ALGEBRAIC STRUCTURES FOR CLASSICAL KNOTS, SINGULAR KNOTS AND VIRTUAL KNOTS

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ABSTRACT

ALGEBRAIC STRUCTURES FOR CLASSICAL KNOTS, SINGULAR KNOTS AND VIRTUAL KNOTS

The purpose of this thesis is to establish algebraic structures on knots. Knot theory is a field in mathematics that investigates the properties and structures of knots. The main objective is to define knot invariants for the purpose of classifying knots. In order to do this, the thesis first provides some fundamental definitions of knots and links. Then we define the colorability of the knot, which we can use as a knot invariant. To further elaborate on this subject, we give definitions of the algebraic structures known as quandle, singquandle, and bondle. Using these determined structures, we provided variables for classifying the circuit topology. Circuit topology refers to a mathematical method used to classify the categorizes of connections between contacts. This thesis aims to classify the structure of proteins using circuit topology and knot theory. Consequently, we define an invariant for the circuit topology. At the end, the thesis determines these structures on virtual knots. In addition, it offers a definition and an example instance of this topic.

ÖZET

Klasik Düğümler, Singüler Düğümler ve Sanal Düğümler İçin Cebirsel Yapılar

Bu tezin amacı, düğümler üzerine cebirsel yapılar kurmaktır. Düğüm teorisi, düğümlerin özelliklerini ve yapısını araştıran matematik dalıdır. Ana hedefimiz, düğümlerin sınıflandırılmasını anlamak için düğüm değişkenlerini tanımlamaktır. Bunu gerçekleştirebilmek için, ilk olarak temel düğümleri tanımlarını sunacağız. Daha sonra, bir düğüm değişmezi olan renklendirilebilirlik kavramını açıklayacağız. Bu konuyla ilgili daha detaylı bilgi sağlamak amacıyla, quandle, singquandle ve bondle olarak bilinen cebirsel yapıların tanımlarını vereceğiz.

Bu cebirsel yapılar, biyoloji alanında proteinlerle ilgili çalışmalara olanak sağlar. Bu yapılar sayesinde, düğüm teorisinde kullandığımız singüler yapılar ile protein yapıları arasında bir ilişki kuruyoruz. Proteinlerin yapılarını ve proteinlerin birbirine bağlanma süreçlerini inceliyoruz. Devre topolojisi, proteinlerin birbirine bağlanmalarını açıklar. Kurduğumuz bu yapılar ile devre topolojisi için değişmezler tanımlayacağız. Son olarak, bu yapıları daha da geliştirmek amacıyla sanal düğümler üzerine çalışmalar yapacağız.

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CHAPTER 1

INTRODUCTION

A knot, in the subject of topology, is a closed, non-self-intersecting curve that is embedded in three-dimensional space. The basic purpose of knot theory is to study and classify these knots. What are the conditions for two knots to be classified as equal or not equal? How can these two knots deform each other under ambient isotopy? The purpose of classifying two knots is to define the invariant of knots. It can be said that two knots are equivalent if and only if they can be transformed into each other by finite Reidemeister moves(Reidemeister, 1926). Those who provide these moves are invariants of the knots. This thesis, will specifically focus on some invariants. It will explain the colorability that is invariant. Colorability is the main idea under the other defined invariant in this thesis. The fundamental idea of coloring a knot involves assigning colors to each arc of a knot diagram according to rules, which remain invariant under Reidemeister moves (Dixon, 2010), (Livingston, 1993), (Carter et al., 2014). This coloring helps classify different knots and provides information on their structural properties. However, coloring cannot be sufficient to classify all knots. Define different invariants, which are the main idea of colorability ared defined in this thesis. These are algebraic structures for classical knots, singular knots, and virtual knots. This algebraic structure be shown in four different ways: quandle, singquandle, bondle, and virtual singquandle.

A quandle is an algebraic structure for classical knots (Kauffman and Manturov, 2004), (Elhamdadi and Nelson, 2015),(Joyce, 1982). It is defined by a set with a binary operation that is invariant under Reidemeister moves. Quandles are important for understanding knot invariants because they provide a way to explain and label the crossings and interactions on a knot diagram. The axioms of the quandle provide coloring information for the knot. This information gives some equations and calculations for knots. All of these provide us with a more comprehensive classification of knots. After the quandle structure, the singquandle structure is defined for singular knots (Ceniceros et al., 2021), (Adams et al., 2020), (Ceniceros et al., 2021),(Joyce, 1982). Singular knots have a singular crossing that has a double crossing (self-intersection) in at least one crossing(Juyumaya and Lambropoulou, 2009). We use these crossings because the other aim this thesis has is to study protein structure. In mathematics, a singular knot structure is similar to the structure of proteins. Due to this, we define a quandle structure for singular knots called a singquandle. Singquandles extend the structure of quandles. This structure is a connection between quandles and bondles. We expand the singular crossing with a bond, and we will get a better understanding of the protein structure with the bondle structure. This study provides the natural flow of proteins with the bondle structure.

Bondles are another algebraic structure in knot theory (Adams et al., 2020), (Ceniceros et al., 2021). Bondles help develop invariants that can distinguish between knots and, similarly, quandles and singquandles. This study falls under circuit topology. Circuit topology analyses connections in biology, electrical engineering, and graph theory(Golovnev and Mashaghi, 2020), (Mashaghi et al., 2014). By representing them as circuits, researchers can find much information about knots. This interdisciplinary approach offers new methods for exploring knot theory and protein structure. Therefore, bondle gives an invariant in circuit topology.

Finally, the quandle structure is defined on virtual knots(Kauffman and Manturov, 2004). Virtual knots expand the classical knot theory by including virtual crossings. Virtual crossings provide a more comprehensive classification of knots and present a new study field. These structures defined by virtual knots give us an invariant. In this section, the aim is combine virtual knots with singular knots. So, the structure of a virtual singquandle was defined in the study. This gives us an invariant for a knot that includes three different crossings.

The thesis is organized as follows. In Section 2, we give the basic definitions and properties of knots and links. In Section 3, we define tricolorable, Fox-n coloring, and coloring matrix. In Section 4, we give algebraic structures for knots. In Section 5, we will define the circuit topology. In Section 6, first define virtual knots, then we define virtual quandles and virtual singquandles.

CHAPTER 2

FUNDAMENTAL NOTIONS OF KNOTS AND LINKS

Mathematicians have been trying to classify and categorize all conceivable knots since the late 1800s, when the first knot tables were produced. Peter Guthrie Tait, a Scottish physicist, aimed to systematically classify knots by their crossing number, taking inspiration from Lord Kelvin's "Vortex Theory of the Atom." After the addition of Reverend Thomas P. Kirkman from England and mathematician Charles Newton Little from America, a table was constructed that listed all prime alternating knots with up to 11 crossings and prime non-alternating knots with up to 10 crossings. While there were important developments in the first half of the 20th century, such as the development of Reidemeister moves and the Alexander polynomial, the most significant developments occurred in the second half. Moreover, these developments continue through expansion. This section will give fundamental definitions (Adams, 1994),(Kauffman, 1987), (Kauffman, 2001), (Rolfsen, 2003).

Definition 1 A knot is a loop that is non self-intersecting. It is embedded in a three-dimensional Euclidean space, represented as $\mathbb{R}^3$.

Definition 2 The simplest knot is the *trivial knot or unknot*, which is just an unknotted circle.

The other simplest knot is called a trefoil knot. In the next figure, unknot and trefoil can be seen.

Figure 2.1. Unknot and trefoil

Definition 3 A *link* is a disjoint union of knots. We denote a link by an L. Each knot forming an L is referred to as the *components* of L.

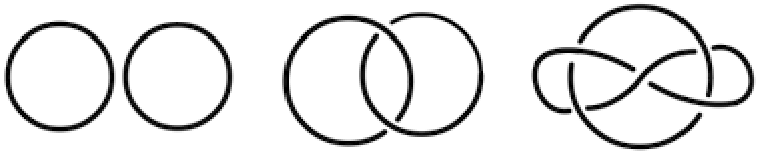


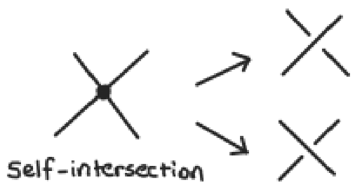
Figure 2.2. Examples of link: Unlink, Hopf link, Whitehead link

Definition 4 Let two knots K_1, K_2 in $\mathbb{R}^3$. These are *ambient isotopy* of $\mathbb{R}^3$ taking K_1 to K_2 is a smooth map h is from $\mathbb{R}^3 \times [0,1]$ to $\mathbb{R}^3$, such that

- $h(x,t_0)$ is a diffeomorphism from $\mathbb{R}^3$ to $\mathbb{R}^3$ for all t_0 in $[0,1]$
- $h(x,0) : \mathbb{R}^3 \rightarrow \mathbb{R}^3$the identity map and $h(K_1,1) = K_2$.

