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The Extended Discrete (G'/G)-expansion Method and Its Application to the Relativistic Toda Lattice System

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Abstract. We propose the extended discrete (G'/G)-expansion method for directly solving nonlinear differential-difference equations. For illustration, we choose the relativistic Toda lattice system. We derive further discrete hyperbolic and trigonometric function traveling wave solutions, as well as discrete rational function solutions.

Keywords: (G'/G)-expansion method; Relativistic Toda lattice system; Lattice equation; Differential-difference equation
PACS: 02.30.Jr; 05.45.Yv; 02.30.Ik

INTRODUCTION

Nonlinear differential-difference equations (NDDEs) can be encountered in many branches of applied physical sciences such as biophysics, atomic chains, condensed matter physics, molecular crystals, quantum physics and discretization in solid state, etc. They also play a crucial role in numerical simulation of soliton dynamics in high energy physics because of their rich structures. Accordingly, researchers have given a wide interest to the study of NDDEs since the original work of Fermi, Pasta and Ulam in the 1950s [1].

While little work is being done to search for exact solutions of NDDEs as far as we could verify, a fairly good number of effective and powerful analytic methods for solving nonlinear evolution equations (NEEs) has been presented in the literature. Recently, Wang et. al [2] developed a powerful analytic expansion method, the so-called (G'/G)-expansion method, to solve NEEs of mathematical physics. Our main goal in this study will be to introduce the extended discrete (G'/G)-expansion method for solving NDDEs.

THE EXTENDED DISCRETE (G'/G)-EXPANSION METHOD

Let us consider a system of M polynomial NDDEs in the form

$$\Delta\left(\mathbf{u}_{\mathbf{n}+\mathbf{p}_1}(\mathbf{x}),\ldots,\mathbf{u}_{\mathbf{n}+\mathbf{p}_k}(\mathbf{x}),\ldots,\mathbf{u}'_{\mathbf{n}+\mathbf{p}_1}(\mathbf{x}),\ldots,\mathbf{u}'_{\mathbf{n}+\mathbf{p}_k}(\mathbf{x}),\ldots,\mathbf{u}^{(r)}_{\mathbf{n}+\mathbf{p}_1}(\mathbf{x}),\ldots,\mathbf{u}^{(r)}_{\mathbf{n}+\mathbf{p}_k}(\mathbf{x})\right)=0\,,\tag{1}$$

in which the dependent variable $\mathbf{u}_{\mathbf{n}}$ have M components $u_{i,\mathbf{n}}$ and so do its shifts; the continuous variable $\mathbf{x}$ has N components x_i; the discrete variable $\mathbf{n}$ has Q components n_j; the k shift vectors $\mathbf{p}_i \in \mathbb{Z}^Q$; and $\mathbf{u}^{(r)}(\mathbf{x})$ denotes the collection of mixed derivative terms of order r.

Step 1. To find traveling wave solutions of Eq. (1), we first make the wave transformation

$$\mathbf{u}_{\mathbf{n}+\mathbf{p}_s}(\mathbf{x}) = \mathbf{U}_{\mathbf{n}+\mathbf{p}_s}(\xi_{\mathbf{n}}),\ \xi_{\mathbf{n}} = \sum_{i=1}^{Q} d_i n_i + \sum_{j=1}^{N} c_j x_j + \zeta\ (s=1,2,\ldots,k),\tag{2}$$

where the coefficients $c_1,c_2,\ldots,c_k,d_1,d_2,\ldots,d_Q$ and the phase ζ are all constants. Then, Eq. (1) changes into

$$\Delta\left(\mathbf{U}_{n+p_1}(\xi_n),\ldots,\mathbf{U}_{n+p_k}(\xi_n),\ldots,\mathbf{U}'_{n+p_1}(\xi_n),\ldots,\mathbf{U}'_{n+p_k}(\xi_n),\ldots,\mathbf{U}^{(r)}_{n+p_1}(\xi_n),\ldots,\mathbf{U}^{(r)}_{n+p_k}(\xi_n)\right)=0 \ . \tag{3}$$

Step 2. We suppose that the solution of Eq. (3) is in the finite series expansion form

$$\mathbf{U}_n(\xi_n)=\sum_{l=-m}^{m}a_i\left(\frac{G'(\xi_n)}{G(\xi_n)}\right)^l \ , \tag{4}$$

where m is a positive integer, a_i's are constants to be determined later, $G(\xi_n)$ is the general solution of the auxiliary equation

$$G''(\xi_n)+\lambda G'(\xi_n)+\mu G(\xi_n)=0 \ , \tag{5}$$

in which λ and μ are constants and prime denotes derivative with respect to ξ_n.

