# RESOLUTIONS IN COTORSION THEORIES

Karen Akinci and Rafail Alizade

Citation: AIP Conference Proceedings **1309**, 761 (2010); doi: 10.1063/1.3525203
View online: http://dx.doi.org/10.1063/1.3525203
View Table of Contents:
http://scitation.aip.org/content/aip/proceeding/aipcp/1309?ver=pdfcov
Published by the AIP Publishing

# RESOLUTIONS IN COTORSION THEORIES

KAREN AKINCI AND RAFAIL ALIZADE

Abstract. We consider the $\lambda$- ($\mu-$) and $\overline{\lambda}$- ($\overline{\mu}$-) dimensions of modules taken under a cotorsion theory $(\mathcal{F},\mathcal{C})$ satisfying the Hereditary Condition, and establish some inequalities between the dimensions of the modules of a short exact sequence, not necessarily $\mathrm{Hom}(\mathcal{F},-)$ exact. We investigate the question of whether the property of having a (special) $\mathcal{F}$- or $\mathcal{C}$-resolution of length $n$ is resolving, closed under extensions or coresolving and establish some inequalities connecting the $\lambda$- ($\mu-$) and $\overline{\lambda}$- ($\overline{\mu}$-) dimensions of modules in a short exact sequence.

## 1. Introduction

Throughout a module will mean a unitary left $R$-module over an arbitrary but fixed ring $R$ with identity.

A *cotorsion theory* (see [8] ) is a pair of classes of modules $(\mathcal{F},\mathcal{C})$ such that

$$\mathcal{F} = {}^{\perp}\mathcal{C} = \left\{F \mid \mathrm{Ext}^1(F,C) = 0 \text{ for all } C \in \mathcal{C}\right\}$$

and

$$\mathcal{C} = \mathcal{F}^{\perp} = \left\{C \mid \mathrm{Ext}^1(F,C) = 0 \text{ for all } F \in \mathcal{F}\right\}.$$

A *partial left $\mathcal{F}$-resolution* (or *partial $\mathcal{F}$-projective resolution*) of a module $M$ of length $n$ is a complex $F_n \overset{d_n}{\to} F_{n-1} \to \ldots \to F_1 \overset{d_1}{\to} F_0 \overset{d_0}{\to} M \to 0$ with each $F_i \in \mathcal{F}$, which is $\mathrm{Hom}(F,-)$ exact for every $F \in \mathcal{F}$. Similarly a *partial right $\mathcal{C}$-resolution* of a module $M$ of length $n$ is a complex $0 \to M \overset{e_0}{\to} C_0 \overset{e_1}{\to} C_1 \to \ldots \to C_{n-1} \overset{e_n}{\to} C_n$ with each $C_i \in \mathcal{C}$, which is $\mathrm{Hom}(-,C)$ exact for every $C \in \mathcal{C}$. Taken under a cotorsion theory $(\mathcal{F},\mathcal{C})$, an $\mathcal{F}$-resolution is normally left and a $\mathcal{C}$-resolution is normally right, this will not be stated where there is no danger of ambiguity. If $\mathrm{Ker}\, d_i \in \mathcal{C}$ for all $i$, then the partial $\mathcal{F}$-resolution is called *special* and similarly the partial $\mathcal{C}$-resolution above is *special* if $\mathrm{Coker}\, e_i \in \mathcal{F}$ for all $i$.

**Definition 1.1.** The $\lambda$-dimension ($\overline{\lambda}$-dimension) of $M$ is defined as follows: $\lambda(M) = n$ ($\overline{\lambda}(M) = n$) if there is a partial $\mathcal{F}$-resolution (special partial $\mathcal{F}$-resolution) $F_n \overset{d_n}{\longrightarrow} F_{n-1} \longrightarrow \ldots \longrightarrow F_1 \overset{d_1}{\longrightarrow} F_0 \overset{d_0}{\longrightarrow} M \longrightarrow 0$ of $M$ of length $n$ and if there is no longer such complex. If there is no partial $\mathcal{F}$-resolution (special partial $\mathcal{F}$-resolution) then we say that $\lambda(M) = -1$ ($\overline{\lambda}(M) = -1$), and if there exists a partial $\mathcal{F}$-resolution (special partial $\mathcal{F}$-resolution) for every $n \geq 0$ we say that $\lambda(M) = \infty$ ($\overline{\lambda}(M) = \infty$). The partial (special

*Key words and phrases.* Cotorsion Theory, Hereditary Condition, Special left $\mathcal{F}$-resolution, Special right $\mathcal{C}$-resolution, $\lambda$-, $\mu-$, $\overline{\lambda}$-, $\overline{\mu}$- dimensions.

partial) $\mathcal{C}$-resolution and $\mu$-dimension ($\overline{\mu}$-dimension) for a class of $\mathcal{C}$ modules are defined dually.