Definition 5 Let K_1 and K_2 be two knots in three-dimensional Euclidean space. K_1 and K_2 are considered equal, represented by $K_1 \sim K_2$, if there is an ambient isotopy that transforms K_1 into K_2.

Definition 6 A *Knot diagram or projection* is a planar projection of a knot in $\mathbb{R}^2$ with extra over/under information endowed at each self-intersection of the projection curve.



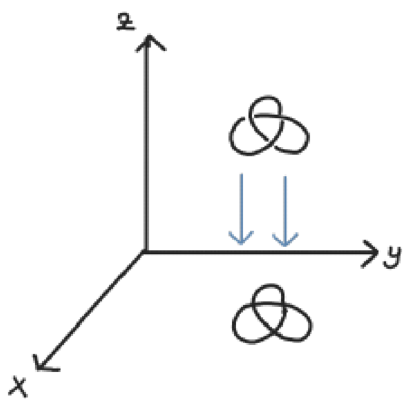


Figure 2.3. Self-intersection Figure 2.4. Projection

Theorem 1 ***Reidemeister Theorem*** *(Kurt Reidemeister,1927)* Two link diagrams in S^3 constitute the same ambient isotopy class if and only if they can be transformed into each other by a finite number of Reidemeister moves Reidemeister (1926).

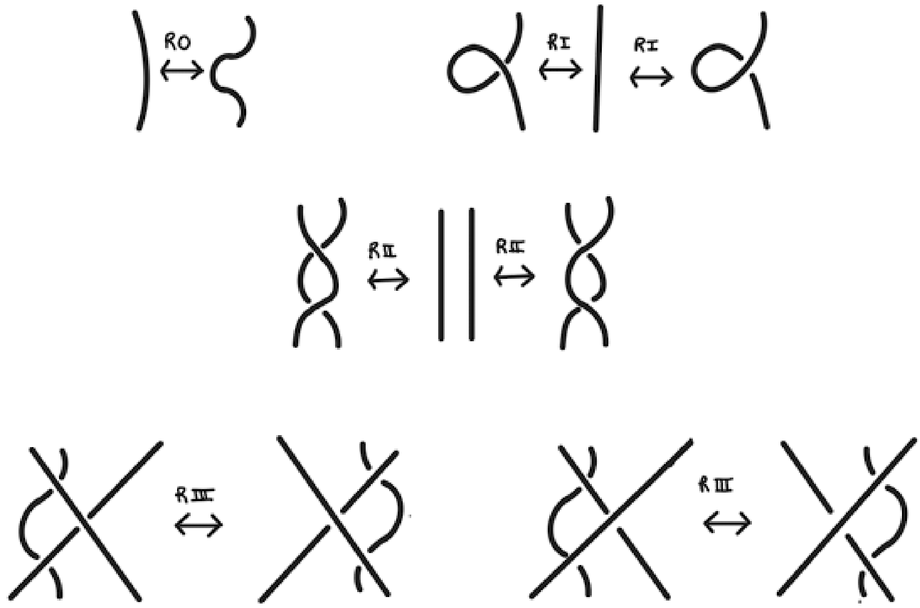


Figure 2.5. Classical Reidemiester Moves

Proof The Reidemeister Theorem extends to polygonal knots and links. Firstly, some definitions will be provided for the purpose of presenting proof.

Definition 7 *A polygonal knot K* is a representation of a knot in three-dimensional space. It is composed of a finite sequence of straight line segments, or edges, which do not have self-intersections. Each edge is connected to its neighboring edges at vertices, creating a continuous and complete loop. A polygonal link in $\mathbb{R}^3$ is a collection of polygonal knots that are limited in number and do not intersect each other.

Example 2.1 We have two examples of polygonal knots and links.

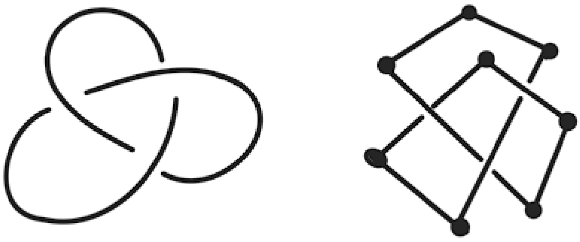


Figure 2.6. An example of a polygonal knot

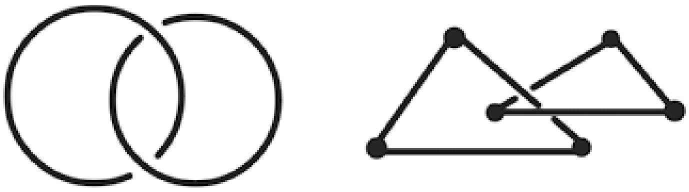


Figure 2.7. An example of polygonal link

Definition 8 *An elementary isotopy* refers to the process of replacing an edge of a link L in $\mathbf{R}^3$ with two edges of a triangle that are not in contact with the other edges of L. An elementary isotopy generated by two moves.

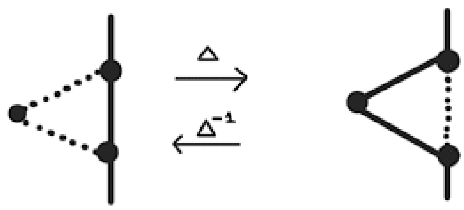


Figure 2.8. The elementary isotopy Δ-move and Δ^{-1}-move

Proof Theorem 1. In order to demonstrate that two link diagrams in S^3 belong to the same ambient isotopy class, it is necessary and sufficient to prove that they can be

transformed into each other by a finite number of Reidemeister moves. We prove only if part induction.
First we start with one strand, no strand intersecting in Δ_0-move

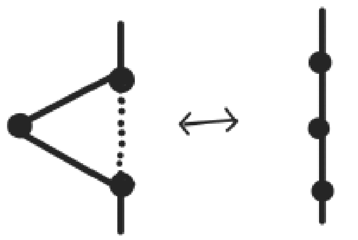


Figure 2.9. The move Δ_0

For one strand, one strand intersecting in Δ_1-move

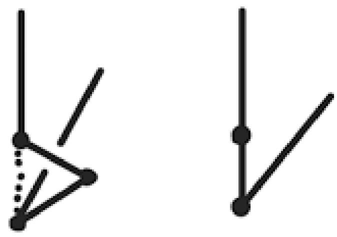


Figure 2.10. The move Δ_1

For two strands, one strand intersecting in Δ_2 - move

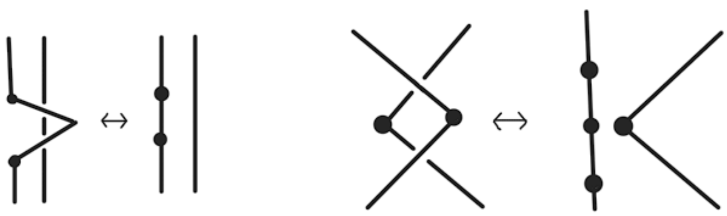


Figure 2.11. The move Δ_2

For three strands in Δ_3 - move

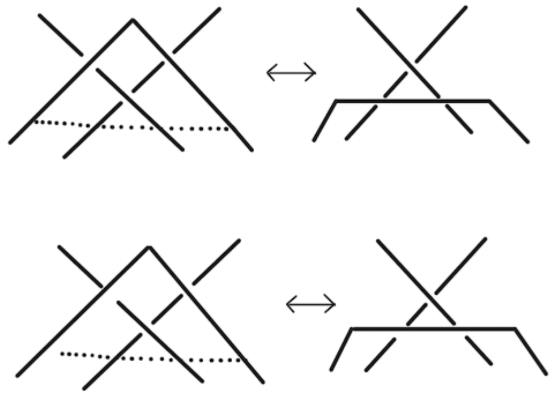


Figure 2.12. The move Δ_3

The moves shown in figures 2.9, 2.10, 2.11, and 2.12 are essentially fundamental moves that connect with classical Reidemeister moves. When there are n strands in the Δ-moves, we use subdivision. This can be seen in figure 2.13.

