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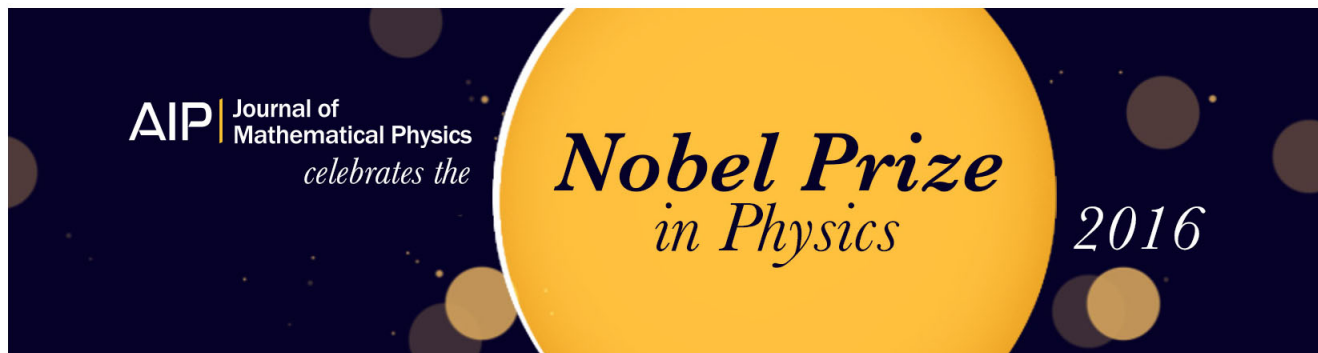
Exactly solvable quantum Sturm–Liouville problems

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The harmonic oscillator with time-dependent parameters covers a broad spectrum of physical problems from quantum transport, quantum optics, and quantum information to cosmology. Several methods have been developed to quantize this fundamental system, such as the path integral method, the Lewis–Riesenfeld time invariant method, the evolution operator or dynamical symmetry method, etc. In all these methods, solution of the quantum problem is given in terms of the classical one. However, only few exactly solvable problems of the last one, such as the damped oscillator or the Caldirola–Kanai model, have been treated. The goal of the present paper is to introduce a wide class of exactly solvable quantum models in terms of the Sturm–Liouville problem for classical orthogonal polynomials. This allows us to solve exactly the corresponding quantum parametric oscillators with specific damping and frequency dependence, which can be considered as quantum Sturm–Liouville problems. © *2009 American Institute of Physics.*
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I. INTRODUCTION

Nonstationary problems of quantum mechanics admit exact solutions quite rarely. From the class of exactly solvable problems, the harmonic oscillator with time-dependent parameters plays the central role due to a broad spectrum of physical applications from quantum transport, quantum optics, and quantum information to cosmology.[1] From recent applications we mention the geometric phase by Berry to study the nonadiabatic geometric phase for coherent states.[2]

Long time ago, for imitation of damped oscillations, the harmonic oscillator with an exponentially increasing mass has been introduced by Caldirola[3] and Kanai.[4] However, it is well known that parametric excitations of a quantum oscillator, resulting from a change of its mass, by proper time reparametrization, can be reduced to the case of a quantum oscillator with variable frequency.[1] Thus, from classical point of view these models are physically equivalent. However, quantization of time-dependent Hamiltonian with exponential mass accretion $m(t) = m_0 e^t$, as a model of quantum damped oscillator,[5] encounters some contradictions with Heisenberg uncertainty relations.[6] The ambiguity problem appearing in quantization of the damped harmonic oscillator has been addressed in Ref. 7.

Several methods have been developed to quantize one dimensional quantum oscillator with variable frequency. It has been considered first by Husimi,[8] who by proper ansatz reduced solution of the Schrödinger equation to nonlinear Riccati equation, linearizable in the form of classical parametric oscillator. Then, other methods such as the path integral method,[9] the Lewis–Riesenfeld quantum time invariant method,[10] the evolution operator or dynamical symmetry method,[11] etc., were found. In all these methods solution of the quantum problem is given in terms

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of the classical one.[12,13] However, only few problems of quantum parametric oscillator, such as the damped oscillator in the Caldirola–Kanai form,[3,4] have been treated exactly.

The goal of the present paper is to introduce a wide class of exactly solvable quantum models in terms of the Sturm–Liouville problem for the classical orthogonal polynomials. This class of functions, called the special functions of mathematical physics, play essential role as a solution of boundary value problems, particularly for the stationary Schrödinger equation.[14] Importance of this class of functions as the hypergeometric function was mentioned first by Klein.[15] For the relation of these functions to the group representations, discovered first by Cartan,[16] see Ref. 17. Application of the hypergeometric function in the form of the Eckart potential, from quantum mechanics,[18] for exact solution of the parametric resonance has been considered by Perelomov, see Refs. 1 and 12 and references therein. It was shown that the classical oscillator is characterized by one parameter, coinciding with the quantum-mechanical reflection coefficient, allowing the use of some known results from stationary quantum mechanics in one dimension. Following in direction of this formal analogy, in the present paper we solve exactly the parametric quantum oscillator with a specific damping and frequency dependence, corresponding to the hypergeometric function and its degenerations. Then, this problem can be considered as the quantum Sturm–Liouville problem.

The paper is organized in the following way. In Sec. II, first we introduce the self-adjoint quantization of the singular Sturm–Liouville problem. Then, we give exact formulas and solutions for the related explicitly time-dependent quantum evolution problem. For this we use the evolution operator method and the Lewis–Riesenfeld invariant method both in the Schrödinger and the Heisenberg pictures. Also, we find exact relation between the Ermakov pairs, which connects the results obtained by the above two methods. In addition, we show that the probability density function is global up to some finite number of removable singularities. In Sec. III, we introduce a wide class of classical singular Sturm–Liouville problems for orthogonal polynomials. The self-adjoint quantization of each problem is considered separately, and the exact solutions are obtained. For some special cases the plots of probability densities, expectation values, and uncertainty relations are constructed explicitly. Section IV is the concluding part were we briefly mention about some relations and applications of this work to different fields of interest.

II. SELF-ADJOINT QUANTIZATION OF STURM–LIOUVILLE PROBLEMS

Consider the classical Sturm–Liouville equation in self-adjoint form,

$$\frac{d}{dt}[\mu(t)\dot{x}(t)] + [s(t) + \lambda r(t)]x(t) = 0, \quad a < t < b, \tag{1}$$

where λ is a spectral parameter, $\mu(t) > 0$ on (a,b), $\mu(t) \in C^1(a,b)$, $s(t) + \lambda r(t) > 0$ on (a,b), and $s(t) + \lambda r(t) \in C(a,b)$ and with allowed singularities occurring at the end points of the fundamental domain (a,b). The reason of the singularities can be the following: (i) (a,b) is an infinite domain, (ii) $\mu(a) = 0$ or/and $\mu(b) = 0$, and (iii) mixture of the first two cases. Then, the boundary conditions are imposed like $x(t)$ must be continuous or bounded or become infinite of an order less than the prescribed, see Ref. 14.

Clearly, the problem (1) can be written in the form of damped parametric oscillator,

$$\ddot{x} + \Gamma(t)\dot{x} + \omega^2(t)x = 0, \quad t \in I = (a,b), \tag{2}$$

with damping coefficient $\Gamma(t) = \dot{\mu}(t)/\mu(t)$ and frequency $\omega^2(t) = (s(t) + \lambda r(t))/\mu(t)$. By transformation of time variable, it is possible to exclude the damping term in Eq. (2), but two classically equivalent systems are not necessarily equivalent at the quantum level. On the other hand, it is known that quantization of the oscillator (2) is not unique. One approach to the problem is based on the Bateman doubled representation when Eq. (2) is complimented with the adjoint one, and thus the total energy in the composed system is conserved. Here, we present the self-adjoint quantization, which leads to Lagrange formulation depending only on the variable $x(t)$ of the

original system. Indeed, the second order differential equation (2) can be written in equivalent, but self-adjoint form,

$$Tx(t) = \frac{d}{dt}[\mu(t)\dot{x}(t)] + \mu(t)\omega^2(t)x(t) = 0, \quad t \in I = (a,b), \tag{3}$$

where

$$T = \frac{d}{dt}\left[\mu(t)\frac{d}{dt}\right] + \mu(t)\omega^2(t)$$

is the corresponding linear differential operator. Then, the action functional corresponding to the formal self-adjoint operator T is

$$S = \frac{1}{2}\langle x(t)|T|x(t)\rangle = \frac{1}{2}\int_a^b (x(t)Tx(t))dt = \frac{1}{2}\int_a^b x(t)\left[\frac{d}{dt}[\mu(t)\dot{x}(t)] + \mu(t)\omega^2(t)x(t)\right]dt$$

$$= \frac{1}{2}\left[\mu(t)x(t)\dot{x}(t)|_a^b - \int_a^b [\mu(t)\dot{x}^2(t) - \mu(t)\omega^2(t)x^2(t)]dt\right].$$

Assuming that the boundary conditions are fixed so that $x(t)$ satisfies $\mu(t)x(t)\dot{x}(t)|_a^b = 0$, we have

$$S = \frac{1}{2}\int_a^b [-\mu(t)\dot{x}^2(t) + \mu(t)\omega^2(t)x^2(t)]dt$$

and the Lagrangian

$$L = \frac{1}{2}\mu(t)[\dot{x}^2(t) - \omega^2(t)x^2(t)],$$

so that the canonical momentum is

$$p = \frac{\partial L}{\partial \dot{x}} = \mu(t)\dot{x}(t).$$

Therefore, the Hamiltonian function for the classical harmonic oscillator (2) with time-dependent damping $\Gamma(t)$ and time-dependent frequency $\omega^2(t)$ is given by

$$H(x,p) = \frac{p^2}{2\mu(t)} + \frac{\mu(t)\omega^2(t)}{2}x^2. \tag{4}$$

The canonical quantization procedure prescribes operator replacement for coordinate $x \to \hat{q}$ and momentum $p \to -i\hbar\, \partial/\partial q$. As a result, we have the quantum Hamiltonian

$$\hat{H}(t) = -\frac{\hbar^2}{2\mu(t)}\frac{\partial^2}{\partial q^2} + \frac{\mu(t)\omega^2(t)}{2}\hat{q}^2 \tag{5}$$

for the related quantum harmonic oscillator. Clearly, if in Eq. (3) we consider $\mu(t)\omega^2(t) \equiv s(t) + \lambda r(t)$, then the self-adjoint quantization of the Sturm–Liouville Eq. (1) leads to the Hamiltonian,

$$\hat{H}(t) = -\frac{\hbar^2}{2\mu(t)}\frac{\partial^2}{\partial q^2} + \left(\frac{s(t) + \lambda r(t)}{2}\right)\hat{q}^2, \tag{6}$$

and the quantum problem corresponding to the classical Sturm–Liouville problem (1) can be called the quantum Sturm–Liouville problem.

For generality and simplicity in notation, in what follows we will work with Hamiltonian (5) and consider the related quantum evolution problem for the Schrödinger equation,

$$i\hbar \frac{\partial}{\partial t}\Psi(q,t) = \hat{H}(t)\Psi(q,t), \tag{7}$$

with initial state

$$\Psi(q,t_0) = \Psi_0(q). \tag{8}$$

We give the solution of this problem using two different methods.

A. The evolution operator method

The evolution operator method, also known as the Wei–Norman algebraic method,[11] is based on the Lie algebraic properties of operators, linear combination of which forms the Hamiltonian. According to this method, the evolution operator can be represented as an ordered product of exponential operators containing single generators of a Lie group. Also, it is well known that there is a relation between solution of the classical equation of motion and the solution of the corresponding quantum problem, see Refs. [1], [11], [19], and [20], etc. Necessary results are adopted and summarized in the following proposition.