For every $F \in \mathcal{F}$ we have special $\mathcal{F}$-resolution $\ldots \to 0 \to \ldots \to 0 \to F \to F \to 0$, so $\lambda(F) = \overline{\lambda}(F) = \infty$. Similarly $\mu(C) = \overline{\mu}(C) = \infty$ for every $C \in \mathcal{C}$.

We study the notions of $\overline{\lambda}$-dimension (and $\overline{\mu}$-dimension) when $(\mathcal{F}, \mathcal{C})$ is a cotorsion theory satisfying the *Hereditary Condition* ($HC$) (see [3]), that is, $\mathrm{Ext}^2(F, C) = 0$ for every $F \in \mathcal{F}$ and $C \in \mathcal{C}$, or equivalently, $\mathcal{F}$ is resolving, or $\mathcal{C}$ is coresolving. Recall that a class $\mathcal{A}$ of modules containing all projective (injective) modules is called *resolving* (*coresolving*) if for every short exact sequence of modules $0 \to A \to B \to C \to 0$ the condition $B, C \in \mathcal{A}$ ($A, B \in \mathcal{A}$) implies $A \in \mathcal{A}$ ($C \in \mathcal{A}$).

The following example of a cotorsion theory not satisfying $HC$ is given in the proof of Proposition 3.6 in [4]. Recall that a module $C$ is called *weakly cotorsion* if it is cotorsion in the Matlis sense, that is, $\mathrm{Ext}^1(Q, C) = 0$ (where $Q$ is the field of fractions of $R$) (see [7]), and a module $F$ is *strongly flat* if $\mathrm{Ext}^1(F, C) = 0$ for all weakly cotorsion modules $C$. Let $R$ be a valuation domain which is not a Matlis domain, i.e. $\mathrm{pr.\,dim}\, Q > 1$ and let $0 \to H \to F \to Q \to 0$ be a free presentation of $Q$. Then $F$ and $Q$ are strongly flat but $H$ is not (see the proof of Prop. 3.6 in [4]). So the cotorsion theory $(\mathcal{S}F, \mathcal{W}C)$, where $\mathcal{S}F$ is the class of strongly flat modules and $\mathcal{W}C$ is the class of the weakly cotorsion modules, does not satisfy $HC$.

## 2. $\overline{\lambda}$- AND $\overline{\mu}$-DIMENSIONS

The following theorem is similar to Theorem 8.6.14 of [5] where the exact sequence $0 \to M' \to M \to M'' \to 0$ is $\mathrm{Hom}(\mathcal{F}, -)$ exact. We can remove this condition and prove the stronger case for any cotorsion theory $(\mathcal{F}, \mathcal{C})$ that satisfies $HC$.

**Theorem 2.1.** *If the cotorsion theory* $(\mathcal{F}, \mathcal{C})$ *satisfies* $HC$ *and* $0 \to M' \to M \to M'' \to 0$ *is exact then* $\overline{\lambda}(M) \geq \min\left(\overline{\lambda}(M'), \overline{\lambda}(M'')\right)$.

*Proof.* Let $\min\left(\overline{\lambda}(M'), \overline{\lambda}(M'')\right) = n$, by induction on $n$ we will prove that $\overline{\lambda}(M) \geq n$. For $n = -1$ there is nothing to prove. If $n = 0$, then both $M'$ and $M''$ have special $\mathcal{F}$-precovers. By Theorem 3.1 of [1], $M$ also has a special $\mathcal{F}$-precover, so $\overline{\lambda}(M) \geq 0$. Assume that for all $n \leq k$ the inequality holds and let $n = k + 1$. Given $\overline{\lambda}(M'), \overline{\lambda}(M'') \geq k + 1$, then there are special $\mathcal{F}$-resolutions: $F'_k \xrightarrow{d'_k} F'_{k-1} \longrightarrow \ldots \longrightarrow F'_1 \xrightarrow{d'_1} F'_0 \xrightarrow{f} M' \longrightarrow 0$ and $F''_k \xrightarrow{d''_k} F''_{k-1} \longrightarrow \ldots \longrightarrow F''_1 \xrightarrow{d''_1} F''_0 \xrightarrow{g} M'' \longrightarrow 0$. By the proof of Theorem 3.1 in [1] we have the following diagram with exact rows