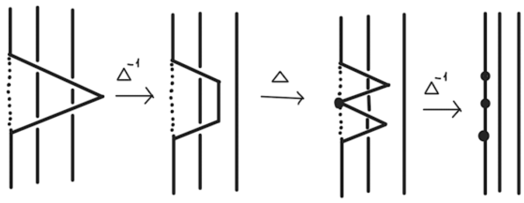


Figure 2.13. Subdivision

Thus, we can show that for n strand, completing the proof. □

Definition 9 An oriented link diagram is one in which each arc is directed in such a way that the oriented crossings.

We can apply Reidemeister moves for oriented link diagrams Polyak (2010). There are four different types for the first Reidemeister move.

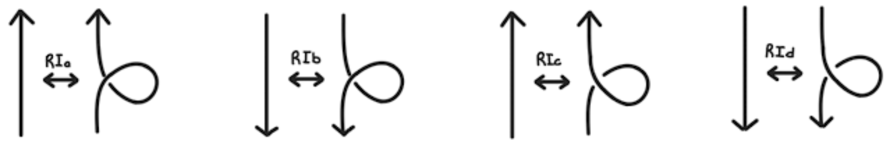


Figure 2.14. The four types of oriented first Reidemeister move

Again, there are four different types for the second Reidemeister move.

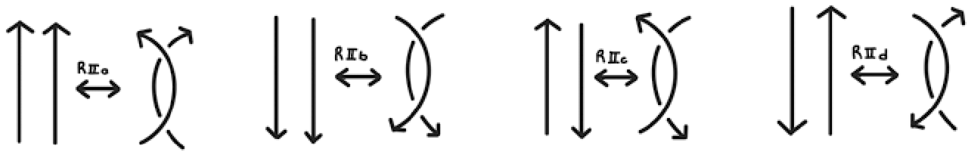


Figure 2.15. The four types of oriented second Reidemeister move

Finally, there are eight different types for the third Reidemeister move.

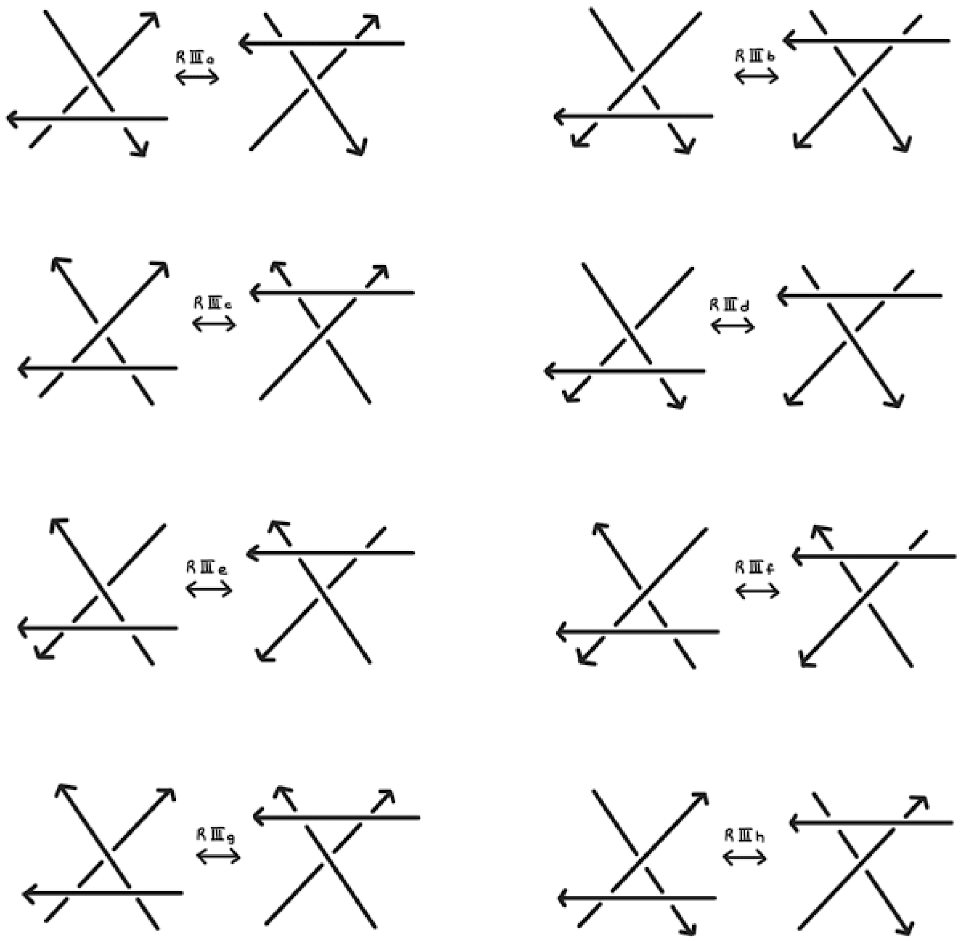


Figure 2.16. The eight types of oriented third Reidemeister move

Theorem 2 (M.PolyakPolyak (2010)) Let L_1 and L_2 be two oriented link diagrams. These diagrams are the same oriented link. There exists a pass from L_1 to L_2 with a finite series of four oriented Reidemeister moves. We see these moves in the figure 2.17.

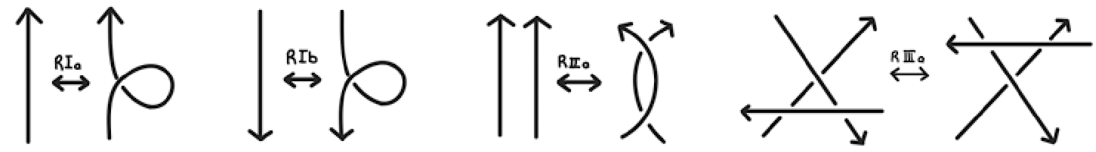


Figure 2.17. Generate oriented Reidemeister Moves

Definition 10 A knot invariant is a function defined as map I knot diagrams go to M where M is a mathematical set such that if K_1 and K_2 are ambient isotopic then $I(K_1) = I(K_2)$.

Definition 11 The *crossing number* $cr(K)$ of a knot K, is the smallest number of crossings of any diagram of the knot K.

Example 2.2 We have two knots: unknot and trefoil.

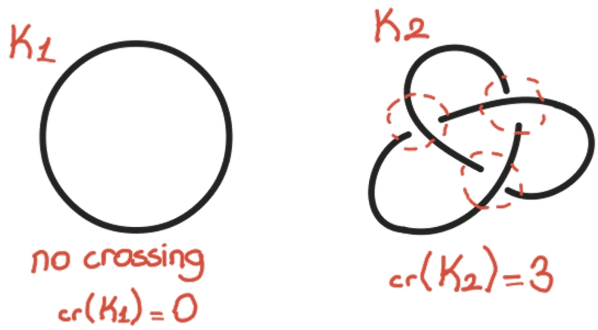


Figure 2.18. Crossing number

The unknot has $cr(K) = 0$ while the trefoil knot has $cr(K) = 3$.

Theorem 3 *The crossing number is an invariant of the knot.*

Proof Assuming that the diagram of K_1 has minimum a regular diagram D_1. It is a diagram of K_1 that has a minimum number of crossings. Let K_2 is equal to K_1 and suppose that D_2 is the minimum regular diagram. We assume K_1 and K_2 are equivalent. Then D_2

as a regular diagram of K_1. We can write these inequalities,

$$cr(D_1) \leq cr(D_2). \tag{2.1}$$

Also, since (D_2) is a diagram of K_2, it follows from the definition that

$$cr(D_2) \leq cr(D_1). \tag{2.2}$$

So, we combine these inequalities,

$$cr(D_1) = cr(D_2). \tag{2.3}$$

That is, (D_1) is the minimum number of crossing points between all knots that are equivalent to K. Therefore, it is a knot invariant. □

Definition 12 L be a link diagram. A crossing c of L is called *shared* if it is shared by two components of L.

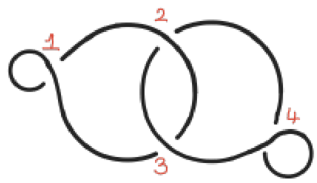


Figure 2.19. Shared crossings are 2 and 3

Definition 13 The *linking number* of $\vec{L}$ defines this formula,

$$L_k(\vec{L}) = \frac{1}{2} \sum \text{sign of shared crossings.} \tag{2.4}$$

Choose an orientation for each of the two components in the link and for every positive crossing, we count +1, and for every negative crossing, we count -1.

