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Solitary wave solution of nonlinear multi-dimensional wave equation by bilinear transformation method

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Abstract

The Hirota method is applied to construct exact analytical solitary wave solutions of the system of multi-dimensional nonlinear wave equation for *n*-component vector with modified background. The nonlinear part is the third-order polynomial, determined by three distinct constant vectors. These solutions have not previously been obtained by any analytic technique. The bilinear representation is derived by extracting one of the vector roots (unstable in general). This allows to reduce the cubic nonlinearity to a quadratic one. The transition between other two stable roots gives us a vector shock solitary wave solution. In our approach, the velocity of solitary wave is fixed by truncating the Hirota perturbation expansion and it is found in terms of all three roots. Simulations of solutions for the one component and one-dimensional case are also illustrated.

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1. Introduction

The nonlinear science is the most important frontier for the fundamental understanding of nature. The solutions of the nonlinear evolution equations are appearing as a travelling waves which play distinctive role in nonlinear phenomena [1,2]. They are observed in various fields ranging from fluids and plasmas to solid-state, chemical, biological and geological systems. In mathematics, a number of techniques have been developed to obtain the travelling wave solution for nonlinear evolution equations [3–5,15]. Although the Lie method is one of the most popular classical method, it is not efficient for some non-integrable nonlinear PDEs (as an example, the well-known Fisher equation) which have poor Lie symmetry being invariant only under

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the time and space translations [4]. The other method [5], based on travelling wave ansatz, reduces nonlinear PDE to a nonlinear ODE. However in ODE system, the travelling wave speed is an unknown parameter that must be fixed by the analysis and choosing a special trial trajectory [2]. The inverse scattering method is also very powerful, but it is the most complicated and additionally needs information about analytical behaviour of scattering data.

From another site, in 1971, the direct method proposed by Hirota has become a powerful tool to construct multi-soliton solutions of integrable systems [3]. This, relatively simple and algebraic rather than analytic method, allows one to avoid many analytic difficulties of the more sophisticated inverse scattering method. Moreover, it is deeply related with Plücker coordinates of Grassmanians, quantum theory of fermions, τ functions and vertex operator representation of infinite-dimensional algebras [8]. The general idea of the method is first to transform the nonlinear equation under consideration into a bilinear equation or system of equations, and then use the formal power series expansion to solve it. For integrable systems the series admit exact truncation for an arbitrary number of solitons. While for periodic solutions it includes an infinite number of terms. We will see below that the truncation of Hirota's perturbation series for non-integrable case, similarly to the Painleve reduction [6], fixes the velocity of soliton.

The purpose of this paper is to demonstrate effectiveness of the Hirota method for constructing shock solitary wave solution of n-component wave equation in three space dimensions. We extended our previous work [16] for this aim. In addition, to our knowledge, the solitary wave solution for the system of multi-dimensional nonlinear wave equation for n-component vector case has not been formulated explicitly by using Hirota method. In this work, we consider equation for the vector order parameter $\boldsymbol{U}(x,y,z) = (U^1(x,y,z), U^2(x,y,z),\ldots,U^n(x,y,z))$ and cubic nonlinear reaction term

$$\alpha\frac{\partial^2\boldsymbol{U}}{\partial t^2} + \gamma\frac{\partial\boldsymbol{U}}{\partial t} = \nabla^2\boldsymbol{U} - (\boldsymbol{U}-\boldsymbol{a}_1,\boldsymbol{U}-\boldsymbol{a}_2)(\boldsymbol{U}-\boldsymbol{a}_3), \tag{1}$$

