Glasgow Math. J. **49** (2007) 127–131. © 2007 Glasgow Mathematical Journal Trust.
doi:10.1017/S0017089507003485. Printed in the United Kingdom

MODULES OVER PRÜFER DOMAINS WHICH SATISFY THE RADICAL FORMULA

DILEK BUYRUK
Department of Mathematics, Abant Izzet Baysal University, 14280 Bolu/Turkey
e-mail: dilekbuyruk@yahoo.com

and DILEK PUSAT-YILMAZ
Department of Mathematics, Izmir Institute of Technology, 35430 Urla, Izmir/Turkey
e-mail: dilekyilmaz@iyte.edu.tr

(Received 6 June, 2006; revised 26 September, 2006; accepted 13 October, 2006)

Abstract. In this paper we prove that if R is a Prüfer domain, then the R-module $R \oplus R$ satisfies the radical formula.

2000 *Mathematics Subject Classification.* 13A15, 13C99, 13F05, 13F30.

1. Introduction. Let M be a module over a commutative ring R and N be a submodule of M. The *prime radical of* N *in* M, $rad_M(N)$, is defined to be the intersection of all prime submodules of M containing N. If there is no prime submodule containing N, then $rad_M(N) = M$. In particular $rad_M(M) = M$.

Let M be an R-module and N a submodule of M. The *envelope of* N *in* M which is denoted by $E_M(N)$ is defined to be the set

$$\{rm : r \in R, m \in M \text{ such that } r^n m \in N \text{ for some } n \in \mathbb{Z}^+\}.$$

We say that the submodule N satisfies the radical formula in M (N s.t.r.f. in M) if $rad_M(N) = \langle E_M(N)\rangle$. A module M s.t.r.f. if for every submodule N of M, the prime radical of N is the submodule generated by its envelope, that is, $rad_M(N) = \langle E_M(N)\rangle$. A ring R s.t.r.f. provided that for every R-module M, M s.t.r.f..

The question of what kind of rings and modules s.t.r.f. has drawn the attention of many authors. In [**2**], Jenkins and Smith proved that Dedekind domains s.t.r.f.. In [**3**], Leung and Man proved that the only Noetherian rings which s.t.r.f. are of dimension at most one and they gave a complete characterization of Noetherian rings which s.t.r.f.. Now we are looking for non-Noetherian rings which s.t.r.f.. For that reason we investigate whether modules over Prüfer domains s.t.r.f.. We prove in Theorem 2.4 that if R is a Prüfer domain, then the R-module $R \oplus R$ s.t.r.f. Throughout $R \oplus R$ will be denoted by R^2. The following is given in [**6**].

PROPOSITION 1.1. *Let* R *be a ring. If the* $R_{\mathcal{M}}$*-module* $R_{\mathcal{M}} \oplus R_{\mathcal{M}}$ *s.t.r.f. for any maximal ideal* $\mathcal{M}$ *of* R*, then the* R*-module* R^2 *s.t.r.f.*.

2. Results.

LEMMA 2.1. *Let* R *be a commutative ring, and* N *be a submodule of* R^2*. If* $N = I \oplus J$ *for some ideals* I, J *of* R*, then* N *s.t.r.f. in* R^2.

Proof. Clearly, for any prime ideal $\mathcal{P}$ of R containing I, $\mathcal{P} \oplus R$ is a prime submodule of R^2 containing N. Thus, $rad_{R^2}(N) \subseteq \sqrt{I} \oplus R$. Similarly, $rad_{R^2}(N) \subseteq R \oplus \sqrt{J}$. Hence $rad_{R^2}(N) \subseteq \sqrt{I} \oplus \sqrt{J}$. Now take $(x, y) \in \sqrt{I} \oplus \sqrt{J}$. That is $x^k \in I$ and $y^t \in J$ for some $k, t \in \mathbb{Z}^+$ and $x^k(1, 0), y^t(0, 1) \in N$. Then $(x, y) = x(1, 0) + y(0, 1) \in \langle E_{R^2}(N) \rangle$. Since we always have the other inclusion, $rad_{R^2}(N) = \sqrt{I} \oplus \sqrt{J} = \langle E_{R^2}(N) \rangle$ and N s.t.r.f. in R^2. □

LEMMA 2.2. *Let D be a valuation domain and let N be a D-submodule of D^2. If (m, n) is an element of N such that $(m^k, 0) \in N$ or $(0, n^{k'}) \in N$ for some $k, k' \in \mathbb{Z}^+$, then $\sqrt{Dm} \oplus \sqrt{Dn} \subseteq \langle E_{D^2}(N) \rangle$.*

Proof. Suppose $(m^k, 0) \in N$ for some $k \in \mathbb{Z}^+$. Then for any $x \in \sqrt{Dm}$ there exists $d \in D$, $l \in \mathbb{Z}^+$ such that $x^l = dm$. Hence $x^{kl}(1, 0) = d^k m^k(1, 0) \in N$ implies that $\sqrt{Dm}(1, 0) \subseteq \langle E_{D^2}(N) \rangle$.

