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A way to get rid of cosmological constant and zero-point energy problems of quantum fields through metric reversal symmetry

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Abstract
In this paper, a framework is introduced to remove the huge discrepancy between the empirical value of the cosmological constant and the contribution to the cosmological constant predicted from the vacuum energy of quantum fields. An extra-dimensional space with metric reversal symmetry and R^2 gravity (that reduces to the usual R gravity after integration over extra dimensions) is considered to this end. The resulting four-dimensional energy–momentum tensor (obtained after integration over extra dimensions) consists of terms that contain off-diagonally coupled pairs of Kaluza–Klein modes. This, in turn, generically results in the vanishing of the vacuum expectation value of the energy–momentum tensor for quantum fields, and offers a way to solve the problem of huge contribution of quantum fields to the vacuum energy density.

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1. Introduction

The observation of the accelerated expansion of the universe [1] boosted the studies on an old cosmological problem, namely the cosmological constant problem [2]. The standard explanation for the accelerated expansion of the universe is a positive definite cosmological constant in Einstein field equations [3, 4]. A cosmological constant (CC) may be considered either as a geometrical object (e.g. as the part of the curvature scalar that depends only on extra dimensions in a higher dimensional space) or as the energy density of a perfect fluid with negative pressure or a combination of both. (Although these two attributions may seem to be really two different manifestations of the same thing this distinction enables a more definite discussion of the problem, as we shall see.) The vacuum expectation values of the energy–momentum tensors of quantum fields (i.e. the energy–momentum tensor due to zero modes of quantum fields) induce energy–momentum tensors that have the form of the CC term in Einstein field equations. This identification is the main origin of the two (probably related) most important cosmological constant problems; (1) why is the energy density ($\sim(10^{-3}$ eV $)^4$) [5] derived from the measurements of acceleration of the universe so small compared to the energy scales associated with quantum phenomena (that is, why is CC so small?), (2) why do the zero modes of quantum fields contribute to the accelerated expansion of the universe so much less than expected?.

There are many attempts, at least partially to answer these questions, namely symmetry principles, anthropic considerations, adjustment mechanisms, quantum cosmology, string landscape, etc [2, 6]. None of these attempts have been wholly satisfactory. One of the main ideas proposed toward the solution of the problem is the use of symmetries such as supersymmetry and supergravity. However these symmetries are badly broken in nature. So it seems that they do not offer a viable solution. Recently, a symmetry principle that does not suffer from such a phenomenological restriction was introduced [7, 10, 11]. This symmetry amounts to invariance under the reversal of the sign of the metric and it has two different realizations. The first realization is implemented through the requirement of the invariance of physics under the multiplication of the coordinates by the imaginary number i [7–9]. The second realization corresponds to invariance under signature reversal [10, 12, 13] and may be realized through extra-dimensional reflections [10]. In this paper, both realizations of the symmetry are named by a common name, 'metric reversal symmetry'. In the previous studies, the symmetry is implemented for a cosmological constant that is geometrical in origin, e.g. a bulk CC or a CC that is induced by the part of the curvature scalar that depends on the extra dimensions only. The aim of the present paper is to extend this symmetry to a possible contribution to CC induced by the vacuum expectation value of the energy–momentum tensor of quantum fields (i.e. quantum zero modes). The main difficulty in applying the symmetry to the contribution of the quantum zero modes is that, in the simple setting considered in the previous studies, it is not possible to impose it so that the matter Lagrangian corresponding to a field is non-vanishing after integration over extra dimensions (i.e. so that the field is observable at the usual four dimensions at the current accessible energies) while the quantum vacuum contributions of the fields are forbidden. This point will be mentioned in more detail in the following section. To this end, in this paper the space is taken to be a union of two $2(2n+1)$-dimensional spaces and the gravitational Lagrangian is taken to be R^2 where R is the curvature scalar. The Robertson–Walker metric is embedded in one of these $2(2n+1)$-dimensional spaces. Both realizations of the metric reversal symmetry are imposed. The four-dimensional Robertson–Walker metric reduces to the Minkowski metric after the symmetry imposed and the action corresponding to matter Lagrangian is forbidden by the requirement of the invariance under $x^A \rightarrow \mathrm{i}x^A$. The requirement of the implementation of (either realization of) the symmetry on each space separately restricts the form of the gravitational action and only some part of the gravitational action survives and it can be identified by the usual Einstein–Hilbert action after integration over extra dimensions. After breaking the $x^A \rightarrow \mathrm{i}x^A$ symmetry (while preserving the signature reversal symmetry) the Minkowski metric converts to the Robertson–Walker metric (with a slowly varying Hubble constant), and results in a small non-vanishing matter Lagrangian (and action). The unbroken signature reversal symmetry imposes the resulting matter Lagrangian generically contain at least one pair of off-diagonally coupled Kaluza–Klein modes in each homogeneous term and hence necessarily contains mixture of different Kaluza–Klein modes. This, in turn, causes the vacuum expectation value of energy–momentum tensor be zero, as we shall see. Then the accelerated expansion of the universe may be attributed to some alternative methods such as quintessence [14, 16], phantoms [15, 16], etc or a small CC may be induced classically after breaking of the $x^A \rightarrow \mathrm{i}x^A$ symmetry, as we shall see.

2. A brief overview of metric reversal symmetry

We consider two different realizations of a symmetry that reverses the sign of the metric

$$ds^2 = g_{AB}\,dx^A\,dx^B \to -ds^2 \tag{1}$$

and leaves the gravitational action

$$S_R = \frac{1}{16\pi G}\int \sqrt{(-1)^S g}\,R\,d^D x \tag{2}$$

invariant, where S and g denote the number of spacelike dimensions and the determinant of the metric tensor, respectively. I call this symmetry 'metric reversal symmetry'.

The first realization of the symmetry [7] is generated by the transformations that multiply all coordinates by the imaginary number i,

$$x^A \to \mathrm{i}x^A, \qquad g_{AB} \to g_{AB}. \tag{3}$$

The second realization [10] is generated by the signature reversal

$$x^A \to x^A, \qquad g_{AB} \to -g_{AB}. \tag{4}$$

The requirement of the invariance of equation (1) under either of the realizations, equations (3) and (4) sets the dimension of the space D to

$$D = 2(2n+1), \qquad n = 0, 1, 2, 3, \ldots. \tag{5}$$

Hence both realizations forbid a bulk cosmological constant (CC) term

$$S_C = \frac{1}{8\pi G}\int \sqrt{g}\,\Lambda\,d^D x \tag{6}$$

(provided that S_G remains invariant) where Λ is the bulk CC.

In fact these conclusions are valid for signature reversal symmetry in a more general setting where the whole space consists of a $2(2n+1)$-dimensional subspace whose metric transforms like (4) and the metric tensor for the rest of the space is even under the symmetry. In other words in a D-dimensional space where

$$x^A \to x^A, \qquad g_{AB} \to -g_{AB}; \qquad A, B = 0, 1, 2, 3, 5, \ldots, 2(2n+1), \tag{7}$$

$$x^A \to x^A, \qquad g_{A'B'} \to g_{A'B'}; \qquad A', B' = 2(2n+1)+1, 2(2n+1)+2, \ldots, D \tag{8}$$

as well S_G is allowed while S_Λ is forbidden.

A higher dimensional metric with local Poincaré invariance may be written as [17]

$$ds^2 = \Omega(y^c)[g_{\mu\nu}(x)\,dx^\mu\,dx^\nu + \tilde{g}_{\tilde{a}\tilde{b}}(y)\,dy^{\tilde{a}}\,dy^{\tilde{b}}] + g_{e'd'}(y)\,dy^{e'}\,dy^{d'}, \tag{9}$$

where x and $\mu\nu = 0, 1, 2, 3$ denote the usual four-dimensional coordinates and indices; y denotes extra-dimensional coordinates, and $\tilde{a}, \tilde{b} = 4, 5, \ldots 2(2n+1)$, $e', d' = 2(2n+1), \ldots, D$ denote the extra-dimensional indices. We let,

$$\Omega \to -\Omega, \qquad g_{\mu\nu} \to g_{\mu\nu}, \qquad g_{\tilde{a}\tilde{b}} \to g_{\tilde{a}\tilde{b}}, \qquad g_{e'd'} \to g_{e'd'}. \tag{10}$$

We take the underlying symmetry that induces (10) be an extra-dimensional reflection symmetry. For example one may take

$$\Omega(y^c) = \cos ky, \qquad y = y^D, \tag{11}$$

where k is some constant and take the symmetry transformation be a reflection about $kz = \frac{\pi}{2}$ given by

$$ky \to \pi - ky. \tag{12}$$

There is a small yet important difference between simply postulating a signature reversal symmetry or realizing it through (9) and (11) although both forbid a cosmological constant (CC). In the case of (9) and (11), one may take a non-vanishing CC from the beginning and it cancels out after integration over extra dimensions while this is not possible if one simply postulates the metric reversal symmetry.

The action functional corresponding to the matter sector is

$$S_M = \int \sqrt{(-1)^S g}\mathcal{L}_M \, \mathrm{d}^D x, \tag{13}$$

where $\mathcal{L}_M$ is the Lagrangian for a matter field. If the symmetry is applicable to the matter sector then the symmetry must leave S_M invariant. One may take the dimension where the field propagates as $D = 2(2n+1)$ so that (at least) the kinetic part of S_M is invariant under the symmetry transformations. For example the kinetic part of the Lagrangian of a scalar field ϕ,

$$\mathcal{L}_{\phi k} = \tfrac{1}{2} g^{AB} \partial_A \phi \partial_B \phi \tag{14}$$

transforms like R under the transformations, (3) and/or (4) so that S_M is invariant under the symmetry if ϕ propagates in a $2(2n+1)$-dimensional space and $\phi \to \pm\phi$ under the symmetry transformation. Meanwhile this allows nonzero contributions to the CC through the vacuum expectation of energy–momentum tensor of quantum fields. The four-dimensional energy–momentum tensor for (14) at low energies, T_μ^ν, is

$$T_\mu^\nu = \int \mathrm{d}^{D-4} y \Omega^{2n} \sqrt{\tilde{g} g_e} \left\{ g^{\nu\tau} \partial_\tau \phi \partial_\mu \phi - \frac{1}{2} \delta_\mu^\nu [g^{\rho\tau} \partial_\rho \phi \partial_\tau \phi + \tilde{g}^{ab} \partial_a \phi \partial_b \phi + \Omega g^{ed} \partial_e \phi \partial_d \phi] \right\}, \tag{15}$$

where we employed the metric (9), and $\tilde{g}$ and g_e denote the determinants of $(\tilde{g}_{\tilde{a}\tilde{b}})$ and $(g_{e'd'})$, and δ_μ^ν denotes the Kronecker delta. If the signature reversal symmetry is imposed through an extra-dimensional reflection, for example, by (11) and (12) then the last term in (15) cancels out while the other terms survive after the integration over the extra dimensions. So the four-dimensional energy–momentum tensor in general gives nonzero contribution to vacuum energy density through its vacuum expectation value after quantization. One may allow $\mathcal{L}_{\phi k}$ by letting ϕ propagates in a $4n$ dimensional but this would allow a bulk CC. In other words one may adjust the dimension of the space where the field propagates so that (13) is allowed and hence the symmetry is true for matter sector but this allows either a bulk CC or the contribution of quantum zero modes. The situation is the same for gauge fields and fermions. So one should consider this as a classical symmetry [8] or one should construct a more sophisticated framework where the symmetry applies both at classical and quantum levels. Constructing such a model will be the aim of the following sections.