where $\boldsymbol{a_j} = (a_j^1, a_j^2,\ldots,a_j^n) \in R^n$, $(j = 1,2,3)$, are three distinct constant vectors, and $(\boldsymbol{U}-\boldsymbol{a}_1,\ \boldsymbol{U}-\boldsymbol{a}_2) \equiv \sum_{i=1}^n (U^i - a_1^i)(U^i - a_2^i)$ means the Euclidean scalar product of vectors $(\boldsymbol{U}-a_1)$ and $(\boldsymbol{U}-a_2)$ and $\nabla^2 = \Delta = \partial^2/\partial x^2 + \partial^2/\partial y^2 + \partial^2/\partial z^2$ is the Laplace operator, α, β are real constants. In the scalar case, when $n = 1$, $\alpha = 0$, and in the one space dimension, for different choices of parameters a_1, a_2, a_3 the model reduces to the well-known nonlinear diffusion equations appearing in a different fields sometimes with different names: the Fitzhugh–Nagumo equation ($a_1 = 0$, $a_2 = 1$, $a_3 = a$) arising in population genetics [9] and models the transmission of nerve impulse [10], autocatalytic chemical reaction model introduced by Schlögl [11,12], generalized Fisher equation [2], Newell–Whitehead equation [13] or Kolmogorov–Petrovsky–Piscounov equation [14] ($a_1 = 0$, $a_2 = 1$, $a_3 = -1$), Huxley equation ($a_1 = a_2 = 0$, $a_3 = 1$).

The paper is organized as follows. In the next section, using the modified Hirota ansatz, Eq. (1) is transformed into the bilinear system of $n+1$ differential equations, which we solve exactly using Hirota's perturbation approach. After finding the the wave vector, we obtain the exact analytical one-solitary wave solution for Eq. (1). The speed of the solitary wave is also found in terms of three distinct constant vectors which determine the reaction part of the equation we considered. Then we present simulations of the solitary waves for one component and one-dimensional case. Finally, we summarize our results and discuss possible extensions to other equations.

2. Vector bilinear forms and solitary waves

The solution of the problem is assumed to have a form of Eq. (2) in order to reduce Eq. (1) with cubic nonlinearity and three distinct roots $\boldsymbol{a}_1$, $\boldsymbol{a}_2$, $\boldsymbol{a}_3$, to the bilinear form. So we have to modify the standard Hirota ansatz by extracting one of the vector roots,

$$\boldsymbol{U} = \boldsymbol{a}_3 + \frac{\boldsymbol{g}}{f}, \tag{2}$$

where $\boldsymbol{g}(\mathbf{x},t)$ is a n-component real vector function and $f(\mathbf{x},t)$ is a real function. All derivatives with respect to the dependent variables in Eq. (1) are expressed in terms of the Hirota's derivatives in the bilinear approach:

$$\frac{\partial \boldsymbol{U}}{\partial t} = \frac{D_t(\boldsymbol{g}\cdot f)}{f^2}, \tag{3a}$$

$$\frac{\partial^2 \boldsymbol{U}}{\partial t^2} = \frac{D_t^2(\boldsymbol{g}\cdot f)}{f^2} - \frac{\boldsymbol{g}}{f}\frac{D_t^2(f\cdot f)}{f^2}, \tag{3b}$$

$$\nabla^2 \boldsymbol{U} = \frac{D_x^2(\boldsymbol{g}\cdot f)}{f^2} + \frac{D_y^2(\boldsymbol{g}\cdot f)}{f^2} + \frac{D_z^2(\boldsymbol{g}\cdot f)}{f^2} - \frac{\boldsymbol{g}}{f}\frac{D_x^2(f\cdot f)}{f^2} - \frac{\boldsymbol{g}}{f}\frac{D_y^2(f\cdot f)}{f^2} - \frac{\boldsymbol{g}}{f}\frac{D_z^2(f\cdot f)}{f^2}, \tag{3c}$$

where the Hirota derivative according to x_i is defined as

$$D_{x_i}^n(a\cdot b) = \left(\frac{\partial}{\partial x_i} - \frac{\partial}{\partial x_i'}\right)^n (a(\mathbf{x})b(\mathbf{x}'))|_{\mathbf{x}=\mathbf{x}'}. \tag{4}$$