If $Dm \subseteq Dn$, then $m/n \in D$. Take a nonzero $s \in \sqrt{Dn}$ that is $s^t = rn$ for some nonzero $r \in D$ and $t \in \mathbb{Z}^+$. Since $s^t((m/n), 1) = (rm, rn) \in N$, and $(s(m/n))^{tk}(1, 0) = r^k(m/n)^{tk-k}(m^k, 0) \in N$ we have

$$(0, s) = s((m/n), 1) - s(m/n)(1, 0) \in \langle E_{D^2}(N) \rangle.$$

Therefore $\sqrt{Dn}(0, 1) \subset \langle E_{D^2}(N) \rangle$.

If $Dn \subseteq Dm$, then $n/m \in D$ and $(0, n^k) = n^{k-1}(m, n) - (n/m)^{k-1}(m^k, 0) \in N$. Thus $\sqrt{Dn}(0, 1) \in \langle E_{D^2}(N) \rangle$.

In any case, $\sqrt{Dm} \oplus \sqrt{Dn} \subseteq \langle E_{D^2}(N) \rangle$.

If $(0, n^{k'}) \in N$, then the proof can be carried out in a similar way. □

THEOREM 2.3. *Let D be a valuation domain with unique maximal ideal $\mathcal{M}$. Then D^2 s.t.r.f.*

Proof. Let N be a nonzero submodule of D^2 where N is generated by $S = \{(a_i, b_i)_{i \in I}\}$. Consider the canonical projections $\pi_\lambda : D^2 \to D$ given by $\pi_\lambda(d_1, d_2) = d_\lambda$ where $d_1, d_2 \in D$, $\lambda = 1, 2$. We have $\pi_1(N) = \langle \{a_i\}_{i \in I} \rangle$ and $\pi_2(N) = \langle \{b_i\}_{i \in I} \rangle$.

Case 1. If $N = \pi_1(N) \oplus \pi_2(N)$, then N satisfies the radical formula in D^2 by Lemma 2.1.

Case 2. If $\pi_1(N) + \pi_2(N)$ is a finitely generated ideal of D, then $\pi_1(N) + \pi_2(N) = \sum_{finite} Da_i + \sum_{finite} Db_i$. Since D is a valuation ring and the ideals of D are totally ordered, we may assume $\pi_1(N) + \pi_2(N) = Db_k$ for some $k \in I$. Then $Da_k \subseteq Db_k$, hence $a_k/b_k \in D$. Note that $\{(a_k/b_k, 1), (1, 0)\}$ forms a basis for D^2.

Define

$$\begin{aligned} \phi : D \oplus D &\to D \oplus D \\ (a_k/b_k, 1) &\to (0, 1) \\ (1, 0) &\to (1, 0). \end{aligned}$$

Clearly ϕ is an isomorphism and $\phi(N) = B \oplus Db_k$ where B is an ideal of D and $B \cong N \cap (D \oplus 0)$. By case 1, $\phi(N)$ s.t.r.f. in D^2. Then by [**5**, Theorem 1.5], N s.t.r.f. in D^2.

Case 3. Suppose $\pi_1(N) + \pi_2(N)$ is not a finitely generated ideal, but $\pi_1(N)$ or $\pi_2(N)$ is finitely generated. We may assume that $\pi_1(N)$ is finitely generated. Clearly $\pi_1(N)$ is nonzero. (If it is zero then N satisfies Case 1 and result is clear.) Then $\pi_1(N) = Da_t$ for some $t \in I$. Since $\pi_1(N) + \pi_2(N)$ is not finitely generated there are infinitely many principal ideals generated by elements of $\pi_2(N)$ containing Da_t. Then there exists an element $(a_s, b_s) \in S$ such that $Da_1 + Db_1 \subsetneqq Db_s, s \in I$. Hence $Da_s b_1 \subseteq Da_1 b_1 \subsetneqq Da_1 b_s$, that is $a_s b_1/a_1 b_s \in \mathcal{M}$ and $1 - a_s b_t/a_t b_s$ is a unit in D. Then

$$b_s(a_t, b_t) - b_t(a_s, b_s) = (1 - a_s b_t/a_t b_s)(a_t b_s, 0) \in N, \text{ so } (a_t b_s, 0) \in N.$$

