Path Generation Synthesis of Planar Double-Slider Linkages via the Elliptic Coupler Curve

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Abstract. The path generation synthesis of a double-slider linkage is performed for a given elliptic curve. It is shown that there are ∞^2 linkages that can trace the same ellipse. The formulation for obtaining all such linkages is presented. The formulation sheds light for the design of the planar slider-crank and four-bar linkages from the given algebraic form of the coupler path curve as well.

Key words: Linkage design, double-slider linkage, path generation, Cardan motion.

1 Introduction

Path generation synthesis problem is one of the main problems of kinematic synthesis of linkages. The synthesis can be performed via graphical or algebraic methods. Although the enhanced computer aided design programs can be fast and efficient means for path generation synthesis, algebraic formulation allows detailed parametric design with several solutions.

The algebraic design of planar linkages have been mainly performed using the dyadic approach for decades [1]. In the dyadic approach, the linkage is divided into couples of links, called dyads and the dyad dimensions are determined using the position and loop closure equations for a given number of positions of the point to be traced on the desired path. In exact synthesis the coupler point of the linkage passes through the selected points for design, which are called precision points. On the other hand, in approximate synthesis, the coupler point does not exactly pass through the points selected for design, which are called design points [2]. While exact synthesis is studied abundantly [1], approximate synthesis methods are less known. For either exact or approximate synthesis, the main approach is to write the position and loop closure equations and then solve these equations to determine the link length dimensions.

An alternative approach makes use of the characteristics of a coupler curve for path generation synthesis. Once the algebraic characteristics of the coupler curve of a linkage is known, one may seek a good fit of such a curve to the desired path. Such a study was first published by Bleschmidt and Uicker in 1986 [3], in which the trinodal tricircular sextic coupler curve of a planar four bar linkage is worked out. The sextic of the four-bar linkages is a special type of sixth order polynomial in x and y coordinates of the coupler point and it has 15 independent coefficients. These 15 coefficients are comprised of 9 independent linkage dimensions, hence an overdetermined system of 15 equations is obtained. In [3] the authors select very specific points on the coupler curve, such as the double points and singular foci, in order to determine the 15 coefficients of the sextic. They propose using an iterative method to solve for the linkage dimensions. In 1993 Ananthasuresh and Kota [4] further worked on [3] and developed an alternative method, which is based on fitting the sextic on 15 points and see if the 9 linkage dimensions can be determined. Obviously such a method is heavily based on trial and error. Recently Bai [5, 6] worked on the dependency among the coefficients for both four-bar and slider-crank linkages in order to obtain a determined system of equations. In case of a slider-crank linkage, the coupler curve is a special quartic with 13 coefficients depending on 8 independent linkage dimensions. Once a proper set of coefficients are given, Bai's formulation allows the designer to determine the corresponding linkage dimensions via numerical solution of a nonlinear set of equations, which results in the cognate solutions. However, this methodology still does not answer the question of how to select the design points for the path generation problem.

The path generation synthesis via the coupler curve equation problem consists of two steps: 1) Fitting the coupler curve with specific algebraic properties on a given set of points on the desired path and 2) Determining the linkage dimensions out of the coefficients of the curve. The curve fitting part is an issue because of the dependent coefficients of the curves of the slider-crank and four-bar linkages.

While the planar slider-crank and four-bar linkages have fourth and sixth order coupler curves, respectively, coupler point of a double-slider linkage traces an ellipse - a second order curve. So, the double-slider linkage is also called an elliptic trammel and it has been used to draw exact ellipses since the times of Archimedes.

Fitting an ellipse to a cluster of points is well studied in the literature (See for ex. [7]). In this study we investigate the determination of the double-slider linkage dimensions for a given ellipse equation. This is also a first step for further investigation of the nature of coupler curves of slider-crank and four-bar linkages.

2 The Double-Slider Linkage

A double-slider linkage is depicted in Fig. 1. One can locate the coupler point P for given X_0, Y_0, α, β, p, q and h. The coupler triangle side lengths a, b and p + q can be determined from p, q and h.
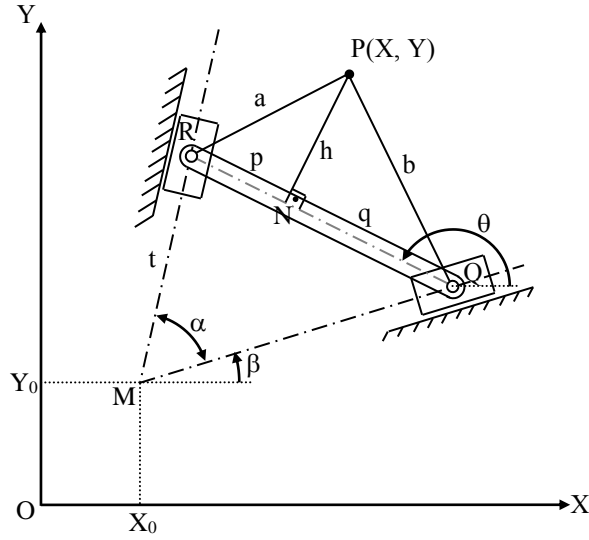
Fig. 1 A double-slider linkage

