Second Order Diffraction of Water Waves by a Bottom Mounted Vertical Circular Cylinder and Some Related Numerical Problems

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A Hankel transformation is used to obtain the second order diffraction solution of vertical cylinder of circular cross section. The improper integral over the free surface is tackled carefully. The singularity at the free surface is overcome effectively using a third order nonlinear transformation. Numerical results for free surface elevations compare well with the published data.
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1 Introduction

Many offshore structures are supported by vertical columns of a circular cross section, which are subjected to wave diffraction. Even with the simplifying assumptions of ideal flow, the interaction of water waves with floating bodies is a difficult nonlinear problem. In this paper, the solution to the second order diffraction problem is resolved using the Hankel transformation, which is suitable for the hydrodynamic interaction problem, and some numerical problems in the evaluation of the potential is discussed.

An expression for the second order diffraction force was, first, derived by Lighthill [1]. He made use of an assisting potential. Molin developed the analysis into finite water depth [2]. Molin's solution has been extended by Eatock Taylor and Hung by developing a method for the evaluation of the free surface integral based on leading asymptotics [3]. Direct solution of the second order diffraction was considered by Kim and Yue [4]. They obtained a Green's theorem integral equation for the second order diffraction potential involving the wave source Green's function. A study similar to the Kim and Yue study was carried out by Chau and Eatock Taylor [5]. Instead of using an integral equation, a direct solution of the boundary value problem was provided by Huang and Eatock Taylor [6]. Numerical solutions of the nonlinear diffraction problem were also studied by several researchers such as Qiu et al. [7]. In this study, a second order diffraction problem of a complete cylinder was resolved using the Hankel transform.

2 Formulation of the Problem

The second order diffraction of a plane monochromatic incident wave of frequency ω, wave number k, determined by the dispersion relation and amplitude A by a fixed circular cylinder of radius a is considered in water of uniform depth H. A cylindrical coordinate system (r,θ,z) is used with (r,θ) on the quiscent free surface and z pointing upward. Assuming irrotational flow and using Stokes perturbation procedure, the velocity potential $\Phi(r,\theta,z,t)$ can be expressed as follows:

$$\Phi(r,\theta,z,t) = \mathrm{Re}\{\phi^{(1)}(r,\theta,z)e^{-i\omega t} + \phi^{(2)}(r,\theta,z)e^{-2i\omega t}\} \quad (1)$$

Second order potential is divided into incident $\phi_i^{(2)}$, "free wave" $\phi_{d1}^{(2)}$ and "locked wave" $\phi_{d2}^{(2)}$ components. The boundary value problem governing the second order potential consists of Laplace's equation in the fluid domain, homogeneous Neumann type conditions at the cylinder surface and the sea bottom, free surface condition, $\partial\phi^{(2)}/\partial z - 4\upsilon\phi = q(r,\theta)$, and a suitable radiation condition. Here, $\upsilon = \omega^2/g$ and g is the gravitational acceleration. q is considered as an effective pressure distribution on the free surface and is a function of first order incident and diffraction potentials. Second order potential consists of incident, homogeneous "free wave" and particular "locked wave" components. Incident wave component $\phi_i^{(2)}$ should satisfy the free surface condition with q replaced by q_I which is a function of first order incident potential alone, together with the Laplace's equation and the sea bottom condition. Free wave component $\phi_{d1}^{(2)}$, apart from Laplace's and sea bottom conditions, is chosen to satisfy the homogeneous form of the free surface condition and also the body condition which is written as follows:

$$\frac{\partial\phi_{d1}^{(2)}}{\partial r} = -\frac{\partial\phi_i^{(2)}}{\partial r} \quad (2)$$

It is obvious that the locked wave component $\phi_{d2}^{(2)}$ should satisfy Laplace's equation in the fluid domain, homogeneous Neumann type conditions at the cylinder surface and the sea bottom, and the free surface condition $\partial\phi^{(2)}/\partial z - 4\upsilon\phi = q(r,\theta) - q_I(r,\theta)$. Second order incident potential and the free wave components can be obtained easily and will not be given here.