Proposition 2.1: *If $x(t)$ is the solution of the classical equation of motion*

$$\ddot{x} + \frac{\dot{\mu}(t)}{\mu(t)}\dot{x} + \omega^2(t)x = 0, \quad x(t_0) = x_0 \neq 0, \quad \dot{x}(t_0) = 0, \tag{9}$$

with $\mu(t) > 0$, $\mu(t) \in C^1(I)$, $\omega^2(t) > 0$, $\omega(t) \in C(I)$, $t_0 \in I \subset R$, then for all $t \in I$, such that $x(t) \neq 0$, the solution of the quantum evolution problem (7) *and* (8) *is given by*

$$\Psi(q,t) = \hat{U}(t,t_0)\Psi_0(q), \tag{10}$$

where

$$\hat{U}(t,t_0) = \exp\left(\frac{i}{2}f(t)q^2\right)\exp\left(h(t)\left(q\frac{\partial}{\partial q} + \frac{1}{2}\right)\right)\exp\left(-\frac{i}{2}g(t)\frac{\partial^2}{\partial q^2}\right) \tag{11}$$

and

$$f(t) = \frac{\mu(t)\dot{x}(t)}{\hbar x(t)}, \tag{12}$$

$$g(t) = -\hbar x^2(t_0)\int^t \frac{d\xi}{\mu(\xi)x^2(\xi)}, \quad g(t_0) = 0, \tag{13}$$

$$h(t) = -\ln|x(t)| + \ln|x(t_0)|. \tag{14}$$

Proof: Conditions $\mu(t) \in C'(I)$, $\omega(t) \in C(I)$ guarantee existence and uniqueness of $x(t)$ in I. To solve the quantum evolution problem (7) and (8) one finds the unitary operator $\hat{U}(t,t_0)$ by solving the operator equation

$$i\hbar \frac{\partial}{\partial t}\hat{U}(t,t_0) = \hat{H}(t)\hat{U}(t,t_0), \quad \hat{U}(t_0,t_0) = \hat{I}, \tag{15}$$

which holds also for time-dependent Hamiltonians. Using the Wei–Norman algebraic procedure, the Hamiltonian (5) is written as time-dependent linear combination,

$$\hat{H}(t) = i\left(-\frac{\hbar^2}{\mu(t)}\hat{K}_- - \mu(t)\omega^2(t)\hat{K}_+\right),$$ (16)

where

$$\hat{K}_- = -\frac{i}{2}\frac{\partial^2}{\partial q^2}, \quad \hat{K}_+ = \frac{i}{2}q^2, \quad \hat{K}_0 = \frac{1}{2}\left(q\frac{\partial}{\partial q} + \frac{1}{2}\right)$$

are operators which satisfy the commutation relations

$$[\hat{K}_-,\hat{K}_+] = 2\hat{K}_0, \quad [\hat{K}_+,\hat{K}_0] = -\hat{K}_+, \quad [\hat{K}_-,\hat{K}_0] = \hat{K}_-$$

of the $SU(1,1)$ algebra. Then, we assume, as an $SU(1,1)$ group element, that

$$\hat{U}(t,t_0) = \exp(f(t)\hat{K}_+)\exp(2h(t)\hat{K}_0)\exp(g(t)\hat{K}_-),$$ (17)

where $f(t)$, $g(t)$, $h(t)$ are real-valued functions to be determined, and that $\hat{U}\hat{U}^* = I$.

The time differentiation gives

$$i\hbar\frac{\partial}{\partial t}\hat{U}(t,t_0) = i\hbar[(\dot{f} - 2f\dot{h} + f^2\dot{g}e^{-2h})\hat{K}_+ + (\dot{g}e^{-2h})\hat{K}_- + (2\dot{h} - 2f\dot{g}e^{-2h})\hat{K}_0]\hat{U}(t,t_0).$$

Therefore, $\hat{U}(t,t_0)$ is a solution of (15), if the auxiliary functions f, g, and h satisfy the nonlinear system of equations,

$$\dot{f} + \frac{\hbar}{\mu(t)}f^2 + \frac{\mu(t)}{\hbar}\omega^2(t) = 0, \quad f(t_0) = 0,$$ (18)

$$\dot{g} + \frac{\hbar e^{2h}}{\mu(t)} = 0, \quad g(t_0) = 0,$$

$$\dot{h} + \frac{\hbar}{\mu(t)}f = 0, \quad h(t_0) = 0.$$

Notice that Eq. (18) is a Riccati equation and by substitution $f(t) = \mu(t)\dot{x}(t)/\hbar x(t)$ can be linearized in the form of the classical damped parametric oscillator (9). Hence, solution $f(t)$ of (18) is obtained explicitly in terms of the solution $x(t)$ of the classical problem. Therefore, the above system has a solution given by (12)–(14). Using that $\Psi(q,t) = \hat{U}(t,t_0)\Psi_0(q)$ and the above results, the proposition is proven.

To obtain an explicit form for evolving in time states $\Psi(q,t)$, we consider the Hamiltonian for the standard harmonic oscillator,

$$\hat{H}_0 = -\frac{\hbar^2}{2}\frac{\partial^2}{\partial q^2} + \frac{\omega_0^2}{2}q^2, \quad \omega_0 = \text{const},$$ (19)

with normalized eigenstates and eigenvalues, respectively,

$$\varphi_k(q) = N_k e^{-(\Omega_0/2)q^2}H_k(\sqrt{\Omega_0}q),$$ (20)

$$E_k = \hbar^2\Omega_0\left(k + \tfrac{1}{2}\right), \quad k = 0,1,2,\ldots,$$

where $H_k(\sqrt{\Omega_0}q)$ are the standard Hermite polynomials, $N_k = (2^k k!)^{-1/2}(\Omega_0/\pi)^{1/4}$ are normalization constants and $\Omega_0 = \omega_0/\hbar$. Since $\{\varphi_k(q)\}_{k=0}^{\infty}$ is an orthonormal basis in the Hilbert space $L_2(R)$, then any initial wave function $\Psi_0(q) \in L_2(R)$ has expansion of the form $\Psi_0(q) = \Sigma_{k=0}^{\infty}\langle\Psi_0,\varphi_k\rangle\varphi_k(q)$,

where $\langle , \rangle$ denotes the standard inner product in $L_2(R)$. Therefore, time-evolved state according to Proposition 2.1 is

$$\Psi(q,t) = \hat{U}(t,t_0)\Psi_0(q) = \sum_{k=0}^{\infty} \langle \Psi_0, \varphi_k \rangle \hat{U}(t,t_0)\varphi_k(q).$$

To find $\Psi_k(q,t) = \hat{U}(t,t_0)\varphi_k(q)$ explicitly, following Ref. 19, we introduce an auxiliary function,

$$\widetilde{\varphi}_k(q;z) = N_k \times \frac{1}{(1+(\Omega_0 z)^2)^{1/4}} \times \exp\left(-i\left(\frac{\Omega_0 z}{1+(\Omega_0 z)^2}\right)\frac{\Omega_0}{2}q^2\right) \times \exp\left(i\left(k+\frac{1}{2}\right)\arctan(\Omega_0 z)\right)$$
$$\times \exp\left(-\left(\frac{1}{1+(\Omega_0 z)^2}\right)\frac{\Omega_0}{2}q^2\right) \times H_k\left(\left(\frac{1}{(1+(\Omega_0 z)^2)^{1/2}}\right)\sqrt{\Omega_0}q\right), \tag{21}$$

with z real and so that $\widetilde{\varphi}_k(q;0) = \varphi_k(q)$. Since $\widetilde{\varphi}_k(q;z)$, satisfies the parabolic wave equation,

$$\left[-\frac{i}{2}\frac{\partial^2}{\partial q^2}\right]\widetilde{\varphi}_k(q;z) = \left[\frac{\partial}{\partial z}\right]\widetilde{\varphi}_k(q;z),$$

then

$$\left[\exp\left(-\frac{i}{2}g(t)\frac{\partial^2}{\partial q^2}\right)\right]\widetilde{\varphi}_k(q;z) = \left[\exp\left(g(t)\frac{\partial}{\partial z}\right)\right]\widetilde{\varphi}_k(q;z),$$

and thus

$$\left[\exp\left(-\frac{i}{2}g(t)\frac{\partial^2}{\partial q^2}\right)\right]\varphi_k(q) = \left[\exp\left(-\frac{i}{2}g(t)\frac{\partial^2}{\partial q^2}\right)\right]\widetilde{\varphi}_k(q;0) = \left[\exp\left(g(t)\frac{\partial}{\partial z}\right)\right]\widetilde{\varphi}_k(q;z)|_{z=0}$$
$$= \widetilde{\varphi}_k(q;z+g(t))|_{z=0} = \widetilde{\varphi}_k(q;g(t)).$$

Hence,

$$\hat{U}(t,t_0)\varphi_k(q) = \exp\left(\frac{i}{2}f(t)q^2\right)\exp\left(h(t)\left(q\frac{\partial}{\partial q}+\frac{1}{2}\right)\right)\exp\left(-\frac{i}{2}g(t)\frac{\partial^2}{\partial q^2}\right)\varphi_k(q)$$
$$= \exp\left(\frac{h(t)}{2}\right)\exp\left(\frac{i}{2}f(t)q^2\right)\widetilde{\varphi}_k(e^{h(t)}q;g(t)).$$

Using definitions of f, g, and h given in Proposition 2.1 and definition (21) of $\widetilde{\varphi}_k(q;z)$, the wave functions are obtained explicitly in terms of the classical solution $x(t)$, that is,

$$\Psi_k(q,t) = N_k\sqrt{R(t)} \times \exp\left(\frac{i}{2}\left(\frac{\mu(t)\dot{x}(t)}{\hbar x(t)}\right)q^2\right) \times \exp\left(-\frac{i}{2}\Omega_0^2 g(t)R^2(t)q^2\right)$$
$$\times \exp\left(i\left(k+\frac{1}{2}\right)\arctan(\Omega_0 g(t))\right) \times \exp\left(-\frac{\Omega_0}{2}R^2(t)q^2\right) \times H_k(\sqrt{\Omega_0}R(t)q), \tag{22}$$

where

$$R(t) = \left(\frac{x_0^2}{x^2(t)+(\Omega_0 x(t)g(t))^2}\right)^{1/2}. \tag{23}$$

The probability density $\rho_k(q,t)$ is then easily found to be

$$\rho_k(q,t) = N_k^2 \times R(t) \times \exp(-(\sqrt{\Omega_0}R(t)q)^2) \times H_k^2(\sqrt{\Omega_0}R(t)q). \tag{24}$$

Notice that the solutions $\Psi_k(q,t)$ and the probability densities $\rho_k(q,t)$ have singularities at the zeros of $x(t)$. However, although the functions $f(t)$, $g(t)$, and $h(t)$ have singularities of an infinite type at the zeros of $x(t)$, we show that under the conditions of Proposition 2.1 and assuming that all zeros of $x(t)$ are simple the probability density $\rho_k(q,t)$ has removable singularities.

Proposition 2.2: *The probability density function $\rho_k(q,t)$, $t \in I$, $q \in R$, has removable singularities at the zeros of $x(t)$.*

Proof: From (24) and definition of $R(t)$, it is clear that all singularity points of $\rho_k(q,t)$ occur at the zeros of $x(t)$. Let us show that they are removable. Suppose τ_0 is a simple zero of $x(t)$, that is, $x(\tau_0)=0$ and $\dot{x}(\tau_0) \neq 0$. Then, for t sufficiently close to τ_0, there is $c_1 \in (\tau_0, t)$ such that $\dot{x}(c_1) \neq 0$ and

$$x(t) = \dot{x}(c_1)(t - \tau_0).$$

On the other hand, since $1/\mu(t) > 0$, $1/\mu(t) \in C'(I)$, then there is $c_2 \in (\tau_0, t)$ such that

$$\frac{1}{\mu(t)} = \frac{1}{\mu(\tau_0)} - \frac{\dot{\mu}(c_2)}{\mu^2(c_2)}(t - \tau_0).$$

Therefore, $\lim_{t\to\tau_0}|g(t)| = \hbar x_0^2 \lim_{t\to\tau_0}|\int^t d\xi/\mu(\xi)x^2(\xi)| = \infty$, and thus we have $\lim_{t\to\tau_0}|x(t)g(t)| = \hbar x_0^2/\mu(\tau_0)|\dot{x}(\tau_0)|$, which implies

$$\lim_{t\to\tau_0} R(t) = \lim_{t\to\tau_0}\left(\frac{x_0^2}{x^2(t) + (\Omega_0 x(t)g(t))^2}\right)^{1/2} = \frac{\mu(\tau_0)|\dot{x}(\tau_0)|}{\hbar\Omega_0|x_0|} < \infty.$$

This shows that the singularities of $\rho_k(q,t)$ are removable.

