Implementation of Partial Synchronization of Different Chaotic Systems by Field Programmable Gate Array

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***Abstract*—In this study, the synchronization of the master-slave systems has been achieved and implemented on Field Programmable Gate Array (FPGA). In this paper, the master system and the slave system have been chosen as Lorenz and Rossler systems, respectively. The feedback control rule has been derived by feedback linearization method. By feedback linearization, the coordinate transformation has been achieved then the control command for synchronization has been obtained. In order to implement designed synchronized system, *Matlab Simulink* design of the system has been translated to *Xilinx System Generator* design to generate *Very-High-Speed Integrated Circuits Hardware Description Language* (VHDL) code which is used to produce bitstream file. By *Xilinx Integrated Software Environment* (ISE) program, VHDL code is converted to bitstream file which has been embedded into FPGA by *Field Upgradeable Systems Environment* (FUSE). Finally, the designed synchronized system has been observed on the HP 54540C oscilloscope.**

I. Introduction

In the nonlinear science, control of chaotic systems have been studied recently owing to its particular importance in applications such as communications, physics etc. [1], [2]. There are several kinds of synchronizations such as generalized synchronization [3], complete synchronization [4], [5], partial synchronization [6], phase synchronization [7] and almost synchronization [8]. The pioneering work [5], has increased the interest in synchronization after having recently found many applications particularly in telecommunications [9], in mechanical systems [10] and in control theory [11]. Partial synchronization problem has been studied in populations of pulse-coupled oscillators [12].

This paper is organized as follows: In Section II the partial synchronization problem is defined and the feedback linearization method is explained. In Section III, as a case study Lorenz system is chosen as the master system and Rossler system is chosen as the slave system then by the suitable feedback, Rossler system is synchronized to Lorenz system. In Section IV, the proposed control system is simulated by *Matlab Simulink* then the simulated design is converted to *Xilinx System Generator* design and the designed synchronized system is implemented by the help of ISE and FUSE programs. Finally, in Section V, conclusions are presented.

II. Synchronization Problem

In this section, the partial synchronization problem in the sense of exact synchronization and practical synchronization will be considered. The definitions given below describes exact synchronization, practical synchronization and partial synchronization, respectively.

Definition 2.1: It is said that two chaotic systems are exactly synchronized if the synchronization error, $e_i = x_i - y_i$, exponentially converges to the origin. This implies that at a finite time $x_i = y_i$ [8].

Definition 2.2: It is said that two chaotic systems are practically synchronized if the trajectories of the synchronization error $e_i = x_i - y_i$ converges to a neighborhood around the origin. This implies that for all time $t \leq t^*$ the trajectories of the slave system are close to the master trajectories, i.e., $x_i \approx y_i$ [8].

Definition 2.3: It is said that two chaotic systems are partially synchronized if, at least, one of the states of the synchronization error system is either practically or exactly synchronized and only if, at least, one of the states of the synchronization error system is neither practically nor exactly synchronized [8].

The difference between master and slave system is called as an error system which can be constructed using the definition given below.

Definition 2.4: Let $\dot{\mathbf{x}} = \mathbf{F}_M(x)$ and $\dot{\mathbf{y}} = \mathbf{F}_S(y) + \mathbf{g}(y)u(y)$ be two chaotic systems in a manifold $\mathbf{M} \subset \mathrm{R}^n$. $\mathbf{F}_M$, $\mathbf{F}_S$ smooth vector fields with scalar output functions $s_M = h(x)$, $s_S = h(y)$ and $\mathbf{x}$, $\mathbf{y} \in \mathrm{R}^n$ and $\mathbf{g}(y) \in \mathrm{R}^n$ is a smooth input vector where subscripts M and S stands for the master and slave, respectively [4].

$$\dot{\mathbf{x}} = \mathbf{F}_M(x), \quad (1)$$
$$\dot{\mathbf{e}} = \mathbf{F}_M(x) - \mathbf{F}_S(x,e) - \mathbf{g}(x,e)u(x,e), \quad (2)$$
$$s_e = h(x,e), \quad (3)$$

where s_e is the output of the synchronization error system and extended synchronization error system can be written in affine form as:

$$\dot{\mathbf{X}} = \mathbf{F}(X) + \mathbf{G}(X)u(X), \quad (4)$$

where $\mathbf{X} = [\mathbf{x}, \mathbf{e}]^T$, $\mathbf{F}(X) = [\mathbf{F}_M(x), \mathbf{F}_M(x) - \mathbf{F}_S(y)]^T$ and $\mathbf{G}(X) = [\mathbf{0}, -\mathbf{g}(x,e)]^T$.