3 "Locked Wave" Component

First the velocity potential due to a concentrated pressure will be obtained and then the integration of this potential over the free surface will give the desired velocity potential which satisfies the nonhomogeneous free surface condition and the homogeneous body condition. In order to determine the velocity potential due to concentrated pressure, the problem will be divided into two parts: (1) Water waves resulting from the pressure concentrated at the origin of the coordinate system in the absence of the cylinder, which satisfies

$$\frac{\partial\tilde{\varphi}}{\partial z} - 4\upsilon\tilde{\varphi} = \delta(x)\delta(y) \quad (z=0) \quad (3)$$

where $\delta(x)$ is the Dirac delta function. (2) Water waves generated by the pressure concentrated at (x_0,y_0) with the cylinder placed at the origin, should satisfy the following boundary conditions:

$$\frac{\partial\varphi}{\partial r} = -\frac{\partial\tilde{\varphi}}{\partial r} \quad (r=a,\ -H<z<0) \quad (4a)$$

$$\frac{\partial\varphi}{\partial z} - 4\upsilon\varphi = 0 \quad (r>0,\ z=0) \quad (4b)$$

where $x_0 = r_0\cos\theta_0$ and $y_0 = r_0\sin\theta_0$. Both velocity potentials must be harmonic and should satisfy the sea bottom condition and a radiation condition. Using the Hankel transform, the first part of the particular solution represented by the boundary value problem (3) is given by the following formula [8]:

$$\tilde{\varphi}(r,z) = \frac{1}{2}iC_0H_0(k_0r)\cosh[k_0(z+H)] + \frac{1}{\pi}\sum_{j=1}^{\infty} C_jK_0(k_jr)\cos[k_j(z+H)] \quad (5)$$

where

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$$C_0 = \frac{\upsilon^2 - k_0^2}{H(\upsilon^2 - k_0^2) - \upsilon}\cosh(k_0 H) \quad C_j = \frac{\upsilon^2 + k_j^2}{H(\upsilon^2 + k_j^2) - \upsilon}\cos(k_j H) \tag{6}$$

The second part of the particular solution represented by Eq. (4) is much easier to derive since the free surface condition is homogeneous. The locked wave component due to the concentrated pressure is the combination of the two components and after some algebraic manipulation it is obtained as follows:

$$\phi(r,\theta,z) = \frac{1}{\pi}\sum_{n=0}^{\infty} \epsilon_n \cos[n(\theta - \theta_0)]\sum_{j=0}^{\infty} C_j G_{jn}(r,r_0)\tilde{Z}_j(z) \tag{7}$$

where $\tilde{Z}_0(z)=\cosh[k_0(z+H)]$ and $\tilde{Z}_j(z)=\cos[k_j(z+H)]$, $j \geq 1$. The function $G_{jn}(r,r_0)$ is defined as follows:

$$\frac{1}{2}i\pi H_n(k_0 r_0)\left[J_n(k_0 r) - \frac{J_n'(k_0 a)}{H_n'(k_0 a)}H_n(k_0 r)\right] = G_{0n}(r,r_0)$$

$$K_n(k_j r_0)\left[I_n(k_j r) - \frac{I_n'(k_j a)}{K_n'(k_j a)}K_n(k_j r)\right] = G_{jn}(r,r_0) \tag{8}$$

Equation (7) gives the velocity potential due to a concentrated pressure at (r_0, θ_0) with a cylinder placed at the origin. If the pressure is distributed

$$\frac{\partial \phi}{\partial z} - 4\upsilon\phi = q(r,\theta) \quad (z=0,\ r=a) \tag{9}$$

then the solution in this case is obtained by integrating the velocity potential in Eq. (7) over the entire free surface (θ_0 is taken zero for simplicity only)

$$\phi_{d2}^{(2)}(r,\theta,z) = 2\sum_{n=0}^{\infty} \epsilon_n \cos(n\theta)\left[\sum_{j=0}^{\infty} C_j \tilde{Z}_j(z)\int_a^{\infty} r_0 q_n(r_0) G_{jn}(r,r_0)dr_0\right] \tag{10}$$

where

$$q(r,\theta) = \sum_{m=0}^{\infty} \epsilon_m q_m(r)\cos(m\theta) \tag{11}$$

This solution is valid when $a < r < r_0$. But whenever $r > r_0$, we just need to change everywhere r for r_0 and vice versa. It can be seen immediately that Eq. (10) is the same as the one given by Huang and Eatock Taylor [6].

4 Numerical Results and Discussion

There are two major numerical problems in evaluating the free surface elevation: First, evaluation of the improper integral, second, singularity at the free surface. To overcome these problems, the infinite integral is divided into two parts, near and far field integrals

$$\int_a^{\infty} r_0 q_n(r_0) G_{jn}(r,r_0)dr_0 = \int_a^{r_s} r_0 q_n(r_0) G_{jn}(r,r_0)dr_0 + \int_{r_s}^{\infty} r_0 q_n(r_0) G_{jn}(r,r_0)dr_0 \tag{12}$$

When $j=0$, the first integral on the right-hand side is calculated using Gaussian ten-point integration method with a high degree of accuracy. However, asymptotic expansions of Hankel functions are used to evaluate the infinite integral for $j=0$. Whenever $j > 0$, near field integral (first one on the right-hand side) becomes singular.