Note that $a_t^2(1, 0) \in D(a_t b_s, 0) \subseteq N$, and hence $(a_i^2, 0) \in N$ for any $i \in I$. By Lemma 2.2, $\sqrt{Da_i} \oplus \sqrt{Db_i} \subseteq \langle E_{D^2}(N)\rangle$ for all $i \in I$.

Hence,

$$rad(N) \subseteq \sqrt{\pi_1(N)} \oplus \sqrt{\pi_2(N)} = \bigcup_{i\in I} \sqrt{Da_i} \oplus \bigcup_{i\in I} \sqrt{Db_i} \subseteq \langle E_{D^2}(N)\rangle.$$

Case 4. Let N be a submodule of $D \oplus D$ where $\pi_1(N)$ and $\pi_2(N)$ are not finitely generated. Recall that $S = \{(a_i, b_i)_{i\in I}\}$ is the set of generators of N. We order the index set I as follows: $i \leqslant j$ if and only if $Da_i \subseteq Da_j$. Define $\mathcal{P}_i = \sqrt{Da_i}$ and $\mathcal{Q}_i = \sqrt{Db_i}$ then $\sqrt{\pi_1(N)} = \bigcup \mathcal{P}_i$ and $\sqrt{\pi_2(N)} = \bigcup \mathcal{Q}_i$.

Subcase 1. For any $i \in I$, if one of the following is satisfied, then $\mathcal{P}_i \oplus \mathcal{Q}_i \subseteq \langle E_{D^2}(N)\rangle$ and N s.t.r.f. in D^2.

(a) There exists $j > i$ such that $Db_j \subsetneqq Db_i$.
(b) There exists $k < i$ such that $Db_i \subsetneqq Db_k$.
(c) There exists $j_0 > i$ such that $Da_i b_{j_0} \neq Da_{j_0} b_i$ while $Db_i \subseteq Db_j$ for all $j > i$,
(d) There exists $j_1 > i$ such that $Da_i b_{j_1} \subseteq D(u_{j_1} - 1)$ while for all indices $j > i$, $Db_i \subseteq Db_j$, and $a_i b_j = u_j a_j b_i$ for some unit u_j.

Proof of Subcase 1. It is enough to prove for any i, $\mathcal{P}_i \oplus \mathcal{Q}_i \subseteq \langle E_{D^2}(N)\rangle$. Assume both a_i and b_i are nonzero, otherwise the result is clear.

Let condition (a) *be satisfied.* Then by assumption we have $(a_j, b_j) \in S$ such that

$$Da_i b_j \subsetneqq Da_i b_i \subseteq Da_j b_i \text{ and } a_i b_j/a_j b_i \in \mathcal{M}.$$

- if $Da_j \subseteq Db_i$, then $b_j(a_i, b_i) - b_i(a_j, b_j) = a_j b_i(a_i b_j/a_j b_i - 1)(1, 0) \in N$. Since $a_i b_j/a_j b_i - 1$ is a unit, $(a_j b_i, 0) \in N$ and so $(a_i^2, 0) \in Da_j b_i(1, 0) \subseteq N$. Hence by Lemma 2.2, $\mathcal{P}_i \oplus \mathcal{Q}_i \subseteq \langle E_{D^2}(N)\rangle$.
- If $Db_i \subseteq Da_j$, then $a_j(a_i, b_i) - a_i(a_j, b_j) = a_j b_i(1 - a_i b_j/a_j b_i)(0, 1) \in N$ and $1 - a_i b_j/a_j b_i$ is a unit implies $a_j b_i(0, 1) \in N$, that is $(0, b_i^2) \in Da_j b_i(0, 1) \subseteq N$ and $\mathcal{Q}_i(0, 1) \subseteq \langle E_{D^2}(N)\rangle$. By Lemma 2.2, $\mathcal{P}_j \oplus \mathcal{Q}_j \subseteq \langle E_{D^2}(N)\rangle$. Hence,

$$\mathcal{P}_i \oplus \mathcal{Q}_i \subseteq \mathcal{P}_j \oplus \mathcal{Q}_j \subseteq \langle E_{D^2}(N)\rangle.$$