$$\begin{array}{ccccccccc}
0 & \longrightarrow & F'_0 & \longrightarrow & F_0 & \longrightarrow & F''_0 & \longrightarrow & 0 \\
 & & \downarrow_f & & \downarrow_e & & || & & \\
0 & \longrightarrow & M' & \longrightarrow & X & \longrightarrow & F''_0 & \longrightarrow & 0 \\
 & & || & & \downarrow_h & & \downarrow_g & & \\
0 & \longrightarrow & M' & \longrightarrow & M & \longrightarrow & M'' & \longrightarrow & 0
\end{array}$$

From this diagram we obtain the following commutative diagram whose columns and rows are exact by the $3 \times 3$ Lemma.

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & C_0' & \longrightarrow & C_0 & \longrightarrow & C_0'' & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow_v & & \downarrow & & \\
0 & \longrightarrow & F_0' & \longrightarrow & F_0 & \longrightarrow & F_0'' & \longrightarrow & 0 \\
 & & \downarrow_f & & \downarrow_{h\circ e} & & \downarrow_g & & \\
0 & \longrightarrow & M' & \longrightarrow & M & \longrightarrow & M'' & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}
$$

where $C_0' = \operatorname{Ker} f \in \mathcal{C}$ and $C_0'' = \operatorname{Ker} g \in \mathcal{C}$. This means $C_0 = \operatorname{Ker}(h \circ e) \in \mathcal{C}$ also. Since $F_0', F_0'' \in \mathcal{F}$ we have that $F_0 \in \mathcal{F}$ also. For $C_0'$ and $C_0''$ there are special $\mathcal{F}$-resolutions: $F_k' \xrightarrow{d_k'} F_{k-1}' \longrightarrow \ldots \longrightarrow F_1' \longrightarrow C_0' \longrightarrow 0$ and $F_k'' \xrightarrow{d_k''} F_{k-1}'' \longrightarrow \ldots \longrightarrow F_1'' \longrightarrow C_0'' \longrightarrow 0$, so $\overline{\lambda}(C_0') \geq k$ and $\overline{\lambda}(C_0'') \geq k$, and by the inductive assumption $\overline{\lambda}(C_0) \geq k$. That is, $C_0$ has a special $\mathcal{F}$-resolution $F_k \longrightarrow F_{k-1} \longrightarrow \ldots \longrightarrow F_1 \xrightarrow{u} C_0 \longrightarrow 0$. Then $F_k \longrightarrow F_{k-1} \longrightarrow \ldots \longrightarrow F_1 \xrightarrow{v\circ u} F_0 \longrightarrow M \longrightarrow 0$, where $v : C_0 \to F_0$ is the inclusion map, gives a special $\mathcal{F}$-resolution of $M$, that is $\overline{\lambda}(M) \geq k+1$ as required. □

The following theorem shows that if $\overline{\lambda}(M) = n > k \geq 0$ then every special $\mathcal{F}$-resolution of length $k$ can be extended to a special $\mathcal{F}$ -resolution of length $n$. Meaning that the $\overline{\lambda}$-dimension of a module $M$ does not depend on the choice of the special $\mathcal{F}$-resolution. The analogous result for $\mathcal{F}$-resolutions was proved in [5] (Prop 8.6.6).

**Theorem 2.2.** *If the cotorsion theory $(\mathcal{F}, \mathcal{C})$ satisfies HC, $\overline{\lambda}(M) \geq n > k \geq 0$ and $F_k \xrightarrow{d_k} F_{k-1} \longrightarrow \ldots \longrightarrow F_1 \xrightarrow{d_1} F_0 \xrightarrow{d_0} M \longrightarrow 0$ is a partial special left $\mathcal{F}$-resolution of length $k$ of $M$, then $\overline{\lambda}(L_k) \geq n-k-1$ where $L_k = \operatorname{Ker} d_k \in \mathcal{C}$. In particular, if $\overline{\lambda}(M) = n$, then $\overline{\lambda}(L_k) = n-k-1$.*

*Proof.* This theorem is again proven by induction on $k$. For $k = 0$, applying Theorem 8.6.16 of [5], to the exact sequence $0 \to L_0 \to F_0 \to M \to 0$, we see that $\overline{\lambda}(L_0) \geq \min\left(\overline{\lambda}(F_0), \overline{\lambda}(M) - 1\right) = \overline{\lambda}(M) - 1 \geq n - 0 - 1$. Assume that $\overline{\lambda}(L_k) \geq n-k-1$. Applying Theorem 8.6.16. of [5] to the exact sequence $0 \to L_{k+1} \to F_{k+1} \to L_k \to 0$, we get $\overline{\lambda}(L_{k+1}) \geq \min\left(\overline{\lambda}(F_{k+1}), \overline{\lambda}(L_k) - 1\right) = \overline{\lambda}(L_k) - 1 \geq n-k-1-1 = n-(k+1)-1$.