Step 3. By a simple calculation, we can get the identity

$$\xi_{n+p_s}=\xi_n+\varphi_s, \quad \varphi_s=p_{s1}d_1+p_{s2}d_2+\cdots+p_{sQ}d_Q \ , \tag{6}$$

where p_{sj} is the j th component of the shift vector $\mathbf{p}_s$. Thus, we derive the uniform formulas

$$\mathbf{U}_{n+p_s}(\xi_n)=\sum_{l=-m}^{m}a_i\left(-\frac{\lambda}{2}+\frac{\frac{\lambda}{2}+\frac{G'(\xi_n)}{G(\xi_n)}+\frac{\sqrt{\lambda^2-4\mu}}{2}\tanh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\varphi_s\right)}{1+\frac{2}{\sqrt{\lambda^2-4\mu}}\left(\frac{\lambda}{2}+\frac{G'(\xi_n)}{G(\xi_n)}\right)\tanh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\varphi_s\right)}\right)^l, \ \lambda^2-4\mu>0, \tag{7a}$$

$$\mathbf{U}_{n+p_s}(\xi_n)=\sum_{l=-m}^{m}a_i\left(-\frac{\lambda}{2}+\frac{\frac{\lambda}{2}+\frac{G'(\xi_n)}{G(\xi_n)}-\frac{\sqrt{4\mu-\lambda^2}}{2}\tan\left(\frac{\sqrt{4\mu-\lambda^2}}{2}\varphi_s\right)}{1+\frac{2}{\sqrt{4\mu-\lambda^2}}\left(\frac{\lambda}{2}+\frac{G'(\xi_n)}{G(\xi_n)}\right)\tan\left(\frac{\sqrt{4\mu-\lambda^2}}{2}\varphi_s\right)}\right)^l, \ \lambda^2-4\mu<0, \tag{7b}$$

$$\mathbf{U}_{n+p_s}(\xi_n)=\sum_{l=-m}^{m}a_i\left(-\frac{\lambda}{2}+\left(\left(\frac{\lambda}{2}+\frac{G'(\xi_n)}{G(\xi_n)}\right)\Big/\left(1+\left(\frac{\lambda}{2}+\frac{G'(\xi_n)}{G(\xi_n)}\right)\varphi_s\right)\right)\right)^l, \ \lambda^2-4\mu=0. \tag{7c}$$

Step 4. Balancing the highest order nonlinear term(s) and the highest-order derivative term in $\mathbf{U}_n(\xi_n)$, we can easily determine the degree m of Eqs. (4) and (7a-c) from Eq. (3). Since $\mathbf{U}_{n+p_s}$ can be interpreted as being of degree zero in $\left(G'(\xi_n)/G(\xi_n)\right)$, the leading terms of $\mathbf{U}_{n+p_s}(\mathbf{p}_s\neq\mathbf{0})$ will not affect the balancing procedure.

Step 5. Substituting the ansätze (4) and (7a-c) together with (5) into Eq. (3), equating the coefficients of $\left(G'(\xi_n)/G(\xi_n)\right)^l$ $(l=0,1,2,\ldots)$ to zero, we obtain a system of nonlinear algebraic equations from which the undetermined constants a_i, d_i, and c_j can be explicitly verified. Substituting these results into (4), we can derive various kinds of exact discrete solutions to Eq. (1).

APPLICATION TO THE RELATIVISTIC TODA LATTICE SYSTEM

The relativistic Toda lattice system [3-6] is known as

$$\frac{\partial u_n}{\partial t} = (1 + \alpha u_n)(v_n - v_{n-1}), \quad \frac{\partial v_n}{\partial t} = v_n(u_{n+1} - u_n + \alpha v_{n+1} - \alpha v_{n-1}), \tag{8}$$

which describes vibrations in mass-spring lattices with an exponential interaction force. To look for traveling wave solutions of Eq. (8), we let $u_n = U_n(\xi_n)$, $v_n = V_n(\xi_n)$, $\xi_n = dn + ct + \zeta$, where c, d and ζ are constants to be determined later. Then, Eq. (8) becomes

