\Psi(x, t) = e^{-\frac{i}{\hbar}(t-t_0)\hat{H}}\Psi(x, t_0)$. But if the Hamiltonian depends explicitly on time, then we have $\hat{U}(t, t_0) = e^{-\frac{i}{\hbar}\int_{t_0}^{t}\hat{H}(t')dt'}$.

## 2.6. Lie Group and Lie Algebra

**Definition 2.18** *A* ***group*** *G is a set together with a binary operation* $\star : G \times G \to G$ *with the following properties:*

*a)* $g \star (h \star k) = (g \star h) \star k$ *for all* $g, h, k \in G$*, that is,* $\star$ *is* ***associative****.*

*b) There exists a unique element* $e \in G$ *called the* ***identity*** *such that* $e \star g = g \star e = g$*.*

*c) For every element* $g \in G$*, there exists an element* $g^{-1}$*, called the* ***inverse*** *of g, such that* $g \star g^{-1} = g^{-1} = g^{-1} \star g = e$*.*

*If* $\star$ *is commutative, i.e.* $g \star h = h \star g$ *for all* $g, h \in G$*, then G is called an* ***abelian*** *group.*

**Definition 2.19** *An* ***algebra*** $\mathcal{A}$ *over* $\mathbb{C}$*(or* $\mathbb{R}$*) is a vector space over* $\mathbb{C}$*(or* $\mathbb{R}$*), together with a binary operation* $\mathcal{A} \times \mathcal{A} \to \mathcal{A}$*, called multiplication. The image of* $(T, S) \in \mathcal{A} \times \mathcal{A}$ *under this mapping is denoted by* $TS$*, and it satisfies the following two relations*

$$T(aS + bU) = aTS + bTU,$$

$$(aS + bU)T = aST + bUT$$

*for all* $T, S, U \in \mathcal{A}$ *and* $a, b \in \mathbb{C}$*(or* $\mathbb{R}$*).*

**Definition 2.20** *A* ***Lie group*** *G is a differentiable manifold endowed with a group structure such that the group operation* $G \times G \to G$ *and the map* $G \to G$ *given by* $g \to g^{-1}$ *are differentiable. If the dimension of the underlying manifold is r, we say that G is an r-****parameter Lie group****.*

**Definition 2.21** *A* ***Lie algebra*** *is a vector space over some field* $\mathbb{F}$ *(*$\mathbb{R}$ *or* $\mathbb{C}$*) together with a binary operation* $[.,.] : L \times L \to L$*, called the Lie bracket, which has the following properties:*

**a)** ***Bilinearity***

$$[aT + bS, U] = a[T, U] + b[S, U],$$

$$[U, aT + bS] = a[U, T] + b[U, S]$$

**b)** ***Jacobi identity***

$$[[T, S], U] + [[U, T], S] + [[S, U], T] = 0$$

**c)** ***Antisymmetry***

$$[T, S] = -[S, T]$$

*for all* $a, b \in \mathbb{F}$ *and* $T, S, U \in L$.

**Example 2.1** *The operators* $\hat{E}_1 = iq$, $\hat{E}_2 = \partial/\partial q$, $\hat{E}_3 = i\hat{I}$ *generate Heisenberg-Weyl algebra with the given commutation relations:*

$$[\hat{E}_1, \hat{E}_2] = -\hat{E}_3, \quad [\hat{E}_1, \hat{E}_3] = 0, \quad [\hat{E}_2, \hat{E}_3] = 0.$$

**Example 2.2** *The operators*

$$\hat{K}_- = -\frac{i}{2}\frac{\partial^2}{\partial q^2}, \qquad \hat{K}_+ = \frac{i}{2}q^2, \qquad \hat{K}_0 = \frac{1}{2}(q\frac{\partial}{\partial q} + \frac{1}{2})$$

*generate su(1,1) algebra with commutation relations*

$$[\hat{K}_-, \hat{K}_+] = 2\hat{K}_0, \quad [\hat{K}_+, \hat{K}_0] = -\hat{K}_+, \quad [\hat{K}_-, \hat{K}_0] = \hat{K}_-.$$

**Example 2.3** *All the operators* $\hat{E}_1$, $\hat{E}_2$, $\hat{E}_3$, $\hat{K}_-$, $\hat{K}_+$ *and* $\hat{K}_0$ *generate a Lie algebra with commutation relations given in the previous two examples and*

$$[\hat{E}_1, \hat{K}_-] = -\hat{E}_2, \quad [\hat{E}_1, \hat{K}_+] = 0, \quad [\hat{E}_1, \hat{K}_0] = -\frac{1}{2}\hat{E}_1,$$
$$[\hat{E}_2, \hat{K}_-] = 0, \quad [\hat{E}_2, \hat{K}_+] = \hat{E}_1, \quad [\hat{E}_2, \hat{K}_0] = \frac{1}{2}\hat{E}_2,$$
$$[\hat{E}_3, \hat{K}_-] = 0, \quad [\hat{E}_3, \hat{K}_+] = 0, \quad [\hat{E}_3, \hat{K}_0] = 0.$$

# CHAPTER 3

# STANDARD HARMONIC OSCILLATOR

In this chapter, we find the eigenstates of the standard Hamiltonian $\hat{H}_0 = \hat{p}^2/2m + (m\omega_0^2/2)\hat{q}^2$ and we give the solution of the IVP for the time-dependent Schrödinger equation corresponding to $\hat{H}_0$. Then coherent states of the standard harmonic oscillator (Glauber coherent states) and their properties are examined.

## 3.1. Solution of the IVP for the Time-Dependent Schrödinger Equation

We consider the IVP for the time-dependent Schrödinger equation

$$\begin{cases} i\hbar\frac{\partial}{\partial t}\Psi(q,t) = \hat{H}_0\Psi, \\ \Psi(q,t_0) = \Psi_0(q), \quad -\infty < q < \infty, \end{cases} \tag{3.1}$$

related with the standard Hamiltonian

$$\hat{H}_0 = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial q^2} + \frac{1}{2}m\omega_0^2\hat{q}^2, \tag{3.2}$$

where $m$ is the constant mass and $\omega_0^2$ is the constant frequency. Since $\hat{H}_0$ does not depend on time, then we can solve the IVP (3.1) by the method of separation of variables. So we look for solutions of the form

$$\Psi(q,t) = \varphi(q)f(t), \tag{3.3}$$

where $\varphi$ is a function of $q$ alone, and $f$ is a function of $t$ alone. Substituting the equation (3.3) into the Schrödinger equation, we obtain

$$i\hbar\varphi\frac{df}{dt} = -\frac{\hbar^2}{2m}\frac{d^2\varphi}{dq^2}f + \frac{m\omega_0^2}{2}q^2\varphi f.$$

Dividing through by $\varphi f$:

$$i\hbar\frac{1}{f}\frac{df}{dt} = -\frac{\hbar^2}{2m}\frac{1}{\varphi}\frac{d^2\varphi}{dq^2} + \frac{m\omega_0^2}{2}q^2. \tag{3.4}$$

Now, the left side of the equation (3.4) is a function of $t$ alone, and the right side is a function of $q$ alone. This is true only if both sides equal to a constant, $E$. Then, we get

$$f(t) = e^{-\frac{i}{\hbar}E(t-t_0)}. \tag{3.5}$$

Also, we have

$$-\frac{\hbar^2}{2m}\frac{d^2\varphi}{dq^2} + \frac{m\omega_0^2}{2}q^2\varphi = E\varphi,$$

which is in fact an eigenvalue equation for $\hat{H}_0$, that is $\hat{H}_0\varphi_k = E_k\varphi_k$.

The eigenstates of the standard Hamiltonian can be determined using the ladder operators given by

$$\hat{a} = \left(\frac{m\omega_0}{2\hbar}\right)^{1/2}\hat{q} + i\left(\frac{1}{2m\omega_0\hbar}\right)^{1/2}\hat{p}, \tag{3.6}$$

$$\hat{a}^\dagger = \left(\frac{m\omega_0}{2\hbar}\right)^{1/2}\hat{q} - i\left(\frac{1}{2m\omega_0\hbar}\right)^{1/2}\hat{p}. \tag{3.7}$$

Here, $\hat{a}$ is called *the annihilation operator*, and $\hat{a}^\dagger$ is called *the creation operator*. The Hamiltonian (3.2) can be expressed in terms of these operators as

$$\hat{a}^\dagger\hat{a} = \frac{m\omega_0}{2\hbar}\hat{q}^2 + \frac{1}{2m\omega_0\hbar}\hat{p}^2 + \frac{i}{2\hbar}[\hat{q},\hat{p}] = \frac{1}{\hbar\omega_0}\hat{H}_0 - \frac{1}{2}.$$

Thus, $\hat{H}_0 = \hbar\omega_0\left(\hat{a}^\dagger\hat{a} + 1/2\right)$. The operators $\hat{a}, \hat{a}^\dagger$ and $\hat{H}_0$ satisfy the following commutation relations which will be useful for us,

$$[\hat{a},\hat{a}^\dagger] = 1, \quad [\hat{a},\hat{H}_0] = \hbar\omega_0\hat{a}, \quad [\hat{a}^\dagger,\hat{H}_0] = -\hbar\omega_0\hat{a}^\dagger.$$

Imagine that $\varphi_k(q)$ is the k-th eigenstate of $\hat{H}_0$ corresponding to eigenvalue $E_k$, then we claim that $\hat{a}^\dagger\varphi_k(q)$ is also an eigenstate:

$$\hat{H}_0(\hat{a}^\dagger\varphi_k) = (\hat{a}^\dagger\hat{H}_0 + \hbar\omega_0\hat{a}^\dagger)\varphi_k = \hat{a}^\dagger(E_k\varphi_k) + \hbar\omega_0\hat{a}^\dagger\varphi_k = (E_k + \hbar\omega_0)\hat{a}^\dagger\varphi_k.$$

This proves our claim, that $\hat{a}^\dagger\varphi_k(q)$ is indeed an eigenstate of $\hat{H}_0$ corresponding to eigenvalue $E_k + \hbar\omega_0$. Likewise, we find that $\hat{a}\varphi_k(q)$ yields $\hat{H}_0(\hat{a}\varphi_k) = (E_k - \hbar\omega_0)\hat{a}^\dagger\varphi_k$. Repeatedly applying creation and annihilation operators to eigenstates of $\hat{H}_0$, we can generate new eigenstates, with $\hat{a}^\dagger$ raising the energy, $\hat{a}$ lowering it and the change in energy is always $\hbar\omega_0$. Because of our choice of potential, the energy must be positive, so there must be a lower limit on the energy, such that $\hat{a}\varphi_0(q) = 0$. Then $\hat{a}^\dagger\hat{a}\varphi_0 = \left(\hat{H}_0/(\hbar\omega_0) - 1/2\right)\varphi_0 = 0$, that is $\hat{H}_0\varphi_0 = (\hbar\omega_0/2)\varphi_0$. Therefore, the minimum energy of the standard harmonic oscillator is $(\hbar\omega_0)/2$. Additionally, for normalized eigenstates $\{\varphi_k\}$, $k = 0, 1, 2, \ldots$, of the Hamiltonian (3.2) we find the following relations

$$\hat{a}\varphi_k = \sqrt{k}\varphi_{k-1}, \quad \hat{a}^\dagger\varphi_k = \sqrt{k+1}\varphi_{k+1}, \quad \hat{a}^\dagger\hat{a}\varphi_k = k\varphi_k.$$

The normalized eigenstates of the standard harmonic oscillator are found by starting with the ground state $\hat{a}\varphi_0(q) = 0$, such as

$$\hat{a}\varphi_0(q) = \left(\sqrt{\frac{m\omega_0}{2\hbar}}\hat{q} + i\sqrt{\frac{1}{2m\omega_0\hbar}}\hat{p}\right)\varphi_0(q) = \left(\sqrt{\frac{m\omega_0}{2\hbar}}q + \sqrt{\frac{\hbar}{2m\omega_0}}\frac{\partial}{\partial q}\right)\varphi_0(q) = 0.$$

Solving this first order differential equation results in the following expression for the ground state $\varphi_0(q) = N_0 e^{-(m\omega_0)/(2\hbar)q^2}$. By doing normalization,

$$1 = ||\varphi_0(q)||^2 = |N_0|^2 \int_{-\infty}^{\infty} e^{-\frac{m\omega_0}{\hbar}q^2} dq,$$

we can find $N_0 = (m\omega_0/\pi\hbar)^{1/4}$. So the ground state of the standard harmonic oscillator is

$$\varphi_0(q) = \left(\frac{m\omega_0}{\pi\hbar}\right)^{1/4} e^{-\frac{m\omega_0}{2\hbar}q^2}. \tag{3.8}$$

Therefore, other states are constructed by applying the raising operator to state (3.8) as follows

$$\varphi_k(q) = (\hat{a}^\dagger)^k \varphi_0(q) = \left(\frac{m\omega_0}{\pi\hbar}\right)^{1/4} \left(\sqrt{\frac{m\omega_0}{2\hbar}}q - \sqrt{\frac{\hbar}{2m\omega_0}}\frac{\partial}{\partial q}\right)^k e^{-\frac{m\omega_0}{2\hbar}q^2}.$$

Substituting $\sqrt{(m\omega_0)/\hbar}q = \xi$, we obtain $\varphi_k(q) = N_k e^{-\xi^2/2} e^{\xi^2/2} (\xi - d/d\xi)^k e^{-\xi^2/2}$, where $e^{\xi^2/2}(\xi - d/d\xi)^k e^{-\xi^2/2} = H_k(\xi)$ represents the $k$-th order Hermite polynomial and $N_k$ is the normalization constant that can be found as $N_k = (2^k k!)^{-1/2} (m\omega_0/\pi\hbar)^{1/4}$. Hence, the normalized eigenstates of the Hamiltonian (3.2) are

$$\varphi_k(q) = \frac{1}{\sqrt{2^k k!}} \left(\frac{m\omega_0}{\pi\hbar}\right)^{1/4} e^{-\frac{m\omega_0}{2\hbar}q^2} H_k\left(\sqrt{\frac{m\omega_0}{\hbar}}q\right), \quad k = 0, 1, 2, \cdots, \tag{3.9}$$

corresponding to eigenvalues $E_k = (k + 1/2)\hbar\omega_0$.

**Proposition 3.1** *The ground state wave function of the standard harmonic oscillator represents the minimum uncertainty state.*

**Proof** Using the relations $\hat{q} = \sqrt{\hbar/(2m\omega_0)}(\hat{a} + \hat{a}^\dagger)$, $\hat{p} = -i\sqrt{(m\omega_0\hbar)/2}(\hat{a} - \hat{a}^\dagger)$, and the fact that the wave functions are orthonormal, we find the following expectation values

$$\langle \hat{q} \rangle_0 = \langle \varphi_0 | \hat{q} | \varphi_0 \rangle = \sqrt{\frac{\hbar}{2m\omega_0}} \langle \varphi_0 | \hat{a} + \hat{a}^\dagger | \varphi_0 \rangle = 0,$$

and

$$\langle \hat{p} \rangle_0 = \langle \varphi_0 | \hat{p} | \varphi_0 \rangle = -i\sqrt{\frac{m\omega_0\hbar}{2}} \langle \varphi_0 | \hat{a} - \hat{a}^\dagger | \varphi_0 \rangle = 0.$$

Furthermore, we compute

$$\begin{aligned}
\langle \hat{q}^2 \rangle_0 &= \langle \varphi_0 | \hat{q}^2 | \varphi_0 \rangle = \frac{\hbar}{2m\omega_0} \langle \varphi_0 | \hat{a}^2 + \hat{a}\hat{a}^\dagger + \hat{a}^\dagger\hat{a} + (\hat{a}^\dagger)^2 | \varphi_0 \rangle \\
&= \frac{\hbar}{2m\omega_0} \langle \varphi_0 | 2\hat{a}^\dagger\hat{a} + 1 | \varphi_0 \rangle = \frac{\hbar}{2m\omega_0}, \\
\langle \hat{p}^2 \rangle_0 &= \langle \varphi_0 | \hat{p}^2 | \varphi_0 \rangle = -\frac{m\omega_0\hbar}{2} \langle \varphi_0 | \hat{a}^2 - \hat{a}\hat{a}^\dagger - \hat{a}^\dagger\hat{a} + (\hat{a}^\dagger)^2 | \varphi_0 \rangle \\
&= -\frac{m\omega_0\hbar}{2} \langle \varphi_0 | -(2\hat{a}^\dagger\hat{a} + 1) | \varphi_0 \rangle = \frac{m\omega_0\hbar}{2}.
\end{aligned}$$

Now, the uncertainties in $\hat{q}$ and $\hat{p}$ are as follows:

$$(\Delta\hat{q})_0^2 = \langle\hat{q}^2\rangle_0 - \langle\hat{q}\rangle_0^2 = \frac{\hbar}{2m\omega_0}, \quad (\Delta\hat{p})_0^2 = \langle\hat{p}^2\rangle_0 - \langle\hat{p}\rangle_0^2 = \frac{m\omega_0\hbar}{2}.$$

Therefore, the Heisenberg uncertainty principle is $(\Delta\hat{q})_0(\Delta\hat{p})_0 = \hbar/2$, which proves the proposition. □

**Proposition 3.2** *The wave functions $\varphi_k(q)$ $(k \neq 0)$, of the standard harmonic oscillator are not minimum uncertainty wave functions.*

**Proof** Since the wave functions are orthonormal, we obtain that

$$\begin{aligned}
\langle\hat{q}\rangle_k &= \langle\varphi_k|\hat{q}|\varphi_k\rangle = \sqrt{\frac{\hbar}{2m\omega_0}}\,\langle\varphi_k|\hat{a} + \hat{a}^\dagger|\varphi_k\rangle \\
&= \sqrt{\frac{\hbar}{2m\omega_0}}(\sqrt{k}\,\langle\varphi_k|\varphi_{k-1}\rangle + \sqrt{k+1}\,\langle\varphi_k|\varphi_{k+1}\rangle) = 0, \\
\langle\hat{p}\rangle_k &= \langle\varphi_k|\hat{p}|\varphi_k\rangle = -i\sqrt{\frac{m\omega_0\hbar}{2}}\,\langle\varphi_k|\hat{a} - \hat{a}^\dagger|\varphi_k\rangle \\
&= -i\sqrt{\frac{m\omega_0\hbar}{2}}(\sqrt{k}\,\langle\varphi_k|\varphi_{k-1}\rangle - \sqrt{k+1}\,\langle\varphi_k|\varphi_{k+1}\rangle) = 0.
\end{aligned}$$

Also we calculate

$$\begin{aligned}
\langle\hat{q}^2\rangle_k &= \langle\varphi_k|\hat{q}^2|\varphi_k\rangle = \frac{\hbar}{2m\omega_0}\,\langle\varphi_k|\hat{a}^2 + \hat{a}\hat{a}^\dagger + \hat{a}^\dagger\hat{a} + (\hat{a}^\dagger)^2|\varphi_k\rangle \\
&= \frac{\hbar}{2m\omega_0}\,\langle\varphi_k|2\hat{a}^\dagger\hat{a} + 1|\varphi_k\rangle = \frac{\hbar}{m\omega_0}\left(k + \frac{1}{2}\right), \\
\langle\hat{p}^2\rangle_k &= \langle\varphi_k|\hat{p}^2|\varphi_k\rangle = -\frac{m\omega_0\hbar}{2}\,\langle\varphi_k|\hat{a}^2 - \hat{a}\hat{a}^\dagger - \hat{a}^\dagger\hat{a} + (\hat{a}^\dagger)^2|\varphi_k\rangle \\
&= -\frac{m\omega_0\hbar}{2}\,\langle\varphi_k| - (2\hat{a}^\dagger\hat{a} + 1)|\varphi_k\rangle = m\omega_0\hbar\left(k + \frac{1}{2}\right).
\end{aligned}$$

Using the above results, the uncertainty relation can be found as

$$\begin{aligned}
(\Delta\hat{q})_k(\Delta\hat{p})_k &= \hbar\left(k + \frac{1}{2}\right), \quad k = 1, 2, 3, \cdots, \\
&> \frac{\hbar}{2}.
\end{aligned}$$

Thus, it is not a minimum uncertainty. □

The eigenstates $\{\varphi_k(q)\}_{k=0}^{\infty}$ of the standard Hamiltonian $\hat{H}_0$ is an orthonormal basis for $L^2(\mathbb{R})$, so any initial wave function $\Psi_0(q) \in L^2(\mathbb{R})$ has expansion of the form $\Psi_0(q) = \sum_{k=0}^{\infty} \langle \Psi_0|\varphi_k \rangle \varphi_k(q)$. Thus, solution of the IVP (3.1) is explicitly

$$\Psi(q,t) = f(t)\Psi_0(q) = \sum_{k=0}^{\infty} \langle \Psi_0|\varphi_k \rangle f(t)\varphi_k(q),$$

where

$$\Psi_k(q,t) = f(t)\varphi_k(q) = e^{-i/\hbar E_k(t-t_0)}\varphi_k(q), \quad k = 0, 1, 2, \cdots .$$

The corresponding probability density functions are then

$$\rho_k(q,t) = |\Psi_k(q,t)|^2 = N_k^2 e^{-\frac{m\omega_0}{\hbar}q^2} H_k^2\left(\sqrt{\frac{m\omega_0}{\hbar}}q\right), \quad k = 0, 1, 2, \cdots .$$

## 3.2. Coherent States of the Standard Harmonic Oscillator

Glauber coherent states have many useful physical and mathematical properties, and they can be defined in different, but equivalent ways, (Glauber, 1963), (Perelomov, 1986), (Nieto, & Simmons, 1979).

### 3.2.1. Minimum Uncertainty Coherent States (MUCS)

Minimum uncertainty coherent states are defined as states $\phi(q)$ satisfying

$$(\Delta\hat{q})_\phi(\Delta\hat{p})_\phi = \frac{\hbar}{2},$$

and therefore they are closest to the classical states. To find $\phi(q)$ in closed form, we first define the Hermitian operators $\hat{Q} = \hat{q} - \langle\hat{q}\rangle_\phi$ , $\hat{P} = \hat{p} - \langle\hat{p}\rangle_\phi$ . Then by the definition (2.15), we obtain that

$$
\begin{aligned}
(\Delta\hat{q})^2_\phi(\Delta\hat{p})^2_\phi &= \langle\phi|\hat{Q}^2|\phi\rangle\,\langle\phi|\hat{P}^2|\phi\rangle \\
&= \langle\hat{Q}\phi|\hat{Q}\phi\rangle\,\langle\hat{P}\phi|\hat{P}\phi\rangle \\
&\geq |\,\langle\hat{Q}\phi|\hat{P}\phi\rangle\,|^2, \quad \text{(by Cauchy-Schwarz inequality)}, \\
&= |\,\langle\phi|\hat{Q}\hat{P}|\phi\rangle\,|^2 \\
&= \left|\left\langle\phi\left|\frac{1}{2}(\hat{Q}\hat{P}+\hat{P}\hat{Q})\right|\phi\right\rangle + \left\langle\phi\left|\frac{1}{2}[\hat{Q},\hat{P}]\right|\phi\right\rangle\right|^2.
\end{aligned}
$$

Now, the inequality in Cauchy-Schwarz becomes equality if and only if both functions are linearly dependent, that is,

$$\hat{P}\phi = \lambda\hat{Q}\phi \tag{3.10}$$

for some $\lambda$. And we want to have

$$(\Delta\hat{q})^2_\phi(\Delta\hat{p})^2_\phi = \left|\left\langle\phi\left|\frac{1}{2}[\hat{Q},\hat{P}]\right|\phi\right\rangle\right|^2 = \frac{\hbar^2}{4},$$

which implies the condition

$$\left\langle\phi\left|\frac{1}{2}(\hat{Q}\hat{P}+\hat{P}\hat{Q})\right|\phi\right\rangle = 0. \tag{3.11}$$

Substituting Eqn. (3.10) into Eqn. (3.11), we get

$$
\begin{aligned}
\left\langle\phi\left|\frac{1}{2}(\hat{Q}\hat{P}+\hat{P}\hat{Q})\right|\phi\right\rangle &= \frac{1}{2}\Big(\langle\phi|\hat{Q}\hat{P}|\phi\rangle + \langle\phi|\hat{P}\hat{Q}|\phi\rangle\Big) \\
&= \frac{1}{2}\Big(\langle\phi|\hat{Q}|\lambda\hat{Q}\phi\rangle + \langle\lambda^*\phi|\hat{Q}^2|\phi\rangle\Big) = \left(\frac{\lambda+\lambda^*}{2}\right)\langle\phi|\hat{Q}^2|\phi\rangle = 0.
\end{aligned}
$$

Then $\lambda + \lambda^* = 0$, that is $\lambda$ is pure imaginary. As a result,

$$(\hat{p} - \langle\hat{p}\rangle_\phi)\phi = \lambda(\hat{q} - \langle\hat{q}\rangle_\phi)\phi, \tag{3.12}$$

where $\langle\hat{p}\rangle_\phi, \langle\hat{q}\rangle_\phi$ are constants and $\lambda$ is pure imaginary. Substituting $\hat{p} = -i\hbar(\partial/\partial q)$ in the equation (3.12), it reduces to an ordinary differential equation and solving this equation, we find $\phi(q)$, depending on $\lambda$ as

$$\phi_\lambda(q) = ce^{i/\hbar\langle\hat{p}\rangle q}e^{-\frac{i\lambda}{2\hbar}(q-\langle\hat{q}\rangle)^2}.$$

Let $\lambda = i\alpha,\ \alpha \in \mathbb{R}$, and by doing normalization we obtain normalized coherent states of the standard harmonic oscillator in closed form

$$\phi_\alpha(q) = \left(\frac{m\omega_0}{\pi\hbar}\right)^{1/4} e^{\frac{i}{\hbar}\langle\hat{p}\rangle_\alpha q}e^{-\frac{m\omega_0}{2\hbar}(q-\langle\hat{q}\rangle_\alpha)^2}. \tag{3.13}$$

### 3.2.2. Annihilation Operator Coherent States (AOCS)

Coherent states are known also as annihilation operator eigenstates, that is for the operator $\hat{a}$, the coherent states satisfy $\hat{a}\phi_\alpha(q) = \alpha\phi_\alpha(q)$, for any complex number $\alpha = \alpha_1 + i\alpha_2, \alpha_1, \alpha_2-$real.