B. The Lewis–Riesenfeld method

Alternative method for solving the Schrödinger equation (7) is the Lewis–Riesenfeld method, see Ref. 10. It provides a solution in terms of the eigenstates of an explicitly time-dependent invariant and a time-dependent phase factor. Here, we give the main results of the method. For the Hamiltonian $\hat{H}(t)$ the invariant operator $\hat{I}(t)$ is defined by

$$\frac{d\hat{I}}{dt} \equiv \frac{\partial\hat{I}}{\partial t} + \frac{1}{ih}[\hat{I},\hat{H}] = 0, \quad \hat{I}^\dagger = \hat{I},$$

and for the model (5) the explicit form of the invariant is

$$\hat{I}(t) = \frac{1}{2}\left[\frac{1}{\varrho^2}\hat{q}^2 + (\varrho\hat{p} - \mu\dot{\varrho}\hat{q})^2\right], \tag{25}$$

where $\varrho(t)$ satisfies the nonlinear auxiliary equation,

$$\ddot{\varrho} + \frac{\dot{\mu}(t)}{\mu(t)}\dot{\varrho} + \omega^2(t)\varrho = \frac{1}{\mu^2(t)\varrho^3}. \tag{26}$$

Eigenvalues λ_k of the Hermitian operator $\hat{I}(t)$ are real, time independent, and the corresponding eigenstates $\Phi_k(q,t)$ form a complete orthonormal system, that is,

$$\hat{I}(t)\Phi_k(q,t) = \lambda_k\Phi_k(q,t),$$

where $\langle\Phi_k,\Phi_{k'}\rangle = \delta_{k,k'}$. Then, as it is well known, the Schrödinger equation (7) with the Hamiltonian (5) has solutions

$$\widetilde{\Psi}_k(q,t) = e^{i\alpha_k(t)}\Phi_k(q,t),$$

where the eigenstates of $\hat{I}(t)$ are

$$\Phi_k(q,t) = \widetilde{N}_k \times \frac{1}{\sqrt{\varrho}} \times \exp\left(\frac{i\mu(t)}{2\hbar}\left(\frac{\dot{\varrho}}{\varrho} + \frac{i}{\mu(t)\varrho^2} \right)q^2 \right) \times H_k\left(\frac{1}{\sqrt{\hbar}}\frac{q}{\varrho} \right), \tag{27}$$

phase factor is $\alpha_k(t) = \left(k+\frac{1}{2}\right)\eta(t)$, $\widetilde{N}_k = (1/\sqrt{\pi}\sqrt{\hbar}k!2^k)^{1/2}$, $\eta(t) = \int^t -d\xi/\mu(\xi)\varrho^2(\xi)$, $\eta(t_0) = 0$. Thus, the Schrödinger equation (7) has orthonormal solutions given explicitly in the form, see Ref. 21,

$$\widetilde{\Psi}_k(q,t) = \widetilde{N}_k \times \frac{1}{\sqrt{\varrho}} \times \exp\left(i\left(k+\frac{1}{2}\right)\eta(t) \right) \times \exp\left(i\frac{\mu(t)}{2\hbar}\frac{\dot{\varrho}}{\varrho}q^2 \right) \times \exp\left(-\frac{1}{2\hbar\varrho^2}q^2 \right) \times H_k\left(\frac{1}{\sqrt{\hbar}}\frac{q}{\varrho} \right), \tag{28}$$

and the corresponding probability densities are

$$\widetilde{\rho}_k(q,t) = \widetilde{N}_k^2 \times \frac{1}{\varrho} \times \exp\left(-\left(\frac{q}{\sqrt{\hbar}\varrho} \right)^2 \right) \times H_k^2\left(\frac{q}{\sqrt{\hbar}\varrho} \right). \tag{29}$$

One can show that the probability density functions (29) and (24) are same.

In fact, the results obtained by the evolution operator method and the Lewis–Riesenfeld method can be compared using that $\eta(t) = \arctan(\Omega_0 g(t))$ and the following proposition for the Ermakov pairs $x(t)$ and $\varrho(t)$.

Proposition 2.3: *If $x(t)$ is the solution of the initial value problem,*

$$\ddot{x} + \frac{\dot{\mu}(t)}{\mu(t)}\dot{x} + \omega^2(t)x = 0, \quad x(t_0) = x_0 \neq 0, \quad \dot{x}(t_0) = 0,$$

then

$$|\varrho(t)| = \frac{1}{\sqrt{\hbar}\Omega_0}\left(\frac{x^2(t) + (\Omega_0 g(t)x(t))^2}{x_0^2} \right)^{1/2}$$

satisfies the nonlinear initial value problem,

$$\ddot{\varrho} + \frac{\dot{\mu}(t)}{\mu(t)}\dot{\varrho} + \omega^2(t)\varrho = \frac{1}{\mu^2(t)\varrho^3}, \quad \varrho(t_0) = \frac{1}{\sqrt{\hbar}\Omega_0}, \quad \dot{\varrho}(t_0) = 0.$$

C. Heisenberg picture

The position and momentum operators in Heisenberg picture are found by the unitary transformation,

$$\hat{q}_H(t) = \hat{U}^\dagger(t,t_0)\hat{q}_S\hat{U}(t,t_0), \quad \hat{q}_H(t_0) = \hat{q}_S,$$

$$\hat{p}_H(t) = \hat{U}^\dagger(t,t_0)\hat{p}_S\hat{U}(t,t_0), \quad \hat{p}_H(t_0) = \hat{p}_S.$$

Hence, using the evolution operator $\hat{U}(t,t_0)$ which was found before and given by (11), we obtain the position and momentum operators in Heisenberg picture explicitly in terms of the solution $x(t)$ of the classical equation of motion (9),

$$\hat{q}_H(t) = \left(\frac{x(t)}{x(t_0)} \right)\hat{q}_H(t_0) - \left(\frac{x(t)}{\hbar x(t_0)}g(t) \right)\hat{p}_H(t_0),$$

$$\hat{p}_H(t) = \left(\frac{\mu(t)\dot{x}(t)}{x(t_0)}\right)\hat{q}_H(t_0) + \left(\frac{x(t_0)}{x(t)} - \frac{\mu(t)\dot{x}(t)}{\hbar x(t_0)}g(t)\right)\hat{p}_H(t_0). \tag{30}$$

These operators satisfy the Heisenberg equations of motion,

$$\frac{d}{dt}\hat{q}_H(t) = \frac{i}{\hbar}[\hat{H}_H(t), \hat{q}_H(t)] = \frac{\partial \hat{H}_H}{\partial \hat{p}_H} = \frac{\hat{p}_H}{\mu(t)},$$

$$\frac{d}{dt}\hat{p}_H(t) = \frac{i}{\hbar}[\hat{H}_H(t), \hat{p}_H(t)] = -\frac{\partial \hat{H}_H}{\partial \hat{q}_H} = -\mu(t)\omega^2(t)\hat{q}_H,$$

so that $\hat{q}_H(t)$ is a solution of the parametric damped oscillator,

$$\frac{d^2\hat{q}_H(t)}{dt^2} + \frac{\dot{\mu}(t)}{\mu(t)}\frac{d\hat{q}_H(t)}{dt} + \omega^2(t)\hat{q}_H(t) = 0.$$

The position and momentum operators in Heisenberg picture can also be obtained using the Lewis–Riesenfeld invariant, one can see Ref. 22. Accordingly, the results are found in terms of the auxiliary function $\varrho(t)$, that is,

$$\hat{q}_H(t) = \sqrt{\hbar\Omega_0}\varrho(t)\cos\,\eta(t)\hat{q}_H(t_0) - \frac{1}{\sqrt{\hbar\Omega_0}}\varrho(t)\sin\,\eta(t)\hat{p}_H(t_0), \tag{31}$$

$$\hat{p}_H(t) = \sqrt{\hbar\Omega_0}\left(\frac{1}{\varrho(t)}\sin\,\eta(t) + \mu(t)\dot{\varrho}(t)\cos\,\eta(t)\right)\hat{q}_H(t_0)$$

$$+ \frac{1}{\sqrt{\hbar\Omega_0}}\left(\frac{1}{\varrho(t)}\cos\,\eta(t) - \mu(t)\dot{\varrho}(t)\sin\,\eta(t)\right)\hat{p}_H(t_0). \tag{32}$$

Again, using Proposition 2.3 and the relations

$$\cos\,\eta(t) = \frac{1}{\sqrt{1 + (\Omega_0 g(t))^2}}, \quad \sin\,\eta(t) = \frac{\Omega_0 g(t)}{\sqrt{1 + (\Omega_0 g(t))^2}},$$

one can show that the equations for $\hat{q}_H(t)$ and $\hat{p}_H(t)$ found first using the evolution operator, and then using the Lewis–Riesenfeld invariant coincide.

D. Coherent states

In previous parts exact solutions of the time-dependent Schrödinger equation were given. Now, we briefly discuss the coherent states for the harmonic oscillator with time-dependent parameters. There are many works on the coherent states in the literature. Here, we mention only a few of them related with our problem. For the standard harmonic oscillator, one can see Refs. 23 and 24, and for oscillators with time-dependent parameters, Refs. 1, 22, and 25. The coherent states can be defined in two ways: as eigenstates of the annihilation operator and as a result of applying the unitary displacement operator on the ground state.

1. Annihilation-operator coherent states

By the Lewis–Riesenfeld approach, the invariant operator $\hat{I}(t)$ defined by (25) can be written as

$$\hat{I}(t) = \hbar\left(\hat{A}^\dagger(t)\hat{A}(t) + \tfrac{1}{2}\right),$$

where the annihilation and creation operators are given, respectively, by

$$\hat{A}(t) = \frac{1}{\sqrt{2\hbar}}\left[\left(\frac{1}{\varrho(t)} - i\mu(t)\dot{\varrho}(t)\right)\hat{q}(t) + i\varrho(t)\hat{p}(t)\right],$$

$$\hat{A}^\dagger(t) = \frac{1}{\sqrt{2\hbar}}\left[\left(\frac{1}{\varrho(t)} + i\mu(t)\dot{\varrho}(t)\right)\hat{q}(t) - i\varrho(t)\hat{p}(t)\right],$$

and the operators $\hat{I}$, $\hat{A}$, and $\hat{A}^\dagger$ generate the Heisenberg–Weyl algebra,

$$[\hat{A}(t),\hat{A}^\dagger(t)] = \hat{I}(t), \quad [\hat{I}(t),\hat{A}(t)] = -\hat{A}(t), \quad [\hat{I}(t),\hat{A}^\dagger(t)] = \hat{A}^\dagger(t).$$

A coherent state $\phi_\alpha(q,t_0)$ at time $t=t_0$ is defined as an eigenstate of the annihilation operator $\hat{A}(t_0)$ with eigenvalue α,

$$\hat{A}(t_0)\phi_\alpha(q,t_0) = \alpha\phi_\alpha(q,t_0). \tag{33}$$

Assume a coherent state of the form $\phi_\alpha(q,t_0) = \Sigma_{k=0}^\infty c_k \Phi_k(q,t_0)$, where $\Phi_k(q,t_0)$ are eigenstates of $\hat{I}(t_0)$. Taking the inner product of (33) by $\Phi_k(q,t_0)$ gives

$$\langle A(t_0)\phi_\alpha(q,t_0), \Phi_k(q,t_0)\rangle = \alpha\langle\phi_\alpha(q,t_0),\Phi_k(q,t_0)\rangle,$$

and using that $\hat{A}^\dagger(t)\Phi_k = \sqrt{k+1}\Phi_{k+1}$, one can find

$$\sqrt{k+1}\langle\phi_\alpha(q,t_0),\Phi_{k+1}(q,t_0)\rangle = \alpha\langle\phi_\alpha(q,t_0),\Phi_k(q,t_0)\rangle.$$

This is a recurrence relation which gives $c_k = (\alpha^k/\sqrt{k!})\langle\phi_\alpha(q,t_0),\Phi_0(q,t_0)\rangle$, and thus

$$\phi_\alpha(q,t_0) = \langle\phi_\alpha(q,t_0),\Phi_0(q,t_0)\rangle\sum_k \frac{\alpha^k}{\sqrt{k!}}\Phi_k(q,t_0).$$

To have $\|\phi_\alpha(q,t_0)\|^2 = 1$, we can choose $\langle\phi_\alpha(q,t_0),\Phi_0(q,t_0)\rangle = -\exp(|\alpha|^2)$. Therefore, the normalized coherent states are

$$\phi_\alpha(q,t_0) = e^{-|\alpha|^2/2}\sum_k \frac{\alpha^k}{\sqrt{k!}}\Phi_k(q,t_0) \tag{34}$$

for arbitrary complex number $\alpha, \alpha = \alpha_1 + i\alpha_2$. The normalized coherent states of the time-dependent oscillator at arbitrary time t can be found as time evolution of the coherent states (34), that is, $\phi_\alpha(q,t) = \hat{U}(t,t_0)\phi_\alpha(q,t_0)$. Using that

$$\hat{U}(t,t_0)\Phi_k(q,t_0) = \hat{U}(t,t_0)\Psi_k(q,t_0) = \Psi_k(q,t) = e^{i\alpha_k(t)}\Phi_k(q,t),$$

we have the coherent states in terms of the eigenstates of $\hat{I}(t)$,

$$\phi_\alpha(q,t) = e^{-|\alpha|^2/2}\sum_k \frac{\alpha^k}{\sqrt{k!}}e^{i\alpha_k(t)}\Phi_k(q,t), \tag{35}$$

or the coherent states in terms of the wave functions of the Schrödinger equation

$$\phi_\alpha(q,t) = e^{-|\alpha|^2/2}\sum_k \frac{\alpha^k}{\sqrt{k!}}\Psi_k(q,t). \tag{36}$$

2. Displacement operator coherent states

The coherent states are also defined as states created by a unitary displacement operator acting on the ground state. The ground state $\Phi_0(q,t_0)$ at $t=t_0$ is defined by $\hat{A}(t_0)\Phi_0(q,t_0)=0$. Then, one has $\phi_\alpha(q,t_0)=\hat{D}_0(\alpha)\Phi_0(q,t_0)$, where $\hat{D}_0(\alpha)=\exp(\alpha\hat{A}^\dagger(t_0)-\bar{\alpha}\hat{A}(t_0))$ is a unitary displacement operator at t_0. Thus, coherent states of the time-dependent oscillator are

$$\phi_\alpha(q,t) = \hat{U}(t,t_0)\hat{D}_0(\alpha)\Phi_0(q,t_0).$$

Using that $\hat{D}_0(\alpha)=\exp(-|\alpha|^2/2)\exp(\alpha\hat{A}^\dagger(t_0))\exp(-\alpha\hat{A}(t_0))$, it is not difficult to verify that the displacement operator approach leads to the same results (35) and (36) for the coherent states.