In order to achieve synchronization, synchronization error system in Eq. 2 should be stabilized around the equilibrium point $e^* = 0$. The definitions and theorems given in the following sequel will be used to find the proper invertible transformation which will be used to derive the control command for synchronization.

Definition 2.5: System (4) is said to have relative degree ρ, $\rho \leq n$ at point $x_0 \in \mathrm{R}^n$ with respect to the output

$$s_e = h(x)$$

if for any $x \in \Omega$ where Ω is some neighborhood of x_0, the following conditions are valid

(i)$L_G L_F^k h(x) = 0$, $k = 0, 1, ...\rho - 2$, $\forall x$ in a neighborhood of x_0 and $k < \rho - 1$,

(ii)$L_G L_F^{\rho-1} h(x_0) \neq 0$.

Recall that $L_\psi \phi(x) = \sum_{i=1}^{n} \frac{\partial \phi}{\partial x_i} \psi(x)$ stands for the Lie derivative of the vector function ϕ along the vector field ψ. Relative degree ρ is exactly equal to the number of times one has to differentiate the output in order to have the input explicitly appearing in the equation which describes the evolution of $s_e^{(r)}(t)$ in the neighborhood of x_0 [2], [13].

Theorem 2.6: System (4) is feedback linearizable in the neighborhood Ω of a point $x_0 \in \mathrm{R}^n$ if and only if there exists a smooth scalar function $h(x)$ defined in Ω such that the relative degree ρ of (3) and (4) is equal to n [2].

Theorem 2.7: Consider the system (4). Suppose that there exist $2n - \rho$ functions $\Phi_i(x,e)$ such that $L_G \Phi_i(x,e) = 0$, $i = \rho + 1, ..., 2n$. This system is feedback linearizable at $(x, 0)$ if and only if there exists a function $h(x,e)$ such that
(i) $< \partial h, ad_F^{k-1}\mathbf{G} > (x,e) = 0$ for $k = 1, ..., \rho - 1; \rho > 1$ and (x,e) in a neighborhood Ω of $(x, 0)$,
(ii) $< \partial h, ad_F^{i}\mathbf{G} > (x,0) \neq 0$ for $i = \rho, ..., n$ at $(x,0)$,
where $\rho = d$ stands for the dimension of the tangent space and the accessibility distribution function $\mathbf{C}(x,e)$ can be expressed as $\mathbf{C}_d = span\{ad_F^{d-1}\mathbf{G}\}$ where $ad_F = [\mathbf{F}, \mathbf{G}]$ and $ad_F^{d-1} = [\mathbf{F}, [\mathbf{F}, [..., [\mathbf{F}, \mathbf{G}], ...,]]]$ for $d = 1, ..., n$ where $[\mathbf{F}, \mathbf{G}]$ is called the Lie bracket of $\mathbf{F}$ and $\mathbf{G}$ [4].

Corollary 2.8: Two chaotic systems with the same order are completely synchronizable if and only if the dynamical error system is feedback linearizable at $(x, 0)$ [4].

Feedback linearization problem: The system in (4) is called feedback linearizable if there exist a smooth reversible change of coordinates $z = \Phi(x,e)$ and smooth transformation of the feedback [13], [14].