**Proposition 3.3** *The oscillator states $\phi_\alpha(q)$ satisfying*

$$\hat{a}\phi_\alpha(q) = \alpha\phi_\alpha(q) \tag{3.14}$$

*can be represented in terms of energy eigenstates $\varphi_k(q)$ of the standard harmonic oscillator as*

$$\phi_\alpha(q) = e^{-|\alpha|^2/2}\sum_{k=0}^{\infty}\frac{\alpha^k}{\sqrt{k!}}\varphi_k(q), \tag{3.15}$$

*for all $\alpha \in \mathbb{C}$.*

**Proof** Assume Eqn. (3.14) has solution of the form $\phi_\alpha(q) = \sum_{k=0}^{\infty} c_k \varphi_k(q)$, where $c_k = \langle \phi_\alpha(q) | \varphi_k(q) \rangle$. Taking the inner product of (3.14) by $\varphi_k(q)$ gives

$$\langle \hat{a}\varphi_k(q) | \varphi_k(q) \rangle = \alpha \langle \phi_\alpha(q) | \varphi_k(q) \rangle .$$

That is,

$$\alpha \langle \phi_\alpha(q) | \varphi_k(q) \rangle = \langle \phi_\alpha(q) | \hat{a}^\dagger \varphi_k(q) \rangle = \langle \phi_\alpha(q) | \sqrt{k+1}\varphi_{k+1}(q) \rangle .$$

Then we get the equality $\langle \phi_\alpha(q) | \varphi_{k+1}(q) \rangle = \alpha(k+1)^{-1/2} \langle \phi_\alpha(q) | \varphi_k(q) \rangle$, which gives $c_k = \langle \phi_\alpha(q) | \varphi_k(q) \rangle = \alpha^k (k!)^{-1/2} \langle \phi_\alpha(q) | \varphi_0(q) \rangle$. Thus, we can write

$$\phi_\alpha(q) = \langle \phi_\alpha(q) | \varphi_0(q) \rangle \sum_{k=0}^{\infty} \frac{\alpha^k}{\sqrt{k!}} \varphi_k(q).$$

By doing normalization,

$$\begin{aligned} \langle \phi_\alpha | \phi_\alpha \rangle &= |\langle \phi_\alpha | \varphi_0 \rangle|^2 \left\langle \sum_{k=0}^{\infty} \frac{(\alpha^*)^k}{\sqrt{k!}} \varphi_k \,\middle|\, \sum_{m=0}^{\infty} \frac{\alpha^m}{\sqrt{m!}} \varphi_m \right\rangle \\ &= |\langle \phi_\alpha | \varphi_0 \rangle|^2 \sum_{k=0}^{\infty} \frac{|\alpha|^{2k}}{k!} = |\langle \phi_\alpha | \varphi_0 \rangle|^2 e^{|\alpha|^2} = 1, \end{aligned}$$

we obtain $\langle \phi_\alpha | \varphi_0 \rangle = e^{-|\alpha|^2/2} e^{i\theta}$, where $e^{i\theta}$ is the phase factor. W.l.o.g. choosing the phase factor as 1 proves our proposition and also shows that coherent states belong to $L^2(\mathbb{R})$. □

**Proposition 3.4** *Coherent states do not form an orthogonal system.*

**Proof** For any two complex numbers $\alpha$ and $\beta$, we have the following

$$\begin{aligned} \langle \phi_\alpha | \phi_\beta \rangle &= e^{-\frac{|\alpha|^2}{2} - \frac{|\beta|^2}{2}} \left\langle \sum_{k=0}^{\infty} \frac{(\alpha^*)^k}{\sqrt{k!}} \varphi_k \,\middle|\, \sum_{m=0}^{\infty} \frac{\beta^m}{\sqrt{m!}} \varphi_m \right\rangle \\ &= e^{-\frac{1}{2}(|\alpha|^2 + |\beta|^2)} \sum_{k=0}^{\infty} \frac{(\alpha^* \beta)^k}{k!} = e^{-\frac{|\alpha|^2}{2} - \frac{|\beta|^2}{2} + \alpha^* \beta} \end{aligned}$$

Similarly, we can write $\langle \phi_\beta | \phi_\alpha \rangle = e^{-\frac{|\alpha|^2}{2} - \frac{|\beta|^2}{2} + \alpha\beta^*}$, and thus obtain the equation $|\langle \phi_\alpha | \phi_\beta \rangle|^2 = e^{-|\alpha-\beta|^2}$, from which we see that $\langle \phi_\alpha | \phi_\beta \rangle \neq 0$ for any $\alpha, \beta \in \mathbb{C}$. Therefore, coherent states are not orthogonal.

□

**Proposition 3.5** *The collection of coherent states $\phi_\alpha(q)$ forms an overcomplete set for any complex number $\alpha$.*

**Proof** The closure relation for coherent states can be found as

$$\begin{aligned}\int |\phi_\alpha\rangle\langle\phi_\alpha|\, d^2\alpha &= \int \left( e^{-|\alpha|^2} \sum_{m,n=0}^{\infty} \frac{\alpha^n(\alpha^*)^m}{\sqrt{n!m!}} |\varphi_n\rangle\langle\varphi_m| \right) d^2\alpha \\ &= \sum_{m,n=0}^{\infty} \left( \int e^{-|\alpha|^2} \alpha^n (\alpha^*)^m d^2\alpha \right) \frac{|\varphi_n\rangle\langle\varphi_m|}{\sqrt{n!m!}}.\end{aligned}$$

Let $I_1 = \int e^{-|\alpha|^2}\alpha^n(\alpha^*)^m d^2\alpha$, then writing $\alpha$ in polar form, $\alpha = re^{i\theta}$ and $d^2\alpha = rdrd\theta$, gives

$$I_1 = \int_0^{2\pi}\int_0^{\infty} e^{-r^2} r^n r^m e^{in\theta} e^{-im\theta} rdrd\theta = \left(\int_0^{\infty} r^{n+m} re^{-r^2} dr\right) \underbrace{\left(\int_0^{2\pi} e^{i(n-m)\theta} d\theta\right)}_{2\pi\delta_{n,m}}.$$

If $n = m$, then $I_1 = 2\pi \int_0^{\infty} r^{2n} re^{-r^2} dr = \pi n!$. But if $n \neq m$, then $I_1 = 0$. It follows that

$$\int |\phi_\alpha\rangle\langle\phi_\alpha|\, d^2\alpha = \sum_{n=0}^{\infty} \pi n! \frac{|\varphi_n\rangle\langle\varphi_n|}{n!}$$

and

$$\frac{1}{\pi}\int |\phi_\alpha\rangle\langle\phi_\alpha|\, d^2\alpha = \hat{I},$$

which shows completeness of coherent states. □

Now, we find the expectation values of position and momentum operators in coherent state $\phi_\alpha(q)$ using the equations (4.52), (4.53) and (3.14) as follows

$$\langle\hat{q}\rangle_\alpha = \langle\phi_\alpha|\hat{q}|\phi_\alpha\rangle = \sqrt{\frac{2\hbar}{m\omega_0}}\alpha_1, \tag{3.16}$$

$$\langle\hat{p}\rangle_\alpha = \langle\phi_\alpha|\hat{p}|\phi_\alpha\rangle = \sqrt{2m\omega_0\hbar}\alpha_2. \tag{3.17}$$

Then, expectations of squares are

$$\langle \hat{q}^2 \rangle_\alpha = \frac{\hbar}{2m\omega_0} \langle \phi_\alpha | (\hat{a} + \hat{a}^\dagger)^2 | \phi_\alpha \rangle = \frac{2\hbar}{m\omega_0}\alpha_1^2 + \frac{\hbar}{2m\omega_0}, \tag{3.18}$$

$$\langle \hat{p}^2 \rangle_\alpha = \frac{m\omega_0\hbar}{2} \langle \phi_\alpha | (\hat{a} - \hat{a}^\dagger)^2 | \phi_\alpha \rangle = 2m\omega_0\hbar\alpha_2^2 + \frac{m\omega_0\hbar}{2}. \tag{3.19}$$

Hence, fluctuations of $\hat{q}$ and $\hat{p}$ are found as $(\Delta\hat{q})_\alpha = \sqrt{\hbar/(2m\omega_0)}$, $(\Delta\hat{p})_\alpha = \sqrt{(m\omega_0\hbar)/2}$, and uncertainty relation is $(\Delta\hat{q})_\alpha(\Delta\hat{p})_\alpha = \hbar/2$. This shows that AOCS are MUCS.

### 3.2.3. Displacement Operator Coherent States (DOCS)

The coherent states $\phi_\alpha(q)$ are also expressed in terms of the displacement operator $\hat{D}_0(\alpha)$, acting on the ground state and is given by $\hat{D}_0(\alpha) = e^{\alpha\hat{a}^\dagger - \alpha^*\hat{a}}$. From Eqn. (3.15), we have

$$\phi_\alpha(q) = e^{-\frac{1}{2}|\alpha|^2} \sum_{k=0}^{\infty} \frac{\alpha^k}{\sqrt{k!}} \varphi_k(q) = e^{-\frac{1}{2}|\alpha|^2} \sum_{k=0}^{\infty} \frac{\alpha^k}{k!} (\hat{a}^\dagger)^k \varphi_0(q) = e^{-\frac{1}{2}|\alpha|^2 + \alpha\hat{a}^\dagger} \varphi_0(q).$$

We rewrite this expression by using the *Baker-Campbell-Hausdorff formula* which states that if $\hat{X}$ and $\hat{Y}$ are any two operators in a Hilbert space, that both commute with $[\hat{X}, \hat{Y}]$, then

$$e^{\hat{X}+\hat{Y}} = e^{-\frac{1}{2}[\hat{X},\hat{Y}]} e^{\hat{X}} e^{\hat{Y}}.$$

Since the commutator of the operators $\alpha\hat{a}^\dagger$ and $-\alpha^*\hat{a}$ is $[\alpha\hat{a}^\dagger, -\alpha^*\hat{a}] = -|\alpha|^2$ and it commutes with both of these operators, Baker-Campbell-Hauusdorff formula gives

$$e^{\alpha\hat{a}^\dagger - \alpha^*\hat{a}} = e^{\alpha\hat{a}^\dagger} e^{-\alpha^*\hat{a}} e^{-\frac{1}{2}|\alpha|^2}.$$

Thus, using $e^{-\alpha^*\hat{a}}\varphi_0 = e^0 = 1$, we obtain

$$\begin{aligned} \phi_\alpha(q) &= e^{-\frac{1}{2}|\alpha|^2} e^{\alpha\hat{a}^\dagger} \varphi_0 = e^{\alpha\hat{a}^\dagger} e^{-\alpha^*\hat{a}} e^{-\frac{1}{2}|\alpha|^2} \varphi_0 \\ &= e^{\alpha\hat{a}^\dagger - \alpha^*\hat{a}} \varphi_0 = \hat{D}_0(\alpha)\varphi_0(q). \end{aligned}$$

### 3.2.4. Time Evolution of Coherent States of Standard Harmonic Oscillator

This section gives the explicit form of time-evolved coherent states of standard harmonic oscillator.

**Proposition 3.6** *Time-evolved coherent state $\phi_\alpha(q,t)$ is also an eigenstate of the annihilation operator $\hat{a}$.*

**Proof** For standard harmonic oscillator the evolution operator is defined as $\hat{U}(t,t_0) = e^{-\frac{i}{\hbar}(t-t_0)\hat{H}_0}$, and we have $\hat{U}^\dagger \hat{a} \hat{U} = e^{-i\omega_0}\hat{a}$. So using these relations, we get

$$\begin{aligned} \hat{a}\phi_\alpha(q,t) &= \hat{a}\hat{U}\phi_\alpha(q,t_0) = \hat{U}\hat{U}^\dagger \hat{a}\hat{U}\phi_\alpha(q,t_0) = \hat{U}(e^{-i\omega_0 t}\hat{a})\phi_\alpha(q,t_0) \\ &= \hat{U}(e^{-i\omega_0 t}\alpha(t_0)\phi_\alpha(q,t_0)) = e^{-i\omega_0 t}\alpha(t_0)\phi_\alpha(q,t). \end{aligned}$$

Denoting $\alpha(t) = e^{-i\omega_0 t}\alpha(t_0)$, we see that $\phi_\alpha(q,t)$ is an eigenstate of $\hat{a}$ corresponding to time-dependent eigenvalue $\alpha(t)$, that is $\hat{a}\phi_\alpha(q,t) = \alpha(t)\phi_\alpha(q,t)$. So it is also a coherent state, showing that coherent states remain coherent under time evolution operator. □

Then, time-evolved coherent states for the standard oscillator are explicitly written as

$$\phi_\alpha(q,t) = e^{-\frac{i\omega}{2}t} e^{-|\alpha(t)|^2/2} \sum_{k=0}^{\infty} \frac{\alpha(t)^k}{\sqrt{k!}} \varphi_k(q),$$

where $\alpha(t) = e^{-i\omega t}\alpha(t_0)$.

# CHAPTER 4

# GENERALIZED TIME-DEPENDENT QUANTUM HARMONIC OSCILLATOR

In this chapter, we introduce the quantum evolution problem related with the generalized quadratic Hamiltonian. Exact explicit solutions to this problem is given by Wei-Norman algebraic approach. This technique is useful for solving evolution problems whose Hamiltonian is a linear combination of generators of a finite dimensional Lie group, so that the evolution operator can be represented as a product of exponential operators. In this process, we obtain exact evolution operator and wave function solutions. Then, time evolution of the Glauber coherent states under the generalized evolution operator are obtained and discussed.

## 4.1. Quantization of the Generalized Quadratic Oscillator

The generalized Hamiltonian for classical oscillator with time-dependent parameters is of the form

$$H_g(x,p,t) = \frac{p^2}{2\mu(t)} + \frac{\mu(t)\omega^2(t)}{2}x^2 + B(t)xp + D(t)x + E(t)p + F(t). \tag{4.1}$$

We note that, since Hamiltonian is a total energy of the system, then the Hamiltonian $H_g(x,p,t)$ in Eq. (4.1) must be written in phase space, i.e. energy space. That means, the dimension of Hamiltonian must be in terms of energy dimension $ML^2/T^2$, where basic quantities are $M$ mass, $T$ time and $L$ position. In (4.1), first and second terms are already of energy dimension. Third terms is right if the dimension $B(t) \to 1/T$ is chosen. In order to have $D(t)$, $E(t)$, $F(t)$ in energy space, they should be chosen in terms of basic quantities as $D(t) \to ML/T^2$ dimension, $E(t) \to L/T^2$ dimension and $F(t) \to ML^2/T^2$ dimension. It is indicated that $D(t)$, $E(t)$, $F(t)$ could not have been chosen only in time dimension.

The corresponding equations of motion are

$$\dot{x} = \frac{\partial H}{\partial p} = B(t)x + \frac{p}{\mu(t)} + E(t), \tag{4.2}$$

$$\dot{p} = -\frac{\partial H}{\partial x} = -(\mu(t)\omega^2(t)x + B(t)p + D(t)). \tag{4.3}$$

Then, we have the classical equation of motion in position space

$$\ddot{x} + \frac{\dot{\mu}}{\mu}\dot{x} + \left(\omega^2(t) - \left(\dot{B} + B^2 + \frac{\dot{\mu}}{\mu}B\right)\right)x = -\frac{1}{\mu}D + \dot{E} + \left(\frac{\dot{\mu}}{\mu} + B\right)E, \tag{4.4}$$

the oscillator equation in momentum space

$$\ddot{p} - \frac{(\dot{\mu\omega^2})}{\mu\omega^2}\dot{p} + \left(\omega^2(t) + \left(\dot{B} - B^2 - \frac{(\dot{\mu\omega^2})}{\mu\omega^2}B\right)\right)p = -\dot{D} + \left(\frac{(\dot{\mu\omega^2})}{\mu\omega^2} + B\right)D - \mu\omega^2 E. \tag{4.5}$$

We notice that, the parameter $B(t)$ of the mixed term in Hamiltonian (4.1) leads to modification of the original frequency $\omega^2(t)$, and the external parameters $D(t)$, $E(t)$, and $F(t)$, all contribute to the forcing term of the oscillator. Replacing the canonical variables in classical Hamiltonian (4.1) by the quantum operators,

$$x \to \hat{q}, \quad p \to \hat{p}, \quad xp \to \frac{\hat{p}\hat{q} + \hat{q}\hat{p}}{2}, \tag{4.6}$$

we consider the evolution problem for the quantum harmonic oscillator

$$\begin{cases} i\hbar\frac{\partial}{\partial t}\Psi(q,t) = \hat{H}_g(t)\Psi(q,t), \\ \Psi(q,t_0) = \Psi_0(q), \quad -\infty < q < \infty, \end{cases} \tag{4.7}$$

with general quadratic Hamiltonian

$$\hat{H}_g(t) = \frac{\hat{p}^2}{2\mu(t)} + \frac{\mu(t)\omega^2(t)}{2}\hat{q}^2 + \frac{B(t)}{2}(\hat{q}\hat{p} + \hat{p}\hat{q}) + D(t)\hat{q} + E(t)\hat{p} + F(t)\hat{I}, \tag{4.8}$$

where $\mu(t) > 0$, $\omega^2(t)$, $B(t)$, $D(t)$, $E(t)$, and $F(t)$ are real parameters depending on time.

## 4.2. The Generalized Evolution Operator

To solve the evolution problem

$$\begin{cases} i\hbar\frac{\partial}{\partial t}\Psi(q,t) = \hat{H}_g(t)\Psi(q,t), \\ \Psi(q,t_0) = \Psi_0(q), \quad -\infty < q < \infty, \end{cases}$$

we use the Lie algebraic approach. Indeed, the Hamiltonian (4.8) can be written as time-dependent linear combination of Lie algebra generators

$$\hat{H}_g(t) = -i\left(\frac{\hbar^2}{\mu(t)}\hat{K}_- + \mu(t)\omega^2(t)\hat{K}_+ + 2\hbar B(t)\hat{K}_0 + D(t)\hat{E}_1 + \hbar E(t)\hat{E}_2 + F(t)\hat{E}_3\right), \tag{4.9}$$

where

$$\hat{E}_1 = iq, \quad \hat{E}_2 = \frac{\partial}{\partial q}, \quad \hat{E}_3 = i\hat{I}$$

are generators of the Heisenberg-Weyl algebra, and

$$\hat{K}_- = -\frac{i}{2}\frac{\partial^2}{\partial q^2}, \qquad \hat{K}_+ = \frac{i}{2}q^2, \qquad \hat{K}_0 = \frac{1}{2}\left(q\frac{\partial}{\partial q} + \frac{1}{2}\right)$$

are generators of the $su(1,1)$ algebra. Then, the evolution operator for the general oscillator can be written as product of exponential operators

$$\hat{U}_g(t,t_0) = \hat{U}_E(t,t_0)\hat{U}_K(t,t_0), \tag{4.10}$$

where

$$\hat{U}_E(t,t_0) \equiv e^{c(t)\hat{E}_3} e^{\frac{a(t)}{\hbar}\hat{E}_1} e^{-b(t)\hat{E}_2}, \quad \hat{U}_K(t,t_0) \equiv e^{f(t)\hat{K}_+} e^{2h(t)\hat{K}_0} e^{g(t)\hat{K}_-}, \tag{4.11}$$

and $f(t), g(t), h(t), a(t), b(t), c(t)$ are unknown real-valued functions to be determined from the IVP, defining the unitary operator $\hat{U}_g$, that is

$$
\begin{aligned}
i\hbar\frac{d}{dt}\hat{U}_g(t,t_0) &= \hat{H}_g(t)\hat{U}_g(t,t_0), \\
\hat{U}_g(t_0,t_0) &= \hat{I}.
\end{aligned} \tag{4.12}
$$

After performing time differentiation we get

$$
\begin{aligned}
\frac{\partial}{\partial t}\hat{U}(t,t_0) = & \Big(\dot{c}\hat{E}_3\Big)e^{c(t)\hat{E}_3}e^{\frac{a(t)}{\hbar}\hat{E}_1}e^{-b(t)\hat{E}_2}e^{f(t)\hat{K}_+}e^{2h(t)\hat{K}_0}e^{g(t)\hat{K}_-} \\
&+e^{c(t)\hat{E}_3}\Big(\frac{\dot{a}}{\hbar}\hat{E}_1\Big)e^{\frac{a(t)}{\hbar}\hat{E}_1}e^{-b(t)\hat{E}_2}e^{f(t)\hat{K}_+}e^{2h(t)\hat{K}_0}e^{g(t)\hat{K}_-} \\
&+e^{c(t)\hat{E}_3}e^{\frac{a(t)}{\hbar}\hat{E}_1}\Big(-\dot{b}\hat{E}_2\Big)e^{-b(t)\hat{E}_2}e^{f(t)\hat{K}_+}e^{2h(t)\hat{K}_0}e^{g(t)\hat{K}_-} \\
&+e^{c(t)\hat{E}_3}e^{\frac{a(t)}{\hbar}\hat{E}_1}e^{-b(t)\hat{E}_2}\Big(\dot{f}\hat{K}_+\Big)e^{f(t)\hat{K}_+}e^{2h(t)\hat{K}_0}e^{g(t)\hat{K}_-} \\
&+e^{c(t)\hat{E}_3}e^{\frac{a(t)}{\hbar}\hat{E}_1}e^{-b(t)\hat{E}_2}e^{f(t)\hat{K}_+}\Big(2\dot{h}\hat{K}_0\Big)e^{2h(t)\hat{K}_0}e^{g(t)\hat{K}_-} \\
&+e^{c(t)\hat{E}_3}e^{\frac{a(t)}{\hbar}\hat{E}_1}e^{-b(t)\hat{E}_2}e^{f(t)\hat{K}_+}e^{2h(t)\hat{K}_0}\Big(\dot{g}\hat{K}_-\Big)e^{g(t)\hat{K}_-}.
\end{aligned} \tag{4.13}
$$

Now, we need to collect the exponentials in the right. For this, we use the Baker-Campbell-Hausdorff relation:

**Proposition 4.1** *If $\hat{A}$ and $\hat{B}$ are two fixed non-commuting operators and $\xi$ is a parameter, then*

$$
e^{\xi\hat{A}}\hat{B}e^{-\xi\hat{A}} = \hat{B} + \xi[\hat{A},\hat{B}] + \frac{\xi^2}{2!}[\hat{A},[\hat{A},\hat{B}]] + \frac{\xi^3}{3!}[\hat{A},[\hat{A},[\hat{A},\hat{B}]]] + \cdots . \tag{4.14}
$$

**Proof** Let $f(\xi) = e^{\xi\hat{A}}\hat{B}e^{-\xi\hat{A}}, \quad f(0) = \hat{B}$. We need to expand $f(\xi)$ in a Maclaurin series in powers of $\xi$, so we first find the derivatives of $f(\xi)$ with respect to $\xi$ as follows:

$$
\frac{df}{d\xi} = \hat{A}e^{\xi\hat{A}}\hat{B}e^{-\xi\hat{A}} - e^{\xi\hat{A}}\hat{B}e^{-\xi\hat{A}}\hat{A} = [\hat{A}, f(\xi)],
$$

then $(df/d\xi)|_{\xi=0} = [\hat{A}, \hat{B}]$. Thus second derivative is found as,

$$\begin{aligned}\frac{d^2 f}{d\xi^2} &= \hat{A}^2 e^{\xi\hat{A}}\hat{B}e^{-\xi\hat{A}} - 2\hat{A}e^{\xi\hat{A}}\hat{B}e^{-\xi\hat{A}}\hat{A} + e^{\xi\hat{A}}\hat{B}e^{-\xi\hat{A}}\hat{A}^2 \\ &= \hat{A}\Big(\hat{A}e^{\xi\hat{A}}\hat{B}e^{-\xi\hat{A}} - e^{\xi\hat{A}}\hat{B}e^{-\xi\hat{A}}\hat{A}\Big) - \Big(\hat{A}e^{\xi\hat{A}}\hat{B}e^{-\xi\hat{A}} - e^{\xi\hat{A}}\hat{B}e^{-\xi\hat{A}}\hat{A}\Big)\hat{A} \\ &= \left[\hat{A}, \frac{df}{d\xi}\right] = [\hat{A}, [\hat{A}, f(\xi)]],\end{aligned}$$

then $(d^2 f/d\xi^2)|_{\xi=0} = [\hat{A}, [\hat{A}, \hat{B}]]$. Continuing in this fashion, we prove the identity (4.14).

□

So, using the above Proposition, we write Eqn. (4.13) in the form

$$\begin{aligned}i\hbar\frac{d}{dt}\hat{U}_g(t,t_0) &= i\hbar\Big[\Big(\dot{c} + \frac{1}{\hbar}a\dot{b} + \frac{1}{2}\dot{f}b^2 + \frac{1}{\hbar}\dot{h}ab - f\dot{h}b^2 \\ &\quad + \dot{g}e^{-2h}\Big(\frac{1}{2\hbar^2}a^2 - \frac{1}{\hbar}fab + \frac{1}{2}f^2b^2\Big)\Big)\hat{E}_3 \\ &\quad + \Big(-\dot{b} - \dot{h}b + \dot{g}e^{-2h}\Big(-\frac{1}{\hbar}a + fb\Big)\Big)\hat{E}_2 \\ &\quad + \Big(\frac{1}{\hbar}\dot{a} - \dot{f}b - \frac{1}{\hbar}\dot{h}a + 2f\dot{h}b + \dot{g}e^{-2h}\Big(\frac{1}{\hbar}fa - f^2b\Big)\Big)\hat{E}_1 \\ &\quad + \Big(\dot{f} - 2f\dot{h} + f^2\dot{g}e^{-2h}\Big)\hat{K}_+ \\ &\quad + 2\Big(\dot{h} - f\dot{g}e^{-2h}\Big)\hat{K}_0 + \Big(\dot{g}e^{-2h}\Big)\hat{K}_-\Big]\hat{U}_g(t,t_0). \end{aligned} \tag{4.15}$$

Using the equations (4.15) and (4.9), we compare both sides of the operator equation (4.12) and obtain that $\hat{U}_g(t, t_0)$ is solution of the problem, if the unknown functions satisfy the nonlinear system of six first-order equations

$$\begin{aligned}\dot{f} + \frac{\hbar}{\mu(t)}f^2 + 2B(t)f + \frac{\mu(t)\omega^2(t)}{\hbar} &= 0, \quad f(t_0) = 0, \\ \dot{g} + \frac{\hbar}{\mu(t)}e^{2h} &= 0, \quad g(t_0) = 0, \\ \dot{h} + \frac{\hbar}{\mu(t)}f + B(t) &= 0, \quad h(t_0) = 0.\end{aligned} \tag{4.16}$$

$$\begin{aligned}
\dot{a} + B(t)a + \mu(t)\omega^2(t)b + D(t) &= 0, \;\; a(t_0) = 0, \qquad (4.17)\\
\dot{b} - B(t)b - \frac{1}{\mu(t)}a - E(t) &= 0, \;\; b(t_0) = 0,\\
\dot{c} + \frac{1}{2\hbar\mu(t)}a^2 + \frac{E(t)}{\hbar}a - \frac{\mu(t)\omega^2(t)}{2\hbar}b^2 + \frac{F(t)}{\hbar} &= 0, \;\; c(t_0) = 0
\end{aligned}$$

In fact, (4.16) and (4.17) are two independent systems, one for $f, g, h$ and second for $a, b, c$. System (4.16) can be easily solved by realizing that first line is an initial value problem for the non-linear Riccatti equation, and using substitution

$$f(t) = \frac{\mu(t)}{\hbar}\left(\frac{\dot{x}}{x} - B(t)\right),$$

it transforms to the classical homogeneous equation of motion

$$\ddot{x} + \frac{\dot{\mu}}{\mu}\dot{x} + \left(\omega^2(t) - (\dot{B} + B^2 + \frac{\dot{\mu}}{\mu}B)\right)x = 0, \qquad (4.18)$$

with initial conditions

$$x(t_0) = x_0 \neq 0, \quad \dot{x}(t_0) = x_0 B(t_0). \qquad (4.19)$$

Denoting by $x_1(t)$, the solution of this IVP (4.18)-(4.19), solution of system (4.16) becomes

$$\begin{aligned}
f(t) &= \frac{\mu(t)}{\hbar}\left(\frac{\dot{x}_1(t)}{x_1(t)} - B(t)\right), \qquad (4.20)\\
g(t) &= -\hbar x_1^2(t_0)\int_{t_0}^{t}\frac{1}{\mu(s)x_1^2(s)}ds,\\
h(t) &= -\ln|x_1(t)| + \ln|x_1(t_0)|.
\end{aligned}$$

Assuming all coefficients in Eq.(4.18) are continuous on time interval containing $t_0$, by $x_2(t)$ we denote a second solution of Eq.(4.18) and using Abel's formula, we obtain that

$$x_2(t) = cx_1(t) \int^t \frac{1}{\mu(s)x_1^2(s)} ds. \tag{4.21}$$

Since

$$g(t) = -\hbar x_1^2(t_0) \int^t \frac{1}{\mu(s)x_1^2(s)} ds, \quad g(t_0) = 0, \tag{4.22}$$

using (4.21), $g(t)$ can be expressed in terms of these two independent solutions in the form

$$g(t) = -\hbar x_1^2(t_0) \left( \frac{x_2(t)}{x_1(t)} \right).$$

Now, we find the initial conditions for $x_2(t)$ so that $g(t_0) = 0$ and $\dot{g}(t_0) = -\hbar/\mu(t_0)$ as

$$x_2(t_0) = 0, \;\; \dot{x}_2(t_0) = 1/\mu(t_0)x_1(t_0).$$

This gives the solution of system (4.16) in terms of two linearly independent solutions $x_1(t)$ and $x_2(t)$ of the homogeneous equation as

$$\begin{aligned} f(t) &= \frac{\mu(t)}{\hbar} \left( \frac{\dot{x}_1(t)}{x_1(t)} - B(t) \right), \\ g(t) &= -\hbar x_1^2(t_0) \left( \frac{x_2(t)}{x_1(t)} \right) \\ h(t) &= -\ln \left| \frac{x_1(t)}{x_1(t_0)} \right|. \end{aligned} \tag{4.23}$$

On the other hand, we realize that in system (4.17), the equations for $b(t)$ and $a(t)$ are the same with the classical equations (4.2) and (4.3) for $x(t)$ and $p(t)$, respectively. Then, using first two equations in system (4.17), we obtain that $b(t)$ is solution of the nonhomogeneous Eq.(4.4), with initial conditions

$$x(t_0) = 0, \;\; \dot{x}(t_0) = E(t_0), \tag{4.24}$$

and we denote this solution by $b(t) = x_p(t)$. Similarly, it follows that $a(t) = p_p(t)$, where $p_p(t)$ is solution of the nonhomogeneous Eq.(4.5) for momentum, with initial conditions

$$p(t_0) = 0, \;\; \dot{p}(t_0) = -D(t_0). \tag{4.25}$$

Then, solution of system (4.17) is found in terms of the two particular solutions $x_p$ and $p_p$ as

$$\begin{aligned} a(t) &= p_p(t), \\ b(t) &= x_p(t), \\ c(t) &= \int^t \left[\frac{-(p_p(s))^2}{2\hbar\mu(s)} - \frac{E(s)}{\hbar}p_p(s) + \frac{\mu(s)\omega^2(s)}{2\hbar}x_p^2(s) - \frac{F(s)}{\hbar}\right] ds. \end{aligned} \tag{4.26}$$

Writing $p_p(t)$ in terms of $x_p(t)$ the solution of this system becomes,

$$\begin{aligned} a(t) &= \mu(t)\left(\dot{x}_p(t) - B(t)x_p(t) - E(t)\right), \\ b(t) &= x_p(t), \\ c(t) &= \frac{-1}{2\hbar}\int_{t_0}^t \mu(s)\Big[\dot{x}_p^2(s) - 2B(s)x_p(s)\dot{x}_p(s) \\ &\quad +\Big(B^2(s) - \omega^2(s)\Big)x_p^2(s) - E^2(s) + \frac{2}{\mu(s)}F(s)\Big] ds, \end{aligned} \tag{4.27}$$

showing that solution of the general oscillator is completely determined by solutions $x_1(t)$, $x_2(t)$ and $x_p(t)$ of the classical oscillator. We note that choosing different ordering of the exponential operators in the evolution operator (4.10), leads to different formulation of the system for the six unknown parameters. In any case, the system can be solved by quadrature, but we can not always easily see its solution in terms of $x_1$, $x_2$ and $x_p$, as in the present case.

