Shock waves, chiral solitons and semiclassical limit of one-dimensional anyons

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Abstract

This paper is devoted to the semiclassical limit of the one-dimensional Schrödinger equation with current nonlinearity and Sobolev regularity, before shocks appear in the limit system. In this limit, the modified Euler equations are recovered. The strictly hyperbolicity and genuine nonlinearity are proved for the limit system wherever the Riemann invariants remain distinct. The dispersionless equation and its deformation which is the quantum potential perturbation of JNLS equation are also derived.
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1. Introduction

Recently, in the study of one-dimensional condensed matter systems with quantum wires and Hall edge states, a new version of the nonlinear Schrödinger equation (JNLS equation for short)

$$i\hbar\psi_t + \frac{\hbar^2}{2}\psi_{xx} + \lambda j(x,t)\psi = 0 \tag{1.1a}$$

with current nonlinearity

$$j = \frac{\hbar}{2i}(\bar{\psi}\psi_x - \psi\bar{\psi}_x) = \hbar\text{Im}(\bar{\psi}\psi_x), \tag{1.1b}$$

($m = 1$) has been derived in [2]. This model that appears in the above works is obtained from a very interesting dimensional reduction from $2 + 1$ dimensional model of anyons, the Chern–Simons gauged nonlinear Schrödinger model (the Jackiw–Pi model) [11,12], taking into account the dynamically active gauge field component $A_2 = B$. The model (1.1a,b) admits novel chiral soliton solution, moving only in one direction, whose mass formula justifies the interpretation of a soliton as a bounded state of elementary particles of the quantized theory in the weak coupling limit [2,3,10]. Evidently, the chiral solitons found in these works may play important role within the context of the practical quantum Hall effect, where chiral excitations are known to appear. Unfortunately it does not pass the Painlevé test [4] and seems not to be integrable in the sense of inverse scattering. Integrable extension of the model admitting N soliton solution appears if one adds the cubic nonlinearity term corresponding to three-particle interaction of bosons [9,25]. But the price for integrability is the lack of chiral solitons. It is why any analytical results related to Eq. (1.1a,b) would be important.

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Particularly in the semiclassical limit, when one could expect creation of the shock wave and interesting properties for chiral solitons and anyons. Therefore, the study of the vanishing Planck number limit is of particular interest.

In this paper we consider the Cauchy problem for JNLS equation (1.1a,b) subject to the rapid oscillating initial condition

$$\psi(x,0) = \psi_0(x) = A_0(x)\exp\left(\frac{\mathrm{i}}{\hbar}S_0(x)\right), \tag{1.2}$$

where $S_0 \in H^s(\mathbb{R})$ for s large enough, and A_0 is a function, polynomial in $\hbar$, with coefficients of Sobolev regularity in x. We will study the behavior of solutions of the problem (1.1a,b) and (1.2) as $\hbar \to 0$ and $-\infty < x < \infty$, $0 \leqslant t \leqslant T$, i.e., within an arbitrary finite time T. The corresponding asymptotic limit is the so-called semiclassical or WKB limit which is to determine the limiting dynamics of any function of the field ψ of the JNLS equation (1.1a,b) as $\hbar \to 0$. The Cauchy problem for a class of nonlinear Schrödinger equations with coupling of derivative type have been studied by Ozawa in [26] using the idea of gauge equivalent.

For defocusing nonlinear Schrödinger equation, the semiclassical limit for initial data with Sobolev regularity in short time has been studied by Grenier [8]. In this limit, the Euler equations for an isentropic compressible flow are recovered. This was proved rigorously by Jin et al. [13] in one dimension using the inverse scattering technique. For derivative nonlinear Schrödinger equation (DNLS for short), the limit behavior is described by the modified Euler equation [5,6,17,21].

The semiclassical limit of the JNLS equation (1.1a,b) can be discussed in the same strategy as Grenier's [8] for NLS equation (see also [5,6,17] for DNLS equations and [19,20] for Schrödinger–Poisson system). Similar to the derivative nonlinear Schrödinger equation [5–7,23], the λ term in JNLS equation (1.1a,b) is not invariant under Galilean transformation, and it flips sign under parity ($x \to -x$), i.e., the JNLS equation does not possess parity and Galilean invariance and therefore the canonical momentum is not conservative [15–17,21,22]. To obtain the conservation law of momentum, we have to introduce the *noncanonical momentum* which is indeed the conservative quantity of the JNLS equation (see (2.12) below). Although JNLS equation is similar to the DNLS equation, they are quit different from each other. For DNLS equation, even the MNLS equation [6,7,17], the associated Riemann invariant forms are given by simple and beautiful formulas. However, for JNLS equation we need to solve a fourth order equation, the associated Riemann form are ugly. This also explains JNLS's lack of integrability.

The rest of the paper is organized as follows. In Section 2 we reformulate the model (1.1a,b) in the Madelung hydrodynamics form and derive the conservative quantities and their conservation laws. The conservation laws can be obtained from the action principle. We also find the formal semiclassical or dispersionless limit of the model (1.1a,b), in the wave equation form as perturbed by so called quantum potential JNLS equation. In Section 3, we study the local smooth solutions of JNLS equation (1.1a,b) and their semiclassical limit based on the modified Madelung's transformation and the classical theory of quasilinear hyperbolic systems. Section 4 is devoted to estimation of the shock wave appearance in the semiclassical limit system. The strict hyperbolicity and geniune nonlinearity is proved for the limit system whenever the Riemann invariants $R_{\pm}$ remain distinct by applying Lax's theory [14,24]. A Riccati equation is found for the evolution of $\partial_x R_{\pm}$ along the associated characteristic and breaking are discussed [18,31].

2. Hydrodynamical structure of JNLS equation

The semiclassical limits are the dynamics obtained by letting $\hbar \to 0$ for the initial value problem (1.1) and (1.2). However, it is not clear directly from (1.1a,b) what form such a dynamics might take. Insight into this question can be gained by considering the conservation laws associated with the JNLS equation. Therefore, this section is devoted to the hydrodynamical structure of the JNLS equation (1.1a,b). We write the complex-valued wave function

$$\psi(x,t) = A(x,t)\exp\left(\frac{\mathrm{i}}{\hbar}S(x,t)\right) \tag{2.1}$$

in terms of the phase and amplitude. This transformation is usually called Madelung's transformation and was originally introduced in the context of the linear Schrödinger equation for quantum mechanics. After substitution in the JNLS equation (1.1a,b) and separation of the real and imaginary parts of the equation, one obtains

$$A_t + \tfrac{1}{2}(AS_{xx} + 2A_xS_x) = 0, \tag{2.2}$$

$$S_t + \frac{1}{2}(S_x)^2 - \lambda S_x A^2 = \frac{\hbar^2}{2}\frac{A_{xx}}{A}. \tag{2.3}$$

In terms of new hydrodynamical variables, the fluid density ρ and the local velocity u, defined as

$$\rho = A^2 = |\psi|^2, \quad u = S_x \tag{2.4}$$

we get

$$\rho_t + (\rho u)_x = 0, \tag{2.5a}$$

$$u_t + u u_x - \lambda(\rho u)_x = \frac{\hbar^2}{2}\left(\frac{(\sqrt{\rho})_{xx}}{\sqrt{\rho}}\right)_x. \tag{2.5b}$$

This system describes the Madelung fluid and is a perturbation (by the quantum potential) of modified compressible Euler equations

$$\rho_t + (\rho u)_x = 0, \tag{2.6a}$$

$$u_t + \left(\frac{1}{2}u^2 - \lambda\rho u\right)_x = 0 \tag{2.6b}$$

while in terms of the velocity potential S given by (2.4), interpreted as the classical action, we obtain the Hamilton–Jacobi equation

$$S_t + \tfrac{1}{2}(S_x)^2 - \lambda S_x \rho = 0 \tag{2.7a}$$

supplied with the Liouville's one

$$\rho_t + (\rho S_x)_x = 0. \tag{2.7b}$$

The reader is cautioned that, even if not explicitly indicated, the solution of the above equations actually is a member of a family of solutions parametrized by $\hbar$ through its dependence on the initial data. The difference between the modified compressible Euler equations and the Eqs. (2.5a,b) lies in the quantum correction term of order $O(\hbar^2)$ on the right hand side of (2.5b). This density dependent term can be interpreted as internal self-potential

$$\frac{\hbar^2}{2}\left(\frac{(\sqrt{\rho})_{xx}}{\sqrt{\rho}}\right)_x \tag{2.8}$$

the so-called Bohm quantum potential [1,30].

Multiplying (2.5b) by ρ and using (2.5a) we find the current density (or canonical momentum) j satisfies

$$j_t + \left(\frac{|j|^2}{\rho}\right)_x - \lambda\rho j_x = \frac{\hbar^2}{4}\partial_x(\rho\partial_x^2 \log \rho), \tag{2.9}$$

which is not a local conservation law except when $\lambda = 0$. This is due to the lack of Galilean symmetry. On the other hand, from the continuity equation we also find

$$\left(\frac{\lambda}{2}\rho^2\right)_t + \lambda\rho j_x = 0. \tag{2.10}$$

Adding (2.9) and (2.10) together yields

$$\frac{\partial}{\partial t}\left(j + \frac{\lambda}{2}\rho^2\right) + \frac{\partial}{\partial x}\left(\frac{|j|^2}{\rho}\right) = \frac{\hbar^2}{4}\frac{\partial}{\partial x}(\rho\partial_x^2 \log \rho). \tag{2.11}$$

Denoting the density of the noncanonical momentum M by

$$M \equiv j + \frac{\lambda}{2}\rho^2 \tag{2.12}$$

the system (2.5a,b) can be rewritten in terms of ρ and M as

$$\frac{\partial}{\partial t}\rho + \frac{\partial}{\partial x}\left(M - \frac{\lambda}{2}\rho^2\right) = 0, \tag{2.13a}$$

$$\frac{\partial}{\partial t}M + \frac{\partial}{\partial x}\left(\frac{M^2}{\rho} - \lambda\rho M + \frac{\lambda^2}{4}\rho^3\right) = \frac{\hbar^2}{4}\frac{\partial}{\partial x}(\rho\partial_x^2 \log \rho). \tag{2.13b}$$

The local conservation laws (2.13a,b) comprise a closed system governing ρ and M which have the form of a perturbation of the modified compressible Euler equations. Similar to the barotropic compressible Euler equation we rewrite (2.13b) as

$$\frac{\partial}{\partial t}M + \frac{\partial}{\partial x}\left(\frac{M^2}{\rho}\right) - \frac{\partial}{\partial x}(\lambda\rho M) + \frac{\partial}{\partial x}P(\rho) = \frac{\hbar^2}{4}\frac{\partial}{\partial x}(\rho\partial_x^2 \log \rho), \tag{2.13b$'$}$$

where $P(\rho)$ is the pressure which is related to the potential energy $\Phi(\rho) = (\lambda^2/8)\rho^3$ by

$$P(\rho) \equiv \rho\Phi'(\rho) - \Phi(\rho) = \frac{\lambda^2}{4}\rho^3. \tag{2.14}$$

Thus, the third term on the left hand side of (2.13b$'$), $-\lambda\rho M$, can be interpreted as the noncanonical pressure created by the background fluid.