Now suppose that $\overline{\lambda}(M) = n$, $\overline{\lambda}(L_k) = s$ and let $G_s \to G_{s-1} \to \ldots \to G_1 \to L_k \to 0$ be a special $\mathcal{F}$ -resolution of $L_k$. Then $G_s \to \ldots \to G_1 \to F_k \to \ldots \to F_0 \to M \to 0$ is a special $\mathcal{F}$-resolution of $M$, so $n = \overline{\lambda}(M) \geq s+k+1$. Therefore, $\overline{\lambda}(L_k) = s \leq n-k-1$. On the other hand $s \geq n-k-1$, so we see that equality holds and $\overline{\lambda}(L_k) = n-k-1$. □

In the case $\overline{\lambda}(M) = \infty$ we have the following corollary.

**Corollary 2.3.** *If the cotorsion theory* $(\mathcal{F},\mathcal{C})$ *satisfies* $HC$ *and* $\overline{\lambda}(M)=\infty$ *then there is an infinite special* $\mathcal{F}$*-resolution* $\ldots \longrightarrow F_n \xrightarrow{d_n} F_{n-1} \longrightarrow \ldots \longrightarrow F_1 \xrightarrow{d_1} F_0 \xrightarrow{d_0} M \longrightarrow 0$ *of* $M$.

*Proof.* Since $\overline{\lambda}(M)\geq 0$, there is a special $\mathcal{F}$-precover $F_0 \xrightarrow{f_0} M \longrightarrow 0$ of $M$. Since $\overline{\lambda}(M)\geq 1$, $\overline{\lambda}(\mathrm{Ker}\, f_0)\geq 0$ by Theorem 2.2, so there is a special $\mathcal{F}$-precover $F_1 \xrightarrow{f_1} \mathrm{Ker}\, f_0 \longrightarrow 0$ of $\mathrm{Ker}\, f_0$. Now $\overline{\lambda}(M)\geq 2$, therefore $\overline{\lambda}(\mathrm{Ker}\, f_0)\geq 0$ and $\mathrm{Ker}\, f_1$ has a special $\mathcal{F}$-precover. Continuing in this way an infinite special $\mathcal{F}$-resolution can be constructed for $M$. □

We would like to give the following result (which follows immediately from Theorem 8.6.16 in [5]) in connection with Theorem 3.8 of [1]. Recall that a cotorsion theory $(\mathcal{F},\mathcal{C})$ is said to satisfy the extended hereditary condition ($EHC$) if it satisfies $HC$, $\mathrm{gl.\,dim}\, R<\infty$ and every module from $\mathcal{C}$ has a special $\mathcal{F}$-precover (or equivalently, every module from $\mathcal{F}$ has a special $\mathcal{C}$-preenvelope) (see [1]). Here the given condition that $EHC$ should hold can now be replaced by the condition that $\overline{\lambda}(M'')\geq 1$.

**Corollary 2.4.** *If the cotorsion theory* $(\mathcal{F},\mathcal{C})$ *satisfies* $HC$ *and in the short exact sequence* $0\to M'\to M\to M''\to 0$, $M$ *has a special* $\mathcal{F}$*-precover and* $\overline{\lambda}(M'')\geq 1$, *then* $M'$ *has a special* $\mathcal{F}$*-precover.*

Now we study the case when every module from $\mathcal{C}$ has a special $\mathcal{F}$-precover.

**Lemma 2.5.** *If every module from* $\mathcal{C}$ *has a special* $\mathcal{F}$*-precover, then for every module* $M$ *either* $\overline{\lambda}(M)=-1$ *or* $\overline{\lambda}(M)=\infty$. *In particular,* $\overline{\lambda}(C)=\infty$ *for every* $C$ *from* $\mathcal{C}$.