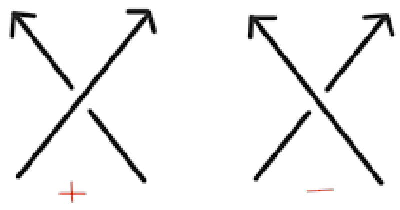


Figure 2.20. Positive and negative crossings

Example 2.3 There are two Hopf links with different orientations, L and L', both crossings are shared. Therefore, the computation includes every crossing. When we have a positive crossing in L, we find that the linking number is equal to 1. However, when there are negative crossings in L', we find the linking number is -1.

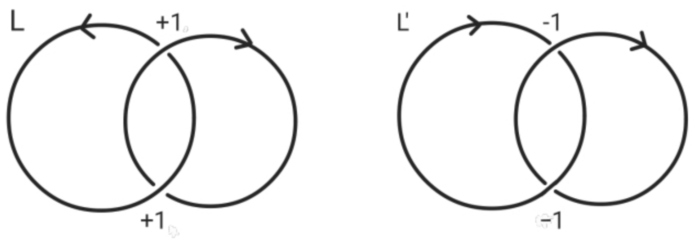


Figure 2.21. Hopf link with two different orientations

Thus, we find $L_k(\vec{L}) = 1$ and $L_k(\vec{L'}) = -1$.

Example 2.4 We have a Whitehead link, L". There are four shared crossings. However, we have that a crossing is not a shared crossing. We show it with a red circle in 2.22. This crossing is not a contribution to the linking number.

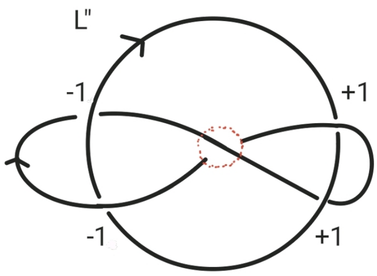


Figure 2.22. Whitehead link

Thus, we find $L_k(\vec{L}'') = 0$.

Theorem 4 *The linking number is an invariant of oriented links.*

Proof We need to verify that the linking number is preserved under oriented Reidemeister moves.

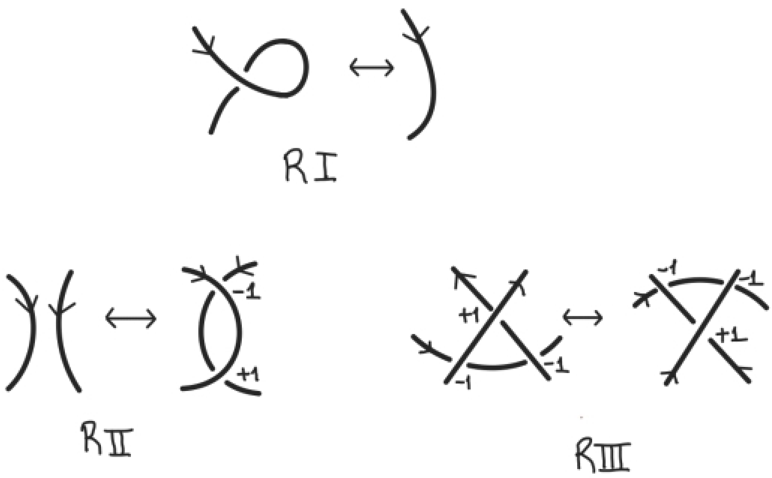


Figure 2.23. Oriented Reidemeister moves

For the first Reidemeister move, there is no shared crossing. It does not contribute to the linking number. In the second Reidemeister move, it has no crossing, so the linking number is 0. When applying the rule, two crossings are shared crossings, and we calculate a linking number that is equal to 0. Finally, the third Reidemeister move contribution is -1 for the linking number. The linking number does not change under the third Reidemeister move. Thus, the linking number is invariant under Reidemeister moves. □

We have seen two of the knots to be invariant so far in Chapter 3, we will see the details of another knot invariant 3-colorable and fox n-coloring.

CHAPTER 3

COLORABILITY OF KNOTS AND LINKS

In this chapter, we examine colorability of knots and links. We will first define a tricolorability of knots. Then we will define a fox-n coloring and we find a coloring matrix for knots. In this section, from (Dixon, 2010), (Livingston, 1993) and (Carter et al., 2014) are used for definitions and examples..

3.1. Tricolorability

Definition 14 A knot is *tricolorable* if each strand shown in the knot diagram can be colored with a different color as long as the rules below are followed:

1. A minimum of two colors must be used; and
2. At every intersection, the three strands are either some color or distinct colors.

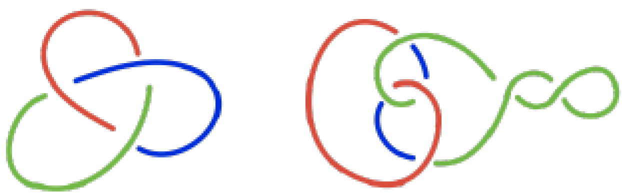


Figure 3.1. Two colored diagrams of trefoil

Theorem 5 *Tricolorability is a knot invariant under Reidemeister moves.*

Proof We want to show this by analyzing each of the three Reidemeister moves.

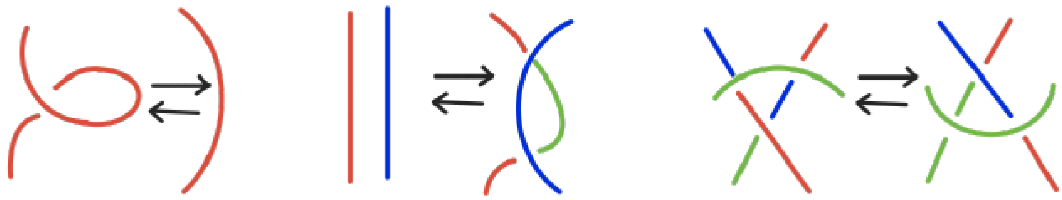


Figure 3.2. Colored Reidemester moves

Since each Reidemeister move can be colored with rules, tricolorability is knot invariant. □

Example 3.1 This rule is explained with two knots: trefoil and figure eight.

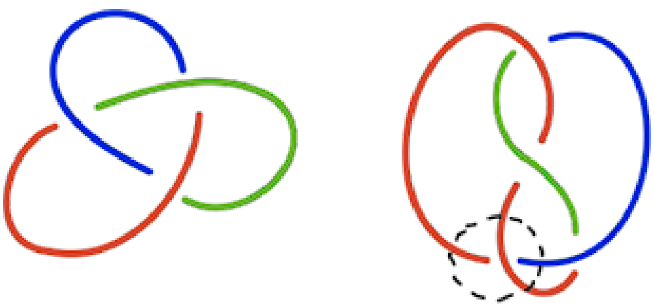


Figure 3.3. Colorable trefoil and non colorable figure eight

Theorem 6 *The unknot is not tricolorable.*

Proof The unknot is a closed loop with no crossings. So, when this knot is to be colored, only one color can be used because there is only a single strand. Therefore, the unknot is not tricolorable since it does not satisfy at least two different color conditions.

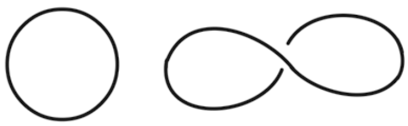


Figure 3.4. Unknot colored with one color

□

Theorem 7 *If a knot is tricolorable, then it is not equivalent to the unknot.*

Proof A knot K that is colorable is taken. Then assume K is equal to unknot. It is a contradiction because we now have a tricolorable knot, and if a knot is colorable, then every projection of a knot is colorable. At the same time, it is known that the unknot is not tricolorable, which can be seen in figure6. Thus, these knots are not equal to some knots. □

Theorem 8 *Trefoil is not equivalent to a trivial knot.*

Proof First, suppose that the trivial knot is not tricolorable, as it is an unknot, which has no crossings. Therefore, we cannot use at least two distinct colors. It has only a single strand so we can use just one color. Also, the trefoil has tree crossings. There are three strands, and these can use three distinct colors. This means that the trefoil is tricolorable. Thus, it shows that trefoil is not equivalent to a trivial knot. □

Let K be a knot diagram. If K is colorable, then either every regular projection of a knot must be tricolorable, or if K is not colorable, every projection of a knot must be non-tricolorable.