$$u = \lambda(x,e) + \mu(x,e)v, \quad (5)$$

where $v \in \mathrm{R}^m$ is the new control if the closed-loop is linear and then the resulting variables z and v satisfy linear dynamical system in the form of

$$\begin{aligned} \dot{\mathbf{z}} &= \mathbf{Az} + \mathbf{bv} \\ \mathbf{z} &= \Phi(x,e) = [h(x,e), L_F h(x,e), ..., L_F^{\rho-1} h(x,e)]^T \\ u &= \frac{1}{L_G L_F^{\rho-1} h(x,e)}(-L_F^{\rho} h(x,e) + v) \end{aligned} \quad (6)$$

$$\begin{aligned} \lambda(x,e) &= \frac{-L_F^{\rho} h(x,e)}{L_G L_F^{\rho-1} h(x,e)} \\ \mu(x,e) &= \frac{1}{L_G L_F^{\rho-1} h(x,e)} \\ \nu &= \mathrm{K}_i(z_i - z_i^*) \end{aligned}$$

where K_i with $i = 1, ..., \rho$ are the control gains and chosen in such a way that the closed-loop subsystem $\dot{z}$ converges to the origin and z_i^*s are the coordinates of the stabilization point. In order to achieve complete synchronization z_i^*s are set to zero.

III. Synchronization of Chaotic Systems with Different Model

Lorenz system [15] is chosen as the master system and its state equations can be written as follows :

$$\begin{aligned} \dot{x_1} &= \alpha(x_2 - x_1) \\ \dot{x_2} &= \beta x_1 - x_2 - x_1 x_3 \\ \dot{x_3} &= x_1 x_2 - \gamma x_3 \end{aligned} \quad (7)$$

where $\alpha = 10$, $\beta = 28$,and $\gamma = 8/3$. Rossler system [16] is chosen as the slave system and is described as:

$$\begin{aligned} \dot{y_1} &= -(y_2 + y_3) + g_1(y)u \\ \dot{y_2} &= y_1 + \theta y_2 + g_2(y)u \\ \dot{y_3} &= \delta + y_3(y_1 - \phi) + g_3(y)u \end{aligned} \quad (8)$$

where $g_1(y)$, $g_2(y)$ and $g_3(y)$ are control inputs and $\theta = 0.2$, $\delta = 0.2$, $\phi = 5.7$. The extended synchronization error system for coupled Lorenz-Rossler system can be written by calculating the error system as:

$$
\begin{aligned}
\dot{e_1} &= \alpha(x_2 - x_1) + (x_2 - e_2) + (x_3 - e_3) - g_1(x,e)u \\
\dot{e_2} &= \beta x_1 - x_2 - x_1x_3 - (x_1 - e_1) - \theta(x_2 - e_2) - g_2(x,e)u \\
\dot{e_3} &= x_1x_2 - \gamma x_3 - \delta - (x_3 - e_3)[(x_1 - e_1) - \phi] - g_3(x,e)u
\end{aligned}
$$

After having found error sytem, the extended synchronization error system can be written as:

$$
\begin{aligned}
\dot{x_1} &= \alpha(x_2 - x_1) \\
\dot{x_2} &= \beta x_1 - x_2 - x_1x_3 \\
\dot{x_3} &= x_1x_2 - \gamma x_3 \\
\dot{e_1} &= \alpha(x_2 - x_1) + (x_2 - e_2) + (x_3 - e_3) - g_1(x,e)u \\
\dot{e_2} &= \beta x_1 - x_2 - x_1x_3 - (x_1 - e_1) - \theta(x_2 - e_2) - g_2(x,e)u \\
\dot{e_3} &= x_1x_2 - \gamma x_3 - \delta - (x_3 - e_3)[(x_1 - e_1) - \phi] - g_3(x,e)u
\end{aligned}
$$

then $\mathbf{F}(X)$ and $\mathbf{G}(X)$ can be found as in Eq. (4)

$$
\mathbf{F} = \begin{bmatrix} \alpha(x_2 - x_1) \\ \beta x_1 - x_2 - x_1x_3 \\ x_1x_2 - \gamma x_3 \\ \alpha(x_2 - x_1) + (x_2 - e_2) + (x_3 - e_3) \\ \beta x_1 - x_2 - x_1x_3 - (x_1 - e_1) - \theta(x_2 - e_2) \\ x_1x_2 - \gamma x_3 - \delta - (x_3 - e_3)[(x_1 - e_1) - \phi] \end{bmatrix} \tag{9}
$$

$$
\mathbf{G} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ -g_1(x,e) \\ -g_2(x,e) \\ -g_3(x,e) \end{bmatrix} \tag{10}
$$

