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The Exp-function approach to the Schwarzian Korteweg–de Vries equation

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ABSTRACT

By means of the Exp-function method and its generalization, we report further exact traveling wave solutions, in a concise form, to the Schwarzian Korteweg–de Vries equation which admits physical significance in applications. Not only solitary and periodic waves but also rational solutions are observed.

1. Introduction

In the last four decades or so, solving nonlinear evolution equations (NEEs) has become a valuable task in many scientific areas including applied mathematics. Thanks to dedicated researchers, some modern analytic methods nowadays are available for handling NEEs in a concise manner. For example, homotopy perturbation method [1], (G'/G)-expansion method [2], variational iteration method [3], homotopy analysis method [4], simplest equation method [5], first integral method [6], multi-exp function method [7], three-wave method [8] and so on.

Recently, He et al. [9] pointed out new directions in nonlinear science by proposing three standard variational iteration algorithms for solving differential equations, integro-differential equations, fractional differential equations, fractal differential equations, differential-difference equations and fractional/fractal differential-difference equations. However, there is no universal method for finding all solutions to all types of differential equations. Each of the existing methods has some advantages and disadvantages over the others when dealing with a specific nonlinear problem.

In 2006, He and Wu [10] introduced the so-called Exp-function method for solving NEEs. The Exp-function method is based on trying rational combinations of exponential functions as an ansatz. It is entirely algorithmic and consist only of algebraic manipulations, which can be carried out by using a computer algebra system. If treated rigorously, it usually provides exact solutions with more arbitrary parameters from which one can construct solitary and periodic waves. The research community responded quite well to the announcement of the method. As a result, it has been extended, generalized and adapted for various kinds of nonlinear problems such as differential-difference equations [11], NEEs with variable coefficients [12], stochastic equations [13], multi-dimensional equations [14,15], three coupled NEEs [16]. Besides, it is generalized to construct n-soliton solutions [17,18], rational solutions [19], double-wave solutions [20].

The basic Korteweg–de Vries (KdV) equation is integrable and possesses a wealth of interesting and crucial properties. It has been the origin of many other integrable equations [21]. Within the family of KdV related equations, probably the most fundamental one is the Schwarzian Korteweg–de Vries equation (SKdV), which first appeared in [22,23], and reads

$$\frac{\phi_t}{\phi_x} + (S\phi)(x) = 0, \tag{1}$$

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where

$$(S\phi)(x) = \left(\frac{\phi_{xx}}{\phi_x}\right)_x - \frac{1}{2}\left(\frac{\phi_{xx}}{\phi_x}\right)^2$$

is the Schwarzian derivative of ϕ, see [24]. Being of great interest in both physics and mathematics, the SKdV equation received a lot of attention. As far as we could verify, no attempt so far has been made to solve this special form equation by implementing most recent analytic methods like the ones mentioned above. We believe that a well-established new method might have the potential of discovering previously unknown solutions which might imply some fascinating physical meanings hidden in the nonlinear problem considered.

Our goal in the present work is to perform an analytic study on the SKdV equation using the Exp-function method. The rest of this paper is organized as follows. In Section 2, we give a short description of our approach to NEEs. In Sections 3 and 4, we apply the method to the SKdV equation for the first time. Finally, a concluding remark is given in Section 5.

2. Methodology

Let us consider a nonlinear partial differential equation for $u(x, t)$ in the form

$$P(u, u_t, u_x, u_{tt}, u_{tx}, u_{xx}, \ldots) = 0, \tag{2}$$

where P is a polynomial in its arguments. The Exp-function method is based on the assumption that traveling wave solutions of Eq. (2) can be expressed as

$$u(x, t) = \frac{\sum_{i=-c}^{d} a_i \exp(i\xi)}{\sum_{j=-p}^{q} b_j \exp(j\xi)}, \quad \xi = kx + wt, \tag{3}$$

where c, d, p and q are positive integers which are known to be specified further; a_i, b_j, k and w are unknown constants to be determined. We remark that Eq. (3) can be rewritten in an alternative form

$$u(x, t) = \frac{a_c \exp(c\xi) + \cdots + a_{-d} \exp(-d\xi)}{b_p \exp(p\xi) + \cdots + b_{-q} \exp(-q\xi)}, \quad \xi = kx + wt. \tag{4}$$

For determining the values of c and p, we balance the linear term of highest order in Eq. (2) with the highest order nonlinear term. Similarly, for determining the values of d and q, we balance the linear term of lowest order in Eq. (2) with the lowest order nonlinear term.