*Proof.* If $\overline{\lambda}(M)\neq -1$, i. e. $M$ has a special $\mathcal{F}$-precover $0\to C_0\to F_0\to M\to 0$, then $C_0$ has a special $\mathcal{F}$-precover $0\to C_1\to F_1\to C_0\to 0$ and so on, $C_n$ has a special $\mathcal{F}$-precover $0\to C_{n+1}\to F_{n+1}\to C_n\to 0$. Yoneda product of these short exact sequences gives an infinite special $\mathcal{F}$-resolution

$$\ldots\to F_n \to F_{n-1}\to\ldots\to F_1\to F_0\to M\to 0$$

of $M$. So $\overline{\lambda}(C)=\infty$. The second statement is obvious. □

**Proposition 2.6.** *If the cotorsion theory* $(\mathcal{F},\mathcal{C})$ *satisfies* $HC$ *and every module from* $\mathcal{C}$ *has a special* $\mathcal{F}$*-precover, then for every module* $M$ *with finite injective dimension,* $\overline{\lambda}(M)=\infty$.

*Proof.* Let $\mathrm{inj.\,dim}\, M=n$ and

$$0\longrightarrow M\longrightarrow I_0\longrightarrow\ldots\longrightarrow I_{n-1}\longrightarrow I_n\longrightarrow 0$$

be an injective resolution of $M$. This sequence can be represented as an Yoneda product of short exact sequences

$$\begin{array}{ccccccccc}
0 & \longrightarrow & M & \longrightarrow & I_0 & \longrightarrow & K_0 & \longrightarrow & 0\\
0 & \longrightarrow & K_0 & \longrightarrow & I_1 & \longrightarrow & K_1 & \longrightarrow & 0\\
 & & \vdots & & \vdots & & \vdots & & \\
0 & \longrightarrow & K_{n-3} & \longrightarrow & I_{n-2} & \longrightarrow & K_{n-2} & \longrightarrow & 0\\
0 & \longrightarrow & K_{n-2} & \longrightarrow & I_{n-1} & \longrightarrow & I_n & \longrightarrow & 0
\end{array}$$

Since $\overline{\lambda}(I_{n-1}) = \overline{\lambda}(I_n) = \infty$, applying Corollary 2.4 to the last row we obtain that $K_{n-2}$ has a special $\mathcal{F}$-precover and by Lemma 2.5 $\overline{\lambda}(K_{n-2}) = \infty$. Similarly Corollary 2.4 and Lemma 2.5 gives $\overline{\lambda}(K_{n-3}) = \infty$. Continuing in this way we see that $\overline{\lambda}(M) = \infty$. □

The dual results hold for $\overline{\mu}$-dimensions.

The following corollary of 2.6 gives an improvement of Proposition 3.7 in [1].

**Corollary 2.7.** *If the cotorsion theory $(\mathcal{F}, \mathcal{C})$ satisfies EHC, then $\overline{\lambda}(M) = \overline{\mu}(M) = \infty$ for every module $M$.*

**Lemma 2.8.** *If the cotorsion theory $(\mathcal{F}, \mathcal{C})$ satisfies* HC*, then $\mathrm{Ext}^n(F, C) = 0$ for every $F \in \mathcal{F}$, $C \in \mathcal{C}$ and $n \geq 1$.*

*Proof.* Let $C \in \mathcal{C}$. By induction on $n$ we prove that $\mathrm{Ext}^n(F, C) = 0$ for every $F \in \mathcal{F}$. For $n = 1, 2$ it satisfies by the definitions. Let $n \geq 3$ and suppose that the equality satisfies for every $k < n$ and let $F \in \mathcal{F}$. Take any short exact sequence $0 \to A \to P \to F \to 0$ with projective $P$. Then $A \in \mathcal{F}$ since $(\mathcal{F}, \mathcal{C})$ satisfies *HC*. Therefore from the exact sequence

$$\ldots \longrightarrow \mathrm{Ext}^{n-1}(A, C) \longrightarrow \mathrm{Ext}^n(F, C) \longrightarrow \mathrm{Ext}^n(P, C) \longrightarrow \ldots$$

we conclude that $\mathrm{Ext}^n(F, C) = 0$. □

**Theorem 2.9.** *Suppose that the cotorsion theory $(\mathcal{F}, \mathcal{C})$ satisfies* HC *and* $\mathrm{gl.dim}\, R = n < \infty$. *If $\lambda(M) \geq n-1$ $(\overline{\lambda}(M) \geq n-1)$, then $\lambda(M) = \infty$ $(\overline{\lambda}(M) = \infty)$.*

