Applications of Graph Coloring

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Abstract. A graph G is a mathematical structure consisting of two sets $V(G)$ (vertices of G) and $E(G)$ (edges of G). Proper coloring of a graph is an assignment of colors either to the vertices of the graphs, or to the edges, in such a way that adjacent vertices / edges are colored differently. This paper discusses coloring and operations on graphs with *Mathematica* and web*Mathematica*. We consider many classes of graphs to color with applications. We draw any graph and also try to show whether it has an Eulerian and Hamiltonian cycles by using our package ColorG.

1 Introduction

Graph theory would not be what it is today if there had been no coloring problems. In fact, a major portion of the 20th-century research in graph theory has its origin in the four color problem [1]. A graph G is a mathematical structure consisting of two sets $V(G)$ (vertices of G) and $E(G)$ (edges of G). Proper coloring of a graph is an assignment of colors either to the vertices of the graphs, or to the edges, in such a way that adjacent vertices / edges are colored differently. Vertex coloring is a hard combinatorial optimization problem.

We apply several operations which act on graphs to give different graphs. In addition to apply graph operations, we color vertices of these obtained graphs properly. Also we developed ColorG package to color the vertices and edges of graphs and to find the Eulerian and the Hamiltonian cycles with web*Mathematica*. Many of these graphs are truly beautiful when drawn properly, and they provide a wide range of structures to manipulate and study.

Before concluding this introduction, we recall some basic definitions.

A complete graph is a simple graph such that every pair of vertices is joined by an edge. A nontrivial closed path is called a cycle. A graph which is obtained by joining a new vertex to every vertices of a cycle is called a wheel. A connected acyclic graph is called a tree [4].

2 Graph Coloring with WebMathematica

One of the most exciting new technologies for dynamic mathematics on the World Wide Web is a web*Mathematica*. This new technology developed by Wolfram research enables instructors to create web sites that allows users to compute

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and visualize results directly from a web browser. web*Mathematica* is based on a standard java technology called servlets. It allows a site to deliver HTML pages that are enhanced by the addition of *Mathematica* commands [5]. When a request is made for one of these pages the *Mathematica* commands are evaluated and the computed result is placed in the page. People who access web*Mathematica* sites do not have to know how to use Mathematica [11].

In this section, we give applications of ColorG package to color the vertices and the edges of the graphs with web*Mathematica*.

2.1 Vertex Coloring

The most applications involving vertex coloring are concerned with determining the minimum number of colors required under the condition that the end points of an edge cannot have the same color. A proper vertex coloring of a graph is an assignment from its vertex set to a color set that the end points of each edge are assigned two different colors. The chromatic number of a graph G, denoted by $\chi(G)$, is the minimum number of different colors required for a proper vertex coloring of G. Applications of vertex coloring include scheduling, assignment of radio frequencies, separating combustible chemical combinations, and computer optimization. We use some commands in the Combinatorica package with Mathematica to color the vertices of graphs and to give web-based examples with webMathematica as in The Fig. 1 [9].

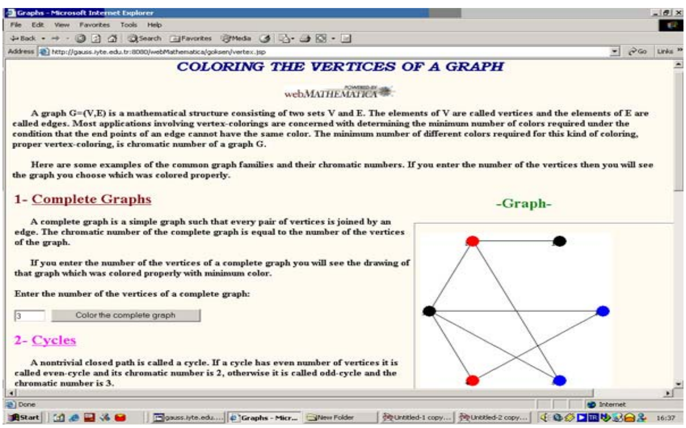

Fig. 1. Vertex Coloring of a Graph with webMathematica

If the users press the CompleteGraph, Cycle, Wheel, Star, RandomTree, and any graph and enter the number of vertices they can get the vertex-colored graph.

