KRULL-SCHMIDT PROPERTIES OVER RINGS OF FINITE CHARACTER

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Ezgi GÜRBÜZ

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We approve the thesis of **Ezgi GÜRBÜZ**

Examining Committee Members:

Assoc. Prof. Dr. Başak AY SAYLAM
Department of Mathematics, İzmir Institute of Technology

Prof. Dr. Engin BÜYÜKAŞIK
Department of Mathematics, İzmir Institute of Technology

Assoc. Prof. Dr. Noyan Fevzi ER
Department of Mathematics, Dokuz Eylül University

27 July 2016

Assoc. Prof. Dr. Başak AY SAYLAM
Supervisor, Department of Mathematics
İzmir Institute of Technology

Prof. Dr. Engin BÜYÜKAŞIK
Head of the Department of
Mathematics

Prof. Dr. Bilge KARAÇALI
Dean of the Graduate School of
Engineering and Sciences

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ABSTRACT

KRULL-SCHMIDT PROPERTIES OVER RINGS OF FINITE CHARACTER

The main purpose of this thesis is to investigate the notion of Krull-Schmidt properties over rings of finite character. In accordance with this aim, we give a survey of necessary and sufficient conditions on an h-local domain for certain Krull-Schmidt properties hold for direct sums of ideals, direct sums of indecomposable submodules of finitely generated free modules and direct sums of rank one torsion-free modules. By using obtained characterizations, some useful results for Krull-Schmidt properties of modules over Noetherian and Prüfer domains are proven. Besides, the characterizations of Noetherian UDI domains are given.

ÖZET

SONLU KARAKTER HALKALARI ÜZERİNDE KRULL-SCHMIDT ÖZELLİKLERİ

Bu tezde sonlu karakter halkaları üzerinde Krull-Schmidt özellikleri incelenmiştir. Bu amaç doğrultusunda, ideallerin dik toplamları, sonlu üretilmiş serbest modüllerin parçalanamaz alt modüllerinin dik toplamları ve bir boyutlu burulmasız modüllerin dik toplamları için Krull-Schmidt özelliklerinin versiyonlarının h-local tamlık bölgelerinde ne zaman geçerli olduğu üzerine inceleme yapıldı. Elde edilen karakterizasyonları kullanarak Noether ve Prüfer tamlık bölgeleri üzerindeki moduller için bazı kullanışlı sonuçlar ispatlanmıştır. Bunların yanı sıra, Noether UDI tamlık bölgelerinin karakterizasyonları verilmiştir.

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LIST OF ABBREVIATIONS

R	a commutative domain with identity 1
Q	quotient field of R
R^*	$R \setminus 0$
$S^{-1}R$	localization of R at S
R_P	localization of R at a prime ideal P
$R_{[M]}$	the intersection of the rings R_N, where $N \neq M$ is maximal ideal of R
$\subseteq$	submodule
$\subset$	proper submodule
$\cong$	isomorphic
$\otimes$	tensor product
$\mathbb{Z}$	the ring of integers
$\bigoplus_{i \in I} M_i$	direct sum of R- modules M_i
$\prod_{i \in I} M_i$	direct product of R- modules M_i
$Ann_R(X)$	annihilator of the set X
$Kerf$	the kernel of the map f
Imf	the image of the map f
$Hom(M, N)$	all R-module homomorphisms from M to N
$End(M)$	the endomorphism ring of a module M
$[Y : X]$	the R-module , $\{q \in Q : qX \subseteq Y\}$
$E(X)$	the endomorphism ring of X where X is R-submodule of Q

CHAPTER 1

INTRODUCTION

Let R be a commutative integral domain and C a class of R-modules. The Krull-Schmidt property holds for C if, whenever

$$G_1 \oplus G_2 \oplus \cdots G_n \cong H_1 \oplus H_2 \oplus \cdots H_m$$

for $G_i, H_j \in C$, then $n = m$ and, possibly after a reindexing, $G_i \cong H_i$ for all $i \leq n$. We say a domain R has unique decomposition into ideals, UDI, if the class of ideals of R has the Krull-Schmidt property.

In Chapter 2 we give the definitions of some basic tools about commutative algebra and their properties which are useful for our further studies.

In Chapter 3 we present two submonoids of R, and then define their connection with h-locality. After giving some properties related with these submonoids, we give a characterization of an h-local integral domains that is very useful for our work.

In Chapter 4 we introduce some types of the Krull-Schmidt property. We examine when the versions of the Krull-Schmidt property hold for direct sums of ideals, direct sums of indecomposable submodules of finitely generated free modules and direct sums of rank one torsion-free modules. In this chapter, we mostly use the notion of the Picard group which is an abelian group consisting of the invertible fractional ideals modulo the principal fractional ideals. P.Goeters and B.Olberding characterized the forms of Krull-Schmidt property for h-local integral domains and this leads to some new results for Krull-Schmidt properties of modules over Noetherian and Prüfer domains.

In Chapter 5 we deal with Noetherian integral domains and give the characterizations of Noetherian UDI domains. Specifically, a Noetherian integral domain R has UDI if and only if R is a PID or R has exactly one nonprincipal maximal ideal M such that R_M has UDI (Goeters & Olberding, 2001). Moreover, P. Goeters and B. Olberding give an explicit description of local UDI domains which is given by Theorem 5.2.

In Chapter 6 we assume that the ring R is of finite character, that is, every non-zero element is contained in only finitely many maximal ideals of R; equivalently, every non-zero ideal is contained in only finitely many maximal ideals of R. We state some new results without giving their proofs and realize that these consequences are useful to obtain new characterizations of domains with finite character which the Krull-Schmidt property holds for some classes of R-modules.

In Conclusion we summarize the main results obtained in this thesis.

CHAPTER 2

PRELIMINARIES

This chapter consists of some basic tools about commutative algebra that are used in this thesis. All rings mentioned below are commutative with identity.

Definition 2.1 *Let $\varphi : R \to S$ be a ring homomorphism.*

- *If I is an ideal in R, then extension I^e of I to S is the ideal $\varphi(I)S$ of S generated by the image of I.*
- *If J is an ideal of S, then the contraction J^c in R of J is the ideal $\varphi^{-1}(J)$.*

In the special case where R is a subring of S and φ is the natural injection, the extension of $I \subseteq R$ is the ideal IS in S and the contraction of $J \subseteq S$ is the ideal $J \cap R$ of R.

It is immediate from the definition that

- $I \subseteq IS \cap R$, more generally, I is contained in the contraction of its extension to S, and
- $(J \cap R)S \subseteq J$, more generally, J contains the extension of its contraction in R.

Definition 2.2 *Let $\varphi : R \to S$ be a ring homomorphism. An ideal I of R is called contracted ideal if $I^{ec} = I$ and an ideal J of S is called extended ideal if $J^{ce} = J$.*

If Q is a prime ideal in S, then its contraction is prime in R. (Although the contraction of a maximal ideal need not be maximal). On the other hand, if P is a prime ideal in R, its extension need not be prime (or even proper) in S; moreover, it is not generally true that P is the contraction of a prime ideal of S.

2.1. Rings and Modules of Fractions

The formation of rings of fractions and the associated process of localization are the most important technical tools in commutative algebra. This section gives the definitions and basic properties of the formulation of fractions.

Let S be a submonoid of R^* (i.e., a multiplicatively closed subset such that $0 \notin S$ and $1 \in S$). The set R_S of equivalence classes of pairs (r, s), $r \in R$, $s \in S$, under the equivalence relation

$$(r, s) \sim (r', s') \text{ if and only if } rs' = r's$$

becomes a ring. If the equivalence class of (r, s) is denoted by $\frac{r}{s}$, then the ring operations are

$$\frac{r_1}{s_1} + \frac{r_2}{s_2} = \frac{r_1 s_2 + r_2 s_1}{s_1 s_2} \text{ and } \frac{r_1}{s_1} \cdot \frac{r_2}{s_2} = \frac{r_1 r_2}{s_1 s_2}$$

where $r_i \in R$, $s_i \in S$. R_S is called the localization of R at S or the quotient ring of R with respect to S. The notation $S^{-1}R$ is an alternative for R_S.

We may think of the canonical embedding $r \mapsto \frac{r}{1}$, $r \in R$ of R in $S^{-1}R$ as the inclusion map. Thereby R becomes a subring of $S^{-1}R$ which is in turn a subring of the quotient field Q of R. Thus localizations are overrings (the converse is not true in general).

Example 2.1 *If S consists of all the non-zero elements of R, $S^{-1}R$ coincides with Q.*

Example 2.2 *For an ideal P of R, $R \setminus P$ is a monoid if and only if P is prime. In this case, the quotient ring of R with respect to $R \setminus P$ is universally notated by R_P. It is called the localization of R at P.*

Proposition 2.1 *((Atiyah & Macdonald, 1969), Proposition 3.11)*

(i) Every ideal in $S^{-1}R$ is an extended ideal.

(ii) If I is an ideal of R, then $I^e = < 1 >$ if and only if $I \cap S \neq \emptyset$.

(iii) The prime ideals of $S^{-1}R$ are in one-to-one correspondence ($P \mapsto S^{-1}P$) with the prime ideals of R such that $P \cap S = \emptyset$.

2.1.1. Modules of quotients and localizations

Let M be an R-module. Form the pairs (m, s) where $m \in M$ and $s \in S$, and view (m, s) and (m', s') equivalent if there exists a $t \in S$ such that

$$t(s'm - sm') = 0.$$

The equivalence class containing (m, s) is denoted by $\frac{m}{s}$. The R-module $S^{-1}M(M_S)$ consisting of the equivalence classes is called the module of quotients of M with respect to S, or the localization of M at S. It becomes an $S^{-1}R$-module by setting

$$\frac{r}{t} \cdot \frac{m}{s} = \frac{rm}{ts}, \ \ (r \in R, s, t \in S, m \in M).$$

The canonical homomorphism $\phi : M \to S^{-1}M$ which sends m into $m/1 \in S^{-1}M$, need not be monic.

As for localization of domains, M_P will stand for the module of quotients of M with respect to the complement of a prime ideal P in R. The canonical homomorphism $M \to M_P$ is denoted by ϕ_P rather than $\phi_{R \setminus P}$.

Proposition 2.2 *Let M be an R-module. Then $S^{-1}R$ modules $S^{-1}M$ and $S^{-1}R \otimes M$ are isomorphic; more precisely, there exists a unique isomorphism $f : S^{-1}R \otimes M \to S^{-1}M$ for which*

$$f((r/s \otimes m)) = rm/s$$

for all $r \in R$, $m \in M$ and $s \in S$.

Several useful conclusions may be drawn for the localization:

(a) A homomorphism of R-modules $\psi : M \to N$ is injective (respectively, surjective) if and only if $\psi_P : M_P \to N_P$ is injective (respectively, surjective) for all maximal ideals P of R.

(b) If $0 \to N \to M \to M/N \to 0$ is an exact sequence of R-modules, then $0 \to N_S \to M_S \to (M/N)_S \to 0$ is an exact sequence of R_S modules. In particular,

$$M_S/N_S \cong (M/N)_S.$$

(c) Localization of modules commutes with finite intersections and arbitrary sums.

(d) $Ann_R M_S = (Ann_R M)_S$ provided M is a finitely generated R-module.

(e) For every R-module M, there is a canonical embedding

$$\phi : M \to \prod_{P \in MaxR} M_P,$$

acting as $\phi(x) = (\ldots, \phi_P(x), \ldots)$. Consequently, $M = 0$ if and only if $M_P = 0$ for all maximal ideals P of R.

2.2. Fractional Ideals

Definition 2.3 *Let R be an integral domain with quotient field Q. A* ***fractional ideal*** *of an integral domain R is an R-submodule J of Q such that $rJ \leq R$ for some non-zero $r \in R$.*

Remark 2.1 *The following can be seen easily.*

1. *An R-submodule of Q is a fractional ideal if and only if it is isomorphic to an ideal of R.*
2. *The ideals of R are clearly fractional ideals.*
3. *A finitely generated submodule of Q is a fractional ideal.*

For R-submodules I and J of Q, we have a binary operation which is called the product:

$$IJ = \{\sum_{i=1}^{n} a_i b_i | a_i \in I, b_i \in J, n < \omega\}$$

Definition 2.4 *A fractional ideal I is said to be invertible if there exists a fractional ideal J of R such that $IJ = R$.*

Remark 2.2 *Every non-zero ideal is invertible if and only if every non-zero fractional ideal is invertible.*

Definition 2.5 *A ring R is said to be semilocal if R has only finitely many maximal ideals.*

Proposition 2.3 *((Fuchs & Salce, 2001), Proposition I.2.5) Let I be an invertible fractional ideal of a domain R. Then:*

(i) $I^{-1} = [R : I]$*;*

(ii) *I is finitely generated;*

(iii) *If R is semilocal, then I is a principal ideal; moreover; if R is local, every generating set of I contains an element generating I;*

(iv) *If I is an ideal and there is an $a \in I$ contained in only finitely many maximal ideals, then $I = aR + bR$ for some $b \in R$.*

Proposition 2.4 *((Fuchs & Salce, 2001), Proposition I.2.7) A finitely generated ideal I of an integral domain R is invertible if and only if R_M is invertible for all maximal ideals M of R.*

2.3. Integral Dependence

Definition 2.6 *Let B be a ring, A a subring of B. An element x of B is said to be integral over A if x is a root of a monic polynomial with coefficients in A, that is if x satisfies an equation of the form*

$$x^n + a_1 x^{n-1} + \ldots + a_n = 0$$

where the a_i are elements of A. Clearly every element of A is integral over A.