3. The need for both realizations of the symmetry and its implications

The requirement of the isotropy and the homogeneity of the usual four-dimensional universe results in the metric

$$\begin{aligned} &\mathrm{d}s^2 = \Omega(y)\big(\mathrm{d}x_0^2 - a(t)\,\mathrm{d}\sigma^2\big) + g_{ab}(y)\,\mathrm{d}y^a\,\mathrm{d}y^b \\ &y \equiv x_5 = y_1, \qquad x_6 = y_2, \ldots, x_D = y_{D-4} \qquad a, b = 1, 2, 3, \ldots, D-4 \\ &\mathrm{d}\sigma^2 = \frac{\mathrm{d}r^2}{1 - K^2 r^2} + r^2\,\mathrm{d}\Omega^2. \end{aligned} \tag{16}$$

Further I impose the symmetry

$$\begin{aligned} &\mathrm{d}s^2 \to -\mathrm{d}s^2 \qquad \text{as} \quad x^A \to \mathrm{i}x^A, \qquad g_{AB} \to g_{AB} \\ &A = 0, 1, 2, 3, 5, \ldots, D. \end{aligned} \tag{17}$$

This requires

$$\Omega(y) \to \Omega(y), \qquad a(t) \to a(t), \qquad K^2 r^2 \to K^2 r^2, \qquad g_{ab} \to g_{ab}. \tag{18}$$

This together with the requirement that after integration over extra dimensions it should correspond to the solution of the four-dimensional Einstein equations with a cosmological constant (as the only source) implies that

$$a(t) = \text{constant}, \qquad K^2 = 0. \tag{19}$$

In other words the first realization of the symmetry, equation (17) requires the four-dimensional part of the metric be the usual Minkowski metric, that is,

$$\mathrm{d}s^2 = \Omega(y)\big(\mathrm{d}x_0^2 - \mathrm{d}x_1^2 - \mathrm{d}x_2^2 - \mathrm{d}x_3^2\big) + g_{ab}(y)\,\mathrm{d}y^a\,\mathrm{d}y^b. \tag{20}$$

Equation (20) suggests that one may get rid of the problem of cosmological constant in the four-dimensional cosmological constant (CC) (provided that extra-dimensional contributions vanish) once the first realization of the metric reversal symmetry or (global) Poincaré symmetry is imposed. Then the smallness of the observational value of CC could be attributed to the breaking of the symmetry by a tiny amount if the renormalized value of CC due to vacuum fluctuations were in the order of the observed value of CC. On the other hand, the renormalized value of CC is proportional to the particle masses [18]. So even a free electron contributes to CC by an amount that is $\sim 10^{33}$ times larger than the observational value of CC. Therefore the first realization of metric reversal symmetry by itself cannot be used to make CC vanish (or tiny). In the following section, we will see how the signature reversal symmetry (realized through extra-dimensional reflections) can be used to make the contribution of the quantum zero modes vanish. However, the first realization has an advantage over the second one especially when the second realization is considered to be an extra-dimensional reflection of the form of (12). Extra-dimensional reflections do not act on the four-dimensional coordinates so they cannot forbid a contribution from the four-dimensional part of the metric, for example through $a(t)$ while the first realization always does by setting it to zero as we have seen. So in the following section we will employ both realizations of the symmetry. The second realization through extra-dimensional reflections will cancel the contributions to CC while the first one will allow a small CC after it is broken by a small amount.

Next see what is the form of the conformal factor Ω when both realizations of the symmetry are imposed. We have obtained in (20) the form of the metric after the first realization of the symmetry is imposed. Equations (17), (18) set the form of the conformal factor Ω in (16) to one of the followings:

$$\Omega(y) = \Omega(|y|) \qquad \text{or} \qquad \Omega(y) = f(y) f(\mathrm{i}y) \quad (\text{e.g.} \cos ky \cosh ky), \tag{21}$$

where $f(y)$ is an even function in y, i.e. $f(-y) = f(y)$. Next apply (12) to (21) and require (10) and take the extra dimension y be an S^1/Z_2 interval. This restricts the form of Ω to

$$\Omega(y) = \cos k|y| \qquad \text{or} \qquad \Omega(y) = \tan k|y|, \tag{22}$$

where $\cot k|z|$ has been excluded because it blows out at the location of the branes at $k|y| = 0$ and $k|y| = \pi$. For simplicity I take

$$\Omega(y) = \cos k|y| \tag{23}$$

in the following section whenever necessary.

4. The model: classical aspects

In this section, we employ both realizations of the metric reversal symmetry in a space that is the sum of two $2(2+1)$-dimensional spaces (where the usual four dimensional is embedded in one of them) and modify the curvature term S_G so that the metric reversal symmetry becomes a good candidate to explain the huge discrepancy between the observed value of cosmological constant (CC) and the theoretically expected contribution to it through quantum zero modes. In this study, I adopt the view that the symmetry forbids both the geometrical and the vacuum energy density contributions to CC. Hence CC is forced to be zero when the symmetry is manifest, and it is tiny when the symmetry is broken by a tiny amount (instead of seeking a solution where both contributions cancel each other up to a very big precession to explain the observed value of CC). In this section, the main classical aspects of a framework to this end are introduced.

Consider the whole space be a sum of two $2(2n + 1)$-dimensional spaces with the metric

$$\begin{aligned} ds^2 &= g_{AB}\, dx^A\, dx^B + g_{A'B'}\, dx^{A'}\, dx^{B'} \\ &= \Omega_z(z)[g_{\mu\nu}(x)\, dx^\mu\, dx^\nu + \tilde{g}_{ab}(y)\, dy^a\, dy^b] + \Omega_y(y)\tilde{g}_{A'B'}(z)\, dz^{A'}\, dz^{B'} \end{aligned} \tag{24}$$

$$\Omega_y(y) = \cos k|y|), \qquad \Omega_z(z) = \cos k'|z| \tag{25}$$

$$A, B = 0, 1, 2, 3, 5, \ldots, N, \qquad N = 2(2n + 1), \qquad A', B' = 1', 2', \ldots N',$$
$$N' = 2(2m + 1) \qquad \mu, \nu = 0, 1, 2, 3,$$
$$a, b = 1, 2, \ldots, N - 4, \qquad n, m = 0, 1, 2, 3 \ldots.$$

The usual four-dimensional space is embedded in the first space $g_{AB}\, dx^A\, dx^B$ as it is evident from (24). We take the action be invariant under both realizations of metric reversal symmetry, that is,

$$ds^2 \to -ds^2 \quad \text{as} \quad x^A \to ix^A, \qquad x^{A'} \to ix^{A'}, \qquad g_{AB} \to g_{AB}, \qquad g_{A'B'} \to g_{A'B'} \tag{26}$$

$$\Rightarrow \Omega_z \to \Omega_z, \qquad \Omega_y \to \Omega_y, \qquad g_{\mu\nu} \to g_{\mu\nu}, \qquad \tilde{g}_{ab} \to \tilde{g}_{ab}, \qquad \tilde{g}_{A'B'} \to \tilde{g}_{A'B'} \tag{27}$$

and

$$ds^2 \to -ds^2 \quad \text{as} \quad ky \to \pi - ky, \qquad k'z \to \pi - k'z, \qquad x^A \to x^A, \qquad x^{A'} \to x^{A'} \tag{28}$$

$$\begin{aligned} &\Rightarrow \Omega_z \to -\Omega_z, \qquad \Omega_y \to -\Omega_y, \qquad g_{\mu\nu} \to g_{\mu\nu}, \\ &\quad \tilde{g}_{ab} \to \tilde{g}_{ab}, \qquad \tilde{g}_{A'B'} \to \tilde{g}_{A'B'}. \end{aligned} \tag{29}$$

As in (20) and (23) the requirements of the homogeneity and isotropy of the four-dimensional space together with the equations (26)–(29) set $g_{\mu\nu}$ to the Minkowski metric $\eta_{\mu\nu} = \mathrm{diag}(1, -1, -1, -1)$ and the conformal factors to (25).

4.1. Curvature sector

We replace the gravitational action in (2) by an R^2 action

$$S_R = \frac{1}{16\pi\tilde{G}} \int dV\, \tilde{R}^2 \tag{30}$$

$$dV = dV_1\, dV_2, \qquad dV_1 = \sqrt{g(-1)^S}\, d^N x, \qquad dV_2 = \sqrt{g'(-1)^{S'}}\, d^{N'} x' \tag{31}$$

$$\tilde{R} = R(x, x') + R'(x, x'), \tag{32}$$

where the unprimed quantities denote those corresponding to the $N = 2(2n+1)$-dimensional space, and the primed quantities denote those corresponding to the $N' = 2(2m+1)$-dimensional space. Under the transformations (28), (29)

$$\mathrm{d}V_1 \to -\mathrm{d}V_1, \quad \mathrm{d}V_2 \to \mathrm{d}V_2 \quad \text{as} \quad ky \to \pi - ky, \quad x^A \to x^A, \quad x^{A'} \to x^{A'} \tag{33}$$

$$\mathrm{d}V_1 \to \mathrm{d}V_1, \quad \mathrm{d}V_2 \to -\mathrm{d}V_2 \quad \text{as} \quad k'z \to \pi - k'z, \quad x^A \to x^A, \quad x^{A'} \to x^{A'} \tag{34}$$

$$R \to R, \quad R' \to -R' \quad \text{as} \quad ky \to \pi - ky, \quad x^A \to x^A, \quad x^{A'} \to x^{A'} \tag{35}$$

$$R \to -R, \quad R' \to R' \quad \text{as} \quad k'z \to \pi - k'z, \quad x^A \to x^A, \quad x^{A'} \to x^{A'}. \tag{36}$$

We observe that

$$\mathrm{d}V = \mathrm{d}V_1\,\mathrm{d}V_2 \to -\mathrm{d}V \tag{37}$$

$$R^2 \to R^2, \qquad R'^2 \to R'^2, \qquad RR' \to -RR' \tag{38}$$

under the action of the symmetry transformations to only one of the spaces, the unprimed or the primed spaces. So, only the cross terms RR' are allowed. In other words only these terms may survive after integration over extra dimensions. In fact it is obvious from the above transformation rules that an Einstein–Hilbert type of action is not allowed directly because each piece R and R' in $\tilde{R}$ is odd while $\mathrm{d}V$ is even under a transformation applied to both subspaces, the unprimed and the primed subspaces. Since only RR' terms are allowed (30) becomes

$$\begin{aligned} S_R &= \frac{M^{N+N'-4}}{16\pi\tilde{G}} \int \sqrt{(-1)^S g}\sqrt{(-1)^{S'} g'}\, 2R(x)R'(x')\,\mathrm{d}^N x\,\mathrm{d}^{N'} x' \\ &= \frac{1}{16\pi G} \int \sqrt{(-1)^S g}\, R(x)\,\mathrm{d}^N x, \end{aligned} \tag{39}$$

where

$$\frac{1}{16\pi G} = M_{pl}^2 \left(\frac{M}{M_{pl}}\right)^2 M^{N+N'-6} \frac{1}{16\pi\tilde{G}} \int \sqrt{(-1)^{S'} g'}\, 2R'(x')\,\mathrm{d}^{D'} x' \tag{40}$$

and $\tilde{G}$ is a dimensionless constant. In other words in the usual four dimensions at low energies (30) is the same as the Einstein–Hilbert action (2). Newton's constant in N dimensions, G is related to Newton's constant in $N + N'$ dimensions through equation (40). The integral in (40) is at the order of $\sim L^{N'-2} \sim \frac{1}{M^{N'-2}}$. Hence equation (40) may explain the smallness of gravitational interaction compared to the other interaction if the energy scale of L' is much smaller than the Planck mass M_{Pl}, i.e. if $L' \gg \frac{1}{M_{Pl}}$ as in the models with large extra dimensions especially when $L(L') < \frac{1}{M}$.