$$cU_n{}'(\xi_n) = (1 + \alpha U_n(\xi_n))(V_n(\xi_n) - V_{n-1}(\xi_n)), \quad cV_n{}'(\xi_n) = V_n(\xi_n)(U_{n+1}(\xi_n) - U_n(\xi_n) + \alpha V_{n+1}(\xi_n) - \alpha V_{n-1}(\xi_n)), \tag{9}$$

where prime denotes derivative with respect to ξ_n. Balancing the linear term of the highest order with the highest nonlinear term in (9) leads to $m = 1$. Thus, we suppose that Eq. (9) has the formal traveling wave solutions expressed in the form

$$U_n(\xi_n) = a_0 + a_1\left(\frac{G'(\xi_n)}{G(\xi_n)}\right) + a_{-1}\left(\frac{G'(\xi_n)}{G(\xi_n)}\right)^{-1}, \quad V_n(\xi_n) = b_0 + b_1\left(\frac{G'(\xi_n)}{G(\xi_n)}\right) + b_{-1}\left(\frac{G'(\xi_n)}{G(\xi_n)}\right)^{-1}, \tag{10}$$

where a_i's and b_i's are constants to be determined later. We have the following cases:

Case 1: When $\lambda^2 - 4\mu > 0$, we obtain discrete hyperbolic function traveling wave solutions of Eq. (8) as

$$u_{n,1}(t) = -\frac{2 + c\alpha\lambda + c\alpha\sqrt{\lambda^2 - 4\mu}\coth\left(\frac{\sqrt{\lambda^2 - 4\mu}}{2}d\right)}{2\alpha} + 2c\mu\left(\lambda - \sqrt{\lambda^2 - 4\mu}\left(\frac{C_1\cosh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}(dn+ct+\zeta)\right) + C_2\sinh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}(dn+ct+\zeta)\right)}{C_1\sinh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}(dn+ct+\zeta)\right) + C_2\cosh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}(dn+ct+\zeta)\right)}\right)^{-1}\right), \tag{11a}$$

$$v_{n,1}(t) = \frac{c}{2\alpha}\left(\lambda + \sqrt{\lambda^2 - 4\mu}\coth\left(\frac{\sqrt{\lambda^2-4\mu}}{2}d\right) - 4\mu\left(\lambda - \sqrt{\lambda^2 - 4\mu}\frac{C_1\cosh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}(dn+ct+\zeta)\right) + C_2\sinh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}(dn+ct+\zeta)\right)}{C_1\sinh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}(dn+ct+\zeta)\right) + C_2\cosh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}(dn+ct+\zeta)\right)}\right)^{-1}\right), \tag{11b}$$

$$u_{n,2}(t) = \frac{1}{2}\left(-\frac{2}{\alpha} - c\sqrt{\lambda^2 - 4\mu}\coth\left(\frac{\sqrt{\lambda^2-4\mu}}{2}d\right) + c\sqrt{\lambda^2 - 4\mu}\frac{C_1\cosh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}(dn+ct+\zeta)\right) + C_2\sinh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}(dn+ct+\zeta)\right)}{C_1\sinh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}(dn+ct+\zeta)\right) + C_2\cosh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}(dn+ct+\zeta)\right)}\right), \tag{12a}$$

$$v_{n,2}(t) = \frac{c\sqrt{\lambda^2 - 4\mu}}{2\alpha}\left(\coth\left(\frac{\sqrt{\lambda^2-4\mu}}{2}d\right) - \frac{C_1\cosh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}(dn+ct+\zeta)\right) + C_2\sinh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}(dn+ct+\zeta)\right)}{C_1\sinh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}(dn+ct+\zeta)\right) + C_2\cosh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}(dn+ct+\zeta)\right)}\right). \tag{12b}$$

Case 2: When $\lambda^2 - 4\mu < 0$, we have discrete trigonometric function traveling wave solutions of Eq. (8) as

$$u_{n,6}(t) = -\frac{2 + c\alpha\lambda + c\alpha\sqrt{4\mu-\lambda^2}\cot\left(\frac{\sqrt{4\mu-\lambda^2}}{2}d\right)}{2\alpha} + 2c\mu\left[\lambda - \sqrt{4\mu-\lambda^2}\left(\frac{-C_1\sin\left(\frac{\sqrt{4\mu-\lambda^2}}{2}(dn+ct+\zeta)\right)+C_2\cos\left(\frac{\sqrt{4\mu-\lambda^2}}{2}(dn+ct+\zeta)\right)}{C_1\cos\left(\frac{\sqrt{4\mu-\lambda^2}}{2}(dn+ct+\zeta)\right)+C_2\sin\left(\frac{\sqrt{4\mu-\lambda^2}}{2}(dn+ct+\zeta)\right)}\right)^{-1}\right], \tag{13a}$$