The hydrodynamical structure also implies the conservation laws of the JNLS equation (1.1a,b).

Theorem 2.1. *Let bar denote the complex conjugate. The following quantities are conservation integrals of the JNLS equation* (1.1a,b);

$$\int_{-\infty}^{\infty} \rho\,\mathrm{d}x = \text{constant} = C_1, \tag{2.15a}$$

$$\int_{-\infty}^{\infty} u\,\mathrm{d}x = \text{constant} = C_2, \tag{2.15b}$$

$$\int_{-\infty}^{\infty} M\,\mathrm{d}x = \text{constant} = C_3, \tag{2.15c}$$

$$\int_{-\infty}^{\infty} E\,\mathrm{d}x = \text{constant} = C_4, \tag{2.15d}$$

where the hydrodynamic variables ρ, u, M and E are given in terms of the wave function ψ as follows;

$$\rho \equiv |\psi|^2 = |A|^2, \tag{2.16a}$$

$$u \equiv S_x = \frac{j}{\rho} = \frac{\hbar}{2i}\left(\frac{\psi_x}{\psi} - \frac{\bar{\psi}_x}{\bar{\psi}}\right), \tag{2.16b}$$

$$M \equiv j + \frac{\lambda}{2}\rho^2 = \frac{\hbar}{2i}(\bar{\psi}\psi_x - \psi\bar{\psi}_x) + \frac{\lambda}{2}|\psi|^4, \tag{2.16c}$$

$$E \equiv \left(\frac{M^2}{2\rho} + \frac{\lambda^2}{8}\rho^3\right) + \frac{\lambda}{2}\rho M + \frac{\hbar^2}{4}\cdot\frac{\rho_x^2}{2\rho} = \frac{\hbar^2}{2}|\psi_x|^2 + \lambda|\psi|^2\frac{\hbar}{2i}(\bar{\psi}\psi_x - \psi\bar{\psi}_x) + \frac{\lambda^2}{2}|\psi|^6. \tag{2.16d}$$

Proof. We only need to prove (2.15d) and (2.16d). The energy is decomposed into classical, noncanonical and quantum parts respectively. Each part propagates according to

$$\frac{\partial}{\partial t}\left(\frac{M^2}{2\rho} + \frac{\lambda^2}{8}\rho^3\right) + \frac{\partial}{\partial x}\left[\left(\frac{M^2}{2\rho} + \frac{\lambda^2}{8}\rho^3 - \frac{\lambda}{2}\rho M + \frac{\lambda^2}{4}\rho^3\right)\cdot\frac{M}{\rho}\right] + \frac{\lambda}{2}\frac{M^2}{\rho}\rho_x - \frac{3\lambda^3}{8}\rho^3\rho_x = \frac{\hbar^2}{4}\frac{M}{\rho}\frac{\partial}{\partial x}(\rho\partial_x^2 \log \rho), \tag{2.17a}$$

$$\frac{\partial}{\partial t}(\rho M) + \frac{\partial}{\partial x}\left[\left(\rho M + \frac{1}{2}\rho M - \lambda\rho^3\right)\cdot\frac{M}{\rho}\right] - \frac{M^2}{\rho}\rho_x + \frac{3\lambda^2}{4}\rho^3\rho_x = \frac{\hbar^2}{4}\rho\frac{\partial}{\partial x}(\rho\partial_x^2 \log \rho) \tag{2.17b}$$

and

$$\frac{\partial}{\partial t}\left(\frac{\rho_x^2}{2\rho}\right)+\frac{\partial}{\partial x}\left(\frac{\rho_x^2}{2\rho}\cdot\frac{M}{\rho}\right)+\frac{\lambda}{2}\frac{\partial}{\partial x}(2\rho_x^2-\rho\rho_{xx})=\frac{\partial}{\partial x}\left(\frac{\rho_{xx}M}{\rho}-\frac{\rho_xM_x}{\rho}\right)-\frac{M}{\rho}\frac{\partial}{\partial x}(\rho\partial_x^2\log\rho)-\frac{\lambda}{2}\rho\frac{\partial}{\partial x}(\rho\partial_x^2\log\rho)$$

(2.17c)

respectively. Therefore the local conservation law of energy is derived by $(2.17a)+(\lambda/2)\,(2.17b)+(\hbar^2/4)\,(2.17c)$;

$$\frac{\partial}{\partial t}E+\frac{\partial}{\partial x}\left(\left(\frac{M^2}{2\rho}+\frac{\lambda}{4}\rho M-\frac{\lambda^2}{8}\rho^3+\frac{\hbar^2}{4}\frac{\rho_x}{2\rho}\right)\cdot\frac{M}{\rho}\right)+\frac{\lambda\hbar^2}{8}\frac{\partial}{\partial x}(2\rho_x^2-\rho\rho_{xx})=\frac{\hbar^2}{4}\frac{\partial}{\partial x}\left(\frac{\rho_{xx}M}{\rho}-\frac{\rho_xM_x}{\rho}\right)$$

or

$$\frac{\partial}{\partial t}E+\frac{\partial}{\partial x}\left(\left(E-P(\rho)-\frac{\lambda}{4}\rho M\right)\cdot\frac{M}{\rho}\right)+\frac{\lambda\hbar^2}{8}\frac{\partial}{\partial x}(2\rho_x^2-\rho\rho_{xx})=\frac{\hbar^2}{4}\frac{\partial}{\partial x}\left(\frac{\rho_{xx}M}{\rho}-\frac{\rho_xM_x}{\rho}\right)$$

(2.18)

where the total energy E is given by (2.16d). Therefore, when fields decrease rapidly at spatial infinity, the energy is time-independent. This completes the proof. $\square$

The conservative quantities of the JNLS equation (1.1a,b) may be recast from the action principle from which it is easier to identify the Hamiltonian (energy), momentum and other constants of motion. In contrast to the NLS equation, JNLS equation (1.1a,b) does not possess a local Lagrangian formulation directly in terms of the field ψ. Consider the action

$$\mathscr{S}=\int\int L\,\mathrm{d}x\,\mathrm{d}t=\int\int\left[\mathrm{i}\hbar\bar{\Psi}\Psi_t-\frac{\hbar^2}{2}\left|\left(\frac{\partial}{\partial x}-\mathrm{i}\frac{\lambda}{2\hbar}\rho\right)\Psi\right|^2\right]\mathrm{d}x\,\mathrm{d}t$$

(2.19)

the Euler–Lagrange equation that follows can be easily shown to be

$$\mathrm{i}\hbar\Psi_t=-\frac{\hbar^2}{2}\left(\frac{\partial}{\partial x}-\mathrm{i}\frac{\lambda}{2\hbar}\rho\right)^2\Psi-\frac{\lambda}{2}J\Psi=-\frac{\hbar^2}{2}\Psi_{xx}+\mathrm{i}\lambda\hbar\rho\Psi_x+\frac{3\lambda^2}{8}\rho^2\Psi.$$

(2.20)

In Eqs. (2.19) and (2.20), ρ represents the density $\bar{\Psi}\Psi$, while

$$J=\frac{\hbar}{2\mathrm{i}}\left[\bar{\Psi}\left(\frac{\partial}{\partial x}-\mathrm{i}\frac{\lambda}{2\hbar}\rho\right)\Psi-\Psi\left(\frac{\partial}{\partial x}+\mathrm{i}\frac{\lambda}{2\hbar}\rho\right)\bar{\Psi}\right]$$

(2.21)

is the corresponding current and the two are linked by the continuity equation

$$\frac{\partial\rho}{\partial t}+\frac{\partial J}{\partial x}=0.$$

(2.22)

Next, we redefine the gauge transformation

$$\Psi(x,t)=\psi(x,t)\exp\left(\mathrm{i}\frac{\lambda}{2\hbar}\int_{-\infty}^x|\psi(y,t)|^2\,\mathrm{d}y\right)$$

(2.23)

which bring Eq. (2.20) into a different, yet equivalent expression

$$\mathrm{i}\hbar\psi_t-\frac{\lambda}{2}\left(\int_{-\infty}^x\partial_t|\psi(y,t)|^2\,\mathrm{d}y\right)\psi=-\frac{\hbar^2}{2}\psi_{xx}-\lambda j\psi.$$

(2.24)

Applying the continuity equation (2.22) and using the identity, $\rho=|\Psi|^2=|\psi|^2$, the resulting equation is just (1.1a,b).