After substituting Eqs. (3) into Eq. (1), the following expression is obtained:

$$\begin{aligned}&\alpha\frac{D_t^2(\boldsymbol{g}\cdot f)}{f^2} - \alpha\frac{\boldsymbol{g}}{f}\frac{D_t^2(f\cdot f)}{f^2} + \gamma\frac{D_t(\boldsymbol{g}\cdot f)}{f^2} - \frac{D_x^2(\boldsymbol{g}\cdot f)}{f^2} - \frac{D_y^2(\boldsymbol{g}\cdot f)}{f^2} - \frac{D_z^2(\boldsymbol{g}\cdot f)}{f^2} + \frac{\boldsymbol{g}}{f}\frac{D_x^2(f\cdot f)}{f^2}\\ &\quad + \frac{\boldsymbol{g}}{f}\frac{D_y^2(f\cdot f)}{f^2} + \frac{\boldsymbol{g}}{f}\frac{D_z^2(f\cdot f)}{f^2} + \frac{\boldsymbol{g}}{f}\left(\left(\frac{\boldsymbol{g}}{f} - (\boldsymbol{a}_1 - \boldsymbol{a}_3)\right), \left(\frac{\boldsymbol{g}}{f} - (\boldsymbol{a}_2 - \boldsymbol{a}_3)\right)\right) = 0.\end{aligned} \tag{5}$$

By introducing n-dimensional vector function $\boldsymbol{g}$ and one more scalar function f instead of one n-dimensional vector function $\boldsymbol{U}$ for Eq. (1), we have freedom to decouple this system of n-equations as the bilinear system of $n+1$ equations

$$(\alpha D_t^2 + \gamma D_t - D_x^2 - D_y^2 - D_z^2)(\boldsymbol{g}\cdot f) = 0, \tag{6a}$$

$$(-\alpha D_t^2 + D_x^2 + D_y^2 + D_z^2 + (\tilde{\boldsymbol{a}}_1, \tilde{\boldsymbol{a}}_2))(f\cdot f) = -(\boldsymbol{g},\boldsymbol{g}) + ((\tilde{\boldsymbol{a}}_1 + \tilde{\boldsymbol{a}}_2), \boldsymbol{g})f, \tag{6b}$$

where $\tilde{\boldsymbol{a}}_1 \equiv \boldsymbol{a}_1 - \boldsymbol{a}_3$ and $\tilde{\boldsymbol{a}}_2 \equiv \boldsymbol{a}_2 - \boldsymbol{a}_3$. To solve this system in the Hirota method, the function f and the vector function $\boldsymbol{g}$ are supposed to have form of the formal perturbation series in a parameter ϵ

$$f = \sum_{i=0}^{\infty} \epsilon^i f_i, \quad \boldsymbol{g} = \sum_{i=0}^{\infty} \epsilon^i \boldsymbol{g}_i, \tag{7}$$

without loss of generality assume that $f_0 = 0$. Substituting (7) into system (6) and equating coefficients of the same powers of ϵ converts (6) into a sequence of the zeroth, first, second and higher order, bilinear equations

$$(\alpha D_t^2 + \gamma D_t - D_x^2 - D_y^2 - D_z^2)(\boldsymbol{g}_0\cdot 1) = 0, \tag{8a}$$

$$(-\alpha D_t^2 + D_x^2 + D_y^2 + D_z^2 + (\tilde{\boldsymbol{a}}_1, \tilde{\boldsymbol{a}}_2))(1\cdot 1) = -(\boldsymbol{g}_0, \boldsymbol{g}_0) + (\tilde{\boldsymbol{a}}_1 + \tilde{\boldsymbol{a}}_2, \boldsymbol{g}_0), \tag{8b}$$

$$(\alpha D_t^2 + \gamma D_t - D_x^2 - D_y^2 - D_z^2)(\boldsymbol{g}_0\cdot f_1 + \boldsymbol{g}_1\cdot 1) = 0, \tag{9a}$$

$$(-\alpha D_t^2 + D_x^2 + D_y^2 + D_z^2 + (\tilde{\boldsymbol{a}}_1, \tilde{\boldsymbol{a}}_2))(2\cdot f_1) = -2(\boldsymbol{g}_0, \boldsymbol{g}_1) + (\tilde{\boldsymbol{a}}_1 + \tilde{\boldsymbol{a}}_2, \boldsymbol{g}_0 f_1 + \boldsymbol{g}_1), \tag{9b}$$