For later use, we will write $x_p(t)$ in terms of solution $x_1(t)$ of the homogeneous IVP (4.18)-(4.19) (For details see Appendix B), which gives

$$
\begin{aligned}
a(t) &= -\frac{z(t)}{\hbar x_1(t)} + \frac{\mu(t)}{\hbar}\left(\dot{x}_1(t) - B(t)x_1(t)\right)\int_{t_0}^{t}\left[-\frac{z(s)}{\mu(s)x_1^2(s)} + \frac{E(s)}{x_1(s)}\right]ds, \quad (4.28)\\
b(t) &= x_1(t)\int_{t_0}^{t}\left[-\frac{z(s)}{\mu(s)x_1^2(s)} + \frac{E(s)}{x_1(s)}\right]ds, \quad (4.29)\\
c(t) &= -\frac{1}{\hbar}\int_{t_0}^{t}\left[\frac{z^2(s)}{2\mu(s)x_1^2(s)} - \frac{E(s)z(s)}{x_1(s)} + F(s)\right]ds \quad (4.30)\\
&- \frac{\mu(t)}{2\hbar}\left(\frac{\dot{x}_1(t)}{x_1(t)} - B(t)\right)x_1^2(t)\left[\int_{t_0}^{t}\left(-\frac{z(s)}{\mu(s)x_1^2(s)} + \frac{E(s)}{x_1(s)}\right)ds\right]^2,
\end{aligned}
$$

where

$$z(t) = \int_{t_0}^{t}\left[\mu(\xi)E(\xi)\Big(\dot{x}_1(\xi) - B(\xi)x_1(\xi)\Big) + D(\xi)x_1(\xi)\right]ds.$$

This formulation looks more complicated, but it gives solution of the systems (4.16) and (4.17) only in terms of the homogeneous solution $x_1(t)$ of IVP (4.18)-(4.19) and the time-dependent parameters of the Hamitonian. This allows us to see more easily the effect of the parameters $B, D, E, F$ to the particular solutions $b(t) = x_p(t)$, $a(t) = p_p(t)$ and can be directly used for exact and numerical calculations.

Now, after finding all unknown functions in (4.10), the exact form of the evolution operator in terms of $x_1(t), x_2(t)$ and $x_p(t), p_p(t)$ becomes

$$
\begin{aligned}
\hat{U}_g(t,t_0) &= \exp\left(\frac{i}{\hbar}\int_{t_0}^{t}\left[\frac{-1}{2\mu(s)}p_p^2(s) - E(s)p_p(s) + \frac{\mu(s)\omega^2(s)}{2}x_p^2(s) - F(s)\right]ds\right)\\
&\times\exp\Big(ip_p(t)q\Big)\times\exp\left(-x_p(t)\frac{\partial}{\partial q}\right)\times\exp\left(i\frac{\mu(t)}{2\hbar}\left(\frac{\dot{x}_1(t)}{x_1(t)} - B(t)\right)q^2\right)\\
&\times\exp\left(\ln\left|\frac{x_1(t_0)}{x_1(t)}\right|\left(q\frac{\partial}{\partial q} + \frac{1}{2}\right)\right)\times\exp\left(\frac{i}{2}\hbar x_1^2(t_0)\left(\frac{x_2(t)}{x_1(t)}\right)\frac{\partial^2}{\partial q^2}\right), \quad (4.31)
\end{aligned}
$$

where $p_p(t) = \mu(t)\left(\dot{x}_p(t) - B(t)x_p(t) - E(t)\right)$. Therefore, with this evolution operator we can solve the quantum oscillator problem (4.7) for given initial data that will be discussed in the following section.

## 4.3. The Wave Functions $\Psi_k(q,t)$

To solve the quantum evolution problem (4.7), as initial functions we choose the normalized eigenstates of the standard Hamiltonian,

$$\varphi_k(q) = N_k e^{-\frac{m\omega_0}{2\hbar}q^2} H_k\left(\sqrt{\frac{m\omega_0}{\hbar}}q\right), \tag{4.32}$$

where $H_k(\sqrt{m\omega_0/\hbar}q)$ are the Hermite polynomials, $N_k = (2^k k!)^{-1/2}(m\omega_0/\pi\hbar)^{1/4}$ are normalization constants, and eigenvalues are $E_k = (\hbar/\omega_0)(k+1/2)$, $k = 0, 1, 2, \ldots$, and applying the evolution operator (4.31) to these initial states we get wave function solutions as

$$\Psi_k(q,t) = \hat{U}_g(t,t_0)\varphi_k(q).$$

Since the evolution operator is formed as products of exponential operators, we need to establish how these operators act on a given function. First of all, for a function $f(q)$, that is continuous and infinitely differentiable, the operator $e^{\lambda(d/dq)}$ produces a shift by $\lambda$, i.e.

$$e^{\lambda(d/dq)}f(q) = \sum_{k=0}^{\infty}\frac{\lambda^k}{k!}\frac{d^k}{dq^k}f(q) = \sum_{k=0}^{\infty}\frac{\lambda^k}{k!}f^{(k)}(q) = f(q+\lambda). \tag{4.33}$$

Accordingly, it is called the *shifting operator*. In addition, the operator $e^{\lambda q(q/dq)}$ so called *dilatation operator* acts on a function $f(q)$ in the following manner;

$$e^{\lambda q(d/dq)}f(q) = f(e^{\lambda}q). \tag{4.34}$$

Equation (4.34) follows from the identity (4.33) by making the substitution $q = e^{\theta}$. Then,

$$e^{\lambda q(d/dq)}f(q) = e^{\lambda(d/dq)}f(e^{\theta}) = f(e^{\theta+\lambda}) = f(e^{\lambda}q).$$

Finally, to find the action of the operator $e^{-i/2g(t)\partial^2/\partial q^2}$ on the initial states $\varphi_k(q)$, we solve free Schrödinger equation

$$\left[-\frac{i}{2}\frac{\partial^2}{\partial q^2}\right]\widetilde{\varphi}_k(q;z) = \left[\frac{\partial}{\partial z}\right]\widetilde{\varphi}_k(q;z), \quad z - \text{real}, \tag{4.35}$$

with initial condition $\widetilde{\varphi}_k(q;0) = \varphi_k(q)$ (see Appendix C), and obtain

$$\begin{aligned}\widetilde{\varphi}_k(q;z) &= \frac{N_k}{\left(1+\left(\frac{m\omega_0}{\hbar}z\right)^2\right)^{\frac{1}{4}}} \times \exp\left(-\frac{i}{2}\left(\frac{\left(\frac{m\omega_0}{\hbar}\right)^2 z}{1+\left(\frac{m\omega_0}{\hbar}z\right)^2}\right)q^2\right)\\ &\times \exp\left(i\left(k+\frac{1}{2}\right)\arctan\left(\frac{m\omega_0}{\hbar}z\right)\right)\times\exp\left(-\left(\frac{\frac{m\omega_0}{2\hbar}}{1+\left(\frac{m\omega_0}{\hbar}z\right)^2}\right)q^2\right)\\ &\times H_k\left(\left(\frac{\sqrt{\frac{m\omega_0}{\hbar}}}{\left(1+\left(\frac{m\omega_0}{\hbar}z\right)^2\right)^{\frac{1}{2}}}\right)q\right).\end{aligned} \tag{4.36}$$

Then, using the equation (4.35), we can write

$$\exp\left(-\frac{i}{2}g(t)\frac{\partial^2}{\partial q^2}\right)\widetilde{\varphi}_k(q;z) = \exp\left(g(t)\frac{\partial}{\partial z}\right)\widetilde{\varphi}_k(q;z),$$

so that

$$\begin{aligned}\exp\left(-\frac{i}{2}g(t)\frac{\partial^2}{\partial q^2}\right)\varphi_k(q) &= \exp\left(-\frac{i}{2}g(t)\frac{\partial^2}{\partial q^2}\right)\widetilde{\varphi}_k(q;0)\\ &= \exp\left(g(t)\frac{\partial}{\partial z}\right)\widetilde{\varphi}_k(q;z)|_{z=0}\\ &= \widetilde{\varphi}_k(q;z+g(t))|_{z=0} = \widetilde{\varphi}_k(q;g(t)).\end{aligned} \tag{4.37}$$

Hence, as a consequence of (4.33), (4.34) and (4.37), we find

$$\begin{aligned}
\hat{U}_g(t,t_0)\varphi_k(q) &= e^{ic(t)}e^{\frac{i}{\hbar}a(t)q}e^{-b(t)\frac{\partial}{\partial q}}e^{\frac{i}{2}f(t)q^2}e^{h(t)\left(q\frac{\partial}{\partial q}+\frac{1}{2}\right)}e^{-\frac{i}{2}g(t)\frac{\partial^2}{\partial q^2}}\varphi_k(q) \\
&= e^{\frac{h(t)}{2}}e^{ic(t)}e^{\frac{i}{\hbar}a(t)q}e^{-b(t)\frac{\partial}{\partial q}}e^{\frac{i}{2}f(t)q^2}e^{h(t)\left(q\frac{\partial}{\partial q}\right)}\widetilde{\varphi}_k(q;g(t)) \\
&= e^{\frac{h(t)}{2}}e^{ic(t)}e^{\frac{i}{\hbar}a(t)q}e^{-b(t)\frac{\partial}{\partial q}}e^{\frac{i}{2}f(t)q^2}\widetilde{\varphi}_k(e^{h(t)}q;g(t)) \\
&= e^{\frac{h(t)}{2}}e^{ic(t)}e^{\frac{i}{\hbar}a(t)q}e^{\frac{i}{2}f(t)(q-b(t))^2}\widetilde{\varphi}_k(e^{h(t)}(q-b(t));g(t)).
\end{aligned}$$

And we obtain exact wave function in the form

$$\begin{aligned}
\Psi_k(q,t) &= N_k\sqrt{R_B(t)}\times\exp\left(i\left(k+\frac{1}{2}\right)\arctan\left(\frac{m\omega_0}{\hbar}g(t)\right)\right) \\
&\times\exp\left[i\left(\frac{f(t)}{2}q^2+\left(-f(t)b(t)+\frac{a(t)}{\hbar}\right)q+\frac{f(t)}{2}b^2(t)+c(t)\right)\right] \\
&\times\exp\left(-\frac{i}{2}\left(\frac{m\omega_0}{\hbar}\right)^2 g(t)R_B^2(t)(q-b(t))^2\right) \\
&\times\exp\left(-\frac{m\omega_0}{2\hbar}R_B^2(t)(q-b(t))^2\right)\times H_k\left(\sqrt{\frac{m\omega_0}{\hbar}}R_B(t)(q-b(t))\right).
\end{aligned}\tag{4.38}$$

Thus, in terms of $x_1(t), x_2(t)$ and $x_p(t), p_p(t)$ the wave function can be found as

$$\begin{aligned}
\Psi_k(q,t) &= N_k\sqrt{R_B(t)}\times\exp\left[i\left(k+\frac{1}{2}\right)\arctan\left(-m\omega_0x_1^2(t_0)\left(\frac{x_2(t)}{x_1(t)}\right)\right)\right] \\
&\times\exp\left\{\frac{i}{2\hbar}\left[\left(\mu(t)\left(\frac{\dot{x}_1(t)}{x_1(t)}-B(t)\right)\right.\right.\right. \\
&\left.\left.\left.-(m\omega_0x_1(t_0))^2\frac{x_2(t)}{x_1(t)}R_B^2(t)\right)(q-x_p(t))^2+2p_p(t)q\right]\right\} \\
&\times\exp\left[\frac{i}{\hbar}\int_{t_0}^{t}\left(\frac{-1}{2\mu(s)}p_p^2(s)-E(s)p_p(s)+\frac{\mu(s)\omega^2(s)}{2}x_p^2(s)-F(s)\right)ds\right] \\
&\times\exp\left[-\frac{1}{2}\left(\sqrt{\frac{m\omega_0}{\hbar}}R_B(t)\left(q-x_p(t)\right)\right)^2\right]\times H_k\left(\sqrt{\frac{m\omega_0}{\hbar}}R_B(t)\left(q-x_p(t)\right)\right),
\end{aligned}\tag{4.39}$$

and the probability density is then

$$\begin{aligned}
\rho_k(q,t) &= N_k^2R_B(t)\exp\left(-\left(\sqrt{\frac{m\omega_0}{\hbar}}R_B(t)(q-x_p(t))\right)^2\right) \\
&\times H_k^2\left(\sqrt{\frac{m\omega_0}{\hbar}}R_B(t)\left(q-x_p(t)\right)\right),
\end{aligned}\tag{4.40}$$

where $R_B(t)$ is the squeezing (or spreading) coefficient given by

$$R_B(t) = \sqrt{\frac{{x_0}^2}{x_1^2(t) + (m\omega_0 {x_0}^2 x_2(t))^2}}, \tag{4.41}$$

and $x_p(t)$ is the displacement of the wave packet. We note that $R_B(t)$ depends on the mixed term coefficient $B(t)$, but does not depend on the external term parameters $D(t), E(t)$ and $F(t)$. On the other hand, the displacement $x_p(t)$, clearly depends on all parameters of the Hamiltonian. Since the wave functions $\Psi_k(q,t)$ are normalized for all $k = 0, 1, 2, \ldots,$ probability density is a conserved quantity, that is

$$\begin{aligned} 1 &= \langle \varphi_k(q) | \varphi_k(q) \rangle = \langle \hat{U}_g^\dagger \hat{U}_g \varphi_k(q) | \varphi_k(q) \rangle = \langle \hat{U}_g \varphi_k(q) | \hat{U}_g \varphi_k(q) \rangle \\ &= ||\Psi_k(q,t)||^2 = \int_{-\infty}^{\infty} |\Psi_k(q,t)|^2 dq = \int_{-\infty}^{\infty} \rho_k(q,t) dq. \end{aligned}$$

By previous assumptions, $x_1(t)$ and $x_2(t)$ are smooth, and can not be simultaneously zero, and if $x_p(t)$ is also smooth, the preceding property can also be shown more precisely as follows. By substitution $\xi = \sqrt{(m\omega_0)/\hbar} R_B(t)(q - x_p(t))$, one has

$$\begin{aligned} \int_{-\infty}^{\infty} \rho_k(q,t) dq &= \lim_{M\to\infty} \int_{-M}^{M} \rho_k(q,t) dq \\ &= \lim_{M\to\infty} \frac{1}{2^k k! \sqrt{\pi}} \int_{-\sqrt{\frac{m\omega_0}{\hbar}} R_B(t)(M + x_p(t))}^{\sqrt{\frac{m\omega_0}{\hbar}} R_B(t)(M + x_p(t))} \exp(-\xi^2) H_k^2(\xi) d\xi = 1, \end{aligned}$$

for any $t$ and each $k$, as a consequence of the well-known integral $\int_{-\infty}^{\infty} \exp(-\xi^2) H_k^2(\xi) d\xi = 2^k k! \sqrt{\pi}$, showing that the probability density is conserved.

In the limiting case, when all external terms are zero, i.e. $B = D = E = F = 0$, the probability density takes the form

$$\rho_k(q,t) = N_k^2 \times R_0(t) \times \exp\left( -\left( \sqrt{\frac{m\omega_0}{\hbar}} R_0(t) q \right)^2 \right) \times H_k^2 \left( \sqrt{\frac{m\omega_0}{\hbar}} R_0(t)\,(q) \right), \tag{4.42}$$

which coincides with the result in (Büyükaşık & Pashaev& Ulaş-Tigrak, 2009).

Finally, we note that, $\varrho(t) = 1/(\sqrt{m\omega_0} R_B(t))$ is solution of the Ermakov-Pinney differential equation

$$\ddot{\varrho} + \frac{\dot{\mu}}{\mu}\dot{\varrho} + \left(\omega^2(t) - \left(\dot{B} + B^2 + \frac{\dot{\mu}}{\mu}B\right)\right)\varrho = \frac{1}{\mu^2\varrho^3}, \tag{4.43}$$

with initial conditions $\varrho(t_0) = 1/\sqrt{m\omega_0}$, $\dot{\varrho}(t_0) = B(t_0)/\sqrt{m\omega_0}$, and that can be used to compare our results by those obtained using the dynamical invariant approach.

## 4.4. Heisenberg Picture

In this section we mention about the Heisenberg picture representation of the position and momentum operators, which will be useful for finding the expectation values in the next section.

An observable operator $\hat{A}$, related with the system

$$\begin{cases} i\hbar\frac{\partial}{\partial t}\Psi(q,t) = \hat{H}_g(t)\Psi(q,t), \\ \Psi(q,t_0) = \Psi_0(q), \quad -\infty < q < \infty, \end{cases}$$

is defined in the Heisenberg picture as

$$\hat{A}_H(t) = \hat{U}_g^\dagger(t,t_0)\hat{A}_S\hat{U}_g(t,t_0), \tag{4.44}$$

where $\hat{U}_g(t,t_0)$ is the evolution operator for this system. Then,

$$\begin{aligned} \frac{d\hat{A}_H}{dt} &= \frac{\partial\hat{U}_g^\dagger}{\partial t}\hat{A}_S\hat{U}_g + \hat{U}_g^\dagger\frac{\partial A_S}{\partial t}\hat{U}_g + \hat{U}_g^\dagger\hat{A}_S\frac{\partial\hat{U}_g}{\partial t} \\ &= \frac{1}{i\hbar}(\hat{U}_g^\dagger\hat{A}_S\hat{H}_g\hat{U}_g - \hat{U}_g^\dagger\hat{H}_g\hat{A}_S\hat{U}_g) + \hat{U}_g^\dagger\frac{\partial A_S}{\partial t}\hat{U}_g \\ &= \frac{1}{i\hbar}(\underbrace{\hat{U}_g^\dagger\hat{A}_S\hat{U}_g}_{\hat{A}_H}\underbrace{\hat{U}_g^\dagger\hat{H}_g\hat{U}_g}_{\hat{H}_H} - \underbrace{\hat{U}_g^\dagger\hat{H}_g\hat{U}_g}_{\hat{H}_H}\underbrace{\hat{U}_g^\dagger\hat{A}_S\hat{U}_g}_{\hat{A}_H}) + \hat{U}_g^\dagger\frac{\partial A_S}{\partial t}\hat{U}_g \\ &= \frac{1}{i\hbar}[\hat{A}_H, \hat{H}_H] + \frac{\partial\hat{A}_H}{\partial t}. \end{aligned} \tag{4.45}$$

The Eq. (4.45) is the Heisenberg equation of motion.

The position and momentum operators in Heisenberg picture defined by

$$\begin{aligned}\hat{q}_H(t) &= \hat{U}_g^{\dagger}(t,t_0)\hat{q}_S\hat{U}_g(t,t_0), \quad \hat{q}_H(t_0)=\hat{q}_S\,,\\ \hat{p}_H(t) &= \hat{U}_g^{\dagger}(t,t_0)\hat{p}_S\hat{U}_g(t,t_0), \quad \hat{p}_H(t_0)=\hat{p}_S\,,\end{aligned}$$

are obtained explicitly using the general evolution operator in the following way:

First, by the definition given above, $\hat{q}_H(t)$ is written as

$$\begin{aligned}\hat{q}_H(t) &= e^{-g(t)\hat{K}_-}e^{-2h(t)\hat{K}_0}e^{-f(t)\hat{K}_+}e^{b(t)\hat{E}_2}e^{-\frac{1}{\hbar}a(t)\hat{E}_1}e^{-c(t)\hat{E}_3}\hat{q}_S \\ &\times e^{c(t)\hat{E}_3}e^{\frac{1}{\hbar}a(t)\hat{E}_1}e^{-b(t)\hat{E}_2}e^{f(t)\hat{K}_+}e^{2h(t)\hat{K}_0}e^{g(t)\hat{K}_-}.\end{aligned} \tag{4.46}$$

Then, by the Proposition (4.1), we obtain the relations

$$e^{b(t)\hat{E}_2}\hat{q}_S e^{-b(t)\hat{E}_2}=\hat{q}_S+b(t),\;\; e^{-2h(t)\hat{K}_0}\hat{q}_S e^{2h(t)\hat{K}_0}=e^{-h(t)}\hat{q}_S\,,\;\; e^{-g(t)\hat{K}_-}\hat{q}_S e^{g(t)\hat{K}_-}=\hat{q}_S-\frac{g(t)}{\hbar}\hat{p}_S\,.$$

So using these equations and the fact that $\hat{q}_S$ commutes $\hat{E}_1$ and $\hat{E}_3$, we rearrange the equation (4.46) and deduce that

$$\hat{q}_H(t)=e^{-h(t)}\left(\hat{q}_H(t_0)-\frac{g(t)}{\hbar}\hat{p}_H(t_0)\right)+b(t). \tag{4.47}$$

By the same way, we write

$$\begin{aligned}\hat{p}_H(t) &= e^{-g(t)\hat{K}_-}e^{-2h(t)\hat{K}_0}e^{-f(t)\hat{K}_+}e^{b(t)\hat{E}_2}e^{-\frac{1}{\hbar}a(t)\hat{E}_1}e^{-c(t)\hat{E}_3}\hat{p}_S \\ &\times e^{c(t)\hat{E}_3}e^{\frac{1}{\hbar}a(t)\hat{E}_1}e^{-b(t)\hat{E}_2}e^{f(t)\hat{K}_+}e^{2h(t)\hat{K}_0}e^{g(t)\hat{K}_-},\end{aligned} \tag{4.48}$$

and from the Proposition (4.1), we get the equalities

$$e^{-\frac{1}{\hbar}a(t)\hat{E}_1}\hat{p}_S e^{\frac{1}{\hbar}a(t)\hat{E}_1}=\hat{p}_S+a(t),\;\; e^{-f(t)\hat{K}_+}\hat{p}_S e^{f(t)\hat{K}_+}=\hat{p}_S+\hbar f(t)\hat{q}_S\,,\;\; e^{-2h(t)\hat{K}_0}\hat{p}_S e^{2h(t)\hat{K}_0}=e^{h(t)}\hat{p}_S\,,$$

by which we have

$$\hat{p}_H(t) = \hbar f(t)e^{-h(t)}\hat{q}_H(t_0) + \left(e^{h(t)} - f(t)g(t)e^{-h(t)}\right)\hat{p}_H(t_0) + a(t). \tag{4.49}$$

We write $\hat{q}_H(t)$ and $\hat{p}_H(t)$ in terms of the functions $x_1(t), x_2(t), x_p(t)$ and $p_p(t)$ as follows:

$$\hat{q}_H(t) = \frac{1}{x_0}x_1(t)\hat{q}_H(t_0) + x_0x_2(t)\hat{p}_H(t_0) + x_p(t), \tag{4.50}$$

$$\begin{aligned}\hat{p}_H(t) = {} & \frac{1}{x_0}\mu(t)(\dot{x}_1(t) - B(t)x_1(t))\hat{q}_H(t_0) \\ & + x_0\mu(t)(\dot{x}_2(t) - B(t)x_2(t))\hat{p}_H(t_0) + p_p(t).\end{aligned} \tag{4.51}$$

Then, it is easy to show that these operators satisfy the Heisenberg equations of motion,

$$\frac{d}{dt}\hat{q}_H(t) = \frac{\hat{p}_H(t)}{\mu(t)} + B(t)\hat{q}_H(t) + E(t),$$

$$\frac{d}{dt}\hat{p}_H(t) = -\left(\mu(t)\omega^2(t)\hat{q}_H(t) + B(t)\hat{p}_H(t) + D(t)\right),$$

and thus $\hat{q}_H(t)$ is solution of the classical equation (4.4), and $\hat{p}_H(t)$ is solution of (4.5).