The corresponding accessibility distribution function $\mathbf{C}_3(x,e)$ can be calculated where $\mathbf{C}_3(x,e) = span\{\mathbf{G}, ad_F\mathbf{G}, ad_F^2\mathbf{G}\}$. Let $\mathbf{G}$ be defined as $\mathbf{G} = [0,\ 0,\ 0,\ -g_1,\ -g_2,\ -g_3]^T$ with g_1, g_2 and g_3 constants, then $ad_F\mathbf{G}$ and $ad_F^2\mathbf{G}$ will be calculated to obtain $\mathbf{C}_3(x,e)$.

$$
\mathbf{C}_3(x,e) = span\left\{\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ -g_1 & -g_2 - g_3 & b^* \\ -g_2 & g_1 + \theta g_2 & c^* \\ -g_3 & a^* & d^* \end{bmatrix}\right\}
$$

$$
\begin{aligned}
a^* &= g_1(x_3 - e_3) + g_3(x_1 - e_1 - \phi) \\
b^* &= g_1(1 + x_3 - e_3) + \theta g_2 + g_3(x_1 - e_1 - \phi) \\
c^* &= -\theta g_1 + g_2(1 - \theta^2) + g_3 \\
d^* &= -g_1(x_1 - e_1 - \phi)(x_3 - e_3 + 1) + g_2(x_3 - e_3) \\
&+ g_3[1 + x_3 - e_3 - (x_1 - e_1 - \phi)^2]
\end{aligned}
$$

For simplicity by setting $g_1 = 0, g_2 = 1$ and $g_3 = 0$ then $C_3(x,0)$ can be written as:

$$
\mathbf{C}_3(x,0) = span\left\{\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -1 & \theta \\ -1 & \theta & 1 - \theta^2 \\ 0 & 0 & x_3 \end{bmatrix}\right\} \tag{11}
$$

In order to derive the control command, we need to determine the dimension of the tangent space d which is generated by the corresponding distribution so the conditions of *Theorem 2.7* must be satisfied. The dimension of the tangent space d is determined to be equal to 2 since the conditions below have been satisfied.

$$
\begin{aligned}
-\frac{\partial h}{\partial e_2} &= 0 \\
-\frac{\partial h}{\partial e_1} + \theta\frac{\partial h}{\partial e_2} &\neq 0
\end{aligned}
$$

For this case $h(x,e) = e_1$ can be chosen as an output function which satisfies conditions of *Theorem 2.7* then $d = \rho = Dim(\mathbf{C}_3(x,0)) = 2$, $\forall x \in \mathrm{R}^3$ and considering *Theorem 2.6*, to have feedback linearizable system for $\rho = 2$, the conditions below must be satisfied

$$
\begin{aligned}
L_G h(x,e) &= 0 \\
L_G L_F h(x,e) &\neq 0
\end{aligned}
$$

then the transformation can be found as

$$
\begin{aligned}
z_1 &= h(x,e) = e_1 \qquad (12) \\
z_2 &= L_F h(x,e) = \dot{e_1} = \alpha(x_2 - x_1) + (x_2 - e_2) + (x_3 - e_3)
\end{aligned}
$$

The complementary functions which should satisfy such that $L_G\Phi_i(x,e) = 0, i = 2n - \rho, ...2n$ can be chosen as

$$
\begin{aligned}
z_3 &= w_1 \\
z_4 &= x_1 \\
z_5 &= x_2 \\
z_6 &= x_3
\end{aligned} \tag{13}
$$