Example 3.2 *Every projection of the trefoil knot is tricolorable.*

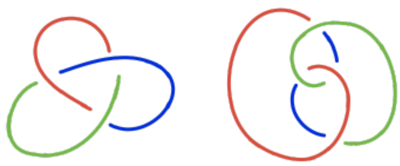


Figure 3.5. Two regular projections of the trefoil

3.2. Fox n-Coloring

After the definition of tricolorable, we will see the fox-n coloring definition. We will find out how many colors can be colored for non-tricolorable knots.

Definition 15 A knot diagram can be labeled mod n if each edge can be labeled with $x_1, x_2, x_3, \ldots, x_n$,

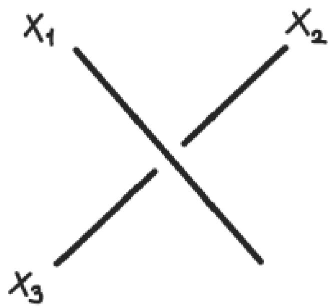


Figure 3.6. Labelled arcs

where x_1 is the label on the overcrossing and x_2 and x_3 are the other two labels. At each crossing the relation

$$2x_1 - x_2 - x_3 = 0 mod(n). \tag{3.1}$$

Theorem 9 *Fox n-coloring is a knot invariant.*

Proof In order to be an invariant, the Fox n-coloring must be provided by Reidemeister moves. So, we will examine the crossing relation under Reidemeister moves.
The first Reidemeister move has a one-crossing relation. This gives a relation,

$$2x_1 - x_1 - x_1 = 0 mod(n). \tag{3.2}$$

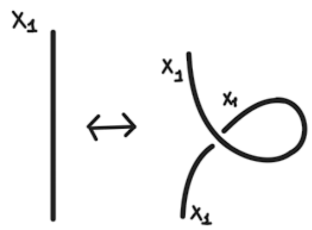


Figure 3.7. Fox-n coloring Reidemeister move 1

The second Reidemeister move has two crossing relations.

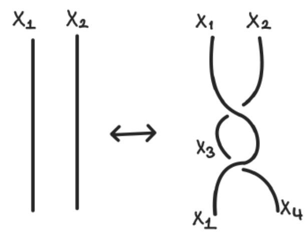


Figure 3.8. Fox-n coloring Reidemeister move 2

This gives the following relations,

$$2X_1 - X_2 - X_3 = 0\,(modn) \tag{3.3}$$

$$2X_1 - X_3 - X_4 = 0\,(modn) \tag{3.4}$$

$$X_2 = X_4\,(modn) \tag{3.5}$$

$$\tag{3.6}$$

And find $x_2 = x_4$

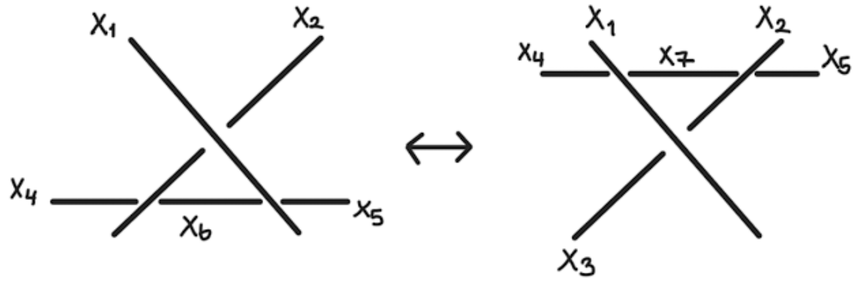


Figure 3.9. Fox-n coloring Reidemeister move 3

The third move has three crossing relations. This gives the following relations,

$$2X_1 - X_2 - X_3 = 0\,(modn) \tag{3.7}$$

$$2X_1 - X_5 - X_6 = 0\,(modn) \tag{3.8}$$

$$2X_3 - X_4 - X_6 = 0\,(modn) \tag{3.9}$$

$$2X_1 - X_2 - X_3 = 0\,(modn) \tag{3.10}$$

$$2X_1 - X_4 - X_7 = 0\,(modn) \tag{3.11}$$

$$2X_2 - X_5 - X_7 = 0\,(modn) \tag{3.12}$$

Hence, it can be shown that n-colorability is a knot invariant under the Reidemester moves.

□

3.3. The coloring matrix

Definition 16 Let K be a knot and M be a coloring matrix of K. The matrix form is K, $n \times n$, such that each column represents an arc in the projection of K and each row represents a crossing in K. There are n crossings in K. Then we find the n equation for K. Each equation becomes with this rule: First, we label all the arcs. The over strand multiplies 2 and the under two strands multiply -1 for each crossing.

Example 3.3 Consider the trefoil knot. We aim to determine a coloring matrix for it. The trefoil knot has three crossings, each corresponding to a distinct crossing relation.

First, label each arc and crossing.

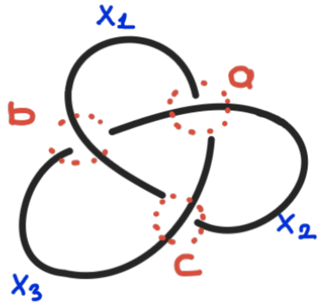


Figure 3.10. Trefoil knot

Then find all the equations for each crossing.

$$a; 2X_2 - X_1 - X_3 = 0 \tag{3.13}$$

$$b; 2X_1 - X_2 - X_3 = 0 \tag{3.14}$$

$$c; 2X_3 - X_1 - X_2 = 0 \tag{3.15}$$

These equations give;

$$\begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix} \tag{3.16}$$

As a result, we obtain linearly dependent rows. When the determinant is calculated, the result received equals to zero. Therefore, we need to make an additional application. We remove one column and one row of the matrix. The remaining matrix form becomes our coloring matrix. Then, the result of the determinant tells us how many colorable these knots can be with. We get this matrix;

$$\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \tag{3.17}$$

$Det(K)$ is free of the choice of the deleted row and column. We conclude with a proposition. We calculate the determinant of this matrix, which equals 3. Therefore, it is possible to say that the trefoil is tricolorable.

Example 3.4 In this example, we will focus on the figure eight knot. In one of the previous sections, we determined that the figure eight knot is not tricolorable. The main objective here is to find its coloring matrix and calculate its determinant. The result will reveal the colorability of the figure eight knot.

The figure eight knot has four crossings and four arcs, resulting in four relations. Let us start by labeling each arc and crossing.

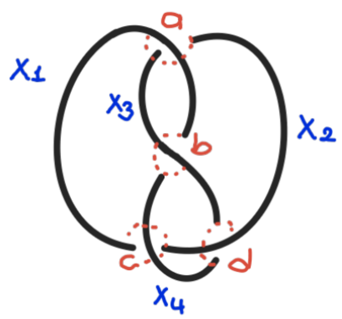


Figure 3.11. Figure eight knot

Then find all the equations for each arc.

$$a; 2X_1 - X_2 - X_3 = 0 \tag{3.18}$$

$$b; 2X_3 - X_1 - X_4 = 0 \tag{3.19}$$

$$c; 2X_4 - X_1 - X_2 = 0 \tag{3.20}$$

$$d; 2X_2 - X_3 - X_4 = 0 \tag{3.21}$$

These equations give;

$$\begin{bmatrix} 2 & -1 & -1 & 0 \\ -1 & 0 & 2 & -1 \\ -1 & -1 & 0 & 2 \\ 0 & 2 & -1 & -1 \end{bmatrix} \tag{3.22}$$

Again, we obtain linearly dependent rows. We remove one column and one row from the matrix. The remaining matrix form becomes the coloring matrix. Then, the result of the determinant tells how many colorable these knots can be with.
The matrix below is obtained:

$$\begin{bmatrix} 2 & -1 & -1 \\ -1 & 0 & 2 \\ -1 & -1 & 0 \end{bmatrix} \tag{3.23}$$

We calculate the determinant of this matrix, which is equal to 5. So we can say that the figure eight knot color with five colors.

Theorem 10 *The determinant is a knot invariant.*

Proof The reader can find the sketch of the proof in (Livingston, 1993). □

In the following chapter, we see a generalization of the colorable idea to the algebraic structure: quandles.