## 4.5. Expectation Values, Fluctuations and Uncertainty Relation at $\Psi_k(q,t)$

The expectations of position and momentum at state $\Psi_k(q,t)$ can be found using that

$$\langle\hat{q}\rangle_k(t) \equiv \langle\Psi_k(q,t)|\hat{q}_S|\Psi_k(q,t)\rangle = \langle\varphi_k(q)|\hat{q}_H(t)|\varphi_k(q)\rangle,$$

$$\langle\hat{p}\rangle_k(t) \equiv \langle\Psi_k(q,t)|\hat{p}_S|\Psi_k(q,t)\rangle = \langle\varphi_k(q)|\hat{p}_H(t)|\varphi_k(q)\rangle,$$

where $\hat{q}_H(t)$ is given by (4.50) and $\hat{p}_H(t)$ is given by (4.51). Indeed, since

$$\hat{q}_H(t_0) = \sqrt{\frac{\hbar}{2m\omega_0}}(\hat{a} + \hat{a}^\dagger), \tag{4.52}$$

$$\hat{p}_H(t_0) = -i\sqrt{\frac{m\omega_0\hbar}{2}}(\hat{a} - \hat{a}^\dagger), \tag{4.53}$$

where $\hat{a}, \hat{a}^\dagger$ are lowering and raising operators of the standard Hamiltonian $\hat{H}_0 = \hbar\omega_0(\hat{a}^\dagger\hat{a}+1/2)$, and $\hat{a}\varphi_k(q) = \sqrt{k}\varphi_{k-1}$, $\hat{a}^\dagger\varphi_k(q) = \sqrt{k+1}\varphi_{k+1}$, $k = 0, 1, 2, \ldots$, we obtain

$$\begin{aligned}
\langle \hat{q} \rangle_k(t) &= \frac{x_1(t)}{x_0} \langle \varphi_k(q)|\hat{q}_H(t_0)|\varphi_k(q)\rangle + x_0 x_2(t) \langle \varphi_k(q)|\hat{p}_H(t_0)|\varphi_k(q)\rangle + x_p(t) \\
&= x_p(t), \\
\langle \hat{p} \rangle_k(t) &= \frac{\mu(t)}{x_0}(\dot{x}_1(t) - B(t)x_1(t)) \langle \varphi_k(q)|\hat{q}_H(t_0)|\varphi_k(q)\rangle \\
&\quad + \mu(t)x_0(\dot{x}_2(t) - B(t)x_2(t)) \langle \varphi_k(q)|\hat{p}_H(t_0)|\varphi_k(q)\rangle + p_p(t) \\
&= p_p(t).
\end{aligned}$$

From the proof of the Proposition (3.2), we know that

$$\langle \varphi_k(q)|\hat{q}_H^2(t_0)|\varphi_k(q)\rangle = \frac{\hbar}{m\omega_0}\left(k + \frac{1}{2}\right), \quad \langle \varphi_k(q)|\hat{p}_H^2(t_0)|\varphi_k(q)\rangle = m\omega_0\hbar\left(k + \frac{1}{2}\right).$$

We also compute

$$\langle \varphi_k(q)|\hat{q}_H(t_0)\hat{p}_H(t_0) + \hat{p}_H(t_0)\hat{q}_H(t_0)|\varphi_k(q)\rangle = -i\hbar \langle \varphi_k(q)|\hat{a}^2 - (\hat{a}^\dagger)^2|\varphi_k(q)\rangle = 0.$$

Then, the expectation values of squares of position and momentum are

$$
\begin{aligned}
\langle \hat{q}^2 \rangle_k(t) &= \langle \varphi_k(q) | \hat{q}_H^2(t) | \varphi_k(q) \rangle \\
&= \left\langle \varphi_k(q) \middle| \left( \frac{x_1(t)}{x_0} \right)^2 \hat{q}_H^2(t_0) + x_1(t) x_2(t) \{ \hat{q}_H(t_0), \hat{p}_H(t_0) \} \right. \\
&\quad \left. + (x_0 x_2(t))^2 \hat{p}_H^2(t_0) + 2 x_p(t) \left( \frac{x_1(t)}{x_0} \hat{q}_H(t_0) + x_0 x_2(t) \right) + x_p^2(t) \middle| \varphi_k(q) \right\rangle \\
&= \left( k + \frac{1}{2} \right) \left( \frac{\hbar}{m\omega_0 R_B^2(t)} \right) + x_p^2(t),
\end{aligned}
$$

$$
\begin{aligned}
\langle \hat{p}^2 \rangle_k(t) &= \langle \varphi_k(q) | \hat{p}_H^2(t) | \varphi_k(q) \rangle \\
&= \left\langle \varphi_k(q) \middle| \mu^2(t) \left[ \frac{1}{x_0^2} (\dot{x}_1(t) - B(t) x_1(t))^2 \hat{q}_H^2(t_0) + x_0^2 (\dot{x}_2(t) - B(t) x_2(t))^2 \hat{p}_H^2(t_0) \right. \right. \\
&\quad \left. + (\dot{x}_1(t) - B(t) x_1(t))(\dot{x}_2(t) - B(t) x_2(t)) \{ \hat{q}_H(t_0), \hat{p}_H(t_0) \} \right] \\
&\quad + 2\mu(t) p_p(t) \left[ \frac{1}{x_0} (\dot{x}_1(t) - B(t) x_1(t)) \hat{q}_H(t_0) \right. \\
&\quad \left. \left. + x_0 (\dot{x}_2(t) - B(t) x_2(t)) \hat{p}_H(t_0) \right] + p_p^2(t) \middle| \varphi_k(q) \right\rangle \\
&= \left( k + \frac{1}{2} \right) (m\omega_0 \hbar R_B^2(t) \left[ 1 + \frac{\mu^2(t)}{(m\omega_0 R_B^2(t))^2} \left( \frac{\dot{R}_B(t)}{R_B(t)} + B(t) \right)^2 \right] + p_p^2(t),
\end{aligned}
$$

and using $(\Delta \hat{q})_k(t) = \sqrt{\langle \hat{q}^2 \rangle_k(t) - \langle \hat{q} \rangle_k(t)}$, $(\Delta \hat{p})_k(t) = \sqrt{\langle \hat{p}^2 \rangle_k(t) - \langle \hat{p} \rangle_k(t)}$, the fluctuations for $\hat{q}$ and $\hat{p}$ are found as

$$
\begin{aligned}
(\Delta \hat{q})_k(t) &= \sqrt{\left( k + \frac{1}{2} \right) \left( \frac{\hbar}{m\omega_0 R_B^2(t)} \right)}, \\
(\Delta \hat{p})_k(t) &= \sqrt{\left( k + \frac{1}{2} \right) (m\omega_0 \hbar R_B^2(t))} \sqrt{1 + \frac{\mu^2(t)}{(m\omega_0 R_B^2(t))^2} \left( \frac{\dot{R}_B(t)}{R_B(t)} + B(t) \right)^2}.
\end{aligned}
$$

This gives the uncertainty relation in the form

$$
(\Delta \hat{q})_k (\Delta \hat{p})_k = \hbar \left( k + \frac{1}{2} \right) \sqrt{1 + \frac{\mu^2(t)}{(m\omega_0 R_B^2(t))^2} \left( \frac{\dot{R}_B(t)}{R_B(t)} + B(t) \right)^2}, \tag{4.54}
$$

from which we have $(\Delta\hat{q})_k(\Delta\hat{p})_k \geq \hbar/2$.

## 4.6. Coherent States of the Generalized Quantum Harmonic Oscillator

In this section, we study time-evolution of initially Glauber coherent states under the influence of the evolution operator (4.31) of the generalized parametric oscillator, that is $\Phi_\alpha(q,t) = \hat{U}_g(t,t_0)\phi_\alpha(q,t_0)$. Using that $\phi_\alpha(q,t_0) = \phi_\alpha(q)$ is given by (3.15), and $\hat{U}_g(t,t_0)\varphi_k(q) = \Psi_k(q,t)$, one directly gets the generalized coherent states in terms of the wave functions of the Schrödinger equation

$$\Phi_\alpha(q,t) = e^{-\frac{|\alpha|^2}{2}} \sum_k \frac{\alpha^k}{\sqrt{k!}} \Psi_k(q,t). \tag{4.55}$$

Or directly applying the evolution operator $\hat{U}_g(t,t_0)$ to $\phi_\alpha(q)$, time evolved coherent states can be found. For this, we first solve the following free Schrödinger equation (see Appendix C)

$$\begin{cases} \frac{\partial}{\partial t}\psi_\alpha(q,t) = -\frac{i}{2}\frac{\partial^2}{\partial q^2}\psi_\alpha(q,t) \\ \psi_\alpha(q,0) = \phi_\alpha(q) = \left(\frac{\Omega_0}{\pi}\right)^{1/4} \exp\left(\frac{i}{\hbar}\langle p\rangle_\alpha q\right)\exp\left(-\frac{\Omega_0}{2}(q-\langle q\rangle_\alpha)^2\right), \end{cases}$$

and obtain

$$\begin{aligned} \psi_\alpha(q,t) \quad = \quad & \left(\frac{m\omega_0}{\pi\hbar}\right)^{1/4} \times \frac{\exp(-\alpha_2^2 + 2i\alpha_1\alpha 2)}{\left(1+\left(\frac{m\omega_0}{\hbar}t\right)^2\right)^{1/4}} \times \exp\left(\frac{i}{2}\arctan\left(\frac{m\omega_0}{\hbar}t\right)\right) \\ & \times \exp\left[\frac{-\frac{m\omega_0}{2\hbar}\left(1+i\frac{m\omega_0}{\hbar}t\right)}{1+\left(\frac{m\omega_0}{\hbar}t\right)^2}\left(q-\sqrt{\frac{2\hbar}{m\omega_0}}\alpha\right)^2\right]. \end{aligned} \tag{4.56}$$

Then, by using the Eqn. (4.56), we write

$$\exp\left(-\frac{i}{2}g(t)\frac{\partial^2}{\partial q^2}\right)\phi_\alpha(q) = \psi_\alpha(q,g(t)). \tag{4.57}$$

So the above result (4.57) gives

$$\begin{aligned}\hat{U}_g(t,t_0)\phi_\alpha(q) &= e^{ic(t)}e^{\frac{i}{\hbar}a(t)q}e^{-b(t)\frac{\partial}{\partial q}}e^{\frac{i}{2}f(t)q^2}e^{h(t)\left(q\frac{\partial}{\partial q}+\frac{1}{2}\right)}e^{-\frac{i}{2}g(t)\frac{\partial^2}{\partial q^2}}\phi_\alpha(q)\\ &= e^{\frac{h(t)}{2}}e^{ic(t)}e^{\frac{i}{\hbar}a(t)q}e^{-b(t)\frac{\partial}{\partial q}}e^{\frac{i}{2}f(t)q^2}\psi_\alpha(e^{h(t)}q,g(t))\\ &= e^{\frac{h(t)}{2}}e^{ic(t)}e^{\frac{i}{\hbar}a(t)q}e^{\frac{i}{2}f(t)(q-b(t))^2}\psi_\alpha(e^{h(t)}(q-b(t)),g(t)).\end{aligned}$$

Thus, time-evolved coherent state is found in the form

$$\begin{aligned}\Phi_\alpha(q,t) &= \left(\frac{m\omega_0}{\pi\hbar}\right)^{1/4}\sqrt{R_B(t)}\times\exp(-\alpha_2^2+2i\alpha_1\alpha 2)\exp\left(\frac{i}{2}\arctan\left(\frac{m\omega_0}{\hbar}t\right)\right)\\ &\times\exp\left[i\left(\frac{f(t)}{2}(q-b(t))^2+\frac{a(t)}{\hbar}q+c(t)\right)\right]\\ &\times\exp\left[-\frac{m\omega_0}{2\hbar}\left(1+i\frac{m\omega_0}{\hbar}g(t)\right)R_B^2(t)\left(q-b(t)-\sqrt{\frac{2\hbar}{m\omega_0}}\frac{x_1(t)}{x_0}\alpha\right)^2\right],\end{aligned}\tag{4.58}$$

and it can be expressed in terms of the functions $x_1(t)$, $x_2(t)$, $x_p(t)$ and $p_p(t)$ as follows

$$\begin{aligned}\Phi_\alpha(q,t) &= \left(\frac{m\omega_0}{\pi\hbar}\right)^{1/4}R_B(t)\times\sqrt{\frac{x_1(t)}{x_0}-i(m\omega_0x_0)x_2(t)}\\ &\times\exp\left[\frac{i}{\hbar}\int_{t_0}^{t}\left(\frac{-1}{2\mu(s)}p_p^2(s)-E(s)p_p(s)+\frac{\mu(s)\omega^2(s)}{2}x_p^2(s)-F(s)\right)ds\right]\\ &\times\exp\left[i(m\omega_0)x_2(t)R_B^2(t)(x_1(t)-i(m\omega_0x_0^2)x_2(t))\alpha^2-\alpha_1^2\right]\times\exp\left(\frac{i}{\hbar}p_p(t)q\right)\\ &\times\exp\left[\alpha\sqrt{\frac{2m\omega_0}{\hbar}}R_B^2(t)\left(\frac{x_1(t)}{x_0}-i(m\omega_0x_0)x_2(t)\right)(q-x_p(t))\right]\\ &\times\exp\left[\left(\frac{-i}{2\hbar}\mu(t)\left(B(t)+\frac{\dot{R}_B(t)}{R_B(t)}\right)-\left(\frac{m\omega_0}{2\hbar}\right)R_B^2(t)\right)(q-x_p(t))^2\right].\end{aligned}\tag{4.59}$$

The corresponding probability density for time evolved coherent states (4.59) is

$$
\begin{aligned}
\rho_\alpha(q,t) &= \sqrt{\frac{m\omega_0}{\pi\hbar}}R_B^2(t)\times\exp\left\{2\left[\left((m\omega_0 x_0)x_2(t)R_B(t)\right)^2(\alpha_1^2-\alpha_2^2)\right.\right. \\
&\quad \left.\left. -2(m\omega_0)x_1(t)x_2(t)R_B^2(t)\alpha_1\alpha_2-\alpha_1^2\right]\right\} \\
&\quad \times\exp\left(2\sqrt{\frac{2m\omega_0}{\hbar}}R_B^2(t)\left(\alpha_1\frac{x_1(t)}{x_0}+\alpha_2(m\omega_0 x_0)x_2(t)\right)(q-x_p(t))\right) \\
&\quad \times\exp\left(-\left(\frac{m\omega_0}{\hbar}\right)R_B^2(t)(q-x_p(t))^2\right).
\end{aligned}
\tag{4.60}
$$

As another approach, we show that $\Phi_\alpha(q,t)$ can be defined also as eigenstates of the annihilation operator $\hat{A}_0(t)$ of a certain dynamical invariant $\hat{I}_0(t)$ of the generalized Hamiltonian system. Indeed, let

$$\hat{A}_0(t) = \hat{U}_g(t,t_0)\hat{a}\hat{U}_g^\dagger(t,t_0)$$

and

$$\hat{A}_0^\dagger(t) = \hat{U}_g(t,t_0)\hat{a}^\dagger\hat{U}_g^\dagger(t,t_0),$$

where $\hat{a}$ and $\hat{a}^\dagger$ are the annihilation and creation operators for $\hat{H}_0$, respectively. We define the operator $\hat{I}_0(t)$

$$\hat{I}_0(t) = \hbar\left(\hat{A}_0^\dagger(t)\hat{A}_0(t)+\frac{1}{2}\right),$$

and we write it in terms of $\hat{H}_0$,

$$
\begin{aligned}
\hat{I}_0(t) &= \hbar\left(\hat{A}_0^\dagger(t)\hat{A}_0(t)+\frac{1}{2}\right) \\
&= \hbar\left(\hat{U}_g(t,t_0)\hat{a}^\dagger\hat{a}\hat{U}_g^\dagger(t,t_0)+\frac{1}{2}\hat{U}_g(t,t_0)\hat{U}_g^\dagger(t,t_0)\right) \\
&= \hbar\left(\hat{U}_g(t,t_0)\left(\hat{a}^\dagger\hat{a}+\frac{1}{2}\right)\hat{U}_g^\dagger(t,t_0)\right) \\
&= \frac{1}{\omega_0}\hat{U}_g(t,t_0)\hat{H}_0\hat{U}_g^\dagger(t,t_0).
\end{aligned}
$$

Then, remembering the fact that $\hat{U}_g(t,t_0)$ satisfies the equation (4.12), we obtain

$$\begin{aligned}\frac{\partial \hat{I}_0}{\partial t} &= \frac{1}{\omega_0}\left(\frac{\partial \hat{U}_g}{\partial t}\hat{H}_0\hat{U}_g^\dagger + \hat{U}_g\hat{H}_0\frac{\partial \hat{U}_g^\dagger}{\partial t}\right)\\ &= \frac{1}{\omega_0}\left(\frac{1}{i\hbar}\hat{H}_g\hat{U}_g\hat{H}_0\hat{U}_g^\dagger + \hat{U}_g\hat{H}_0\left(-\frac{1}{i\hbar}\hat{U}_g^\dagger\hat{H}_g\right)\right)\\ &= \frac{1}{i\omega_0\hbar}\Big(\hat{H}_g\underbrace{\hat{U}_g\hat{H}_0\hat{U}_g^\dagger}_{=\omega_0\hat{I}_0} - \underbrace{\hat{U}_g\hat{H}_0\hat{U}_g^\dagger}_{=\omega_0\hat{I}_0}\hat{H}_g\Big)\\ &= \frac{1}{i\hbar}[\hat{H}_g,\hat{I}_0].\end{aligned}$$

Therefore, the operator $\hat{I}_0(t)$ satisfies

$$\frac{d\hat{I}_0}{dt} \equiv \frac{\partial \hat{I}_0}{\partial t} + \frac{1}{i\hbar}[\hat{I}_0,\hat{H}_g] = 0,$$

which shows that it is an invariant for the system. Knowing the evolution operator and the relations,

$$\begin{aligned}e^{-\frac{i}{2}g(t)\frac{\partial^2}{\partial q^2}}\hat{q}e^{\frac{i}{2}g(t)\frac{\partial^2}{\partial q^2}} &= \hat{q} + \frac{g(t)}{\hbar}\hat{p},\\ e^{h(t)q\frac{\partial}{\partial q}}\hat{q}e^{-h(t)q\frac{\partial}{\partial q}} &= e^{h(t)}\hat{q},\\ e^{h(t)q\frac{\partial}{\partial q}}\hat{p}e^{-h(t)q\frac{\partial}{\partial q}} &= e^{-h(t)}\hat{p},\\ e^{\frac{i}{2}f(t)q^2}\hat{p}e^{-\frac{i}{2}f(t)q^2} &= \hat{p} - \hbar f(t)\hat{q},\\ e^{-b(t)\frac{\partial}{\partial q}}\hat{q}e^{b(t)\frac{\partial}{\partial q}} &= \hat{q} - b(t),\\ e^{\frac{i}{\hbar}a(t)\hat{q}}\hat{p}e^{-\frac{i}{\hbar}a(t)\hat{q}} &= \hat{p} - a(t),\end{aligned}$$

obtained by using the Proposition (4.1), the lowering operator $\hat{A}_0(t)$ is explicitly found as

$$\begin{aligned}\hat{A}_0(t) &= \hat{U}_g(t,t_0)\hat{a}\hat{U}_g^\dagger(t,t_0)\\ &= e^{c(t)\hat{E}_3}e^{\frac{1}{\hbar}a(t)\hat{E}_1}e^{-b(t)\hat{E}_2}e^{f(t)\hat{K}_+}e^{2h(t)\hat{K}_0}e^{g(t)\hat{K}_-}\\ &\quad\times\left(\sqrt{\frac{m\omega_0}{2\hbar}}\hat{q} + \frac{i}{\sqrt{2m\omega_0\hbar}}\hat{p}\right)\hat{U}_g^\dagger(t,t_0)\\ &= \left[\sqrt{\frac{m\omega_0}{2\hbar}}\left(\frac{x_0}{x_1(t)} - f(t)g(t)\frac{x_1(t)}{x_0}\right) - i\sqrt{\frac{\hbar}{2m\omega_0}}f(t)\frac{x_1(t)}{x_0}\right](\hat{q} - b(t))\\ &\quad+\left[\sqrt{\frac{m\omega_0}{2\hbar}}\frac{g(t)x_1(t)}{\hbar x_0} + i\sqrt{\frac{\hbar}{2m\omega_0}}\frac{x_1(t)}{\hbar x_0}\right](\hat{p} - a(t)).\end{aligned}$$

By doing the same procedure, the raising operator $\hat{A}_0^\dagger(t)$ is found as

$$\begin{aligned}\hat{A}_0^\dagger(t) &= \left[\sqrt{\frac{m\omega_0}{2\hbar}}\left(\frac{x_0}{x_1(t)} - f(t)g(t)\frac{x_1(t)}{x_0}\right) + i\sqrt{\frac{\hbar}{2m\omega_0}}f(t)\frac{x_1(t)}{x_0}\right](\hat{q} - b(t)) \\ &+\left[\sqrt{\frac{m\omega_0}{2\hbar}}\frac{g(t)x_1(t)}{\hbar x_0} - i\sqrt{\frac{\hbar}{2m\omega_0}}\frac{x_1(t)}{\hbar x_0}\right](\hat{p} - a(t)).\end{aligned}$$

Using the definition of $R_B(t)$ given by the equation (4.41), the lowering and raising operators found above are written in terms of the functions $x_1(t), x_2(t), x_p(t)$ and $p_p(t)$

$$\begin{aligned}\hat{A}_0(t) &= \left\{\left[\sqrt{\frac{m\omega_0}{2\hbar}}R_B(t) + \frac{i\mu(t)}{\sqrt{2m\omega_0\hbar}R_B^2(t)}(B(t)R_B(t) + \dot{R}_B(t))\right](\hat{q} - x_p(t))\right. \\ &\left.+\frac{i}{\sqrt{2m\omega_0\hbar}R_B(t)}(\hat{p} - p_p(t))\right\}\exp\left(i\arctan\left(m\omega_0 x_0^2\frac{x_2(t)}{x_1(t)}\right)\right),\end{aligned}$$

$$\begin{aligned}\hat{A}_0^\dagger(t) &= \left\{\left[\sqrt{\frac{m\omega_0}{2\hbar}}R_B(t) - \frac{i\mu(t)}{\sqrt{2m\omega_0\hbar}R_B^2(t)}(B(t)R_B(t) + \dot{R}_B(t))\right](\hat{q} - x_p(t))\right. \\ &\left.-\frac{i}{\sqrt{2m\omega_0\hbar}R_B(t)}(\hat{p} - p_p(t))\right\}\exp\left(-i\arctan\left(m\omega_0 x_0^2\frac{x_2(t)}{x_1(t)}\right)\right).\end{aligned}$$

and $\hat{I}_0(t)$ becomes

$$\begin{aligned}\hat{I}_0(t) &= \frac{m\omega_0}{2}R_B^2(t)(\hat{q} - x_p(t))^2 \\ &+\frac{1}{2m\omega_0 R_B^2(t)}\left[(\hat{p} - p_p(t)) + \frac{\mu(t)}{R_B(t)}[B(t)R_B(t) + \dot{R}_B(t)](\hat{q} - x_p(t))\right]^2.\end{aligned}$$

Then, by construction we have $\hat{A}_0(t)\Phi_\alpha(q,t) = \alpha\Phi_\alpha(q,t)$, showing that coherent states $\Phi_\alpha(q,t)$ are the eigenstates of $\hat{A}_0(t)$. It is known that for a given Hamiltonian the invariants depend on the initial wave functions. The invariant $\hat{I}_0(t)$ found here corresponds to the initial state $\varphi_k(q)$. We note also that, the time-dependent invariant operator constructed above is special in the sense that, its eigenstates are $\Psi_k(q,t)$ corresponding to the time-independent eigenvalues $E_k = (\hbar/\omega_0)(k+1/2)$ of the standard Hamiltonian $\hat{H}_0$, that's $\hat{I}_0(t)\Psi_k(q,t) = E_k\Psi_k(q,t),\ k = 0, 1, 2, ....$

## 4.7. Expectations and Uncertainties at Coherent States

Using that $\langle\hat{q}\rangle_\alpha(t) \equiv \langle\Phi_\alpha(q,t)|\hat{q}|\Phi_\alpha(q,t)\rangle = \langle\phi_\alpha(q,t_0)|\hat{q}_H(t)|\phi_\alpha(q,t_0)\rangle$, and the equations (3.16) and (3.17) the expectation value of position at coherent state $\Phi_\alpha(q,t)$ is

$$\begin{aligned}\langle\hat{q}\rangle_\alpha(t) &= \left\langle\phi_\alpha(q,t_0)\left|\frac{1}{x_0}x_1(t)\hat{q}_H(t_0) + x_0x_2(t)\hat{p}_H(t_0) + x_p(t)\right|\phi_\alpha(q,t_0)\right\rangle \\ &= \sqrt{\frac{2\hbar}{m\omega_0}}\left(\frac{\alpha_1}{x_0}x_1(t) + \alpha_2(m\omega_0x_0)x_2(t)\right) + x_p(t). \qquad (4.61)\end{aligned}$$

Similarly, using that $\langle\hat{p}\rangle_\alpha(t) = \langle\phi_\alpha(q,t_0)|\hat{p}_H(t)|\phi_\alpha(q,t_0)\rangle$, the expectation value of the momentum is

$$\begin{aligned}\langle\hat{p}\rangle_\alpha(t) &= \Big\langle\phi_\alpha(q,t_0)\Big|\mu(t)\Big[\frac{1}{x_0}(\dot{x}_1(t) - B(t)x_1(t))\hat{q}_H(t_0) + x_0(\dot{x}_2(t) - B(t)x_2(t))\hat{p}_H(t_0)\Big] \\ &\quad + p_p(t)\Big|\phi_\alpha(q,t_0)\Big\rangle \\ &= \sqrt{\frac{2\hbar}{m\omega_0}}\mu(t)\Big[\frac{\alpha_1}{x_0}(\dot{x}_1(t) - B(t)x_1(t)) + \alpha_2(m\omega_0x_0)(\dot{x}_2(t) - B(t)x_2(t))\Big] \qquad (4.62) \\ &\quad + p_p(t).\end{aligned}$$

After that, it is not difficult to show that the expectation values at coherent states satisfy the classical equation of motion.