$$(\alpha D_t^2 + \gamma D_t - D_x^2 - D_y^2 - D_z^2)(\boldsymbol{g}_0\cdot f_2 + \boldsymbol{g}_1\cdot f_1 + \boldsymbol{g}_2\cdot 1) = 0, \tag{10a}$$

$$\begin{aligned}&(-\alpha D_t^2 + D_x^2 + D_y^2 + D_z^2 + (\tilde{\boldsymbol{a}}_1, \tilde{\boldsymbol{a}}_2))(2\cdot f_2 + f_1\cdot f_1)\\ &\quad = -2(\boldsymbol{g}_0, \boldsymbol{g}_2) - (\boldsymbol{g}_1, \boldsymbol{g}_1) + (\tilde{\boldsymbol{a}}_1 + \tilde{\boldsymbol{a}}_2, \boldsymbol{g}_0 f_2 + \boldsymbol{g}_1 f_1 + \boldsymbol{g}_2).\end{aligned} \tag{10b}$$

We assume that all components of vector $\boldsymbol{g}_0$ are constants, then the first Eq. (8) is satisfied automatically. From the second equation we get

$$(\boldsymbol{g}_0, \boldsymbol{g}_0) - (\tilde{\boldsymbol{a}}_1 + \tilde{\boldsymbol{a}}_2, \boldsymbol{g}_0) + (\tilde{\boldsymbol{a}}_1, \tilde{\boldsymbol{a}}_2) = (\boldsymbol{g}_0 - \tilde{\boldsymbol{a}}_1, \boldsymbol{g}_0 - \tilde{\boldsymbol{a}}_2) = 0. \tag{11}$$

Solutions of the last equation can be found as: (1) $\boldsymbol{g}_0 = \tilde{\boldsymbol{a}}_1$, (2) $\boldsymbol{g}_0 = \tilde{\boldsymbol{a}}_2$ and (3) $(\boldsymbol{g}_0 - \tilde{\boldsymbol{a}}_1) \perp (\boldsymbol{g}_0 - \tilde{\boldsymbol{a}}_2)$. For simplicity, we assume that $\boldsymbol{g}_0 = \tilde{\boldsymbol{a}}_1$. As a next step we are going to find the first-order solutions, $\boldsymbol{g}_1$ and f_1. Then Eqs. (9) can be reduced to a linear system

$$\tilde{\boldsymbol{a}}_1(\alpha\partial_t^2 - \gamma\partial_t - \Delta)f_1 + (\alpha\partial_t^2 + \gamma\partial_t - \Delta)\boldsymbol{g}_1 = 0, \tag{12a}$$

$$-2\alpha\partial_t^2 f_1 + 2\Delta f_1 + 2(\tilde{\boldsymbol{a}}_1, \tilde{\boldsymbol{a}}_2)f_1 + 2(\tilde{\boldsymbol{a}}_1, \boldsymbol{g}_1) - (\tilde{\boldsymbol{a}}_1 + \tilde{\boldsymbol{a}}_2, \tilde{\boldsymbol{a}}_1 f_1 + \boldsymbol{g}_1) = 0. \tag{12b}$$

where Δ denotes the three-dimensional Laplace operator. Nontrivial solution of this system is supposed to have the form

$$\boldsymbol{g}_1 = \boldsymbol{a}\mathrm{e}^{\eta_1}, \quad f_1 = b\mathrm{e}^{\eta_1}, \tag{13}$$

where $\boldsymbol{a} = (a^1, a^2, \ldots, a^n)$ and b are $n+1$ constants, $\eta_1 = \boldsymbol{k}\mathbf{x} + \omega t + \delta_0$. Here $\boldsymbol{k}\mathbf{x} = k_x x + k_y y + k_z z$ means the three-dimensional scalar product. Unknown constants, the wave vector $\boldsymbol{k}$ and frequency ω are fixed by a dispersion relation, while n constants $(a^1, a^2, \ldots, a^n)$ are fixed by Eqs. (12). After substituting (13) into system (12) for variables $(a^1, a^2, \ldots, a^n)$ and b we have $(n+1)\times(n+1)$ homogeneous linear algebraic system:

$$\mathbf{A}\begin{bmatrix} a^1 \\ a^2 \\ \vdots \\ a^n \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix}$$

where

$$\mathbf{A} = \begin{bmatrix} (\alpha\omega^2 + \gamma\omega - \boldsymbol{k}^2) & 0 & 0 & \cdot & \tilde{a}_1^1(\alpha\omega^2 - \gamma\omega - \boldsymbol{k}^2) \\ 0 & (\alpha\omega^2 + \gamma\omega - \boldsymbol{k}^2) & 0 & \cdot & \tilde{a}_1^2(\alpha\omega^2 - \gamma\omega - \boldsymbol{k}^2) \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdot & (\alpha\omega^2 + \gamma\omega - \boldsymbol{k}^2) & \tilde{a}_1^n(\alpha\omega^2 - \gamma\omega - \boldsymbol{k}^2) \\ (\tilde{a}_1^1 - \tilde{a}_2^1) & \cdot & \cdot & (\tilde{a}_1^n - \tilde{a}_2^n) & -2(\alpha\omega^2 - \boldsymbol{k}^2) + (\tilde{\boldsymbol{a}}_1, \tilde{\boldsymbol{a}}_2 - \tilde{\boldsymbol{a}}_1) \end{bmatrix}.$$

Nontrivial solution of this system of equations exists only if $\mathrm{Det}(\boldsymbol{A}) = 0$. This determinant can be evaluated by expansion along the last row, so that we have expression

$$\begin{aligned} &\mathrm{Det}(\mathbf{A}) = (\alpha\omega^2 + \gamma\omega - \boldsymbol{k}^2)^{n-1}, \\ &\left[(\alpha\omega^2 + \gamma\omega - \boldsymbol{k}^2)(-2\alpha\omega^2 + 2\boldsymbol{k}^2 + (\tilde{\boldsymbol{a}}_1, \tilde{\boldsymbol{a}}_2 - \tilde{\boldsymbol{a}}_1)) - (\alpha\omega^2 - \gamma\omega - \boldsymbol{k}^2)(\tilde{\boldsymbol{a}}_1, \tilde{\boldsymbol{a}}_1 - \tilde{\boldsymbol{a}}_2)\right]. \end{aligned} \tag{14}$$

It gives us the following dispersion relations:

$$\alpha\omega^2 + \gamma\omega - \boldsymbol{k}^2 = 0, \tag{15}$$

and

$$\alpha\omega^2 + \gamma\omega - \boldsymbol{k}^2 - (\tilde{\boldsymbol{a}}_1, \tilde{\boldsymbol{a}}_2 - \tilde{\boldsymbol{a}}_1) = 0. \tag{16}$$

The dispersion relation (15) gives us a trivial solution, thus we use dispersion (16) for the further calculations. Solving the first n equations in the above $(n+1)\times(n+1)$ linear algebraic system we have

$$\boldsymbol{a} = b\gamma_1\tilde{\boldsymbol{a}}_1, \tag{17}$$

where

$$\gamma_1 = \frac{2\boldsymbol{k}^2 - 2\alpha\omega^2 + (\tilde{\boldsymbol{a}}_1, \tilde{\boldsymbol{a}}_2 - \tilde{\boldsymbol{a}}_1)}{(\tilde{\boldsymbol{a}}_1, \tilde{\boldsymbol{a}}_2 - \tilde{\boldsymbol{a}}_1)}, \tag{18}$$

or combining this with Eq. (16), we find the other representation for γ_1

$$\gamma_1 = \frac{2\gamma\omega - (\tilde{\boldsymbol{a}}_1, \tilde{\boldsymbol{a}}_2 - \tilde{\boldsymbol{a}}_1)}{(\tilde{\boldsymbol{a}}_1, \tilde{\boldsymbol{a}}_2 - \tilde{\boldsymbol{a}}_1)}. \tag{19}$$

Thus, we have

$$\boldsymbol{g}_1 = b\gamma_1\tilde{\boldsymbol{a}}_1\mathrm{e}^{\eta_1}, \quad f_1 = b\mathrm{e}^{\eta_1}. \tag{20}$$