3. Schrödinger's coherent states

The first attempt to construct nonspreading wave packets was made by Schrödinger.[26] Considering linear harmonic oscillator, he succeeded to find the wave packet, localized at point $q=A\cos\omega t$, in the form of the Gaussian function. It was the first example of the coherent states, with maximally classical properties. In this part, we apply Schrödinger's procedure but for the damped parametric oscillator with time-dependent parameters. The wave function (22) can be written in the form

$$\Psi_k(q,t) = N_k B(q,t) e^{ik\eta} e^{-\xi^2/2} H_k(\xi), \tag{37}$$

where $B(q,t)=\sqrt{R(t)}\times\exp(i(\mu x/2\hbar x-\Omega_0^2 gR^2/2)q^2+(i/2)\eta)$ and $\xi(q,t)=\sqrt{\Omega_0}R(t)q$. Substituting (37) in (36), we have

$$\phi_\alpha(q,t) = \widetilde{B}(q,t)e^{-\xi^2/2}\sum_{k=0}^{\infty}\frac{z^k}{k!}H_k(\xi), \tag{38}$$

where $\widetilde{B}(q,t)=(\Omega_0/\pi)^{1/4}e^{-|\alpha|^2/2}B(q,t)$ and $z(t)=\alpha e^{i\eta}/\sqrt{2}$. Using generating function for the Hermite polynomials, $\sum_{k=0}^{\infty}(z^k/k!)H_k(\xi)=e^{-z^2+2z\xi}$, the above equation (38) can be written as

$$\phi_\alpha(q,t) = C(q,t)\exp\left[-\left(\alpha\cos\eta-\frac{\xi}{\sqrt{2}}\right)^2\right], \tag{39}$$

where $C(q,t)=\widetilde{B}(q,t)\exp[\alpha^2/2]\exp[-i((\alpha^2/2)\sin 2\eta+\sqrt{2}\alpha(\sin\eta)\xi)]$.

In expression (39) the Gaussian packet is localized at the roots of the equation $\alpha\cos\eta-\xi/\sqrt{2}=0$. It describes the relation between the classical position $x(t)$ and the quantum position q in the form

$$q = \frac{\alpha}{\sqrt{\Omega_0}}\frac{x(t)}{x(t_0)}, \tag{40}$$

so that the wave packet follows the classical trajectory.

E. Expectation values and uncertainty relations

Using that $\langle\hat{q}\rangle=\langle\phi_\alpha(q,t_0)|\hat{q}_H(t)|\phi_\alpha(q,t_0)\rangle$, the expectation value of the position operator at the coherent state $\phi_\alpha(q,t)$ is

$$\langle\hat{q}\rangle = \sqrt{2\hbar}|\alpha|\varrho(t)\sin(\eta(t)+\delta), \tag{41}$$

where $\tan\delta=\alpha_1/\alpha_2$. Similarly using that $\langle\hat{p}\rangle=\langle\phi_\alpha(q,t_0)|\hat{p}_H(t)|\phi_\alpha(q,t_0)\rangle$, the expectation value of the momentum operator at the coherent state $\phi_\alpha(q,t)$ is

$$\langle \hat{p} \rangle = \frac{-\sqrt{2\hbar}|\alpha|}{\varrho(t)} \cos(\eta(t) + \delta) + \sqrt{2\hbar}|\alpha|\mu(t)\dot{\varrho}(t)\sin(\eta(t) + \delta). \tag{42}$$

One can find also

$$(\Delta \hat{q})^2 = \frac{\hbar}{2}\varrho^2(t), \quad (\Delta \hat{p})^2 = \frac{\hbar}{2}\left(\frac{1}{\varrho^2} + \mu\dot{\varrho}^2\right),$$

and therefore, the uncertainty relation for the damped oscillator with time-dependent coefficients is

$$\Delta \hat{q}\Delta \hat{p} = \frac{\hbar}{2}\sqrt{1 + \mu^2\varrho^2\dot{\varrho}^2}, \tag{43}$$

from which it follows also that $\Delta \hat{q}\Delta \hat{p} \geq \hbar/2$. This is a well known result, and one can see, for example, Refs. 21 and 27.

III. INTEGRABLE QUANTUM STURM–LIOUVILLE PROBLEMS

A. Quantization of Hermite equation

Consider the classical Sturm–Liouville problem given by the Hermite differential equation in standard form,

$$\ddot{x} - 2t\dot{x} + \lambda x = 0, \quad -\infty < t < \infty,$$

or in self-adjoint form,

$$\frac{d}{dt}(e^{-t^2}\dot{x}) + \lambda e^{-t^2}x = 0, \quad -\infty < t < \infty,$$

with boundary conditions that the eigenfunctions should not become infinite at $t = \pm\infty$ of an order higher than a finite power of t. Then, the eigenvalues are $\lambda = 2n$, $n = 0,1,2,\ldots$ with the corresponding eigenfunctions $x_n(t) = H_n(t)$, where

$$H_n(t) = (-1)^n e^{t^2} \frac{d^n}{dt^n} e^{-t^2}$$

are the Hermite polynomials. They satisfy the known relations

$$H_{n+1}(t) = 2tH_n - 2nH_{n-1}, \quad \dot{H}_n(t) = 2nH_{n-1}(t).$$

For quantization, we consider the oscillator form of Hermite equation,

$$\ddot{H}_n - 2t\dot{H}_n + 2nH_n = 0, \quad H_n(t_0) = x_{n0} \neq 0, \quad \dot{H}_n(t_0) = 0, \tag{44}$$

where $\Gamma(t) = -2t$ is the damping, $\omega^2 = 2n$ is the frequency, and $\mu(t) = e^{-t^2}$ is the integrating factor. According to the general discussion made before, the corresponding Lagrangian for the coordinate $x(t) = H_n(t)$ is

$$L = \frac{1}{2}e^{-t^2}(\dot{x}^2(t) - 2nx^2(t)),$$

the classical Hamiltonian is

$$H(x,p) = \frac{1}{2}(e^{t^2}p^2 + 2ne^{-t^2}x^2),$$

and the corresponding quantum Hamiltonian is

$$\hat{H}(t) = -\hbar^2 \frac{e^{t^2}}{2} \frac{\partial^2}{\partial q^2} + n e^{-t^2} q^2. \tag{45}$$

Therefore, for each fixed $n = 0, 1, 2, \ldots$, we can find the wave function solutions $\Psi_k^n(q, t), k = 0, 1, 2, \ldots$, of the quantum evolution problem with Hamiltonian (45) in terms of the Hermite polynomial $H_n(t)$. The corresponding probability density is then

$$\rho_k^n(q, t) = N_k^2 \times R_n(t) \times \exp(-(R_n(t)\sqrt{\Omega_0}q)^2) \times H_k^2(R_n(t)\sqrt{\Omega_0}q),$$

$$R_n(t) = \left(\frac{H_n^2(t_0)}{H_n^2(t) + [\Omega_0 g_n(t) H_n(t)]^2} \right)^{1/2},$$

$$g_n(t) = -\hbar H_n^2(t_0) \int^t \frac{e^{\xi^2} d\xi}{H_n^2(\xi)}, \quad g(t_0) = 0.$$

Also, using the relations

$$|\varrho_n(t)| = \frac{1}{\sqrt{\hbar \Omega_0}} \sqrt{\frac{H_n^2(t) + (\Omega_0 g_n(t) H_n(t))^2}{H_0^2}},$$

$$\cos \eta(t) = \frac{1}{\sqrt{1 + (\Omega_0 g_n(t))^2}}, \quad \sin \eta(t) = \frac{\Omega_0 g_n(t)}{\sqrt{1 + (\Omega_0 g_n(t))^2}},$$

and formulas (41)–(43), one can find the expectation values and the uncertainty relation as follows:

$$\langle \hat{q} \rangle = \sqrt{2\hbar} |\alpha| \varrho_n(t) \sin(\eta(t) + \delta), \quad \langle \hat{p} \rangle = \frac{-\sqrt{2\hbar}|\alpha|}{\varrho_n(t)} \cos(\eta(t) + \delta) + \sqrt{2}|\alpha|\mu(t)\dot{\varrho}_n(t)\sin(\eta(t) + \delta),$$

$$\Delta \hat{q} \Delta \hat{p} = \frac{\hbar}{2} \sqrt{1 + \mu^2 \varrho_n^2 \dot{\varrho}_n^2}.$$

Clearly, when n is even, for quantization we can take $t_0 = 0$, since the initial conditions $H_n(0) \neq 0$ and $\dot{H}_n(0) = 0$ hold for even n. When n is odd there exists t_0 such that the initial conditions $H_n(t_0) \neq 0$ and $\dot{H}_n(t_0) = 0$ are again satisfied. Hence, using the above formulas and remarks about the initial conditions, one can obtain the plots for desired cases. As an example, we consider plots for the ground state case $k = 0$ and $n = 4$, ($\hbar = 1$, $\Omega_0 = 1$, and $x_0 = 1$ in all following plots).

From Fig. 1 for probability density, we see that essentially nontrivial localization of the particle takes place at time interval $|t| \leq 1$, and at $|t| \gg 1$, the probability density spreads along q-coordinate. This can be understood from Hamiltonian (45), where we have the potential energy of the form $V(q, t) \sim e^{-t^2} q^2$ and the mass $\mu(t) = e^{-t^2}$. Then, strength of the potential is exponentially decreasing in time and the potential influence is essential only at times $|t| \leq 1$. For $|t| \gg 1$ and finite q, the potential vanishes and thus the particle motion becomes free.

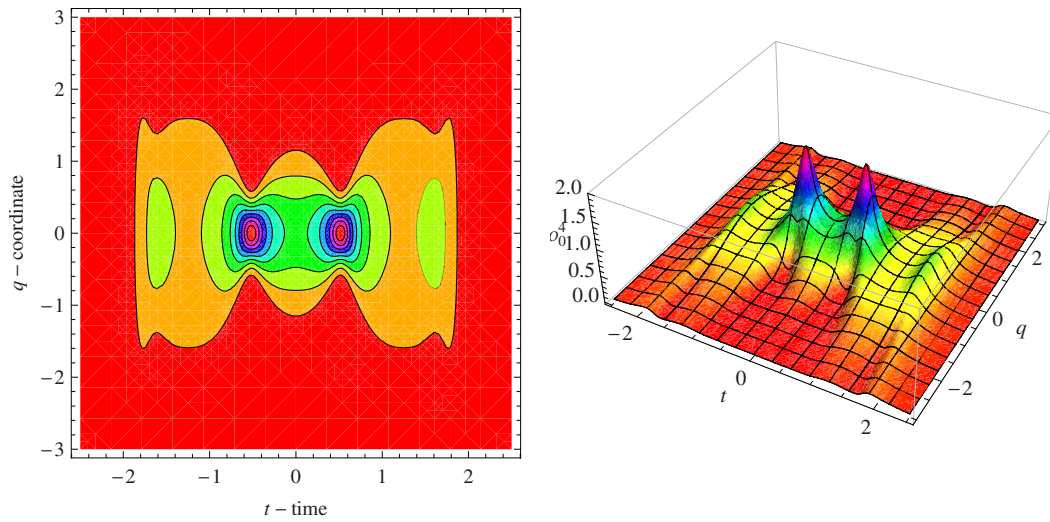In general, the number and shape of the peaks, appearing in the probability density plot,

FIG. 1. (Color online) Contour plot and $3d$-plot of the probability density for Hermite oscillator, $k=0$ and $n=4$.

depend both on the zeros of the classical solution $x_n(t)$ and the moving zeros of the Hermite polynomial $H_k(R_n(t)\sqrt{\Omega_0}q)$. Notice that in Fig. 1, for $k=0$ and $n=4$, there are four peaks related with zeros of the Hermite polynomial $x_4(t)=H_4(t)$.