Proposition 2.5 *(Atiyah & Macdonald, 1969) The following are equivalent:*

(i) *$x \in B$ is integral over A;*

(ii) *$A[x]$ is a finitely generated A-module;*

(iii) *$A[x]$ is contained in a subring C of B such that C is a finitely generated A-module;*

(iv) *there exists a faithful $A[x]$-module M which is finitely generated as an A-module.*

Definition 2.7 *An integral domain is said to be integrally closed if it is integrally closed in its field of fractions.*

Corollary 2.1 *((Atiyah & Macdonald, 1969), Corollary 5.8) Let $A \subseteq B$ be rings, B integral over A; let Q be a prime ideal of B and let $P = Q^c = Q \cap A$. Then Q is maximal if and only if P is maximal.*

Theorem 2.1 *((Atiyah & Macdonald, 1969), Theorem 5.10) Let $A \subseteq B$ be rings, B integral over A; let P be a prime ideal of A. Then there exists a prime ideal Q of B such that $Q \cap A = P$.*

Proposition 2.6 *((Atiyah & Macdonald, 1969), Proposition 5.13) Let A be an integral domain. Then the following are equivalent:*

(i) *A is integrally closed;*

(ii) *A_P is integrally closed for each prime ideal P;*

(iii) *A_M is integrally closed for each maximal ideal M.*

2.4. Valuation Rings and Dedekind Domains

Definition 2.8 *A domain R is a* ***valuation ring*** *if it is not a field and for any $x \in Q$, $x \in R$ or $x^{-1} \in R$.*

Lemma 2.1 *If I and J are ideals in a valuation ring R, then either $I \subset J$ or $J \subset I$. In particular, R is local.*

Proof Let $x \in I$ and $x \notin J$. For $y \neq 0$ in J, $x/y \notin R$, hence $y/x \in R$ and $y = (y/x)x \in I$. □

Lemma 2.2 *((Fuchs & Salce, 2001), Lemma 1.3) For a valuation ring R with a unique maximal ideal P, we have:*

(i) *finitely generated ideals are principal;*

(ii) *the only principal ideals that can possibly be primes are P and 0;*

(iii)) *for a proper ideal I, either $I^n = 0$ for some $n < \omega$ or the intersection $J = \bigcap_{n<\omega} I^n$ is a prime ideal of R.*

Definition 2.9 *An R-module M is called uniserial if all submodules of M are totally ordered by inclusion, equivalently; for all $m_1, m_2 \in M$, either $m_1R \subseteq m_2R$ or $m_2R \subseteq m_1R$.*

Lemma 2.3 *((Fuchs & Salce, 2001), Lemma 1.4) If R is a valuation domain, then*

(a) its field of quotients Q is a uniserial R-module;

(b) every proper submodule of Q is a fractional ideal of R.

Definition 2.10 *An overring of R is a subring of the quotient field Q of R which contains R.*

The next result gives a full characterization of the overrings of a valuation domain.

Proposition 2.7 *Let R be a valuation domain. A subring S of Q is an overring of R if and only if $S = R_L$ for some prime ideal L of R. It is necessarily a valuation domain.*

Definition 2.11 *Let R^* denote the subgroup of units in a ring R. A discrete valuation on a field K is a function $v : K \setminus \{0\} \to \mathbb{Z}$ that satisfies the following properties:*

1. *v is surjective,*

2. *$v(xy) = v(x) + v(y)$,*

3. *$v(x + y) \geq min\{v(x), v(y)\}$.*

The subring $\{x \in K | v(x) \geq 0\} \cup \{0\}$ is called the valuation ring of v.

Definition 2.12 *An integral domain R is called a **Discrete Valuation Ring (D.V.R)** if R is the valuation ring of a discrete valuation v on the field of fractions of R.*

The valuation v is often extended to all of K by defining $v(0) = \infty$, in which 2nd and 3rd cases hold for all $a, b \in K$.

Example 2.3 *The localization $\mathbb{Z}_{<p>}$ of $\mathbb{Z}$ at any nonzero prime ideal $< p >$ is a D.V.R. with respect to the discrete valuation v_p on $\mathbb{Q}$ defined as follows:*
Every element $\frac{a}{b} \in \mathbb{Q}^*$ can be written uniquely in the form $\frac{a}{b} = p^n \frac{a_1}{b_1}$, where $n \in \mathbb{Z}$, $\frac{a_1}{b_1} \in \mathbb{Q}^*$ and both a_1 and b_1 are relatively prime. Define

$$v_p\Big(\frac{a}{b}\Big) = v_p\Big(p^n \frac{a_1}{b_1}\Big) = n.$$

Then v_p is discrete valuation. The corresponding valuation ring is the set of rational numbers $\frac{a}{b}$ where b is not divisible by p, which is $\mathbb{Z}_{<p>}$.

Theorem 2.2 *The following properties of a ring R are equivalent:*

(i) *R is a DVR.*

(ii) *R is a principal ideal domain with a unique maximal ideal $P \neq 0$.*

(iii) *R is a unique factorization domain with a unique (up to associates) irreducible element t.*

(iv) *R is a Noetherian integral domain that is also a local ring whose unique maximal ideal is nonzero and principal.*

(v) *R is a Noetherian, integrally closed, integral domain that is also a local ring of Krull dimension 1, i.e. R has a unique nonzero prime ideal.*

Definition 2.13 *An integral domain R is a **Dedekind domain** if every non-zero ideal of R is invertible.*

Theorem 2.3 *The following statements are equivalent for an integral domain R which is not a field.*

(i) *Every non-zero ideal of R is invertible.*

(ii) *R is Noetherian and each localization R_P at a prime ideal P is a DVR.*

(iii) *R is Noetherian, integrally closed, and of Krull dimension 1.*

(iv) *Every non-zero proper ideal of R is a product of maximal ideals.*

(v) *Every non-zero proper ideal of R is a product of prime ideals.*

Moreover, the product decomposition in (4) is then unique.

2.5. Prüfer Domains

An integral domain R is a ***Prüfer domain*** if all its localizations at maximal ideals are valuation domains; thus, Prüfer domains are those domains which are locally valuation domains. Clearly, if R is a Prüfer domain and L is a non-zero prime ideal of R, then R_L is a valuation domain.

Theorem 2.4 *((Fuchs & Salce, 2001), Theorem 1.1) For a domain R, the following conditions are equivalent:*

(a) *R is Prüfer domain;*

(b) *every finitely generated non-zero fractional ideal is invertible;*

(c) *the lattice of the fractional ideals of R is distributive: for fractional ideals I, J, K of R,*

$$I \cap (J + K) = (I \cap J) + (I \cap K);$$

(d) *every overring of R is a Prüfer domain.*

CHAPTER 3

H-LOCAL DOMAINS

First, we introduce two submonoids of R^*, and then we will define their connection with h-locality. It is an easy matter to verify that, for any domain R, the following two subsets of R^* are saturated submonoids, and so is their intersection $S_0 = S_1 \cap S_2$:

$$S_1 = \{r \in R^* | \text{r is of finite character}\}$$

$$S_2 = \{r \in R^* | \text{any prime containing r is contained in only one maximal ideal}\}.$$

The term ***'of finite character'*** is used to indicate that every non-zero element (equivalently, every non-zero ideal) of R is contained in but a finite number of maximal ideals.

Definition 3.1 *A domain R is h-**local domain** if $S_0 = R^*$; equivalently, R is h-local if and only if the following conditions are satisfied:*

(i) every nonzero element of R is contained in only finitely many maximal ideals of R, and

(ii) each nonzero prime ideal of R is contained in a unique maximal ideal of R.

We observe that (i) means that R/I is semilocal for every ideal $I \neq 0$, while (ii) asserts that R/I is even local if $I \neq 0$ is a prime ideal.

Example 3.1 *Local domains are h-local.*

Example 3.2 *If R is a domain of Krull dimension 1, then R is an h-local domain. In fact, for any nonzero ideal I of R, R/I is Noetherian ring with Krull dimension 0, so R/I is an Artinian ring. Thus, R/I is semilocal, which yields that I is contained in only finitely many maximal ideals. Clearly, any nonzero prime ideal of R is maximal ideal. Thus, R satisfies* (i) *and* (ii).

Example 3.3 *Dedekind domains are h-local since every prime ideal of a Dedekind domain is maximal and every nonzero ideal is product of prime ideals.*

Lemma 3.1 *A domain R is of finite character if and only if, for maximal ideals P, P_i, $i \in I$, where P_i's are distinct, the inclusion $0 \neq \bigcap_{i \in I} P_i \subseteq P$ implies that $P = P_i$ for some i.*

Proof Suppose R is of finite character and $0 \neq \bigcap_{i \in I} P_i \subseteq P$ for maximal ideals P, P_i $i \in I$. If I is infinite, then the ideal $0 \neq \bigcap_{i \in I} P_i$ is contained in infinitely many maximal ideals of R which contradicts the assumption that R is of finite character. So, I must be a finite set. Thus, the desired property follows.

For the converse, assume R satisfies the stated condition. Take any $a \in R^*$ and let $\{P_i, i \in I\}$ be the set of maximal ideals containing a. Set $A_i = \bigcap_{j \neq i} P_j$. Then A_i is an ideal of R which is not contained in P_i, so $\sum_{i \in I} A_i = R$. If $\sum_{i \in I} A_i \neq R$, then $\sum_{i \in I} A_i \subseteq M$ for some maximal ideal M of R. By assumption $M = P_i$ for some $i \in I$. So, A_i is contained in P_i, which is a contradiction. Thus, there must be a finite subset $\{1, \ldots, m\}$ of I such that $A_1 + \ldots + A_m = R$. This shows that R is contained in each P_j for $j \neq 1, \ldots, m$. Consequently, no such P_j can exist, i.e, $I = \{1, \ldots, m\}$. □

Lemma 3.2 *(Matlis) A domain R satisfies (ii) if and only if*

$$R_P \otimes_R R_{P'} \cong Q$$

for any two different maximal ideals P and P' of R.

Proof We first note that $R_P \otimes_R R_{P'} = (R_P)_{P'} = S^{-1}R$ where $S = (R \setminus P)(R \setminus P')$. Suppose $R_P \otimes_R R_{P'} \cong Q$. Let I be a prime ideal of R such that I is contained in two different maximal ideals P and P'. We know that $S^{-1}I$ is prime in $S^{-1}R$ if and only if $I \cap S = \emptyset$. Now take $x \in I \cap S = I \cap [(R \setminus P)(R \setminus P')]$. Then $x = ab$ for some $a \in R \setminus P$ and $b \in R \setminus P'$. Since $x \in I$ and I is a prime ideal we get a contradiction, and so no such x exists. Hence, $I \cap S = \emptyset$ which shows that $S^{-1}I$ is a prime ideal in $S^{-1}R$ but since $S^{-1}R = Q$ is a field by assumption, this is impossible. So, any prime ideal of R cannot be contained in two different maximal ideals.

For the converse, suppose R satisfies (ii). Let I be a prime ideal of R. If $I \subseteq P$, then $I \nsubseteq P'$ where P and P' are the maximal ideals of R. Since $I \nsubseteq P'$, there exists an element $x \in I$ and $x \notin P'$. So, $x \in I \cap [(R \setminus P)(R \setminus P')]$ and so $S^{-1}I = S^{-1}R$ for any prime ideal I of R which shows $S^{-1}R$ does not have any proper prime ideal. So, $S^{-1}R$ must be a field. □

It will be illuminating to compare particular properties of S-torsion modules for submonoids S of the monoid S_1 or S_2. First we consider the submonoids of S_1.

Definition 3.2 *Let M be an R-module. S-torsion part of M is defined as*

$$S(M) = \{x \in M | \, sx = 0 \, for \; some s \in S\}.$$

*A module M is S-**torsion** if $S(M) = M$.*

Proposition 3.1 *For a submonoid S of R^*, the following conditions are equivalent:*

- ***(a)*** *S is contained in S_1;*
- ***(b)*** *every S-torsion module M canonically embeds in the direct sum $\bigoplus_P M_P$ of its localizations at maximal ideals P;*
- ***(c)*** *there is a canonical embedding*

$$\phi : R_S/R \to \bigoplus_P (R_S/R)_P.$$

Proof **(a) ⇒ (b)** Suppose $S \subseteq S_1$, and let M be a S-torsion module. We know that there is a canonical embedding $\phi : M \to \prod_{P \in MaxR} M_P$, via $\phi(x) = (\ldots, \frac{x}{1}, \ldots)$, $(x \in M)$. We show that the Pth coordinate of $\phi(x)$ vanishes if and only if $Ann_R x$ is not contained in P. In order to see this, assume $Ann_R x$ is not contained in P, and take an element $y \in Ann_R x$ such that $y \notin P$, and see $\phi_P(x) = 0$ for some x. Conversely, suppose the Pth coordinate of $\phi(x)$ vanishes, then $\phi_P(x) = \frac{x}{1} = 0$ which implies $xt = 0$ for some $t \in R \setminus P$, so $t \in Ann_R x$. Since M is S-torsion, $Ann_R x$ contains some $s \in S$, hence is contained but in a finite number of maximal ideals of R. Thus almost all the coordinates of $\phi(x)$ vanish, i.e, $Im\phi \leq \bigoplus_{P \in MaxR} M_P$.
(b) ⇒ (c) is trivial since R_S/R is S-torsion R-module.
(c) ⇒ (a) If ϕ is an embedding, then for each $s \in S$, the Pth coordinate of $\phi(s^{-1} + R)$ vanishes for almost all P. So, $Ann(s^{-1} + R)$ is contained but in a finite number of maximal ideals. Since $sR = Ann_R(s^{-1} + R)$, s is contained in only finitely many maximal ideals which shows $S \subseteq S_1$. □

Definition 3.3 *For a submonoid S of R^*, an R-module M is said to be S-**divisible** if $sM = M$ for all $s \in S$. Clearly, R_S and M_S (for an R-module) are S-divisible. Furthermore, $M \otimes_R N = 0$ whenever M is S-divisible and N is S-torsion.*

Proposition 3.2 *For a submonoid S of R^*, the following conditions are equivalent:*

(a) *S is contained in S_2;*

(b) *for every pair of distinct maximal ideals P and $P^{'}$, the prime ideals contained in $P \cap P^{'}$ are disjoint from S;*

(c) *for every pair of distinct maximal ideals P and $P^{'}$, $R_P \otimes R_{P^{'}}$ is S-divisible.*

Proof **(a) ⇒ (b)** Suppose $S \subseteq S_2$. Let I be a prime ideal of R such that $I \subseteq P \cap P^{'}$ where P and $P^{'}$ are maximal ideals of R. Take $x \in I \cap S$. Since $x \in S$, any prime ideal containing x is contained in a unique maximal ideal which contradicts with the assumption $I \subseteq P \cap P^{'}$. So, such an x cannot exist. Hence $I \cap S = \emptyset$.