CHAPTER 4

ALGEBRAIC STRUCTURE FOR KNOTS

In this chapter, we give definitions of kei, quandles, singquandles, and bondles. Then we give some examples.
Quandle theory is a developing subject in abstract algebra that has applications to various other areas of mathematics, such as knot theory. Quandle theory dates back to the 1940s, when Mituhisa Takasaki introduced the notion of kei. The quandle theory was introduced in the doctoral dissertation of David Joyce in 1990. The Quandle variations ideas have been studied by Conway, Brieskorn, Matveev, and Kauffman, Nelson see these papers, (Joyce, 1982)(Kamada, 2002), (Carter, 2010), (Kauffman and Manturov, 2004), (Elhamdadi and Nelson, 2015).

4.1. Kei

In this section, we define the kei for non-oriented knots. Mituhisa Takasaki chose the term "Kei".

Definition 17 Consider a set X with the binary operation $*: X \times X \to X$. This structure is called a *Kei* if the following axioms hold for any elements $x, y, z \in X$.

1. $x * x = x$ (idempotent)
2. $(x * y) * y = x$ (involution)
3. $(y * z) * x = (y * x) * (z * x)$ (right self-distributive)

The first kei axiom is known as idempotency. This can be explained as the following: there is a matrix A that is idempotent if $A^2 = A$, a projection map onto a coordinate axis. This axiom can mean that every element acted like 0, since operations like addition. The second axiom means that elements affect other elements through involutions. It can be explained as having a function $\beta_y : X \to X$ that has its own inverse defined by $\beta_y(x) = x * x$. The last axiom is self-distributive. Similar multiplication distributes over addition.

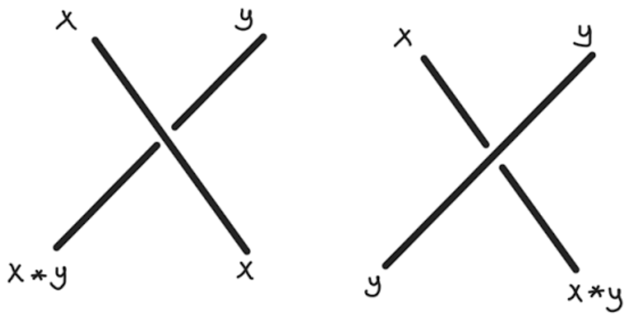


Figure 4.1. Operation on crossings

The fundamental idea of this structure is colorability. One can consider each element of set X as the color assignment of a knot diagram. We have a diagram and arcs. We define an operation as $*$. The $x * y$ operation corresponds to one arc x passing under another arc y to become $x * y$. A new arc $x * y$ is present.
Now, we see Reidemeister moves with kei axioms;

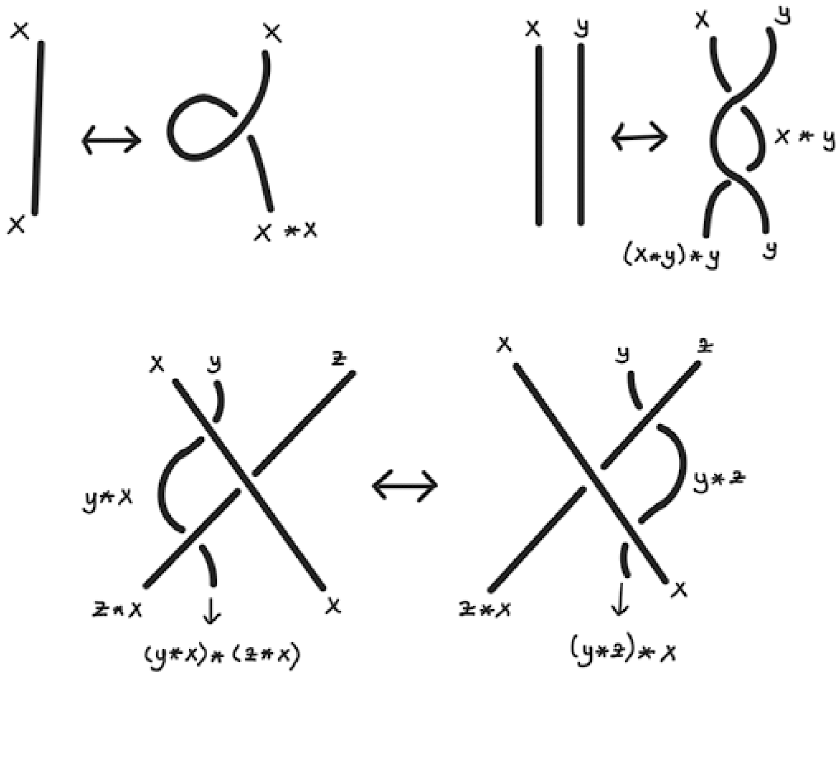


Figure 4.2. Kei Reidemeister moves

There is one arc, x. Twisting is applied on an arc in the first Reidemeister move. There is one crossing in the first move. The crossing gives us $x * x = x$. In the second Reidemeister move, there are two arcs, then pass one arc over or under another arc. After this move, we result obtained is $(x * y) * y = x$. In this part, the twice quandle operation is applied. The third Reidemeister move is the right self-distributive one. When we apply the quandle operation, we find $(y * z) * x = (y * x) * (z * x)$.

We have an example,

Example 4.1 Takasaki kei is a different term that describes the kei operation, which is also known as cyclic kei or dihedral quandle. Let $X = \mathbb{Z}$ and define

$$x * y = 2y - x. \tag{4.1}$$

To demonstrate that this $*$ operation is indeed a kei operation, one needs to confirm that it satisfies all three kei axioms.
In first axiom : $x * x = x$, and we apply Takasaki kei,

$$x * x = 2x - x = x. \tag{4.2}$$

In second axiom : $(x * y) * y = x$, and there is;

$$(x * y) * y = (2y - x) * y = 2y - (2y - x) = 2y - 2y + x = x. \tag{4.3}$$

In third axiom: $(x * y) * z = (x * z) * (y * z)$, and there is;

$$(x * y) * z = (2y - x) * z = 2z - (2y - x) = 2z - 2y + x, \tag{4.4}$$

$$(x * z) * (y * z) = (2z - x) * (2z - y) = 2(2z - y) - (2z - x), \tag{4.5}$$

$$4z - 2y - 2z + x = 2z - 2y + x. \tag{4.6}$$

Therefore, the fact that Takasaki Kei satisfies the Kei axioms are proved.

Definition 18 Let **K** be a knot, link, or tangle that has a corresponding associated kei called the *fundamental kei*, denoted by $\mathcal{K}(K)$. This fundamental kei can be obtained from a diagram of K using principal universal algebra.
Let K be a set, where the elements of $X = \{x_1, \ldots, x_n\}$ are called generators.

Now, consider $X = \{x_1, \ldots, x_n\}$ to be a set containing one element for each arc in a knot diagram, K. All crossings in our diagram K yield an equation, referred to as a crossing relation, representing Kei form $x * y = z$.

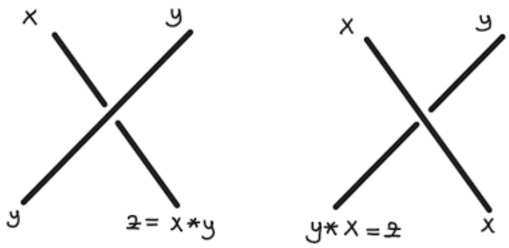


Figure 4.3. Relation of the Kei

K be a knot, and the fundamental kei of the knot can be defined as the set of equivalence classes of elements in the kei on X, where the crossing relations determine equivalence. Usually, one represents this using a *kei presentation* that describes the components of X, referred to as generators, along with the crossing relations.

Definition 19 Let $\mathcal{K}(K)$ be the fundamental kei, which is constant regardless of the chosen diagram of K. The set of all kei homomorphisms can be identified by *counting the number* of colorings of every diagram of K using a kei X.
We have $f(x * y) = f(x) * f(y)$ and,

$$\mathrm{Hom}(\mathcal{K}(K), X) = \{f : \mathcal{K}(K) \to X \mid f(x * y) = f(x) * f(y)\} \tag{4.7}$$

The cardinality of this set is

$$\mathrm{Hom}(\mathcal{K}(K), X). \tag{4.8}$$

This equation means the kei counting invariant is a computable link invariant.

Example 4.2 Consider the trefoil knot. The aim is to determine a fundamental kei presentation for it. The trefoil knot has three crossings, each corresponding to a distinct crossing relation.