The invariance of the action $\mathscr{S}$ under space/time translations reflects itself into the presence of the conservative quantities;

$$M=\hbar\mathrm{Im}(\bar{\Psi}\Psi_x)=\frac{\hbar}{2\mathrm{i}}(\bar{\Psi}\Psi_x-\bar{\Psi}_x\Psi)=j+\frac{\lambda}{2}\rho^2,$$

(2.25a)

$$E=\frac{\hbar^2}{2}\left|\left(\frac{\partial}{\partial x}-\mathrm{i}\frac{\lambda}{2\hbar}\rho\right)\Psi\right|^2=\frac{\hbar^2}{2}|\psi_x|^2+\lambda|\psi|^2\frac{\hbar}{2\mathrm{i}}(\bar{\psi}\psi_x-\psi\bar{\psi}_x)+\frac{\lambda^2}{2}|\psi|^6$$

(2.25b)

the momentum and energy respectively (comparing with (2.16c,d)). The associated momentum and energy fluxes are given respectively by

$$\widetilde{M} = \hbar^2 \left| \left(\frac{\partial}{\partial x} - \mathrm{i}\frac{\lambda}{2\hbar}\rho \right) \Psi \right|^2 - \frac{\hbar^2}{4}\frac{\partial^2}{\partial x^2}\rho, \tag{2.26a}$$

$$\widetilde{E} = -\frac{\hbar^2}{2}\left[\bar{\Psi}_t \left(\frac{\partial}{\partial x} - \mathrm{i}\frac{\lambda}{2\hbar}\rho \right)\Psi + \Psi_t \left(\frac{\partial}{\partial x} + \mathrm{i}\frac{\lambda}{2\hbar}\rho \right)\bar{\Psi} \right] \tag{2.26b}$$

and they verify the continuity equations

$$\frac{\partial}{\partial t}M + \frac{\partial}{\partial x}\widetilde{M} = 0, \quad \frac{\partial}{\partial t}E + \frac{\partial}{\partial x}\widetilde{E} = 0 \tag{2.27}$$

which are equivalent to (2.13b) and (2.18) respectively. Notice that Eq. (2.12) or (2.25a) shows that M possesses the dynamical contribution $(\lambda/2)\rho^2$ in addition to the usual kinematical term j; therefore JNLS (1.1a,b) is not Galileo-invariant.

Furthermore, we can reformulate the JNLS equation (1.1a,b) as a *linear* dispersive perturbation of a symmetric hyperbolic system with the help of the modified Madelung's transformation. Indeed, we will look for solutions $\psi^\hbar$ of the form

$$\psi^\hbar(x,t) = A^\hbar(x,t)\exp\left(\frac{\mathrm{i}}{\hbar}S^\hbar(x,t) \right), \tag{2.28}$$

where the complex-value function $A^\hbar = a^\hbar + \mathrm{i}b^\hbar$ represents the amplitude and the real-valued $S^\hbar$ represents the phase. Unlike the usual WKB method to look for solution of the form $\psi^\hbar(x,t) = A^\hbar(x,t)\exp\left(\frac{\mathrm{i}}{\hbar}S(x,t) \right)$, where S is independent of $\hbar$, we allow $S^\hbar$ to depend on $\hbar$. Now insert (2.28) in JNLS equation (1.1a,b), we obtain

$$\mathrm{i}\hbar A_t^\hbar - A^\hbar S_t^\hbar + \frac{\hbar^2}{2}A_{xx}^\hbar - \frac{1}{2}A^\hbar(S_x^\hbar)^2 + \frac{\mathrm{i}\hbar}{2}(A^\hbar S_{xx}^\hbar + 2A_x^\hbar S_x^\hbar) + \lambda|A^\hbar|^2 A^\hbar S_x^\hbar = 0$$

then we split into

$$S_t^\hbar + \frac{1}{2}(S_x^\hbar)^2 - \lambda|A^\hbar|^2 S_x^\hbar = 0,$$
$$\mathrm{i}A_t^\hbar + \frac{\mathrm{i}}{2}(A^\hbar S_{xx}^\hbar + 2A_x^\hbar S_x^\hbar) + \frac{\hbar}{2}A_{xx}^\hbar = 0. \tag{2.29}$$

Considering the change of variable $w^\hbar = S_x^\hbar$ and using the fact that $A^\hbar = a^\hbar + \mathrm{i}b^\hbar$ we have the equivalent form;

$$a_t^\hbar + w^\hbar a_x^\hbar + \frac{1}{2}a^\hbar w_x^\hbar = -\frac{\hbar}{2}b_{xx}^\hbar, \tag{2.30a}$$

$$b_t^\hbar + w^\hbar b_x^\hbar + \frac{1}{2}b^\hbar w_x^\hbar = \frac{\hbar}{2}a_{xx}^\hbar, \tag{2.30b}$$

$$w_t^\hbar - 2\lambda a^\hbar w^\hbar a_x^\hbar - 2\lambda b^\hbar w^\hbar b_x^\hbar + (w^\hbar - \lambda((a^\hbar)^2 + (b^\hbar)^2))w_x^\hbar = 0 \tag{2.30c}$$

with initial data

$$a^\hbar(x,0) = a_0^\hbar(x), \quad b^\hbar(x,0) = b_0^\hbar(x), \quad w^\hbar(x,0) = w_0^\hbar(x) \tag{2.30d}$$

satisfying

$$(a_0^\hbar(x))^2 + (b_0^\hbar(x))^2 = |A_0^\hbar(x)|^2, \quad w_0^\hbar(x) = \frac{\mathrm{d}}{\mathrm{d}x}S_0^\hbar(x). \tag{2.30e}$$

This system can be written in the vector form

$$U_t^\hbar + \mathscr{A}(U^\hbar)U_x^\hbar = \frac{\hbar}{2}\mathscr{L}(U^\hbar), \quad U^\hbar = (a^\hbar, b^\hbar, w^\hbar)^t, \quad U^\hbar(0,x) = U_0^\hbar(x) = (a_0^\hbar(x), b_0^\hbar(x), w_0^\hbar(x))^t, \tag{2.31}$$

where

$$\mathscr{A}(U^\hbar) \equiv \begin{pmatrix} w^\hbar & 0 & \frac{1}{2}a^\hbar \\ 0 & w^\hbar & \frac{1}{2}b^\hbar \\ -2\lambda a^\hbar w^\hbar & -2\lambda b^\hbar w^\hbar & w^\hbar - \lambda((a^\hbar)^2 + (b^\hbar)^2) \end{pmatrix} \tag{2.32}$$

and

$$\mathscr{L}(U^\hbar) = \begin{pmatrix} 0 & -\partial_x^2 & 0 \\ \partial_x^2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} a^\hbar \\ b^\hbar \\ w^\hbar \end{pmatrix} = \begin{pmatrix} -b_{xx}^\hbar \\ a_{xx}^\hbar \\ 0 \end{pmatrix} \tag{2.33}$$

is an antisymmetric matrix.

In addition to the linear dispersive perturbation of the quasilinear symmetric hyperbolic system nature, the modified Madelung transformation also give us more information about the phase transportation. Since $A^\hbar$ is complex-valued, we introduce the polar coordinates:

$$A^\hbar = a^\hbar + \mathrm{i}b^\hbar = \sqrt{\rho^\hbar}\mathrm{e}^{\mathrm{i}\vartheta^\hbar}. \tag{2.34}$$

Apply the chain rule to obtain

$$a^\hbar b_{xx}^\hbar - a_{xx}^\hbar b^\hbar = \partial_x(\rho^\hbar \vartheta_x^\hbar) \tag{2.35}$$

then from (2.30a–c) we derive the system

$$\rho_t^\hbar + (\rho^\hbar w^\hbar + \hbar \rho^\hbar \vartheta_x^\hbar)_x = 0, \tag{2.36a}$$

$$\vartheta_t^\hbar + w^\hbar \vartheta_x^\hbar + \frac{\hbar}{2}|\vartheta_x^\hbar|^2 = \frac{\hbar}{2}\frac{(\sqrt{\rho^\hbar})_{xx}}{\sqrt{\rho^\hbar}}, \tag{2.36b}$$

$$w_t^\hbar + w^\hbar w_x^\hbar - \lambda(\rho^\hbar w^\hbar)_x = 0. \tag{2.36c}$$

The quantum effect in this system is of order $\mathrm{O}(\hbar)$ not $\mathrm{O}(\hbar^2)$ as (2.5b). Note the transport equation for $\rho^\hbar$ has an extra term of order $\mathrm{O}(\hbar)$ comparing with the typical equation of continuity. Formally letting $\hbar \to 0$, we have

$$\rho_t + (\rho w)_x = 0, \tag{2.37a}$$

$$\theta_t + w\theta_x = 0, \tag{2.37b}$$

$$w_t + ww_x - \lambda(\rho w)_x = 0. \tag{2.37c}$$

Since (2.37b) is the pure transport equation then $\theta = 0$ for the trivial initial data, thus we have the same limit system as (2.5a,b).

3. Semiclassical limit, dispersionless wave equation and its deformation

Consider the family, parameterized by $\hbar > 0$, of solutions $\psi^\hbar(x,t)$ to the JNLS equation

$$\mathrm{i}\hbar\psi_t^\hbar + \frac{\hbar^2}{2}\psi_{xx}^\hbar + \lambda j^\hbar(x,t)\psi^\hbar = 0 \tag{3.1a}$$

with current nonlinearity

$$j^\hbar(x,t) = \frac{\hbar}{2\mathrm{i}}(\bar{\psi}^\hbar\psi_x^\hbar - \psi^\hbar\bar{\psi}_x^\hbar) \tag{3.1b}$$

with rapidly oscillating initial data

$$\psi^\hbar(x,0) = \psi_0^\hbar(x) = A_0(x)\exp\left(\frac{\mathrm{i}}{\hbar}S_0(x)\right), \tag{3.1c}$$

where the (nonnegative) amplitude $A_0(x)$ and (real) phase $S_0(x)$ are assumed to be smooth and independent of $\hbar$. The initial conserved densities are then

$$\rho^\hbar(x,0) = |A_0(x)|^2,$$
$$M^\hbar(x,0) = |A_0(x)|^2\partial_x S_0(x) + \frac{\lambda}{2}|A_0(x)|^4. \tag{3.2}$$

The semiclassical limit is then to determine the limiting behavior of any function of the field $\psi^\hbar$ as $\hbar \to 0$. Arguing formally, it is natural to conjecture that the $O(\hbar^2)$ dispersive term appearing in (2.13a,b) is negligible as $\hbar \to 0$, and the limiting densities ρ and M satisfy the modified compressible Euler system

$$\frac{\partial}{\partial t}\rho + \frac{\partial}{\partial x}\left(M - \frac{\lambda}{2}\rho^2\right) = 0, \tag{3.3a}$$

$$\frac{\partial}{\partial t}M + \frac{\partial}{\partial x}\left(\frac{M^2}{\rho} - \lambda\rho M + \frac{\lambda^2}{4}\rho^3\right) = 0 \tag{3.3b}$$

with initial condition inferred from (3.2) given by

$$\rho(x,0) = |A_0(x)|^2,$$
$$M(x,0) = |A_0(x)|^2 \partial_x S_0(x) + \frac{\lambda}{2}|A_0(x)|^4. \tag{3.3c}$$