(b) ⇒ (a) Suppose for every pair of distinct maximal ideals P and $P^{'}$, the prime ideals in $P \cap P^{'}$ are disjoint from S. If $S \nsubseteq S_2$, then there exists $s \in S$ such that $s \notin S_2$. Since $s \notin S_2$, any prime ideal containing s can be contained in more than one maximal ideal. If I is a prime ideal containing s and I is contained in P and P' where P and P' are maximal ideals of R, then $I \subseteq P \cap P^{'}$. So, by assumption, $I \cap S = \emptyset$ but $s \in I \cap S$ which is a contradiction.

(b) ⇒ (c) First we observe that $R_P \otimes R_{P^{'}} = (S^*)^{-1}R$, $S^* = (R \setminus P)(R \setminus P^{'})$ is S-divisible if $S \subseteq S^*$. Equivalently, if every prime ideal of R disjoint from S^* is disjoint from S. But a prime ideal is disjoint from S^* if it is contained in $P \cap P^{'}$.

(c) ⇒ (b) Suppose for every pair of distinct maximal ideals P and $P^{'}$, $R_P \otimes R_{P^{'}} = (S^*)^{-1}R$, $S^* = (R \setminus P)(R \setminus P^{'})$, is S-divisible. Let I be a prime ideal such that $I \subseteq P \cap P^{'}$. We claim that $I \cap S = \emptyset$. Take $x \in I \cap S$. Since $I \subseteq P \cap P^{'}$, there exists $p \in P$ such that $p \notin I$. Then, $\frac{p}{b} \in (S^*)^{-1}R$ where $b \in S^*$. We have $\frac{p}{b} = x\frac{c}{d}$, $c \in R$, $d \in S^*$ since $R_P \otimes R_{P^{'}}$ is S-divisible. Then, $pd = bxc$ implies $pd \in I$, and hence $d \in I$. Since $d \in S^*$, $d = mn$, $m \in R \setminus P$, $n \in R \setminus P^{'}$. Thus, we have $d = mn \in I$ which yields $m \in I$ or $n \in I$ and this is not possible. So, $d \in I$ is not true which shows $I \cap S = \emptyset$. □

Proposition 3.3 *Let S be a submonoid of S_2 and M an S-torsion R-module. The localization map $\phi_P : M \to M_P$ is surjective for every maximal ideal P of R.*

Proof The cokernel of the localization map, $M_P/Im\phi_P$, is isomorphic to $(R_P/R) \otimes_R M$. We show that $R_{P^{'}} \otimes_R (R_P/R) \otimes_R M = 0$ for all maximal ideals P'. If $P^{'} = P$, this is trivial since $R_P \otimes_P (R_P/R) = 0$. If $P^{'} \neq P$, then since $R_{P^{'}} \otimes_R (R_P/R) \otimes_R M \cong (R_P \otimes R_{P^{'}})/R_{P^{'}} \otimes_R M_{P^{'}} = 0$ by Proposition (3.2) and the fact that $M_{P^{'}}$ is S-torsion. □

Theorem 3.1 *Let S be a submonoid of R^*. For every S-torsion module M, there is a canonical isomorphism*

$$M \longrightarrow \bigoplus_{P \in MaxR} M_P$$

if and only if S is contained in $S_0 = S_1 \cap S_2$.

Proof Suppose S is contained in $S_0 = S_1 \cap S_2$. By Proposition (3.1) and Proposition (3.3), we conclude that every S-torsion module M is a subdirect sum of its localizations: $M \leq \oplus_{P \in MaxR} M_P$. To show that this is not a proper containment, we prove that localizations of the two sides at any $P' \in MaxR$ coincide. But this is obvious from the fact that $(M_P)_{P'} = M_P$ or 0 according as $P' = P$ or not. In fact $(M_P)_{P'} = M \otimes R_P \otimes R_{P'} = 0$ since M is S-torsion and $R_P \otimes R_{P'}$ is S-divisible by Proposition (3.2)(c).

Conversely, by hypothesis, $R_S/R \cong \oplus P(R_S/R)_P$ holds for the S-torsion module R_S/R. From Proposition (3.1) we obtain the inclusion $S \subseteq S_1$. Localizing both sides of the last isomorphism at a maximal ideal P', we deduce that $(R_S/R)_P \otimes_R R_{P'} = 0$ for $P' \neq P$. Thus $R_S \otimes_R R_P \otimes_R \otimes R_{P'} = R_P \otimes R_{P'}$, so the S-divisibility of $R_P \otimes_R R_{P'}$ is evident. The inclusion $S \subseteq S_2$ now follows from Proposition (3.2)(c). □

Theorem 3.2 *((Matlis)) For a domain R, the following conditions are equivalent:*

(a) *R is h-local;*

(b) *every torsion R-module M is canonically isomorphic to $\bigoplus_{P \in MaxR} M_P$;*

(c) *Q/R is canonically isomorphic to $\bigoplus_{P \in MaxR} (Q/R)_P$;*

(d) *$R_{[P]} \otimes R_P \cong Q$ for every maximal ideal P of R.*

Proof **(a)** $\Leftrightarrow$ **(b)** is an immediate consequence of the definition of h-local domains and Theorem (3.1).

(b) $\Rightarrow$ **(c)** is trivial since Q/R is torsion R-module.

(a) $\Rightarrow$ **(d)** Suppose R is h-local domain and let P be a maximal ideal of R. Let $A = \prod_{N \neq P} (Q/R)_N$ and $B = \bigoplus_{N \neq P} (Q/R)_N$, where N ranges over the maximal ideals of R different from M. Define $\varphi : Q \to A$ by $\varphi(x) =< x + R_N >$ for $x \in Q$. Now, $x = a/b$, where $a, b \in R$ and $b \neq 0$. Since b is contained in only a finitely many maximal ideals of R, we have $x \in R_N$ in only finitely many maximal ideals of R. Hence, we have $Im\varphi \subset B$. Since $Ker\varphi = R_{[P]}$, we have $Q/R_{[P]} \cong Im\varphi \subset B$. Thus, we have an exact sequence:

$$0 \to Q/R_{[P]} \to B.$$

Now, if N is a maximal ideal of R different from P, we have by Lemma (4.8) that $R_P \otimes R_N \cong Q$. Thus, $R_P \otimes (Q/R)_N \cong R_P \otimes (Q/R_N) \cong Q/R_P \otimes R_N = 0$ which yields $(Q/R)_P = 0$. Since $Q/R_{[P]} \subset B$, we have $R_{[P]} \otimes R_P \cong Q$.

(d) $\Rightarrow$ **(c)** Assume that $R_{[P]} \otimes R_P \cong Q$ for every maximal ideal P of R. Let M be a maximal ideal of R, and let $A = \sum_{N \neq M} R_{[N]}$, the sum of all of the $R_{[N]}$ for N a maximal ideal different from M. Then $R \subset A \subset R_M$, and hence $A_M = R_M$. On the other hand, by assumption, we have $A_N = Q$ for all $N \neq M$. Thus $A = \bigcap_{N \neq M} A_N \cap A_M = Q \cap R_M = R_M$. Consequently, we have $(R_M + R_{[M]})_N = (A + R_{[M]})_N = Q$ for all maximal ideals N of R including M. Therefore, $R_M + R_{[M]} = Q$. Since $R_M \cap R_{[M]} = R$, we have $(Q/R)_M = Q/R_M = (R_M + R_{[M]})/R_M \cong R_{[M]}/(R_M \cap R_{[M]}) = R_{[M]}/R$.

Now we have $Q = A + R_{[M]} = \sum_N R_{[N]}$, where the sum ranges over all maximal ideals N of R including M. If $N_1, \ldots, N_t$ is any finite set of maximal ideals of R different from M, then $R \subset (\sum_{i=1}^t R_{[N_i]} \cap R_{[M]}) \subset A \cap R_{[M]} = R$. Thus $\sum_{i=1}^t R_{[N_i]} \cap R_{[M]} = R$. From these facts it follows that $Q/R = \bigoplus (R_{[N]}/R)$, where N ranges over all maximal ideals of R. Since we have shown that $R_{[N]}/R \cong (Q/R)_N$, we have $Q/R \cong \bigoplus (Q/R)_N$.

(c) $\Rightarrow$ **(a)** Suppose $Q/R \cong \bigoplus_{P \in MaxR} (Q/R)_P$. Then $(Q/R)_P \otimes R_{P'} = 0$ for maximal ideals P, P' such that $P \neq P'$. Thus, $Q/R_P \otimes R_{P'} = Q/R_P \otimes R_{P'} = 0$ and so $R_P \otimes R_{P'} \cong Q$. The property (ii) of h-locality holds by Lemma (4.8), so the property (i) just follows from Proposition (3.1) taking $S = R^*$. □

Lemma 3.3 *Let R be an h-local domain and P a maximal ideal of R.*

(i) *If A_i $(i \in I)$ are submodules of Q with $\bigcap_{i \in I} A_i \neq 0$, then*

$$\Big(\bigcap_{i \in I} A_i\Big)_P = \bigcap_{i \in I} \Big(A_i\Big)_P.$$

(ii) *If $B \leq A$ are submodules of Q, then*

$$(A : B)_P = A_P : B_P.$$

Proof **(i)** Without loss of generality, $1 \in \bigcap_{i \in I} A_i$ may be assumed. Form the exact sequence

$$0 \longrightarrow \Big(\bigcap_{i \in I} A_i\Big)_P \longrightarrow Q \longrightarrow \Big(Q/\bigcap_{i \in I} A_i\Big)_P \longrightarrow 0.$$

Because of the structure of Q/R and the full invariance of its submodules $(Q/\cap_{i \in I})_P$ is just $Q/\bigcap_{i \in I}(A_i)_P$. Hence the claim is evident.

(ii) We start by observing that the claim is obvious if B is singly generated (even if it is finitely generated). The general case follows from (i):

$$\begin{aligned}(A : B)_P = (A : \Sigma_{b\in B} bR)_P = (\cap_{b\in B}(A : bR))_P &= \cap_{b\in B}(A_P : bR_P)\\ &= (A_P : \Sigma_{b\in B} bR_P) = (A_P : B_P)\end{aligned}$$

□

CHAPTER 4

KRULL-SCHMIDT PROPERTY FOR IDEALS AND MODULES OVER INTEGRAL DOMAINS

4.1. Krull-Schmidt and Pic(R)

In this chapter Krull-Schmidt properties for certain classes of indecomposable torsion-free modules over commutative integral domains are examined. All rings mentioned below are commutative with identity.

Definition 4.1 *Let R be a commutative integral domain and C a class of R-modules. The Krull-Schmidt property holds for C if, whenever*

$$G_1 \oplus G_2 \oplus \cdots G_n \cong H_1 \oplus H_2 \oplus \cdots H_m$$

for $G_i, H_j \in C$, then $n = m$ and, after a possible reindexing, $G_i \cong H_i$ for all $i \leq n$. If, instead of $G_i \cong H_i$, it is required only that there exists $k > 0$ such that $G_i^{(k)} \cong H_i^{(k)}$ for all i, then we say that the weak Krull-Schmidt property holds for C. ($G^{(k)}$ represents direct sum of k copies of a module G.)

Definition 4.2 *A torsionless module over a domain R is a submodule of a finitely generated free R-module.*

Definition 4.3 *An integral domain R has the torsion-free Krull-Schmidt property, TFKS, ,if the class of indecomposable torsionless R-modules has the Krull-Schmidt property; R has weak TFKS if this class has the weak Krull-Schmidt property.*

We also study the uniqueness of decomposition for ideals.

Definition 4.4 *A domain R has unique decomposition into ideals, UDI, if the class of ideals of R has the Krull-Schmidt property. Similarly, R has weak UDI if the class of ideals of R has the weak Krull-Schmidt property.*

Definition 4.5 *Two R-modules G and H are locally isomorphic if* $G_M \cong H_M$ *for all maximal ideals M of R, and G and H are power isomorphic if* $n > 0$ *exists such that* $G^{(n)} \cong H^{(n)}$. *If G is locally isomorphic to H,we write* $G \cong_l H$. *If G is power isomorphic to H, we write* $G \cong_\wp H$.

We use the notion of the Picard group of an integral domain to distinguish between weak UDI and UDI. If R is an integral domain, the ***Picard group*** of R is the abelian group consisting of the invertible fractional ideals of R modulo the principal fractional ideals of R. In the following sections we show that the Picard group of a weak UDI domain R measures how close R is to having UDI. We denote the Picard group of by $Pic(R)$.

For the purpose of proving some technical lemmas, we use the following notion.

Definition 4.6 *Let R be a domain and S an overring of R. Then* (R, S) *is a weak UDI pair if every overring T of R such that* $R \subseteq T \subseteq S$ *has weak UDI. Similarly,* (R, S) *is a weak TFKS pair if every overring of R contained in S has weak TFKS.*

Furthermore, we consider Krull-Schmidt for rank one modules, those torsion-free modules that are isomorphic to submodules of the quotient field.