In Fig. 2 (left), the phase-plane trajectory $\langle q(t)\rangle$, $\langle p(t)\rangle$ for the Hermite oscillator, shows bounded motion of the particle at finite time interval near the origin. For $|t|\to\infty$, the particle is almost free, $\langle q\rangle\to\infty$ (due to vanishing potential), with close to zero momentum $\langle p\rangle$ (due to vanishing mass). The right, Fig. 2, shows that the minimum uncertainty relation is greater than zero, more precisely $\Delta\hat{q}\Delta\hat{p}\geq\frac{1}{2}$. Also, in a finite time interval near $t=0$ the uncertainty have oscillatory behavior, while for $|t|\to\infty$, the uncertainty grows.
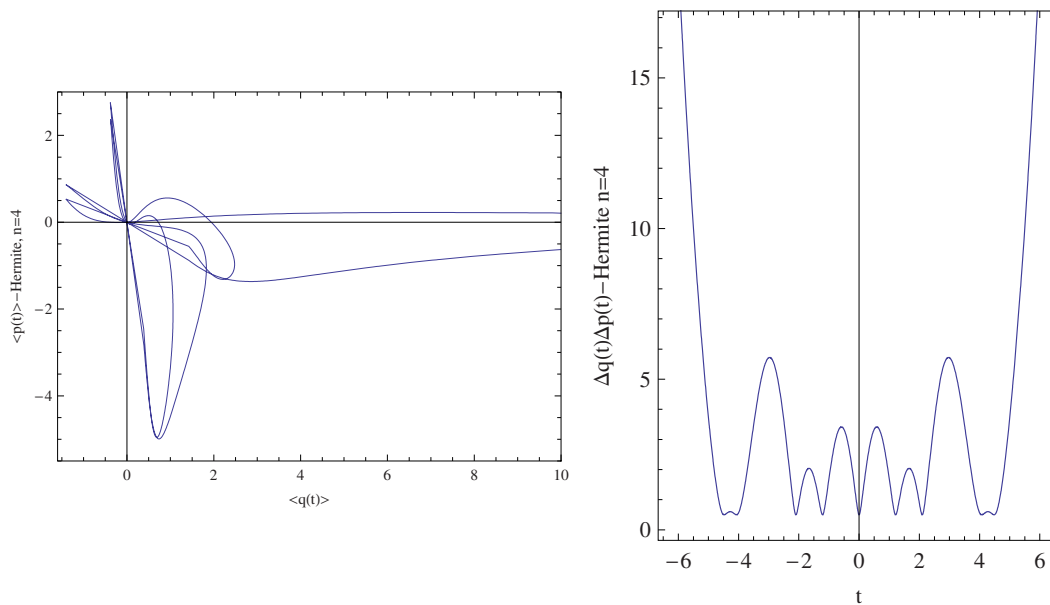

FIG. 2. (Color online) Phase-plane diagram (left) and uncertainty relation (right) for Hermite oscillator, $n=4$ and $\alpha=1/\sqrt{2}-i1/\sqrt{2}$.

B. Quantization of first kind Chebyshev equation

Consider the classical Sturm–Liouville problem for the first kind Chebyshev equation given in standard form,

$$(1 - t^2)\ddot{x} - t\dot{x} + \lambda x = 0, \quad -1 < t < 1,$$

or in self-adjoint form,

$$\frac{d}{dt}(\sqrt{1 - t^2}\dot{x}) + \frac{\lambda}{\sqrt{1 - t^2}}x = 0, \quad -1 < t < 1,$$

with boundary conditions that the eigenfunctions are regular at $t = \pm 1$. Then, the eigenvalues are $\lambda = n^2$, $n = 0, 1, 2, \ldots$ with the corresponding eigenfunctions—the first kind Chebyshev polynomials $x_n(t) = T_n(t)$ defined as

$$T_n(t) = [(-1)^{n/2}]\cos(n \arccos(t)), \quad t \in (-1, 1). \tag{46}$$

These polynomials satisfy the relations

$$T_{n+1}(t) = 2tT_n(t) - T_{n-1}(t), \quad (1 - t^2)\dot{T}_n(t) = -ntT_n(t) + nT_{n-1}(t).$$

For quantization, we consider the oscillator form of first kind Chebyshev equation,

$$\ddot{T}_n - \frac{t}{1 - t^2}\dot{T}_n + \frac{n^2}{1 - t^2}T_n = 0, \quad T_n(t_0) = x_{n0} \neq 0, \quad \dot{T}_n(t_0) = 0, \tag{47}$$

where $\Gamma(t) = -t/(1 - t^2)$ is the damping term, $\omega^2(t) = n^2/(1 - t^2)$ is the frequency, and $\mu(t) = \sqrt{1 - t^2}$. The related Lagrangian for the coordinate $x(t) = T_n(t)$ is

$$L = \frac{1}{2}\sqrt{1 - t^2}\left(\dot{x}^2 - \frac{n^2}{1 - t^2}x^2\right),$$

the classical Hamiltonian is

$$H(x, p) = \frac{1}{2\sqrt{1 - t^2}}p^2 + \frac{n^2}{2\sqrt{1 - t^2}}x^2,$$

and the quantum Hamiltonian is

$$\hat{H}(t) = -\frac{\hbar^2}{2\sqrt{1 - t^2}}\frac{\partial^2}{\partial q^2} + \frac{n^2}{2\sqrt{1 - t^2}}q^2. \tag{48}$$

Hence, for each fixed $n = 0, 1, 2, \ldots$, we can find the wave function solutions, $\Psi_k^n(q, t)$, $k = 0, 1, 2, \ldots$, of the quantum evolution problem for the Hamiltonian (48) in terms of the first kind Chebyshev polynomial $T_n(t)$, with properly chosen initial time t_0. The probability density is
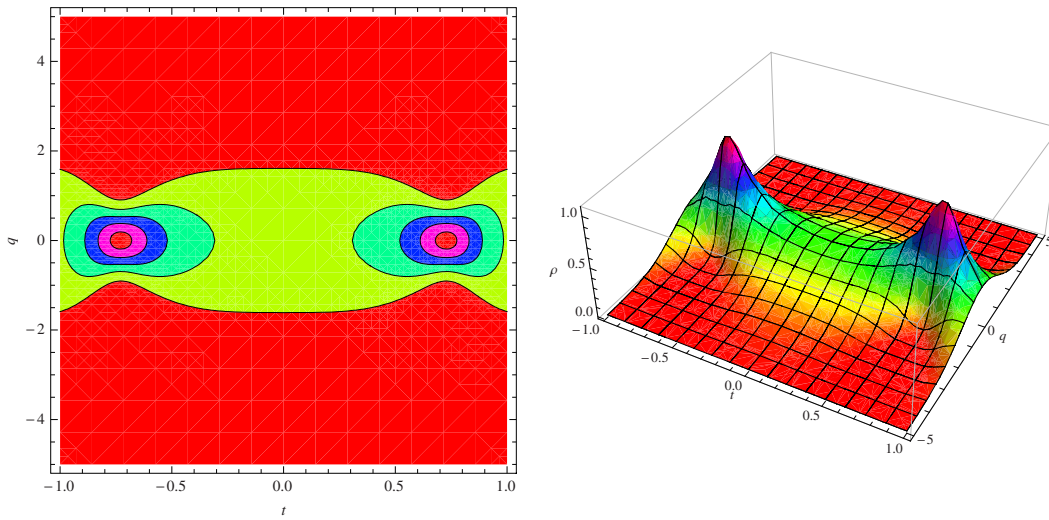
FIG. 3. (Color online) Contour plot and $3d$-plot of the probability density for first kind Chebyshev oscillator $k=0$ and $n=2$.
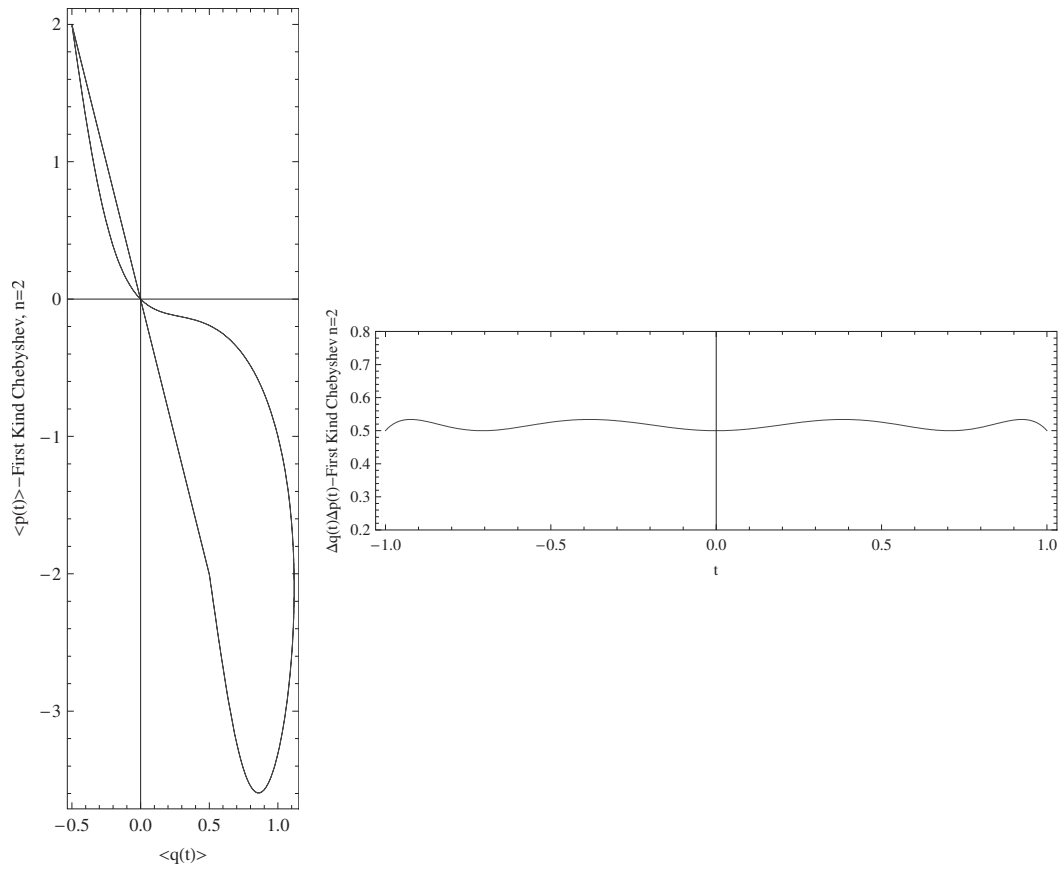

FIG. 4. Phase-plane diagram (left) and uncertainty relation (right) for first kind Chebyshev oscillator, $n=2$ and $\alpha=1/\sqrt{2}$ $-i1/\sqrt{2}$.

$$\rho_k^n(q,t) = N_k^2 \times R_n(t) \times \exp(-(R_n(t)\sqrt{\Omega_0}q)^2) \times H_k^2(R_n(t)\sqrt{\Omega_0}q),$$

where

$$R_n(t) = \left(\frac{T_n^2(t_0)}{T_n^2(t) + [\Omega_0 g(t)T_n(t)]^2}\right)^{1/2}, \quad g_n(t) = -\hbar T_n^2(t_0)\int^t \frac{d\xi}{\sqrt{1-\xi^2 T_n^2(\xi)}}, \quad g_n(t_0) = 0.$$

Using that $|\varrho_n(t)| = 1/\sqrt{\hbar\Omega_0}\sqrt{T_n^2(t) + (\Omega_0 g_n(t)T_n(t))^2/T_0^2}$, $\cos\eta(t) = 1/\sqrt{1+(\Omega_0 g_n(t))^2}$, $\sin\eta(t) = \Omega_0 g_n(t)/\sqrt{1+(\Omega_0 g_n(t))^2}$, and formulas (41)–(43), one can find the expectation values and the uncertainty relation.

For Hamiltonian (48), the potential is of the form $V(q,t) \sim 1/\sqrt{1-t^2}q^2$, and the mass is $\mu(t) = \sqrt{1-t^2}$. Then, the potential influence becomes essential in the neighborhood of $t = \pm 1$, where the strength of the potential becomes infinite and the mass approaches zero. As we see in Fig. 3 for the probability density, this shows strong localization of the particle near the origin at times $t = \pm 0.7$. The phase-plane diagram for the first kind Chebyshev oscillator in the left Fig. 4 represents bounded motion of the particle, and the right, Fig. 4, shows oscillatory bounded uncertainty on the whole time interval $(-1,1)$.