Let condition (b) *be satisfied.* That is there is a $k < i$ such that $Db_k \supsetneqq Db_i$. Assume $a_k \neq 0$ (otherwise result is clear), then

$$Da_k b_i \subsetneqq Da_k b_k \subseteq Da_i b_k \text{ and } a_k b_i/a_i b_k \in \mathcal{M}.$$

- if $Da_i \subseteq Db_k$, then $b_k(a_i, b_i) - b_i(a_k, b_k) = a_i b_k(1 - a_k b_i/a_i b_k)(1, 0) \in N$. Since $1 - a_k b_i/a_i b_k$ is a unit, $(a_i b_k, 0) \in N$. Then $(a_i^2, 0) \in Da_i b_k(1, 0) \subseteq N$ and $\mathcal{P}_i(1, 0) \in \langle E_{D^2}(N)\rangle$. By Lemma 2.2, $\mathcal{P}_k \oplus \mathcal{Q}_k \subseteq \langle E_{D^2}(N)\rangle$ and hence $\mathcal{P}_i \oplus \mathcal{Q}_i \subseteq \mathcal{P}_i \oplus \mathcal{Q}_k \subseteq \langle E_{D^2}(N)\rangle$.
- If $Db_k \subseteq Da_i$, then $a_k(a_i, b_i) - a_i(a_k, b_k) = a_i b_k(a_k b_i/a_i b_k - 1)(0, 1) \in N$ and $1 - a_k b_i/a_i b_k$ is a unit implies $a_i b_k(0, 1) \in N$ and $(0, b_i^2) \in Da_i b_k(0, 1) \subseteq N$. Hence we have $\mathcal{P}_i \oplus \mathcal{Q}_i \subseteq \langle E_{D^2}(N)\rangle$ by Lemma 2.2.

Let (c) *be satisfied.* That is there exists $j_0 \in I$ such that $j_0 > i$ and $Da_ib_{j_0} \neq Da_{j_0}b_i$. Then one of these ideals of D is contained in the other. Let $Da_ib_{j_0} \subsetneq Da_{j_0}b_i$ that is $a_ib_{j_0}/a_{j_0}b_i \in \mathcal{M}$ and $(a_{j_0}b_i, 0) \in N$; again we have two cases, such that $Da_{j_0} \subseteq Db_i$ or $Db_i \subseteq Da_{j_0}$. $Da_{j_0} \subseteq Db_i$ implies $(a_i^2, 0) \in Da_{j_0}b_i(1,0) \subseteq N$ and $Db_i \subseteq Da_{j_0}$ implies $(0, b_i^2) \in N$. Using Lemma 2.2, we have $\mathcal{P}_i \oplus \mathcal{Q}_i \in \langle E_{D^2}(N)\rangle$.

Let (d) *be satisfied.* That is, $a_i/a_j = u_j(b_i/b_j)$ for some unit u_j for all $j > i$. By the above argument, $a_ib_j(1-u_j)(1,0)$ and $a_ib_j(1-u_j)(0,1) \in N$ for all $j > i$. By assumption, there is $j_1 \in I$ with $j_1 > i$ such that $Da_ib_{j_1} \subseteq D(1-u_{j_1})$ for some $j_1 > i$. Then $(a_i^4, 0)$ or $(0, b_i^4) \in N$. By Lemma 2.2, we have $\mathcal{P}_i \oplus \mathcal{Q}_i \subseteq \langle E_{D^2}(N)\rangle$ Since this equality holds for all $i \in I$, we have

$$rad(N) = \sqrt{\pi_1(N)} \oplus \sqrt{\pi_2(N)} = \bigcup \mathcal{P}_i \oplus \bigcup \mathcal{Q}_i \subseteq \langle E_{D^2}(N)\rangle.$$

Subcase 2. Suppose the possibilities in subcase 1 do not occur for some $i \in I$, then there exists $j > i$ such that $Db_i \subsetneq Db_j$, $a_ib_j = u_ja_jb_i$ for some unit u_j and $D(1-u_j) \subseteq Da_ib_j$. In this case N s.t.r.f. in D^2.