Definition 4.7 *A maximal ideal M of an integral domain R is complemented if every ideal of R not contained in M is invertible.*

Definition 4.8 *Two ideals* I, J *are said to be comaximal (or coprime) if* $I + J = R$.

For comaximal ideals we have $IJ = I \cap J$. Clearly, two ideals I, J are comaximal if and only there exists $x \in I$ and $y \in J$ such that $x + y = 1$.

Remark 4.1 *Let R be an integral domain and suppose that every prime ideal of R is invertible. Then R is of Krull dimension 1. In fact, let P be a nonzero prime ideal of R, then there is a maximal ideal M of R that contains P. Note that M is invertible by assumption. Consequently* $M^{-1}P$ *is a fractional ideal of R with* $M^{-1}P \subset M^{-1}M = R$, *so* $M^{-1}P$ *is an ideal in R. Since* $M(M^{-1}P) = RP = P$ *and P is prime; either* $M \subset P$ *or* $M^{-1}P \subset P$. *But if* $M^{-1}P \subset P$, *then* $R \subset M^{-1} = M^{-1}R = M^{-1}PP^{-1} \subset PP^{-1} \subset R$, *so* $M^{-1} = R$. *Thus* $R = MM^{-1} = MR = M$, *which contradicts the fact that M is maximal. Therefore* $M \subset P$ *and so* $M = P$. *Hence, P is maximal.*

Lemma 4.1 *((Goeters & Olberding, 2002), Lemma 2.1) Let R be an h-local domain with a complemented maximal ideal M.*

(i) *If P and Q are comaximal prime ideals, then P or Q is a maximal ideal.*

(ii) *For all maximal ideals $N \neq M$ of R, then R_N is a DVR.*

(iii) *If S is a fractional overring of R and $N \neq M$ is a maximal ideal of R, then SN is a maximal ideal of S.*

(iv) *$Pic(R) = 0$ if and only if every maximal ideal of R distinct from M is principal.*

Proof *(i)* Suppose P and Q are comaximal prime ideals, then

$$P + Q = R.$$

If P and Q are not maximal, they are contained in some maximal ideal. Since they are comaximal, they must be contained in different maximal ideals, say $P \subseteq N_1$ and $Q \subseteq N_2$, $N_1 \neq N_2$. If $N_1 \neq M$, then P is invertible because R is h-local. Similarly, if $N_2 \neq M$, then Q is invertible. Since $N_1 = N_2$ is impossible, without loss of generality, we may assume P is invertible prime ideal and $P \subseteq N_1 = N$, $N \neq M$.

Now consider R_N. For the reason that every prime ideal of R_N is invertible, R_N is of Krull dimension 1 by Remark 4.1. Since PR_N is a prime ideal in R_N and NR_N is the unique maximal ideal of R_N, we get $PR_N = NR_N$. Thus we obtain $N = NR_N \cap R = PR_N \cap R = P$, and hence $P = N$. So, P must be maximal.

(ii) By the proof of (i), every ideal of R_N, where $N \neq M$, is invertible. So R_N is a local ring such that every ideal of R_N is invertible, and so R_N is a DVR.

(iii) If $N \neq M$ is a maximal ideal of R, by (i), R_N is a DVR, so since S_N is a fractional overring of R_N, $R_N = S_N$. Then $SN_N = N_N$ and $S_N/SN_N = R_N/N_N$ so SN_N is a maximal ideal of S_N. By local verification, we see that $SN = SN_N \cap S$. Thus, SN is a maximal ideal of S.

(iv) Suppose $Pic(R) = 0$. Let N be a maximal ideal distinct from M. Then since $N \nsubseteq M$, N is invertible, and by assumption N must be principal.

For the converse, suppose every maximal ideal of R distinct from M is principal. Let I be an invertible ideal of R. Since I is invertible, there exists a fractional ideal J of R such that $IJ = R$. This implies that for some nonzero $x \in J$, $xI \nsubseteq M$. Since $xI \cong I$, we may assume without loss of generality $I \nsubseteq M$. Since R is h-local, I is contained in at most finitely many maximal ideals, say $N_1, \ldots, N_k$. So, for each $i \leq k$, R_{N_i} is a DVR which implies

$$IR_{N_i} = N_i^{j(i)} R_{N_i}$$

for some $j(i) > 0$. Local verification shows that

$$I = N_1^{j(1)} \cdots N_k^{j(k)}$$

Thus, I is principal. $\square$

Remark 4.2 ***(Splitting map)*** *Let $f : M \to N$ and $f' : N \to M$ be homomorphisms such that $ff' = 1_N$. Then f is an epimorphism, f' is a monomorphism and $M = Ker(f) \oplus Im(f')$. We say f is split by the map f', in this case.*

Lemma 4.2 *((Goeters & Olberding, 2002), Lemma 2.2) Let S be an overring of an integral domain R. If (R, S) is a weak UDI pair and I and J are comaximal ideals of S, then $S = R + I$ or $S = R + J$.*

Proof If either $I = S$ or $J = S$, then the claim is clear. If S is quasilocal, then $I + J \neq S$. So, suppose S is not quasilocal and $I, J \neq S$. Observe that

$$S = I + J \subseteq [R + I : S] + [R + J : S].$$

Since $1 \in S$, $1 \in [R + I : S] + [R + J : S]$. Thus, there exists $a \in [R + I : S]$, $b \in [R + J : S]$ such that $a + b = 1$. Define a homomorphism

$$\Phi : (R + I) \oplus (R + S) \to S$$

by $\Phi((x, y)) = x + y$ for all $x \in R + I, y \in R + J$. Define a homomorphism

$$\Psi : S \to (R + I) \oplus (R + S)$$

by $\Psi(s) = (as, bs)$ for all $s \in S$. Since $\Phi(\Psi(s)) = \Phi((as, bs)) = as + bs = s$, Ψ is a splitting map for Φ. So, by splitting map property we can write

$$(R + I) \oplus (R + J) \cong S \oplus Ker\Phi.$$

Let $T := (R + I) \cap (R + J)$, then T is an overring of R such that $R \subset T \subset S$. Since T has weak UDI and S is a fractional ideal of T, $S^{(n)} \cong (R + I)^{(n)}$ or $S^{(n)} \cong (R + J)^{(n)}$ for some $n > 0$. Taking the nth exterior power of each side with respect to $R + I$ and $R + J$, respectively, yields $S \cong R+I$ or $S \cong R+J$ since $R+I$ and $R+J$ are subrings of S. Finally, isomorphism can be replaced by equality, again since $R + I$ and $R + J$ are rings. □

Lemma 4.3 *((Goeters & Olberding, 2002), Lemma 2.3) Let R be an integral domain. If S is an overring of R such that (R, S) is a weak UDI pair, then S is quasilocal or $S = R+N$ for some maximal ideal N of S. Furthermore, for each maximal ideal M of R, there are at most three maximal ideals of S lying over M, and if (R, S) is a weak TFKS pair, then there are at most two maximal ideals of S lying over M.*

Proof If S has at least two distinct maximal ideals N and N', then since they are comaximal, by Lemma (4.2), $S = R + N$ or $S = R + N'$. Suppose that there are four distinct maximal ideals N_1, N_2, N_3, N_4 of S lying over a maximal ideal M of R. Define $I = N_1N_2$ and $J = N_3N_4$. Since I and J are comaximal ideals of S, then again by Lemma (4.2), without loss of generality we can assume $S = R + I$. There exists $n \in N_1$ such that $n \notin I$ since $I = N_1N_2 \neq N_1$. By using $S = R + I$ we can write $n = r + i$ for some $r \in R$ and $i \in I$. Since $n - i \in R$ and $n - i \in N_1$, we have $n - i \in M$ because $N_1 \cap R = M$. Thus, $n \in I + M = I$, which is a contradiction. So, there cannot be more than three maximal ideals of S lying over M.

Now assume (R, S) is a weak TFKS pair and S has three distinct maximal ideals, N_1, N_2, N_3. Define $I_1 = N_2N_3$, $I_2 = N_1N_3$ and $I_3 = N_1N_2$. Then since $S = I_1 + I_2 + I_3 \subseteq [R + I_1 : S] + [R + I_2 : S] + [R + I_3 : S]$, there exists $u_i \in [R + I_i : S], i = 1, 2, 3$ such that $u_1 + u_2 + u_3 = 1$. Define a homomorphism

$$\sigma : (R + I_1) \oplus (R + I_2) \oplus (R + I_3) \to S$$

by $\sigma((a, b, c)) = a + b + c$. This homomorphism is split by the map

$$\varphi : S \to (R + I_1) \oplus (R + I_2) \oplus (R + I_3)$$

where $\varphi(s) = (su_1, su_2, su_3)$ for all $s \in S$. Setting $T = (R + I_1) \cap (R + I_2) \cap (R + I_3)$, we get an overring of R. Since $R \subset T \subset S$ and (R, S) is a weak TFKS pair and Ker(σ) is a torsionless T-module, and the $R + I_j$ are fractional ideals of T, it follows that $S \cong_{\wp} R + I_i$

for some $i = 1, 2, 3$ which implies $S = R + I_i$. Now, applying a similar argument to the one above, we get a contradiction which implies S cannot have more than two maximal ideals lying over M.

□

Lemma 4.4 *((Goeters & Olberding, 2002), Lemma 2.4) If R has weak UDI, then R has a complemented maximal ideal M and the Picard group of R is torsion. If R has UDI, then* $Pic(R) = 0$.

Proof Let I be an invertible ideal of R. Then since I is a projective R-module, it is a summand of a free R-module. So, $I^{(n)} \cong R^{(n)}$ for some $n > 0$. Taking the nth exterior power of both sides yields $I^n \cong R$. Thus, for every $IJ \in Pic(R)$ where I is an invertible ideal and J is a principal ideal, we can write $(I^{n-1})(IJ) \cong RJ$ for some $n > 0$. Since RJ is also a principal ideal, we conclude that $Pic(R)$ is torsion. If R has UDI, by a similar consideration, I is principal which implies $Pic(R) = 0$.

It remains to show that R has a complemented maximal ideal when R has weak UDI. If R is Dedekind domain, then every ideal of R is invertible, so every maximal ideal is complemented. If R is quasilocal, then the claim is clear. So, suppose R is not Dedekind and R is not quasilocal. If I and J are comaximal ideals of R, then since $I + J = R$, it follows that $I \oplus J \cong R \oplus (I \cap J)$. Since R has weak UDI, there exists $n > 0$ such that $I^{(n)} \cong R$ or $J^{(n)} \cong R$. Thus, $I^{(n)}$ or $J^{(n)}$ is free R-module which implies I or J is projective R-module, and hence invertible ideal of R.

Now, let A be the sum of all noninvertible ideals of R. By assumption, R is not Dedekind, so there is at least one noninvertible ideal. We have showed that $I + J = R$ implies I or J is invertible. So, $A = R$ is impossible. Thus, it follows that a maximal ideal M containing A exists. If B is an ideal of R such that B is not contained in M, then B is not contained in A. Thus, B must be invertible, and M is complemented maximal ideal. □

Lemma 4.5 *((Goeters & Olberding, 2002), Lemma 2.5) Let R be an h-local domain with complemented maximal ideal M, and suppose S is an overring of R and* (R_M, S_M) *is a weak UDI pair. If B is an invertible fractional ideal of S, then* $B = SA$ *for some invertible fractional ideal A of R.*

Proof First, we will show that S_M has at most four maximal ideals. If S_M is quasilocal, then the claim is clear, so suppose S_M has at least two maximal ideals. If N is a maximal ideal of S_M such that $S_M = R_M + N$, then $S_M/N \cong R_M/(R_M \cap N)$ which shows that $R_M \cap N$ is a maximal ideal of R_M. Thus, $MR_M = R_M \cap N$, and so N lies over the maximal ideal M of R. By Lemma (4.3), there are at most three maximal ideals of S_M lying over M. If L is a maximal ideal of S_M such that $S_M \neq R_M + L$, then by Lemma (4.3), $S_M = R_M + N$ for every maximal ideal N of S_M distinct from L. Thus, all the maximal ideals of S_M except possibly one contract to maximal ideals of R_M. Therefore, S_M has at most four maximal ideals. Since B is an invertible fractional ideal of S, BS_M is an invertible fractional ideal of S_M. It follows that $BS_M = aS_M$ for some $a \in B$ because S_M is semilocal.

Now let M_α denote the set of all maximal ideals of R such that $M_\alpha \neq M$ and $S_{M_\alpha} \neq Q$ where Q is the quotient field of R. By Lemma (4.1) (ii), R_{M_α} is a DVR, so $S_{M_\alpha} = R_{M_\alpha}$. So for all α,

$$BR_{M_\alpha} = a_\alpha R_{M_\alpha}.$$

Define

$$A = aR_M \cap (\cap_\alpha a_\alpha R_{M_\alpha}) \cap (\cap_N R_N),$$

where N ranges over the maximal ideals of R not in $\{M_\alpha\} \cup \{M\}$. Since R is h-local, $R_{M_\alpha} R_N = Q$ if N is a maximal ideal of R distinct from a particular M_α. Also, since R is h-local, localizations commute with infinite intersections, so $AR_M = aR_M, AR_{M_\alpha} = a_\alpha R_{M_\alpha}$ for all α and $AR_N = R_N$ for all maximal ideals N of R not in $\{M_\alpha\} \cup \{M\}$. Furthermore, $SA = B$ and A is a locally free R-submodule of Q. In fact, since B is a finitely generated S-submodule of Q and and every nonzero element of R is contained in at most finitely many maximal ideals of R, it follows that $BR_{M_\alpha} = a_\alpha R_{M_\alpha}$ for all but finitely many α. This implies that A is a fractional ideal of R. Since R is h-local, A is a finitely generated fractional ideal (**(Matlis), Theorem 26**). Hence A is an invertible fractional ideal of R. □

Lemma 4.6 *((Goeters & Olberding, 2002), Lemma 2.6) Let R be an h-local domain, and let X and Y be rank one R-modules such that $(R, E(X))$ is a weak UDI pair. Then $X \cong_\wp Y$ if and only if $XA = Y$ for some invertible fractional ideal A of R.*

Proof Suppose $X^{(n)} \cong Y^{(n)}$. Then the canonical homomorphism, $X \otimes_R Hom_R(X, Y) \to Y$ is surjective, and it follows that $X[Y : X] = Y$. The existence of a splitting map for the induced surjection $X^{(n)} \to Y$ shows that $1 \in [Y : X][X : Y] \subseteq E(X)$ where $E(X)$ denotes the R-module $[X : X]$, and $E(X) \cong [X : X]$. In particular, $[Y : X][X : Y] = E(X)$, and it follows that $[Y : X]$ is an invertible fractional ideal of $E(X)$. (Indeed, if $q \in [X : Y] \cap [Y : X]$, then $q[Y : X] \subseteq E(X)$ and $q[X : Y] \subseteq E(X)$.) Set $B := [Y : X]$ and $S := E(X)$. By Lemma (4.5), a fractional invertible ideal A of R exists such that $SA = B$. Thus $XA = XSA = XB = Y$.