To construct rational solutions to Eq. (2), as suggested in [19], we consider the following modified form of the ansatz (3)

$$u(x, t) = \frac{\sum_{i=-c}^{d} a_i (\mu_1 \exp(\xi) + \mu_2 \xi)^i}{\sum_{j=-p}^{q} b_j (\mu_1 \exp(\xi) + \mu_2 \xi)^j}, \quad \xi = kx + wt, \tag{5}$$

where μ_1 and μ_2 are two embedded constants. It is easy to see that when $\mu_1 = 1$ and $\mu_2 = 0$, the ansatz (5) becomes the ansatz (3). In this case also, we follow the same solution procedure.

3. Solitary and periodic solutions

By means of the transformation $u(x, t) = U(\xi)$, $\xi = kx + wt$, Eq. (1) can be reduced to the ODE

$$w(U')^2 + k^3 U' U''' - \frac{3}{2} k^3 (U'')^2 = 0 \tag{6}$$

where the primes denote derivatives with respect to ξ. We initially guess that the solution of Eq. (6) is of the form (4). Then we have the following case analysis:

Case 1: When $p = c = 1$ and $d = q = 1$, the solution of Eq. (6) can be expressed as

$$U(\xi) = \frac{a_1 \exp(\xi) + a_0 + a_{-1} \exp(-\xi)}{b_1 \exp(\xi) + b_0 + b_{-1} \exp(-\xi)}. \tag{7}$$

Substituting (7) into Eq. (6), we get an equation of the form

$$\left(2\left(b_{-1} + b_0 \exp(\xi) + b_1 \exp(2\xi)\right)^3\right)^{-1} \sum_{n=2}^{6} C_n \exp(n\xi) = 0, \tag{8}$$

where C_n $(2 \le n \le 6)$ are polynomial expressions in terms of a_1, a_0, a_{-1}, b_1, b_0, b_{-1}, k, and w. To save space, we illustrate just one of the them, corresponding to $n = 2$, as

$$C_2 = -k^3a_0^2b_{-1}^2 + 2wa_0^2b_{-1}^2 + 2k^3a_{-1}a_0b_{-1}b_0 - 4wa_{-1}a_0b_{-1}b_0 - k^3a_{-1}^2b_0^2 + 2wa_{-1}^2b_0^2.$$

Thus, solving the resulting system $C_n = 0$ $(2 \le n \le 6)$ simultaneously, we obtain the solution set

$$\left\{w = 2k^3, b_0 = 0, a_0 = 0\right\} \tag{9}$$

which yields an exponential function solution to Eq. (1) as

$$u_1(x,t) = \frac{a_{-1} + a_1 \exp\left(2kx + 4k^3t\right)}{b_{-1} + b_1 \exp\left(2kx + 4k^3t\right)} \tag{10}$$

where a_1, a_{-1}, b_1, b_{-1}, and k are arbitrary constants.

Case 2: When $p = c = 2$ and $d = q = 1$, the trial function (4) becomes

$$U(\xi) = \frac{a_2 \exp(2\xi) + a_1 \exp(\xi) + a_0 + a_{-1}\exp(-\xi)}{b_2 \exp(2\xi) + b_1 \exp(\xi) + b_0 + b_{-1}\exp(-\xi)}. \tag{11}$$

Since we repeat the same procedure, here and thereafter, we omit some of the details and just present the results. In this case, we get the solution sets

$$\left\{w = \frac{k^3}{2}, a_{-1} = 0, a_0 = \frac{1}{2}\left(a_1b_1 + a_2\left(2b_0 - b_1^2\right) \mp (a_1 - a_2b_1)\sqrt{b_1^2 - 4b_0}\right), b_{-1} = 0, b_2 = 1\right\}. \tag{12}$$

which corresponds to exponential function solutions to Eq. (1) as

$$u_2^{\mp}(x,t) = \frac{a_1 \exp\left(kx + \frac{k^3}{2}t\right) + \frac{1}{2}\left(a_1b_1 + a_2\left(2b_0 - b_1^2\right) \mp (a_1 - a_2b_1)\sqrt{b_1^2 - 4b_0}\right) + a_2 \exp\left(2kx + k^3t\right)}{\exp\left(2kx + k^3t\right) + b_0 + b_1 \exp\left(kx + \frac{k^3}{2}t\right)}, \tag{13}$$

where a_2, a_1, b_1, b_0, and k are arbitrary constants.