4.1 Singularity at Free Surface. We consider the near field integral for $j > 0$. It has been shown by Fenton that the near field integral becomes singular as $j \to \infty$ [9]. To overcome this difficulty, we first subtract out the singularity and then integrate the singularity separately

$$I_s = \sum_{j=1}^{\infty} 2C_j\tilde{Z}_j(0)\int_a^{r_s} r_0 q_n(r_0) G_{jn}(r,r_0)dr_0$$
$$= \sum_{j=1}^{\infty}\int_a^{r_s}\left(2C_j\tilde{Z}_j(0) r_0 q_n(r_0) G_{jn}(r,r_0) - \frac{2}{\pi\sqrt{r_0 a}}\frac{e^{-j\pi/H(r_0-a)}}{j}\right)dr_0$$
$$- \int_a^{r_s}\frac{2}{\pi\sqrt{r_0 a}}\ln[1 - e^{-\pi/H(r_0-a)}]dr_0 \tag{13}$$

A third order nonlinear transformation is used to integrate the singular integral (second one on the right-hand side of Eq. (13)) [10]

$$I_s = -\frac{r_s - a}{\pi\sqrt{a}}\int_{-1}^{1}\frac{3(\gamma+1)^2}{4}\sqrt{\frac{r_s - a}{2}\frac{(\gamma+1)^3 - 4}{4} + \frac{r_s + a}{2}}$$
$$\times q_n\left(\frac{r_s - a}{2}\frac{(\gamma+1)^3 - 4}{4} + \frac{r_s + a}{2}\right)\ln\left[\frac{\pi}{H}\frac{r_s - a}{2}\frac{(\gamma+1)^3}{4}\right]d\gamma \tag{14}$$

The first rational expression in the integrand is the Jacobian and it is clear that it smoothes out the singularity at −1. Two successive transformations are used to obtain Eq. (14). First transformation changes the limits of the integration to −1 and 1

$$r_0 = \frac{r_s - a}{2}\zeta + \frac{r_s + a}{2}$$

and the second transformation removes the singularity at −1

$$\zeta = \frac{(\gamma+1)^3}{4} - 1 \tag{15}$$

Second order transformation can also be used to evaluate, Eq. (13) but with third order transformation convergence is faster and the expression (15) is simpler than the corresponding second order transformation expression, $\zeta = 1/2\gamma^2 + \gamma - 1/2$.

4.2 Comparison of Numerical Results. Numerical results of free surface elevations are compared with those of Kim and Yue [4]. In Table 1, $\eta_D^{(2)}$ is the free surface elevation due to the second order "locked wave" and "free wave" components

Table 1 Second order surface elevation angular modal amplitudes normalized by A^2/a on the vertical cylinder ($a/h=1$) for various υa values

υa	1.2		2.0		2.8	
n	$\eta_D^{(2)}$	$\eta_D^{(2)*}$	$\eta_D^{(2)}$	$\eta_D^{(2)*}$	$\eta_D^{(2)}$	$\eta_D^{(2)*}$
0	0.3242	0.3254	1.1600	1.1515	0.4404	0.4352
1	0.8133	0.8165	1.1375	1.1319	2.3280	2.3037
2	0.7719	0.7721	1.1666	1.1516	1.9287	1.9093
3	0.3300	0.3247	0.9640	0.9561	1.6163	1.6078
4	1.4545	1.4473	0.2415	0.2431	1.2758	1.2661
5	1.4183	1.4129	0.8139	0.8040	0.6720	0.6665
6	0.5586	0.5567	1.2414	1.2304	0.3541	0.3482
7	0.1152	0.1147	1.0073	0.9994	1.0061	1.0015
8	0.0194	0.0193	0.5904	0.5862	1.0968	1.0833
9	0.0029	0.0029	0.2228	0.2214	0.7860	0.7767
10	0.0004	0.0004	0.0490	0.0487	0.4555	0.4500
11	0.0001	0.0001	0.0086	0.0085	0.2263	0.2233
12	0.6×10^{-5}	0.6×10^{-5}	0.0014	0.0014	0.0862	0.0851
13	0.6×10^{-6}	0.6×10^{-6}	0.0002	0.0002	0.0208	0.0209
14	0.5×10^{-7}	0.7×10^{-7}	0.0000	0.0000	0.0036	0.0039