Substituting $\boldsymbol{g}_1, f_1$ to the first system of Eqs. (10) and using property of bilinear Hirota operators [7]

$$(D_t^2 + D_t - D_x^2)(\boldsymbol{g}_1 \cdot f_1) = b^2 \gamma_1 \tilde{\boldsymbol{a}}_1 (D_t^2 + D_t - D_x^2)(\mathrm{e}^{\eta_1} \cdot \mathrm{e}^{\eta_1}) = 0 \tag{21}$$

for $\boldsymbol{g}_2, f_2$ we find the following set of equations:

$$\tilde{\boldsymbol{a}}_1(\alpha\partial_t^2 - \gamma\partial_t - \Delta)f_2 + (\alpha\partial_t^2 + \gamma\partial_t - \Delta)\boldsymbol{g}_2 = 0, \tag{22}$$

similar to the first equation in system (12). The simplest solution for these equations is the trivial one $\boldsymbol{g}_2 = 0$ and $f_2 = 0$. Then, from the second equation in (10) we find additional constraint on $\boldsymbol{g}_1, f_1$:

$$(\tilde{\boldsymbol{a}}_1, \tilde{\boldsymbol{a}}_2) f_1^2 = -(\boldsymbol{g}_1, \boldsymbol{g}_1) + (\tilde{\boldsymbol{a}}_1 + \tilde{\boldsymbol{a}}_2, \boldsymbol{g}_1) f_1. \tag{23}$$

Substituting solution (20) into Eq. (23) results in the following relation:

$$(\gamma_1 - 1)(\tilde{\boldsymbol{a}}_1, (\gamma_1 \tilde{\boldsymbol{a}}_1 - \tilde{\boldsymbol{a}}_2)) = 0. \tag{24}$$

Assuming $\gamma_1 = 1$, leads to the trivial result $\boldsymbol{k} = 0$. Thus, if $\gamma_1 \neq 1$ from Eq. (24) the following nontrivial solution can be found:

$$\gamma_1 = \frac{(\tilde{\boldsymbol{a}}_1, \tilde{\boldsymbol{a}}_2)}{(\tilde{\boldsymbol{a}}_1, \tilde{\boldsymbol{a}}_1)}. \tag{25}$$

From Eqs. (25) and (19), we can find the frequency explicitly

$$\omega = \frac{(\tilde{\boldsymbol{a}}_1, \tilde{\boldsymbol{a}}_2 - \tilde{\boldsymbol{a}}_1)(\tilde{\boldsymbol{a}}_1, \tilde{\boldsymbol{a}}_2 + \tilde{\boldsymbol{a}}_1)}{2\gamma(\tilde{\boldsymbol{a}}_1, \tilde{\boldsymbol{a}}_1)}. \tag{26}$$

Combining this equation with Eq. (18), we find restrictions on allowed values of the length for the wave vector $\boldsymbol{k}$

$$\boldsymbol{k}^2 = \frac{(\tilde{\boldsymbol{a}}_1, \tilde{\boldsymbol{a}}_2 - \tilde{\boldsymbol{a}}_1)^2}{2(\tilde{\boldsymbol{a}}_1, \tilde{\boldsymbol{a}}_1)} \left(1 + \alpha \frac{(\tilde{\boldsymbol{a}}_1, \tilde{\boldsymbol{a}}_2 + \tilde{\boldsymbol{a}}_1)^2}{2\gamma^2} \right). \tag{27}$$

Finally, the velocity vector is given by formula

$$\boldsymbol{v} = -\omega \frac{\boldsymbol{k}}{|\boldsymbol{k}|^2}. \tag{28}$$

With the wave vector $\boldsymbol{k}$ given by Eq. (27) and frequency ω given by Eq. (26) for the speed of solitary wave we have the expression in terms of three vectors $\boldsymbol{a}_1$, $\boldsymbol{a}_2$, $\boldsymbol{a}_3$ as

$$|\boldsymbol{v}| = \frac{\chi}{|\boldsymbol{a}_1 - \boldsymbol{a}_3| \sqrt{\alpha\chi^2 + \gamma^2}}, \tag{29}$$

where

$$\chi = \frac{(\boldsymbol{a}_1 - \boldsymbol{a}_3, \boldsymbol{a}_1 + \boldsymbol{a}_2 - 2\boldsymbol{a}_3)}{\sqrt{2}}.$$