In this case the limiting energy density will be given by

$$E = \left(\frac{M^2}{2\rho} + \frac{\lambda^2}{8}\rho^3\right) + \frac{\lambda}{2}\rho M \tag{3.4}$$

and satisfies

$$\frac{\partial}{\partial t}E + \frac{\partial}{\partial x}\left(\left(\frac{M^2}{2\rho} + \frac{\lambda}{4}\rho M - \frac{\lambda^2}{8}\rho^3\right) \cdot \frac{M}{\rho}\right) = 0. \tag{3.5}$$

This argument is self-consistent only if the solution of the modified Euler system (3.3a,b) remains classical. Equivalently, we can employ (2.30a–c) or (2.31,33) to discuss the semiclassical limit of the JNLS equation (1.1a,b). First, the matrix $\mathscr{A}(U^\hbar)$ can be symmetrized by

$$\mathscr{S}(U^\hbar) = \begin{pmatrix} -4\lambda w^\hbar & 0 & 0 \\ 0 & -4\lambda w^\hbar & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{3.6}$$

which is symmetric and positive if $-\lambda w^\hbar > 0$, for all $U^\hbar$. Thus, we write (1.1a,b) as a dispersive perturbation of a quasilinear symmetric hyperbolic system:

$$\mathscr{S}(U)U_t + \widetilde{\mathscr{A}}(U)U_x = \frac{\hbar}{2}\widetilde{\mathscr{L}}(U), \tag{3.7}$$

where $\widetilde{\mathscr{A}} = \mathscr{S}\mathscr{A}$ is symmetric (we omit $\hbar$ for convenience). The importance of symmetry is that it leads to simple L^2 and more generally H^s estimates which are often related to physical quantities like energy or entropy. The antisymmetric operator $(\hbar/2)\widetilde{\mathscr{L}} = (\hbar/2)\mathscr{S}\mathscr{L}$ reflects the dispersive nature of the equations. Moreover, the classical energy estimate shows that this term contributes nothing to the estimate, i.e., the singular perturbation does not create energy. Therefore, the existence of the classical solutions and its semiclassical limit proceed along the lines of the classical theory of quasilinear symmetric hyperbolic systems (with some modifications). First, we have the existence and uniqueness of the classical solution of the dispersive perturbation of the quasilinear symmetric system (2.30a–d).

Theorem 3.1. *Assume the initial data $U_0^\hbar = (a_0^\hbar, b_0^\hbar, w_0^\hbar)^t \in H^s \times H^s \times H^s, s > (1/2) + 2$ satisfies the uniform bound*

$$\|U_0^\hbar\|_{H^s} = \|a_0^\hbar\|_{H^s} + \|b_0^\hbar\|_{H^s} + \|w_0^\hbar\|_{H^s} < C_1 \tag{3.8}$$

and

$$-\lambda w^\hbar(x,t) > 0, \quad (x,t) \in \mathbf{R} \times [0,\infty). \tag{3.9}$$

Then there is a time interval $[0,T]$ with $T > 0$, so that the quasilinear symmetric hyperbolic system (2.30) or (2.31) has a unique classical solution $U^\hbar = (a^\hbar, b^\hbar, w^\hbar)^t$;

$$(a^\hbar(x,t), b^\hbar(x,t)) \in (C^1(\mathbf{R} \times [0,T]) \wedge C^1([0,T]; C^2(\mathbf{R})))^2$$
$$w^\hbar(x,t) \in C^1(\mathbf{R} \times [0,T]). \tag{3.10}$$

Furthermore,

$$U^\hbar \in C([0,T]; H^s) \wedge C^1([0,T]; H^{s-2}) \tag{3.11}$$

and T depends on the bound C_1 in (3.8) and in particular, not on $\hbar$. The solution $U^\hbar = (a^\hbar, b^\hbar, w^\hbar)^t$ satisfies the estimate

$$\|U^\hbar\|_{H^s} = \|a^\hbar\|_{H^s} + \|b^\hbar\|_{H^s} + \|w^\hbar\|_{H^s} < C_2 \tag{3.12}$$

for all $t \in [0,T]$. The constant C_2 is also independent of $\hbar$. In addition, if $\rho_0^\hbar(x) = (a_0^\hbar)^2 + (b_0^\hbar)^2 > 0$ then $\rho^\hbar(x,t) > 0$ for all $t \geqslant 0$; if $\rho_0^\hbar$ has a compact support, then $\rho^\hbar(\cdot, t)$ does too for any $t \in [0,T]$ and

$$R\{\rho^\hbar(t, \cdot)\} \leqslant R\{\rho_0^\hbar\} + (1 + \hbar)C_2 T,$$

where $R\{u\} \equiv \sup\{|x| : u(x) \neq 0\}$ for $u \in C$.

Proof. To show that $\rho^\hbar(x,t) = (a^\hbar(x,t))^2 + (b^\hbar(x,t))^2 > 0$ for all $0 \leqslant t < \infty$, we will employ the polar coordinates; $A^\hbar = a^\hbar + ib^\hbar = \sqrt{\rho^\hbar}e^{i\vartheta^\hbar}$. Applying the chain rule to obtain

$$a^\hbar b_{xx}^\hbar - b^\hbar a_{xx}^\hbar = (\rho^\hbar \vartheta_x^\hbar)_x$$

then from (2.30a,b) we derive the continuity equation for $\rho^\hbar$

$$\frac{\partial}{\partial t}\rho^\hbar + \frac{\partial}{\partial x}(\rho^\hbar w^\hbar + \hbar \rho^\hbar \vartheta_x^\hbar) = 0$$

which has an extra term of order $O(\hbar)$ comparing with the usual continuity equation. Let (ξ, τ) be an arbitrary fixed space–time point in $\Omega \times [0,T]$. Since $w^\hbar + \hbar \vartheta_x^\hbar \in C^1(\mathbf{R} \times [0,T])$, the well-known theorem for ordinary differential equations guarantees that the problem

$$\frac{dx}{dt} = w^\hbar(x,t) + \hbar \vartheta_x^\hbar(x,t), \quad x|_{t=\tau} = \xi$$

has a unique solution $x = \Psi(t) \in C^1([0,T]; \mathbf{R})$. The continuity equation implies

$$\frac{d}{dt}\rho^\hbar(\Psi(t), t) = -\frac{\partial}{\partial x}(w^\hbar + \hbar \vartheta_x^\hbar)\rho^\hbar.$$

Integrating over $[0, \tau]$ we have

$$\rho^\hbar(\xi, \tau) = \rho^\hbar(\Psi(0), 0) \exp\left[-\int_0^\tau \frac{\partial}{\partial x}(w^\hbar(\Psi(t), t) + \hbar \theta_x^\hbar(\Psi(t), t)) \, dt \right].$$

Thus $\rho^\hbar(\xi, \tau) \geqslant 0$ if $\rho^\hbar(\Psi(0), 0) = \rho_0^\hbar(\Psi(0)) \geqslant 0$. If $\rho^\hbar(\xi, \tau) \neq 0$ then $\rho_0^\hbar(\Psi(0)) \neq 0$ so that $|\Psi(0)| \leqslant R\{\rho_0^\hbar\}$, and

$$|\xi| = |\Psi(\tau)| = \left| \Psi(0) + \int_0^\tau w^\hbar(\Psi(t), t) + \hbar \nabla \theta^\hbar(\Psi(t), t) \, dt \right|$$

$$\leqslant |\Psi(0)| + \int_0^\tau |w^\hbar|_\infty + \hbar |\nabla \theta^\hbar|_\infty \, dt \leqslant R\{\rho_0^\hbar\} + (1 + \hbar)C_2 \tau. \quad \square$$

Theorem 3.2. *Under the same assumption of Theorem 3.1. In addition, suppose $(A_0^\hbar, S_0^\hbar) \in H^s \times H^{s+1}$. The initial value problem of the JNLS equation (3.1a–c) has a unique classical solution in $C^1([0,T] \times \Omega) \wedge C^1([0,T]; C^2)$ of the form $\psi^\hbar(x,t) = A^\hbar(x,t) \exp\left(\frac{i}{\hbar}S^\hbar(x,t)\right)$ on the time interval $[0,T]$. Moreover, $A^\hbar$ and $S_x^\hbar$ are bounded in $L^\infty([0,T]; H^s)$.*

Proof. Since $A^\hbar = a^\hbar + ib^\hbar$ and $w^\hbar = S_x^\hbar$ it follows from (3.10)–(3.12) that

$$A^\hbar \in C([0,T]; H^s) \wedge C^1([0,T]; H^{s-2}),$$
$$S^\hbar \in C([0,T]; H^{s+1}) \wedge C^1([0,T]; H^s)$$

and thus

$$A^\hbar \in C^1(\Omega \times [0,T]) \wedge C^1([0,T]; C^2),$$
$$S^\hbar \in C^1([0,T]; C^2)$$

by Sobolev embedding theorem. Due to the expression of $\psi^\hbar$ in the short wave form (2.28), $\psi^\hbar(x,t) = A^\hbar(x,t)\exp\left(\frac{i}{\hbar}S^\hbar(x,t)\right)$, $\psi^\hbar$ has the same regularity as $A^\hbar$ thus

$$\psi^\hbar \in C([0,T];H^s) \wedge C^1([0,T];H^{s-2})$$

hence

$$\psi^\hbar \in C^1([0,T] \times \Omega) \wedge C^1([0,T];C^2).$$

For classical solution, the JNLS equation (1.1a,b) is equivalent to the dispersive quasilinear hyperbolic system (2.30a–c) or (2.31)–(2.33). Applying this equivalent relation the theorem follows immediately by Theorem 3.1. □

The limiting system of (2.30) or (2.31) is the quasilinear symmetric hyperbolic system (formally letting $\hbar \to 0$)

$$U_t + \mathscr{A}(U)U_x = 0, \quad U(x,t) = (a,b,w)^t, \tag{3.13a}$$

$$U(x,0) = U_0(x) = (a_0(x), b_0(x), w_0(x))^t, \tag{3.13b}$$

which is equivalent to (3.3a–c) as long as the solutions are smooth. It is possible to pass to the limit $\hbar \to 0$ in (2.30) or (2.31). Indeed, by the classical compactness arguments, Arzela–Ascoli theorem (applied in the time variable), the Rellich lemma (applied in space variable) and using the fact that $\mathscr{L}(U^\hbar)$ is uniformly bounded in H^s we have the following corollary.