Proof of Subcase 2. Similar to the proof of subcase 1, we have $a_ib_j(1-u_j)(1,0)$ and $a_ib_j(1-u_j)(0,1) \in N$ that is $((1-u_j)^2, 0) \in N$ for all $j > i$. Now we may assume that $Da_i \subseteq Db_i$ then define

$$\begin{array}{rcl} \phi : D \oplus D & \to & D \oplus D \\ (\frac{a_i}{b_i}, 1) & \to & (0,1) \\ (1,0) & \to & (1,0) \end{array}$$

Consider $N = N_1 + N_2$ where $N_1 = \langle\{(a_k, b_k)\}_{k \leqslant i}\rangle$ and $N_2 = \langle\{(a_k, b_k)\}_{k > i}\rangle$. If $k \leqslant i$, then $\phi(a_k, b_k) = \phi(a_k - (b_k/b_i)a_i, 0) + \phi((b_k/b_i)a_i, b_k) = (a_k - (b_k/b_i)a_i, 0) + b_k(0,1)$, and $\phi(a_i, b_i) = b_i\phi(a_i/b_i, 1) = b_i(0,1) = (0, b_i) \in \phi(N_1)$ that implies $0 \oplus Db_i \subseteq \phi(N_1)$ and $(0, b_k) = (b_k/b_i)(0, b_i) \in \phi(N_1)$ where $k \leqslant i$ and $(a_k - (b_k/b_i)a_i, 0) = \phi(a_k, b_k) - b_k(0,1) \in \phi(N_1)$. Thus

$$\phi(N_1) = B \oplus Db_i$$

where B is an ideal of D such that $B \cong (D \oplus 0) \cap \langle (a_s, b_s)_{s \leqslant i}\rangle$ by case 2. If $k > i$, then $\phi(a_k, b_k) = \phi(a_k - u_ka_k, 0) + \phi(u_ka_k, b_k) = (a_k - u_ka_k, 0) + b_k\phi(a_i/b_i, 1) = (a_k - u_ka_k, b_k)$. Hence

$$\phi(N_2) = \langle (a_k(1-u_k), b_k)\}_{i<k}\rangle.$$

So $\phi(N) = (B \oplus Db_i) + \langle (a_k(1-u_k), b_k)_{k>i}\rangle$. Since $((1-u_j)^2, 0) \in N$ we have $(a_j(1-u_j)^2, 0) = \phi(a_j(1-u_j)^2, 0) \in \phi(N)$. By Lemma 2.2,

$$\sqrt{Da_j(1-u_j)} \oplus \sqrt{Db_j} \subseteq \langle E_{D^2}(\phi(N))\rangle,$$

for all $j > i$. Combining this result by Lemma 2.1, we have

$$rad(\phi(N)) = \sqrt{\pi_1(\phi(N))} \oplus \sqrt{\pi_2(\phi(N))} \subseteq \langle E_{D^2}(\phi(N))\rangle.$$

Thus $\phi(N)$ s.t.r.f. in D^2 and by [**5**, Theorem 1.5], N s.t.r.f. in D^2. □

THEOREM 2.4. *Let R be a Prüfer domain, then the R-module R^2 satisfies the radical formula.*

Proof. For any maximal ideal $\mathcal{M}$ of R, $R_{\mathcal{M}}$ is a valuation ring. Then by Theorem 2.3, the $R_{\mathcal{M}}$-module $R_{\mathcal{M}} \oplus R_{\mathcal{M}}$ satisfies the radical formula. By Proposition 1.1, R^2 satisfies the radical formula. □

REFERENCES

1. R. Gilmer, *Multiplicative Ideal Theory*, Queen's Papers Pure Appl. Math. **90** (Kingston, Ontario, 1992).

2. J. Jenkins and P. F. Smith, On the prime radical of a module over a commutative ring, *Comm. Algebra* **20** (1992), 3593–3602.

3. Ka Hin Leung and S. H. Man, On commutative Noetherian rings which satisfy the radical formula, *Glasgow Math. J.* **39** (1997), 285–293.

4. S. H. Man, One dimensional domains which satisfy the radical formula are Dedekind domains, *Arch. Math. (Basel)* **66** (1996), 276–279.

5. R. L. McCasland and M. E. Moore, On radicals of submodules, *Comm. Algebra* **19**(5) (1991), 1327–1341.

6. H. Sharif, Y. Sharifi and S. Namazi, Rings satisfying the radical formula, *Acta Math. Hungar.* **71** (1996), 103–108.