C. Quantization of second kind Chebyshev equation

The classical Sturm–Liouville problem for the second kind Chebyshev equation is

$$(1-t^2)\ddot{x} - 3t\dot{x} + \lambda x = 0, \quad -1 < t < 1,$$

or in self-adjoint form

$$\frac{d}{dt}[(1-t^2)^{3/2}\dot{x}] + \lambda(1-t^2)^{1/2}x = 0,$$

with the boundary conditions that the eigenfunctions are regular at $t = \pm 1$. Then, the eigenvalues are $\lambda = n(n+2)$, $n = 0, 1, 2, \ldots$, and the corresponding eigenfunctions $x_n(t) = U_n(t)$ are the second kind Chebyshev polynomials,

$$U_n(t) = [(-1)^{n/2}]\frac{\sin[(n+1)\arccos(t)]}{\sqrt{1-t^2}}, \quad -1 < t < 1,$$

which satisfy the known relations,

$$U_{n+1}(t) = 2tU_n(t) - U_{n-1}(t), \quad (1-t^2)\dot{U}_n(t) = -ntU_n(t) + (n+1)U_{n-1}(t).$$

For quantization, we consider the oscillator form of second kind Chebyshev equation,

$$\ddot{U}_n - \frac{3t}{1-t^2}\dot{U}_n + \frac{n(n+2)}{1-t^2}U_n = 0, \quad U_n(t_0) = x_{n0} \neq 0, \quad \dot{U}_n(t_0) = 0, \tag{49}$$

where $\Gamma(t) = -3t/(1-t^2)$ is the damping term, $\omega^2(t) = n(n+2)/(1-t^2)$ is the frequency, and $\mu(t) = (1-t^2)^{3/2}$ is the integration factor. Hence, we have the Lagrangian for the coordinates $x(t) = U_n(t)$,

$$L = \frac{1}{2}(1-t^2)^{3/2}\left(\dot{x}^2(t) - \frac{n(n+2)}{1-t^2}x^2(t)\right),$$
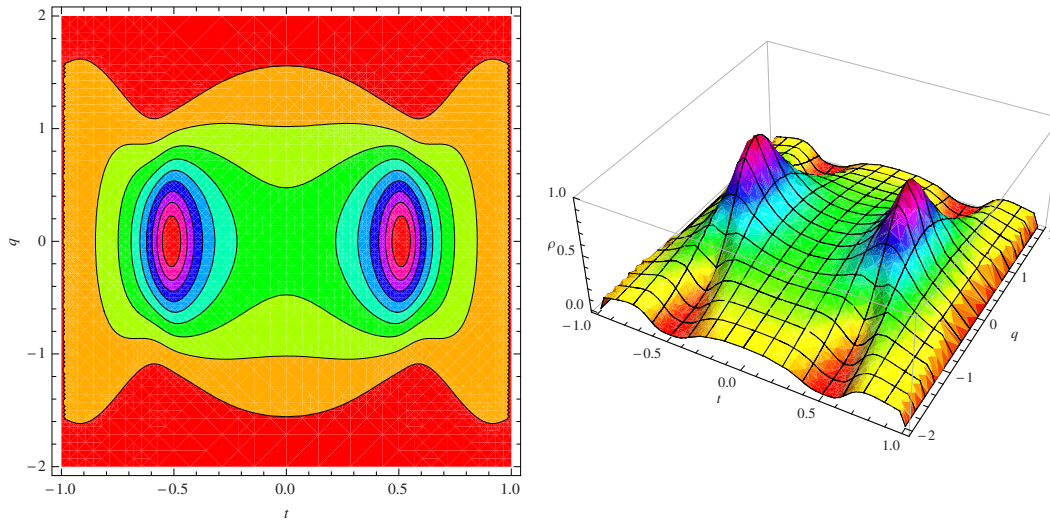
the classical Hamiltonian,

FIG. 5. (Color online) Contour plot and 3d-plot of the probability density for second kind Chebyshev oscillator, k $=0$, $n=2$.

$$H(x,p) = \frac{1}{2}\left(\frac{1}{(1-t^2)^{3/2}}p^2 + n(n+2)\sqrt{1-t^2}x^2\right),$$

and then the corresponding quantum Hamiltonian,

$$\hat{H}(t) = -\frac{\hbar^2}{2(1-t^2)^{3/2}}\frac{\partial^2}{\partial q^2} + \frac{n(n+2)\sqrt{1-t^2}q^2}{2}. \tag{50}$$
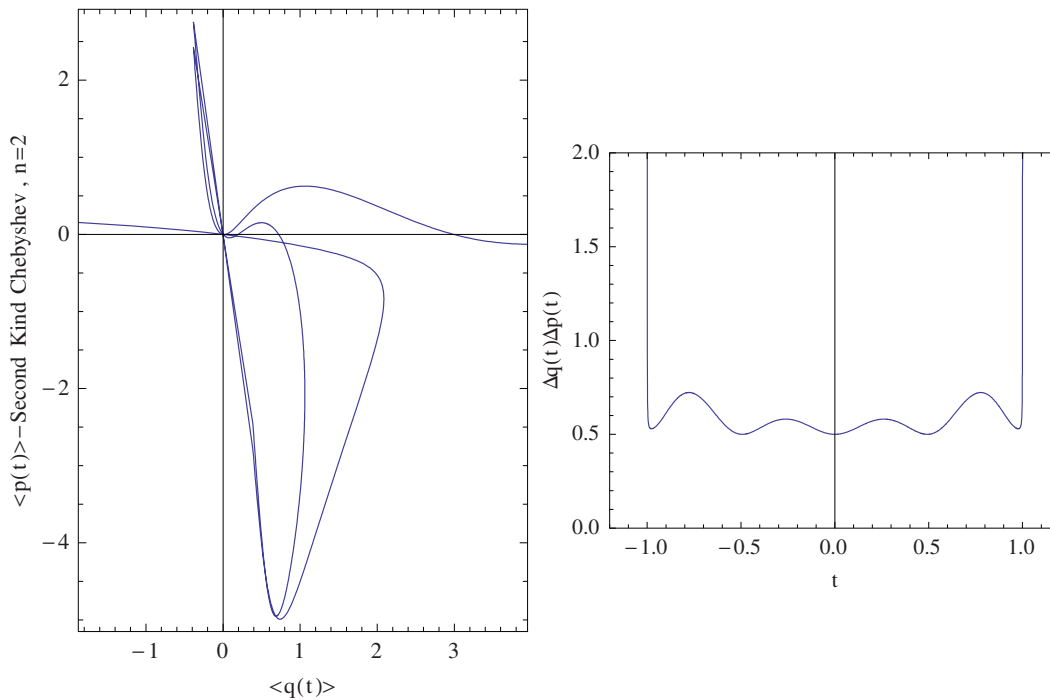

FIG. 6. (Color online) Phase-plane diagram (left) and uncertainty relation (right) for second kind Chebyshev oscillator, $n=2$ and $\alpha = 1/\sqrt{2} - i1/\sqrt{2}$, $(-1 < t < 1)$.

Computation of probability densities, expectation values, and uncertainty relations can be done as in the previous cases, so we omit writing the formulas and give just plots illustrating the behavior of the second kind Chebyshev oscillator in case $k=0$, $n=2$.

From the Hamiltonian (50) for the second kind Chebyshev oscillator, the potential and mass are, respectively, $V(q,t) \sim \sqrt{1-t^2}q^2$ and $\mu(t)=(1-t^2)^{3/2}$. Clearly, for $t \to \pm 1$ and finite q, both potential and mass approach zero, so that the particle exhibits a free motion in the neighborhood of $t= \pm 1$. The essential localization of the particle is concentrated near the origin at times $t = \pm 0.5$, see Fig. 5. The left part of Fig. 6 shows that for $t \to \pm 1$, the particle motion becomes unbounded, while in the rest of the time interval, the motion is bounded. Except for $t \to \pm 1$, where the uncertainty sharply grows, the uncertainty exhibits small oscillations near the minimum uncertainty value of 0.5, as seen in the right part of Fig. 6.

D. Quantization of shifted Chebyshev equation

The classical Sturm–Liouville problem for the shifted Chebyshev equation is defined by

$$t(1-t)\ddot{x} + \frac{1-2t}{2}\dot{x} + \lambda x = 0, \quad 0 < t < 1,$$

or in self-adjoint form,

$$\frac{d}{dt}[\sqrt{t(1-t)}\dot{x}] + \lambda \sqrt{t(1-t)}x = 0,$$

and the boundary conditions that the eigenfunctions are regular at $t=0$ and $t=1$. Then, the eigenvalues are $\lambda=n^2$, $n=0,1,2,\ldots$ with the corresponding eigenfunctions—the Shifted Chebyshev polynomials,

$$T_n^* = [(-1)^{n/2}]\cos(n \arccos(2t-1)),$$

which satisfy the recurrence relation,

$$T_{n+1}^*(t) = (-2+4t)T_n^*(t) - T_{n-1}^*(t),$$

and the differential relation,

$$4t(1-t)\dot{T}_n^*(t) = -(2t-1)T_n^*(t) + T_{n-1}^*(t).$$

Note that the Shifted Chebyshev polynomials can also be represented in terms of the first kind Chebyshev polynomials as follows:

$$T_n^*(t) = T_n(2t-1),$$

where T_n^* is the shifted Chebyshev and T_n is the first kind Chebyshev polynomial.

For quantization, we consider the shifted Chebyshev equation in oscillator form,

$$\ddot{T}_n^* + \frac{1-2t}{2t(1-t)}\dot{T}_n^* + \frac{n^2}{t(1-t)}T_n^* = 0, \quad T_n^*(t_0) = x_{n0} \neq 0, \quad \dot{T}_n^*(t_0) = 0, \tag{51}$$

where $\Gamma(t)=(1-2t)/2t(1-t)$, $\omega^2(t)=n^2/t(1-t)$, $\mu(t)=\sqrt{t(1-t)}$. Hence, we have the Lagrangian,

$$L = \frac{1}{2}\sqrt{t(1-t)}\left(\dot{x}^2(t) - \frac{n^2}{t(1-t)}x^2(t)\right),$$
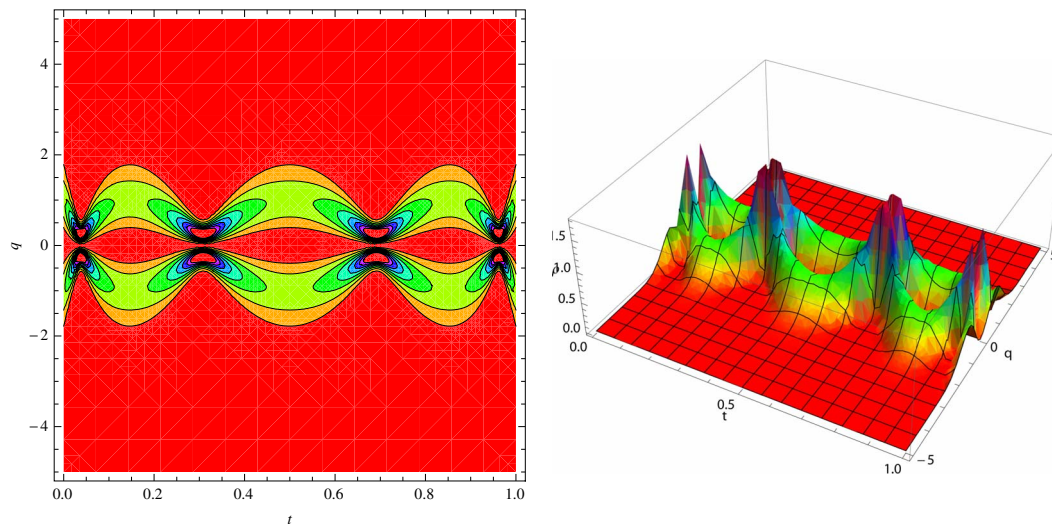
the classical Hamiltonian,

FIG. 7. (Color online) Contour plot and $3d$-plot of the probability density for shifted Chebyshev oscillator, $k=1$ and $n=4$.
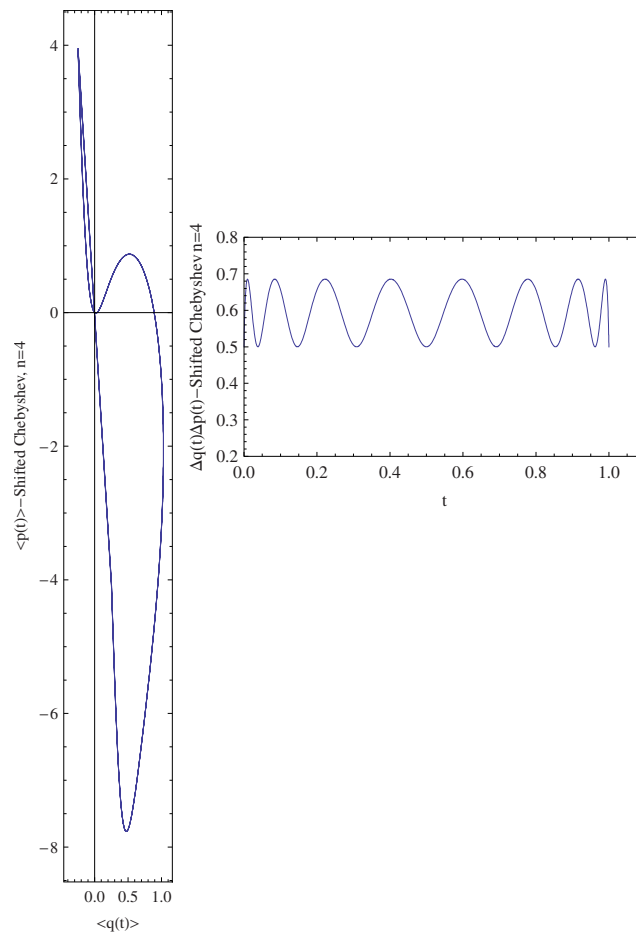

FIG. 8. (Color online) Phase-plane diagram (left) and uncertainty relation (right) for shifted Chebyshev oscillator, $n=4$, $\alpha=1/\sqrt{2}-i1/\sqrt{2}$, $(0<t<1)$.

$$H(x,p) = \frac{1}{2}\left(\frac{1}{\sqrt{t(1-t)}}p^2 + \frac{n^2}{\sqrt{t(1-t)}}x^2 \right),$$

and the corresponding quantum Hamiltonian,

$$\hat{H}(t) = -\frac{\hbar^2}{2\sqrt{t(1-t)}}\frac{\partial^2}{\partial q^2} + \frac{n^2}{2\sqrt{t(1-t)}}q^2. \tag{52}$$

From Hamiltonian (52) for shifted Chebyshev oscillator, the potential is $V(q,t) \sim 1/\sqrt{t(1-t)}$ and mass is $\mu(t) = \sqrt{t(1-t)}$.