*Proof.* Suppose that $\lambda(M) \geq n-1$, i.e. we have a partial $\mathcal{F}$-resolution $F_{n-1} \xrightarrow{d_{n-1}} F_{n-2} \longrightarrow \ldots \longrightarrow F_1 \xrightarrow{d_1} F_0 \xrightarrow{d_0} M \longrightarrow 0$ with each $F_i \in \mathcal{F}$. Then we have the following short exact sequences:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K_{n-1} & \longrightarrow & F_{n-1} & \longrightarrow & K_{n-2} & \longrightarrow & 0 \\ 0 & \longrightarrow & K_{n-2} & \longrightarrow & F_{n-2} & \longrightarrow & K_{n-3} & \longrightarrow & 0 \\ & & \vdots & & \vdots & & \vdots & & \\ 0 & \longrightarrow & K_1 & \longrightarrow & F_1 & \longrightarrow & K_0 & \longrightarrow & 0 \\ 0 & \longrightarrow & K_0 & \longrightarrow & F_0 & \longrightarrow & M & \longrightarrow & 0 \end{array}$$

where $K_i = \mathrm{Ker}\, d_i$ for $i = 0, 1, 2, \ldots, n-1$. For every $C \in \mathcal{C}$ we have the following exact sequences:

$$\begin{array}{ccccccc} \mathrm{Ext}^1(F_{n-1}, C) & \longrightarrow & \mathrm{Ext}^1(K_{n-1}, C) & \longrightarrow & \mathrm{Ext}^2(K_{n-2}, C) & \longrightarrow & \mathrm{Ext}^2(F_{n-1}, C) \\ \mathrm{Ext}^2(F_{n-2}, C) & \longrightarrow & \mathrm{Ext}^2(K_{n-2}, C) & \longrightarrow & \mathrm{Ext}^3(K_{n-3}, C) & \longrightarrow & \mathrm{Ext}^3(F_{n-2}, C) \\ & & \vdots & & \vdots & & \vdots \\ \mathrm{Ext}^{n-1}(F_1, C) & \longrightarrow & \mathrm{Ext}^{n-1}(K_1, C) & \longrightarrow & \mathrm{Ext}^n(K_0, C) & \longrightarrow & \mathrm{Ext}^n(F_1, C) \\ \mathrm{Ext}^n(F_0, C) & \longrightarrow & \mathrm{Ext}^n(K_0, C) & \longrightarrow & \mathrm{Ext}^{n+1}(M, C) & \longrightarrow & \mathrm{Ext}^{n+1}(F_0, C) \end{array}$$

Since $\mathrm{Ext}^1(F_{n-1}, C) = \mathrm{Ext}^2(F_{n-1}, C) = \mathrm{Ext}^2(F_{n-2}, C) = \mathrm{Ext}^3(F_{n-2}, C) = \ldots = \mathrm{Ext}^n(F_0, C) = \mathrm{Ext}^{n+1}(F_0, C) = 0$ we have the isomorphisms

$$\mathrm{Ext}^1(K_{n-1}, C) \cong \mathrm{Ext}^2(K_{n-2}, C) \cong \mathrm{Ext}^3(K_{n-3}, C) \cong \ldots \cong \mathrm{Ext}^{n-1}(K_1, C) \cong$$
$$\cong \mathrm{Ext}^n(K_0, C) \cong \mathrm{Ext}^{n+1}(M, C).$$

But $\mathrm{gl.dim}\, R = n$ , so $\mathrm{Ext}^1(K_{n-1}, C) \cong \mathrm{Ext}^{n+1}(M, C) \cong 0$. Therefore $K_{n-1} \in \mathcal{F}$. So we have an infinite $\mathcal{F}$-resolution of $M$:

$$\ldots \to 0 \longrightarrow K_{n-1} \longrightarrow F_{n-1} \longrightarrow \ldots \longrightarrow F_1 \xrightarrow{d_1} F_0 \xrightarrow{d_0} M \longrightarrow 0$$

The proof of the equality $\overline{\lambda}(M) = \infty$ is similar. □

The dual results hold for for the $\mu-$ and $\overline{\mu}-$ dimensions of modules.