Conversely, suppose $XA = Y$ for some invertible fractional ideal of R. By Lemma (4.4), $Pic(R)$ is torsion, so $A^n \cong R$ for some $n > 0$. It follows that $A^{(n)} \cong R^{(n)}$. Since A and

R are flat R-modules, it follows that $Y^{(n)} \cong (XA)^{(n)} \cong (X \otimes_R A)^{(n)} \cong X \otimes_R A^{(n)} \cong X \otimes_R R^{(n)} \cong (X \otimes_R R)^{(n)} \cong X^{(n)}$, and the claim is proved. □

Theorem 4.1 *((Goeters & Olberding, 2002), Theorem 2.7) An h-local integral domain R has UDI if and only if R has weak UDI and $Pic(R) = 0$.*

Proof Suppose R has weak UDI, $Pic(R) = 0$ and $I_1 \oplus I_2 \oplus I_n \cong J_1 \oplus J_2 \oplus J_n$ for some ideals $I_1, I_2 \ldots I_n, J_1 \ldots J_n$ of R. Since R has weak UDI, after reindexing, we may assume that $I_k \cong_{\wp} J_k$ for each $k \leq n$. Since $E(I_k)$ is a fractional ideal of R for all $k \leq n$, it follows that $(R, E(I_k))$ is a weak UDI pair. By Lemma (4.6), $I_k = AJ_k$ for some invertible fractional ideal A of R. By assumption, since $Pic(R) = 0$, A is a principal ideal of R, so $I_k \cong J_k$ which implies R has UDI. The converse is clear from Lemma (4.4). □

4.2. Main Reductions

In this section, reduction theorems for the various Krull-Schmidt properties are proven.

Lemma 4.7 *((Goeters & Olberding), Proposition 2.8, Theorems 2.11 and 2.13) Let R be an h-local integral domain and G and H be torsionless R-modules.*

- ***(i)*** *If $Pic(R) = 0$, then $G \cong_l H$ if and only if $G \oplus R \cong H \oplus R$.*
- ***(ii)*** *If $Pic(R) = 0$, $G \cong_l H$ and G has a summand isomorphic to an ideal of R, then $G \cong H$.*
- ***(iii)*** *If $Pic(R)$ is torsion and $G \cong_l H$, then $G \cong_{\wp} H$.*

Lemma 4.8 *Let R be an h-local domain with a complemented maximal ideal. If $G \cong_l H$, then G is indecomposable if and only if H is indecomposable.*

Lemma 4.9 *Let R be an h-local domain with complemented maximal ideal M. If $G := G_1 \oplus \cdots G_n$ and $H := H_1 \oplus \cdots H_m$ are direct sums of torsionless R_M-modules such that $G \cong H$, then torsionless R-modules $G' := G'_1 \oplus \cdots G'_n$ and $H' := H'_1 \oplus \cdots H'_m$ exist such that $G' \cong_l H'$ and, for all $i \leq n$, $j \leq m$, $G_i = (G'_i)_M$ and $H_j = (H'_j)_M$.*

Proof We may assume that for each for each $i \leq n$ and $j \leq m$, free R-modules exist such that $G_i \subseteq (E_i)_M$ and $H_j \subseteq (F_j)_M$ since G_i and H_j are torsionless R_M-modules. In fact, $G_i \subseteq R_M^{(k(i))}$ and $H_j \subseteq R_M^{(t(j))}$ for some $k(i) > 0$, $t(j) > 0$ and $E_i = R^{k(i)}$, $H_j = R^{t(j)}$. Define

$G'_i = G_i \cap E_i$ and $H'_j = G_i \cap F_j$ for each $i \leq n$ and $j \leq m$, then G'_i and H'_j are torsionless R-modules. Then, for all i, j, $G_i = (G'_i)_M$ and $H_j = (H'_j)_M$. By Lemma (4.1)(ii), R_N is a DVR for each maximal ideal N distinct from M. Thus, the torsionless R_N-modules are free of the same rank for all maximal ideals $N \neq M$. It follows that $G' \cong_l H'$. □

In the proof of the next theorem (v), we use the following terminology from the theory of torsion-free abelian groups. Two rank one modules are *quasi-isomorphic* if each is isomorphic to a submodule of the other. A *type* τ is the quasi-isomorphism class of a rank one module. The collection of types is partially ordered by the relation $\tau_1 \leq \tau_2$ whenever U_1 is isomorphic to a submodule of U_2 (where τ_i is the type of the R-module U_i). If A is a torsion-free module and $a \in A$, then the *type* of a is the quasi-isomorphism class of the pure submodule of A generated by a, i.e., the submodule $\{b \in A : rb = sa \text{ for some } r, s \in R\}$. Given a type τ, define $A(\tau) = \{a \in A : \text{type of } a \geq \tau\}$.

Theorem 4.2 *((Goeters & Olberding, 2002), Theorem 3.4) Let R be an h-local domain. The following statements hold for R.*

- ***(i)*** *R has weak UDI if and only if* $Pic(R)$ *is torsion and R has a complemented maximal ideal M such that* R_M *has UDI.*
- ***(ii)*** *R has UDI if and only if* $Pic(R) = 0$ *and R has a complemented maximal ideal M such that* R_M *has UDI.*
- ***(iii)*** *R has weak TFKS if and only if* $Pic(R)$ *is torsion and R has a complemented maximal ideal M such that* R_M *has weak TFKS.*
- ***(vi)*** *R has TFKS if and only if locally isomorphic torsionless modules are isomorphic and R has a complemented maximal ideal M such that* R_M *has TFKS.*
- ***(v)*** *R has the Krull-Schmidt property for rank one modules if and only if* $Pic(R) = 0$ *and R has a complemented maximal ideal M such that* R_M *has the Krull-Schmidt property for rank one modules.*

Proof **(i)** Suppose first that R has weak UDI. By Lemma (4.4), $Pic(R)$ is torsion and a complemented ideal M exists. Assume that

$$I_1R_M \oplus I_2R_M \oplus \cdots I_nR_M \cong J_1R_M \oplus J_2R_M \oplus \cdots J_nR_M,$$

where I_i, J_k are ideals of R for $i \leq n, k \leq m$.

Define $G := I_1R_M \oplus I_2R_M \oplus \cdots I_nR_M$ and $H := J_1R_M \oplus J_2R_M \oplus \cdots J_nR_M$. Then G and H are completely decomposable torsionless R_M-modules such that $G \cong H$. By Lemma (4.9), torsionless R-modules $G' = I'_1 \oplus I'_2 \ldots \oplus I'_n$ and $H' = H'_1 \oplus H'_2 \ldots \oplus H'_m$ exist such that $G' \cong_l H'$ and $I_iR_M = (I'_i)_M$, $J_iR_M = (J'_k)_M$ for each $i \leq n, j \leq m$. By the proof of Lemma (4.9), I'_i's and J'_k's are chosen as $I'_i = I_iR_M \cap R = I_i$, $J'_k = J_kR_M \cap R = J_k$. So, $I_1 \oplus I_2 \oplus \cdots \oplus I_n \cong J_1 \oplus J_2 \oplus \cdots \oplus J_m$. Since R has weak UDI, $m = n$ and after reindexing $I_i^{(k)} \cong J_i^{(k)}$ for some $k > 0$ which implies $(I_iR_M)^{(k)} \cong (J_iR_M)^{(k)}$. Thus, R_M has weak UDI and since R_M is local, R_M has UDI.

To prove the converse, assume that M is a complemented maximal ideal of R, R_M has UDI and the Picard group of R is torsion. Suppose that

$$I_1 \oplus I_2 \oplus \cdots \oplus I_n \cong J_1 \oplus J_2 \oplus \cdots \oplus J_m,$$

where I_i, J_k are ideals of R and $i = 1, 2, \ldots, n, j = 1, 2, \ldots, m$. Define $G := I_1 \oplus I_2 \oplus \cdots \oplus I_n$ and $H := J_1 \oplus J_2 \oplus \cdots \oplus J_m$, then G and H are torsionless R-modules such that $G \cong H$. Since R_M has UDI, $n = m$ and after reindexing $(I_j)_M \cong (J_j)_M$ for each j. For each maximal ideal N distinct from M, R_N is a DVR, so $(I_j)_N \cong (J_j)_N$ which implies $I_j \cong_l J_j$. By Lemma (4.7), $I_j \cong_\wp J_j$ which shows that R has weak UDI.

(ii) Suppose R has UDI, by Lemma (4.4), $Pic(R) = 0$ and a complemented maximal ideal M exists. By (i), R_M has UDI. Conversely, suppose R has a complemented maximal ideal M such that R_M has UDI and $Pic(R) = 0$. By (1), R has weak UDI and by Theorem (4.1) R has UDI.

(iii) Suppose R has weak TFKS. Since ideals of R are torsionless R-modules, R has weak UDI. By Lemma (4.4), R has a complemented maximal ideal M and the Picard group of R is torsion. We need to show that R_M has weak TFKS. Assume that

$$G_1 \oplus G_2 \oplus \cdots \oplus G_n \cong H_1 \oplus H_2 \oplus \cdots \oplus H_m$$

where G_i, H_j are torsionless R_M-modules and $i = 1, 2, \ldots, n, j = 1, 2, \ldots, m$. Let $G := G_1 \oplus G_2 \oplus \cdots \oplus G_n$ and $H := H_1 \oplus H_2 \oplus \cdots \oplus H_m$ be direct sum of torsionless R_M-modules. Then since $G \cong H$, by Lemma (4.9), there exist torsionless R-modules $G' = G'_1 \oplus G'_2 \oplus \cdots \oplus G'_n$ and $H' = H'_1 \oplus H'_2 \oplus \cdots \oplus H'_m$ such that $G' \cong_l H'$ and $(G'_i)_M = G_i$,

$(H'_j)_M = H_j$. By Lemma (4.7), $G' \cong_\wp H'$. Since R has weak TFKS, we have $n = m$ and after reindexing, $G'_i \cong_p H'_i$ for all i. So, $(G'_i)_M \cong_\wp (H'_i)_M$ which yields $G_i \cong_\wp H_i$. Thus, R_M has weak TFKS.

For the converse, suppose R has complemented maximal ideal M, R_M has weak TFKS and the Picard group of R is torsion. Let $G_1, G_2, \cdots, G_n, H_1, H_2, \cdots, H_m$ be indecomposable torsionless R-modules such that $G_1 \oplus G_2 \oplus \cdots G_n \cong H_1 \oplus H_2 \oplus \cdots H_m$. Passing to R_M, each $(G_i)_M$ and $(H_i)_M$ remains indecomposable because by Lemma (4.8) if G is an indecomposable torsionless R-module, then G_M is indecomposable. Thus, the assumption that R_M has weak TFKS implies that $n = m$ and after reindexing $(G_i)_M \cong_\wp (H_i)_M$ for all $i \leq n$. Since R is h-local domain, R_N is a DVR for each maximal ideal $N \neq M$, $(G_i)_N$ and $(H_i)_N$ are free R_N-modules of the same rank. Thus, $G^{(k)} \cong_l H^{(k)}$ for some $k > 0$, and by Lemma (4.7), $G_i \cong_p H_i$ for all $i \leq n$. Hence, R has weak TFKS.

(iv) Suppose R has TFKS and G and H are locally isomorphic R-modules. Since ideals of R are torsionless R-modules, R has UDI. By *(ii)*, $Pic(R) = 0$. So, by Lemma (4.7), $G \oplus R \cong H \oplus R$. Since R has TFKS, $G \cong H$ and we conclude that locally isomorphic torsionless R-modules are isomorphic. By (ii), R has a complemented maximal ideal M. Suppose $G_1 \oplus G_2 \oplus \cdots G_n \cong H_1 \oplus H_2 \oplus \cdots H_m$ for indecomposable torsionless R_M-modules G_i and H_j. Set $G = G_1 \oplus G_2 \oplus \cdots G_n$ and $H = H_1 \oplus H_2 \oplus \cdots H_m$. By Lemma (4.9), torsionless R-modules $G' := G'_1 \oplus \cdots G'_n$ and $H' := H'_1 \oplus \cdots H'_m$ exist such that $G' \cong_l H'$ and $G_i = (G_i)_M$ and $H_j = (H'_j)_M$. So, $G'_1 \oplus G'_2 \oplus \cdots G'_n \cong H'_1 \oplus H'_2 \oplus \cdots H'_m$. Since R has TFKS, we have $n = m$ and after reindexing, $G'_i \cong H'_i$ for all $i \leq n$. Thus, $(G'_i)_M \cong (H'_i)_M$ which shows $G_i \cong H_i$ for all i. Hence, R_M has TFKS.