Clearly, the qualitative behavior of the potential and mass of the shifted Chebyshev oscillator is similar to the behavior of the first kind Chebyshev oscillator. Thus, one can read Fig. 7 and 8 according to the previous discussions.

Notice that, in this case ($k=1$, $n=4$), the number and shape of the peaks in the probability density diagram are related with the zeros of the shifted Chebyshev polynomial $T_4^*(t)$ and the moving zeros of the Hermite polynomial $H_1(\sqrt{\Omega_0}R_4(t)q)$.

E. Quantization of Laguerre equation

The Sturm–Liouville problem for the Laguerre equation is defined by

$$t\ddot{x} + (1-t)\dot{x} + \lambda x = 0, \quad 0 < t < \infty,$$

or in self-adjoint form,

$$\frac{d}{dt}(te^{-t}\dot{x}) + \lambda e^{-t}x = 0,$$

and the boundary conditions that the eigenfunctions must be finite at $t=0$, and, as $t \to \infty$, they must not become infinite of an order higher than a positive power of t. Then, the eigenvalues are $\lambda = n$, $n = 0,1,2,\ldots$ with eigenfunctions—the Laguerre polynomials,

$$L_0(t) = 1, \quad L_n(t) = \frac{e^t}{n!}\frac{d^n}{dt^n}(t^n e^{-t}),$$

for which one has the relations

$$(n+1)L_{n+1}(t) = (2n+1-t)L_n(t) - nL_{n-1}(t), \quad t\dot{L}_n(t) = nL_n(t) - nL_{n-1}(t).$$

For quantization, we consider the oscillator form of the Laguerre equation,

$$\ddot{L}_n + \frac{(1-t)}{t}\dot{L}_n + \frac{n}{t}L_n = 0, \quad L_n(t_0) = x_{n0} \neq 0, \quad \dot{L}_n(t_0) = 0, \tag{53}$$

where $\Gamma(t) = (1-t)/t$, $\omega^2(t) = n/t$, and $\mu(t) = te^{-t}$. Then, we have the Lagrangian,

$$L = \frac{1}{2}te^{-t}\left(\dot{x}^2 - \frac{n}{t}x^2 \right),$$

the classical Hamiltonian,

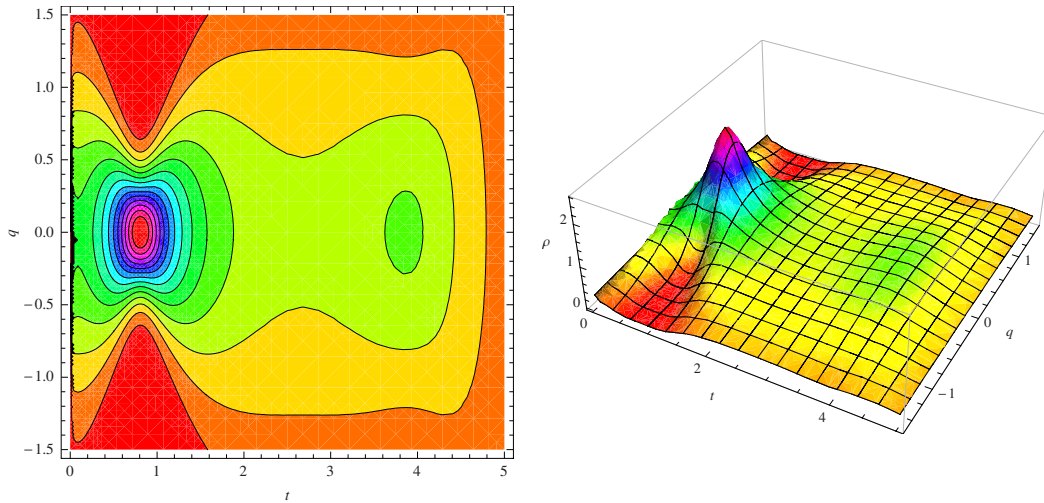$$H(x,p) = \frac{e^t}{2t}p^2 + \frac{n}{2e^t}x^2,$$

and the quantum Hamiltonian,

FIG. 9. (Color online) Contour plot and 3d-plot of the probability density for Laguerre oscillator, $k=0$, $n=2$.

$$\hat{H}(t) = -\hbar^2 \frac{e^t}{2t} \frac{\partial^2}{\partial q^2} + \frac{n}{2e^t} q^2. \tag{54}$$

The probability density for the Laguerre damped oscillator is therefore

$$\rho_k^n(q,t) = |\Psi_k^n(q,t)|^2 = N_k^2 \times R_n(t) \times \exp(-(R_n(t)\sqrt{\Omega_0}q)^2) \times H_k^2(R_n(t)\sqrt{\Omega_0}q),$$

where

$$R_n(t) = \left(\frac{L_n^2(t_0)}{L_n^2(t) + [\Omega_0 g_n(t) L_n(t)]^2} \right)^{1/2}, \quad g_n(t) = -\hbar L_n^2(t_0) \int^t \frac{e^\xi d\xi}{\xi L_n^2(\xi)}, \quad g_n(t_0) = 0.$$

For calculating the expectation values and the uncertainty relation one can use

$$|\varrho_n(t)| = \frac{1}{\sqrt{\hbar \Omega_0}} \sqrt{\frac{L_n^2(t) + (\Omega_0 g_n(t) L_n(t))^2}{L_0^2}},$$

$$\cos \eta(t) = \frac{1}{\sqrt{1 + (\Omega_0 g_n(t))^2}}, \quad \sin \eta(t) = \frac{\Omega_0 g_n(t)}{\sqrt{1 + (\Omega_0 g_n(t))^2}}.$$

The potential energy for the Laguerre oscillator is of the form $V(q,t) \sim e^{-t}q^2$ and the mass is $\mu(t) = te^{-t}$. Then, the influence of the potential is essential at a finite time interval and the particle is strongly localized near the origin at time $t=0.6$ (see Fig. 9). For $t \geq 1$ and finite q, the potential strength vanishes, the particle motion becomes free and probability density spreads along the
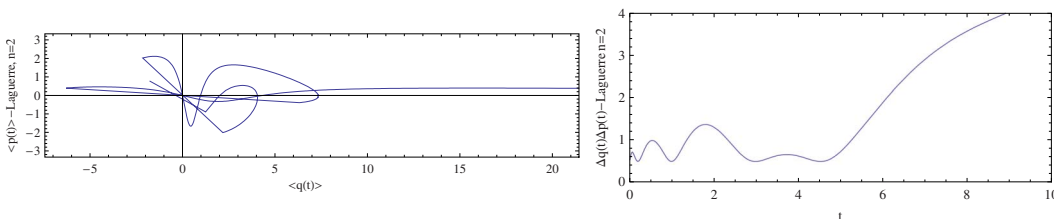

FIG. 10. (Color online) Phase-plane diagram (left) and uncertainty relation (right) for Laguerre oscillator, $n=2$, $\alpha = 1/\sqrt{2} - i1/\sqrt{2}$.

q-coordinate. In Fig. 10 (left), the phase-plane trajectory shows bounded motion of the particle at times close to zero, while for $t \gg 1$ the particle is almost free with close to zero momentum. This explains also the bounded uncertainty near $t=0$ and the growing uncertainty when $t \to \infty$ [Fig. 10 (right)].

F. Quantization of associated Laguerre equation

The Sturm–Liouville problem for the associated Laguerre differential equation is defined by

$$t\ddot{x} + (m + 1 - t)\dot{x} + \lambda x = 0, \quad 0 < t < \infty,$$

or in the self-adjoint form,

$$\frac{d}{dt}(t^{m+1}e^{-t}\dot{x}) + \lambda t^{m+1}e^{-t}x = 0,$$

with the boundary conditions that the eigenfunctions must be finite at $t=0$, and, as $t \to \infty$, they must be of a polynomial order. Then, the eigenvalues are $\lambda = n$, $n = 0,1,2,\dots$ and the corresponding eigenfunctions are the associated Laguerre polynomials,

$$L_n^m(t) = \frac{e^t t^{-m}}{n!} \frac{d^n}{dx^n}(e^{-t}t^{n+m}), \quad m > -1,$$

which also can be obtained by differentiating regular Laguerre polynomials, that is,

$$L_n^m(t) = (-1)^n \frac{d^m}{dt^m} L_{n+m}(t).$$

These polynomials satisfy the recurrence relation,

$$(n+1)L_{n+1}^m(t) = (2n + m + 1 - t)L_n^m(t) - (n+m)L_{n-1}^m(t),$$

and the differential relation,

$$t\dot{L}_n^m(t) = nL_n^m(t) - (n+m)L_{n-1}^m(t).$$

For quantization, we consider the associated Laguerre equation in oscillator form,

$$\ddot{L}_n^m + \frac{(m+1-t)}{t}\dot{L}_n^m + \frac{n}{t}L_n^m = 0, \quad L_n^m(t_0) = x_0 \neq 0, \quad \dot{L}_n^m(t_0) = 0, \tag{55}$$

where $\Gamma(t) = (m+1-t)/t$, $\omega^2(t) = n/t$, and $\mu(t) = t^{m+1}e^{-t}$. Then, we have the Lagrangian

$$L = \frac{1}{2}t^{m+1}e^{-t}\left(\dot{x}^2(t) - \frac{n}{t}x^2(t)\right),$$

the classical Hamiltonian,

$$H(x,p) = \frac{1}{2}\left(t^{-(m+1)}e^t p^2 + \frac{nt^m}{e^t}x^2\right),$$

and thus the corresponding quantum Hamiltonian is

$$\hat{H}(t) = -\hbar^2 \frac{e^t}{2t^{m+1}} \frac{\partial^2}{\partial q^2} + \frac{nt^m}{2e^t}q^2. \tag{56}$$

G. Quantization of Legendre equation

Consider the Sturm–Liouville problem for the Legendre equation given in standard form,

$$(1 - t^2)\ddot{x} - 2t\dot{x} + \lambda x = 0, \quad -1 < t < 1,$$

or in self-adjoint form,

$$\frac{d}{dt}[(1 - t^2)\dot{x}] + \lambda x = 0, \quad -1 < t < 1,$$

with boundary conditions that the eigenfunctions are finite at the singular points $t = \pm 1$. Then, the eigenvalues are $\lambda = n(n+1)$, $n = 0, 1, 2, \ldots$ and the corresponding eigenfunctions $x_n(t) = P_n(t)$ are the Legendre polynomials,

$$P_0 = 1, \quad P_n(t) = \frac{1}{2^n n!} \frac{d^n}{dt^n} (t^2 - 1)^n,$$

which satisfy the relations

$$(1 - t^2)\dot{P}_n(t) = -ntP_n(t) + nP_{n-1}(t),$$

$$(n + 1)P_{n+1}(t) = (2n + 1)tP_n(t) - nP_{n-1}(t).$$

For quantization, we consider the Legendre equation in oscillator form,

$$\ddot{P}_n - \frac{2t}{1 - t^2}\dot{P}_n + \frac{n(n + 1)}{1 - t^2}P_n = 0, \quad P_n(t_0) = x_{n0} \neq 0, \quad \dot{P}_n(t_0) = 0, \tag{57}$$

where the damping is $\Gamma(t) = -2t/(1 - t^2)$, the frequency is $\omega^2(t) = n(n+1)/(1-t^2)$, and the integrating factor is $\mu(t) = (1 - t^2)$. Hence, we have the related Lagrangian,

$$L = \frac{1}{2}(1 - t^2)\left(\dot{x}^2 - \frac{n(n + 1)}{1 - t^2}x^2\right),$$

the classical Hamiltonian,

$$H(x, p) = \frac{1}{2(1 - t^2)}p^2 + \frac{n(n + 1)}{2}x^2,$$

and the quantum Hamiltonian,

$$\hat{H}(t) = -\frac{\hbar^2}{2(1 - t^2)}\frac{\partial^2}{\partial q^2} + \frac{n(n + 1)}{2}q^2. \tag{58}$$