Corollary 3.3. *Given $U_0^\hbar, U_0 \in H^s \times H^s \times H^s$, $s > \frac{1}{2} + 2$ and $U_0^\hbar(x)$ converges $U_0(x)$ in H^s as $\hbar$ tends to 0. Let $[0,T]$ be the fixed time interval determined in Theorem 3.1. Then as $\hbar \to 0$ there exists $U(t,x) \in L^\infty([0,T];H^s)$ so that*

$$U^\hbar(x,t) \to U(x,t), \quad \text{in } C([0,T];H^{s-\epsilon}), \ \forall \epsilon > 0.$$

The function $U(x,t)$ belongs to $C([0,T];H^s) \wedge C^1([0,T];H^{s-1})$ and is a classical solution of (3.13a,b) with initial data $U(x,0) = U_0(x)$.

Concerning the Cauchy problem at finite time, we also give sufficient conditions for the well-posedness in Sobolev space H^s, $s \geqslant 3$. Indeed, to ensure the strong convergence of $\psi^\hbar$ to a classical solution of the modified Euler system (3.3a–c) we require the hypothesis that we are near the JNLS equation (1.1a,b) initially.

Theorem 3.4. *Let (ρ, M) be a solution of the quasilinear hyperbolic system (3.3a–c) for $0 \leqslant t \leqslant T$. Then there exists a critical value of $\hbar$, $\hbar_c$, and a constant $C > 0$ such that under the assumption*

(1) *$A_0^\hbar(x)$ converges strongly to A_0 in H^s as $\hbar$ tends to 0,*
(2) *$\|\rho_0\|_{H^s} < \infty$, $\|M_0\|_{H^s} < \infty$, $s \geqslant 3$,*
(3) *$0 < \hbar < \hbar_c$,*

the IVP for the Eq. (3.1a–c) has a unique classical solution of the form $\psi^\hbar(x,t) = A^\hbar(x,t)\exp\left(\frac{i}{\hbar}S^\hbar(x,t)\right)$ on $[0,T]$. Moreover, $A^\hbar$ and $S_x^\hbar$ are bounded in $L^\infty([0,T];H^s)$ uniformly in $\hbar$.

Proof. We consider the difference of (2.31) and (3.13). Setting $V^\hbar = U^\hbar - U$ then we have

$$V_t^\hbar + \mathscr{A}(U + V^\hbar)V_x^\hbar = F^\hbar, \tag{3.14}$$

where

$$F^\hbar = \frac{\hbar}{2}(\mathscr{L}(V^\hbar) + \mathscr{L}(U)) - (\mathscr{A}(U + V^\hbar) - \mathscr{A}(U))U_x. \tag{3.15}$$

Once the symmetrizer $\mathscr{S}(U + V^\hbar)$ is positive definite for all $\hbar$, the energy estimates of the quasilinear hyperbolic theory is applicable to (3.14) and (3.15). The matrix $\mathscr{A}(U + V^\hbar)$ is symmetrizable. The energy associated with (3.14) is

$$\|V^\hbar(t)\|_{E^\hbar}^2 \equiv \int \langle \mathscr{S}(U + V^\hbar)V^\hbar, V^\hbar \rangle \, dx \tag{3.16}$$

and the associated energy equality is

$$\frac{\mathrm{d}}{\mathrm{d}t}\|V^{\hbar}(t)\|_{E^{\hbar}}^2 = \int \langle \Gamma^{\hbar} V^{\hbar}, V^{\hbar}\rangle \,\mathrm{d}x + 2\int \langle \mathscr{S}(U+V^{\hbar})F^{\hbar}, V^{\hbar}\rangle \,\mathrm{d}x, \tag{3.17}$$

where

$$\Gamma^{\hbar} = \frac{\partial}{\partial t}(\mathscr{S}(U+V^{\hbar})) + \frac{\partial}{\partial x}(\mathscr{S}(U+V^{\hbar})\mathscr{A}(U+V^{\hbar})). \tag{3.18}$$

We will estimate separately. First from the antisymmetry of $\mathscr{L}$ we have

$$\frac{\hbar}{2}\int \langle \mathscr{S}(U+V^{\hbar})\mathscr{L}(V^{\hbar}), V^{\hbar}\rangle \,\mathrm{d}x = 0.$$

The Cauchy–Schwarz inequality implies

$$\frac{\hbar}{2}\int \langle \mathscr{S}(U+V^{\hbar})\mathscr{L}(U), V^{\hbar}\rangle \,\mathrm{d}x \leqslant \hbar C\|U\|_{H^2}\|V^{\hbar}\|_{L^2},$$

$$\int \langle \mathscr{S}(U+V^{\hbar})(\mathscr{A}(U+V^{\hbar}) - \mathscr{A}(U))U_x, V^{\hbar}\rangle \,\mathrm{d}x \leqslant C\|V^{\hbar}\|_{L^2}^2.$$

By applying Gronwall inequality and the strict positivity of $\mathscr{S}(U+V^{\hbar})$, we deduce the energy inequality

$$\|V^{\hbar}\|_{L^2} \leqslant C(\hbar)$$

with $C(\hbar) \to 0$ as $\hbar \to 0$. This completes the proof of the theorem. $\square$

Next, the bounds uniform in $\hbar$ on the solution $\psi^{\hbar}$ also allow us to justify the WKB expansion on the same time interval $[0, T]$. We look for formally asymptotic solutions of (2.31) in the form

$$U^{\hbar} = U^{(0)} + \hbar U^{(1)} + \hbar^2 U^{(2)} + \cdots + \hbar^N U^{(N)} + \cdots \tag{3.19}$$

The hierarchy now reads

$$U_t^{(0)} + \mathscr{A}(U^{(0)})U_x^{(0)} = 0, \tag{3.20a}$$

$$U_t^{(1)} + \mathscr{A}(U^{(0)})U_x^{(1)} + \mathscr{A}'(U^{(0)})U_x^{(0)} = \tfrac{1}{2}\mathscr{L}(U^{(0)}), \tag{3.20b}$$

$$U_t^{(2)} + \mathscr{A}(U^{(0)})U_x^{(2)} + \mathscr{A}'(U^{(0)})U_x^{(1)} + \mathscr{A}''(U^{(0)})U_x^{(0)} = \tfrac{1}{2}\mathscr{L}(U^{(1)}), \tag{3.20c}$$

$$U_t^{(N)} + \mathscr{A}(U^{(0)})U_x^{(N)} + \mathscr{A}'(U^{(0)})U_x^{(N-1)} + \cdots + \mathscr{A}^{(N)}(U^{(0)})U_x^{(0)} = \tfrac{1}{2}\mathscr{L}(U^{(N-1)}). \tag{3.20d}$$

We consider the zeroth order first. It follows from the Theorem 3.1 that $(A^{\hbar}, w^{\hbar})$ is bounded in $C([0, T]; H^s) \wedge C^1([0, T]; H^{s-2})$ thus, applying the Arzela–Ascoli theorem, there exists $(A^{(0)}, w^{(0)})$ such that

$$(A^{\hbar}, w^{\hbar}) \to (A^{(0)}, w^{(0)}) \quad \text{in } C([0, T]; H^{s-\epsilon}), \ \forall 0 < \epsilon < 2$$

and satisfies the quasilinear hyperbolic system

$$A_t^{(0)} + w^{(0)}A_x^{(0)} + \tfrac{1}{2}A^{(0)}w_x^{(0)} = 0, \tag{3.21a}$$

$$w_t^{(0)} + w^{(0)}w_x^{(0)} - \lambda(|A^{(0)}|^2 w^{(0)})_x = 0. \tag{3.21b}$$

The initial condition is complemented by

$$A^{(0)}(x, 0) = \lim_{\hbar \to 0} A_0^{\hbar}(x), \quad w^{(0)}(x, 0) = \partial_x S_0(x). \tag{3.21c}$$

We can also discuss the first order approximation. For convenience we set

$$\widetilde{U}_1^{\hbar} \equiv \frac{1}{\hbar}[U^{\hbar} - U^{(0)}]. \tag{3.22}$$

Since $U^{(0)}$ is bounded in $C([0, T]; H^s) \wedge C^1([0, T]; H^{s-2})$. The energy estimate implies that

$$\widetilde{U}_1^{\hbar} \text{ is bounded in } C([0, T]; H^{s-2}) \wedge C^1([0, T]; H^{s-4}) \tag{3.23}$$

hence

$$\widetilde{U}_1^\hbar \to U^{(1)} \quad \text{in } C([0,T]; H^{s-2-\epsilon}) \tag{3.24}$$

by passing to a subsequence in $\hbar$. Taking the limit of the equation of $\widetilde{U}_1^\hbar$, we deduce that $U^{(1)}$ is *the* unique solution of the linearized equation (3.20b) with initial condition

$$U^{(1)}(x,0) = \lim_{\hbar \to 0} \frac{U^\hbar(x,0) - U^{(0)}(x,0)}{\hbar}. \tag{3.25}$$

Notice that no extraction of subsequence is needed for $\tilde{U}_1^\hbar$ due to the uniqueness. In fact, the whole sequence converges to $U^{(1)}$. Similarly, the N-th order approximation is obtained as follows. Suppose the first $N-1$ terms have been obtained, we can define

$$\tilde{U}_N^\hbar \equiv \frac{1}{\hbar^N}\left[U^\hbar - (U^{(0)} + \hbar U^{(1)} + \cdots + \hbar^{N-1}U^{(N-1)})\right], \tag{3.26}$$

where $(k = 0, 1, \ldots, N-1)$

$$U^{(k)} \in C([0,T]; H^{s-2k}) \wedge C^1([0,T]; H^{s-2k-2}) \tag{3.27}$$

then by the energy estimate again;

$$\tilde{U}_N^\hbar \text{ is bounded in } C([0,T]; H^{s-2N}) \wedge C^1([0,T]; H^{s-2N-2}) \tag{3.28}$$

as soon as the initial data $\tilde{U}_N^\hbar(x,0)$ is bounded in H^{s-2N}. Therefore, there exists $U^{(N)}$ such that

$$\tilde{U}_N^\hbar \to U^{(N)} \text{ in } C([0,T]; H^{s-2N-\epsilon}) \tag{3.29}$$

and satisfies (3.20d) with initial data

$$U^{(N)}(x,0) = \lim_{\hbar \to 0} \frac{1}{\hbar^N}\left[U^\hbar(x,0) - (U^{(0)}(x,0) + \cdots + \hbar^{N-1}U^{(N-1)}(x,0))\right]. \tag{3.30}$$

Thus, we have proved the following theorem.