Case 3: When $p = c = 2$ and $d = q = 2$, the ansatz (4) turns out to be

$$U(\xi) = \frac{a_2 \exp(2\xi) + a_1 \exp(\xi) + a_0 + a_{-1}\exp(-\xi) + a_{-2}\exp(-2\xi)}{b_2 \exp(2\xi) + b_1 \exp(\xi) + b_0 + b_{-1}\exp(-\xi) + b_{-2}\exp(-2\xi)}. \tag{14}$$

In this case, we get the solution sets

$$\left\{\begin{array}{l} a_0 = \dfrac{4a_2\sqrt{b_{-2}}b_0 \mp \sqrt{2}a_1\sqrt{\sqrt{b_0^2 - 4b_{-2}} - b_0}\left(b_0 + \sqrt{b_0^2 - 4b_{-2}}\right)}{4\sqrt{b_{-2}}}, a_{-2} = a_2b_{-2} \mp \dfrac{a_1\sqrt{b_{-2}}\sqrt{\sqrt{b_0^2 - 4b_{-2}} - b_0}}{\sqrt{2}}, \\ w = \dfrac{k^3}{2}, a_{-1} = \dfrac{1}{2}a_1\left(b_0 - \sqrt{b_0^2 - 4b_{-2}}\right), b_2 = 1, b_{-1} = b_1 = 0 \end{array}\right\} \tag{15}$$

which gives rise to exponential function solutions to Eq. (1) as

$$u_3^{\mp}(x,t) = \frac{\left[\begin{array}{l} a_2 \exp\left(2kx + k^3t\right) + \frac{1}{2}a_1\left(b_0 - \sqrt{b_0^2 - 4b_{-2}}\right)\exp\left(-kx - \frac{k^3}{2}t\right) + \left(a_2b_{-2} \mp \frac{a_1\sqrt{b_{-2}}\sqrt{\sqrt{b_0^2 - 4b_{-2}} - b_0}}{\sqrt{2}}\right)\exp\left(-2kx - k^3t\right) \\ + \frac{4a_2\sqrt{b_{-2}}b_0 \mp \sqrt{2}a_1\sqrt{\sqrt{b_0^2 - 4b_{-2}} - b_0}\left(b_0 + \sqrt{b_0^2 - 4b_{-2}}\right)}{4\sqrt{b_{-2}}} + a_1 \exp\left(kx + \frac{k^3}{2}t\right) \end{array}\right]}{b_0 + \exp\left(2kx + k^3t\right) + b_{-2}\exp\left(-2kx - k^3t\right)} \tag{16}$$

where a_2, a_1, b_0, b_{-2}, and k are arbitrary constants.

Case 4: When $p = c = 3$ and $d = q = 3$, the ansatz (4) becomes

$$U(\xi) = \frac{a_3 \exp(3\xi) + a_2 \exp(2\xi) + a_1 \exp(\xi) + a_0 + a_{-1}\exp(-\xi) + a_{-2}\exp(-2\xi) + a_{-3}\exp(-3\xi)}{b_3 \exp(3\xi) + b_2 \exp(2\xi) + b_1 \exp(\xi) + b_0 + b_{-1}\exp(-\xi) + b_{-2}\exp(-2\xi) + b_{-3}\exp(-3\xi)}. \tag{17}$$