Eqs. (14)and (15) constitute an invertible transformation around $(x,0)$ which means that there is a proper control command $u(x,e)$ such that the slave system trajectory tracks the master system trajectory. The transformed system can then be written as follows:

$$
\begin{aligned}
\dot{z_1} &= z_2 \\
\dot{z_2} &= L_F^2 h(x,e) + L_G L_F u(x,e) \\
\dot{z_3} &= \dot{w_1} \\
\dot{z_4} &= \dot{x_1} \\
\dot{z_5} &= \dot{x_2} \\
\dot{z_6} &= \dot{x_3}
\end{aligned}
$$

where the control command can be found as:

$$u(x,e) = \frac{1}{L_G L_F h(x,e)} (-L_F^2 h(x,e) + \kappa_1 z_1 + \kappa_2 z_2)$$

$$u(x,e) = \alpha \dot{x_1} - (1+\alpha)\dot{x_2} - \dot{x_3} + \dot{e_2} + \dot{e_3} + \kappa_1 e_1 + \kappa_2 \dot{e_1}$$

where $\dot{x_1}$, $\dot{x_2}$, $\dot{x_3}$, $\dot{e_1}$, $\dot{e_2}$ and $\dot{e_3}$ can be written in terms of x_1, x_2, x_3, e_1, e_2 and e_3.

IV. Simulation and Implementation Results of the Designed Synchronized System

After simulating the designed system, in order to generate VHDL code, *Xilinx System Generator* blocks can be used. *System Generator* is a very useful tool since *Simulink* design is easily converted to *System Generator* blocks and it works under *Matlab* as a toolbox which provides to generate VHDL code of the designed system. Then, this *VHDL* code is used to produce *bitstream* file by using *Xilinx ISE* program. At the last step, *bitstream* file is embedded into FPGA by *FUSE* program.

The results of designed system can be observed on HP54540 scope as in Fig. (1). The state $e_1(t) = x_1(t) - y_1(t)$ is exactly synchronized while the state $e_2(t) = x_2(t) - y_2(t)$ is practically synchronized and the state $e_3(t) = x_3(t) - y_3(t)$ is not synchronized. Therefore, Lorenz system is said to be partially synchronized with Rossler system.

V. Conclusion

The control system, Lorenz's attractor and Rossler's attractor have been embedded into FPGA. The controller has been designed to synchronize Rossler's attractor to Lorenz's attractor by feedback linearization method and partial synchronization of Lorenz system with Rossler system has been achieved.

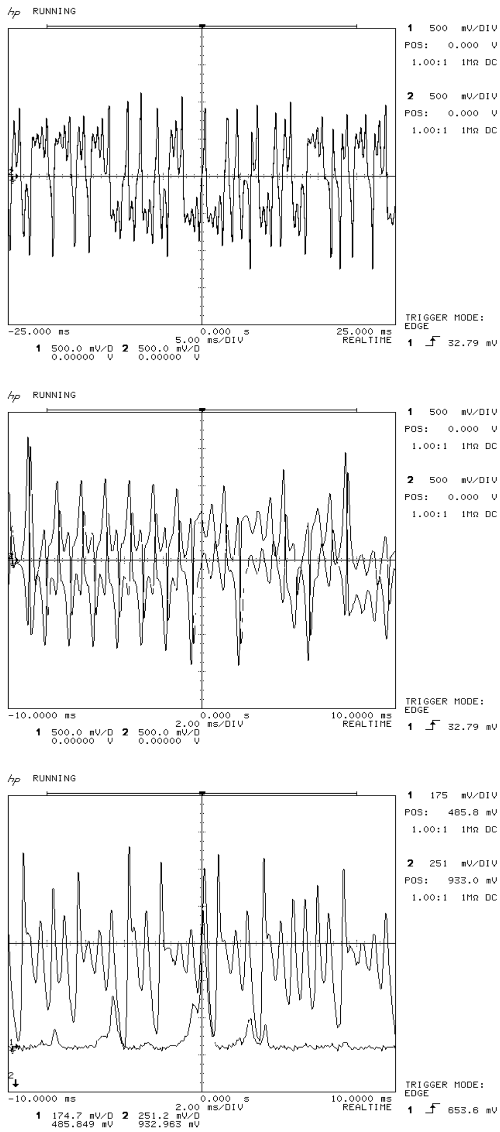

Fig. 1. Partial Practical Synchronization of Lorenz system with Rossler system. From the top to the bottom Channel 1 represetns $x_1(t)$ and Channel 2 represents $y_1(t)$; Channel 1 represetns $x_2(t)$ and Channel 2 represents $y_2(t)$; Channel 1 represetns $x_3(t)$ and Channel 2 represents $y_3(t)$, respectively.

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