Theorem 3.5 (WKB expansion). *Under the assumption of Theorem 3.1, suppose the initial amplitude $A_0^\hbar(x)$ admits the following expansion;*

$$A_0^\hbar(x) = \sum_{k=0}^{N} \hbar^k A_0^{(k)}(x) + R_N^{in}(x,\hbar)\hbar^N, \tag{3.31}$$

where

$$\lim_{\hbar \to 0} \|R_N^{in}(x,\hbar)\|_{H^\sigma} = 0 \tag{3.32}$$

for $N \in \mathbf{N}$ and $\sigma > 2N + 2 + \frac{1}{2}$, then the solutions of the JNLS equation (1.1) can be represented as

$$\psi^\hbar(x,t) = A^\hbar(x,t) \exp\left(\frac{\mathrm{i}}{\hbar}S^\hbar(x,t)\right) = \sum_{k=0}^{N} \hbar^k A^{(k)}(x,t) \exp\left(\frac{\mathrm{i}}{\hbar}S(x,t)\right) + \hbar^N R_N(x,t,\hbar) \tag{3.33}$$

and where

$$\lim_{\hbar \to 0} \|R_N(x,t,\hbar)\|_{C([0,T]; H^{\sigma-2N-\epsilon})} = 0, \ \forall \epsilon > 0. \tag{3.34}$$

There are different formulations of the semiclassical limit of the JNLS equation (1.1a) when the solutions are smooth. We will look at the typical Hamilton–Jacobi equation (2.7a) and the Liouville one (2.7b) which is the dispersionless wave equation associated with JNLS equation (1.1a). Introducing the new wave function

$$\chi(x,t) = A(x,t)\exp(\mathrm{i}S(x,t)) \tag{3.35}$$

we can rewrite the system (2.7a,b)

$$i\chi_t + \frac{1}{2}\chi_{xx} + \lambda j(x,t)\chi = \frac{1}{2}\frac{|\chi|_{xx}}{|\chi|}\chi, \tag{3.36a}$$

$$j = \frac{1}{2i}(\bar{\chi}\chi_x - \chi\bar{\chi}_x), \tag{3.36b}$$

as perturbed by quantum potential (QP) JNLS (1.1a,b). The quantum potential contribution to the r.h.s. with fixed strength completely compensates U(1) gauge invariant contribution to dispersion on the l.h.s. [1]. This potential, the so-called Bohm potential or internal self-potential was introduced by de Broglie and explored by Bohm to make a hidden-variable theory and is responsible for producing the quantum behavior, so that all quantum features are related to its special properties. The role of the quantum potential is to change the dispersion of the JNLS equation (1.1a,b) [27,28]. If the strength of the QP deviates from the critical value as given in dispersionless Eq. (3.36a,b), then we have the deformed wave equation [29]

$$i\chi_t + \frac{1}{2}\chi_{xx} + \lambda j(x,t)\chi = (1 - \hbar^2)\frac{1}{2}\frac{|\chi|_{xx}}{|\chi|}\chi, \tag{3.37}$$

which is equivalent to the original JNLS equation (1.1a,b).

Theorem 3.6. *The Hamilton–Jacobi and the Liouville equations* (2.7a,b) *can be converted into* (3.36a,b) *which is the quantum potential perturbation of the JNLS equation* (1.1a,b) *through the wave function* (3.35). *Furthermore, the same wave function applied to deform* (3.36a,b), *the equivalent formulation of JNLS equation* (1.1a,b), *into* (3.37) *such that* (3.36a,b) *is its formal dispersionless limit.*

4. Semiclassical limit and shock formation

As mentioned in Section 2, the JNLS equation (1.1a,b) has four integrals of motion. There, it has to appear that the canonical momentum

$$\int_{-\infty}^{\infty} j\,\mathrm{d}x \tag{4.1}$$

is not conserved quantity due to the lack of Galilean symmetry. The velocity cannot be arbitrarily reduced. Instead of this, the modified momentum

$$\int_{-\infty}^{\infty}\left(j + \frac{\lambda}{2}\rho^2\right)\mathrm{d}x \tag{4.2}$$

is conserved [10]. Therefore, we discuss the semiclassical limit with the system (2.13a,b) in terms of (ρ, M) rather than (ρ, u). The semiclassical limit of (2.13a,b) as $\hbar \to 0$ is given by (3.3a–c). This system has the form

$$V_t + BV_x = 0, \quad V = (\rho, M)^t \tag{4.3}$$

with

$$B = \begin{pmatrix} -\lambda\rho & 1 \\ -\frac{M^2}{\rho^2} - \lambda M + \frac{3\lambda^2}{4}\rho^2 & \frac{2M}{\rho} - \lambda\rho \end{pmatrix} \tag{4.4}$$

and can be represented in the Riemann invariant form. First, the eigenvalues and the associated right eigenvectors of B are given respectively by

$$\mu_{\pm} = -\lambda\rho + \frac{M}{\rho} \pm \sqrt{\frac{3\lambda^2}{4}\rho^2 - \lambda M}, \tag{4.5}$$

$$\mathbf{r}_{\pm} = \left(1, \frac{M}{\rho} \pm \sqrt{\frac{3\lambda^2}{4}\rho^2 - \lambda M}\right)^t. \tag{4.6}$$

Consider the Riemann invariants $R_\pm(\rho, M)$ corresponding to $\mu_\pm$. We know that it satisfies the equation

$$\nabla R_\pm \cdot \mathbf{r}_\pm = 0,$$

where the gradient is taken with respect to (ρ, M); hence

$$\frac{\partial R_\pm}{\partial \rho} + \left(\frac{M}{\rho} \pm \sqrt{\frac{3\lambda^2}{4}\rho^2 - \lambda M} \right) \frac{\partial R_\pm}{\partial M} = 0. \tag{4.7}$$

We can solve this first order differential equation by characteristic:

$$\frac{d\rho}{1} = \frac{dM}{\frac{M}{\rho} \pm \sqrt{\frac{3\lambda^2}{4}\rho^2 - \lambda M}} = \frac{d\eta}{\frac{\eta}{\rho} \mp \lambda\sqrt{\eta} + \frac{3}{4}\lambda^2\rho}, \tag{4.8}$$

where

$$\eta = \frac{3\lambda^2}{4}\rho^2 - \lambda M. \tag{4.9}$$

Eq. (4.8) can be rewritten as ($\eta := \rho^2 \zeta^2$)

$$\frac{d\rho}{2\rho} = \frac{\zeta \, d\zeta}{\lambda^2 - (\zeta \pm \frac{\lambda}{2})^2} = \frac{\zeta \, d\zeta}{\left(\frac{3\lambda}{2} \pm \zeta\right)\left(\frac{\lambda}{2} \mp \zeta\right)}, \tag{4.10}$$

i.e.,

$$\frac{d\rho}{2\rho} = \frac{\zeta \, d\zeta}{\left(\frac{3\lambda}{2} + \zeta\right)\left(\frac{\lambda}{2} - \zeta\right)} = -\frac{3}{4}\frac{d\zeta}{\frac{3\lambda}{2} + \zeta} + \frac{1}{4}\frac{d\zeta}{\frac{\lambda}{2} - \zeta},$$

$$\frac{d\rho}{2\rho} = \frac{\zeta \, d\zeta}{\left(\frac{3\lambda}{2} - \zeta\right)\left(\frac{\lambda}{2} + \zeta\right)} = \frac{3}{4}\frac{d\zeta}{\frac{3\lambda}{2} - \zeta} - \frac{1}{4}\frac{d\zeta}{\frac{\lambda}{2} + \zeta}.$$

Integration yields

$$\frac{1}{2}\log\rho + \frac{3}{4}\log\left(\zeta + \frac{3\lambda}{2}\right) + \frac{1}{4}\log\left(\zeta - \frac{\lambda}{2}\right) = \text{constant},$$

$$\frac{1}{2}\log\rho + \frac{3}{4}\log\left(\zeta - \frac{3\lambda}{2}\right) + \frac{1}{4}\log\left(\zeta + \frac{\lambda}{2}\right) = \text{constant}.$$

Therefore the Riemann invariants can be chosen as (after taking the exponential)

$$R_\pm = \rho^2 \lambda^4 \left(\xi^4 \pm 4\xi^3 + \frac{9}{2}\xi^2 - \frac{27}{16} \right), \tag{4.11}$$

where $\xi \equiv \zeta/\lambda$. We also have

$$\mu_\pm = \frac{1}{\lambda\rho}\left(-\xi^2 \pm \xi - \frac{1}{4} \right). \tag{4.12}$$

From (4.11) we have

$$\lambda\rho = \frac{\xi^{-3/2}}{2\lambda}\sqrt{\frac{R_+ - R_-}{2}} \tag{4.13}$$

and for ξ we have the fourth order algebraic equation

$$\xi^4 - 4\frac{R_+ + R_-}{R_+ - R_-}\xi^3 + \frac{9}{2}\xi^2 - \frac{27}{16} = 0 \tag{4.14}$$

which fortunately according general theory of algebraic equation has solution in radicals. Introducing

$$\begin{aligned} Q_+ &= R_+ + R_-, \quad Q_- = R_+ - R_-, \\ \eta_+ &= \mu_+ + \mu_-, \quad \eta_- = \mu_+ - \mu_-. \end{aligned} \tag{4.15}$$

Then we find four equations from (4.11) and (4.12) correspondingly

$$Q_+ = 2\rho^2\lambda^4\left(\xi^4 + \frac{9}{2}\xi^2 - \frac{27}{16}\right), \tag{4.16a}$$

$$Q_- = 8\rho^2\lambda^4\xi^3, \tag{4.16b}$$

$$\eta_+ = -\frac{2}{\lambda\rho}\left(\xi^2 + \frac{1}{4}\right), \tag{4.16c}$$

$$\eta_- = \frac{2}{\lambda\rho}\xi. \tag{4.16d}$$

Substituting $\xi^2 + 1/4$ from (4.16c) to (4.16a) we obtain the quadratic equation for η_+

$$(\lambda\rho)^2\eta_+^2 - 8(\lambda\rho)\eta_+ - 11 - \frac{2Q_+}{\rho^2\lambda^4} = 0. \tag{4.17}$$