2.2 Edge Coloring

Edge coloring is an optimization problem: An edge-coloring of a graph G is an assignment of colors to the edges of G such that edges with a common endpoint

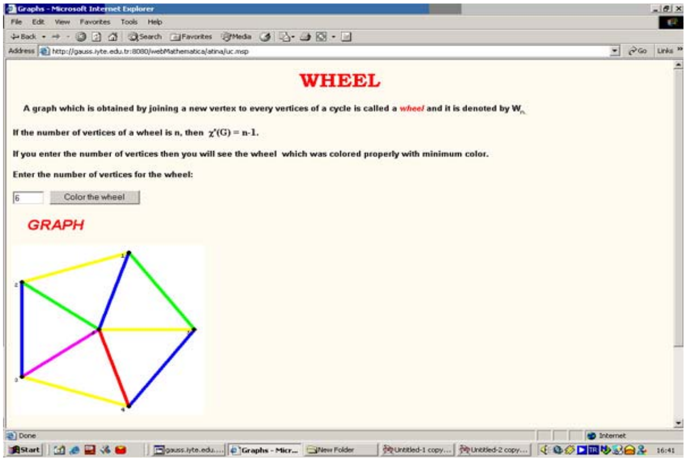

Fig. 2. Edge Coloring of a Graph with webMathematica

have different colors. Let $\chi(G')$ denote the chromatic index of G, that is the minimum number of colors necessary to color the edges of G. Vizing [10] proved that $\chi(G')$ is either $\triangle(G)$ or $\triangle(G)+1$ for each graph G, where $\triangle(G)$ denotes the maximum degree of a vertex in G. Then the graph G belongs to one of two classes; either to class 1 or to class 2. This classification problem is NP-complete, and this implies that there are no polynomial-time algorithms for this problem. We use some commands in the ColorG package with Mathematica to color the edges of graphs and to give web-based examples with webMathematica as in The Fig. 2 [9].

If the user press the Wheel button and enter the number of vertices, he/she can get the edge-colored graph.

3 Generating Graphs with WebMathematica

The most important operations on graphs are sum, union, join, and product of two graphs. We will do these operations with the common graphs; complete graph, random tree, wheel, and cycle.

The join of two graphs is their union with the addition of edges between all pairs of vertices from different graphs. To take the join of two graphs the user should enter the number of the graphs and their vertex numbers into the boxes then he/she sees the join of those graphs and also its proper vertex coloring.

This operation above is extended for the other graph operations: sum, join, and product, also.

The union of two graphs is formed by taking the union of the vertices and edges of the graphs. Thus the union of graphs is always disconnected. The sum operation of two graphs is to take the edges of the second graph and add them to the first graph. The product of two graphs $G \times H$ has a vertex set defined by the Cartesian product of the vertex sets of G and H. There is an edge between

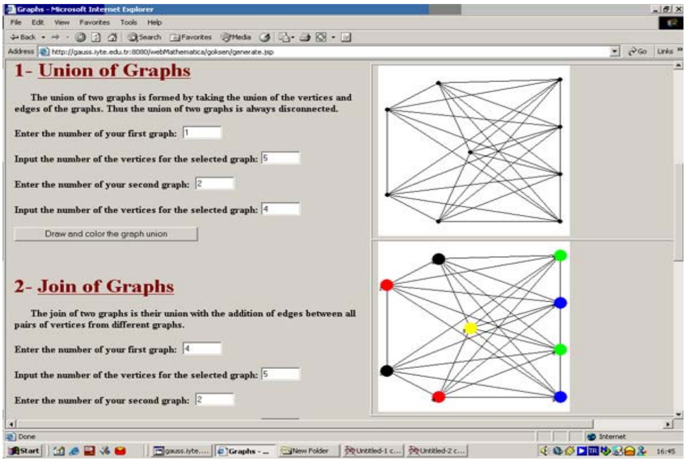

Fig. 3. A Generated Graph with webMathematica

(u, v) and (s, t) if $u = s$ and v is adjacent to t in H or $v = t$ and u is adjacent to s in G.