4.2. Matter sector

In this subsection, we consider the matter action

$$S_M = \int \mathrm{d}V\,\mathcal{L}_M \qquad \mathrm{d}V = \sqrt{(-1)^S g}\sqrt{(-1)^{S'} g'}\,\mathrm{d}^D x\,\mathrm{d}^{D'} x' \tag{41}$$

and we consider the four-dimensional form of S_M after integration over extra-dimensional spaces. Then we study the vacuum expectation value of the energy–momentum tensor induced by the corresponding Lagrangian in the section after the following section.

It is evident that under the first realization of the symmetry

$$\mathrm{d}V \to \mathrm{d}V \qquad \text{as} \quad x^{A(A')} \to \mathrm{i}x^{A(A')}, \qquad g_{AB(A'B')} \to g_{AB(A'B')} \tag{42}$$

for a space consisting of the sum of two $2(2n+1)$-dimensional spaces as in (24). The kinetic part of $\mathcal{L}_M$ is not invariant under the transformations $x^{A(A')} \to \mathrm{i}x^{A(A')}$ for the usual fields [8]. So S_M is not invariant under the symmetry generated by $x^{A(A')} \to \mathrm{i}x^{A(A')}$. In other words the first realization of the metric reversal symmetry is maximally broken in the matter sector (and hence the scale factor $a(t)$ in the Robertson–Walker metric may be time dependent). On the other hand, I take a higher dimensional version of the PT symmetry $x^{A(A')} \to -x^{A(A')}$ be almost exact and broken by a tiny amount. In other words I adopt

$$x^A \to -x^A, \qquad x^{A'} \to -x^{A'}, \tag{43}$$

which is a subgroup of the group generated by

$$x^{A(A')} \to \mathrm{i}x^{A(A')} \to \mathrm{i}(\mathrm{i}x^{A(A')}) = -x^{A(A')} \to \mathrm{i}(\mathrm{i}(\mathrm{i}x^{A(A')})) = -\mathrm{i}x^{A(A')} \\ \to \mathrm{i}(\mathrm{i}(\mathrm{i}(\mathrm{i}x^{A(A')}))) = x^{A(A')}. \tag{44}$$

The symmetries in (43) are imposed on each subspace separately. Next I impose an additional four-dimensional PT symmetry generated by

$$x \to -x. \tag{45}$$

Equations (44), (45) together imply that a PT symmetry in the four dimensions and an additional PT-like symmetry in the extra-dimensional sector are assumed. One observes that $\mathcal{L}_M$ is invariant under equations (44), (45) because S_M and $\mathrm{d}V$ are invariant under these symmetries. The eigenvectors of equations (44), (45) do not mix because the Lagrangian (so the Hamiltonian) is invariant under these symmetries. So the fields ϕ in the Lagrangian should be eigenvectors of these symmetries.

To make the argument more concrete consider the Fourier decomposition (i.e. Kaluza–Klein decomposition) of a general field ϕ (where possible spinor or vector indices are suppressed). For simplicity we take $\tilde{g}_{ab} = -\delta_{ab}, g_{A'B'} = -\delta_{A'B'}$, and consider only the Fourier decomposition of ϕ corresponding to single dimensions y and z from each of the subspaces, the unprimed and the primed ones. We show that the Fourier expansions given below are the eigenvectors of equations (44), (45),

$$\phi_{AA}(x, y, z) = \sum_{n,m} \phi^{AA}_{n,m}(x) \sin(nky) \sin(mk'z) \tag{46}$$

$$\phi_{AS}(x, y, z) = \sum_{n,m} \phi^{AS}_{n,m}(x) \sin(nky) \cos(mk'z) \tag{47}$$

$$\phi_{SA}(x, y, z) = \sum_{n,m} \phi^{SA}_{n,m}(x) \cos(nky) \sin(mk'z) \tag{48}$$

$$\phi_{SS}(x, y, z) = \sum_{n,m} \phi^{SS}_{n,m}(x) \cos(nky) \cos(mk'z) \\ k = \frac{\pi}{L}, \qquad k' = \frac{\pi}{L'}, \qquad 0 \leqslant y \leqslant L, \qquad 0 \leqslant z \leqslant L', \qquad n, m = 0, 1, 2, \ldots, \tag{49}$$

where we have used $k = \frac{\pi}{L}, k' = \frac{\pi}{L'}$ since $0 \leqslant y \leqslant L, 0 \leqslant z \leqslant L'$. In the case of fermions the integers n, m in (46), (49) should be replaced by $\frac{1}{2}n, \frac{1}{2}m$, respectively. One observes that

$$n(m) \to -n(m) \qquad \text{as} \quad y(z) \to -y(z), \tag{50}$$

since $n(m)$ are the eigenvalues of $\frac{\partial}{\partial y}\left(\frac{\partial}{\partial z}\right)$, i.e. they are the momenta corresponding to the directions y and z. There are two eigenvalues, i.e. ± 1 of the each transformation in (50) since the application of the transformations twice the results in the identity transformation.

Now we show that the fields (46)–(49) are the eigenstates of the transformations (50). First consider (46). Applying the transformation (43) and using (50), ϕ_{AA} in (46) transforms to

$$\phi_{AA}(x,y,z) \to \phi'(x,y',z) = \sum_{n,m} \phi^{AA}_{-n,m}(x)\sin(nky)\sin(mk'z) \qquad \text{as} \quad y \to -y \tag{51}$$

$$\to \phi'(x,y'z') = \sum_{n,m} \phi^{AA}_{n,-m}(x)\sin(nky)\sin(mk'z) \qquad \text{as} \quad z \to -z. \tag{52}$$

There will be no mixture of the eigenstates of (43) in the Lagrangian because the Lagrangian is invariant under (43). So ϕ_{AA} is either odd or even under (43). In the light of (50), (52) the eigenstates of ϕ_{AA} under the transformation are determined by $\phi^{AA}_{n,m}(x)$. The same conclusion is true for all ϕ's (46), (49). So, for all ϕ's (46)–(49) we have two cases for each symmetry in (50)

$$\phi_{-n,m}(-x) = \pm\phi_{-n,m}(x) = \pm\phi_{n,m}(x) \tag{53}$$

$$\phi_{n,-m}(-x) = \pm\phi_{n,m}(x) = \pm\phi_{n,m}(x). \tag{54}$$

Meanwhile one may write (46)–(49) in the following form as well:

$$\begin{aligned}\phi_{AA}(x,y,z) &= \frac{1}{2}\sum_{n,m}\left(\phi^{AA}_{n,m}(x) - \phi^{AA}_{-n,m}(x)\right)\sin(nky)\sin(mk'z)\\ &= \frac{1}{2}\sum_{n,m}\left(\phi^{AA}_{n,m}(x) - \phi^{AA}_{n,-m}(x)\right)\sin(nky)\sin(mk'z)\end{aligned} \tag{55}$$

$$\begin{aligned}\phi_{AS}(x,y,z) &= \frac{1}{2}\sum_{n,m}\left(\phi^{AS}_{n,m}(x) - \phi^{AS}_{-n,m}(x)\right)\sin(nky)\cos(mk'z)\\ &= \frac{1}{2}\sum_{n,m}\left(\phi^{AS}_{n,m}(x) + \phi^{AS}_{n,-m}(x)\right)\sin(nky)\cos(mk'z)\end{aligned} \tag{56}$$

$$\begin{aligned}\phi_{SA}(x,y,z) &= \frac{1}{2}\sum_{n,m}\left(\phi^{SA}_{n,m}(x) + \phi^{SA}_{-n,m}(x)\right)\cos(nky)\sin(mk'z)\\ &= \frac{1}{2}\sum_{n,m}\left(\phi^{SA}_{n,m}(x) - \phi^{SA}_{n,-m}(x)\right)\cos(nky)\sin(mk'z)\end{aligned} \tag{57}$$

$$\begin{aligned}\phi_{SS}(x,y,z) &= \frac{1}{2}\sum_{n,m}\left(\phi^{SS}_{n,m}(x) + \phi^{SS}_{-n,m}(x)\right)\cos(nky)\cos(mk'z)\\ &= \frac{1}{2}\sum_{n,m}\left(\phi^{SS}_{n,m}(x) + \phi^{SS}_{n,-m}(x)\right)\cos(nky)\cos(mk'z).\end{aligned} \tag{58}$$

It is evident from equations (55)–(58) that ϕ_{AA} is antisymmetric under both of $n \to -n$, $m \to -m$, ϕ_{AS} is antisymmetric under $n \to -n$ while it is symmetric under $m \to -m$, ϕ_{SA} is symmetric under $n \to -n$ while it is antisymmetric under $m \to -m$, and ϕ_{SS} is symmetric under both of $n \to -n$, $m \to -m$. This result will be important in the value of S_M after integration over extra dimensions.

4.2.1. Scalar field. First consider $\mathcal{L}_{\phi k}$, the kinetic part of the Lagrangian $\mathcal{L}_{Mk}$ for a scalar field (in the space given in (24))

$$\mathcal{L}_{\phi k} = \mathcal{L}_{\phi k1} + \mathcal{L}_{\phi k2} \tag{59}$$

$$\mathcal{L}_{\phi k1} = \tfrac{1}{2} g^{AB} \partial_A \phi \partial_B \phi, \qquad \mathcal{L}_{\phi k2} = \tfrac{1}{2} g^{A'B'} \partial_{A'} \phi \partial_{B'} \phi. \tag{60}$$

Once the breaking of the first realization of the symmetry in the matter sector is granted we may go on to seek the implications of the manifestations of the residual symmetry (43), (45) and the second realization of the symmetry given by equations (28), (29) that remains unbroken. $\mathcal{L}_M$ (i.e. $\mathcal{L}_{\phi k}$ in this case) is even under the simultaneous application of the signature reversal symmetry to both subspaces because dV is even under the symmetry and we require the invariance of S_M (i.e. $S_{\phi k}$ in this case). So any ϕ may be written as a sum of the eigenstates of the symmetry. The eigenvalues of the symmetry transformation $k^{(\prime)} y(z) \to \pi - k^{(\prime)} y(z)$ are ± 1 because the application of the transformation twice results in the identity transformation. Because $g^{AB} (g^{A'B'})$ is odd then the terms $\partial\phi\partial\phi$ are odd as well under the symmetry transformation. So the kinetic term in (59) contains mixed eigenstates of the symmetry. In the following paragraphs, we will identify these eigenstates with odd and even terms in the Fourier decomposition (i.e. Kaluza–Klein decomposition) of ϕ. Then this result will have important consequences in the following paragraphs. In the following paragraph we see, through an example, explicitly how S_M contains mixing of different Kaluza–Klein modes off-diagonally. This result, in turn, will be crucial in ensuring vanishing of the vacuum expectation value of energy–momentum tensors of quantum fields in the section after the following section.