Solving this equation and choosing only one root (according to the sign in (4.16c)) we obtain

$$\eta_+ = \frac{1}{\lambda\rho}(4 - s), \quad s = \sqrt{27 + \frac{2Q_+}{\rho^2\lambda^4}}. \tag{4.18}$$

This root's existence also pose constraint on the Riemann invariant

$$Q_+ > -\frac{27}{2}\rho^2\lambda^4 \tag{4.19}$$

which also follows immediately from (4.16a). (Only in this case exist solution of the problem.) Eq. (4.16d) can be written in terms of η_+ and η_- as

$$Q_- = -\lambda^5\rho^3(1 + 2\lambda\rho\eta_+)\eta_- \tag{4.20}$$

from which substituting for η_+ in (4.17) above we obtain

$$\eta_- = \frac{-Q_-}{\lambda^5\rho^3(9 - 2s)}. \tag{4.21}$$

Thus from (4.16d) and (4.21) we have

$$\xi = \frac{-Q_-}{2\lambda^4\rho^2(9 - 2s)}. \tag{4.22}$$

Since ξ and Q_- have the same sign (from (4.16b)) we have a stronger constraint then (4.19)

$$Q_+ > -\frac{27}{8}\rho^2\lambda^4 \tag{4.23a}$$

or (since $R_+ > R_-$)

$$R_+ > -\frac{27}{16}\rho^2\lambda^4. \tag{4.23b}$$

Later on we will have stronger constraint then (4.23a,b) in the following theorem. Due to the fourth algebraic equation has solution by radical, we can represent all related parameters in terms of $R_\pm$ explicitly. Thus

$$\mu_+ = \mu_+(R_+, R_-) = \tfrac{1}{2}(\eta_+ + \eta_-), \tag{4.24a}$$

$$\mu_- = \mu_-(R_+, R_-) = \tfrac{1}{2}(\eta_+ - \eta_-) \tag{4.24b}$$

which shows that $\mu_\pm$ can be represented explicitly (radically) in terms of the Riemann invariants $R_\pm$. We rewrite the system (4.3) in the Riemann invariant form

$$\partial_t R_+ + \mu_+\partial_x R_+ = 0, \tag{4.25a}$$

$$\partial_t R_- + \mu_-\partial_x R_- = 0 \tag{4.25b}$$

with $\mu_\pm$ given above.

Theorem 4.1. *Subject to the ordering condition*

$$Q_- = R_+ - R_- > 0 \Longleftrightarrow \xi > 0 \tag{4.26}$$

and

$$Q_+ > \frac{1}{2} \left[\left(\frac{27 + \sqrt{170}}{4} \right)^2 - 27 \right] \rho^2 \lambda^4, \quad Q_- > \frac{85}{16} \rho^2 \lambda^4 \tag{4.27}$$

then for $\lambda > 0$ the system (4.25a,b) is strictly hyperbolic, with the characteristic speed ordered as

$$\mu_+(R_+, R_-) > \mu_-(R_+, R_-) \tag{4.28a}$$

and genuinely nonlinear, with

$$\frac{\partial \mu_+}{\partial R_+} < 0 \text{ and } \frac{\partial \mu_-}{\partial R_-} < 0. \tag{4.28b}$$

Proof. Define s the same as (4.18) by $s = (27 + (2Q_+/\rho^2\lambda^4))^{1/2}$. Since $\partial Q_+/\partial R_+ = \partial Q_+/\partial R_- = \partial Q_-/\partial R_+ = 1$, $\partial Q_-/\partial R_- = -1$ and $\partial s/\partial Q_+ = 1/\rho^2\lambda^4 s$ we have

$$\frac{\partial \mu_+}{\partial R_+} = \frac{1}{2} \left(\frac{\partial \eta_+}{\partial R_+} + \frac{\partial \eta_-}{\partial R_+} \right) = \frac{1}{2} \left(\frac{\partial \eta_+}{\partial Q_+} + \frac{\partial \eta_+}{\partial Q_-} + \frac{\partial \eta_-}{\partial Q_+} + \frac{\partial \eta_-}{\partial Q_-} \right) = -\frac{1}{2\lambda^5\rho^3 s} \times \frac{A}{\rho^2\lambda^4(9 - 2s)^2},$$

where

$$A = \rho^2\lambda^4 \left[2s^2 - 27s + 81 + \frac{2|Q_-|}{\rho^2\lambda^4} \right].$$

Denote D the discriminant, $D = 85 - (16|Q_-|/\rho^2\lambda^4)$, then by (4.27) we conclude that $D < 0$ hence $A > 0$. Thus $\partial\mu_+/\partial R_+ < 0$. Similarly

$$\frac{\partial \mu_-}{\partial R_-} = \frac{1}{2} \left(\frac{\partial \eta_+}{\partial R_-} - \frac{\partial \eta_-}{\partial R_-} \right) = \frac{1}{2} \left(\frac{\partial \eta_+}{\partial Q_+} - \frac{\partial \eta_+}{\partial Q_-} - \frac{\partial \eta_-}{\partial Q_+} + \frac{\partial \eta_-}{\partial Q_-} \right) = -\frac{1}{2\lambda^5\rho^3 s} \times \frac{\widetilde{A}}{\rho^2\lambda^4(9 - 2s)^2},$$

where

$$\tilde{A} = \rho^2\lambda^4 \left[2s^2 - 27s + 81 - \frac{2|Q_-|}{\rho^2\lambda^4} \right].$$

The discriminant, $\tilde{D} = 85 + (16|Q_-|/\rho^2\lambda^4) > 0$. Let s_1 and s_2 be the two real roots

$$s_1 = \frac{1}{4} \left[27 + \sqrt{85 + \frac{16Q_-}{\rho^2\lambda^4}} \right], \quad s_2 = \frac{1}{4} \left[27 - \sqrt{85 + \frac{16Q_-}{\rho^2\lambda^4}} \right].$$

Since we already have $s > 4$ and $s > 9/2$. Thus if we require (using (4.27)$_2$)

$$s = \sqrt{27 + \frac{2Q_+}{\rho^2\lambda^4}} > s_1 = \frac{1}{4} \left[27 + \sqrt{85 + \frac{16Q_-}{\rho^2\lambda^4}} \right] > \frac{1}{4}(27 + \sqrt{170})$$

which turns out to be (4.27)$_1$ then for $\lambda > 0$ we conclude

$$\frac{\partial \mu_-}{\partial R_-} = -\frac{1}{2\lambda^5\rho^3 s} \times \frac{(s - s_1)(s - s_2)}{\rho^2\lambda^4(9 - 2s)^2} < 0. \quad \square$$

Since the system (4.25a,b) is genuinely nonlinear, the Riemann invariant, R_+ or R_-, may in a finite time develop an infinite spatial derivative at a point while maintaining the ordering condition (4.26). To study the breaking of solutions to the Riemann invariant equations given by (4.25a,b), we can show that $Z_\pm = \partial_x R_\pm$ satisfies the Riccati equation.

Theorem 4.2. *Let* (R_+, R_-) *be a solution of (4.25a,b) subject to the ordering condition (4.26). There exist* $h_\pm$ *such that* $Z_\pm = \partial_x R_\pm$ *satisfies the Riccati equation*

$$(\partial_t + \mu_\pm \partial_x)\left(\frac{Z_\pm}{e^{-h_\pm}}\right) + \left(e^{-h_\pm}\frac{\partial \mu_\pm}{\partial R_\pm}\right)\left(\frac{Z_\pm}{e^{-h_\pm}}\right)^2 = 0, \tag{4.29}$$

where $h_\pm = h_\pm(R_+, R_-)$ *satisfies*

$$h'_+ = \frac{\partial \mu_+}{\partial R_-} \cdot \frac{R'_-}{\mu_+ - \mu_-}, \quad \dot{h}_- = \frac{\partial \mu_-}{\partial R_+} \cdot \frac{\dot{R}_+}{\mu_- - \mu_+}, \tag{4.30}$$

where ' and the dot "·" denote the differentiation in the μ_+-*characteristic and* μ_--*characteristic directions respectively*

$$h'_+ = \partial_t h_+ + \mu_+ \partial_x h_+, \quad \frac{dx_+}{dt}(t) = \mu_+(x_+(t), t), \tag{4.31a}$$

$$\dot{h}_- = \partial_t h_- + \mu_- \partial_x h_-, \quad \frac{dx_-}{dt}(t) = \mu_-(x_-(t), t). \tag{4.31b}$$

Proof. We start from the Riemann invariant form (4.25a,b);

$$R'_+ = \partial_t R_+ + \mu_+ \partial_x R_+ = 0, \tag{4.32a}$$

$$\dot{R}_- = \partial_t R_- + \mu_- \partial_x R_- = 0. \tag{4.32b}$$

Differentiate (4.32a) with respect to x and set $Z_+ \equiv \partial_x R_+$ we have

$$(\partial_t + \mu_+ \partial_x)Z_+ + \frac{\partial \mu_+}{\partial R_+}Z_+^2 + \frac{\partial \mu_+}{\partial R_-}\partial_x R_- Z_+ = 0. \tag{4.33}$$

From (4.32b) we deduce that

$$R'_- = \partial_t R_- + \mu_+ \partial_x R_-$$

hence

$$\partial_x R_- = \frac{R'_-}{\mu_+ - \mu_-}. \tag{4.34}$$

Substituting this relation into (4.33) we obtain

$$Z'_+ + \frac{\partial \mu_+}{\partial R_-}\frac{R'_-}{\mu_+ - \mu_-}Z_+ + \frac{\partial \mu_+}{\partial R_+}Z_+^2 = 0. \tag{4.35}$$

Multiplying (4.35) by e^{h_+} and using the first identity of (4.30) leads to the Riccati equation

$$(\partial_t + \mu_+ \partial_x)\left(\frac{Z_+}{e^{-h_+}}\right) + \left(e^{-h_+}\frac{\partial \mu_+}{\partial R_+}\right)\left(\frac{Z_+}{e^{-h_+}}\right)^2 = 0. \tag{4.36}$$