4 Cycle Structure in Graphs with WebMathematica

A cycle in a graph is a simple closed path. We will represent a cycle in G as a list of vertices $C = v_1, v_2, ..., v_1$ such that there is an edge of G from each vertex to the next in G.

4.1 Eulerian Cycle

Euler initiated the study of graph theory in 1736 with the famous Seven Bridges of Königsberg problem. The town of Königsberg straddled the Pregel River with a total of seven bridges connecting the two shores and two islands. The townsfolk were interested in crossing every bridge exactly once and returning to the starting point. An Eulerian cycle is a complete tour of all the edges of a graph. The term circuit is often used instead of cycle, since each vertex can be visited more than once.

We use ColorG package with Mathematica to find the Eulerian cycle and to give web-based examples with webMathematica. If the number of the vertices is entered, it is possible to see the Eulerian cycle in that graph if there exists.

4.2 Hamiltonian Cycle

A Hamiltonian cycle of a graph G is a cycle which visits every vertex in G exactly once, as opposed to an Eulerian cycle which visits each edge exactly once. A Hamiltonian path is like a Hamiltonian cycle, except that it is a path. The problem of computing a Hamiltonian cycle or a Hamiltonian path is fundamen-

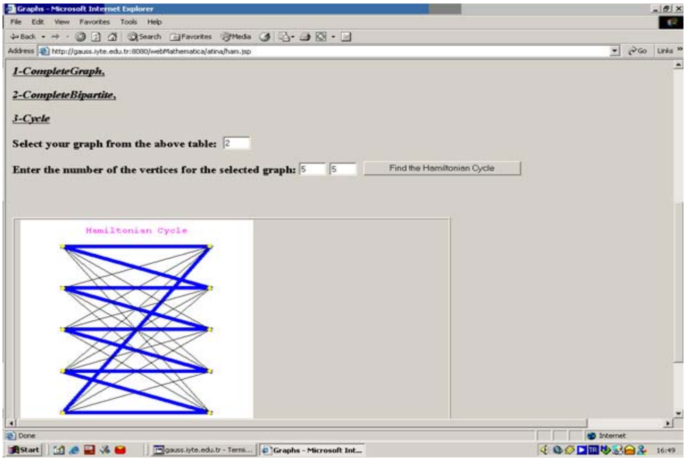

Fig. 4. A Hamiltonian Cycle in a Complete Bipartite Graph with webMathematica

tally different from the problem of computing an Eulerian cycle, because testing whether a graph is Hamiltonian is NP-complete.

We use ColorG package with Mathematica to find the Hamiltonian cycle and to give web-based examples with web*Mathematica*. If the number of the vertices is entered, it is possible to see the Hamiltonian cycle in that graph if there exists. The Fig. 4. shows the Hamiltonian Cycle for the Complete Bipartite Graph.

4.3 An Application

Some scheduling problems induce a graph coloring, i.e., an assignment of positive integers (colors) to vertices of a graph. We discuss a simple example for coloring the vertices of a graph with a small number k of colors and present computational results for calculating the chromatic number, i.e., the minimal possible value of such a k. Draw up an examination schedule involving the minimum number of days for the following problem.

Example: Set of students: $S_1, S_2, S_3, S_4, S_5, S_6, S_7, S_8, S_9$ Examination subjects for each group: {algebra, real analysis, and topology}, {algebra, operations research, and complex analysis}, {real analysis, functional analysis, and topology}, {algebra, graph theory, and combinatorics}, {combinatorics, topology, and functional analysis}, {operations research, graph theory, and coding theory}, {operations research, graph theory, and number theory}, {algebra, number theory, and coding theory}, {algebra, operations research, and real analysis}.