It is easy to show that each bilinear equation, which has order greater than 2, has simple solution as, $\boldsymbol{g}_i = 0$ and $f_i = 0$, for $i > 2$. Therefore, we have only finite number of terms in the expansion (7). After substituting f and $\boldsymbol{g}$ in Eq. (2), we find the following exact solution of our problem:

$$\boldsymbol{U} = \boldsymbol{a}_3 + (\boldsymbol{a}_1 - \boldsymbol{a}_3) \frac{1 + \gamma_1 \mathrm{e}^{\eta_1}}{1 + \mathrm{e}^{\eta_1}}, \tag{30}$$

where $\gamma_1 = (\tilde{\boldsymbol{a}}_1, \tilde{\boldsymbol{a}}_2)/(\tilde{\boldsymbol{a}}_1, \tilde{\boldsymbol{a}}_1)$. We note that the constant b appearing only in front of exponential terms can be absorbed by the arbitrary constant δ_0 in Eq. (13) and leads just to shift of the soliton's origin. In terms of original vectors $\boldsymbol{a}_1$, $\boldsymbol{a}_2$, $\boldsymbol{a}_3$ solution acquires the final form

$$\boldsymbol{U} = \boldsymbol{a}_3 + \frac{\boldsymbol{a}_1 - \boldsymbol{a}_3}{|\boldsymbol{a}_1 - \boldsymbol{a}_3|^2} \frac{((\boldsymbol{a}_1 - \boldsymbol{a}_3), (\boldsymbol{a}_1 - \boldsymbol{a}_3 + (\boldsymbol{a}_2 - \boldsymbol{a}_3)\mathrm{e}^{\eta_1}))}{1 + \mathrm{e}^{\eta_1}}. \tag{31}$$

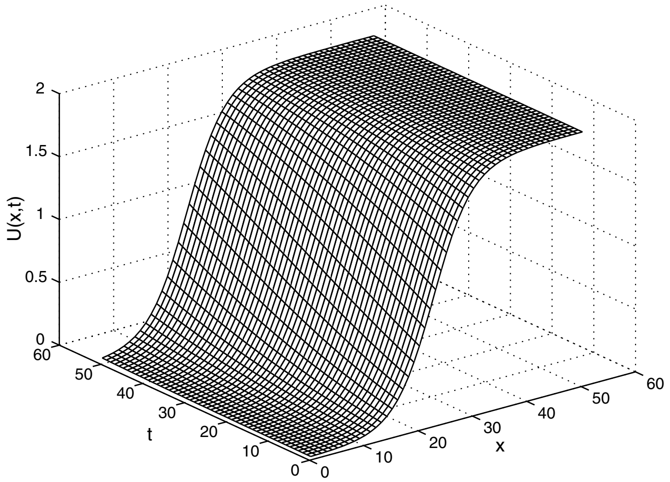

Fig. 1. Evolution of the travelling wave with $v \neq 0$. The parameters $a_1 = 0$, $a_2 = 2$, $a_3 = 1.5$, $\alpha = 1$, $\gamma = 10$.