The coordinates of P are found as follows:

$$X + iY = \overrightarrow{OP} = \overrightarrow{OM} + \overrightarrow{MR} + \overrightarrow{RP} = X_0 + iY_0 + te^{i(\alpha+\beta)} - (p + ih)e^{i\theta}$$
$$\Rightarrow X = X_0 + tc(\alpha+\beta) - pc\theta + hs\theta \; ; \; Y = Y_0 + ts(\alpha+\beta) - ps\theta - hc\theta \tag{1}$$

where c and s are short for cos and sin, respectively. The variables t = |MR| and θ depend on each other due to loop closure. Using sine theorem in triangle MQR

$$\frac{p+q}{s\alpha} = \frac{t}{s(\theta-\beta)} \Rightarrow t = (p+q)\frac{s(\theta-\beta)}{s\alpha} \tag{2}$$

Substituting Eq. (2) in Eq. (1) and eliminating θ from X and Y terms:

$$AX^2 + BXY + CY^2 + DX + EY + F = 0 \tag{3}$$

where

$$A = \left[(p+q)\frac{s(\alpha+\beta)s\beta}{s\alpha} + h\right]^2 + \left[(p+q)\frac{s(\alpha+\beta)c\beta}{s\alpha} - p\right]^2$$

$$B = -2\left\{(p+q)^2\frac{c(\alpha+\beta)s(\alpha+\beta)}{s^2\alpha} + \frac{p+q}{s\alpha}\left[s(\alpha+2\beta)h - c(\alpha+2\beta)p\right]\right\}$$

$$C = \left[(p+q)\frac{c(\alpha+\beta)s\beta}{s\alpha} + p \right]^2 + \left[(p+q)\frac{c(\alpha+\beta)c\beta}{s\alpha} + h \right]^2$$

$$D = -2AX_0 - BY_0 \quad ; \quad E = -BX_0 - 2CY_0$$

$$F = AX_0{}^2 + BX_0 Y_0 + CY_0{}^2 - \left[p^2 + h^2 - \frac{(p+q)(ps\alpha - hc\alpha)}{s\alpha} \right]^2$$

Eq. (3) is a quadratic polynomial in X and Y, so it is a conic section. Checking the discriminant $\delta = B^2 - 4AC$, it can be seen that it is an ellipse equation:

$$\delta = B^2 - 4AC = -4\left(h^2 + \frac{c\alpha}{s\alpha}(p+q)h - pq \right)^2 < 0 \qquad (4)$$

Also we have a real ellipse if $C\Delta < 0$, where

$$\Delta = \begin{vmatrix} A & B/2 & D/2 \\ B/2 & C & E/2 \\ D/2 & E/2 & F \end{vmatrix} = \frac{BED - CD^2 - AE^2 - \delta F}{4} \qquad (5)$$

δ and Δ are invariants (under coordinate transformations) of the quadratic equation (3). For an ellipse $\delta < 0$ and if $\Delta = 0$, it is a point ellipse and if $C\Delta > 0$, it is an imaginary ellipse [8]. Eq. (3) is a homogeneous equation with 6 coefficients, so only 5 of the coefficients can be arbitrary. Considering that the coefficients depend on 7 linkage dimensions, two of the linkage dimensions can be selected arbitrarily. Therefore, infinitely many different double-slider linkages can draw the same ellipse, unlike the finitely many cognates of slider-crank or four-bar linkages drawing the same curve. Having noticed this, we can simplify the problem by assuming two of the linkage dimensions. Let $\alpha = \pi/2$ and $h = 0$. This gives us the elliptic trammel originally used by Archimedes. In this case, the dimensions $|RP| = a = p$ and $|QP| = b = q$ give the semi-major and semi-minor axes of the ellipse, respectively [9]. The coefficients simplify to

$$\begin{aligned} A &= a^2 s^2\beta + b^2 c^2\beta & D &= -2AX_0 - BY_0 \\ B &= (b^2 - a^2)s2\beta & E &= -BX_0 - 2CY_0 \\ C &= a^2 c^2\beta + b^2 s^2\beta & F &= AX_0{}^2 + BX_0 Y_0 + CY_0{}^2 - a^2 b^2 \end{aligned} \qquad (6)$$

In Section 3, we explain how to determine the linkage dimensions from Eq. (6) for given coefficients. Then in Section 4, we generalize the solution for $\alpha \neq \pi/2$ and $h \neq 0$.