To illustrate the idea I avoid unnecessary complications and consider the simplest realistic case; $N = 6$, $N' = 2$. The kinetic part of S_M (i.e. S_ϕ in this case) for ϕ_{SS} of equation (49) in the space (24) where the conformal factors are of the form (25) is given by (see appendix A)

$$\begin{aligned} S_{\phi k} = \frac{1}{8}(LL')^2 \int \mathrm{d}^4x \Bigg\{ & 4\partial_\mu[\phi_{1,2}(x) + \phi_{1,0}(x)]\partial_\nu(\phi_{0,0}(x)) \\ & + 4\partial_\mu[\phi_{0,2}(x) + \phi_{0,0}(x) + \phi_{2,2}(x) + \phi_{2,0}(x)]\partial_\nu(\phi_{1,0}(x)) \\ & + 4\eta^{\mu\nu} \sum_{r=1,s=1}^{\infty} \partial_\mu[\phi_{|r-1|,|s-2|}(x) + \phi_{|r-1|,s+2}(x) \\ & + 2\phi_{|r-1|,s}(x) + \phi_{r+1,|s-2|}(x) + \phi_{r+1,s+2}(x) + 2\phi_{r+1,s}(x)]\partial_\nu(\phi_{r,s}(x)) \\ & - 4k^2 \sum_{r=1,s=0} r[(|r-1|)(\phi_{|r-1|,|s-2|}(x) + \phi_{|r-1|,s+2}(x) + 2\phi_{|r-1|,s}(x)) \\ & + (r+1)(\phi_{r+1,|s-2|}(x) + \phi_{r+1,s+2}(x) + 2\phi_{r+1,s}(x)) - \phi_{r+1,s}(x))]\phi_{r,s}(x) \\ & - 4\frac{1}{2}k'^2 \sum_{r=0,s=1} s[(|s-3|)\phi_{r,|s-3|}(x) + (s+3)\phi_{r,s+3}(x) \\ & + 3(|s-1|)\phi_{r,|s-1|}(x) + 3(s+1)(\phi_{r,s+1}(x)]\phi_{r,s}(x) \Bigg\}. \end{aligned} \tag{61}$$

The expressions for $\phi_{AS}, \phi_{SA}, \phi_{AA}$ are the same as (A.3) up to minus and pluses in front of the ϕ_{mn} terms. Hence the expressions for $\phi_{AS}, \phi_{SA}, \phi_{AA}$ are the same as (61) because the change in the sign of the coefficients of ϕ_{mn} are compensated by the change of the sign due to the symmetry properties of ϕ_{mn}'s under $n \to -nm \to -m$. Although the expressions for $S_{\phi k}$ for all $\phi_{AA}, \phi_{AS}, \phi_{SA}, \phi_{SS}$ are essentially the same and given by (61), in fact the $S_{\phi k}$ for ϕ_{SS} has an important difference than the others because only that result contains the zero mode

$\phi_{0,0}$ that is identified by the usual particles. So I take ϕ_{SS} as the only physically relevant state for ϕ. One observes that equation (61) contains only off-diagonal mixing of Kaluza–Klein modes. One may easily see that a bulk mass term for ϕ results in essentially the same form as the four-dimensional kinetic term in (61) where the derivatives are absent. Any other power of ϕ necessarily contains off-diagonal mixings of Kaluza–Klein modes. These observations are important when the vacuum expectation of energy–momentum tensor is obtained to give zero in the exact manifestation of extra-dimensional reflection symmetry. A more detailed analysis of equation (61) and these points will be given in the following section.

Next consider a bulk mass term (for ϕ_{SS})

$$\begin{aligned} S_{\phi m} &= \frac{1}{2}m\int\sqrt{(-1)^S g}\sqrt{(-1)^{S'}g'}\,\mathrm{d}^D x\,\mathrm{d}^{D'}x'\phi^2 \\ &= \frac{1}{2}mLL'\int\mathrm{d}^4x\left\{\sum_{n,m,r,s}\phi_{n,m}(x)\phi_{r,s}(x)\int_0^L\mathrm{d}y\cos ky\cos(nk|y|)\cos(rk|y|)\right. \\ &\quad\times\int_0^{L'}\mathrm{d}z\cos^3 k'z\cos(mk'|z|))\cos(sk'|z|)) \\ &= \frac{1}{64}m(LL')^2\int\mathrm{d}^4x\left\{\sum_{n,m,r,s}\phi_{n,m}(x)\phi_{r,s}(x)[(\delta_{n,-r-1}+\delta_{n,1-r})+\delta_{n,r-1}+\delta_{n,1+r})\right. \\ &\quad\times(\delta_{m,-s-3}+\delta_{m,3-s}+\delta_{m,s-3}+\delta_{m,s+3} \\ &\quad\left.+3\delta_{m,-s-1}+3\delta_{m,1-s}+3\delta_{m,s-1}+3\delta_{m,s+1})]\right\}. \end{aligned} \tag{62}$$

The common aspect of the equations (61) and (62) are that the Kaluza–Klein modes mix in such a way that there are no diagonal terms, i.e. the terms of the form $\phi_{n,m}\phi_{n,m}$. In fact this is a generic property of all possible terms for all kinds of fields, i.e. scalars, fermions, gauge fields or any other kind of field. All terms necessarily contain at least a pair of Kaluza–Klein modes that couple in a non-diagonal way. This can be seen as follows: a pair of fields that mix in a diagonal way (i.e. as $\phi_{n,m}\phi_{n,m}$) is even under either of the transformations in (28) since it corresponds to the terms of the form $\cos^2 nky\sin^2 mk'z$. If the whole terms consists of such pairs then the whole term is even under (28). However the volume element is odd under either of the transformations in (28). So such a term cannot exist, i.e. it must contain at least one pair of fields that couple in a off-diagonal way. This fact plays a crucial role in making the vacuum expectation value of the energy–momentum tensor zero in the exact manifestation of the metric reversal symmetry. In the following subsection we consider one additional example, that is, the kinetic term for fermions because it is not a straightforward generalization of the scalar case. We will see that the same conclusion also holds in that case as expected.

4.2.2. Fermionic fields. The kinetic term of the Lagrangian for fermionic fields in the space given by (24) in the presence of the signature reversal symmetry (where the conformal factors and the unprimed space are given by (25) and (20)) is

$$\mathcal{L}_{fk} = \mathrm{i}\bar{\psi}\Gamma^A\partial_A\psi + \mathrm{i}\bar{\psi}\Gamma^{A'}\partial_{A'}\psi. \tag{63}$$

For simplicity I take

$$g_{\mu\nu} = \eta_{\mu\nu}, \qquad \tilde{g}_{ab} = -\delta_{ab}, \qquad \tilde{g}_{A'B'} = -\delta_{A'B'}. \tag{64}$$

In fact $g_{\mu\nu} = \eta_{\mu\nu}$ is enforced by the symmetry, the four-dimensional homogeneity and isotropy of the metric as we have discussed in the previous section. So

$$\Gamma^A = \left(\cos\frac{kz}{2}\tau_3 + \mathrm{i}\sin\frac{kz}{2}\tau_1\right)^{-1} \otimes \gamma^A$$
$$\Gamma^{A'} = \left(\cos\frac{ky}{2}\tau_3 + \mathrm{i}\sin\frac{ky}{2}\tau_1\right)^{-1} \otimes \gamma^{A'}, \tag{65}$$

where

$$\{\Gamma^{A(A')}, \Gamma^{B(B')}\} = 2g^{AB(A'B')}, \qquad \{\gamma^A, \gamma^B\} = 2\eta^{AB}, \qquad \{\gamma^{A'}, \gamma^{B'}\} = -2\delta^{A',B'} \tag{66}$$

and τ_3, τ_1 are the diagonal and the off-diagonal real Pauli matrices, and $\otimes$ denotes tensor product. In the case of fermions one should use the complex expansion for the Fourier expansion

$$\psi(x, y, z) = \sum_{n,m} \psi_{n,m}(x)\,\mathrm{e}^{\frac{\mathrm{i}}{2}nky}\,\mathrm{e}^{\frac{\mathrm{i}}{2}mk'z}$$
$$= \sum_{n,m}\left(\psi^{nS}_{n,m}(x)\cos\left(\frac{1}{2}nky\right) + \psi^{nA}_{n,m}(x)\sin\left(\frac{1}{2}nky\right)\right)\mathrm{e}^{\frac{\mathrm{i}}{2}mk'z}$$
$$= \sum_{n,m}\left(\left(\psi^{mS}_{n,m}(x)\cos\left(\frac{1}{2}mk'z\right) + \psi^{mA}_{n,m}(x)\sin\left(\frac{1}{2}mk'z\right)\right)\mathrm{e}^{\frac{\mathrm{i}}{2}nky}, \tag{67}$$

where

$$\psi^{nS}_{n,m}(x) = \frac{1}{2}(\psi_{n,m}(x) + \psi_{-n,m}(x)), \qquad \psi^{nA}_{n,m}(x) = \frac{\mathrm{i}}{2}(\psi_{n,m}(x) - \psi_{-n,m}(x))$$
$$\psi^{mS}_{n,m}(x) = \frac{1}{2}(\psi_{n,m}(x) + \psi_{n,-m}(x)), \qquad \psi^{mA}_{n,m}(x) = \frac{\mathrm{i}}{2}(\psi_{n,m}(x) - \psi_{n,-m}(x)). \tag{68}$$

Next we substitute (67) into (63) to get S_{fk}. To be specific we take $N = 6$ and $N' = 2$ as in the previous subsubsection. Then (63) becomes (see appendix B)

$$S_{fk} = \frac{1}{32}(LL')^2 \int \mathrm{d}^4x \left\{ \sum_{n,m,r,s} [\mathrm{i}\psi_{n,m}(x)\tau_3 \otimes \gamma^\mu \partial_\mu(\psi_{r,s}(x)\right.$$
$$\times (\delta_{n,r+2} + \delta_{n,r-2})(\delta_{m,s+5} + \delta_{m,s-3} + 2\delta_{m,s+1}(\delta_{m,s-5} + \delta_{m,s+3} + 2\delta_{m,s-1})$$
$$- \psi_{n,m}(x)\tau_3 \otimes \gamma^\mu \partial_\mu(\psi_{r,s}(x)$$
$$\times (\delta_{n,r+2} + \delta_{n,r-2})(\delta_{m,s+3} + \delta_{m,s-5} - \delta_{m,s+5} - \delta_{m,s-3} + 2\delta_{m,s-1} - 2\delta_{m,s+11})$$
$$- \frac{1}{2}\psi_{n,m}(x)(r-n)\tau_3 \otimes \gamma^y \psi_{r,s}(x)$$
$$\times (\delta_{n,r+2} + \delta_{n,r-2})(\delta_{m,s+3} + \delta_{m,s-5} - \delta_{m,s+5} - \delta_{m,s-3} + 2\delta_{m,s-1} - 2\delta_{m,s+11})$$
$$+ \frac{1}{2}(r-n)\psi_{n,m}(x)\tau_1 \otimes \gamma^y \psi_{r,s}(x)$$
$$\times (\delta_{n,r+2} + \delta_{n,r-2})(\delta_{m,s+3} + \delta_{m,s-5} - \delta_{m,s+5} - \delta_{m,s-3} + 2\delta_{m,s-1} - 2\delta_{m,s+11})$$
$$- \frac{1}{2}\psi_{n,m}(x)(s-m)\tau_3 \otimes \gamma^y \psi_{r,s}(x)$$
$$\times (\delta_{n,r+1} + \delta_{n,r-1})(\delta_{m,s+6} + \delta_{m,s-6} + 3\delta_{m,s+2} + 3\delta_{m,s-2})$$
$$+ (s-m)\frac{1}{2}\psi_{n,m}(x)\tau_1 \otimes \gamma^y \psi_{r,s}(x)$$
$$\left. \times (\delta_{n,r-1} - \delta_{n,r+1})(\delta_{m,s+6} + \delta_{m,s-6} + 3\delta_{m,s+2} + 3\delta_{m,s-2}) \right\}, \tag{69}$$

where we have used the identity $\cos u\left(\cos\frac{u}{2}\tau_3 + \mathrm{i}\sin\frac{u}{2}\tau_1\right)^{-1} = \left(\cos\frac{u}{2}\tau_3 + \mathrm{i}\sin\frac{u}{2}\tau_1\right)$. We see that, in this case as well, each homogeneous term consists of one off-diagonally coupled pair of Kaluza–Klein modes.