Conversely, suppose that locally isomorphic torsionless modules are isomorphic and R has a complemented maximal ideal M such that R_M has TFKS. Let I be an invertible ideal of R. Then there exists a fractional ideal J of R such that $IJ = R$. There exists an element $x \in J$ such that $xI \nsubseteq M$. Since $xI \cong I$, without loss of generality we may assume that $I \nsubseteq M$. So, $IR_M = R_M$. Since R is h-local, for every maximal ideal $N \neq M$, $IR_N \cong R_N$. Thus, $I \cong_l R$ and I is principal which yields $Pic(R) = 0$. By (iii), R has weak TFKS. It sufficies to check that if $G \cong_\wp H$ for torsionless indecomposable R-modules G and H, then $G \cong H$. By Lemma (4.8), G_M and H_M are indecomposable R_M-modules. Since R_M has TFKS, $G_M \cong H_M$. Moreover, since R_N is a DVR for all maximal ideals $N \neq M$, $G_N \cong H_N$. So, $G \cong_l H$. By assumption, $G \cong H$.

(v) If R has the Krull-Schmidt property for rank one modules, then, by Lemma (4.4),

$Pic(R) = 0$ and R has a complemented maximal ideal M. Since rank one R_M-modules are rank one R-modules, R_M has the Krull-Schmidt property for rank one modules.

Conversely, suppose M is a complemented maximal ideal of R, R_M has the Krull-Schmidt property for rank one modules and $Pic(R) = 0$. The proof is modeled on a classical proof of a theorem of Baer for abelian groups as in (**(Goeters & Olberding, 2001), Theorem 4.3**). Suppose $G := X_1 \oplus \cdots \oplus X_n$ and $H := Y_1 \oplus \cdots Y_n$ are direct sums rank one R-modules X_i and Y_j such that $G \cong H$. Since types are preserved under isomorphism, $G(\tau) \cong H(\tau)$ and $G/G(\tau) \cong H/H(\tau)$ for all types τ. Select a type τ that is maximal with respect to the types of X_i and Y_j. Then $G(\tau) \cong H(\tau)$ and, without loss of generality, we may assume $X_1 \oplus \cdots \oplus X_k \cong Y_1 \oplus \cdots \oplus Y_k$, where $k \leq n$ and each X_i and Y_i has type τ. After reindexing, we may assume $(X_i)_M \cong (Y_i)_M$ for all $i \leq k$. Moreover, since the X_i and Y_i have the same type and R_N is a DVR for all $N \neq M$, it follows that $(X_i)_N \cong (Y_i)_N$ for all $N \neq M$. Thus $X_i \cong_l Y_i$ for all $i \leq k$ and X_i and Y_i are quasi-isomorphic rank one modules. In particular, $[X_i : Y_i][Y_i : X_i] = E(Y_i)$, since localizations commute with brackets of quasi-isomorphic rank one modules over h-local domains (see (Fuchs & Salce, 2001), Lemma IV.3.10). Set $B = [X_i : Y_i]$ and $S = E(Y_i)$. Then B is an invertible fractional ideal of S, and (R_M, S_M) is a weak UDI pair, so by Lemma (4.5), $B = SA$ for some invertible fractional ideal A of R. Since $Pic(R) = 0$, B is a principal ideal of S, and it follows that $X_i \cong Y_i$ for all $i \leq k$. Since $G/G(\tau) \cong H/H(\tau)$, an inductive argument completes the proof that R has the Krull-Schmidt property for rank one modules. □

4.3. Non-Noetherian case

In this section, we treat the non-Noetherian case of UDI and TFKS, with special emphasis on the Prüfer case. First, we need to show R is a quasilocal Prüfer domain if and only if R is a valuation domain. Let R be a quasilocal Prüfer domain and M be its unique maximal ideal. Taking any two ideals I_1, I_2 of R, we see that I_1R_M and I_2R_M are two ideals of R_M. Since R is Prüfer domain, R_M is a valuation domain and so $I_1R_M \subseteq I_2R_M$ or $I_2R_M \subseteq I_1R_M$ which implies $I_1 \subseteq I_2$ or $I_2 \subseteq I_1$. Hence R is a valuation domain. Conversely, assume R is a valuation domain. Then clearly, R is quasiolocal. Since finitely generated ideals are principal and hence invertible in a valuation domain, R is a Prüfer domain. Secondly, we need to show all quasilocal Prüfer domains(=valuation domains) have UDI. Let R be a valuation domain, and suppose $I_1 \oplus I_2 \oplus \cdots \oplus I_n \cong J_1 \oplus J_2 \oplus \cdots \oplus J_n$, where I_i, J_k are ideals of R. For any ideal of I of R, I is rank one R-module and $End(I) = [I : I]$ is an overring of R. Since R is a valuation domain, $[I : I] = R_L$ for some prime ideal L of R and

and hence $[I : I]$ is local. So, all ideals of R have local endomorphism rings which yields R has UDI. Moreover, since the rank one modules of a valuation domain are isomorphic to ideals, the equivalence of UDI and Krull-Schmidt for rank one modules is immediate.

Definition 4.9 *A ring is a Bezout ring if its finitely generated ideals are principal.*

Proposition 4.1 *The following statements hold for R, an h-local Prüfer domain.*

(i) *R has UDI if and only if R has the Krull-Schmidt property for rank one modules; if and only if R is a Bezout domain with complemented maximal ideal.*

(ii) *R has weak UDI if and only if $Pic(R)$ is torsion and R has a complemented maximal ideal.*

Proof **(i)** Suppose R has UDI, then R has a complemented maximal ideal by Theorem (4.2). Let I be a finitely generated ideal of R. Since R is Prüfer domain, I is invertible, and so I must be principal because $Pic(R) = 0$. So, R is a B$\acute{e}$zout domain. Conversely, suppose R is a B$\acute{e}$zout domain with complemented maximal ideal M. We need to show that R_M has UDI and $Pic(R) = 0$. Suppose $I_1 \oplus \cdots \oplus I_n \cong J_1 \oplus \cong \oplus J_m$ where I_i, J_k are ideals of R_M. Since $End(I)$ is an overring of R_M and R_M is a valuation domain, $End(I)$ is a local ring for any ideal I of R_M. Thus, R_M has UDI. Let I be an invertible fractional ideal of R. Then since R is a Bezout domain, I must be principal. So, $Pic(R) = 0$. Now, assume that R has the Krull-Schmidt property for rank one modules. Since ideals are also rank one modules, R has UDI. For the converse, suppose R has UDI, then R has a complemented maximal ideal M and $Pic(R) = 0$ by Theorem (4.2). We need to show that R_M has Krull-Schmidt property for rank one modules which is equivalent to showing that R_M has UDI. By the same consideration, $End(I)$ is local for any ideal of R_M which yields R_M has UDI. **(ii)** By arguments similar to those in the proof of (i) and by Theorem (4.2), the result follows.

□

Combining the Prüfer description of UDI with results on decompositions of torsion-free modules, we obtain characterizations of some strong forms of the Krull-Schmidt property. By way of application, we are interested in some of the following variations of the Krull-Schmidt property. In order to characterize these properties, we recall several related notions. A Prüfer domain R satisfies ($\sharp\sharp$) if every prime ideal of R is the radical of a finitely generated ideal of R (see **(Gilmer & Heinzer, 1967),Theorem 3**). In particular, if every nonzero ideal of a Prüfer domain R is contained in at most finitely many maximal ideals of R, then R satisfies ($\sharp\sharp$). A Prüfer domain R is h-local if and only if R

satisfies ($\sharp\sharp$) and each nonzero prime ideal of R is contained in a unique maximal ideal of R (**(Olberding, 1998)**).

We also need the notion of an almost maximal ring, that is, a ring for which R/I is a linearly compact R-module for all nonzero ideals I. A Prüfer domain R is almost maximal if and only if Q/R is an injective R-module and R is h-local (see **(Brandal, 1973)**).

Definition 4.10 *A D-ring is an integral domain R for which every torsion-free finite rank R-module decomposes into a direct sum of rank one R-modules.*

Proposition 4.2 *((Goeters & Olberding, 2002), Proposition 5.2) Let R be an integral domain.*

(i) *R is an h-local Bezout domain with a complemented maximal ideal if and only if, for each R-module $G := I_1 \oplus \cdots I_n$, that is a direct sum of ideals of R, every pure submodule of G is a summand of G that is isomorphic to a direct sum of the I_js.*

(ii) *R is an almost maximal Bezout domain with complemented maximal ideal if and only if, for each torsionless R-module G, $G \cong I_1 \oplus \cdots I_n$ for some ideals I_j of R, and every pure submodule of G is a summand of G that is isomorphic to a direct sum of the I_js.*

(iii) *R is an almost maximal Bezout domain with complemented maximal ideal if and only if R is a Prüfer ($\sharp\sharp$) domain such that every torsionless R-module decomposes uniquely, up to isomorphism, into a direct sum of rank one modules.*

(iv) *R is a quasilocal D-ring if and only if every torsion-free finite rank R-module decomposes uniquely, up to isomorphism, into a direct sum of rank one R-modules.*

Proof **(i)** An integral domain R is an h-local Prüfer domain if and only if pure submodules of completely decomposable torsionless R-modules are summands (**(Olberding, 1999)**, Theorem 3.2). This implies h-local Prüfer domains have the property that pure submodules of completely decomposable torsionless modules are completely decomposable. Thus, the asserted property holds if and only if R is an h-local Prüfer domain with UDI.

(ii) Suppose R is an almost maximal Bezout domain with complemented maximal ideal. Let G be a torsionless R-module. Since a Prüfer domain R is almost maximal if and only if R is h-local and every torsionless R-module is completely decomposable ($*$) **(Fuchs &**

Salce, 2001), G is completely decomposable. Also, since R is an h-local Bezout domain, by using (i) the result follows. For the converse, we just use $(*)$ and (i).

(iii) Assume torsionless R-modules decompose uniquely into a direct sum of rank one modules and R is a Prüfer ($\sharp\sharp$)domain. R is an almost maximal domain exactly if R is locally almost maximal and R is h-local. First, we show that R is almost maximal. Let M be a maximal ideal of R and G a torsionless R_M-module. Then there is a free R-module F such that $G \subseteq F_M$. Define $G' = G \cap F$, then G' is a torsionless R-module and $(G')_M = G$. By assumption G', hence G, is completely decomposable. Thus, any torsionless R_M-module is completely decomposable which shows R_M is almost maximal valuation domain.

Now, we need to show that R is h-local. Since R satisfies ($\sharp\sharp$), it is enough to check that each nonzero prime ideal of R is contained in a unique maximal ideal of R. Suppose P is a prime ideal of R contained in at least two maximal ideals N_1 and N_2 of R. Set $S = R \setminus N_1 \cup N_2$, $T = R_S$, $A = (N_1)_S$, $B = (N_2)_S$, $L = P_S$. Then $LT_L = L$ and T/L has quotient field T_L/L. Since R_{N_1} and R_{N_2} are almost maximal valuation domains, T_A/L and T_B/L are independent maximal valuations with common quotient field T_L/L. As such, each must have a divisible value group (see **(Vamos), Theorem A**). But since R has UDI, R has a complemented maximal ideal and at least one of A and B is principal. In particular, the value group of T_A/L or T_B/L must have a copy of $\mathbb{Z}$ as a summand. This contradiction implies each nonzero prime ideal of R is contained in a unique maximal ideal of R. Consequently, since R satisfies ($\sharp\sharp$), R is an h-local locally almost maximal domain; hence R is almost maximal(**(Fuchs & Salce, 2001), Theorem IV.3.9**). Since R has UDI, R must be a Bezout domain. This proves the claim. The converse follows from (ii).

(iv) Assume R is a quasilocal D-ring. Then since every torsion-free finite rank R-module decomposes into a direct sum of rank one modules, if Krull-Schmidt property for rank one modules holds, then we are done. Since R is a quasilocal D-ring, the integral closure of R is a valuation domain (see **(Matlis)**). We claim that every overring of R is quasilocal. Let S be an overring of R and $\overline{S}$ the integral closure of S. Take any $x \in \overline{R}$, then x is integral over R and so x is integral over S. So, $\overline{R} \subset \overline{S}$. Since $\overline{R}$ is a valuation domain, $\overline{S} = \overline{R}_L$ for some prime ideal L of $\overline{R}$ which shows $\overline{S}$ is quasilocal. Now, suppose S has at least two distinct maximal ideals, say M_1 and M_2. Since $S \subseteq \overline{S}$, there exists prime ideals Q_1 and Q_2 of $\overline{S}$ such that $Q_1 \cap S = M_1$ and $Q_2 \cap S = M_2$. By assumption, $Q_1 \neq Q_2$ but Q_1 and

Q_2 are maximal ideals of $\overline{S}$ which yields $Q_1 = Q_2$ and so $M_1 = M_2$. Thus, S is quasilocal too. In particular, rank one modules have quasilocal endomorphism rings because endomorphism rings of rank one modules are overrings of R. Thus, Krull-Schmidt property holds for rank one modules when R is a D-ring.

Conversely, suppose torsion-free finite rank R-modules decompose uniquely into direct sums of rank one R-modules. Then $\overline{R}$ is a D-ring with UDI. An integrally closed D-ring is the intersection of at most 2 maximal valuation domains **(Matlis)**. However, $\overline{R}$ has a complemented maximal ideal, so if $\overline{R}$ has two maximal ideals M and N, one of these ideals, say N, is principal. In particular, the maximal valuation domain $\overline{R}_N$ has a nondivisible value group. As in the proof of (iii), this is in contradiction to the fact that two independent maximal valuation domains having the same quotient field each have divisible value group. Thus $\overline{R}$ is quasilocal; hence, R is quasilocal and a D-ring. □

Using the following proposition, one can construct examples of non-Noetherian UDI domains for which h-locality fails in a strong way.