3 Determining Linkage Dimensions from Coefficients

For a given general ellipse equation with coefficients A, B, C, D, E and F, we can determine the linkage dimensions X_0, Y_0, β, a and b from Eq. (6). However notice that the ellipse equation (3) is homogeneous, i.e. multiplying all coefficients with a constant does not alter the equation. We shall solve this problem as follows: The six coefficients given in Eq. (6) depend on five parameters, so there should be a relation among the six parameters. This relation is found by eliminating the linkage dimensions from Eq. (6). Solving for X_0 and Y_0 from D and E:

$$\left.\begin{array}{l} 2AX_0 + BY_0 = -D \\ BX_0 + 2CY_0 = -E \end{array}\right\} \begin{bmatrix} 2A & B \\ B & 2C \end{bmatrix} \begin{bmatrix} X_0 \\ Y_0 \end{bmatrix} = \begin{bmatrix} -D \\ -E \end{bmatrix}$$

$$\Rightarrow X_0 = \frac{2CD - BE}{B^2 - 4AC} \; ; \; Y_0 = \frac{2AE - BD}{B^2 - 4AC} \tag{7}$$

Substituting X_0 and Y_0 solutions of Eq. (7) in F expression in Eq. (6) and using the δ and Δ definitions given in Eqs. (4) and (5):

$$a^2 b^2 = 4\Delta/\delta \tag{8}$$

For $\alpha = \pi/2$ and $h = 0$ from Eq. (4) $\delta = -4a^2b^2$, so from Eq. (8):

$$\delta^2 = -16\Delta \tag{9}$$

Since $C > 0$ from Eq. (6), we can conclude that $C\Delta < 0$ from Eq. (9). So it is guaranteed that the ellipse is real. Eq. (9) gives a relationship among the coefficients of the ellipse equation (3). With this relationship we can deal with the homogeneity issue.

For given link length dimensions X_0, Y_0, β, a and b, we can uniquely determine the ellipse equation coefficients A, B, C, D, E and F by Eq. (6). Lets call these the normalized coefficients. However, when an ellipse equation is given, generically the coefficients A′, B′, C′, D′, E′, F′ are not normalized. In [5, 6], the homogeneity problem is solved by dividing the equation by the leading coefficient, but this results in rather complicated rational expressions in terms of the linkage dimensions. We follow an alternative approach here. Let $A = kA'$, $B = kB'$, $C = kC'$, $D = kD'$, $E = kE'$ and $F = kF'$ for a nonzero constant k. Substituting these in Eq. (9) we get:

$$k^4 \left(B'^2 - 4A'C' \right)^2 = -4k^3 \left[B'E'D' - C'D'^2 - A'E'^2 - F' \left(B'^2 - 4A'C' \right) \right]$$

$$\Rightarrow k = -4 \frac{B'E'D' - C'D'^2 - A'E'^2 - F' \left(B'^2 - 4A'C' \right)}{\left(B'^2 - 4A'C' \right)^2} = -\frac{16\Delta'}{\delta'^2} \tag{10}$$

Given a general ellipse equation, the normalized coefficients are obtained by multiplying all original coefficients with k given in Eq. (10). After normalization, we can determine the linkage dimensions. X_0 and Y_0 are already found in Eq. (7). β, a and b can be determined from the A, B and C expressions given in Eq. (6):

$$
\begin{aligned}
A + C &= a^2 + b^2 \\
\sqrt{B^2 + (A-C)^2} &= a^2 - b^2
\end{aligned}
\quad \Rightarrow \quad
\begin{aligned}
a &= \sqrt{\dfrac{A + C + \sqrt{B^2 + (A-C)^2}}{2}} \\
b &= \sqrt{\dfrac{A + C - \sqrt{B^2 + (A-C)^2}}{2}}
\end{aligned}
\tag{11}
$$

Noting that b < a, because a is the semi-major axis and b is the semi-minor axis of the ellipse, β can be determined as follows:

$$
\begin{aligned}
C - A &= (a^2 - b^2)c2\beta \\
-B &= (a^2 - b^2)s2\beta
\end{aligned}
\quad \Rightarrow \quad
\beta = \frac{1}{2}\operatorname{atan2}(C - A, -B)
\tag{12}
$$

4 The General Case ($\alpha \neq \pi/2$ and h ≠ 0)

In this section we demonstrate that the general case where $\alpha \neq \pi/2$ and h ≠ 0 can be analyzed using the results from Section 3. It is well known that the coupler link of a planar double-slider linkage performs the Cardan (or elliptic) motion. In Cardan motion the moving and fixed centrodes are both circles. The moving circle has half the radius of the fixed circle and rolls inside. A general point has an elliptic path, while a point selected on the moving circle has a linear path [10]. In Fig. 1, point M is the center of the fixed circle and also a point on the moving circle. Also points Q and R are on the moving circle due to their linear path. So, for a given double-slider linkage, the moving centrode is simply the circle passing through the slider points Q, R and the intersection point M of the slider lines. Since all points on the moving circle have linear paths, one can relocate the slider points Q and R on the moving circle without disturbing the motion of the coupler link.