## 3. Relations between $\overline{\lambda}$- and $\lambda$-dimensions, and $\overline{\mu}$- and $\mu$-dimensions

In this section we aim to give inequalities between the $\overline{\lambda}$- and $\lambda$-dimensions of modules in a short exact sequence. These inequalities are similar to the inequalities involving only the $\lambda$-dimensions, or the $\overline{\lambda}$-dimensions in [6]. We use Theorem 3.1 of [1] to prove the following theorem which is similar to Theorem 8.6.9 of [5]. In our case the complex is not necessarily $\mathrm{Hom}(\mathcal{F}, -)$ exact, but the given cotorsion theory satisfies $HC$.

**Theorem 3.1.** *If the cotorsion theory* $(\mathcal{F}, \mathcal{C})$ *satisfies* $HC$ *and* $0 \to M' \to M \to M'' \to 0$ *is exact then:*

*(1)* $\lambda(M) \geq \min\left(\overline{\lambda}(M'), \lambda(M'')\right)$,

*(2)* $\lambda(M') \geq \min\left(\lambda(M), \overline{\lambda}(M'') - 1\right)$.

*Proof.* (1) We modify the proof of Theorem 2.1 as follows. Let $\min\left(\overline{\lambda}(M'), \lambda(M'')\right) = n$. By induction on $n$ we prove that $\lambda(M) \geq n$. Again for $n = -1$ there is nothing to prove. Assume that for $n \leq k$ the inequality holds and let $n = k+1$. There is a special $\mathcal{F}$ -resolution $F'_k \to F'_{k-1} \to \ldots \to F'_1 \to F'_0 \to M' \to 0$ for $M'$ and an $\mathcal{F}$-resolution $F''_k \to F''_{k-1} \to \ldots \to F''_1 \to F''_0 \to M'' \to 0$ for $M''$. For every $F \in \mathcal{F}$ applying $\mathrm{Hom}(F, -)$ to the commutative exact diagram:

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & C''_0 & = & C''_0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & M' & \longrightarrow & X & \longrightarrow & F''_0 & \longrightarrow & 0 \\
 & & || & & \downarrow_h & & \downarrow_g & & \\
0 & \longrightarrow & M' & \longrightarrow & M & \longrightarrow & M'' & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

we have the following diagram with exact rows:

$$\begin{array}{ccccccc}
0 \longrightarrow & \mathrm{Hom}(F, M') & \longrightarrow & \mathrm{Hom}(F, X) & \longrightarrow & \mathrm{Hom}(F, F'') & \longrightarrow & Ext^1(F, M') \\
 & || & & \downarrow_{h_*} & & \downarrow_{g_*} & & || \\
0 \longrightarrow & \mathrm{Hom}(F, M') & \longrightarrow & \mathrm{Hom}(F, M) & \longrightarrow & \mathrm{Hom}(F, M'') & \longrightarrow & Ext^1(F, M')
\end{array}$$

Since $g_*$ is epic, $h_*$ is also epic by the Five Lemma. Furthermore since $\mathrm{Ker}\, e \cong \mathrm{Ker}\, f = C'_0 \in \mathcal{C}$, applying $\mathrm{Hom}(F, -)$ to the exact sequence $0 \to C'_0 \longrightarrow F_0 \xrightarrow{e} X \longrightarrow 0$ we obtain the following exact sequence

$$\ldots \longrightarrow \mathrm{Hom}\,(F, F_0) \xrightarrow{e_*} \mathrm{Hom}\,(F, X) \longrightarrow Ext^1\,(F, C_0') = 0$$

from which we conclude that $e_*$ is also epic. Then $(h \circ e)_* = h_* \circ e_*$ is epic and therefore $0 \longrightarrow C_0 \longrightarrow F_0 \xrightarrow{h \circ e} M \longrightarrow 0$ is $\mathrm{Hom}\,(\mathcal{F}, -)$ exact.

Here $F_0'$ and $F_0''$ in $\mathcal{F}$ gives us that $F_0 \in \mathcal{F}$. Now if $k = 0$ then $F_0 \xrightarrow{h \circ e} M$ is an $\mathcal{F}$-precover of $M$, so $\lambda\,(M) \geq 0$. If $k \geq 1$, then $\min\left(\overline{\lambda}\,(C_0'), \lambda\,(C_0'')\right) \geq k$, and so by the inductive assumption $\lambda\,(C_0) \geq k$, that is, $C_0$ has a $\mathcal{F}$-resolution $F_k \longrightarrow F_{k-1} \longrightarrow \ldots \longrightarrow F_1 \xrightarrow{u} C_0 \longrightarrow 0$, therefore $F_k \longrightarrow F_{k-1} \longrightarrow \ldots \longrightarrow F_1 \xrightarrow{v \circ u} F_0 \longrightarrow M \longrightarrow 0$ forms an $\mathcal{F}$-resolution of $M$. That is, $\lambda\,(M) \geq k+1$.