Proposition 4.3 *Let R be an integral domain with a prime ideal P such that $PR_P = P$. Then the Krull-Schmidt property holds for rank one modules of R if R_P is a DVR and the Krull-Schmidt property holds for rank one modules of R/P.*

Proof For each ideal I of R, there is an exact sequence,

$$Hom_R(I, P) \to Hom_R(I, R_P) \to Hom_R(I, R_P/P) \to Ext_R(I, P).$$

Now P is a principal ideal of R_P, so $Ext_R(I, P) \cong Ext_{R_P}(IR_P, P) = 0$. Also, we have that $Hom_R(I, R_P/P) \cong Hom_R(I/IP, R_P/P)$, hence $Hom_R(I, P) \to Hom(I, R_P)$ is surjective if and only if $I = IP$. Note that, if $I = IP$, then clearly $Hom_R(I/IP, R_P/P) = 0$. On the other hand, if $Hom_R(I/IP, R_P/P) = 0$, then, since R_P/P is the quotient field of R/P and I/IP is a torsion-free R/P-module, it must be the case that $I = IP$.

Tensoring both sides of $I_1 \oplus \cdots \oplus I_n \cong J_1 \oplus \cdots J_n$ with R/P yields $I_1/PI_1 \oplus \cdots I_n/PI_n \cong J_1/PJ_1 \oplus \cdots J_n/PJ_n$. After reindexing, we may assume that $m, m' \leq n$ exist such that $I_1/PI_1 \oplus \cdots I_m/PI_m \cong J_1/PJ_1 \oplus \cdots J_{m'}/PJ_{m'}$ and no I_k/PI_k or J_l/PJ_l is trivial for $k \leq m$, $l \leq m'$. The preeceding argument shows that, for all $k \leq m$, $Hom_R(I_k, P) \to Hom_R(I_k, R_P)$ is not surjective, so I_k is isomorphic to an R-submodule of R_P that is not contained in P. Similarly, for all $l \leq m'$, J_l is isomorphic to an R-submodule of R_P that is not contained in P. Thus we assume that $I_k, J_l \subseteq R_P$ but $I_k, J_l \nsubseteq P$ for all $k \leq m$, $l \leq m'$. In particular, $I_kR_P = J_lR_P = R_P$ implies $I_kPR_P = J_lPR_P = PR_P = P$ for all $k \leq m$, $l \leq m'$. Since

$I_k/P, J_l/P \subseteq R_P/P$, each $I_k/P, J_l/P$ is a rank one R/P-module. Thus $m = m'$ and, after reindexing, we may conclude that $I_k/P \cong J_k/P$ for each $k \leq m$. It follows that $a, b \in R$ exists with $b \notin P$ such that $aI_k + P = bJ_k$. If $a \in P$, then $aI_k \subseteq PI_k = P$ and $P = bJ_k$. However, this implies that $R_PJ_k = J_k$, hence $PJ_k \neq P$ since R_P is a DVR. This contradiction forces $a \notin P$. Thus, since $P = PI_k \subseteq I_k$ and $a^{-1}P = P$, we have $P \subseteq aI_k$ proving that, for all $k \leq m$, $I_k \cong J_k$. If $k > m$, then $I_kP = I_k$ and $J_kP = J_k$. Since R_P is a DVR, I_k and J_k are principal ideals of R_P, hence $I_k \cong J_k$. It follows that R has UDI.

Finally, observe that if X is a proper submodule of Q, the quotient field of R, then $X_P \neq Q$ since $X_P = Q$ would force $Q = XP \subseteq X$. Thus X is a fractional ideal of the DVR R_P and since, R_P is a fractional ideal of R, X is a fractional ideal of R. It follows that every proper rank one R-module is a fractional ideal of R. This proves the claim. □

Example 4.1 *There exist domains that are not h-local but that satisfy the Krull-Schmidt property for rank one modules. Define* $S := \mathbb{Z} + XQ[X]_{(X)}$. *Then by Proposition (4.3),* S *has Krull-Schmidt property for rank one modules. In fact,* $X + Q[X]_{(X)}$ *is a prime ideal of* S *such that it is contained in infinitely many maximal ideals of* S*, which shows* S *is not h-local.*

CHAPTER 5

UNIQUE DECOMPOSITONS INTO IDEALS FOR NOETHERIAN DOMAINS

A ring R is said to be Noetherian if it satisfies the following three equivalent conditions:

- Every non-empty set of ideals in R has a maximal element.
- Every ascending chain of ideals in R is stationary.
- Every ideal in R is finitely generated.

Proposition 5.1 *If R is Noetherian and P is a prime ideal of R, then R_P is Noetherian.*

Remark 5.1 *The following characterizations are useful for the proofs of the next theorems.*

- *(Kaplansky): For a commutative Noetherian ring R, R is a principal ideal ring if and only if every maximal ideal is principal.*
- *(Cohen): For a commutative ring R, R is Noetherian if and only if every prime ideal is finitely generated.*
- *(Cohen-Kaplansky): A commutative ring R is a principal ideal ring if and only if every prime ideal is principal.*

Throughout this chapter, R always represents a Noetherian integral domain, unless otherwise is stated, with quotient field Q, and $\overline{R}$ denotes the integral closure of R in Q. All of the modules mentioned below are torsion-free and have finite rank, where the rank of A is defined as the dimension of the Q-vector space $Q \otimes_R A$.

5.1. Reduction to the local case

Lemma 5.1 *((Goeters & Olberding, 2001), Lemma 2.1) If R has UDI, then at most one maximal ideal of R is non-principal.*

Proof If R is local, then there is nothing to consider. So, suppose R is not local and M_1 and M_2 are distinct maximal ideals of R. Then M_1 and M_2 are comaximal ideals that is $M_1 + M_2 = R$. The exact sequence

$$0 \longrightarrow M_1 \cap M_2 \longrightarrow M_1 \oplus M_2 \longrightarrow M_1 + M_2 \longrightarrow 0$$

splits since R is a projective R-module. Thus, $M_1 \oplus M_2 \cong R \oplus (M_1 \cap M_2)$. Since R has UDI, $M_1 \cong R$ or $M_2 \cong R$ which shows one of M_1 or M_2 is principal. Thus, there cannot be two nonprincipal maximal ideals of R. □

Definition 5.1 *If I is any ideal of R, the radical of I is*

$$r(I) = \{x \in R \mid x^n \in I \text{ for some } n > 0\}$$

Definition 5.2 *An ideal Q in a ring R is primary if $Q \neq R$ and if*

$$xy \in Q \Rightarrow \text{either } x \in Q \text{ or } y^n \in Q \text{ for some } n > 0$$

Proposition 5.2 *((Atiyah & Macdonald, 1969), , Proposition 4.1) Let Q be a primary ideal in a ring R. Then $r(Q)$ is the smallest prime ideal containing Q.*

Remark 5.2 *Suppose R is Noetherian and each nonzero prime ideal of R is contained in a unique maximal ideal of R. Then R is of finite character because Noetherian rings have the property that every ideal can be written as a finite intersection of primary ideals. In fact, assume R is Noetherian, take any $a \in R$, set the ideal $I =< a >$. Then $I = \cap_{i=1}^{n} Q_i$ where Q_i are primary ideals for $i = 1, 2, \dots n$. Suppose $r(Q_i) \subseteq M_i$, $i = 1, 2, \dots n$, M_i's are the maximal ideals of R. Then $I \subseteq r(I) = r(\cap_{i=1}^{n} Q_i) = \cap_{i=1}^{n} r(Q_i) \subseteq \cap_{i=1}^{n} M_i \subseteq M_i$ for every i, so $I \subseteq M_i$. Now, suppose $I \subseteq M$ where M is a maximal ideal of R distinct from M_i for each i. Since $I \subseteq M$, $r(I) \subseteq r(M) = M$, and so $r(I) = \cap_{i=1}^{n} r(Q_i) \subseteq M$ implies $r(Q_i) \subseteq M$ for some i which contradicts with assumption that every prime ideal is contained in a unique maximal. Thus, I is contained in only the maximal ideals $M_1, \dots, M_n$ which implies a is contained in finitely many maximal ideals of R. Hence, R is of finite character.*

Lemma 5.2 *((Goeters & Olberding, 2001), Lemma 2.2) If M is a maximal ideal of R*

such that every maximal ideal other than M is principal, then R is h-local, and $R_{[M]}$ is a PID that is also a flat R-module.

Proof If P is a prime ideal contained in a principal maximal ideal N, then N has height at most one by the Krull Principal theorem which yields $P = N$. Otherwise, P is contained in M exclusively, and it follows that R is h-local. $R_{[M]}$ is a flat R-module if and only if localizations of $R_{[M]}$ at maximal ideals N of R is flat R_N module. The localizations of $R_{[M]}$ at maximal ideals of R are either $R_{[M]}R_M = Q$ or $R_{[M]}R_N = R_N$ where $N \neq M$. Since R is Noetherian, its localizations are also Noetherian. The maximal ideals of $R_{[M]}$ are $N \cdot R_{[M]}$, where N is a principal maximal ideal of R, so $R_{[M]}$ is a PID. □

Definition 5.3 *Two torsion-free modules G and H are called nearly isomorphic, if for each ideal $I \neq 0$, there is an embedding $f : G \to H$ such that $J = annCokerf$ is a nonzero ideal of R comaximal with I.*

Clearly, if G and H are isomorphic torsion-free modules, then G and H are nearly isomorphic. Also, near isomorphism implies local isomorphism.

Lemma 5.3 *((Goeters & Olberding, 2001), Lemma 2.3) Suppose R has UDI and that G and H are finitely generated torsion-free modules. If $G_M \cong H_M$ for all maximal ideals M of R, then G is nearly isomorphic to H.*

Proposition 5.3 *((Goeters & Olberding, 2001), Proposition 2.4) Assume that R has UDI and that R has exactly one nonprincipal maximal ideal M. Suppose G is a finitely generated torsion-free module and H is completely decomposable. Then the following statements are equivalent:*

1. *$G_M \cong H_M$.*
2. *G and H are nearly isomorphic.*
3. *G and H are isomorphic.*

Corollary 5.1 *((Goeters & Olberding, 2001), Corollary 2.5) If R has UDI, then R_N has UDI for every maximal ideal N.*

Proof Let M be the nonprincipal maximal ideal of R and $I_1', \ldots, I_n'$, $J_1', \ldots, J_n'$ be the ideals of R_M such that $I_1' \oplus \cdots \oplus I_n' \cong J_1' \oplus \cdots \oplus J_n'$. Set $I_j = I_j' \cap R$ and $J_i = J_i' \cap R$. Since I_j, J_i are ideals of R and R is Noetherian, the modules $G = \bigoplus_j I_j$ and $H = \bigoplus_i J_i$ are finitely generated and torsion-free R-modules that satisfy $G_M \cong H_M$. Thus, G and H are isomorphic and since R has UDI, $I_j \cong J_j$ for all j after reindexing. Hence, $I_j' \cong J_j'$ for all j which proves R_M has UDI. If $N \neq M$ is a maximal ideal of R, NR_N is the principal maximal ideal of Noetherian ring R_N. So, R_N is a PID and have UDI. $\square$

Lemma 5.4 *((Goeters & Olberding, 2001), Lemma 2.6) If there exists a maximal ideal M of R such that every other maximal ideal of R is principal, then every ideal I not contained in M is principal. In addition, every invertible ideal is principal.*

Proof Let I be an ideal not contained in M. Since R is h-local, I is contained in only finitely many maximal ideals of R, call these ideals as $N_1, \ldots N_k$. Then $IR_{N_i} = a_i^{e_i} R_{N_i}$ for some $e_i \geq 1$ since R_{N_i} is a DVR and $N_i = a_i R$. Then $IR_{[M]} = aR_{[M]}$ where $a = a_1^{e_1} \cdots a_n^{e_n}$. By local verification we see that $I = aR$ because $(a \notin M)$. Now, let J be an invertible ideal. Consider the exact sequence

$$0 \longrightarrow M \overset{f}{\hookrightarrow} R \twoheadrightarrow R/M \longrightarrow 0.$$

Since J is projective as an R-module, $Hom(J, \square)$ is an exact functor. Thus, we get an exact sequence as

$$0 \longrightarrow Hom(J, M) \xrightarrow{g} Hom(J, R) \longrightarrow Hom(J, R/M) \longrightarrow 0,$$

where f is an embedding from M to R and $g = f \circ h$ for every $h \in Hom(J, M)$. If g is an epimorphism, then f is an epimorphism which yields a contradiction. So, g cannot be an epimorphism and there exists a $q \in Hom(J, R)$ such that $q(J) \nsubseteq M$. Thus, $q(J) \subseteq R$ is principal. $\square$

Lemma 5.5 *((Ay & Klingler, 2011), Lemma 1.3) Let R be any ring and P be a prime ideal of R and $Q(R_P)$ the total quotient ring of R_P. Then $Q_P \subseteq Q(R_P)$.*

Proof Let S_1 be the set of all the regular elements of R and $S_2 = R \setminus P$. Then $Q_P = S_2^{-1}(S_1^{-1}R) = (S_1 S_2)^{-1}R = (\overline{S}_1)^{-1}(S_2^{-1}R) = (\overline{S}_1)^{-1}(R_P)$, where $\overline{S}_1$ denotes the image of S_1 in R_P. Since regular elements of R remain regular in R_P, we get that $\overline{S}_1$ is contained in the set of regular elements of R_P, and hence $(\overline{S}_1)^{-1}(R_P) \subseteq Q(R_P)$. $\square$

Lemma 5.6 *((Ay & Klingler, 2011), Lemma 3.1) Let R' be an overring of R, and let N be a maximal ideal of R' which contracts to a principal maximal ideal of R. Then N is principal.*

Proof Let $P = N \cap R$ be the principal maximal ideal such that $P = xR$ for some $x \in P$. Let $Q(R_P)$ be the ring of fractions of R_P. Since $Q_P \subseteq Q(R_P)$ and $R \subset R' \subset Q$, we get R'_P is an overring of R_P. Since P is principal and R is Noetherian, R_P is a DVR, so $Q(R_P)$ is a field. Therefore, $R'_P = R_P$ or $R'_P = Q(R_P)$. Since N is prime ideal of R' and disjoint from $R \setminus P$, N_P is proper ideal of R'_P. Also, $P = N \cap R$ implies $PR_P = NR_P \cap R_P \subseteq NR_P$ and since $PR_P \neq 0$, $NR_P \neq O$ which yields R'_P cannot be a field, and hence $R'_P = R_P$.