Furthermore, using the equations (3.18), (3.19) and

$$\langle\phi_\alpha(q,t_0)|\hat{q}_H(t_0)\hat{p}_H(t_0) + \hat{p}_H(t_0)\hat{q}_H(t_0)|\phi_\alpha(q,t_0)\rangle = 4\hbar\alpha_1\alpha_2,$$

we find the expectations of squares:

$$
\begin{aligned}
\langle \hat{q}^2 \rangle_\alpha(t) &= \langle \phi_\alpha(q,t_0)|\hat{q}_H^2(t)|\phi_\alpha(q,t_0)\rangle \\
&= \Big\langle \phi_\alpha(q,t_0)\Big| \left(\frac{x_1(t)}{x_0}\right)^2 \hat{q}_H^2(t_0) + x_1(t)x_2(t)\{\hat{q}_H(t_0),\hat{p}_H(t_0)\} + (x_0 x_2(t))^2 \hat{p}_H^2(t_0) \\
&\quad +2x_p(t)\left(\frac{x_1(t)}{x_0}\hat{q}_H(t_0) + x_0 x_2(t)\right) + x_p^2(t)\Big|\phi_\alpha(q,t_0)\Big\rangle \\
&= \left[\sqrt{\frac{2\hbar}{m\omega_0}}\left(\alpha_1\frac{x_1(t)}{x_0} + \alpha_2(m\omega_0 x_0)x_2(t)\right) + x_p(t)\right]^2 + \frac{\hbar}{2m\omega_0 R_B^2(t)}, \qquad (4.63)
\end{aligned}
$$

$$
\begin{aligned}
\langle \hat{p}^2 \rangle_\alpha(t) &= \langle \phi_\alpha(q,t_0)|\hat{p}_H^2(t)|\phi_\alpha(q,t_0)\rangle \\
&= \Big\langle \phi_\alpha(q,t_0)\Big|\mu^2(t)\Big[\frac{1}{x_0^2}(\dot{x}_1(t) - B(t)x_1(t))^2\hat{q}_H^2(t_0) + x_0^2(\dot{x}_2(t) - B(t)x_2(t))^2\hat{p}_H^2(t_0) \\
&\quad +(\dot{x}_1(t) - B(t)x_1(t))(\dot{x}_2(t) - B(t)x_2(t))\{\hat{q}_H(t_0),\hat{p}_H(t_0)\}\Big] \\
&\quad +2\mu(t)p_p(t)\Big[\frac{1}{x_0}(\dot{x}_1(t) - B(t)x_1(t))\hat{q}_H(t_0) + x_0(\dot{x}_2(t) - B(t)x_2(t))\hat{p}_H(t_0)\Big] \\
&\quad +p_p^2(t)\Big|\phi_\alpha(q,t_0)\Big\rangle \\
&= \Bigg\{\sqrt{\frac{2\hbar}{m\omega_0}}\mu(t)\Big[\frac{\alpha_1}{x_0}(\dot{x}_1(t) - B(t)x_1(t)) + \alpha_2(m\omega_0 x_0)(\dot{x}_2(t) - B(t)x_2(t))\Big] \\
&\quad +\hbar\alpha(t)\Bigg\}^2 + \frac{\hbar}{2m\omega_0}\left[(m\omega_0 R_B(t))^2 + \frac{\mu^2(t)}{R_B^2(t)}\left(\frac{\dot{R}_B(t)}{R_B(t)} + B(t)\right)^2\right]. \qquad (4.64)
\end{aligned}
$$

Thus, the fluctuations for $\hat{q}$ and $\hat{p}$ become

$$
\begin{aligned}
(\Delta\hat{q})_\alpha(t) &= \sqrt{\langle \hat{q}^2\rangle_\alpha(t) - \langle \hat{q}\rangle_\alpha^2(t)} \\
&= \sqrt{\frac{\hbar}{2m\omega_0}}\frac{1}{R_B(t)}, \qquad (4.65) \\
(\Delta\hat{p})_\alpha(t) &= \sqrt{\langle \hat{p}^2\rangle_\alpha(t) - \langle \hat{p}\rangle_\alpha^2(t)} \\
&= \sqrt{\frac{m\omega_0\hbar}{2}}R_B(t)\sqrt{1 + \frac{\mu^2(t)}{(m\omega_0 R_B^2(t))^2}\left(\frac{\dot{R}_B(t)}{R_B(t)} + B(t)\right)^2}. \qquad (4.66)
\end{aligned}
$$

We note that the expectation values depend on all parameters of the Hamiltonian, however the fluctuations depend only on $\mu(t), \omega^2(t)$ and parameter $B(t)$. In other words, uncertainties does not depend on the external linear terms, which contribute only to displacement of the wave packet. Finally, the uncertainty relation for the generalized harmonic oscillator with time dependent parameters is

$$(\Delta\hat{q})_\alpha(\Delta\hat{p})_\alpha = \frac{\hbar}{2}\sqrt{1 + \frac{\mu^2(t)}{(m\omega_0 R_B^2(t))^2}\left(\frac{\dot{R}_B(t)}{R_B(t)} + B(t)\right)^2}, \tag{4.67}$$

where clearly $(\Delta\hat{q})_\alpha(\Delta\hat{p})_\alpha \geq \hbar/2$. This relation coincides with the uncertainty (4.54) for the Gaussian ground state k=0.

As a result we can say that, our formulas confirm the well known properties, such as coherent states of the generalized parametric oscillator are displaced Gaussian wave packets, they are eigenstates of the annihilation operator of a dynamical invariant, and follow the classical trajectory. However, they are spreading or squeezing in time, since $(\Delta\hat{q})_\alpha$ depends on time, and are no longer minimum uncertainty states.

# CHAPTER 5

# HERMITE TYPE GENERALIZED QUANTUM OSCILLATOR

## 5.1. Exactly Solvable Models

Since the solution of the generalized time-dependent quadratic oscillator is completely determined by the corresponding classical equation of motion, it is interesting to consider cases for which this equation has exact closed form solutions. Here, we introduce generalized oscillator models related with the classical orthogonal polynomials, which are eigenfunctions of certain singular Sturm-Liouville problems, and are also solutions of the classical oscillator

$$\ddot{x} + \frac{\dot{\mu}(t)}{\mu(t)}\dot{x} + \Omega^2(t)x = 0. \tag{5.1}$$

Precisely, we shall consider problems in which the damping $\Gamma(t) = \dot{\mu}(t)/\mu(t)$ and the modified frequency

$$\Omega^2(t) = \omega^2(t) - (\dot{B} + B^2 + \frac{\dot{\mu}}{\mu}B)$$

are coefficients of the classical Hermite, Laquerre and Legendre differential equations. Clearly, this requires a special relation between the original frequency $\omega^2(t)$ and the parameter $B(t)$. To see this relation, we denote by

$$\Lambda^2(t) = -(\dot{B} + B^2 + \frac{\dot{\mu}}{\mu}B) \tag{5.2}$$

the modification of the original frequency $\omega^2(t)$. Then, substitution $B(t) = \dot{y}/y$ in (5.2), gives differential equation for classical oscillator with frequency $\Lambda^2(t)$

$$\ddot{y} + \frac{\dot{\mu}}{\mu}\dot{y} + \Lambda^2(t)y = 0. \tag{5.3}$$

This suggests that, it is possible to obtain exact solutions of the classical oscillator (5.1), when for given parameters $\mu(t)$, $\omega^2(t)$ and $B(t)$, equations (5.1) and (5.3) are related with the same Sturm-Liouville problem, that is the frequencies $\Omega^2(t)$ and $\Lambda^2(t)$ are compatible. According to this, in next sections we introduce generalized Hermite, Laguerre and Jacobi type oscillators.

## 5.2. Quantization of Hermite Type Generalized Oscillator

We define the Hermite type generalized quantum oscillator by the Hamiltonian

$$\hat{H}_g(t) = \frac{e^{t^2}}{2}\hat{p}^2 + ne^{-t^2}\hat{q}^2 + \left(\frac{\dot{H}_r(t)}{H_r(t)}\right)\frac{(\hat{q}\hat{p}+\hat{p}\hat{q})}{2} + D(t)\hat{q} + E(t)\hat{p} + F(t), \tag{5.4}$$

with variable mass $\mu(t) = e^{-t^2}$, constant frequency $\omega^2(t) = 2n$, $n = 0, 1, 2, \ldots$, mixed term parameter $B(t) = \dot{H}_r/H_r$ , where

$$H_r(t) = r!\sum_{k=0}^{\lfloor r/2 \rfloor}\frac{(-1)^k}{k!(r-2k)!}(2t)^{r-2k},\ \ r = 0, 1, 2, \ldots$$

are the standard Hermite polynomials, and external parameters $D(t),\ E(t),\ F(t)$. Then, the classical equation of motion is a forced Hermite differential equation

$$\ddot{x} - 2t\dot{x} + 2(n+r)x = -\frac{1}{\mu}D + \dot{E} + \left(\frac{\dot{\mu}}{\mu} + \frac{\dot{H}_r}{H_r}\right)E, \qquad -\infty < t < \infty. \tag{5.5}$$

with time-variable damping $\Gamma(t) = \dot{\mu}/\mu = -2t$, and modified frequency $\Omega^2(t) = \omega^2(t) + \Lambda^2(t) = 2(n+r)$, where $r = 0$ corresponds to the case $B(t) = 0$. Note that coefficients of the homogeneous equation are continuous, despite that $B(t)$ has singularities at the zeros of $H_r(t)$. Then, essential properties of the particular solution will depend on the total forcing term in Eq.(5.5). By special choice of $E(t)$ it is possible to remove the singularities in the total force, so that it also becomes continuous. Then, solution of the quantum oscillator with Hamiltonian (5.4) can be written in terms of two independent homogeneous solutions

$x_1(t)$ and $x_2(t)$ of (5.5), satisfying the initial conditions

$$x_1(t_0) = x_0 \neq 0, \quad \dot{x}_1(t_0) = x_0 \dot{H}_r(t_0)/H_r(t_0), \quad H_r(t_0) \neq 0,$$

$$x_2(t_0) = 0, \quad \dot{x}_2(t_0) = 1/\mu(t_0)x_0$$

respectively, and a particular solution $x_p(t)$ of (5.5) satisfying: $x_p(t_0) = 0$, $\dot{x}_p(t_0) = E(t_0)$. When these solutions are smooth, probability densities of the wave functions and the coherent states (4.59) will be also smooth. However, singularities of $B(t)$ will be reflected in momentum expectation values (4.62) and fluctuations (4.66), and in the uncertainty relation (4.67), as we will show in the examples. Before this, we recall that, solution of the homogeneous Eq.(5.5) with given initial conditions, in general will be a linear combination of Hermite polynomial and a confluent hypergeometric function of first kind ${}_1F_1(a, b; t)$, which is represented by the series

$${}_1F_1(a, b; t) = \sum_{n=0}^{\infty} \frac{(a)_n}{(b)_n} \frac{t^n}{n!}, \quad b \neq 0, -1, -2, \ldots$$

where $(a)_n$ and $(b)_n$ are Pochhammer symbols that are given by the relation $(a)_n = \Gamma(a + n)/\Gamma(a)$. However, there are some cases which can be easily treated:

**(i)** when $n$ is an odd positive integer and $r$ is an even positive integer, that is $n = 2k + 1$ and $r = 2s$, $k, s = 0, 1, 2, 3, \ldots$, $t_0 = 0$ and $x_0 = 1/\dot{H}_{2(k+s)+1}(0)$, then $x_1(t) = x_0({}_1F_1(-(2(k+s)+1)/2, 1/2; t^2))$ and second solution is the Hermite polynomial $x_2(t) = H_{2(k+s)+1}(t)$.

**(ii)** when $n$ and $r$ are both positive even integers, that is $n = 2k$ and $r = 2s$, $k, s = 0, 1, 2, 3, \ldots$, $t_0 = 0$ and $x_0 = H_{2(k+s)}(0)$, first solution is the Hermite polynomial, $x_1(t) = H_{2(k+s)}(t)$, and second linearly independent solution is $x_2(t) = t/x_0({}_1F_1(-(k+s-1/2), 3/2; t^2))$.

On the other hand, the particular solution of Eq.(5.5) will depend on the choice of the external parameters. We write some special cases which could be of interest:

**a)** When $B(t) = \dot{H}_r/H_r$, $D(t) = [(d/dt)(e^{-t^2} H_r E(t))]/H_r(t)$, and $E(t_0) = 0$, then the total force in (5.5) is zero, so that $x_p(t) = 0$, and $p_p(t) = -\mu(t)E(t)$.

**b)** When $B(t) = 0$ and $D(t) = -\mu(t)\omega^2(t) \int_{t_0}^{t} E(t')dt'$, then $x_p(t) = \int_{t_0}^{t} E(t')dt'$ and $p_p(t) = 0$.

**c)** When $B(t) = \dot{H}_r/H_r$, $D(t) \neq 0$, $E(t) = 0$, then

$$x_p(t) = -x_1(t)\int^t \frac{1}{\mu(s)x_1^2(s)}\int_{t_0}^s D(\xi)x_1(\xi)d\xi ds,\ \ x_p(t_0) = 0.$$

## 5.3. Exact Solutions

In this section, we give concrete examples of generalized parametric oscillator of Hermite type with and without linear external terms.

**Example 5.1** ***a)*** *Let* $n = 2,\ r = 2$, *and* $B(t) = \dot{H}_2(t)/H_2(t)$, *but* $D(t) = E(t) = F(t) = 0$, *so that there is no linear force. Then, the Hamiltonian is*

$$\hat{H}_g(t) = \frac{e^{t^2}}{2}\hat{p}^2 + 2e^{-t^2}\hat{q}^2 + \left(\frac{\dot{H}_2(t)}{H_2(t)}\right)\frac{(\hat{q}\hat{p}+\hat{p}\hat{q})}{2},$$

*and the corresponding classical equation of motion becomes* $\ddot{x} - 2t\dot{x} + 8x = 0$. *For* $t_0 = 0$, *two linearly independent homogeneous solutions, satisfying the initial conditions* $x_1(0) = 12,\ \dot{x}_1(0) = 0$, *and* $x_2(0) = 0,\ \dot{x}_2(0) = 1/12$, *are*

$$x_1(t) = H_4(t),$$

$$x_2(t) = \frac{t}{12}{}_1F_1\left(\frac{-3}{2},\frac{3}{2};t^2\right) = \frac{1}{192}\left[e^{t^2}\left(\frac{-H_3(t)}{2} + 2H_1(t)\right) + \frac{\sqrt{\pi}}{4}H_4(t)\mathrm{erfi}(t)\right],$$

*where erfi(t) is the imaginary error function defined by* $\mathrm{erfi}(t) = (2/\sqrt{\pi})\int_0^t e^{s^2}ds$, *and*

$$R_B(t) = \sqrt{\frac{144}{H_4^2(t) + \left(12m\omega_0 t\left({}_1F_1\left(\frac{-3}{2},\frac{3}{2};t^2\right)\right)\right)^2}}\ , \tag{5.6}$$

*which is smooth and oscillatory in a finite time-interval near* $t = 0$, *and* $R_B(t) \to 0$ *as* $t \to \infty$. *Then, the probability density* $\rho_k^{n,r}(q,t) = |\Psi_k^{n,r}(q,t)|^2$ *for* $n = 2,\ r = 2$ *becomes*

$$\rho_k^{2,2}(q,t) = N_k^2 R_B(t)\exp\left(-\frac{m\omega_0}{\hbar}R_B^2(t)q^2\right)H_k^2\left(\sqrt{\frac{m\omega_0}{\hbar}}R_B(t)q\right),$$

*and in Fig.* 5.1(*i*) *we plot it for* $k = 2$, *where one can see that it is smooth, and since* $k = 2$, *it has two moving zeros. Also, the essentially nontrivial localization of the particle takes place for* $|t| \leq 2$, *and for* $|t| \geq 2$ *the probability density spreads along q-coordinate. The probability density in coherent state* $\rho_\alpha^{n,r}(q,t) = |\Phi_\alpha^{n,r}(q,t)|^2$ *for* $n = 2$, $r = 2$ *is*

$$\begin{aligned}
\rho_\alpha^{2,2}(q,t) &= \sqrt{\frac{m\omega_0}{\pi\hbar}} R_B(t) \times \exp\left\{2\left[\left(12m\omega_0\left({}_1F_1\left(\frac{-3}{2},\frac{3}{2};t^2\right)\right)R_B(t)\right)^2(\alpha_1^2-\alpha_2^2)\right.\right. \\
&\quad \left.\left. -2m\omega_0 H_4(t)\left({}_1F_1\left(\frac{-3}{2},\frac{3}{2};t^2\right)\right)R_B^2(t)\alpha_1\alpha_2 - \alpha_1^2\right)\right\} \\
&\quad \times \exp\left[\sqrt{\frac{2m\omega_0}{\hbar}} R_B^2(t)\left(\alpha_1\frac{H_4(t)}{6} + \alpha_2(24m\omega_0)\left({}_1F_1\left(\frac{-3}{2},\frac{3}{2};t^2\right)\right)\right)q\right] \\
&\quad \times \exp\left(-\left(\frac{m\omega_0}{\hbar}\right)R_B^2(t)q^2\right).
\end{aligned}$$

*In Fig.* 5.1(*ii*) *we plot it for* $\alpha = 1/\sqrt{2} + i(1/\sqrt{2})$. *We observe that it is a Gaussian type wave packet following the classical trajectory described by the expectation values*

$$\langle \hat{q} \rangle_\alpha(t) = \sqrt{\frac{2\hbar}{m\omega_0}}\left[\alpha_1\frac{H_4(t)}{12} + \alpha_2(m\omega_0 t)\left({}_1F_1\left(\frac{-3}{2},\frac{3}{2};t^2\right)\right)\right], \tag{5.7}$$

$$\begin{aligned}
\langle \hat{p} \rangle_\alpha(t) &= \sqrt{\frac{2\hbar}{m\omega_0}} e^{-t^2}\left\{\frac{\alpha_1}{12}\left(\dot{H}_4 - \frac{\dot{H}_2}{H_2}H_4(t)\right)\right. \\
&\quad \left. +\alpha_2(12m\omega_0)\left[\left(1 - t\frac{\dot{H}_2(t)}{H_2(t)}\right)\left({}_1F_1\left(\frac{-3}{2},\frac{3}{2};t^2\right)\right) - t\left({}_1F_1\left(\frac{-1}{2},\frac{5}{2};t^2\right)\right)\right]\right\}.
\end{aligned} \tag{5.8}$$

*With* $R_B(t)$ *as found in (5.6), fluctuations for* $\hat{q}$ *and* $\hat{p}$ *and uncertainty relation at coherent states take the form*

$$(\Delta\hat{q})_\alpha(t) = \sqrt{\frac{\hbar}{2m\omega_0}}\frac{1}{R_B(t)}, \tag{5.9}$$

$$(\Delta\hat{p})_\alpha(t) = \sqrt{\frac{m\omega_0\hbar}{2}} R_B(t)\sqrt{1 + \frac{e^{-2t^2}}{(m\omega_0 R_B^2(t))^2}\left(\frac{\dot{R}_B(t)}{R_B(t)} + \frac{\dot{H}_2(t)}{H_2(t)}\right)^2}, \tag{5.10}$$

$$(\Delta\hat{q})_\alpha(\Delta\hat{p})_\alpha(t) = \frac{\hbar}{2}\sqrt{1 + \frac{e^{-2t^2}}{(m\omega_0 R_B^2(t))^2}\left(\frac{\dot{R}_B(t)}{R_B(t)} + \frac{\dot{H}_2(t)}{H_2(t)}\right)^2}. \tag{5.11}$$

*Since coefficients of the classical equation are continuous, the expectations (5.7) and fluctuations (5.9) of the position are smooth. In a finite time interval near the origin* $(\Delta\hat{q})_\alpha(t)$ *oscillates, and for* $|t| \to \infty$ *we have increasing* $(\Delta\hat{q})_\alpha(t)$*, showing spreading in position. On the other hand, the singularities of the coefficient* $B(t)$ *at zeros of* $H_2(t)$*, are reflected in the expectations (5.8) and fluctuations (5.10) of the momentum. Then, as shown in Fig. 5.2, the uncertainty (5.11) is oscillatory in a finite time interval near the origin, but it has singularities at the two zeros of the Hermite polynomial* $H_2(t)$*. As* $|t| \to \infty$, $(\Delta\hat{q})_\alpha(\Delta\hat{p})_\alpha(t) \to \infty$*, which shows that the uncertainties do not compensate each other in the limiting case.*

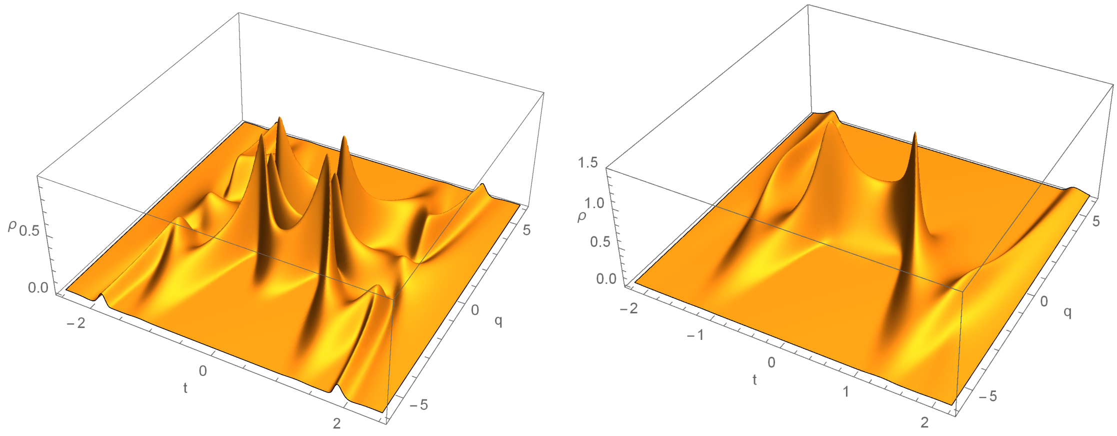



Figure 5.1. Hermite type generalized oscillator, when $D(t) = E(t) = F(t) = 0$.
($i$) Probability density $\rho_2^{2,2}(q,t) = |\Psi_2^{2,2}(q,t)|^2$ , $n = r = k = 2$.
($ii$) Probability density in coherent state $\rho_\alpha^{2,2}(q,t) = |\Phi_\alpha^{2,2}(q,t)|^2$ for $\alpha = 1/\sqrt{2} + i(1/\sqrt{2})$, $n = r = 2$.

***b)*** *Now, we consider the oscillator in part (a) under the influence of linear external terms. That is,* $n = 2, r = 2$*, and we choose* $D(t) = tH_2(t)$, $E(t) = -((1 + 2t^2)H_2(t)e^{t^2})/4$, $F(t) = 0$*. Then, the Hamiltonian becomes*

$$H_g(t) = \frac{e^{t^2}}{2}\hat{p}^2 + 2e^{-t^2}\hat{q}^2 + \left(\frac{\dot{H}_2(t)}{H_2(t)}\right)\frac{(\hat{q}\hat{p} + \hat{p}\hat{q})}{2} + tH_2(t)\hat{q} - \frac{1}{4}(1 + 2t^2)H_2(t)e^{t^2}\hat{p},$$

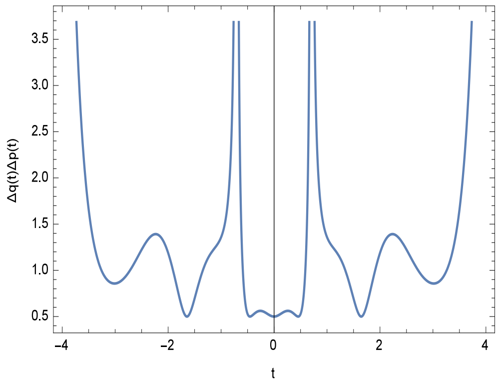



Figure 5.2. Uncertainty relation for generalized Hermite oscillator, $n = r = 2$.

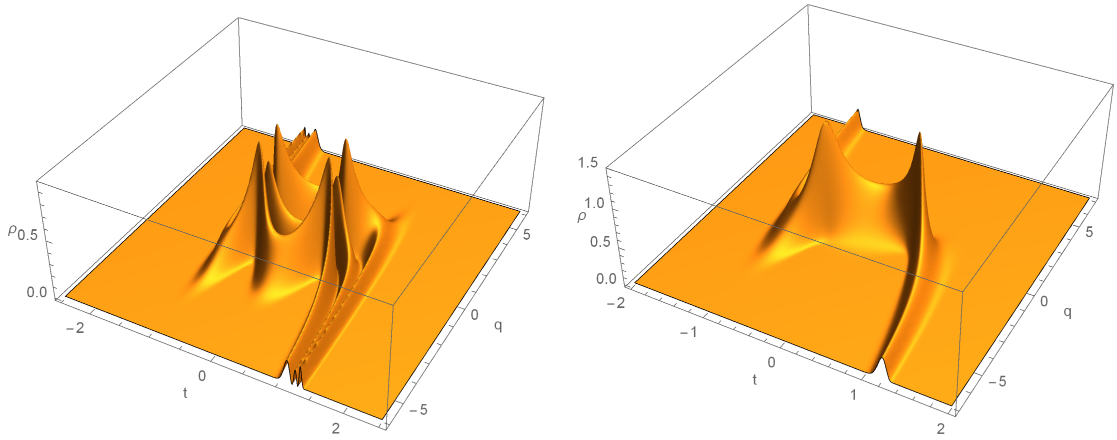
*and the corresponding classical equation is*

$$\ddot{x} - 2t\dot{x} + 8x = -\left(2tH_2(t) + \frac{1}{2}(1 + 2t^2)\dot{H}_2(t)\right)e^{t^2}. \tag{5.12}$$

*We note that, by above choice of $D(t)$ and $E(t)$, $p_p(t)$ is zero and $E(t)$ compensates the singularities coming from $B(t)$, so that the forcing term in Eq.(5.12) is continuous. For $t_0 = 0$, two homogeneous solutions $x_1(t)$ and $x_2(t)$ of (5.12) are as given in part (a), and the particular solution satisfying $x_p(0) = 0,\ \dot{x}_p(0) = 1/2$, is $x_p(t) = -(tH_2(t)e^{t^2})/4$. This gives new probability density*

$$\rho_k^{2,2}(q,t) = N_k^2 R_B(t) \exp\left(-\left(\sqrt{\frac{m\omega_0}{\hbar}} R_B(t)\left(q + \frac{1}{4}tH_2(t)e^{t^2}\right)\right)^2\right) \\ \times H_k^2\left(\sqrt{\frac{m\omega_0}{\hbar}} R_B(t)\left(q + \frac{1}{4}tH_2(t)e^{t^2}\right)\right),$$

*with $R_B(t)$ as found in part (a), but position coordinate displaced by $x_p(t)$. The influence of this displacement can be seen in Fig. 5.3($i$). On the other hand, the new*




*(i)* *(ii)*

Figure 5.3. Hermite type generalized oscillator, when $D(t) = tH_2(t)$, $E(t) = -((1 + 2t^2)H_2(t)e^{t^2})/4$, $F(t) = 0$.
($i$) Probability density $\rho_2^{2,2}(q,t) = |\Psi_2^{2,2}(q,t)|^2$ , $n = r = k = 2$.
($ii$) Probability density in coherent states $\rho_\alpha^{2,2}(q,t) = |\Phi_\alpha^{2,2}(q,t)|^2$ for $\alpha = 1/\sqrt{2} + i(1/\sqrt{2})$, $n = r = 2$.