5. The relation to Linde's model

It is evident from (61) that the four-dimensional kinetic term contains the zero mode ϕ_{00} while the other terms, i.e. the mass terms do not contain the zero mode. This implies that there is a zero mass eigenstate that contains ϕ_{00}. However the form of (61) is rather involved since it involves, in general, mixing of all Kaluza–Klein modes. An important aspect of this mixing is the absence of diagonal terms in the mixing terms. We will see in the following section how this plays a crucial role in making the vacuum expectation value of energy–momentum tensor zero. Before passing to this issue, first we should make the form of (61) more manageable. In any case one should diagonalize (61) so that, at least, the fields in the four-dimensional kinetic term couple to each other diagonally, i.e. we should pass to the interaction basis. One observes due to the signature reversal symmetry (induced through extra-dimensional reflections) that all the terms in the four-dimensional kinetic term in (61) are mixed so that the terms with odd n's mix with the even n's, and the odd m's with odd m's, the even m's with even m's. There is the same behavior for the terms with the coefficient k^2, and a similar behavior for the terms with the coefficient k'^2 (the odd n's mix with the odd n's, the even n's mix with the even n's and the odd m's mix with the even m's and vice versa). So the form given by the four-dimensional part of (61) may be only induced by the mixture of either of

$$\phi_{SS}^{OO}(x,y,z) = \sum_{j,l=0} \phi_{2j+1,2l+1}^{OOSS}(x)\cos(2j+1)ky\cos(2l+1)k'z$$

and

$$\phi_{SS}^{EO}(x,y,z) = \sum_{j,l=0} \phi_{2j,2l+1}^{EOSS}(x)\cos(2j)ky\cos(2l+1)k'z \tag{70}$$

or

$$\phi_{SS}^{EE}(x,y,z) = \sum_{j=1,l=0} \phi_{2j,2l}^{EESS}(x)\cos(2j)ky\cos(2l)k'z$$

and

$$\phi_{SS}^{OE}(x,y,z) = \sum_{j,l=0} \phi_{2j+1,2l}^{OESS}(x)\cos(2j+1)ky\cos(2l)k'z. \tag{71}$$

The each sum may be an infinite series if all modes are mixed or it may correspond to a set of finite sums if the modes mix with each other in a set of subsets of r and s in (61). In the expansion of ϕ_{SS}^{EE} the sum over j starts from one because we take the zero mode ϕ_{00} in a different eigenstate as we will see. The requirement that the internal symmetries that may be induced by extra-dimensional symmetries and the usual spacetime symmetries are independent requires the whole space be a direct product of the four-dimensional space with the extra-dimensional space. This, in turn, requires all $\phi_{n,m}(x)$'s in the above equations be the same up to constant coefficients, that is,

$$\phi_{SS,n,m}^{XY} = C_{n,m}^{XYSS}\phi^{XY}(x), \tag{72}$$

where X, Y may take the values O, E, and $C_{n,m}^{XYSS}$ is some constant with the condition that it leads to a finite series. For example, one may take

$$C_{n,m} = \frac{|n-2||m-2|}{(n^2+1)(m^2+1)}, \tag{73}$$

where $|n-2||m-2|$ is included to make the analysis of the zero mass eigenstate more manageable as will see. Then equations (70), (71) become

$$\phi^{OO}(x,y,z) = \left[\sum_{j,l=0} C^{OO}_{2j+1,2l+1}\cos(2j+1)ky\cos(2l+1)k'z\right]\phi^{OO}(x)$$
$$= \left[\sum_{j,l=0}\frac{|2j-1||2l-1|}{((2j+1)^2+1)((2l+1)^2+1)}\cos(2j+1)ky\cos(2l+1)k'z\right]\phi^{OO}(x)$$

and

$$\phi^{EO}(x,y,z) = \left[\sum_{j,l=0} C^{EO}_{2j,2l+1}\cos(2j)ky\cos(2l+1)k'z\right]\phi^{EO}(x)$$
$$= \left[\sum_{j,l=0}\frac{|2j-2||2l-1|}{((2j)^2+1)((2l+1)^2+1)}\cos(2j)ky\cos(2l+1)k'z\right]\phi^{EO}(x) \qquad (74)$$

or

$$\phi^{EE}(x,y,z) = \sum_{j=1,l=0} C^{EE}_{2j,2l}\cos(2j)ky\cos(2l)k'z]\phi^{EE}(x)$$
$$= \left[\sum_{j,l=0}\frac{|2j-2||2l-2|}{((2j)^2+1)((2l)^2+1}\cos(2j)ky\cos(2l)k'z\right]\phi^{EE}(x)$$

and

$$\phi^{OE}(x,y,z) = \sum_{j,l=0} C^{OE}_{2j+1,2l}\cos(2j+1)ky\cos(2l)k'z\phi^{OE}(x)$$
$$= \left[\sum_{j,l=0}\frac{|2j-1||2l-2|}{((2j+1)^2+1)((2l)^2}+1)\cos(2j+1)ky\cos(2l)k'z\right]\phi^{OE}(x), \qquad (75)$$

where the SS indices are suppressed. In the light of (74), (75) equation (61) becomes

$$S_{\phi k} = \frac{1}{2}(LL')^2\int d^4x\left\{2\eta^{\mu\nu}\partial_\mu(\phi_{1,0})\partial_\nu(\phi_{0,0}) + 2C_1C_2\eta^{\mu\nu}\partial_\mu(\phi^{EO}(x))\partial_\nu(\phi^{OO}(x))\right.$$
$$+2C_3C_4\eta^{\mu\nu}\partial_\mu(\phi^{EE}(x))\partial_\nu(\phi^{OE}(x))$$
$$-k^2[2C_5C_6\phi^{OO}(x)\phi^{EO}(x) + 2C_7C_8\phi^{EE}(x)\phi^{OE}(x)]$$
$$\left.-\frac{1}{2}k'^2[2C_9C_{10}\phi^{OO}(x)(\phi^{OE}(x) + 2C_{11}C_{12}\phi^{EE}(x)\phi^{EO}(x)]\right\}, \qquad (76)$$

where the form of the coefficients $C_i, i = 1,2,3,\ldots,12$ are given in appendix C. The diagonalization of (76) results in

$$S_{\phi k} = \frac{1}{2}(LL')^2\int d^4x\left\{\eta^{\mu\nu}(\partial_\mu\phi_1)\partial_\nu(\phi_1) - \eta^{\mu\nu}(\partial_\mu\phi_2)\partial_\nu(\phi_2)\right.$$
$$+C_1C_2(\eta^{\mu\nu}(\partial_\mu\phi_3(x))(\partial_\nu\phi_3(x)) - \eta^{\mu\nu}\partial_\mu(\phi_4(x))\partial_\nu(\phi_4(x)))$$
$$+C_3C_4(\eta^{\mu\nu}\partial_\mu(\phi_5(x))\partial_\nu(\phi_5(x)) - \eta^{\mu\nu}\partial_\mu(\phi_6(x))\partial_\nu(\phi_6(x))))$$
$$-k^2[C_5C_6(\phi_3(x)\phi_3(x) - \phi_4(x)\phi_4(x))$$
$$+C_7C_8(\phi_5(x)\phi_5(x) - \phi_6(x)\phi_6(x))]$$

$$- \frac{1}{2}k'^2[C_9C_{10}(\phi_7(x)\phi_7(x) - \phi_8(x)\phi_8(x)) + C_{11}C_{12}(\phi_9(x)\phi_9(x) - \phi_{10}(x)\phi_{10}(x))]\Bigg\}, \tag{77}$$

where

$$\begin{array}{llll} \phi_1 = \phi_{0,0} + \phi_{1,0}, & \phi_2 = \phi_{0,0} - \phi_{1,0}, & \phi_3 = \phi^{EO} + \phi^{OO}, & \phi_4 = \phi^{EO} - \phi^{OO} \\ \phi_5 = \phi^{EE} + \phi^{OE}, & \phi_6 = \phi^{EE} - \phi^{OE}, & \phi_7 = \phi^{OO} + \phi^{OE}, & \phi_8 = \phi^{OO} - \phi^{OE} \\ \phi_9 = \phi^{EE} + \phi^{EO}, & \phi_{10} = \phi^{EE} - \phi^{EO}. & & \end{array} \tag{78}$$

It is evident from (77) that the scalar kinetic Lagrangian (61) is equivalent to a Lagrangian that consists of a set of usual scalars and a set of ghost scalars. In fact this conclusion is valid for all quadratic terms for all fields, e.g. $\bar{\psi}_{n,m}\psi_{r,s}$ where $n \neq r$ and/or $m \neq s$ due to the symmetry and this term is equivalent to $\frac{1}{2}(\bar{\psi}_1\psi_1 - \bar{\psi}_2\psi_2)$ where $\bar{\psi}_1 = \psi_{n,m} + \psi_{r,s}$, $\bar{\psi}_2 = \psi_{n,m} - \psi_{r,s}$. This setting is similar to Linde's model [19] and its variants [20]. Only mixing between the usual particles and ghost sector may be induced through quartic and higher order terms. A detailed analysis of such possible mixings and suppressing these couplings needs a separate study by its own.