In this case, we obtain the solution set

$$\left\{\begin{array}{l} a_{-3}=a_3b_{-3}-\dfrac{a_1b_{-3}^{1/3}\left(\sqrt{b_0^2-4b_{-3}}-b_0\right)^{2/3}}{2^{2/3}},\ a_0=a_3b_0+\dfrac{2^{1/3}a_1b_{-3}^{1/3}}{\left(\sqrt{b_0^2-4b_{-3}}-b_0\right)^{1/3}},\ a_{-2}=\dfrac{1}{2}a_1\left(b_0-\sqrt{b_0^2-4b_{-3}}\right),\\ a_{-1}=-\dfrac{a_1\left(\sqrt{b_0^2-4b_{-3}}-b_0\right)^{4/3}}{2^{4/3}b_{-3}^{1/3}},\ a_2=\dfrac{a_1\left(\sqrt{b_0^2-4b_{-3}}-b_0\right)^{1/3}}{2^{1/3}b_{-3}^{1/3}},\ w=\dfrac{k^3}{2},\ b_3=1,\ b_2=b_1=b_{-1}=b_{-2}=0 \end{array}\right\} \tag{18}$$

which provides an exponential function solution to Eq. (1) as

$$u_4(x,t)=\frac{\left(\begin{array}{l} a_1\exp\left(kx+\dfrac{k^3}{2}t\right)+a_3\exp\left(3kx+\dfrac{3k^3}{2}t\right)+a_3b_0+\dfrac{1}{2}a_1\left(b_0-\sqrt{b_0^2-4b_{-3}}\right)\exp\left(-2kx-k^3t\right)+\dfrac{2^{1/3}a_1b_{-3}^{1/3}}{\left(\sqrt{b_0^2-4b_{-3}}-b_0\right)^{1/3}}\\ +\dfrac{a_1\left(\sqrt{b_0^2-4b_{-3}}-b_0\right)^{1/3}\exp\left(2kx+k^3t\right)}{2^{1/3}b_{-3}^{1/3}}-\dfrac{a_1\left(\sqrt{b_0^2-4b_{-3}}-b_0\right)^{4/3}\exp\left(-kx-\frac{k^3}{2}t\right)}{2^{4/3}b_{-3}^{1/3}}\\ +\left(a_3b_{-3}-\dfrac{a_1b_{-3}^{1/3}\left(\sqrt{b_0^2-4b_{-3}}-b_0\right)^{2/3}}{2^{2/3}}\right)\exp\left(-3kx-\dfrac{3k^3}{2}t\right) \end{array}\right)}{b_0+\exp\left(3kx+\frac{3k^3}{2}t\right)+b_{-3}\exp\left(-3kx-\frac{3k^3}{2}t\right)}, \tag{19}$$

where a_3, a_1, b_0, b_{-3}, and k are arbitrary constants.

Remark. We have obtained a wide class of traveling wave solutions to Eq. (1). By setting special values to the arbitrary parameters, we can construct formal solitary and periodic wave solutions. For instance, taking "$a_{-1}=-a_1$ and $b_{-1}=b_1$" or "$a_{-1}=a_1$ and $b_{-1}=-b_1$" in (10) reveals solitary waves to Eq. (1) as

$$u_5(x,t)=\frac{a_1}{b_1}\tanh\left(kx+2k^3t\right), \tag{20}$$

$$u_6(x,t)=\frac{a_1}{b_1}\coth\left(kx+2k^3t\right) \tag{21}$$

where a_1, b_1, and k are arbitrary parameters.

Moreover, when k and w are imaginary numbers in the complex variation $\xi=kx+wt$, say $k=\mathrm{i}K$, $w=\mathrm{i}W$, $\mathrm{i}^2=-1$, then using the transformations

$$\exp(\pm\xi)=\exp\left(\pm\mathrm{i}\left(Kx+Wt\right)\right)=\cos\left(Kx+Wt\right)\pm\mathrm{i}\sin\left(Kx+Wt\right), \tag{22}$$

one can convert the obtained solutions into periodic solutions. Hence, taking (22) into consideration, (13) becomes

$$u_7^{\mp}(x,t)=\frac{2a_1+(a_1(b_1\mp\sqrt{b_1^2-4b_0})+a_2(2+2b_0-b_1(b_1\mp\sqrt{b_1^2-4b_0})))\cos(Kx-\frac{K^3}{2}t)-\mathrm{i}(a_1(b_1\mp\sqrt{b_1^2-4b_0})-a_2(2-2b_0+b_1(b_1\mp\sqrt{b_1^2-4b_0})))\sin(Kx-\frac{K^3}{2}t)}{2(\mathrm{i}(1-b_0)\sin(Kx-\frac{K^3}{2}t)+(1+b_0)\cos(Kx-\frac{K^3}{2}t)+b_1)}. \tag{23}$$