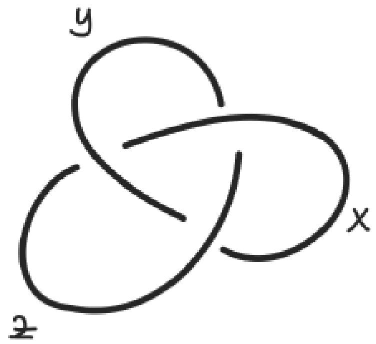


Figure 4.4. Labelled a diagram of trefoil

$$\mathcal{K}(K) = \langle x, y, z \mid x * y = z, y * z = x, z * x = y \rangle. \tag{4.9}$$

Example 4.3 Take the Hopf link with the Takasaki kei $\mathbb{Z}_4$. And the relations R_1 and R_2 are $x * y = x$ and $y * x = y$ for crossings. Let us calculate the kei counting invariant for the Hopf link.

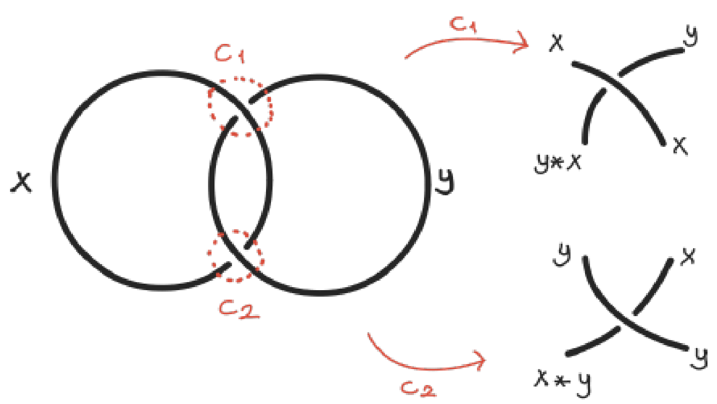


Figure 4.5. Examine two crossings for Hopf Link

First, the crossings are called C_1 and C_2. The C_1 crossing gives the $y*x = y$ relation, while C_2 gives us the $x*y = x$ relation. The four elements Takasaki kei $\mathbb{Z}_4$ are used for relations. This can be explained as follows: $y*x = y$ provides $2x - y = y$, and $x*y = x$ provides $2y - x = x$ according to the elements of $\mathbb{Z}_4$.

In the table, the column represents x, while the line represents y. We take 3 for x and 1 for y and calculate with $x*y = x$; this relation must result $3*1 = 3$ according to the table. Calculated with Takasaki kei, $2*1-3 = -1$ and $-1 = 3mod(4)$. We find all the results in the table.

*	0	1	2	3
0	0	2	0	2
1	3	1	3	1
2	2	0	2	0
3	1	3	1	3

(4.10)

When we analyze the table, we want to see if the relations $y*x = y$ and $x*y = x$ are satisfied. If it is satisfying, then we color with these elements.

$f(x)$	$f(y)$	R_1	R_2	$f(x)$	$f(y)$	R_1	R_2
0	0	✓	✓	2	0	✓	✓
0	1			2	1		
0	2	✓	✓	2	2	✓	✓
0	3			2	3		
1	0			3	0		
1	1	✓	✓	3	1	✓	✓
1	2			3	2		
1	3	✓	✓	3	3	✓	✓

(4.11)

Therefore, the cardinality of $\mathrm{Hom}(\mathcal{K}(K))$ is equal to 8.

Example 4.4 As a last example, in this part, we have figure eight knot. The counting invariant is computed.

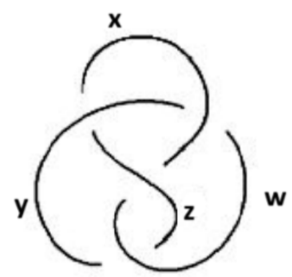


Figure 4.6. Labelled a diagram of figure eight knot

Then the following applies for crossing relation,

$$\mathcal{K}(K) = \langle x, y, z, w \mid x * y = z, y * w = z, y * x = w, x * z = w \rangle. \tag{4.12}$$

First, eliminating the generator w,

$$\mathcal{K}(K) = \langle x, y, z \mid x * y = z, y * (y * x) = z, x * z = y * x \rangle. \tag{4.13}$$

Second, eliminating the generator z,

$$\mathcal{K}(K) = \langle x, y \mid y*(y*x) = x*y, x*(x*y) = y*x\rangle. \tag{4.14}$$

As a result one obtains two crossing relation R_1 and R_2 that are equal to these equations, respectively, $y*(y*x) = x*y$ and $x*(x*y) = y*x$.

$*$	0	1	2	3
0	0	2	0	2
1	3	1	3	1
2	2	0	2	0
3	1	3	1	3

(4.15)

$f(x)$	$f(y)$	R_1	R_2	$f(x)$	$f(y)$	R_1	R_2
0	0	✓	✓	2	0		
0	1			2	1		
0	2			2	2	✓	✓
0	3			2	3		
1	0			3	0		
1	1	✓	✓	3	1		
1	2			3	2		
1	3			3	3	✓	✓

(4.16)

4.2. Quandles

In this section, we will define a quandle. Kei is also called the involutary quandle. Quandles define oriented knots as opposed to kei.

Definition 20 A *quandle* is a set X and with an operation $*: X \times X \to X$ that satisfies the following three conditions for all $x, y, z \in X$.

1. $x * x = x$.

2. A function $\beta_y : X \to X$ defined by $\beta_y(x) = x * y$ is invertible.

3. $(x * y) * z = (x * z) * (y * z)$.

We will use $x *^{-1} y$ instead of $\beta_y^{-1}(x)$. We explain second axiom: it can be written $\beta_y(\beta_y(x)) = x$. Then, β_y is its own inverse function, and write $\beta_y^{-1}(x) = \beta_y(x)$.

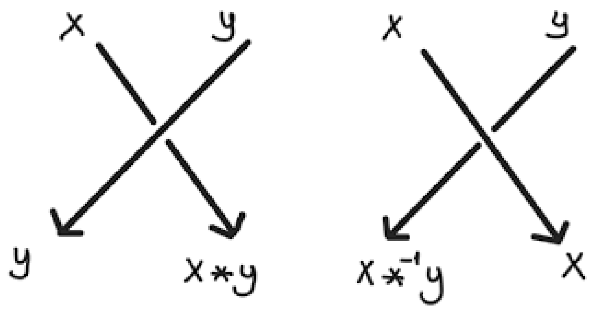


Figure 4.7. Quandle coloring at a positive and negative crossing respectively

When we color each arc of an oriented knot diagram with a quandle element of X, one can that the axiom assumes giving invariant quandle coloring under Reidemister moves.

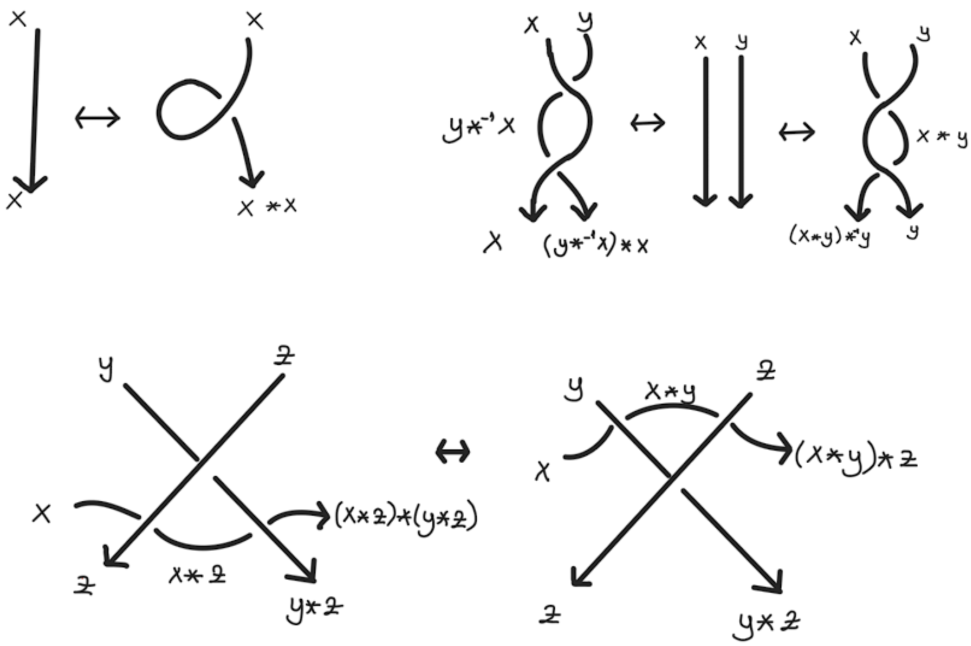


Figure 4.8. Quandle Reidemeister moves

Examples of quandles:

Example 4.5 Let X be set, with the operation $*$ called the trivial quandle. Such that $x * y = x$ for all $x, y \in X$. The trivial quandle with n elements is denoted by T_n.