6. The vacuum expectation value of the energy–momentum tensor in the presence of metric reversal symmetry

The four-dimensional energy–momentum tensor corresponding to the action (76) is

$$\begin{aligned} T_\mu^\nu = {} & \frac{2}{\sqrt{(-1)^S g}\sqrt{(-1)^{S'} g'}} g_{\mu\rho}\frac{\delta S_M}{\delta g_{\nu\rho}} = 2\partial_\mu\phi_{1,0}(x)\partial^\nu\phi_{0,0}(x) \\ & + 2C_1C_2\partial_\mu(\phi^{EO}(x))\partial^\nu(\phi^{OO}(x)) + 2C_3C_4\partial_\mu[(\phi^{EE}(x))\partial^\nu(\phi^{OE}(x)) \\ & - \delta_\mu^\nu\Bigg\{\eta^{\mu\nu}\partial_\mu(\phi_{1,0})\partial_\nu(\phi_{0,0}) + C_1C_2\eta^{\mu\nu}\partial_\mu(\phi^{EO}(x))\partial_\nu(\phi^{OO}(x)) \\ & + C_3C_4\eta^{\mu\nu}\partial_\mu(\phi^{EE}(x))\partial_\nu(\phi^{OE}(x)) \\ & - k^2[C_5C_6\phi^{OO}(x)\phi^{EO}(x) + C_7C_8\phi^{EE}(x)\phi^{OE}(x)] \\ & - \frac{1}{2}k'^2[C_9C_{10}\phi^{OO}(x)(\phi^{OE}(x) + C_{11}C_{12}\phi^{EE}(x)\phi^{EO}(x)]\Bigg\}. \end{aligned} \tag{79}$$

It is evident from (79) that all terms consist of off-diagonally coupled Kaluza–Klein modes. As we have remarked before any four-dimensionally Lagrangian term (after integration over extra dimensions) necessarily contains at least a pair of Kaluza–Klein modes that are off-diagonally coupled in the space given by (24). (As we have remarked in the previous section, this is due to the fact that if a term wholly consists of pairs of diagonally coupled Kaluza–Klein modes then that term is even under the signature reversal symmetry in contradiction with the invariance of the action under the signature reversal symmetry.) This, in turn, leads to cancellation of the vacuum expectation value of T_μ^ν since it is proportional to terms of the form

$$\langle 0|T_\mu^\nu|0\rangle \propto \langle 0|a_{n,m}a_{r,s}^\dagger|0\rangle = 0, \qquad \langle 0|a_{r,s}^\dagger a_{r,s}|0\rangle = 0 \quad n \neq r \quad \text{and/or} \quad m \neq s \tag{80}$$

(because $a_{r,s}|0\rangle = 0$, and $\left[a_{n,m}, a_{r,s}^\dagger\right] = 0$ for $n \neq r$ and/or $m \neq s$) where $a_{n,m}$, $a_{n,m}^\dagger$ are the creation and annihilation operators in the expansion of the quantum fields (in Minkowski space) given by

$$\phi_{n,m}(x) = \sum_{\vec{k}}[a_{n,m}(\vec{k})\,\mathrm{e}^{-\mathrm{i}Et}\,\mathrm{e}^{\mathrm{i}\vec{k}.\vec{x}} + a_{n,m}^\dagger(\vec{k})\,\mathrm{e}^{\mathrm{i}Et}\,\mathrm{e}^{-\mathrm{i}\vec{k}.\vec{x}}]. \tag{81}$$

The same reasoning is true for all fields. Therefore the vacuum energy density of all fields in this scheme is zero.

In this scheme the Casimir effect can be seen as follows. The introduction of (metallic) boundaries into the vacuum results in a change in the vacuum configuration for the usual particles while the ghost sector vacuum remains the same. This point can be seen better when one considers the energy–momentum tensor written in terms of the usual and ghost fields by using (77)

$$\begin{aligned}
T_\mu^\nu = {} & (\partial_\mu\phi_1(x)\partial^\nu\phi_1(x)) - \partial_\mu\phi_2(x)\partial^\nu\phi_2(x)) + C_1C_2(\partial_\mu\phi_3(x)\partial^\nu\phi_3(x) - \partial_\mu\phi_4(x)\partial^\nu\phi_4(x)) \\
& + C_3C_4(\partial_\mu\phi_5(x)\partial^\nu\phi_5(x)) - \partial_\mu\phi_6(x)\partial^\nu\phi_6(x)) \\
& - \frac{1}{2}\delta_\mu^\nu\Big\{\eta^{\mu\nu}(\partial_\mu\phi_1)\partial_\nu(\phi_1) - \eta^{\mu\nu}(\partial_\mu\phi_2)\partial_\nu(\phi_2) \\
& + C_1C_2(\eta^{\mu\nu}(\partial_\mu\phi_3(x))(\partial_\nu\phi_3(x)) - \eta^{\mu\nu}\partial_\mu(\phi_4(x))\partial_\nu(\phi_4(x))) \\
& + C_3C_4(\eta^{\mu\nu}\partial_\mu(\phi_5(x))\partial_\nu(\phi_5(x)) - \eta^{\mu\nu}\partial_\mu(\phi_6(x))\partial_\nu(\phi_6(x)))] \\
& - k^2[C_5C_6(\phi_3(x)\phi_3(x) - \phi_4(x)\phi_4(x)) + C_7C_8(\phi_5(x)\phi_5(x) - \phi_6(x)\phi_6(x))] \\
& - \frac{1}{2}k'^2[C_9C_{10}(\phi_7(x)\phi_7(x) - \phi_8(x)\phi_8(x)) \\
& + C_{11}C_{12}(\phi_9(x)\phi_9(x) - \phi_{10}(x)\phi_{10}(x))]\Big\}. \qquad (82)
\end{aligned}$$

To see the situation better let us consider a simple case, for example the part of the energy–momentum tensor that contains the zero mode. After introduction of the (metallic) boundary the vacuum expectation value of the corresponding part of the energy–momentum tensor changes as follows:

$$\begin{aligned}
\langle 0|T_\mu^\nu|0\rangle_0 &= \langle 0|(\partial_\mu\phi_1)\partial^\nu(\phi_1)|0\rangle_0 - \langle 0|(\partial_\mu\phi_2)\partial^\nu(\phi_2)|0\rangle_0 = 0 \to \langle 0|T_\mu^\nu|0\rangle_{\Sigma_1} \\
&= \langle 0|(\partial_\mu\phi_1)\partial^\nu(\phi_1)|0\rangle_{\Sigma_1} - \langle 0|(\partial_\mu\phi_2)\partial^\nu(\phi_2)|0\rangle_0 \neq 0, \qquad (83)
\end{aligned}$$

where the subscript 0 denotes complete vacuum (without any boundary) and the subscript Σ_1 denotes the vacuum in the presence of the (metallic) boundaries. It is evident that this scheme results in an automatic application of the usual subtraction prescription in the calculation of Casimir energies, i.e. an automatic subtraction of the zero-point energy from the total vacuum energy in the presence of a boundary.

To summarize, I have shown that the quantum zero modes do not contribute to cosmological constant (CC) in the scheme presented here in the presence of metric reversal symmetry. Now, for the sake of completeness, I discuss the other possible contributions to CC. The first additional contribution is a bulk CC (that is geometric in origin). The transformations (33) and/or (34) (or equivalently the form of the conformal factors given in (25)) forbid a bulk CC (or equivalently make it vanish after integration over extra dimensions). The second possible contribution is a four-dimensional CC that may be induced by the part of the scalar curvature that depends only on extra dimensions. Equation (39) implies that such a contribution vanishes provided that the half of the extra dimensions in the $2(2n + 1)$-dimensional space (embedding the usual four-dimensional space) are spacelike and half are timelike as in [7]. The next possible contribution is the vacuum energy induced by the vacuum expectation value of Higgs field, and is about $\sim 10^{55}$ times the observational value of CC. This contribution has the form of a bulk CC, and hence vanishes provided that Higgs field propagates in the whole space or in its a $2(2k + 1)$-dimensional subspace. Another possible standard contribution is the vacuum expectation value of the QCD vacuum (that is about 10^{44} times the observational value of CC). At classical level the same condition as Higgs field applies to the space where the corresponding condensate forms. However a rigorous conclusion needs an analysis at quantum level. There are many phenomenological and/or nonperturbative schemes aiming to explain the formation and value of QCD condensates (hence of QCD vacuum energy) that only partially can give insight into the problem [21]. So a definite conclusion about this point needs further additional study. However this issue is not as urgent as the issue of zero-point energies because the problem of zero-point energies arises as soon as the fields are introduced (and quantized) even in the case of free fields while QCD vacuum is present only inside the hadrons and is not perfectly well understood. Another important issue to be studied in future is: although I have shown that quantum fields do not induce non-vanishing vacuum energy at fundamental Lagrangian level (i.e. quantum zero modes do not contribute to vacuum energy) in the presence of metric reversal symmetry there is no guarantee of nonzero contributions to vacuum energy due to higher dimension operators (than those of the fundamental Lagrangian). If this is the case the resulting vacuum energy due to quantum fields will be scale dependent through renormalization group equations. The most reasonable consequence of this, in turn, would be a time-varying cosmological constant [22]. Time varying cosmological constant scenarios together with quintessence models have an additional virtue of explaining cosmic coincidence, i.e. the energy density of matter and dark energy being in the same order of magnitude, that is not addressed by the scheme in this paper. All these studies must be studied in future for a clearer picture of the cause and dynamics of the accelerated expansion of the universe.

7. Inducing a small cosmological constant by breaking the symmetry by a small amount

We have seen that contribution of quantum fields to the energy–momentum tensor is always zero in the manifestation of signature reversal symmetry. However this is not true for classical fields. For example consider a classical field that depends only on extra dimensions and has a Fourier expansion as in (46)–(49). This field gives nonzero contribution to four-dimensional cosmological constant (CC) after integration over extra dimensions. For example one may take

$$\mathcal{L}_{cl} = \alpha v_{1,0} v_{0,1} \cos ky \cos k'z, \tag{84}$$

where $\alpha \ll 1$ is a constant that reflects that $\mathcal{L}_{cl}$ is small since it corresponds to the breaking of the $x^A \to \mathrm{i}x^A, x^{A'} \to \mathrm{i}x^{A'}$ symmetries separately by a small amount, and $v_{1,0}$, $v_{0,1}$ are some constants. If one takes the same space as in section 4 and take $N = 6$, $N' = 2$ (as before) then $\mathcal{L}_{cl}$ in (84) after integration over extra-dimensions results in a four-dimensional CC given by

$$\Lambda^{(4)} = \frac{3\alpha v_{1,0} v_{0,1}}{16} (LL')^2. \tag{85}$$

For $\alpha v_{1,0} v_{0,1} \simeq 1$ (85) results in the observed value of $\Lambda \simeq (10^{-3}\ \mathrm{eV})^4$ for L, L' in the millimeter scale and for $\alpha v_{1,0} v_{0,1} \simeq \frac{M_{ew}}{M_{pl}} \simeq 10^{-17}$, for example, $L(L') < 10^{-7} m$. In any case a nonzero CC if exists is a classical phenomena in this scheme. Another point is that the energy density due to CC obtained in a way similar to (85) may be argued to be in the order of matter (i.e. the usual matter plus dark matter) density since both are induced by matter Lagrangian that corresponds to breaking of the $x^A \to \mathrm{i}x^A, x^{A'} \to \mathrm{i}x^{A'}$ symmetries. However there is a difference between the two cases. The induction of S_M corresponds to breaking the symmetry that corresponds to the simultaneous application of $x^A \to \mathrm{i}x^A$ and $x^{A'} \to \mathrm{i}x^{A'}$ while $\mathcal{L}_{cl}$ in equation (84) corresponds to breaking of $x^A \to \mathrm{i}x^A$ and $x^{A'} \to \mathrm{i}x^{A'}$ separately.