*probability density in coherent state becomes*

$$
\begin{aligned}
\rho_\alpha^{2,2}(q,t) \quad = \quad & \sqrt{\frac{m\omega_0}{\pi\hbar}} R_B(t) \times \exp\left\{2\left[\left(12m\omega_0\left({}_1F_1\left(\frac{-3}{2},\frac{3}{2};t^2\right)\right)R_B(t)\right)^2(\alpha_1^2-\alpha_2^2)\right.\right. \\
& \left.\left. -2m\omega_0 H_4(t)\left({}_1F_1\left(\frac{-3}{2},\frac{3}{2};t^2\right)\right)R_B^2(t)\alpha_1\alpha_2-\alpha_1^2\right]\right\} \\
& \times\exp\left[\sqrt{\frac{2m\omega_0}{\hbar}}R_B^2(t)\Big(\alpha_1\frac{H_4(t)}{6}\right. \\
& \left. +\alpha_2(24m\omega_0)\Big({}_1F_1\Big(\frac{-3}{2},\frac{3}{2};t^2\Big)\Big)\Big)\Big(q+\frac{1}{4}tH_2(t)e^{t^2}\Big)\right] \\
& \times\exp\left(-\left(\frac{m\omega_0}{\hbar}\right)R_B^2(t)\left(q+\frac{1}{4}tH_2(t)e^{t^2}\right)^2\right),
\end{aligned}
$$

*and its evolution is shown in Fig.* 5.3(*ii*). *Clearly, expectation of position (5.7) will be also displaced by* $x_p(t)$, *and it takes the form*

$$
\langle\hat{q}\rangle_\alpha(t) \quad = \quad \sqrt{\frac{2\hbar}{m\omega_0}}\left[\alpha_1\frac{H_4(t)}{12}+\alpha_2(m\omega_0 t)\left({}_1F_1\left(\frac{-3}{2},\frac{3}{2};t^2\right)\right)\right]-\frac{1}{4}tH_2(t)e^{t^2},
$$

*but the expectation of momentum* $\langle \hat{p} \rangle_\alpha(t)$ *given by Eq.(5.8) does not change, since in this example* $p_p(t) = 0$. *From the general results we know that, the fluctuations and uncertainty relation obtained in part (a) do not change under the influence of the linear external terms.*

# CHAPTER 6

# ASSOCIATED LAGUERRE TYPE GENERALIZED QUANTUM OSCILLATOR

We define a generalized associated Laguerre type oscillator by the Hamiltonian

$$\hat{H}_g(t) = \frac{e^t}{2t^{m+1}}\hat{p}^2 + \frac{nt^m}{2e^t}\hat{q}^2 + \left(\frac{\dot{L}_r^m(t)}{L_r^m(t)}\right)\frac{(\hat{q}\hat{p}+\hat{p}\hat{q})}{2} + D(t)\hat{q} + E(t)\hat{p} + F(t), \tag{6.1}$$

with variable mass $\mu(t) = t^{m+1}e^{-t}$, $m > -1$, variable frequency $\omega^2(t) = n/t$, $n = 0, 1, 2, ...,$ and $B(t) = \dot{L}_r^m(t)/L_r^m(t)$, $r = 0, 1, 2, ...,$ where $L_r^m(t) = e^t t^{-m}(r!)^{-1} d^r\left(e^{-t}t^{r+m}\right)/dt^r$ are the associated Laguerre polynomials. The corresponding classical oscillator is a forced associated Laguerre differential equation

$$\ddot{x} + \frac{(m+1-t)}{t}\dot{x} + \frac{(n+r)}{t}x = -\frac{e^t}{t^{m+1}}D + \dot{E} + \left(\frac{m+1-t}{t} + \frac{\dot{L}_r^m}{L_r^m}\right)E, \qquad 0 < t < \infty, \tag{6.2}$$

with damping $\Gamma(t) = (m+1-t)/t$, and modified frequency $\Omega^2(t) = (n+r)/t$. Here, we shall examine and give example for the case when $m = 0$.

## 6.1. Quantization of Laguerre Type Generalized Oscillator

For m=0, the Hamiltonian for a Laguerre type generalized oscillator is

$$\hat{H}_g(t) = \frac{e^t}{2t}\hat{p}^2 + \frac{n}{2e^t}\hat{q}^2 + \left(\frac{\dot{L}_r(t)}{L_r(t)}\right)\frac{(\hat{q}\hat{p}+\hat{p}\hat{q})}{2} + D(t)\hat{q} + E(t)\hat{p} + F(t), \tag{6.3}$$

where $\mu(t) = te^{-t}$, $\omega^2(t) = n/t$, $n = 0, 1, 2, ...,$ $t \in (0, \infty)$, $B(t) = \dot{L}_r(t)/L_r(t)$, and

$$L_r(t) = \sum_{k=0}^{r}\binom{r}{k}\frac{(-1)^k}{k!}t^k, \;\; r = 0, 1, 2, \ldots$$

are the standard Laguerre polynomials. Then, the corresponding classical oscillator is a forced Laguerre differential equation

$$\ddot{x} + \frac{(1-t)}{t}\dot{x} + \frac{(n+r)}{t}x = -\frac{e^t}{t}D + \dot{E} + \left(\frac{1-t}{t} + \frac{\dot{L}_r}{L_r}\right)E, \qquad 0 < t < \infty, \tag{6.4}$$

with $\Gamma(t) = (1-t)/t$, and $\Omega^2(t) = (n+r)/t$. Since coefficients of the homogeneous equation are continuous for $t > 0$, assuming the total force is also continuous for $t > 0$, solution of the quantum oscillator with Hamiltonian (6.3) can be written in terms of two independent homogeneous solutions $x_1(t)$ and $x_2(t)$ of (6.4), satisfying the initial conditions

$$x_1(t_0) = x_0 \neq 0, \quad \dot{x}_1(t_0) = x_0\frac{\dot{L}_r(t_0)}{L_r(t_0)}, \quad L_r(t_0) \neq 0,$$

$$x_2(t_0) = 0, \quad \dot{x}_2(t_0) = 1/\mu(t_0)x_0$$

respectively, and a particular solution $x_p(t)$ of (6.4) satisfying: $x_p(t_0) = 0$, $\dot{x}_p(t_0) = E(t_0)$.

## 6.2. Concrete Examples and Discussions

In this section, we give some examples of Laguerre type generalized oscillator with exact solutions.

**Example 6.1** ***a)*** *Let* $n = r = 1$, $B(t) = \dot{L}_1(t)/L_1(t)$ *and* $D(t) = E(t) = F(t) = 0$. *Then, the Hamiltonian becomes*

$$\hat{H}_g(t) = \frac{e^t}{2t}\hat{p}^2 + \frac{1}{2e^t}\hat{q}^2 + \left(\frac{\dot{L}_1(t)}{L_1(t)}\right)\frac{(\hat{q}\hat{p} + \hat{p}\hat{q})}{2},$$

*and the classical equation is* $\ddot{x} + (1-t)/t\dot{x} + 2/tx = 0$. *For* $t_0 = 2$, *two solutions satisfying the conditions* $x_1(2) = 1$, $\dot{x}_1(2) = 1$, *and* $x_2(2) = 0$, $\dot{x}(2) = e^2/2$ *respectively, are*

$$x_1(t) = \frac{1}{e^2}[e^t(t-3) - 2L_2(t)(e^2 - Ei(2) + Ei(t))], \tag{6.5}$$

*and*

$$x_2(t) = \frac{1}{2}[e^t(t-3) - L_2(t)(e^2 - 2Ei(2) + 2Ei(t))], \tag{6.6}$$

*where $Ei(t)$ is the exponential integral defined by $Ei(t) = -\int_{-t}^{\infty}(e^{-s}/s)ds$. With above $x_1(t)$ and $x_2(t)$, we have*

$$R_B(t) = \sqrt{\frac{1}{x_1^2(t) + (m\omega_0 x_2(t))^2}}, \tag{6.7}$$

*which is smooth for $t > 0$ and $R_B(t) \to 0$ as $t \to \infty$. Then, the corresponding probability density for $n = 1,\ r = 1$ is*

$$\rho_k^{1,1}(q,t) = N_k^2 R_B(t) \exp\left(-\frac{m\omega_0}{\hbar} R_B^2(t) q^2\right) H_k^2\left(\sqrt{\frac{m\omega_0}{\hbar}} R_B(t) q\right),$$

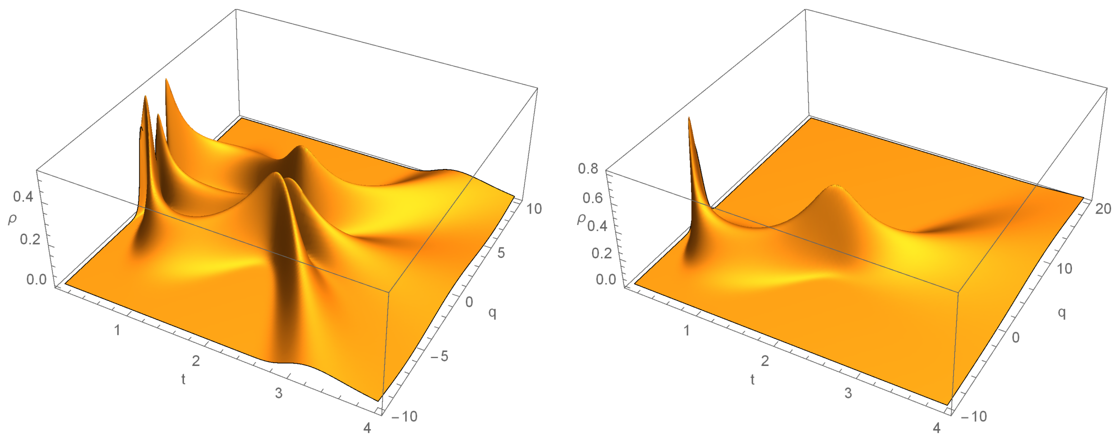
*and in Fig. 6.1$(i)$ we plot it for $k = 2$. The probability density is a smooth function, which has two moving zeros, since $k = 2$. It shows oscillatory behavior in finite time interval near $t = 0$, and then spreads along the $q$-coordinate. The probability density in coherent state, for $n = 1,\ r = 1$ is*

$$\begin{aligned}\rho_\alpha^{1,1}(q,t) &= \left(\frac{m\omega_0}{\pi\hbar}\right)^{1/2} R_B(t) \times \exp\left\{2\left[\left((m\omega_0)x_2(t)R_B(t)\right)^2(\alpha_1^2 - \alpha_2^2)\right.\right. \\ &\left.\left. -2(m\omega_0)x_1(t)x_2(t)R_B^2(t)\alpha_1\alpha_2 - \alpha_1^2\right]\right\} \\ &\times \exp\left(2\sqrt{\frac{2m\omega_0}{\hbar}} R_B^2(t)\left(\alpha_1 x_1(t) + \alpha_2(m\omega_0)x_2(t)\right) q\right) \\ &\times \exp\left(-\left(\frac{m\omega_0}{\hbar}\right) R_B^2(t) q^2\right),\end{aligned}$$

*where $x_1(t),\ x_2(t)$ are defined by (6.5), (6.6), and in Fig. 6.1$(ii)$ one can explicitly see that it is a Gaussian type wave packet. Then, the expectation values at coherent states are*

$$\langle\hat{q}\rangle_\alpha(t) = \sqrt{\frac{2\hbar}{m\omega_0}}\left(\alpha_1 x_1(t) + \alpha_2(m\omega_0)x_2(t)\right), \tag{6.8}$$

$$\begin{aligned}\langle\hat{p}\rangle_\alpha(t) &= \sqrt{\frac{2\hbar}{m\omega_0}} t e^{-t}\left[\alpha_1\left(\dot{x}_1(t) - \frac{\dot{L}_1(t)}{L_1(t)} x_1(t)\right)\right. \\ &\left. +\alpha_2(m\omega_0)\left(\dot{x}_2(t) - \frac{\dot{L}_1(t)}{L_1(t)} x_2(t)\right)\right],\end{aligned} \tag{6.9}$$




(*i*) (*ii*)

Figure 6.1. Laguerre type generalized oscillator, when $D(t) = E(t) = F(t) = 0$.
(*i*) Probability density $\rho_2^{1,1}(q,t) = |\Psi_2^{1,1}(q,t)|^2$, $n = r = 1$, and $k = 2$.
(*ii*) Probability density $\rho_\alpha^{1,1}(q,t) = |\Psi_\alpha^{1,1}(q,t)|^2$ in coherent states for $\alpha = 1/\sqrt{2} + i(1/\sqrt{2})$, $n = r = 1$.

*which shows that the wave packet of the coherent state follows the trajectory of the classical particle. With $R_B(t)$ calculated from (6.7), the fluctuations and uncertainty relation become*

$$(\Delta\hat{q})_\alpha(t) = \sqrt{\frac{\hbar}{2m\omega_0}}\frac{1}{R_B(t)}, \tag{6.10}$$

$$(\Delta\hat{p})_\alpha(t) = \sqrt{\frac{m\omega_0\hbar}{2}}R_B(t)\sqrt{1 + \frac{t^2e^{-2t}}{(m\omega_0 R_B^2(t))^2}\left(\frac{\dot{R}_B(t)}{R_B(t)} + \frac{\dot{L}_1(t)}{L_1(t)}\right)^2}, \tag{6.11}$$

$$(\Delta\hat{q})_\alpha(\Delta\hat{p})_\alpha = \frac{\hbar}{2}\sqrt{1 + \frac{t^2e^{-2t}}{(m\omega_0 R_B^2(t))^2}\left(\frac{\dot{R}_B(t)}{R_B(t)} + \frac{\dot{L}_1(t)}{L_1(t)}\right)^2}. \tag{6.12}$$

*Since the solution of the classical oscillator is given in terms of Laguerre polynomials and exponenial functions, $(\Delta\hat{q})_\alpha(t)$ shows oscillatory behavior in a finite time interval near $t = 0$, while for $|t| \to \infty$, $(\Delta\hat{q})_\alpha(t)$ goes to infinity, which confirms spreading in position coordinate. However, the singularity in parameter $B(t)$ at the zero of $L_1(t)$ appears both in the expectation (6.9) and fluctuation (6.11) of the momentum, where it becomes undefined. Consequently, the uncertainty relation also has singularity at finite time, where $L_1(t) = 0$, and for $t \to 0$ and $t \to \infty$, one has $(\Delta\hat{q})_\alpha(\Delta\hat{p})_\alpha \to \infty$, as one can see in Fig. 6.2.*

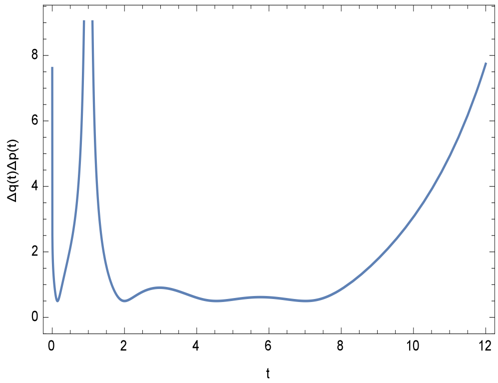



Figure 6.2. Uncertainty relation for generalized Laguerre oscillator, $n = r = 1$.

***b)*** *Now, we consider the system in part (a) under the influence of linear external terms. That is, let* $n = 1, r = 1$, *and* $D(t) = (t-2)L_2(t)$, $E(t) = (1-t)L_2(t)e^t$, $F(t) = 0$. *Then, the corresponding Hamiltonian is*

$$\hat{H}_g(t) = \frac{e^t}{2t}\hat{p}^2 + \frac{1}{2e^t}\hat{q}^2 + \left(\frac{\dot{L}_1(t)}{L_1(t)}\right)\frac{(\hat{q}\hat{p}+\hat{p}\hat{q})}{2} + (t-2)L_2(t)\hat{q} + (1-t)L_2(t)e^t\hat{p},$$

*and the classical equation becomes*

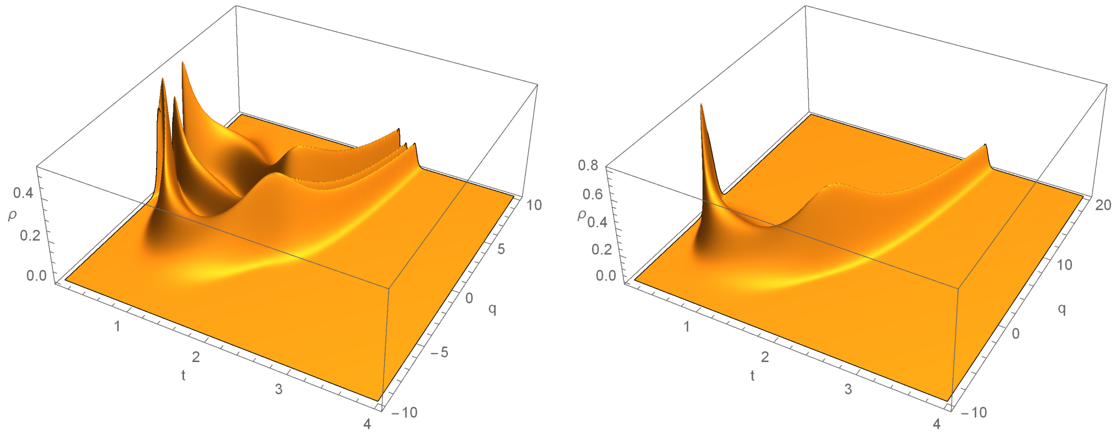
$$\ddot{x} + \frac{1-t}{t}\dot{x} + \frac{2}{t}x = \frac{e^t}{t}(L_1^2(t)-1) + 3e^tL_1(t)\dot{L}_1(t), \quad 0 < t < \infty. \tag{6.13}$$

*For* $t_0 = 2$, *solutions* $x_1(t)$ *and* $x_2(t)$ *of Eq.(6.13) are same as in part (a), and the particular solution satisfying the initial conditions* $x_p(2) = 0$, $\dot{x}_p(2) = e^2$, *is* $x_p(t) = (2-t)L_1(t)e^t$. *Thus, the new probability density is*

$$\rho_k^{1,1}(q,t) = N_k^2 R_B(t)\exp\left(-\frac{m\omega_0}{\hbar}R_B^2(t)\left(q+(t-2)L_1(t)e^t\right)^2\right) \times H_k^2\left(\sqrt{\frac{m\omega_0}{\hbar}}R_B(t)\left(q+(t-2)L_1(t)e^t\right)\right) \tag{6.14}$$

*with* $R_B(t)$ *as found in part (a), and position coordinate displaced by* $x_p(t)$, *see Fig.*

6.1($i$).




($i$) ($ii$)

Figure 6.3. Laguerre type generalized oscillator, when $D(t) = (t-2)L_2(t)$, $E(t) = (1-t)L_2(t)e^t$.
($i$) Probability density $\rho_2^{1,1}(q,t) = |\Psi_2^{1,1}(q,t)|^2$, $n = r = 1$, and $k = 2$.
($ii$) Probability density $\rho_\alpha^{1,1}(q,t) = |\Psi_\alpha^{1,1}(q,t)|^2$ in coherent states for $\alpha = 1/\sqrt{2} + i(1/\sqrt{2})$, $n = r = 1$.

*The new probability density in time evolved coherent state is*

$$\begin{aligned}\rho_\alpha^{1,1}(q,t) \;=\; & \left(\frac{m\omega_0}{\pi\hbar}\right)^{1/2} R_B(t)\times\exp\Big\{2\Big[\left((m\omega_0)x_2(t)R_B(t)\right)^2(\alpha_1^2-\alpha_2^2)\\ & -2(m\omega_0)x_1(t)x_2(t)R_B^2(t)\alpha_1\alpha_2-\alpha_1^2\Big]\Big\}\\ & \times\exp\left(2\sqrt{\frac{2m\omega_0}{\hbar}}R_B^2(t)\left(\alpha_1x_1(t)+\alpha_2(m\omega_0)x_2(t)\right)\left(q+(t-2)L_1(t)e^t\right)\right)\\ & \times\exp\left(-\left(\frac{m\omega_0}{\hbar}\right)R_B^2(t)(q+(t-2)L_1(t)e^t)^2\right),\end{aligned}$$

*which was plotted in Fig. 6.1($ii$). Comparing the probability densities, found in part (a) and part (b) of this example, one can explicitly see the change in the evolution of the wave packets under the displacement of the position coordinate by $x_p(t)$.*

# CHAPTER 7

# JACOBI TYPE GENERALIZED QUANTUM OSCILLATOR

We define a generalized Jacobi type oscillator by a Hamiltonian of the form

$$\begin{aligned} \hat{H}_g(t) &= \frac{\hat{p}^2}{2(1-t)^{a+1}(1+t)^{a+1}} + \frac{[n\,(n+a+b+1)]\,(1-t)^a(1+t)^b}{2}\hat{q}^2 + \left(\frac{\dot{P}_r^{a,b}(t)}{P_r^{a,b}(t)}\right)\frac{(\hat{q}\hat{p}+\hat{p}\hat{q})}{2} \\ &+ D(t)\hat{q} + E(t)\hat{p} + F(t), \end{aligned} \tag{7.1}$$

with mass $\mu(t) = (1-t)^{a+1}(1+t)^{b+1}$, $a, b > -1$, frequency $\omega^2(t) = [n\,(n+a+b+1)]/(1-t^2)$, $-1 < t < 1$, and $B(t) = \dot{P}_r^{a,b}(t)/P_r^{a,b}(t)$, where

$$P_n^{a,b}(t) = \frac{(-1)^n}{2^n n!}(1-t)^{-a}(1+t)^{-b}\frac{d^n}{dt^n}\left[(1-t)^{a+n}(1+t)^{b+n}\right],$$

are the Jacobi polynomials. Then, the corresponding classical oscillator is a forced Jacobi differential equation

$$\begin{aligned} &\ddot{x} + \frac{(b-a-(a+b+2)t)}{1-t^2}\dot{x} + \frac{n(n+a+b+1)+r(r+a+b+1)}{1-t^2}x \\ &= -\frac{1}{\mu}D + \dot{E} + \left(\frac{(b-a-(a+b+2)t)}{(1-t^2)} + \frac{\dot{P}_r^{a,b}(t)}{P_r^{a,b}(t)}\right)E, \quad -1 < t < 1, \end{aligned}$$

where $\Gamma(t) = [(b-a-(a+b+2)t)]/(1-t^2)$ is the damping coefficient, and

$$\Omega^2(t) = \frac{n(n+a+b+1)}{1-t^2} + \frac{r(r+a+b+1)}{1-t^2}$$

is the modified frequency. Thus, to preserve the structure after the modification, for given $n, r = 0, 1, 2, ....$ and $a, b > -1$, we need to find nonnegative integer $m$, for which the equation $n(n+a+b+1) + r(r+a+b+1) = m(m+a+b+1)$ holds. We shall treat explicitly two special cases: for $a = b = 0$ the Legendre generalized oscillators and for

$a = b = -1/2$ the first-kind Chebyshev (FKC) oscillator.

## 7.1. Legendre Type Generalized Quantum Oscillator with Examples

The Hamiltonian for a Legendre type generalized oscillator is

$$\hat{H}_g(t) = \frac{1}{2(1-t^2)}\hat{p}^2 + \frac{n(n+1)}{2}\hat{q}^2 + \left(\frac{\dot{P}_r(t)}{P_r(t)}\right)\frac{(\hat{q}\hat{p}+\hat{p}\hat{q})}{2} + D(t)\hat{q} + E(t)\hat{p} + F(t) \qquad (7.2)$$

where $\mu(t) = (1-t^2)$, $\omega^2(t) = n(n+1)/\left(1-t^2\right)$, $n = 0, 1, 2, ...$, $t \in (-1, 1)$, $B(t) = \dot{P}_r(t)/P_r(t)$, and

$$P_r(t) = \frac{1}{2^r}\sum_{k=0}^{\lfloor r/2 \rfloor}(-1)^k \frac{(2r-2k)!}{k!(r-k)!(r-2k)!}t^{r-2k}, \; r = 0, 1, 2, ...$$

are the Legendre polynomials. Then, the classical equation is a forced Legendre differential equation

$$\ddot{x} - \frac{2t}{1-t^2}\dot{x} + \frac{n(n+1)+r(r+1)}{1-t^2}x = -\frac{1}{\mu}D + \dot{E} + \left(-\frac{2t}{1-t^2} + \frac{\dot{P}_r(t)}{P_r(t)}\right)E, \qquad (7.3)$$

where $-1 < t < 1$, with $\Gamma(t) = -2t/(1-t^2)$ and $\Omega^2(t) = [n(n+1) + r(r+1)]/(1-t^2)$. Here, if for given $n$ and $r\,(r \neq 1)$, $m$ is a positive integer satisfying the equation $n(n+1) + r(r+1) = m(m+1)$, then the homogeneous part of Eq.(7.3) has solution in the form $x(t) = c_1P_m(t) + c_2Q_m(t), \quad t \in (-1, 1)$, where $P_m(t)$ are Legendre polynomials, and $Q_m(t)$ are the Legendre functions of the second kind given by the formula

$$Q_m(t) = \frac{1}{2}P_m(t)\ln\frac{1+t}{1-t} - \sum_{k=1}^{m}\frac{1}{k}P_{k-1}(t)P_{m-k}(t).$$

**Example 7.1** *Let* $n = 2$, $r = 2$, *and* $B(t) = \dot{P}_2(t)/P_2(t)$, $D(t) = tP_2(t)$, $E(t) = -P_2(t)/6$,

*$F(t) = 0$. Then the Hamiltonian becomes*

$$\hat{H}_g(t) = \frac{1}{2(1-t^2)}\hat{p}^2 + 3\hat{q}^2 + \left(\frac{\dot{P}_2(t)}{P_2(t)}\right)\frac{(\hat{q}\hat{p} + \hat{p}\hat{q})}{2} + tP_2(t)\hat{q} - \frac{P_2(t)}{6}\hat{p},$$

*and the corresponding classical equation is*

$$\ddot{x} + \frac{-2t}{1-t^2}\dot{x} + \frac{12}{1-t^2}x = \frac{-1}{3}\left(\frac{2t}{1-t^2}P_2(t) + \dot{P}_2(t)\right). \tag{7.4}$$

*For $t_0 = 0$, two homogeneous solutions $x_1(t)$ and $x_2(t)$ of Eq.(7.4), satisfying the initial conditions $x_1(0) = -2/3$, $\dot{x}_1(0) = 0$, and $x_2(0) = 0$, $\dot{x}_2(0) = -3/2$, are*

$$x_1(t) = -Q_3(t) = \frac{5t^3 - 3t}{4}\ln\left(\frac{1-t}{1+t}\right) + \frac{5t^2}{2} - \frac{2}{3}, \quad x_2(t) = P_3(t) = \frac{1}{2}(5t^3 - 3t),$$

*since for $n = 2$, $r = 2$ we have $m = 3$. Then the particular solution satisfying the initial conditions $x_p(0) = 0$, $\dot{x}_p(0) = 1/12$, is*

$$x_p(t) = -\frac{t}{6}P_2(t) = -\frac{t}{12}(3t^2 - 1),$$

*and we calculate*

$$R_B(t) = \frac{2}{3}\sqrt{\frac{1}{Q_3^2(t) + \left(\frac{4m\omega_0}{9}P_3^2(t)\right)^2}}, \tag{7.5}$$

*which is bounded and has oscillatory behavior for $t \in (-1, 1)$, and $R_B(t) \to 0$, when $t \to \pm 1$. Then, the probability density in state $\Psi_k^{2,2}(q,t)$ is*

$$\rho_k^{2,2}(q,t) = N_k^2 R_B(t)\exp\left(-\frac{m\omega_0}{\hbar}R_B^2(t)\left(q + \frac{tP_2(t)}{6}\right)^2\right)H_k^2\left(\sqrt{\frac{m\omega_0}{\hbar}}R_B(t)\left(q + \frac{tP_2(t)}{6}\right)\right),$$

*which is plotted for $k = 2$ in the Fig. 7.1($i$), and the probability density in coherent state* *is*

$$\begin{aligned}\rho_{\alpha}^{2,2}(q,t) &= \frac{3}{2}\sqrt{\frac{m\omega_0}{\pi\hbar}}R_B(t)\times\exp\left\{2\left[\left(\frac{2m\omega_0}{3}P_3(t)R_B(t)\right)^2(\alpha_1^2-\alpha_2^2)\right.\right.\\ &\left.\left.+2m\omega_0P_3(t)Q_3(t)R_B^2(t)\alpha_1\alpha_2-\alpha_1^2\right]\right\}\\ &\times\exp\left[3\sqrt{\frac{2m\omega_0}{\hbar}}R_B^2(t)\left(\alpha_1Q_3(t)-\alpha_2\left(\frac{4m\omega_0}{9}\right)P_3(t)\right)\left(q+\frac{tP_2(t)}{6}\right)\right]\\ &\times\exp\left[-\frac{m\omega_0}{\hbar}R_B^2(t)\left(q+\frac{tP_2(t)}{6}\right)^2\right],\end{aligned}$$

*which is shown in Fig.* 7.1(*ii*) *for* $\alpha = 1/\sqrt{2}+i(1/\sqrt{2})$. *Then, we compute the expectation values,*

$$\langle\hat{q}\rangle_\alpha(t) = \sqrt{\frac{2\hbar}{m\omega_0}}\left[\alpha_1\frac{3Q_3(t)}{2}-\alpha_2\left(\frac{2m\omega_0}{3}\right)P_3(t)\right]-\frac{tP_2(t)}{6}, \tag{7.6}$$

$$\begin{aligned}\langle\hat{p}\rangle_\alpha(t) &= \sqrt{\frac{2\hbar}{m\omega_0}}(1-t^2)\left[\frac{3}{2}\alpha_1\left(\dot{Q}_3(t)-\frac{\dot{P}_2(t)}{P_2(t)}Q_3(t)\right)\right.\\ &\left.+\alpha_2\left(\frac{2m\omega_0}{3}\right)\left(\dot{P}_3(t)-\frac{\dot{P}_2(t)}{P_2(t)}P_3(t)\right)\right],\end{aligned} \tag{7.7}$$

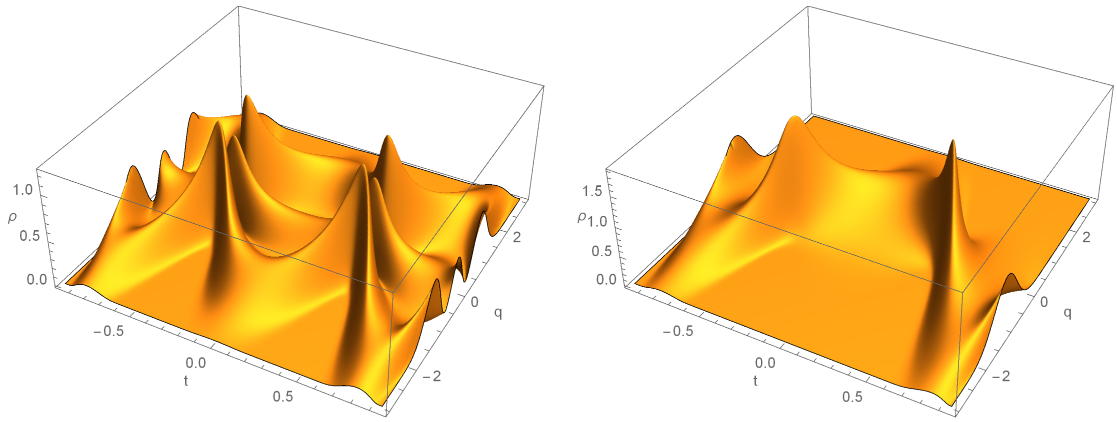
*and with* $R_B(t)$ *given by (7.5) we get the fluctuation for* $\hat{q}$ *and* $\hat{p}$, *and uncertainty relation at coherent states as follows:*

$$(\Delta\hat{q})_\alpha(t) = \sqrt{\frac{\hbar}{2m\omega_0}}\frac{1}{R_B(t)}, \tag{7.8}$$

$$(\Delta\hat{p})_\alpha(t) = \sqrt{\frac{m\omega_0\hbar}{2}}R_B(t)\sqrt{1+\frac{(1-t^2)^2}{(m\omega_0R_B^2(t))^2}\left(\frac{\dot{R}_B(t)}{R_B(t)}+\frac{\dot{P}_2(t)}{P_2(t)}\right)^2}, \tag{7.9}$$

$$(\Delta\hat{q})_\alpha(\Delta\hat{p})_\alpha = \frac{\hbar}{2}\sqrt{1+\frac{(1-t^2)^2}{(m\omega_0R_B^2(t))^2}\left(\frac{\dot{R}_B(t)}{R_B(t)}+\frac{\dot{P}_2(t)}{P_2(t)}\right)^2}. \tag{7.10}$$

*Because the coefficients of the forced oscillator (7.4) are continuous, the expectations (7.6) and fluctuations (7.8) of the position are smooth on the interval* $t \in (-1,1)$. *But the expectations (7.7) and fluctuations (7.9) of the momentum are not defined at zeros of* $P_2(t)$. *The uncertainty relation is bounded on (-1,1), except in the neighborhoods of the zeros of* $P_2(t)$, *where it tends to infinity, see Fig.* 7.2.