Now, we claim that $N = xR'$. If $M \neq P$ is a maximal ideal of R, then since $x \notin M$, it follows that x is a unit in R'_M, and so $xR'_M = R'_M$. On the other other hand, $x \in P \subseteq N$ implies that $R'_M = xR'_M \subseteq N_M \subseteq R'_M$, and so $N_M = xR'_M$. Additionally, $P_P = (xR)_P = xR_P = xR'_P \subseteq N_P \subsetneq R'_P$, where P_P is the maximal ideal of $R_P = R'_P$ which yields $P_P = N_P$. Therefore, $N_P = xR'_P$. Thus, N and xR' are locally equal at every maximal ideal of R, and hence they are equal. □

Lemma 5.7 *((Goeters & Olberding, 2001), Lemma 2.7) Suppose R has a unique nonprincipal maximal ideal M. If S is an overring of R and N is a maximal ideal of S lying over M such that N_M is principal, then N is principal.*

Theorem 5.1 *((Goeters & Olberding, 2001), Theorem 2.8) R has UDI if and only if R is PID, or, there exists a lone nonprincipal maximal ideal M of R and R_M has UDI.*

Proof One direction is clear from the previous lemmas. For the converse, assume that R has a unique nonprincipal maximal ideal M and R_M has UDI. First; we need to show that a finitely generated overring S of R has at most one nonprincipal maximal ideal. So, let S be a finitely generated overring of R and N be a maximal ideal of S. Since S is finitely generated over R, $N \cap R$ is a maximal ideal of R. If $N \cap R \neq M$, then $N \cap R$ is principal, and so N is principal. Thus, we assume that $N \cap R = M$ that is N lies over M.

Since R_M has UDI and S_M is a finitely generated overring of R_M, S_M has UDI. Therefore, S_M has at most one nonprincipal maximal ideal. If N_M is a principal maximal ideal of S_M, then N is a principal ideal of S, so S has at most one nonprincipal maximal ideal.

Now, suppose $I_1 \oplus \cdots I_n \cong J_1 \oplus \cdots J_n$ where $I_1, \ldots I_n$, $J_1, \ldots, J_n$ are ideals of R. After localizing at M and after reindexing, we have $(I_j)_M \cong (J_j)_M$ for each $j = 1, \ldots, n$. To simplfy notation, fix $j \leq n$ and set $I = I_j$, $J = J_j$ and $S = E(J)$.

Now, we observe that $Hom(I, J)$ is a fractional ideal of S that is locally invertible, since $I_N \cong J_N$ for each maximal ideal N of R. Since S has at most one nonprincipal maximal ideal, $Hom(I, J)$ must be a principal fractional ideal of S. If $Hom(I, J) = g \cdot S$, then $I_N \cong J_N$ for all maximal ideals N and g is an isomorphism. □

5.2. Local domains with UDI

Remark 5.3 *The following facts are used in the proofs.*

- *If R is a Noetherian ring, then every finitely generated R-module is Noetherian as an R-module.*
- *Every finitely generated R-submodule of Q is a fractional ideal and if R is Noetherian these are all the fractional ideals of R.*
- *M is indecomposable R-module if and only if $End_R(M)$ has no nontrivial idempotents.*

Lemma 5.8 *((Ay & Klingler, 2011), Lemma 1.2) Let R be a ring. If A and B are torsion-free S-modules, where S is an overring of R, then $Hom_S(A, B) = Hom_R(A, B)$.*

Proof Take $f \in Hom_S(A, B)$, then f is an S-module homomorphism, and clearly an R-module homomorphism. Now, take any $\phi \in Hom_R(A, B)$, we need to show $\phi(sa) = s\phi(a)$ for every $a \in A$, $s \in S$. Note that since $R \subseteq S \subseteq Q$, for any $s \in S$, $s = x/y$ for some x, y in R and y is not a zero divisor. If $a \in A$, then $sa \in A$ since A is an S-module. Then $ys\phi(a) = x\phi(a) = \phi(xa) = \phi(ysa) = y\phi(sa)$, so $ys\phi(a) - y\phi(sa) = 0$ and $y(s\phi(a) - \phi(sa)) = 0$. Since y is not a zero divisor, $s\phi(a) - \phi(sa) = 0$ which yields $s\phi(a) = \phi(sa)$. Hence, every R-homomorphism is also an S-module homomorphism.

□

Lemma 5.9 *((Ay & Klingler, 2011), Proposition 3.4) Let R be a ring and S be an overring of R and suppose that S is finitely generated as an R-module. If R has the UDI property, then so does S.*

Proof Assume that $I_1 \oplus I_2 \oplus \cdots I_n \cong J_1 \oplus J_2 \oplus \cdots J_m$ for indecomposable ideals $I_1, I_2, \cdots I_n$, $J_1, J_2, \cdots J_m$ of S. Since S is a finitely generated overring of R, every ideal of S becomes a finitely generated fractional ideal of R. Since I_j is indecomposable as an S-module, $End_S(I_j)$ contains no nontrivial idempotents. And since $End_S(I_j) = End_R(I_j)$ does not contain any nontrivial idempotents either which shows I_j is indecomposable as an R-module. Similarly, each J_k remains indecomposable as an R-module. So, since R has UDI, $m = n$ and after reindexing $I_j \cong J_j$ as R-modules for all indices j. But these isomorphisms are also S-homomorphisms, and hence each $I_j \cong J_j$ as S-modules. □

Lemma 5.10 *((Goeters & Olberding, 2001), Lemma 3.1) If R is local with UDI, then $\overline{R}$ has at most 3 maximal ideals.*

Proof Let R be a local domain with unique maximal ideal M. Suppose $\overline{R}$ has at least 4 distinct maximal ideals. Then there exists a finitely generated overring S of R with least 4 distinct maximal ideals: Since $\overline{R}$ is the integral closure of R, for some $x \in \overline{R}$, x is integral over R, so $R[x]$ finitely generated R-module. We choose S to be $R[x]$, then we see that S is finitely generated overring of R. Since contraction of the maximal ideals of S are maximal in $\overline{R}$, S has at least 4 distinct maximal ideals. Call these maximal ideals as M_1, M_2, M_3 and M_4. Then the map

$$\sigma : (R + M_1M_2) \oplus (R + M_3M_4) \rightarrow S$$

defined by $\sigma((a, b)) = a + b$ is split by the map

$$\phi : S \rightarrow (R + M_1M_2) \oplus (R + M_3M_4)$$

defined by $\phi(t) = (tx, ty)$ where $x \in M_1M_2$ and $y \in M_3M_4$ such that $x + y = 1$. Such x and y exist because $M_1M_2 + M_3M_4 = S$. By splitting homomorphism property

$$Ker\sigma \oplus Im\phi = (R + M_1M_2) \oplus (R + M_3M_4)$$

and ϕ is one-to-one and σ is onto, so $Im\phi \cong S$. Thus, we have

$$Ker\sigma \oplus S = (R + M_1M_2) \oplus (R + M_3M_4).$$

Now, since R has UDI and S is finitely generated as an R-module, S has UDI. So, $S \cong R + M_1M_2$ or $S \cong R + M_3M_4$, without loss of generality say $S \cong R + M_1M_2$. Since both objects are rings, we must have $S = R + M_1M_2$ but this is impossible because $M \subseteq M_1M_2$ implies $(R + M_1M_2)/M_1M_2 \cong R/M$. This yields M_1M_2 is prime in S but S/M_1M_2 is not an integral domain. Hence, $\overline{R}$ cannot have more than 3 maximal ideals. □

Theorem 5.2 *((Goeters & Olberding, 2001), Theorem 3.2) Assume that R is local with* *maximal ideal M. The following are equivalent for R:*

(1) R has UDI.

(2) There exists a fractional overring R' of R with $\mid max(R') \mid = \mid max(\overline{R}) \mid$ such that one of the following occurs:

(i) R' is local.

(ii) R' has exactly 2 distinct maximal ideals M'_1, M'_2 such that M'_1 is principal with $M \nsubseteq (M'_1)^2$, and $R'/M'_1 \cong R/M$.

(iii) $R' = \overline{R}$ has exactly 3 distinct maximal ideals M'_1, M'_2, M'_3; all are principal and satisfy $M \nsubseteq (M'_j)^2$ and $R'/M'_j \cong R/M$.

(3) Every fractional overring R' of R such that $\mid max(R') \mid = \mid max(\overline{R}) \mid$ satisfies $(i), (ii)$ or (iii) above.

Corollary 5.2 *((Goeters & Olberding, 2001), Corollary 3.3) Let R be a local ring. If R has UDI, then one of the three possibilities occurs:*

(i) *$\overline{R}$ is quasilocal.*

(ii) *$\overline{R}$ has exactly 2 distinct maximal ideals P_1, P_2 such that P_1 is principal, $M \nsubseteq P_1^2$, and $\overline{R}/P_1 \cong R/M$.*

(iii) *$\overline{R}$ has exactly 3 distinct maximal ideals P_1, P_2, P_3, that are is principal and satisfy $\overline{R}/P_i \cong R/M$ and $M \nsubseteq P_i^2$ for each i.*

The converse holds if $\overline{R}$ is a finite R-module.

Corollary 5.3 *((Goeters & Olberding, 2001), Corollary 3.4) The following are equivalent:*

(1) R has UDI.

(2) R is a PID, or, R has a unique nonprincipal maximal ideal M such that R_M satisfies the conditions of Theorem 5.2.

CHAPTER 6

KRULL-SCHMIDT PROPERTIES OVER RINGS OF FINITE CHARACTER

In this chapter we assume that the ring R is of finite character, that is, every non-zero element is contained in only finitely many maximal ideals of R; equivalently, every non-zero ideal is contained in only finitely many maximal ideals of R. We will state some useful results without giving their proofs.

In Chapter 3 we gave a characterization of h-local domains which the versions of the Krull-Schmidt property hold for direct sums of ideals, direct sums of indecomposable submodules of finitely generated free modules and direct sums of rank are torsion-free modules. While proving this characterization, we mostly used Lemma (4.7). The results stated below, which are given by Ay and Klingler in an upcoming paper, are some other versions of this lemma, but the most important detail is that the ring is not necessarily an h-local domain.

Lemma 6.1 *Let R be an integral domain of finite character and F a finitely generated free module of rank n. If G is a rank n R-submodule of F and H is a torsionless R-module that is locally isomorphic to G, then there exists a finitely generated projective R-submodule P of QH such that $H \subseteq P$ and $F/G \cong P/H$.*

Proposition 6.1 *Let R be an integral domain of finite character. Suppose that G and H are locally isomorphic torsionless R-modules. If G is a finitely generated R-module, and G has a direct summand isomorphic to a nonzero ideal of R, then $G \cong H$.*

Proposition 6.2 *Let R be an integral domain of finite character. Suppose that G and H are locally isomorphic torsionless R-modules. If G is finitely generated, then there exists $n > 0$ such that $G^{(n)} \cong H^{(n)}$.*

We think that new characterizations of domains with finite character which Krull-Schmidt property holds for some classes of R-modules may be preved by using Lemma (6.1), Proposition (6.1) and Proposition (6.2). Since we drop the second condition from h-locality, it will be worthwile to extend this study.

CHAPTER 7

CONCLUSION

Let R be a commutative integral domain and C a class of R-modules. The Krull-Schmidt property holds for C if, whenever

$$G_1 \oplus G_2 \oplus \cdots G_n \cong H_1 \oplus H_2 \oplus \cdots H_m$$

for $G_i, H_j \in C$, then $n = m$ and, after indexing, $G_i \cong H_i$ for all $i \leq n$. If, instead of $G_i \cong H_i$, it is required only that there exists $k > 0$ such that $G_i^{(k)} \cong H_i^{(k)}$ for all i, then we say that the weak Krull-Schmidt property holds for C. ($G^{(k)}$ represents direct sum of k copies of a module G.) We say a domain R has unique decompositions into ideals, UDI, if the class of ideals of R has the Krull-Schmidt property.

We explicitly give a characterization, shown by P. Goeters and B. Olberding, for an h-local domain when the Krull-Schmidt properties hold:
((Goeters & Olberding, 2002), Theorem 3.4) Let R be an h-local domain. The following statements hold for R.

(i) R has weak UDI if and only if $Pic(R)$ is torsion and R has a complemented maximal ideal M such that R_M has UDI.

(ii) R has UDI if and only if $Pic(R) = 0$ and R has a complemented maximal ideal M such that R_M has UDI.

(iii) R has weak TFKS if and only if $Pic(R)$ is torsion and R has a complemented maximal ideal M such that R_M has weak TFKS.

(vi) R has TFKS if and only if locally isomorphic torsionless modules are isomorphic and R has a complemented maximal ideal M such that R_M has TFKS.

(v) R has the Krull-Schmidt property for rank one modules if and only if $Pic(R) = 0$ and R has a complemented maximal ideal M such that R_M has the Krull-Schmidt property for rank one modules.

Moreover, we gave a characterization of Noetherian UDI domains, shown by P. Goeters and B. Olberding: A Noetherian integral domain R has UDI if and only if R is a PID or R has exactly nonprincipal maximal ideal M such that R_M has UDI **(Goeters & Olberding, 2001)**. We also observed that there might be a possibility to extend these Krull-Schmidt properties for domains of finite character.

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