Table 2 Effect of evanescent modes on second order surface elevation for νa=2.8

	N_e=5	10	15	25	50	100	150
n=0	0.41393	0.43141	0.43388	0.43486	0.43516	0.43520	0.43521
1	2.21975	2.28895	2.29874	2.30260	2.30375	2.30391	2.30393
2	1.83821	1.89678	1.90507	1.90834	1.90932	1.90947	1.90949
3	1.54996	1.59775	1.60440	1.60699	1.60775	1.60787	1.60788
4	1.22139	1.25847	1.26355	1.26550	1.26607	1.26615	1.26615
5	0.64182	0.66246	0.66516	0.66618	0.66647	0.66652	0.66652
6	0.35313	0.34898	0.34844	0.34824	0.34818	0.34817	0.34817
7	0.98772	0.99883	1.00059	1.00131	1.00152	1.00156	1.00156
8	1.06668	1.08015	1.08221	1.08303	1.08327	1.08330	1.08331
9	0.76636	0.77476	0.77603	0.77654	0.77669	0.77671	0.77671
10	0.44567	0.44923	0.44977	0.44999	0.45005	0.45006	0.45006
11	0.22218	0.22309	0.22324	0.22329	0.22331	0.22331	0.22331
12	0.08520	0.08516	0.08515	0.08515	0.08515	0.08515	0.08515
13	0.02102	0.02090	0.02088	0.02088	0.02087	0.02087	0.02087
14	0.00397	0.00392	0.00391	0.00391	0.00391	0.00391	0.00391
CPU	7.72	10.8	13.80	20.20	38.26	67.82	100
Error	4.9	0.87	0.31	0.08	0.01	0.0	0.0

$$\eta_D^{(2)} = \frac{2i\omega}{g}(\phi_{d1}^{(2)} + \phi_{d2}^{(2)})$$

Superscript "*" in Table 1 denotes the values obtained by the present method.

Agreement with the results of Kim and Yue [4] is good. Present calculations are carried out for 50 evanescent modes and the near field integration is taken from a to r_s=15a. This optimum value of r_s for the present geometry is determined by performing many calculations; r_s should be large enough so that in the far field integral we do not have to consider the evanescent modes, at the same time r_s should be small enough so that the near field integral can be calculated to a high degree of accuracy with reasonable CPU time spent. To achieve the desired accuracy, near field integral is carried out in 25 equal intervals. Infinite summations are truncated after 15 terms which give quite a good convergence. Also $q_s(r)$ is truncated at 14 terms.

Next, convergence tests for the evanescent modes and for different r_s values are presented. First, free surface calculations are repeated for different number of evanescent modes (Table 2). Before the last line of Table 2, relative CPU times are given for the different number of evanescent modes N_e considered in the calculations. Maximum relative errors are given in the last line of the table. By choosing N_e=50, relative error is only 1/10,000 and the CPU time spent is one third of that of the case with N_e=150. In Table 3, the value of r_s is varied to investigate the effects of near and far field integrals on the result. In Table 3, relative CPU times are shown in the CPU row and the number of subintervals N_p used in the near field integral is given in the fourth row of the table. As the upper limit of the near field integral increases, the value of N_p increases accordingly to achieve a desired accuracy of 1/10,000. Also the number of terms used to calculate the asymptotic values of the Hankel functions N_s decreases as r_s increases. That means as r_s increases we need fewer terms of the asymptotic expansions of the Hankel functions in the far field integral but we need more subintervals for the near field integral. It seems that we optimize the CPU time by selecting r_s as 15a, in that case the desired accuracy is satisfied and also the relative CPU time is kept at two thirds of the maximum CPU used in the calculations.

Table 3 Effect of r_s on second order surface elevation for νa=2.8

(r_s/a)	2	5	10	15	20	25	30	50
n=0	0.5859	0.4353	0.4352	0.4352	0.4352	0.4352	0.4352	0.4352
n=1	2.4383	2.3041	2.3040	2.3039	2.3040	2.3040	2.3040	2.3040
CPU	32.8	51.7	65.8	66.9	74.9	90.1	100.0	100.0
N_p	15	20	25	30	40	50	60	60
N_s	30	30	30	30	30	30	15	15

5 Concluding Remarks

A second order diffraction problem of a vertical cylinder of circular cross sections is treated. The locked wave component of the second order potential is divided into two parts: potential due to concentrated source at the free surface with no cylinder present and potential with a cylinder placed at the free surface with homogeneous boundary condition. This form of locked wave potential will be advantageous when hydrodynamic interaction among multiple cylinders is considered. Apart from this, with present formulation, singularity at the free surface is tackled effectively with a third order nonlinear transformation, which saves some CPU time compared with the calculations which use a second order transformation. Numerical results for the second order free surface elevation compare well with those of Kim and Yue (Table 1). The effect of angular and evanescent modes is investigated and it was found that 14 angular and 50 evanescent modes are sufficient to reach an accuracy of 1/10,000 (Tables 1 and 2). In Table 3 r_s, the cut-off point in the infinite integral, is varied and it was observed that when r_s=15a CPU time is optimized.

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