Similarly, we also have the Riccati equation for $Z_- \equiv \partial_x R_-$. This complete the proof of the theorem. $\square$

For convenience, we write the Riccati equation (4.29) as

$$q'_+ + k_+ q_+^2 = 0, \quad \dot{q}_- + k_- q_-^2 = 0, \tag{4.37}$$

by using the abbreviation

$$q_\pm \equiv e^{h_\pm}Z_\pm = e^{h_\pm}\partial_x R_\pm, \quad k_\pm \equiv \frac{\partial \mu_\pm}{\partial R_\pm}e^{-h_\pm}. \tag{4.38}$$

The solution of (4.37) is given by

$$q_\pm(x, t) = \frac{q_\pm^0}{1 + q_\pm^0 K_\pm(t)}, \quad q_\pm^0 = q_\pm(x(0), 0), \tag{4.39}$$

where

$$K_{\pm}(t) = \int_0^t k_{\pm}(\tau, x_{\pm}(\tau)) \, \mathrm{d}\tau \tag{4.40}$$

the integration along the $\mu_{\pm}$-characteristic. By the genuine nonlinearity assertion (4.28) of Theorem 4.1, the coefficient of the quadratic term in the Riccati equation (4.37) is nonnegative. So $q_{\pm}$ is a strictly increasing quantity along the characteristic provided $Z_{\pm} \neq 0$ because $\mathrm{e}^{h_{\pm}} > 0$. Now if $Z_{\pm} = \partial_x R_{\pm} < 0$, the only way to have $Z_{\pm} \to -\infty$ in finite time is for $\mathrm{e}^{-h_{\pm}} \to \infty$ such that $q_{\pm}$ is increasing. But if the ordering condition (4.26) holds, continuity implies $\mathrm{e}^{-h_{\pm}}$ must bounded above. Thus, we conclude:

Corollary 4.3. *If $|\partial_x R_{\pm}|$ becomes unbounded in finite time along the $\mu_{\pm}$-characteristic, then either*

(a) $\partial_x R_{\pm} > 0$, *or*
(b) *the ordering condition* (4.26) *is violated in finite time.*

Moreover, the detail study of the Riccati equation also leads to the break-time t_b.

Theorem 4.4. *The break-time t_b for* (3.3a–c) *can be estimated in the following way*:

$$t_b = \min\{t_{+,b}, t_{-,b}\}, \tag{4.41}$$

where

$$t_{\pm,b} \leqslant \inf_{x_0 \in \Omega_{\pm}} \{t : G_{\pm}(t, x_0) = 0\}, \quad \Omega_{\pm} = \{x_0 : \partial_x R_{\pm}(x_0) \leqslant 0\} \tag{4.42}$$

with

$$G_{\pm} = 1 + \mathrm{e}^{h_{\pm}(x_{\pm}^0)} \partial_x R_{\pm}(x_{\pm}^0) \int_0^t K_{\pm}(x_{\pm}(\tau), \tau) \, \mathrm{d}\tau \tag{4.43}$$

and particle path $x_{\pm} = x_{\pm}(t)$ satisfies the differential equation

$$\frac{\mathrm{d}x_{\pm}}{\mathrm{d}t} = \mu_{\pm}(R_+, R_-), \quad x_{\pm}(0) = x_{\pm}^0. \tag{4.44}$$

Proof. It follows immediately from (4.39) that at initial time if

$$\partial_x R_{\pm}(x_{\pm}(0), 0) = \partial_x R_{\pm}(x_{\pm}^0) < 0,$$

i.e., $q_{\pm}^0 < 0$ then $q_{\pm}(x, t)$ must become unbounded in finite time. This means that $q_{\pm}(x, t)$ must blow up at some later time t, where

$$1 + q_{\pm}^0 K_{\pm}(t) = 0.$$

Therefore, the break t_b can be estimated by the following rules. Let $t_{\pm,b}$ satisfy (4.42) and (4.43). The particle path $x_{\pm} = x_{\pm}(t)$ satisfies $x_{\pm}(0) = x_{\pm}^0$ and the differential equation (4.44). $\quad \square$

Remark 4.5. It is interesting to mention that when $\lambda \to 0$ the semiclassical limit of the JNLS equation reduces to (formally let $\lambda = 0$ in (2.6a,b))

$$\rho_t + (\rho u)_x = 0, \tag{4.45a}$$

$$(\rho u)_t + (\rho u^2)_x = 0. \tag{4.45b}$$

This system of conservation laws arising in the model of adhesion particle dynamics, more precisely, the system of free particles which stick under collision. It was proposed by Zeldovich [32] that a possible model for the description of large scale dynamics of the mass distribution, in the early stage of evolution of the universe. For smooth solutions, (4.45a,b) is equivalent to the dispersionless Burgers equation

$$u_t + \left(\frac{u^2}{2} \right) = 0 \tag{4.46a}$$

together with a scalar transport equation

$$\rho_t + (\rho u)_x = 0. \tag{4.46b}$$

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References

[1] Auberson G, Sabatier PC. On a class of homogeneous nonlinear Schrödinger equations. J Math Phys 1994;35:4028–40.
[2] Aglietti U, Griguolo L, Jackiw R, Pi S-Y, Seminara D. Anyons and chiral solitons on a line. Phys Rev Lett 1996;77:4406–9.
[3] Bronski JC, Jerrard RL. Soliton dynamics in a potential. Math Res Lett 2000;7:329–42.
[4] Clarkson PA, Cosgrove CM. Painlevé analysis of nonlinear Schrödinger family of equation. J Phys A 1987;20:2003–24.
[5] Desjardins B, Lin C-K, Tso T-C. Semiclassical limit of the derivative nonlinear Schrödinger equation. Math Models Meth Appl Sci 2000;10:261–85.
[6] Desjardins B, Lin C-K. On the semiclassical limit of the general modified NLS equation. J Math Anal Appl 2001;260:546–71.
[7] Gasser I, Lin C-K, Markowich P. A review of dispersive limit of the (non)linear Schrödinger-type equation. Taiwan J Math 2000;4:501–29.
[8] Grenier E. Semiclassical limit of the nonlinear Schrödinger equation in small time. Proc Am Math Soc 1998;126:523–30.
[9] Griguolo L, Seminara D. Chiral solitons from dimensional reduction of Chern–Simons gauged nonlinear Schrödinger equation: classical and quantum aspects. Nucl Phys B 1998;516:467–98.
[10] Jackiw R. A nonrelativistic chiral soliton in one dimension. J Nonl Math Phys 1997;4:261–70.
[11] Jackiw R, Pi S-Y. Soliton solutions to the gauged nonlinear Schrödinger equation on the plane. Phys Rev Lett 1990;64:2969–72.
[12] Jackiw R, Pi S-Y. Time-dependent Chern–Simons solitons and their quantization. Phys Rev D 1991;44:2524–32.
[13] Jin S, Levermore CD, McLaughlin DW. The semiclassical limit of the defocusing NLS hierarchy. Commun Pure Appl Math 1999;52:613–54.
[14] Lax PD. Hyperbolic Systems of Conservation Laws and the Mathematical Theory of Shock Waves. SIAM; 1973.
[15] Lee J-H. Global solvability of the derivative nonlinear Schrödinger equation. Trans Am Math Soc 1989;314:107–18.
[16] Lee J-H, Lin C-K, Pashaev OK. Equivalent relation and bilinear representation for derivative nonlinear Schrödinger type equation. In: Boiti M, Martina L, Pempinelli F, Prinari B, Soliani G, editors. Proceeding of Workshop on Nonlinearity, Integrability and all that: Twenty Years after NEEDS'79. Singapore: World Scientific; 2000. p. 175–81.
[17] Lee J-H, Lin C-K. The behavior of solutions of NLS equation of derivative type in the semiclassical limit. Chaos, Solitons & Fractals 2002;13:1475–92.
[18] Levermore CD. The hyperbolic nature of the zero dispersion KdV limit. Commun Partial Diff Equat 1988;13:495–514.
[19] Li H-L, Lin C-K. Semiclassical limit and well-posedness of Schrödinger–Poisson and quantum hydrodynamics, preprint (2001).
[20] Li H-L, Lin C-K, Masmoudi N. From Schrödinger–Poisson system to the incompressible Euler for general initial data, preprint (2002).
[21] Lin C-K. On the fluid dynamical analogue of the general nonlinear Schrödinger equation. Southeast Asia Bull Math 1998;22:45–56.
[22] Lin C-K. Remark on the singularity of the nonlinear Schrödinger equation. Southeast Asia Bull Math 1998;22:161–70.
[23] Lin C-K. Singular limit of the modified nonlinear Schrödinger equation. In: Sigal IM, Sulem C, editors. Nonlinear Dynamics and Renormalization Group. CRM Proceeding & Lecture Note, vol. 27. American Mathematical Society; 2001. p. 97–109.
[24] Majda A. In: Compressible Fluid Flow and Systems of Conservation Laws in Several Space Variables. Applied Mathematical Science, vol. 53. Springer-Verlag; 1984.
[25] Min H, Park Q-H. Scattering of solitons in the derivative nonlinear Schrödinger model. Phys Lett B 1996;388:621–5.
[26] Ozawa T. On the nonlinear Schrödinger equations of derivative type. Indiana Univ Math J 1996;45:137–63.
[27] Pashaev OK, Lee J-H. Soliton resonances black holes and Madelung fluid. J Nonl Math Phys 2001;8(Suppl):230–4.

[28] Pashaev OK, Lee J-H. Resonance soliton as black holes in Madelung fluid. Mod Phys Lett A 2002;17:1601–13.

[29] Pashaev OK, Lee J-H. Black holes and solitons of quantized dispersionless NLS and DNLS. ANZIAM J Appl Math 2002;44:73–81.

[30] Sulem C, Sulem P-L. The Nonlinear Schrödinger Equation. In: Applied Mathematical Science, 139. Springer-Verlag; 1999.

[31] Whitham GB. Linear and Nonlinear Waves. New York: Wiley-Interscience; 1974.

[32] Zeldovich YaB. Gravitational instability: an approximate theory for large density perturbations. Astron Astrophys 1970;5:84–9.