8. Conclusion

We have considered a space that is a sum of two $2(2n+1)$-dimensional spaces with R^2 gravity and metric reversal symmetry. The usual four-dimensional space is embedded in one of these subspaces. We have shown that the curvature sector reduces to the usual Einstein–Hilbert action, and the four-dimensional energy–momentum tensor of matter fields generically mixes different Kaluza–Klein modes so that each homogeneous term contains at least one pair of off-diagonally coupled Kaluza–Klein modes. This, in turn, results in vanishing of the vacuum expectation value of the energy–momentum tensor of quantum fields. I have also shown that such a model is equivalent to a variation of Linde's model (where the universe consists of the usual universe plus a ghost one). There may be some relation between this scheme and the Pauli–Villars regularization scheme [23] (that employs ghost-like auxiliary fields for regularization), and also between this scheme and Lee–Wick quantum theory [24]. In my opinion all these points need further and detailed studies in future.

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Appendix A. Calculation of $S_{\phi k}$

$$\begin{aligned}
S_{\phi k} &= \int \mathrm{d}V\,\mathcal{L}_{\phi k} \\
&= \frac{1}{2}\int \sqrt{(-1)^S g}\sqrt{(-1)^{S'} g'}\,\mathrm{d}^D x\,\mathrm{d}^{D'} x' \left[\frac{1}{2} g^{AB}\partial_A\phi\partial_B\phi + \frac{1}{2} g^{A'B'}\partial_{A'}\phi\partial_{B'}\phi\right] \\
&= \frac{1}{2}\int \mathrm{d}^4x\,\mathrm{d}y_1\,\mathrm{d}y_2\,\mathrm{d}z_1\,\mathrm{d}z_2\Omega_z^3\Omega_y \left\{\Omega_z^{-1}\left[\eta^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - \left(\frac{\partial\phi}{\partial y_1}\right)^2 - \left(\frac{\partial\phi}{\partial y_2}\right)^2\right]\right. \\
&\quad \left. - \Omega_y\left[\left(\frac{\partial\phi}{\partial z_1}\right)^2 + \left(\frac{\partial\phi}{\partial z_2}\right)^2\right]\right\} \\
&= \frac{1}{2}LL'\int \mathrm{d}^4x \int_0^L\int_0^{L'} \mathrm{d}y\,\mathrm{d}z\,\cos^3 k'z\cos ky \\
&\quad \times \left\{\cos^{-1}k'z\left[\eta^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - \left(\frac{\partial\phi}{\partial y}\right)^2\right] - \cos^{-1}ky\left(\frac{\partial\phi}{\partial z}\right)^2\right\}
\end{aligned} \tag{A.1}$$

First evaluate (A.1) for (46)

$$\begin{aligned}
S_{Mk} = \frac{1}{2}LL'\int \mathrm{d}^4x &\left\{\eta^{\mu\nu}\sum_{n,m,r,s}\partial_\mu(\phi_{n,m}(x))\partial_\nu(\phi_{r,s}(x))\right. \\
&\times \int_0^L \mathrm{d}y\cos ky\sin(nk|y|)\sin(rk|y|)\int_0^{L'}\mathrm{d}z\cos^2 k'z\sin(mk'|z|))\sin(sk'|z|)) \\
&- k^2\sum_{n,m,r,s} nr\phi_{n,m}(x)\phi_{r,s}(x)\int_0^L \mathrm{d}y\cos ky\cos(nk|y|)\cos(rk|y|) \\
&\left.\times \int_0^{L'}\mathrm{d}z\cos^2 k'z\sin(mk'|z|))\sin(sk'|z|))\right\}
\end{aligned}$$

$$- k'^2 \sum_{n,m,r,s} ms\phi_{n,m}(x)\phi_{r,s}(x) \int_0^L \mathrm{d}y \sin(nk|y|)\sin(rk|y|)$$
$$\times \int_0^{L'} \mathrm{d}z \cos^3 k'z \cos(mk'|z|)) \cos(sk'|z|))$$
$$= \frac{1}{32}(LL')^2 \int \mathrm{d}^4x \Bigg\{ \eta^{\mu\nu} \sum_{n,m,r,s} \partial_\mu(\phi_{n,m}(x))\partial_\nu(\phi_{r,s}(x))$$
$$\times (\delta_{n,r-1} + \delta_{n,r+1} - \delta_{n,-r-1} - \delta_{n,1-r})$$
$$\times (\delta_{m,s-2} + \delta_{m,s+2} - \delta_{m,-s-2} - \delta_{m,2-s} + 2\delta_{m,s} - 2\delta_{m,-s})$$
$$- k^2 \sum_{n,m,r,s} nr\phi_{n,m}(x)\phi_{r,s}(x)(\delta_{n,r-1} + \delta_{n,r+1} + \delta_{n,-r-1} + \delta_{n,1-r})$$
$$\times (\delta_{m,s-2} + \delta_{m,s+2} - \delta_{m,-s-2} - \delta_{m,2-s} + 2\delta_{m,s} - 2\delta_{m,-s})$$
$$- \frac{1}{2}k'^2 \sum_{n,m,r,s} ms\phi_{n,m}(x)\phi_{r,s}(x)(\delta_{n,r} - \delta_{n,-r})$$
$$\times (\delta_{m,s-3} + \delta_{m,s+3} + \delta_{m,-s-3} + \delta_{m,3-s}$$
$$+ 3\delta_{m,s-1} + 3\delta_{m,s+1} + 3\delta_{m,-s-1} + 3\delta_{m,1-s}) \Bigg\} \tag{A.2}$$

$$= \frac{1}{32}(LL')^2 \int \mathrm{d}^4x \Bigg\{ \eta^{\mu\nu} \sum_{r,s} \partial_\mu[\phi_{r-1,s-2}(x) + \phi_{r-1,s+2}(x) - \phi_{r-1,-s-2}(x)$$
$$- \phi_{r-1,2-s}(x) + 2\phi_{r-1,s}(x) - 2\phi_{r-1,-s}(x) + \phi_{r+1,s-2}(x) + (\phi_{r+1,s+2}(x)$$
$$- \phi_{r+1,-s-2}(x) - \phi_{r+1,2-s}(x) + 2\phi_{r+1,s}(x) - 2\phi_{r+1,-s}(x) - \phi_{-r-1,s-2}(x))$$
$$- \phi_{-r-1,s+2}(x) + \phi_{-r-1,-s-2}(x) + \phi_{-r-1,2-s}(x)$$
$$- 2\phi_{-r-1,s}(x) + 2\phi_{-r-1,-s}(x) - \phi_{1-r,s-2}(x) - \phi_{1-r,s+2}(x) + \phi_{1-r,-s-2}(x)$$
$$+ \phi_{1-r,2-s}(x) - 2\phi_{1-r,s}(x) + 2\phi_{1-r,-s}(x)]\partial_\nu(\phi_{r,s}(x))$$
$$- k^2 \sum_{r,s} r[(r-1)(\phi_{r-1,s-2}(x) - \phi_{1-r,s-2}(x))$$
$$+ (r-1)(\phi_{r-1,s+2}(x) - \phi_{1-r,s+2}(x)) - (r-1)(\phi_{r-1,-s-2}(x)$$
$$- \phi_{1-r,-s-2}(x)) - (r-1)(\phi_{r-1,2-s}(x) - \phi_{1-r,2-s}(x))$$
$$+ 2(r-1)(\phi_{r-1,s}(x) - \phi_{1-r,s}(x)) - 2(r-1)(\phi_{r-1,-s}(x) - \phi_{1-r,-s}(x))$$
$$+ (r+1)(\phi_{r+1,s-2}(x) - \phi_{-r-1,s-2}(x)) + (r+1)(\phi_{r+1,s+2}(x) - \phi_{-r-1,s+2}(x))$$
$$- (r+1)(\phi_{r+1,-s-2}(x) - \phi_{-r-1,-s-2}(x)) - (r+1)(\phi_{r+1,2-s}(x)$$
$$- \phi_{-r-1,2-s}(x)) + 2(r+1)(\phi_{r+1,s}(x) - \phi_{-r-1,s}(x))$$
$$- 2(r+1)(\phi_{r+1,-s}(x) - \phi_{-r-1,-s}(x))]\phi_{r,s}(x) - \frac{1}{2}k'^2 \sum_{r,s} s[(s-3)(\phi_{r,s-3}(x)$$
$$- \phi_{r,3-s}(x)) + (s+3)(\phi_{r,s+3}(x) - \phi_{r,-s-3}(x)) + 3(s-1)(\phi_{r,s-1}(x)$$
$$- \phi_{r,1-s}(x)) + 3(s+1)(\phi_{r,s+1}(x) - \phi_{r,-s-1}(x)) + (3-s)(\phi_{-r,s-3}(x)$$
$$- \phi_{-r,3-s}(x)) + (s+3)(\phi_{-r,-s-3}(x) - \phi_{-r,s+3}(x)) + 3(1-s)(\phi_{-r,s-1}(x)$$
$$- \phi_{-r,1-s}(x)) - 3(s+1)(\phi_{-r,s+1}(x) - \phi_{-r,-s-}(x))]\phi_{r,s}(x) \Bigg\}, \tag{A.3}$$

where I have used (23) and taken $y = y_2, z = z_2$ and $\phi_{mn} = \phi^{AA}_{mn}$ in (46). After using the antisymmetry of ϕ^{AA}_{mn} under both of $n \to -n$ $m \to -m$, (A.3) may be written in a simplified form as (46). In fact this result is essentially the same as those of $\phi_{AS}, \phi_{SA}, \phi_{SS}$ as explained after equation (61).