(*i*) (*ii*)

Figure 7.1. Legendre type generalized oscillator, when $D(t) = tP_2(t)$, $E(t) = -P_2(t)/6, F(t) = 0$.
(*i*) Probability density $\rho_2^{2,2}(q,t) = |\Psi_2^{2,2}(q,t)|^2$.
(*ii*) Probability density $\rho_\alpha^{2,2}(q,t) = |\Phi_\alpha^{2,2}(q,t)|^2$ in coherent states for $\alpha = 1/\sqrt{2} + i(1/\sqrt{2})$.

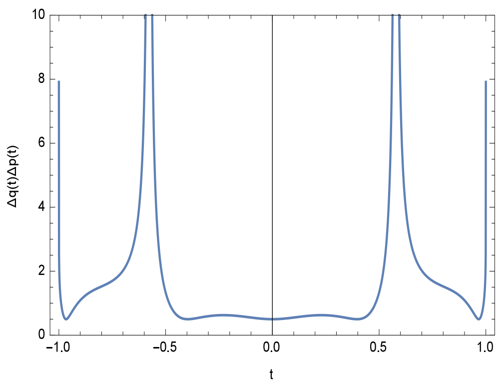



Figure 7.2. Uncertainty relation for generalized Legendre oscillator, $n = r = 2$.

## 7.2. First-kind Chebyshev Type Generalized Quantum Oscillator with Examples

The Hamiltonian for a FKC generalized oscillator is

$$\hat{H}_g(t) = \frac{\hat{p}^2}{2\sqrt{1-t^2}} + \frac{n^2}{2\sqrt{1-t^2}}\hat{q}^2 + \left(\frac{\dot{T}_r(t)}{T_r(t)}\right)\frac{(\hat{q}\hat{p}+\hat{p}\hat{q})}{2} + D(t)\hat{q} + E(t)\hat{p} + F(t) \tag{7.11}$$

where $\mu(t) = \sqrt{1-t^2}$, $\omega^2(t) = n^2/\left(1-t^2\right)$, $n = 0, 1, 2, ...$, $t \in (-1, 1)$, $B(t) = \dot{T}_r(t)/T_r(t)$, and

$$T_r(t) = \frac{r}{2}\sum_{k=0}^{\lfloor r/2 \rfloor}(-1)^k \frac{(r-k-1)!}{k!(r-2k)!}(2t)^{r-2k}, \; r = 0, 1, 2, \ldots$$

are the first-kind Chebyshev polynomials. Then, the classical equation is a forced FKC differential equation

$$\ddot{x} - \frac{t}{1-t^2}\dot{x} + \frac{(n^2+r^2)}{1-t^2}x = -\frac{1}{\mu}D + \dot{E} + \left(-\frac{t}{1-t^2} + \frac{\dot{T}_r(t)}{T_r(t)}\right)E, \quad -1 < t < 1, \tag{7.12}$$

with $\Gamma(t) = -t/(1-t^2)$ and $\Omega^2(t) = (n^2+r^2)/(1-t^2)$. We note that, when $n^2+r^2 = m^2$, where $m$ is also a positive integer, that's when $(n, r, m)$ are Pythagorean triples, the corresponding homogeneous equation has solution of the form

$$x(t) = c_1 T_m(t) + c_2\sqrt{1-t^2}U_{m-1}(t),$$

where

$$U_m(t) = \sum_{k=0}^{\lfloor m/2 \rfloor}(-1)^k \frac{(m-k)!}{k!(m-2k)!}(2t)^{m-2k}, \; m = 0, 1, 2, \ldots$$

are the Chebyshev polynomials of the second kind.

**Example 7.2** *Let* $n = 3, r = 4$, $B(t) = \dot{T}_4(t)/T_4(t)$, $D(t) = tT_4(t), E(t) = T_6(t)(2t^2 - 1)/(64\sqrt{1-t^2})$, $F(t) = 0$. *Then, the Hamiltonian becomes*

$$\hat{H}_g(t) = \frac{\hat{p}^2}{2\sqrt{1-t^2}} + \frac{9}{2\sqrt{1-t^2}}\hat{q}^2 + \left(\frac{\dot{T}_4(t)}{T_4(t)}\right)\frac{(\hat{q}\hat{p}+\hat{p}\hat{q})}{2} + tT_4(t)\hat{q} + \frac{T_4(t)}{9}\frac{(2t^2-1)}{\sqrt{1-t^2}}\hat{p},$$

*and the corresponding classical equation is*

$$\ddot{x} - \frac{t}{1-t^2}\dot{x} + \frac{5}{1-t^2}x = \frac{1}{9\sqrt{1-t^2}}(5tT_4(t) + 2(2t^2-1)\dot{T}_4(t)), \tag{7.13}$$

*where by the above choice of* $E(t)$*, the singularities in* $B(t)$ *are removed, so that the forcing in Eq.(7.13) becomes continuous. For* $t_0 = 0$*, homogeneous solutions* $x_1(t)$ *and* $x_2(t)$ *of the Eq.(7.13), satisfying the initial conditions* $x_1(0) = 1/5,\ \dot{x}_1(0) = 0$*, and* $x_2(0) = 0,\ \dot{x}_2(0) = 5$*, respectively are*

$$x_1(t) = \frac{1}{5}\sqrt{1-t^2}U_4(t) = \frac{1}{5}\sqrt{1-t^2}(16t^4 - 12t^2 + 1), \quad x_2(t) = T_5(t) = 16t^5 - 20t^3 + 5t,$$

*and the particular solution satisfying the initial conditions* $x_p(0) = 0,\ \dot{x}_p(0) = -1/9$ *is*

$$x_p(t) = -\frac{t}{9}\sqrt{1-t^2}T_4(t) = -\frac{t}{9}\sqrt{1-t^2}(8t^4 - 8t^2 + 1).$$

*Then, we calculate*

$$R_B(t) = \sqrt{\frac{1}{(1-t^2)U_4^2(t) + \frac{1}{25}(m\omega_0 T_5(t))^2}}, \tag{7.14}$$

*where* $R_B(t)$ *is bounded and oscillating in* $t \in (-1, 1)$*, but does not approach zero for* $t \to \pm 1$*, as in the case of the Legendre oscillator, since Chebyshev polynomials are defined at* $t = \pm 1$*. Then, the probability density for* $n = 3,\ r = 4$ *is in the form*

$$\rho_k^{3,4}(q,t) = N_k^2 R_B(t) \exp\left(-\left(\sqrt{\frac{m\omega_0}{\hbar}}R_B(t)\left(q + \frac{t\sqrt{1-t^2}}{9}T_4(t)\right)\right)^2\right)$$
$$\times H_k^2\left(\sqrt{\frac{m\omega_0}{\hbar}}R_B(t)\left(q + \frac{t\sqrt{1-t^2}}{9}T_4(t)\right)\right),$$

*and in Fig.* 7.3($i$) *we plot it for* $k = 2$*. The probability density in coherent state is*

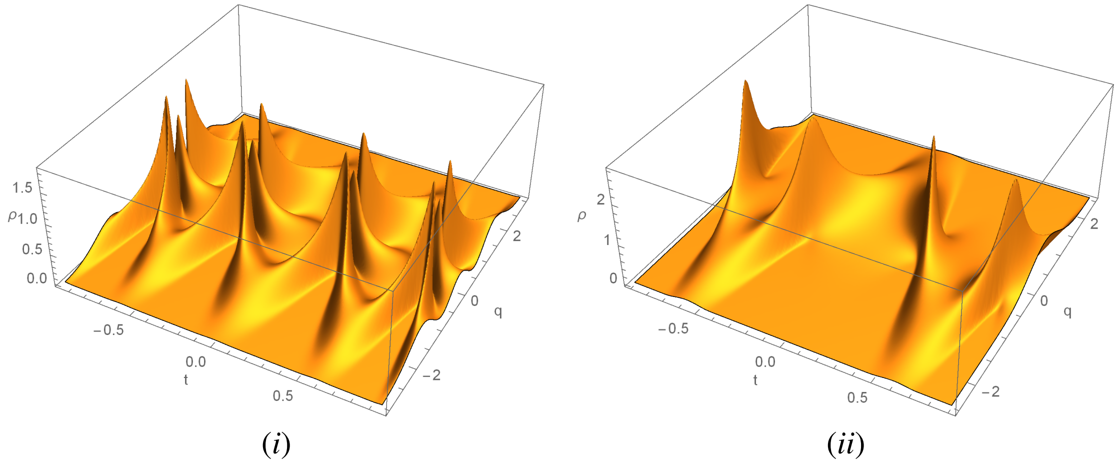



Figure 7.3. FKC type generalized oscillator, when $D(t) = tT_4(t)$, $E(t) = T_6(t)(2t^2 - 1)/(64\sqrt{1-t^2})$, $F(t) = 0$. $(i)$ Probability density $\rho_2^{3,4}(q,t)$. $(ii)$ Probability density $\rho_\alpha^{3,4}(q,t)$ in coherent states for $\alpha = 1/\sqrt{2} + i(1/\sqrt{2})$.

$$
\begin{aligned}
\rho_\alpha^{3,4}(q,t) &= \sqrt{\frac{m\omega_0}{\pi\hbar}} R_B(t) \times \exp\left\{2\left[\left(\left(\frac{m\omega_0}{5}\right)T_5(t)R_B(t)\right)^2(\alpha_1^2-\alpha_2^2)\right.\right.\\
&\left.\left.-\left(\frac{2m\omega_0}{5}\right)\sqrt{1-t^2}T_5(t)U_4(t)R_B^2(t)\alpha_1\alpha_2-\alpha_1^2\right]\right\}\\
&\times\exp\left[2\sqrt{\frac{2m\omega_0}{\hbar}}R_B^2(t)\left(\alpha_1\sqrt{1-t^2}U_4(t)\right.\right.\\
&\left.\left.+\alpha_2\left(\frac{m\omega_0}{5}\right)T_5(t)\right)\left(q+\frac{t\sqrt{1-t^2}}{9}T_4(t)\right)\right]\\
&\times\exp\left[-\left(\frac{m\omega_0}{\hbar}\right)R_B^2(t)\left(q+\frac{t\sqrt{1-t^2}}{9}T_4(t)\right)^2\right],
\end{aligned}
$$

*and we plot it for* $\alpha = 1/\sqrt{2} + i(1/\sqrt{2})$ *in the Fig. 7.3(ii). The expectation values for* $\hat{q}$ *and* $\hat{p}$ *are found as*

$$
\langle\hat{q}\rangle_\alpha(t) = \sqrt{\frac{2\hbar}{m\omega_0}}\left[\alpha_1\sqrt{1-t^2}U_4(t)+\alpha_2\left(\frac{m\omega_0}{5}\right)T_5(t)\right]-\frac{t\sqrt{1-t^2}}{9}T_4(t), \quad (7.15)
$$

$$
\begin{aligned}
\langle\hat{p}\rangle_\alpha(t) &= \sqrt{\frac{2\hbar}{m\omega_0}}\left\{\frac{\alpha_1}{5}\left[(1-t^2)\dot{U}_4(t)-\left(t+(1-t^2)\frac{\dot{T}_4(t)}{T_4(t)}\right)U_4(t)\right]\right. \quad (7.16)\\
&\left.+\alpha_2\left(\frac{m\omega_0}{5}\right)\sqrt{1-t^2}\left(\dot{T}_5(t)-\frac{\dot{T}_4(t)}{T_4(t)}T_5(t)\right)\right\},
\end{aligned}
$$

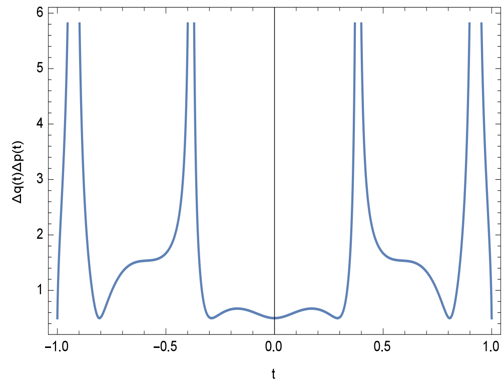



Figure 7.4. Uncertainty relation for generalized FKC oscillator, $n = 3, r = 4$.

*and with $R_B(t)$ given by (7.14) we have*

$$(\Delta\hat{q})_\alpha(t) = \sqrt{\frac{\hbar}{2m\omega_0}\frac{1}{R_B(t)}}, \tag{7.17}$$

$$(\Delta\hat{p})_\alpha(t) = \sqrt{\frac{m\omega_0\hbar}{2}}R_B(t)\sqrt{1+\frac{1-t^2}{(m\omega_0R_B^2(t))^2}\left(\frac{\dot{R}_B(t)}{R_B(t)}+\frac{\dot{T}_4(t)}{T_4(t)}\right)^2}, \tag{7.18}$$

$$(\Delta\hat{q})_\alpha(\Delta\hat{p})_\alpha = \frac{\hbar}{2}\sqrt{1+\frac{1-t^2}{(m\omega_0R_B^2(t))^2}\left(\frac{\dot{R}_B(t)}{R_B(t)}+\frac{\dot{T}_4(t)}{T_4(t)}\right)^2}. \tag{7.19}$$

*We see that the expectations (7.15) and fluctuations (7.17) of the position are smooth, but at the singularities of $B(t)$ the expectations (7.16) and fluctuations (7.18) of momentum are not defined. Since $B(t)$ has singularities at the four zeros of the FKC polynomial $T_4(t)$, the uncertainty relation is also singular at these points. On the other hand, when $|t| \to \pm 1$, uncertainty approaches minimum, that is $(\Delta\hat{q})_\alpha(\Delta\hat{p})_\alpha(t) \to \hbar/2$, see Fig.* 7.4.

# CHAPTER 8

# CONCLUSION

In the present thesis, we solved quantum system with the generalized quantum Hamiltonian and time-variable parameters using Wei-Norman Lie algebraic approach. Since the quantum Hamiltonian of this system could be written in terms of the generators of su(1,1) and Heisenberg-Weyl Lie algebra, the exact form of its evolution operator was found by means of two linearly independent homogeneous solutions and a particular solution of the corresponding forced classical equation of motion. Using the evolution operator, we obtained wave function solutions of time-dependent Schrödinger equation, time evolution of Glauber coherent states, corresponding probability densities, expectation values and uncertainties.

To get better insight to this problem, we also examined exactly solvable models. We studied quantum parametric oscillator related with the classical orthogonal polynomials of Hermite, Laguerre and Jacobi type, under the influence of external forces. We realized that the mixed term parameter $B(t)$ modifies the original frequency, and by a special choice of this parameter we were able to preserve the structure of the original oscillator. However, in Hermite, Laguerre and Jacobi type oscillators, this choice of $B(t)$ develops finite time singularities at the zeros of the related orthogonal polynomials. Since the coefficients of the classical oscillators are continuous, the expectations and fluctuations of the position are smooth but the singularities of $B(t)$ are reflected in the expectations and fluctuations of the momentum and the uncertainty relation. Thus, as time approaches these singularities, uncertainty relation tends to infinity. The probability densities of all models are smooth and oscillatory in a finite time interval near the initial point. For Hermite and Laguerre oscillators, which are defined on infinite time intervals, the spreading coefficient $R_B(t)$ of the wave packets tends to zero with increasing time. Therefore, the amplitude of the wave packets is decreasing and approaching zero when time goes to infinity, and wave packets are spreading along $q$-coordinate. For the Legendre oscillator, defined on finite time interval $(-1, 1)$, we have $R_B(t) \to 0$, as $t \to \pm 1$, so that wave amplitudes approach zero in the neighborhood of $t = \pm 1$, and wave packets are spreading with respect to q . However, for the first-kind Chebyshev model $R_B(t)$ is bounded in $t \in (-1, 1)$, but does not approach zero for $t \to \pm 1$, as in the case of the Legendre oscillator.

Moreover, we have seen that the linear external terms $D(t)$ and $E(t)$ lead to displacement in the position coordinate of the wave packets. So expectation values of position and momentum were shifted by the particular solutions $x_p(t)$ and $p_p(t)$ of the classical equations of motion. We also gave some examples with and without external forces to see the influence of the linear external terms and made comparison. We observed the difference in the evolution of the wave packets under the influence of the external forces. Nevertheless, uncertainty relations do not depend on the linear external terms, they depend only on the mass $\mu(t)$, frequency $\omega^2(t)$ and the mixed parameter $B(t)$ of the oscillator.

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# APPENDIX A

# COMMUTATION RELATIONS OF THE LIE GROUP GENERATORS

The generators $\hat{E}_1, \hat{E}_2$ and $\hat{E}_3$ of Heisenberg-Weyl algebra and the generators $\hat{K}_-, \hat{K}_+$ and $\hat{K}_0$ of the $su(1,1)$ algebra generate a Lie algebra. For any function $f(q)$ in a Hilbert space $H$, we prove the commutation relations which we used in Chapter 3 as follows:

$$[\hat{E}_1, \hat{E}_2]f = \left[iq, \frac{\partial}{\partial q}\right]f = i\left(q\frac{\partial f}{\partial q} - \frac{\partial}{\partial q}(qf)\right) = -if = -\hat{E}_3 f, \tag{A.1}$$

$$\Rightarrow [\hat{E}_1, \hat{E}_2] = -\hat{E}_3.$$

$$[\hat{E}_1, \hat{K}_-]f = \left[iq, -\frac{i}{2}\frac{\partial^2}{\partial q^2}\right]f = \frac{1}{2}\left(q\frac{\partial^2 f}{\partial q^2} - \frac{\partial^2}{\partial q^2}(qf)\right) = -\frac{\partial f}{\partial q} = -\hat{E}_2 f, \tag{A.2}$$

$$\Rightarrow [\hat{E}_1, \hat{K}_-] = -\hat{E}_2.$$

$$[\hat{E}_1, \hat{K}_+]f = \left[iq, \frac{i}{2}q^2\right]f = 0, \tag{A.3}$$

$$\Rightarrow [\hat{E}_1, \hat{K}_+] = 0.$$

$$
\begin{aligned}
[\hat{E}_1, \hat{K}_0]f &= \left[iq, \frac{1}{2}\left(q\frac{\partial}{\partial q} + \frac{1}{2}\right)\right]f \qquad (A.4)\\
&= \frac{i}{2}\left[q, q\frac{\partial}{\partial q}\right]f = \frac{i}{2}\left(q^2\frac{\partial f}{\partial q} - q\frac{\partial}{\partial q}(qf)\right)\\
&= -\frac{i}{2}qf = -\frac{1}{2}\hat{E}_1 f,
\end{aligned}
$$

$$
\Rightarrow [\hat{E}_1, \hat{K}_0] = -\frac{1}{2}\hat{E}_1.
$$

$$
[\hat{E}_2, \hat{K}_-]f = \left[\frac{\partial}{\partial q}, -\frac{i}{2}\frac{\partial^2}{\partial q^2}\right]f = 0, \qquad (A.5)
$$

$$
\Rightarrow [\hat{E}_2, \hat{K}_-] = 0.
$$

$$
\begin{aligned}
[\hat{E}_2, \hat{K}_+]f = \left[\frac{\partial}{\partial q}, \frac{i}{2}q^2\right]f &= \frac{i}{2}\left(\frac{\partial}{\partial q}(q^2 f) - q^2\frac{\partial f}{\partial q}\right) \qquad (A.6)\\
&= iqf = \hat{E}_1 f,
\end{aligned}
$$

$$
\Rightarrow [\hat{E}_2, \hat{K}_+] = \hat{E}_1.
$$

$$
\begin{aligned}
[\hat{E}_2, \hat{K}_0]f &= \left[\frac{\partial}{\partial q}, \frac{1}{2}\left(q\frac{\partial}{\partial q} + \frac{1}{2}\right)\right]f \qquad (A.7)\\
&= \frac{1}{2}\left[\frac{\partial}{\partial q}, q\frac{\partial}{\partial q}\right]f = \frac{1}{2}\left(\frac{\partial}{\partial q}\left(q\frac{\partial f}{\partial q}\right) - q\frac{\partial^2 f}{\partial q^2}\right)\\
&= \frac{1}{2}\frac{\partial f}{\partial q} = \frac{1}{2}\hat{E}_2 f,
\end{aligned}
$$

$$
\Rightarrow [\hat{E}_2, \hat{K}_0] = \frac{1}{2}\hat{E}_2.
$$

$$
\begin{aligned}
[\hat{K}_-, \hat{K}_+]f &= \left[-\frac{i}{2}\frac{\partial^2}{\partial q^2}, \frac{i}{2}q^2\right] f = \frac{1}{4}\left(\frac{\partial^2}{\partial q^2}(q^2 f) - q^2\frac{\partial^2 f}{\partial q^2}\right) \qquad \text{(A.8)}\\
&= \left(\frac{1}{2} + q\frac{\partial}{\partial q}\right) f = 2\hat{K}_0 f,
\end{aligned}
$$

$$
\Rightarrow [\hat{K}_-, \hat{K}_+] = 2\hat{K}_0.
$$

$$
\begin{aligned}
[\hat{K}_+, \hat{K}_0] &= \left[\frac{i}{2}q^2, \frac{1}{2}\left(q\frac{\partial}{\partial q} + \frac{1}{2}\right)\right] f \qquad \text{(A.9)}\\
&= \frac{i}{4}\left[q^2, q\frac{\partial}{\partial q}\right] f = \frac{i}{4}\left(q^3\frac{\partial f}{\partial q} - q\frac{\partial}{\partial q}(q^2 f)\right)\\
&= -\frac{i}{2}q^2 f = -\hat{K}_+ f,
\end{aligned}
$$

$$
\Rightarrow [\hat{K}_+, \hat{K}_0] = -\hat{K}_+.
$$

$$
\begin{aligned}
[\hat{K}_-, \hat{K}_0]f &= \left[-\frac{i}{2}\frac{\partial^2}{\partial q^2}, \frac{1}{2}\left(q\frac{\partial}{\partial q} + \frac{1}{2}\right)\right] f \qquad \text{(A.10)}\\
&= -\frac{i}{4}\left[\frac{\partial^2}{\partial q^2}, q\frac{\partial}{\partial q}\right] f = -\frac{i}{4}\left(\frac{\partial^2}{\partial q^2}\left(q\frac{\partial}{\partial q}\right) - q\frac{\partial^3 f}{\partial q^3}\right)\\
&= -\frac{i}{2}\frac{\partial^2 f}{\partial q^2} = \hat{K}_- f,
\end{aligned}
$$

$$
\Rightarrow [\hat{K}_-, \hat{K}_0] = \hat{K}_-.
$$

We note that, since $\hat{E}_3 = i\hat{I}$, it commutes all the other operators.

# APPENDIX B

# EVOLUTION OPERATOR WITH DIFFERENT ORDERING OF THE LIE GROUP GENERATORS

Here, we solve the Schrödinger equation (4.7) with the general Hamiltonian (4.8) by the evolution operator method as in Chapter 3. But now, we write the evolution operator as product of exponential operators in a different order from (4.10) such that

$$\hat{U}_{0g}(t,t_0) = e^{c_0(t)\hat{E}_3} e^{\frac{a_0(t)}{\hbar}\hat{E}_1} e^{f_0(t)\hat{K}_+} e^{-b_0(t)\hat{E}_2} e^{2h_0(t)\hat{K}_0} e^{g_0(t)\hat{K}_-}, \tag{B.1}$$

where $f_0(t), g_0(t), h_0(t), a_0(t), b_0(t), c_0(t)$ are real valued functions to be determined so that $\hat{U}_{0g}(t,t_0)$ is a solution to the operator equation (4.12) and one also has $\hat{U}_{0g}\hat{U}_{0g}^{\dagger} = \hat{I}$.