$$v_{n,6}(t) = \frac{c}{2\alpha}\left[\lambda + \sqrt{4\mu-\lambda^2}\cot\left(\frac{\sqrt{4\mu-\lambda^2}}{2}d\right)\right] - 4\mu\left[\lambda - \sqrt{\lambda^2-4\mu}\left(\frac{-C_1\sin\left(\frac{\sqrt{4\mu-\lambda^2}}{2}(dn+ct+\zeta)\right)+C_2\cos\left(\frac{\sqrt{4\mu-\lambda^2}}{2}(dn+ct+\zeta)\right)}{C_1\cos\left(\frac{\sqrt{4\mu-\lambda^2}}{2}(dn+ct+\zeta)\right)+C_2\sin\left(\frac{\sqrt{4\mu-\lambda^2}}{2}(dn+ct+\zeta)\right)}\right)^{-1}\right], \tag{13b}$$

$$u_{n,7}(t) = -\frac{1}{\alpha} - \frac{c\sqrt{4\mu-\lambda^2}}{2}\left[\cot\left(\frac{\sqrt{4\mu-\lambda^2}}{2}d\right) - \left(\frac{-C_1\sin\left(\frac{\sqrt{4\mu-\lambda^2}}{2}(dn+ct+\zeta)\right)+C_2\cos\left(\frac{\sqrt{4\mu-\lambda^2}}{2}(dn+ct+\zeta)\right)}{C_1\cos\left(\frac{\sqrt{4\mu-\lambda^2}}{2}(dn+ct+\zeta)\right)+C_2\sin\left(\frac{\sqrt{4\mu-\lambda^2}}{2}(dn+ct+\zeta)\right)}\right)\right], \tag{14a}$$

$$v_{n,7}(t) = \frac{c\sqrt{4\mu-\lambda^2}}{2\alpha}\left[\cot\left(\frac{\sqrt{4\mu-\lambda^2}}{2}d\right) - \left(\frac{-C_1\sin\left(\frac{\sqrt{4\mu-\lambda^2}}{2}(dn+ct+\zeta)\right)+C_2\cos\left(\frac{\sqrt{4\mu-\lambda^2}}{2}(dn+ct+\zeta)\right)}{C_1\cos\left(\frac{\sqrt{4\mu-\lambda^2}}{2}(dn+ct+\zeta)\right)+C_2\sin\left(\frac{\sqrt{4\mu-\lambda^2}}{2}(dn+ct+\zeta)\right)}\right)\right]. \tag{14b}$$

Case 3: When $\lambda^2 - 4\mu = 0$, we obtain discrete rational function traveling wave solutions of Eq. (8) as

$$u_{n,11}(t) = -\frac{c}{d} - \frac{1}{\alpha} + \frac{c\lambda C_1}{(\lambda(dn+ct+\zeta)-2)C_1 + \lambda C_2}, \quad v_{n,11}(t) = \frac{c}{\alpha}\left(\frac{1}{d} - \frac{\lambda C_1}{(-2+\lambda(dn+ct+\zeta))C_1 + \lambda C_2}\right), \tag{15}$$

$$u_{n,12}(t) = -\frac{c}{d} - \frac{1}{\alpha} + \frac{cC_1}{C_1(dn+ct+\zeta)+C_2}, \quad v_{n,12}(t) = \frac{c}{d\alpha} - \frac{cC_1}{\alpha(C_1(dn+ct+\zeta)+C_2)}, \tag{16}$$

where in all cases C_1 and C_2 are arbitrary constants.

CONCLUSION

In summary, we proposed the extended discrete (G'/G)-expansion method for handling NDDEs in a direct manner. Three types of discrete traveling wave solutions with more free parameters for the relativistic Toda lattice system are obtained. Besides, we witnessed that some of the previously known results are particular cases of our results. We assured the correctness of our solutions by putting them back into the original equations with the help of Mathematica. It is natural to state that we are unable to give more details regarding the applications of our analytic solutions in reality due to the lack of theoretical and experimental basis related to these solutions. As a final remark, we believe that the extended discrete (G'/G)-expansion method is worthy of further study and it may allow one to discover new physically interesting exact discrete solutions to other NDDEs of applied sciences.

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