Let S be a set of students, $P = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$ be the set of examinations respectively algebra,real analysis, topology, operational research, complex analysis, functional analysis, graph theory, combinatorics, coding theory, and number theory. $S(p)$ be the set of students who will take the examination $p \in P$. Form a graph $G = G(P, E)$, where $a, b \in P$ are adjacent if and only if $S(a) \bigcap S(b) \neq \emptyset$. Then each proper vertex coloring of G yields an examination schedule with the vertices in any color class representing the schedule on a particular day. Thus $\chi(G)$ gives the minimum number of days required for the examination schedule. The Mathematica commands for this solution are as follows:

```
<< DiscreteMath`ColorG`
k = Input["Input the number of the students"];
S = Table[Input["Input number of the lessons which the student
will choose"],  k];
b = Union[Flatten[Table[KSubsets[S[[i]], 2], i, k], 1]];
ColorVertices[t = DrawG[b]];
h = VertexColoring[t]; d=ChromaticNumber[t];
Print[d"days are required and you can see below the lessons in the
same  parenthesis which are on the same day"]
Table[Flatten[Position[h, i], 2], i, Max[h]]
```

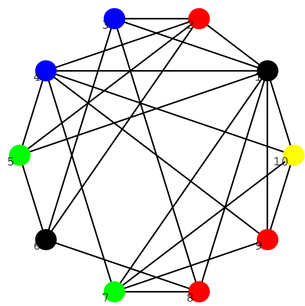

Fig. 5. The colored graph of the example

5 days are required and you can see below the lessons in the same parenthesis which are on the same day

$$\{\{1,6\},\{2,8,9\},\{3,4\},\{5,7\},\{10\}\}$$

It was very exciting to take 100-year old ideas, simple as they are, and implement them in *Mathematica* and web*Mathematica* for anybody. But, there is more work to be done, both of a theoretical and practical nature. If we consider a coloring problem posed as a two-person game, with one person (Alice) trying to color the graph, and the other (Bob) trying to prevent this from happening. Alice and Bob alternate turns, with Alice having the first move. A move consisting of selecting an uncolored vertex x and assigning it a color from the color set X distinct from the colors assigned previously to neighbors of x. If after $n = |V(G)|$ moves, the graph G is colored, Alice is the winner. Bob wins if an impasse is reached before all vertices in the graph are colored. The game chromatic number of a graph G, denoted by $\chi_g(G)$, is the least cardinality of a color set X for which Alice has a winning strategy [3]. We believe that it is possible to play this game online with web*Mathematica*. Of course, this leads to the rich area of game coloring and the many difficult and intriguing questions there.

The following URL address shows our works on graph coloring: http://gauss.iyte.edu.tr:8080/webMathematica/goksen/

References

1. Appel,K.I., Haken,W., and Koch,J.: Every planar map is four colorable I: Discharing, Illinois J.Math, (1977)
2. Birkhoff,G.D. and Lewis,D.C.: Chromatic polinomials, Trans.Amer.Math.Soc., 60:355-451, (1946)
3. Bodlaender, H.L.: On the complexity of some coloring games, Lecture Notes in Computer Science, (1991)
4. Jonathan,G and Jay,Y.: Graph Theory and Its Applications,CRC Press, (1999)
5. Pemmaraju,S. and Skiena,S.: Computational Discrete Mathematics, Cambrigde Univ. Press, (2003)
6. Skiena,S.: Implementing Discrete Mathematics-Combinatorics and Graph Theory with Mathematica, Addison-Wesley Publishing Company,(1990)
7. Soaty,T.L., and Kainen,P.C.: The four color problem, Dover, New York, (1986)
8. Thulasiraman, K. and Swamy,M.N.S.: Graphs: Theory and Algorithms, John Wiley and Sons, Inc., (1992)
9. Ufuktepe, U., Bacak, G., and Beseri, T. : Graph Coloring with webMathematica, Lecture Notes in Computer Science, Springer-Verlag, (2003)
10. Vizing, V.G.: On an estimate of the chromatic class of a p-graph, Discret. Analiz, (1964)
11. Wickham,T.:webMathematica A User Guide,Wolfram Research, Inc., (2002)
12. Wolfram,S.:The Mathematica Book, Cambrigde Univ. Press, (1996)