Then, for each fixed $n = 0, 1, 2, \ldots$, we can find the wave function solutions $\Psi_k^n(q, t)$, $k = 0, 1, 2, \ldots$, of the quantum evolution problem in terms of the classical Legendre polynomial P_n, and the probability density is

$$\rho_k^n(q, t) = N_k^2 \times R_n(t) \times \exp(-(R_n(t)\sqrt{\Omega_0}q)^2) \times H_k^2(R_n(t)\sqrt{\Omega_0}q),$$
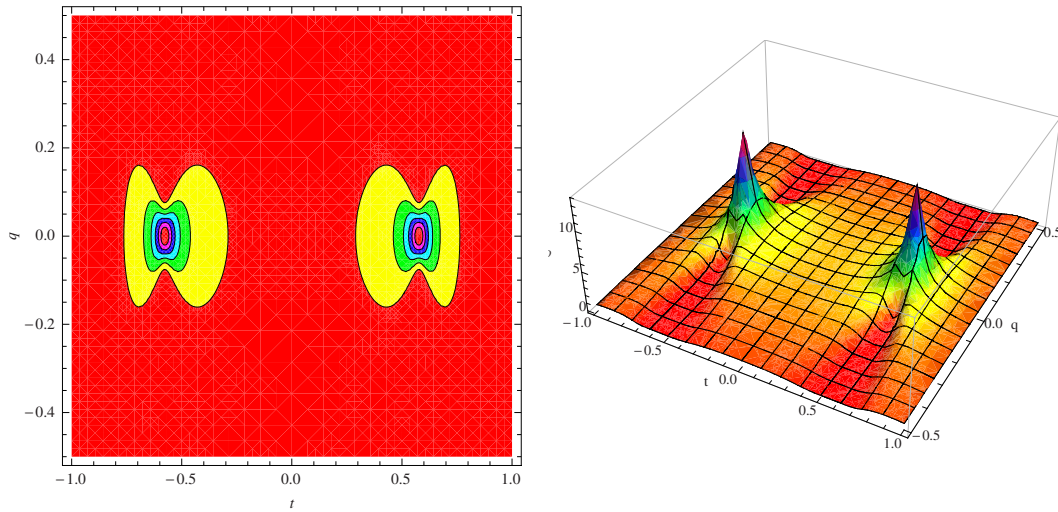
where

FIG. 11. (Color online) Contour plot and $3d$-plot of the probability density for Legendre oscillator, $k=0$, $n=2$.

$$R_n(t) = \left(\frac{P_n^2(t_0)}{P_n^2(t) + [\Omega_0 g_n(t) P_n(t)]^2} \right)^{1/2}, \quad g_n(t) = -\hbar P_n^2(t_0) \int^t \frac{d\xi}{(1-\xi^2) P_n^2(\xi)}, \quad g_n(t_0) = 0.$$

The potential energy of the Legendre oscillator is just the potential of the stationary Harmonic oscillator with coupling coefficient $n(n+1)$. The mass of the particle is $\mu(t) = 1 - t^2$, which vanishes for $|t| \to \pm 1$. The contour and $3d$-plot for the probability density in case $k=0$, $n=2$ are shown in Fig. 11. Similarly to the first kind Chebyshev case, the uncertainty relation is not growing with time, but shows small oscillations near value of 0.5 (see Fig. 12).
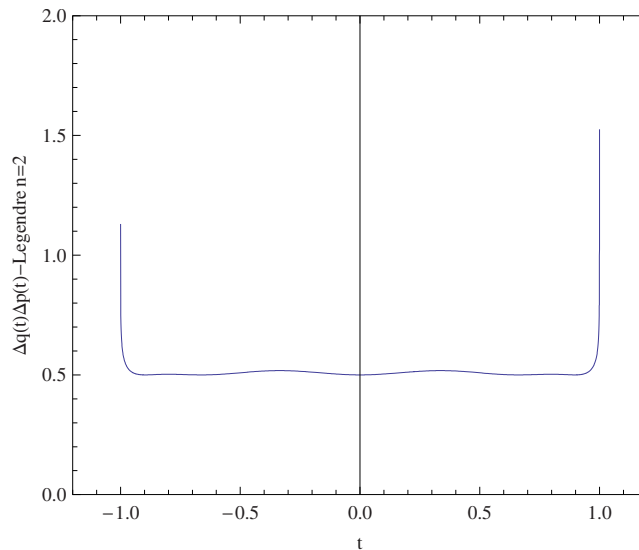

FIG. 12. (Color online) Uncertainty relation for Legendre oscillator, $n=2$, $(-1 < t < 1)$.

H. Quantization of Jacobi equation

The Sturm–Liouville problem for the Jacobi differential equation is given by

$$(1 - t^2)\ddot{x} + (\beta - \alpha - (\alpha + \beta + 2)t)\dot{x} + \lambda x = 0, \quad -1 < t < 1,$$

or in self-adjoint form,

$$\frac{d}{dt}[(1 - t)^{\alpha+1}(1 + t)^{\beta+1}\dot{x}] + \lambda(1 - t)^{\alpha}(1 + t)^{\beta}x = 0,$$

and the boundary conditions that the eigenfunctions are finite at the singular points $t = \pm 1$. Then the eigenvalues are $\lambda = n(n + \alpha + \beta + 1)$, $n = 0, 1, 2, \ldots$, $\alpha, \beta > -1$, and the corresponding eigenfunctions are the Jacobi polynomials,

$$P_n^{\alpha,\beta}(t) = \frac{(-1)^n}{2^n n!}(1 - t)^{-\alpha}(1 + t)^{-\beta}\frac{d^n}{dt^n}[(1 - t)^{\alpha+n}(1 + t)^{\beta+n}]. \tag{59}$$

They satisfy the recurrence relation,

$$2(n + 1)(n + \alpha + \beta + 1)(2n + \alpha + \beta)P_{n+1}^{(\alpha,\beta)}(t) = [(2n + \alpha + \beta + 1)(\alpha^2 - \beta^2) + (2n + \alpha + \beta)t]P_n^{(\alpha,\beta)}(t)$$
$$+ 2(n + \alpha)(n + \beta)(2n\alpha + \beta + 2)P_{n-1}^{(\alpha,\beta)}(t),$$

and the differential relation,

$$(2n + \alpha + \beta)(1 - t^2)\frac{d}{dx}P_n^{(\alpha,\beta)} = n[\alpha - \beta - (2n + \alpha + \beta)t]P_n^{(\alpha,\beta)} + 2(n + \alpha)(n + \beta)P_{n-1}^{(\alpha,\beta)}.$$

For quantization, we consider the Jacobi equation in oscillator form

$$\ddot{P}_n^{\alpha,\beta} + \frac{(\beta - \alpha - (\alpha + \beta + 2)t)}{1 - t^2}\dot{P}_n^{\alpha,\beta} + \frac{n(n + \alpha + \beta + 1)}{1 - t^2}P_n^{\alpha,\beta} = 0, \quad -1 < t < 1,$$

where the damping is $\Gamma(t) = (\beta - \alpha - (\alpha + \beta + 2)t)/(1 - t^2)$, frequency is $\omega^2(t) = n(n + \alpha + \beta + 1)/(1 - t^2)$, and the integration factor is $\mu(t) = (1 - t)^{\alpha+1}(1 + t)^{\beta+1}$.

The corresponding Lagrangian, classical Hamiltonian, and quantum Hamiltonian are, respectively,

$$L = \frac{1}{2(1 - t)^{\alpha+1}(1 + t)^{\beta+1}}\left(\dot{x}^2(t) - \frac{n(n + \alpha + \beta + 1)}{1 - t^2}x^2(t)\right),$$

$$H(x,p) = \frac{1}{2}\left(\frac{p^2}{(1 - t)^{\alpha+1}(1 + t)^{\beta+1}} + (1 - t)^{\alpha}(1 + t)^{\beta}(1 - t^2)[n(n + \alpha + \beta + 1)]x^2\right),$$

$$\hat{H}(t) = -\frac{\hbar^2}{2(1 - t)^{\alpha+1}(1 + t)^{\beta+1}}\frac{\partial^2}{\partial q^2} + \frac{(1 - t)^{\alpha}(1 + t)^{\beta}(1 - t^2)[n(n + \alpha + \beta + 1)]}{2}q^2.$$

I. Quantization of hypergeometric equation

The standard form for the Hypergeometric equation is

$$t(1 - t)\ddot{x} + [c - (a + b + 1)t]\dot{x} - abx = 0, \quad 0 < t < 1, \tag{60}$$

Where, in general, the independent variables t and a, b, c can be complex numbers. With certain restrictions on the parameters a, b, c and the variable t, solutions of the hypergeometric differential equation are $_2F_1(a, b, c; t)$, $t^{1-c} {}_2F_1(a + 1 - c, b + 1 - c, 2 - c; t)$, $c \neq 0, -1, -2, -3 \ldots$, where

$$_2F_1(a,b,c;t) = 1 + \frac{ab}{c}\frac{t}{1!} + \frac{a(a+1)b(b+1)}{c(c+1)}\frac{t^2}{2!} + \cdots = \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_n}\frac{t^n}{n!}$$

denotes the hypergeometric functions, known also as the hypergeometric series. If $t \in (0,1)$ and α, β are real numbers, such that $c = \alpha+1$, $a+b = \alpha+\beta+1$, $ab = -\lambda$, one obtains Sturm–Liouville form of the hypergeometric equation,

$$\frac{d}{dt}(t^{\alpha+1}(1-t)^{\beta+1}\dot{x}) + \lambda t^{\alpha}(1-t)^{\beta}x = 0. \tag{61}$$

Its oscillator form is

$$\ddot{x} + \left[\frac{(\alpha+1)+(\beta-\alpha)t}{t(1-t)}\right]\dot{x} + \frac{\lambda}{t(1-t)}x = 0, \tag{62}$$

where $\Gamma(t) = ((\alpha+1)+(\beta-\alpha)t)/t(1-t)$ is the damping term, $\omega^2(t) = \lambda/t(1-t)$ is the frequency, and $\mu(t) = t^{\alpha+1}(1-t)^{\beta+1}$. Then, we have the Lagrangian,

$$L = \frac{1}{2}t^{\alpha+1}(1-t)^{\beta+1}\left[\dot{x}^2 - \frac{\lambda}{t(1-t)}x^2\right],$$

the classical Hamiltonian,

$$H(x,p) = \frac{p^2}{2t^{\alpha+1}(1-t)^{\beta+1}} + \frac{\lambda t^{\alpha}(1-t)^{\beta}}{2}x^2,$$

and the quantum Hamiltonian,

$$\hat{H}(t) = -\frac{\hbar^2}{2t^{\alpha+1}(1-t)^{\beta+1}}\frac{\partial^2}{\partial q^2} + \frac{\lambda t^{\alpha}(1-t)^{\beta}}{2}q^2.$$

J. Quantization of confluent hypergeometric equation

The standard form of the confluent Hypergeometric equation is

$$t\ddot{x} + (c-t)\dot{x} - ax = 0, \quad t > 0, \tag{63}$$

which can be written in self-adjoint form,

$$\frac{d}{dt}[t^c e^{-t}\dot{x}] - at^c e^{-t}x = 0.$$

The solutions of (63) can be written in terms of the functions $_1F_1(a;c;t)$, $U(a,c,t)$, where $_1F_1(a;c;t)$ is a confluent hypergeometric function of the first kind and $U(a,c,t)$ of second kind. The oscillator representation of the confluent hypergeometric equation is

$$\ddot{x} + \frac{(c-t)}{t}\dot{x} + \frac{a}{t}x = 0, \tag{64}$$

where $\Gamma(t) = (c-t)/t$, $\omega^2(t) = a/t$, and $\mu(t) = t^c e^{-t}$. The corresponding Lagrangian, classical Hamiltonian, and quantum Hamiltonian are, respectively,

$$L = \frac{e^t}{2t^c}\left(\dot{x}^2(t) - \frac{a}{t}x^2(t)\right), \quad H(x,p) = \frac{1}{2}\left(\frac{e^t}{t^c}p_x^2 + \frac{at^{c-1}}{e^t}x^2\right),$$

$$\hat{H}(t) = -\hbar^2\frac{e^t}{2t^c}\frac{\partial^2}{\partial q^2} + \frac{at^{c-1}}{2e^t}q^2.$$

IV. CONCLUSION

Quantization of the classical Sturm–Liouville problems, which has been considered in this paper, provides a reach set of exactly solvable quantum-mechanical problems of harmonic oscillator with a specific time-dependent frequency and damping. Explicit solutions for probability densities, uncertainty relations, and phase space trajectories were given in terms of classical orthogonal polynomials. So, we found new application for the hypergeometric equation to a harmonic oscillator with variable parameters and its quantization.

The formal analogy between the Sturm–Liouville equation and the parametric damped oscillator considered in this paper could have some direct physical application in the general relativity theory. As it is well known, the Schwarzschild metric inside of black hole interchanges the minus sign from the time coordinate t to the space coordinate r. This means that inside the event horizon, r is the timelike coordinate and t is the spacelike coordinate. Then, the time is defined by the r coordinate and the space is defined by t coordinate. Thus, at the event horizon, when $r=2M$, the swapping of space and time, changing the role of space and time occurs.

Finally, we like to mention some application of our results to the nonlinear dynamics. There exists a relation between the linear time-dependent Schrödinger equation and the nonlinear complex Burgers equation,[28] which is a complex version of the known in the soliton theory the Cole–Hopf transformation. But, in fact, in quantum mechanics, it is just the Madelung hydrodynamic representation. Then, motion of the zeroes of the wave function corresponds to moving poles in the hydrodynamic representation. Application of these relations to the exactly solvable systems considered in the present paper provides dynamics of poles in the complex Burgers–Schrödinger equation. Results of that work would be published elsewhere.

ACKNOWLEDGMENTS

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