(2) Let $\min\left(\lambda\,(M), \overline{\lambda}\,(M'') - 1\right) = n$. Then there is an exact sequence, $0 \to C_0'' \to F_0'' \to M_0'' \to 0$ with $F_0'' \in \mathcal{F}$ and $C_0'' \in \mathcal{C}$ and $\overline{\lambda}\,(C_0'') \geq n$. We have an exact commutative diagram:

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & C_0'' & = & C_0'' & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & M' & \longrightarrow & X & \longrightarrow & F_0'' & \longrightarrow & 0 \\
 & & || & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & M' & \longrightarrow & M & \longrightarrow & M'' & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

By 1) we have that $\lambda\,(X) \geq n$. Then we have a $\mathrm{Hom}\,(\mathcal{F}, -)$ exact sequence $0 \to A_0 \to F_0 \to X \to 0$ with $F_0 \in \mathcal{F}$ and $\lambda\,(A_0) \geq n-1$. From this we get the following exact commutative diagram;

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & A_0 & = & A_0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & F_0' & \longrightarrow & F_0 & \longrightarrow & F_0'' & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & || & & \\
0 & \longrightarrow & M' & \longrightarrow & X & \longrightarrow & F_0'' & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

One can easily verify by means of the Five Lemma (using the techniques of the proof of the first part), that the sequence $0 \to A_0 \to F_0' \to M' \to 0$ is $\mathrm{Hom}\,(\mathcal{F}, -)$ exact. $F_0$, $F_0'' \in \mathcal{F}$ means that because of $HC$, $F_0' \in \mathcal{F}$. Now if $F_{n-1}' \to \ldots \to F_1' \to A_0 \to 0$ is a left $\mathcal{F}$-resolution of $A_0$, then $F_{n-1}' \to \ldots \to F_1' \to F_0 \to M' \to 0$ is a left $\mathcal{F}$-resolution of $M'$. Therefore $\lambda\,(M') \geq n$. $\square$

The dual results for $\overline{\mu}$- and $\mu$-dimensions derived in a similar way and we state the theorem without proof.

**Theorem 3.2.** *If the cotorsion theory* $(\mathcal{F}, \mathcal{C})$ *satisfies* $HC$ *and* $0 \to M' \to M \to M'' \to 0$ *is exact then;*
*1)* $\mu(M) \geq \min(\mu(M'), \overline{\mu}(M''))$,
*2)* $\mu(M'') \geq \min(\overline{\mu}(M') - 1, \mu(M))$.

## References


[1] K. D. Akıncı and R. Alizade, *Special precovers in cotorsion theories*, Proc. Edin. Math. Soc. **45:2** (2002), 411–420.

[2] ———, *Cotorsion theories and resolutions*, International Congress of Math., Abstracts of Short Communications and Poster Sessions (2002), 9.

[3] R. Alizade, *Inheritance of the properties of coprojectivity and coinjectivity for certain proper classes,(Russian)*, Izv. Akad. Nauk Azerbaijan. SSR Ser. Fiz.-Tekhn. Mat. Nauk **4:5** (1983), 3–7.

[4] S. Bazzoni and L. Salce, *On strongly flat modules over integral domains*, Rocky Mountan J. Math. **34:2** (2004), 417–439.

[5] E. Enochs and O. M. G. Jenda, *Relative homological algebra*, Walter de Gruyter-New York, 2000.

[6] E.Enochs, O. M. G. Jenda, and L. Oyonarte, *$\lambda$ and $\mu$-Dimensions of Modules*, Rend. Sem. Mat. Univ. Padova **105** (2001), 111–123.

[7] E.Matlis, *Torsion-Free Modules*, Chicago Lectures in Mathematics, University of Chicago Press, Chicago-London, 1972.

[8] L.Salce, *Cotorsion theories for abelian groups*, Symp. Math. **23** (1972), 12–32.

[9] J.Xu, *Flat covers of modules*, Lecture Notes in Mathematics, 1634, Springer Verlag, 1996.



E-MAIL:KARENAKINCI@YAHOO.CO.UK

IZMIR INSTITUTE OF TECHNOLOGY,, DEPARTMENT OF MATHEMATICS,, GÜLBAHÇEKÖYÜ, 35437, URLA,, IZMIR, TURKEY, E-MAIL: RAFAILALIZADE@IYTE.EDU.TR