Now, letting $b_0=1$, $b_1=0$, $a_2=\pm ia_1/2$ in (23) gives periodic solutions to Eq. (1) as

$$u_8^{\mp}(x,t)=\frac{a_1}{2}\sec\left(Kx-\frac{K^3}{2}t\right)\left(1\mp\sin\left(Kx-\frac{K^3}{2}t\right)\right) \tag{24}$$

where a_1 and K are arbitrary parameters.

4. Rational solutions

According to the ansatz (5), for the case $p=c=1$ and $d=q=1$, the solution of Eq. (6) can be expressed as

$$U(\xi)=\frac{a_1\left(\mu_1\exp\left(\xi\right)+\mu_2\xi\right)+a_0+a_{-1}\left(\mu_1\exp\left(\xi\right)+\mu_2\xi\right)^{-1}}{b_1\left(\mu_1\exp\left(\xi\right)+\mu_2\xi\right)+b_0+b_{-1}\left(\mu_1\exp\left(\xi\right)+\mu_2\xi\right)^{-1}}. \tag{25}$$

Substituting (25) into Eq. (6) and equating the coefficients of $\xi^m \exp(n\xi)$ $(0 \le m \le 4, 0 \le n \le 6)$ to zero and solving the resulting nonlinear algebraic system for a_1, a_0, a_{-1}, b_1, b_0, b_{-1}, k, w, μ_1, and μ_2, we get the solution sets

$$\left\{w = 0, a_{-1} = b_{-1}\left(2a_1b_{-1} - a_0\left(b_0 \pm \sqrt{b_0^2 - 4b_{-1}b_1}\right)\right) \bigg/ \left(2b_{-1}b_1 - b_0\left(b_0 + \sqrt{b_0^2 - 4b_{-1}b_1}\right)\right), \mu_1 = 0, \mu_2 = 1\right\} \tag{26}$$

which leads to the rational solutions to Eq. (19) as

$$u_9^{\mp}(x,t) = \frac{\begin{array}{l} \mp 2a_1b_{-1}^2 \pm a_0b_{-1}b_0 + a_0b_{-1}\sqrt{b_0^2 - 4b_{-1}b_1} + k\left(\pm a_0b_0^2 \mp 2a_0b_{-1}b_1 + a_0b_0\sqrt{b_0^2 - 4b_{-1}b_1}\right)x \\ + k^2\left(\pm a_1b_0^2 \mp 2a_1b_{-1}b_1 + a_1b_0\sqrt{b_0^2 - 4b_{-1}b_1}\right)x^2 \end{array}}{\begin{array}{l} \pm b_{-1}b_0^2 \mp 2b_{-1}^2b_1 + b_{-1}b_0\sqrt{b_0^2 - 4b_{-1}b_1} + k\left(\pm b_0^3 \mp 2b_{-1}b_0b_1 + b_0^2\sqrt{b_0^2 - 4b_{-1}b_1}\right)x \\ + k^2\left(\pm b_0^2b_1 \mp 2b_{-1}b_1^2 + b_0b_1\sqrt{b_0^2 - 4b_{-1}b_1}\right)x^2 \end{array}} \tag{27}$$

where a_1, a_0, b_1, b_0, b_{-1}, and k are arbitrary constants. We note that (27) represent non-constant steady-state (time independent) solutions to Eq. (1). We omit to discuss other cases since the calculation becomes tedious and more complicated.

5. Conclusion

We successfully derived various kinds of exact solutions to the SKdV equation via the Exp-function method and one of its generalizations. Our results might be of great importance to explain the physical phenomena related to the equation discussed here. We verified the correctness of the solutions by substituting them back into the original equation with the aid of MATHEMATICA, it gives an extra measure of confidence in the results. For future work, our plan is to investigate n-soliton solutions to other types of NEEs using the Exp-function method since the wide applications of the method indicate that it is currently well established.

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