Differentiating $\hat{U}_{0g}(t,t_0)$ with respect to $t$, we obtain

$$\begin{aligned}
\frac{\partial}{\partial t}\hat{U}_{0g}(t,t_0) \;=\; & \Big(\dot{c}_0\hat{E}_3\Big) e^{c_0(t)\hat{E}_3} e^{\frac{a_0(t)}{\hbar}\hat{E}_1} e^{f_0(t)\hat{K}_+} e^{-b_0(t)\hat{E}_2} e^{2h_0(t)\hat{K}_0} e^{g_0(t)\hat{K}_-} \\
& + e^{c_0(t)\hat{E}_3}\Big(\frac{1}{\hbar}\dot{a}_0\hat{E}_1\Big) e^{\frac{a_0(t)}{\hbar}\hat{E}_1} e^{f_0(t)\hat{K}_+} e^{-b_0(t)\hat{E}_2} e^{2h_0(t)\hat{K}_0} e^{g_0(t)\hat{K}_-} \\
& + e^{c_0(t)\hat{E}_3} e^{\frac{a_0(t)}{\hbar}\hat{E}_1}\Big(\dot{f}_0\hat{K}_+\Big) e^{f_0(t)\hat{K}_+} e^{-b_0(t)\hat{E}_2} e^{2h_0(t)\hat{K}_0} e^{g_0(t)\hat{K}_-} \\
& + e^{c_0(t)\hat{E}_3} e^{\frac{a_0(t)}{\hbar}\hat{E}_1} e^{f_0(t)\hat{K}_+}\Big(-\dot{b}_0\hat{E}_2\Big) e^{-b_0(t)\hat{E}_2} e^{2h_0(t)\hat{K}_0} e^{g_0(t)\hat{K}_-} \\
& + e^{c_0(t)\hat{E}_3} e^{\frac{a_0(t)}{\hbar}\hat{E}_1} e^{f_0(t)\hat{K}_+} e^{-b_0(t)\hat{E}_2}\Big(2\dot{h}_0\hat{K}_0\Big) e^{2h_0(t)\hat{K}_0} e^{g_0(t)\hat{K}_-} \\
& + e^{c_0(t)\hat{E}_3} e^{\frac{a_0(t)}{\hbar}\hat{E}_1} e^{f_0(t)\hat{K}_+} e^{-b_0(t)\hat{E}_2} e^{2h_0(t)\hat{K}_0}\Big(\dot{g}_0\hat{K}_-\Big) e^{g_0(t)\hat{K}_-}.
\end{aligned} \tag{B.2}$$

Using the Theorem (4.1) and the commutation relations of the operators $\hat{E}_1, \hat{E}_2, \hat{E}_3$, and

$\hat{K}_-, \hat{K}_+, \hat{K}_0$, we rearrange the Eqn. (B.2) in the form

$$\begin{aligned}
\frac{\partial}{\partial t}\hat{U}_{0g}(t,t_0) &= \Bigg[\Big(\dot{c}_0 + \frac{1}{\hbar}a_0\dot{b}_0 + \frac{1}{\hbar}a_0 b_0\dot{h}_0 + \frac{1}{2\hbar^2}a_0^2\dot{g}_0 e^{-2h_0}\Big)\hat{E}_3 \\
&+\Big(-\dot{b}_0 - \dot{h}_0 b_0 - \frac{1}{\hbar}a_0\dot{g}_0 e^{-2h_0}\Big)\hat{E}_2 \\
&+\Big(\frac{1}{\hbar}\dot{a}_0 + f_0\dot{b}_0 - \frac{1}{\hbar}a_0\dot{h}_0 + f_0 b_0\dot{h}_0 + \frac{1}{\hbar}a_0 f_0\dot{g}_0 e^{-2h_0}\Big)\hat{E}_1 \\
&+\Big(\dot{f}_0 - 2f_0\dot{h}_0 + f^2\dot{g}_0 e^{-2h_0}\Big)\hat{K}_+ \\
&+2\Big(\dot{h}_0 - f_0\dot{g}_0 e^{-2h_0}\Big)\hat{K}_0 + \Big(\dot{g}_0 e^{-2h_0}\Big)\Bigg]\hat{U}_{0g}(t,t_0).
\end{aligned} \tag{B.3}$$

If we substitute (B.3) in the operator equation (4.12), we obtain a non-linear system of first-order equations

$$\begin{aligned}
\dot{f}_0 + \frac{\hbar}{\mu(t)}f_0^2 + 2B(t)f_0 + \frac{\mu(t)\omega^2(t)}{\hbar} &= 0, \quad f_0(t_0) = 0, \\
\dot{g}_0 + \frac{\hbar}{\mu(t)}e^{2h_0} &= 0, \quad g_0(t_0) = 0, \\
\dot{h}_0 + \frac{\hbar}{\mu(t)}f_0 + B(t) &= 0, \quad h_0(t_0) = 0,
\end{aligned} \tag{B.4}$$

$$\begin{aligned}
\dot{a}_0 + \left(\frac{\hbar}{\mu(t)}f_0 + B(t)\right)a_0 + \hbar E(t)f_0 + D(t) &= 0, \quad a_0(t_0) = 0, \\
\dot{b}_0 - \left(\frac{\hbar}{\mu(t)}f_0 + B(t)\right)b_0 - \frac{a_0}{\mu(t)} - E(t) &= 0, \quad b_0(t_0) = 0, \\
\dot{c}_0 + \frac{a_0^2}{2\hbar\mu(t)} + \frac{E(t)}{\hbar}a_0 + \frac{F(t)}{\hbar} &= 0, \quad c_0(t_0) = 0.
\end{aligned} \tag{B.5}$$

Then comparing the systems (B.4) and (4.16), we see that the functions $f_0(t), g_0(t), h_0(t)$ and $f(t), g(t), h(t)$ satisfy the same differential equations with same initial conditions.

Thus we get

$$
\begin{aligned}
f_0(t) &= f(t) = \frac{\mu(t)}{\hbar}\left(\frac{\dot{x}_1(t)}{x_1(t)} - B(t)\right), \qquad \text{(B.6)}\\
g_0(t) &= g(t) = -\hbar x_1^2(t_0)\int_{t_0}^{t}\frac{1}{\mu(s)x_1^2(s)}ds,\\
h_0(t) &= h(t) = -\ln|x_1(t)| + \ln|x_1(t_0)|,
\end{aligned}
$$

where $x_1(t)$ is the solution of the homogeneous equation of motion (4.18) with initial conditions (4.19). Furthermore, since (B.4) and (B.5) are dependent systems, substituting $f(t)$ in the system (B.5), we obtain its solution in terms of $x_1(t)$ as

$$
\begin{aligned}
a_0(t) &= -\frac{1}{x_1(t)}\int_{t_0}^{t}\Big(\mu(s)E(s)(\dot{x}_1(s) - B(s)x_1(s)) + D(s)x_1(s)\Big)ds, \qquad \text{(B.7)}\\
b_0(t) &= x_1(t)\int_{t_0}^{t}\frac{1}{\mu(s)}\left(\frac{a_0(s)}{\mu(s)} + E(s)\right)ds,\\
c_0(t) &= -\int_{t_0}^{t}\left(\frac{a_0^2(s)}{2\hbar\mu(s)} + \frac{E(s)}{\hbar}a_0(s) + \frac{F(s)}{\hbar}\right)ds.
\end{aligned}
$$

Now, to solve the evolution problem (4.7), we take the initial function as normalized eigenstates of the standard harmonic oscillator

$$
\varphi_k(q) = \frac{1}{\sqrt{2^k k!}}\left(\frac{m\omega_0}{\pi\hbar}\right)^{1/4} e^{-\frac{m\omega_0}{2\hbar}q^2} H_k\left(\sqrt{\frac{m\omega_0}{\hbar}}q\right), \quad k = 0, 1, 2, \cdots,
$$

and apply the evolution operator (B.1) to these states as follows

$$
\begin{aligned}
\hat{U}_{0g}(t,t_0)\varphi_k(q) &= e^{c_0(t)}e^{\frac{i}{\hbar}a_0(t)q}e^{-\frac{i}{2}f(t)q^2}e^{-b_0(t)\frac{\partial}{\partial q}}e^{h(t)\left(q\frac{\partial}{\partial q}+\frac{1}{2}\right)}e^{-\frac{i}{2}g(t)\frac{\partial^2}{\partial q^2}}\varphi_k(q)\\
&= e^{\frac{h(t)}{2}}e^{ic_0(t)}e^{\frac{i}{\hbar}a_0(t)q}e^{\frac{i}{2}f(t)q^2}e^{-b_0(t)\frac{\partial}{\partial q}}e^{h(t)q\frac{\partial}{\partial q}}\widetilde{\varphi}_k(q;g(t))\\
&= e^{\frac{h(t)}{2}}e^{ic_0(t)}e^{\frac{i}{\hbar}a_0(t)q}e^{\frac{i}{2}f(t)q^2}e^{-b_0(t)\frac{\partial}{\partial q}}\widetilde{\varphi}_k(e^{h(t)}q;g(t))\\
&= e^{\frac{h(t)}{2}}e^{ic_0(t)}e^{\frac{i}{\hbar}a_0(t)q}e^{\frac{i}{2}f(t)q^2}\widetilde{\varphi}_k(e^{h(t)}(q-b_0(t));g(t)),
\end{aligned}
$$

where $\widetilde{\varphi}_k(q;z)$ is given by (4.36).

Therefore, we find the wave function solutions of the Schrödinger equation in the form

$$\begin{aligned}\Psi_k(q,t) &= N_k\sqrt{R_B(t)}\times\exp\left(i\left(k+\frac{1}{2}\right)\arctan\left(\frac{m\omega_0}{\hbar}g(t)\right)\right) \qquad \text{(B.8)}\\ &\times\exp\left(i\left(\frac{f(t)}{2}q^2+\frac{a_0(t)}{\hbar}q+c_0(t)\right)\right)\\ &\times\exp\left(-\frac{i}{2}\left(\frac{m\omega_0}{\hbar}\right)^2 g(t)R_B^2(t)(q-b_0(t))^2\right)\\ &\times\exp\left(-\frac{m\omega_0}{2\hbar}R_B^2(t)(q-b_0(t))^2\right)\times H_k\left(\sqrt{\frac{m\omega_0}{\hbar}}R_B(t)(q-b_0(t))\right).\end{aligned}$$

Since the wave function solutions (B.8) and (4.38) found in two ways are equal, comparing them we find a relation between the auxiliary functions $a_0(t), b_0(t), c_0(t)$ and $a(t), b(t), c(t)$:

$$a(t) = a_0(t) + \hbar f(t)b(t), \quad b(t) = b_0(t), \quad c(t) = c_0(t) - \frac{f(t)}{2}b^2(t).$$

Thus, by this way we get a solution of the system (4.17) in terms of $x_1(t)$:

$$a(t) = -\frac{z(t)}{\hbar x_1(t)} + \frac{\mu(t)}{\hbar}\left(\dot{x}_1(t) - B(t)x_1(t)\right)\int_{t_0}^{t}\left[-\frac{z(s)}{\mu(s)x_1^2(s)} + \frac{E(s)}{x_1(s)}\right]ds, \qquad \text{(B.9)}$$

$$b(t) = x_1(t)\int_{t_0}^{t}\left[-\frac{z(s)}{\mu(s)x_1^2(s)} + \frac{E(s)}{x_1(s)}\right]ds, \qquad \text{(B.10)}$$

$$\begin{aligned}c(t) &= -\frac{1}{\hbar}\int_{t_0}^{t}\left[\frac{z^2(s)}{2\mu(s)x_1^2(s)} - \frac{E(s)z(s)}{x_1(s)} + F(s)\right]ds \qquad \text{(B.11)}\\ &- \frac{\mu(t)}{2\hbar}\left(\frac{\dot{x}_1(t)}{x_1(t)} - B(t)\right)x_1^2(t)\left[\int_{t_0}^{t}\left(-\frac{z(s)}{\mu(s)x_1^2(s)} + \frac{E(s)}{x_1(s)}\right)ds\right]^2,\end{aligned}$$

where

$$z(t) = \int_{t_0}^{t}\left[\mu(\xi)E(\xi)\Big(\dot{x}_1(\xi) - B(\xi)x_1(\xi)\Big) + D(\xi)x_1(\xi)\right]ds.$$

# APPENDIX C

# FREE SCHRODINGER EQUATION

In sections 3.4 and 4.3 there are two free Schrödinger equations. In this part, we solve these problems using Fourier transform.

## C.1. The Fourier Transform

In this section, we introduce the Fourier transform and discuss its basic properties.

**Definition C.1** *Let $f : \mathbb{R} \to \mathbb{R}$. The **Fourier transform** of $f \in L^1(\mathbb{R})$, denoted by $\hat{f}$, is given by the integral*

$$\hat{f}(\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i\xi x} f(x) dx.$$

Instead of $\hat{f}$ the notation "$\mathcal{F}\{f(x)\}$" is also used.

**Theorem C.1** ***(Inversion of the Fourier Transform)*** *Let $f : \mathbb{R} \to \mathbb{R}$. Suppose that $f \in L^1(\mathbb{R})$ and in any finite interval, $f, f'$ are piecewise continuous. Then if $f$ is continuous at $x \in \mathbb{R}$, we have*

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i\xi x} \hat{f}(\xi) d\xi.$$

**Theorem C.2** ***(Linearity)*** *Let $f, g \in L^1(\mathbb{R})$ and $\alpha, \beta \in \mathbb{C}$. Then*

$$\mathcal{F}\{\alpha f(x) + \beta g(x)\} = \alpha \mathcal{F}\{f(x)\} + \beta \mathcal{F}\{g(x)\}.$$

**Theorem C.3** *Let $f \in L^1(\mathbb{R})$. Then*

**(a)** $\mathcal{F}\{e^{i\alpha x} f(x)\} = \hat{f}(\xi - \alpha)$ *(translation),*

**(b)** $\mathcal{F}\{f(x - x_0)\} = \hat{f}(\xi) e^{-ix_0\xi}, \quad x_0 \in \mathbb{R}$ *(shifting),*

**(c)** $\mathcal{F}\{f(\alpha x)\} = \frac{1}{|\alpha|} \hat{f}\left(\frac{\xi}{\alpha}\right), \quad \alpha \in \mathbb{R}$ *(scaling),*

**(d)** $\mathcal{F}\{\bar{f}(x)\} = \overline{\mathcal{F}\{f(-x)\}}$ *(conjugate).*

**Theorem C.4** *If $f$ is a continuous function, n-times piecewise differentiable, $f, f', \dots, f^{(n)}$ are in $L^1(\mathbb{R})$, and*

$$\lim_{|x|\to\infty} f^{(k)}(x) = 0 \;\; for \;\; k = 0, 1, 2, \dots, n-1,$$

*then*

$$\mathcal{F}\{f^{(n)}(x)\} = (i\xi)^n \mathcal{F}\{f(x)\}.$$

**Proposition C.1** *The Fourier transform of the Gaussian function $g(x)$, defined as*

$$g(x) = e^{\frac{-x^2}{2}}$$

*is again a Gaussian.*

**Proof** Taking the Fourier transform of $g(x)$, we get

$$\begin{aligned} \hat{g}(\xi) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ix\xi} e^{\frac{-x^2}{2}} dx \\ &= \frac{1}{\sqrt{2\pi}} e^{\frac{-\xi^2}{2}} \int_{-\infty}^{\infty} e^{\frac{-(x+i\xi)^2}{2}} dx. \end{aligned}$$

Let $u = x + i\xi$, then $dx = du$. So

$$\hat{g}(\xi) = \frac{1}{\sqrt{2\pi}} e^{\frac{-\xi^2}{2}} \int_{-\infty}^{\infty} e^{\frac{-u^2}{2}} du = e^{\frac{-\xi^2}{2}},$$

where we used the well-known integral

$$\int_{-\infty}^{\infty} e^{\frac{-u^2}{2}} du = \sqrt{2\pi}.$$

This shows $\hat{g}(\xi)$ is also a Gaussian function. □

**Proposition C.2** *If $f(x) = e^{-x^2/2} H_n(x)$, where $H_n(x)$ represents the n-th Hermite polynomial, then*

$$\mathcal{F}\{f(x)\} = (-i)^n \exp\left(-\frac{\xi^2}{2}\right) H_n(\xi) \tag{C.1}$$

*for all* $n = 0, 1, 2, \ldots$.

**Proof** From the definition of Hermite polynomials, we know that

$$\exp(+2xt - t^2) = \sum_{n=0}^{\infty} H_n(x)\frac{t^n}{n!}. \tag{C.2}$$

Multiplying both sides of the Equation (C.2) by $e^{-x^2/2}$, gives

$$\exp\left(-\frac{x^2}{2} + 2xt - t^2\right) = \sum_{n=0}^{\infty} \exp\left(-\frac{x^2}{2}\right) H_n(x)\frac{t^n}{n!}. \tag{C.3}$$

The Fourier transform of the LHS of the Equation (C.3) is then

$$\begin{aligned}
\mathcal{F}\left\{\exp(-\frac{x^2}{2} + 2xt - t^2)\right\} &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp(-ix\xi)\exp\left(-\frac{x^2}{2} + 2xt - t^2\right) dx \\
&= \frac{1}{\sqrt{2\pi}} \exp\left(\frac{1}{2}(2t - i\xi)^2 - t^2\right) \\
&\quad \times \underbrace{\int_{-\infty}^{\infty} \exp\left[-\frac{1}{2}(x - (2t - i\xi))^2\right] dx}_{I}.
\end{aligned}$$

If we change the variable $x - (2t - i\xi) = u$, it follows that $dx = du$. Then the integral $I$ becomes

$$I = \int_{-\infty}^{\infty} \exp\left(-\frac{u^2}{2}\right) du = \sqrt{2\pi}.$$

Therefore,

$$\mathcal{F}\left\{\exp\left(-\frac{x^2}{2} + 2xt - t^2\right)\right\} = \exp\left(t^2 - 2it\xi - \frac{\xi^2}{2}\right). \tag{C.4}$$

Taking the Fourier transform of the RHS of the Equation (C.3),

$$\mathcal{F}\left\{\sum_{n=0}^{\infty} \exp\left(-\frac{x^2}{2}\right) H_n(x)\frac{t^n}{n!}\right\} = \sum_{n=0}^{\infty} \mathcal{F}\left\{\exp\left(-\frac{x^2}{2}\right) H_n(x)\right\} \frac{t^n}{n!}, \tag{C.5}$$

and equating the Equations (C.4) and (C.5) gives

$$\begin{aligned}\sum_{n=0}^{\infty} \mathcal{F}\left\{\exp\left(-\frac{x^2}{2}\right)H_n(x)\right\}\frac{t^n}{n!} &= \exp\left(t^2 - 2it\xi - \frac{\xi^2}{2}\right)\\ &= \sum_{n=0}^{\infty}\exp\left(-\frac{\xi^2}{2}\right)H_n(\xi)\frac{(-it)^n}{n!}.\end{aligned}$$

Using the above results, the proposition is proven.

□

## C.2. Solution of the Free Schrödinger Equation

The first problem is given as

$$\begin{cases} -\frac{i}{2}\frac{\partial^2}{\partial q^2}\phi_k(q;z) = \frac{\partial}{\partial z}\phi_k(q;z), \quad z \in \mathbb{R}, \\ \phi_k(q;0) = \varphi_k(q) = N_k \exp\left(-\frac{\Omega_0}{2}q^2\right)H_k(\sqrt{\Omega_0}q), \end{cases} \tag{C.6}$$

where $N_k = (2^k k!)^{-1/2}(m\omega_0/\pi\hbar)^{1/4}$ and $\Omega_0 = (m\omega_0)/\hbar$. To solve the initial value problem, we take Fourier transform of (C.6) and obtain

$$\begin{cases} \frac{\partial}{\partial z}\hat{\phi}_k(\xi;z) = \frac{i}{2}\xi^2\hat{\phi}_k(\xi;z) \\ \hat{\phi}_k(\xi;0) = N_k\mathcal{F}\left\{\exp\left(-\frac{\Omega_0}{2}q^2\right)H_k(\sqrt{\Omega_0}q)\right\}. \end{cases} \tag{C.7}$$

By using the Proposition (C.2) and the third property of the Theorem (C.3), we can calculate

$$\mathcal{F}\left\{\exp\left(-\frac{\Omega_0}{2}q^2\right)H_k(\sqrt{\Omega_0}q)\right\} = \frac{(-i)^k}{\sqrt{\Omega_0}}\exp\left(-\frac{\xi^2}{2\Omega_0}\right)H_k\left(\frac{\xi}{\sqrt{\Omega_0}}\right).$$

Then solving the IVP (C.7), we get $\hat{\phi}_k(\xi; z)$ as in the form

$$
\begin{aligned}
\hat{\phi}_k(\xi; z) &= \exp\left(\frac{i}{2}\xi^2 z\right)\hat{\phi}_k(\xi; 0) \\
&= N_k \frac{(-i)^k}{\sqrt{\Omega_0}} \exp\left(-\frac{\xi^2}{2\Omega_0}(1 - i\Omega_0 z)\right) H_k\left(\frac{\xi}{\sqrt{\Omega_0}}\right). \qquad \text{(C.8)}
\end{aligned}
$$

If we take the inverse Fourier transform of (C.8), then

$$
\begin{aligned}
\phi_k(q; z) &= \mathcal{F}^{-1}\{\hat{\phi}_k(\xi; z)\} \\
&= N_k \frac{(-i)^k}{\sqrt{2\pi}\sqrt{\Omega_0}} \int_{-\infty}^{\infty} \exp(iq\xi) \exp\left(-\frac{\xi^2}{2\Omega_0}(1 - i\Omega_0 z)\right) H_k\left(\frac{\xi}{\sqrt{\Omega_0}}\right) d\xi \\
&= \frac{N_k}{\sqrt{1 - i\Omega_0 z}} \left(\frac{1 + i\Omega_0 z}{\sqrt{1 + (\Omega_0 z)^2}}\right)^k \times \exp\left(-\left(\frac{1 + i\Omega_0 z}{1 + (\Omega_0 z)^2}\right)\frac{\Omega_0}{2} q^2\right) \\
&\quad \times H_k\left(\sqrt{\frac{\Omega_0}{1 + (\Omega_0 z)^2}}\, q\right).
\end{aligned}
$$

Finally, using the relation

$$
\frac{\omega}{|\omega|} = \exp\left(i \arctan\left(\frac{v}{u}\right)\right),
$$

for any complex number $\omega = u + iv$, we obtain explicitly the function $\phi_k(q; z)$:

$$
\begin{aligned}
\phi_k(q; z) &= \frac{N_k}{(1 + (\Omega_0 z)^2)^{1/4}} \times \exp\left(i\left(k + \frac{1}{2}\right)\arctan(\Omega_0 z)\right) \qquad \text{(C.9)} \\
&\quad \times \exp\left(-\left(\frac{1 + i\Omega_0 z}{1 + (\Omega_0 z)^2}\right)\frac{\Omega_0}{2} q^2\right) \times H_k\left(\sqrt{\frac{\Omega_0}{1 + (\Omega_0 z)^2}}\, q\right).
\end{aligned}
$$

The second initial value problem is

$$
\begin{cases}
\frac{\partial}{\partial t}\psi_\alpha(q, t) = -\frac{i}{2}\frac{\partial^2}{\partial q^2}\psi_\alpha(q, t) \\
\psi_\alpha(q, 0) = \phi_\alpha(q) = \left(\frac{\Omega_0}{\pi}\right)^{1/4} \exp\left(\frac{i}{\hbar}\langle p\rangle_\alpha q\right) \exp\left(-\frac{\Omega_0}{2}(q - \langle q\rangle_\alpha)^2\right),
\end{cases} \qquad \text{(C.10)}
$$

where $\langle p\rangle_\alpha = \sqrt{2m\omega_0\hbar}\alpha_2$, $\langle q\rangle_\alpha = \sqrt{2\hbar/(m\omega_0)}\alpha_1$, and $\alpha = \alpha_1 + i\alpha_2$. Taking the Fourier transform of both sides of the equations in (C.10) gives

$$\begin{cases} \frac{\partial}{\partial t}\hat{\psi}_\alpha(\xi,t) = \frac{i}{2}\xi^2\hat{\psi}_\alpha(\xi,t) \\ \hat{\psi}_\alpha(\xi,0) = \mathcal{F}\{\phi_\alpha(q)\}. \end{cases} \tag{C.11}$$

Now, completing the square in the function $\phi_\alpha(q)$, we find its Fourier transform by using the Proposition (C.1) and the properties ($b$) and ($c$) of the Theorem (C.3) as

$$\begin{aligned} \mathcal{F}\{\phi_\alpha(q)\} &= \left(\frac{\Omega_0}{\pi}\right)^{1/4} \exp\left(2i\alpha_1\alpha_2 - \alpha_2^2\right)\mathcal{F}\left\{\exp\left(-\frac{\Omega_0}{2}\left(q - \sqrt{\frac{2}{\Omega_0}}\alpha\right)^2\right)\right\} \\ &= \frac{\exp\left(2i\alpha_1\alpha_2 - \alpha_2^2\right)}{(\Omega_0\pi)^{1/4}} \exp\left(-\frac{\xi^2}{2\Omega_0} - i\sqrt{\frac{2}{\Omega_0}}\alpha\xi\right). \end{aligned}$$

Solving the IVP (C.11), we obtain

$$\begin{aligned} \hat{\psi}_\alpha(\xi,t) &= \exp\left(\frac{i}{2}\xi^2 t\right)\hat{\psi}_\alpha(\xi,0) \\ &= \frac{\exp\left(2i\alpha_1\alpha_2 - \alpha_2^2\right)}{(\Omega_0\pi)^{1/4}} \exp\left(\xi^2\left(\frac{i}{2}t - \frac{1}{2\Omega_0}\right) - i\sqrt{\frac{2}{\Omega_0}}\alpha\xi\right). \end{aligned} \tag{C.12}$$

Then we take inverse Fourier transform of (C.12) and the exact form of $\psi_\alpha(q,t)$ follows

$$\begin{aligned} \psi_\alpha(q,t) &= \frac{\exp\left(2i\alpha_1\alpha_2 - \alpha_2^2\right)}{(\Omega_0\pi)^{1/4}}\mathcal{F}^{-1}\left\{\exp\left(\xi^2\left(\frac{i}{2}t - \frac{1}{2\Omega_0}\right) - i\sqrt{\frac{2}{\Omega_0}}\alpha\xi\right)\right\} \\ &= \frac{\exp\left(2i\alpha_1\alpha_2 - \alpha_2^2\right)}{(\Omega_0\pi)^{1/4}}\sqrt{\frac{\Omega_0}{1 - i\Omega_0 t}} \exp\left(-\frac{\Omega_0}{2(1 - i\Omega_0 t)}\left(q - \frac{2}{\Omega_0}\alpha\right)^2\right). \end{aligned}$$

Hence,

$$\begin{aligned} \psi_\alpha(q,t) = {} & \left(\frac{\Omega_0}{\pi}\right)^{1/4} \times \frac{\exp\left(2i\alpha_1\alpha_2 - \alpha_2^2\right)}{(\Omega_0\pi)^{1/4}} \times \exp\left(\frac{i}{2}\arctan(\Omega_0 t)\right) \\ & \times \exp\left(-\frac{\Omega_0}{2}\left(\frac{1 + i\Omega_0 t}{1 + (\Omega_0 t)^2}\right)\left(q - \sqrt{\frac{2}{\Omega_0}}\alpha\right)^2\right). \end{aligned} \tag{C.13}$$