This solution is the shock-solitary wave with asymptotics $\boldsymbol{U} \to \boldsymbol{a}_1$, for $\eta_1 \to -\infty$ and $\boldsymbol{U} \to (1-\gamma_1)\boldsymbol{a}_3 + \gamma_1 \boldsymbol{a}_1$, for $\eta_1 \to +\infty$, at a fixed time. As easy to see, these asymptotic solutions are the stationary ($\boldsymbol{U}_t = 0$) homogeneous ($\boldsymbol{U}_x = \boldsymbol{U}_y = \boldsymbol{U}_z = 0$) solutions of Eq. (1). The parameter γ_1 in this solution has meaning of the ratio $\gamma_1 = (|\tilde{\boldsymbol{a}}_2|/|\tilde{\boldsymbol{a}}_1|)\cos\alpha$, where $0 \leqslant \alpha \leqslant \pi$ is the angle between vectors $\tilde{\boldsymbol{a}}_1, \tilde{\boldsymbol{a}}_2$. Since Eq. (24) has infinite number of solutions, our shock soliton interpolates between the "vacuum" solution determined by vector $\boldsymbol{a}_1$ and the solution with vector $\boldsymbol{a}_3 + (\tilde{\boldsymbol{a}}_1, \tilde{\boldsymbol{a}}_2)\tilde{\boldsymbol{a}}_1/|\tilde{\boldsymbol{a}}_1|^2$ which is valued in the continuum set.

If we choose another root for $\boldsymbol{g}_0$ ($\boldsymbol{g}_0 = \tilde{\boldsymbol{a}}_2$) in Eq. (11), then we have another shock soliton solution of Eq. (1)

$$\boldsymbol{U} = \boldsymbol{a}_3 + (\boldsymbol{a}_2 - \boldsymbol{a}_3)\frac{1+\gamma_2 e^{\eta_1}}{1+e^{\eta_1}}, \tag{32}$$

where $\gamma_2 = (\tilde{\boldsymbol{a}}_1, \tilde{\boldsymbol{a}}_2)/(\tilde{\boldsymbol{a}}_2, \tilde{\boldsymbol{a}}_2)$. Solution (32) has asymptotic $\boldsymbol{U} \to \boldsymbol{a}_2$, for $\eta_1 \to -\infty$ and $\boldsymbol{U} \to (1-\gamma_2)\boldsymbol{a}_3 + \gamma_2 \boldsymbol{a}_2$, for $\eta_1 \to +\infty$, at a fixed time. The parameter γ_2 has meaning of the ratio $\gamma_2 = (|\tilde{\boldsymbol{a}}_1|/|\tilde{\boldsymbol{a}}_2|)\cos\alpha$, where α as above for the γ_1, is the angle between vectors $\tilde{\boldsymbol{a}}_1, \tilde{\boldsymbol{a}}_2$.

In the scalar case, when $n = 1$, the wave number (27), frequency (26) and velocity (28) (as well as the form of our solution (34)) reduces to following expression:

$$k^2 = \frac{(a_2 - a_1)^2}{2}\left[1 + \alpha\frac{(a_2 + a_1)^2}{2\gamma^2}\right], \quad \omega = \frac{a_2^2 - a_1^2}{2\gamma}, \quad |v| = \frac{\frac{a_1 + a_2 - 2a_3}{\sqrt{2}}}{\sqrt{\alpha\left(\frac{a_1 + a_2 - 2a_3}{\sqrt{2}}\right)^2 + \gamma^2}}, \tag{33}$$

and

$$u = a_3 + \frac{(a_1 - a_3) + e^{\eta_1}}{(a_2 - a_3) + e^{\eta_1}}(a_2 - a_3), \tag{34}$$

where $\eta_1 = kx + wt + \gamma$. Fig. 1 shows a typical one-soliton solution with $a_1 = 0$, $a_2 = 2$, $a_3 = 1.5$, $\alpha = 1$, $\gamma = 10$.

3. Summary and discussion

In this paper, we systematically found the exact analytical solitary wave solution for the system of multidimensional nonlinear wave equation by using the modified Hirota technique. Proposed modified Hirota ansatz allows us to construct a bilinear representation for the equation we considered. The system of bilinear equation can be easily solved by Hirota's approach. Truncating of the perturbation series in our second-order calculations restricts value of wave number and velocity of the travelling wave, and, in this sense, works similarly to the way of Ablowitz and Zeppetella [6] who obtained an exact travelling wave solution of Fisher's equation by finding the special wave speed for which the resulting ODE is of the Painleve type.

Finally, we showed that Hirota method is very efficient and systematic procedure to obtain exact solutions of such nonlinear equations. We hope that the results obtained in this work will allow one to construct nonlinear wave configurations of amazing complexity, like circular or curved solitons, scroll-waves and vortex tubes.

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