The specific case we worked on in Section 3 ($\alpha = \pi/2$ and h = 0) corresponds to the case where |RQ| = a – b = 2r (r: radius of the moving circle) and the midpoint of RQ is the center of the moving circle. For a given elliptic equation (3), one can first obtain the specific linkage solution with $\alpha = \pi/2$ and h = 0. Then the moving circle can be obtained with center at midpoint T of RQ and radius |TR| = |TQ| = r = (a – b)/2. Then the slider points Q and R can be relocated to Q′ and R′ on the moving circle (Fig. 2). The new locations Q′ and R′ gives us the two arbitrary parameters for the infinitely many solutions of the linkage.
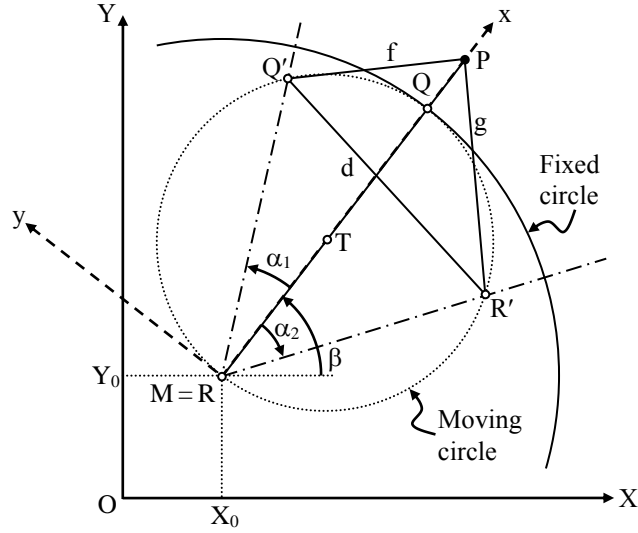
Fig. 2 Cardan circles and the coupler link

In Fig. 2, a moving frame is attached to the coupler link such that x-axis is along RQ and the configuration where joint R is coincident with the fixed circle center M is shown. At this configuration, the new locations Q′ and R′ can be parameterized by angles α_1 and α_2 measured from the x-axis such that α_1, $\alpha_2 \in (-\pi/2, \pi/2]$ (CCW is positive). α_1 and α_2 are two parameters to be used instead of α and h of Fig. 1. Coordinates of P, Q′ and R′ with respect to xy-frame:

$$\vec{P} = a \quad ; \quad \vec{Q'} = 2rc\alpha_1 e^{i\alpha_1} = r(1 + c2\alpha_1) + irs2\alpha_1 \quad ; \quad \vec{R'} = r(1 + c2\alpha_2) + irs2\alpha_2 \quad (13)$$

Recalling that r = (a − b)/2, the new link lengths of the coupler link Q′R′P are

$$d = |Q'R'| = r\sqrt{(c2\alpha_2 - c2\alpha_1)^2 + (s2\alpha_2 - s2\alpha_1)^2} = (a-b)|s(\alpha_1 - \alpha_2)|$$

$$f = |Q'P| = r\sqrt{(c2\alpha_1 - a)^2 + s^2 2\alpha_1} = \sqrt{a^2 s^2 \alpha_1 + b^2 c^2 \alpha_1} \qquad (14)$$

$$g = |R'P| = r\sqrt{(c2\alpha_2 - a)^2 + s^2 2\alpha_2} = \sqrt{a^2 s^2 \alpha_2 + b^2 c^2 \alpha_2}$$

X_0, Y_0, β, a (semi-major axis) and b (semi-minor axis) are determined in Section 3. Eq. (14) gives d, e and f in terms of a, b and selected α_1, α_2. Once the link lengths are found, the kinematic analysis of the linkage can be performed and it can be verified that the coupler point P traces the given ellipse equation. We performed this verification and also animated the motion of the linkage for any given ellipse equation using Microsoft Excel®.

5 Conclusions

In order to determine the planar double-slider linkage dimensions for a given algebraic equation of an elliptic curve, the scaling factor of the homogeneous equation is determined from the dependency of the ellipse equation coefficients and it is shown that the scale factor can be written in terms of the coordinate-free invariants of the curve. Similarly, one can analyze the coupler curves of planar slider-crank and four-bar linkages in a coordinate-free representation leading to determination of the invariants of the curve. We already solved the coordinate-free part for the slider-crank mechanism and there remains only the inclusion of coordinate transformations. Application of the approach for the four-bar linkage is also promising.

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