Appendix B. Calculation of S_{fk}

$$
\begin{aligned}
S_{fk} &= \frac{1}{2}LL' \int \mathrm{d}^4x \int_0^L \int_0^{L'} \mathrm{d}y\,\mathrm{d}z \cos^3 k'z \cos ky \\
&\quad\times \left[\mathrm{i}\bar{\psi} \left(\cos\frac{kz}{2}\tau_3 + \mathrm{i}\sin\frac{kz}{2}\tau_1 \right)^{-1} \otimes \gamma^A \partial_A \psi \right. \\
&\quad \left. + \mathrm{i}\bar{\psi} \left(\cos\frac{ky}{2}\tau_3 + \mathrm{i}\sin\frac{ky}{2}\tau_1 \right)^{-1} \otimes \gamma^{A'} \partial_{A'} \psi \right] \\
&= \frac{1}{2}LL' \int \mathrm{d}^4x \int_0^L \int_0^{L'} \mathrm{d}y\,\mathrm{d}z \left[\mathrm{i}\bar{\psi}\cos^2 k'z \cos ky \left(\cos\frac{kz}{2}\tau_3 + \mathrm{i}\sin\frac{kz}{2}\tau_1 \right) \otimes \gamma^A \partial_A \psi \right. \\
&\quad \left. + \mathrm{i}\bar{\psi}\cos^3 k'z \left(\cos\frac{ky}{2}\tau_3 + \mathrm{i}\sin\frac{ky}{2}\tau_1 \right) \otimes \gamma^{A'} \partial_{A'} \psi \right] \\
&= \frac{1}{2}LL' \int \mathrm{d}^4x \left\{ \sum_{n,m,r,s} \left[\mathrm{i}\psi_{n,m}(x)\tau_3 \otimes \gamma^\mu \partial_\mu(\psi_{r,s}(x) \int_0^L \mathrm{d}y \cos(k|y|) \exp\left(\frac{\mathrm{i}}{2}(r-n)k|y| \right) \right. \right. \\
&\quad \times \int_0^{L'} \mathrm{d}z \cos^2 k'|z| \cos\left(\frac{1}{2}k'|z| \right) \exp\left(\frac{\mathrm{i}}{2}(s-m)k'|z| \right) \\
&\quad - \psi_{n,m}(x)\tau_1 \otimes \gamma^\mu \partial_\mu(\psi_{r,s}(x) \int_0^L \mathrm{d}y \cos(k|y|) \exp\left(\frac{\mathrm{i}}{2}(r-n)k|y| \right) \\
&\quad \times \int_0^{L'} \mathrm{d}z \cos^2 k'|z| \sin\left(\frac{1}{2}k'|z| \right) \exp\left(\frac{\mathrm{i}}{2}(s-m)k'|z| \right) \\
&\quad - \frac{1}{2}\psi_{n,m}(x)(r-n)\tau_3 \otimes \gamma^y \psi_{r,s}(x) \int_0^L \mathrm{d}y \cos(k|y|) \exp\left(\frac{\mathrm{i}}{2}(r-n)k|y| \right) \\
&\quad \times \int_0^{L'} \mathrm{d}z \cos^2 k'|z| \cos\left(\frac{1}{2}k'|z| \right) \exp\left(\frac{\mathrm{i}}{2}(s-m)k'|z| \right) \\
&\quad - \mathrm{i}\frac{1}{2}(r-n)\psi_{n,m}(x)\tau_1 \otimes \gamma^y \psi_{r,s}(x) \int_0^L \mathrm{d}y \cos(k|y|) \exp\left(\frac{\mathrm{i}}{2}(r-n)k|y| \right) \\
&\quad \times \int_0^{L'} \mathrm{d}z \cos^2 k'|z| \sin\left(\frac{1}{2}k'|z| \right) \exp\left(\frac{\mathrm{i}}{2}(s-m)k'|z| \right) \\
&\quad - \frac{1}{2}\frac{1}{2}\psi_{n,m}(x)(s-m)\tau_3 \otimes \gamma^y \psi_{r,s}(x) \int_0^L \mathrm{d}y \cos\left(\frac{1}{2}k|y| \right) \exp\left(\frac{\mathrm{i}}{2}(r-n)k|y| \right) \\
&\quad \times \int_0^{L'} \mathrm{d}z \cos^3 k'|z| \exp\left(\frac{\mathrm{i}}{2}(s-m)k'|z| \right) \\
&\quad - \mathrm{i}\frac{1}{2}(s-m)\psi_{n,m}(x)\tau_1 \otimes \gamma^y \psi_{r,s}(x) \int_0^L \mathrm{d}y \sin\left(\frac{1}{2}k|y| \right) \exp\left(\frac{\mathrm{i}}{2}(r-n)k|y| \right) \\
&\quad \left. \times \int_0^{L'} \mathrm{d}z \cos^3 k'|z| \exp\left(\frac{\mathrm{i}}{2}(s-m)k'|z| \right) \right\}
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{1}{2} L L' \int \mathrm{d}^4 x \left\{ \sum_{n,m,r,s} \left[\mathrm{i} \bar{\psi}_{n,m}(x) \tau_3 \otimes \gamma^\mu \partial_\mu (\psi_{r,s}(x) \right.\right. \\
&\quad \times \int_0^L \mathrm{d}y \cos(k|y|) \cos\left(\frac{1}{2}(r-n)k|y|\right) \\
&\quad \times \int_0^{L'} \mathrm{d}z \cos^2 k'|z| \cos\left(\frac{1}{2}k'|z|\right) \cos\left(\frac{1}{2}(s-m)k'|z|\right) \\
&\quad - \bar{\psi}_{n,m}(x) \tau_1 \otimes \gamma^\mu \partial_\mu (\psi_{r,s}(x) \int_0^L \mathrm{d}y \cos(k|y|) \cos\left(\frac{1}{2}(r-n)k|y|\right) \\
&\quad \times \int_0^{L'} \mathrm{d}z \cos^2 k'|z| \sin\left(\frac{1}{2}k'|z|\right) \sin\left(\frac{1}{2}(s-m)k'|z|\right) \\
&\quad - \frac{1}{2} \bar{\psi}_{n,m}(x)(r-n) \tau_3 \otimes \gamma^y \psi_{r,s}(x) \int_0^L \mathrm{d}y \cos(k|y|) \cos\left(\frac{1}{2}(r-n)k|y|\right) \\
&\quad \times \int_0^{L'} \mathrm{d}z \cos^2 k'|z| \cos\left(\frac{1}{2}k'|z|\right) \cos\left(\frac{1}{2}(s-m)k'|z|\right) \\
&\quad + \frac{1}{2}(r-n) \bar{\psi}_{n,m}(x) \tau_1 \otimes \gamma^y \psi_{r,s}(x) \int_0^L \mathrm{d}y \cos(k|y|) \cos\left(\frac{1}{2}(r-n)k|y|\right) \\
&\quad \times \int_0^{L'} \mathrm{d}z \cos^2 k'|z| \sin\left(\frac{1}{2}k'|z|\right) \sin\left(\frac{1}{2}(s-m)k'|z|\right) \\
&\quad - \frac{1}{2} \bar{\psi}_{n,m}(x)(s-m) \tau_3 \otimes \gamma^y \psi_{r,s}(x) \int_0^L \mathrm{d}y \cos\left(\frac{1}{2}k|y|\right) \cos\left(\frac{1}{2}(r-n)k|y|\right) \\
&\quad \times \int_0^{L'} \mathrm{d}z \cos^3 k'|z| \cos\left(\frac{1}{2}(s-m)k'|z|\right) \\
&\quad + (s-m) \frac{1}{2} \bar{\psi}_{n,m}(x) \tau_1 \otimes \gamma^y \psi_{r,s}(x) \int_0^L \mathrm{d}y \sin\left(\frac{1}{2}k|y|\right) \sin\left(\frac{1}{2}(r-n)k|y|\right) \\
&\quad \left. \times \int_0^{L'} \mathrm{d}z \cos^3 k'|z| \cos\left(\frac{1}{2}(s-m)k'|z|\right) \right\}. \qquad \text{(B.1)}
\end{aligned}
$$

After integration over y and z this equation results in (69).

Appendix C. Explicit forms of C_K, K = 1, 2, 3, . . . , 12

After inserting equations (74) and (75) (in the light of equations (70) and (71)) into equation (A.1) and integrating over the extra dimensions it should be equal to (61). Hence after comparing the result of the integration with equation (61) we obtain the following results for the constants:

$$
\begin{aligned}
2C_1C_2 = \sum_{j,l} \left\{ \frac{|2j-1||2l-1|}{((2j+1)^2+1)((2l+1)^2+1)} \left[\frac{|2j-2||2l-3|}{((2j)^2+1)((2l-1)^2+1)} \right.\right. \\
+ \frac{|2j-2||2l+1|}{((2j)^2+1)((2l+3)^2+1)} + 2\frac{|2j-2||2l-1|}{(2j)^2(2l+1)^2} \\
+ \frac{|2j||2l-1|}{((2j+2)^2+1)((2l-1)^2+1)} + \frac{|2j||2l+1|}{(2j+2)^2(2l+3)^2} \\
\left.\left. + 2\frac{|2j||2l-1|}{((2j+2)^2+1)((2l+1)^2+1)} \right] \right\}
\end{aligned}
$$

$$
\begin{aligned}
2C_3C_4 = \sum_{j,l} \Bigg\{ & \frac{|2j-1||2l-2|}{((2j+1)^2+1)((2l)^2+1)} \Bigg[\frac{|2j-2||2l-4|}{((2j)^2+1)((2l-2)^2+1)} \\
& + \frac{|2j-2||2l|}{((2j)^2+1)((2l+2)^2+1)} + 2\frac{|2j-2||2l-2|}{((2j)^2+1)((2l)^2+1)} \\
& + \frac{|2j||2l-4|}{((2j+2)^2+1)((2l-2)^2+1)} + \frac{|2j||2l|}{((2j+2)^2+1)((2l+2)^2+1)} \\
& + 2\frac{|2j||2l-2|}{((2j+2)^2+1((2l)^2+1)} \Bigg] \Bigg\} \\
2C_5C_6 = \sum_{j,l} \Bigg\{ & (2j)\frac{|2j-2||2l-1|}{((2j)^2+1)(2l+1)^2+1)} \Bigg[(2j-1)\Bigg(\frac{|2j-3||2l-3|}{((2j-1)^2+1)((2l-1)^2+1)} \\
& + \frac{|2j-3||2l+1|}{((2j-1)^2+1)((2l+3)^2+1)} + 2\frac{|2j-3||2l-1|}{((2j-1)^2+1)((2l+1)^2+1)} \Bigg) \\
& + (2j+1)\Bigg(\frac{|2j-1||2l-3|}{((2j+1)^2+1)((2l-1)^2+1)} \\
& + \frac{|2j-1||2l+1|}{((2j+1)^2+1)((2l+3)^2+1)} + 2\frac{|2j-1||2l-1|}{((2j+1)^2+1)((2l+1)^2+1)} \Bigg) \Bigg] \Bigg\} \\
2C_7C_8 = \sum_{j,l} \Bigg\{ & (2j+1)\Bigg(\frac{|2j-1||2l-2|}{((2j+1)^2+1)((2l)^2+1)} \Bigg[(2j)\Bigg(\frac{|2j-2||2l-4|}{((2j)^2+1)((2l-2)^2+1)} \\
& + \frac{|2j-2||2l|}{((2j)^2+1)((2l+2)^2+1)} + 2\frac{|2j-2||2l-2|}{((2j)^2+1)((2l)^2+1)} \\
& + (2j+2)\Bigg(\frac{|2j||2l-4|}{((2j+2)^2+1)((2l-2)^2+1)} + \frac{|2j||2l|}{((2j+2)^2+1)((2l+2)^2+1)} \\
& + 2\frac{|2j||2l-2|}{((2j+2)^2+1)((2l)^2+1)} \Bigg] \Bigg\} \\
2C_9C_{10} = \sum_{j,l} \Bigg\{ & (2l)\Bigg(\frac{|2j-1||2l-2|}{((2j+1)^2+1)((2l)^2+1)} \Bigg[(2l-3)\frac{|2j-1||2l-5|}{((2j+1)^2+1)((2l-3)^2+1)} \\
& + (2l+3)\frac{|2j-1||2l+1|}{((2j+1)^2+1)((2l+3)^2+1)} \\
& + 3(2l-1)\frac{|2j-1||2l-3|}{((2j+1)^2+1)((2l-1)^2+1)} \\
& + 3(2l+1)\frac{|2j-1||2l-1|}{((2j+1)^2+1)((2l+1)^2+1)} \Bigg] \Bigg\} \\
2C_{11}C_{12} = \sum_{j,l} \Bigg\{ & (2l+1)\frac{|2j-1||2l-2|}{((2j)^2+1)((2l+1)^2+1)} \Bigg[(2l-2)\frac{|2j-2||2l-4|}{((2j)^2+1)((2l-2)^2+1)} \\
& + (2l+4)\frac{|2j-2||2l+2|}{((2j)^2+1)((2l+4)^2+1)} + 3(2l)\frac{|2j-2||2l-2|}{((2j)^2+1)((2l)^2+1)} \\
& + 3(2l+2)\frac{|2j-2||2l|}{((2j)^2+1)((2l+2)^2+1)} \Bigg] \Bigg\}.
\end{aligned}
\qquad \text{(C.1)}
$$

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