Example 4.6 G is any group; then it defines a quandle with conjugation operation such that

$$x * y = y * x^{-1}. \tag{4.17}$$

Another example of a quandle is the Alexander quandle. This part will be examined in detail.

4.2.1. Alexander Quandles

Definition 21 Consider a module M over the ring of Laurent polynomial $L = Z[t^{\pm 1}]$. The action of L on M induces an Alexander quandle structure on M, where the quandle operation $*$ is given by:

$$x * y = tx + (1-t)y. \tag{4.18}$$

Theorem 11 Alexander quandle is invariant.

Proof Alexander Quandle verifies the axioms of a quandle. Let us consider the first axiom,

$$x * x = tx + (1-t)x = (t+1-t)x = x. \tag{4.19}$$

Then we have second axiom,

$$x = tw + (1-t)y, \tag{4.20}$$

$$x - (1-t)y = tw, \tag{4.21}$$

$$t^{-1}x - t^{-1}(1-t)y = w, \tag{4.22}$$

$$t^{-1}x - \left(t^{-1} - 1\right)y = w, \tag{4.23}$$

$$t^{-1}x + \left(1 - t^{-1}\right)y = w, \tag{4.24}$$

Thus, we have

$$x *^{-1} y = t^{-1}x + \left(1 - t^{-1}\right)y. \tag{4.25}$$

Finally, we have third axiom:

$$(x * y) * z = t(x * y) + (1-t)z \tag{4.26}$$

$$= t(tx + (1-t)y) + (1-t)z \tag{4.27}$$

$$= t^2x + t(1-t)y + (1-t)z, \tag{4.28}$$

since

$$(x*z)*(y*z) = t(x*z)+(1-t)(y*z) \tag{4.29}$$
$$= t(tx+(1-t)z)+(1-t)(ty+(1-t)z) \tag{4.30}$$
$$= t^2x+t(1-t)y+\left[t(1-t)+(1-t)^2\right]z \tag{4.31}$$
$$= t^2x+t(1-t)y+\left[t-t^2+1-2t+t^2\right]z \tag{4.32}$$
$$= t^2x+t(1-t)y+(1-t)z. \tag{4.33}$$

Thus, it can be shown that relations satisfy all axioms.

□

Definition 22 An oriented knot diagram K. Let us label all arcs with $x_1, x_2, \ldots, x_n$. And we get a quandle relation $x_a * x_b = x_c$ for each crossing. Let's consider this as an Alexander Quandle relation, where we can write it as the equation $tx_a + (1-t)x_b = x_c$ or $tx_a + (1-t)x_b - x_c = 0$. Therefore, we have a system of linear equations in which all the equations are of identical form, and it can be represented it as a matrix equation: $A\vec{x} = \overrightarrow{0}$. In matrix rows, for each crossing, write t, $1-t$, or -1 for the arcs. If arcs involve crossing, one enters 0.

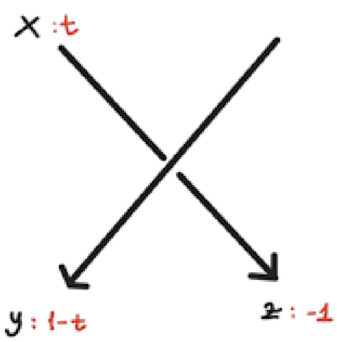


Figure 4.9. Alexander quandle relation

The Alexander Module of K is the kernel of matrix A. We explain this statement by construing the relationship between the fundamental quandle of knot K and an Alexander quandle. Matrix A is referred as the presentation matrix for the Alexander module.

Example 4.7 There is a figure-eight knot. If we want to find a presentation matrix so that we can label the arcs in a diagram with the crossing relations according to Alexander Quandle operations:

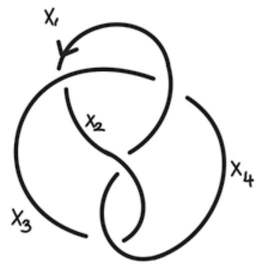


Figure 4.10. Figure eight knot

$$\begin{vmatrix} -1 & t & 1-t & 0 \\ 1-t & 0 & -1 & t \\ -1 & 1-t & 0 & t \\ 0 & t & -1 & 1-t \end{vmatrix} \tag{4.34}$$

We can remove one row and one column; the specific choice does not affect the end result. Following that, we compute the determinant. In this example, eliminate row 1 and column 1:

$$\begin{vmatrix} 0 & -1 & t \\ 1-t & 0 & t \\ t & -1 & 1-t \end{vmatrix} = 0-(-1)\begin{vmatrix} 1-t & t \\ t & 1-t \end{vmatrix} + t\begin{vmatrix} 1-t & 0 \\ t & -1 \end{vmatrix} \tag{4.35}$$

$$= 0-(-1)\left[(1-t)^2-t^2\right]+t(t-1) \tag{4.36}$$

$$= \left[1-2t+t^2-t^2\right]+t^2-t \tag{4.37}$$

$$= [1-2t]+t^2-t \tag{4.38}$$

$$= 1-2t+t^2-t \tag{4.39}$$

$$= 1-3t+t^2 \tag{4.40}$$

$$\tag{4.41}$$

And at the same time this equation represents that of Alexander Polynomial for figure eight.

4.3. Singquandles

In this section, we will introduce the structure of a singquandle, which is used to describe singular knots. Unlike quandles, which are associated with classical knots, singquandles are defined as singular knots. We will begin by providing a brief overview of singular knots.

4.3.1. A Review of singular knot

In this section, first we give a fundamental definition and theorem for singular knots. We will use the articles (Juyumaya and Lambropoulou, 2009), (Ceniceros et al., 2021).

Definition 23 A *singular link* with n components is created by immersed circles in three-dimensional space. This embedding allows for a finite number of self-intersections, called *singular crossings*. These crossings are restricted to simple double points, meaning that at any given point, only two strands of the link intersect. In essence, a singular link resembles a classical link, but with the added flexibility of a finite number of permitted transversal self-intersections. When a singular link consists of a single component, it is called a *singular knot*.

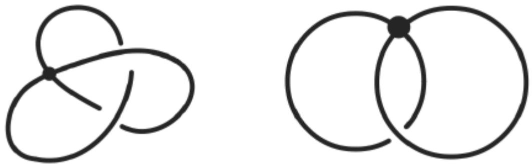


Figure 4.11. Two examples of singular knot and link

K_1, K_2 be two singular links that are isotopic, which means that if there is a homeomorphism of S^3 carrying one to the other.

Theorem 12 *Two singular link diagrams are isotopic if and only if they can be transformed into each other using planar isotopy and a finite number of classical Reidemeister movements see in figure 2.5 and singular Reidemeister motions, see in figure 4.12.*

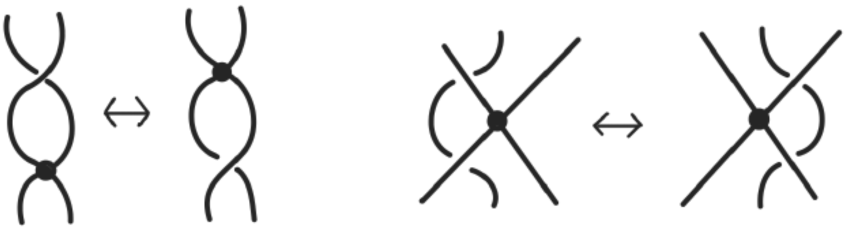


Figure 4.12. Singular Reidemeister moves

4.3.2. Singquandles

We see the definition of a singular knot, we can now introduce the concept of singquandles. Unlike classical knots, singular knots lack over- and under-information at their crossings. Therefore, we will define two maps to examine the singquandle structure on these knots.

Definition 24 Let $(X,*)$ be a quandle. There are R_1 and R_2 be two mapp, from the set $X \times X$ to X. An oriented singquandle is defined as the quadruple $(X,*,R_1,R_2)$ that satisfies the following axioms for any $x,y,z \in X$:

$$R_1(x*y,z)*^{-1}y = R_1(x,z*^{-1}y) \tag{4.42}$$

$$R_2(x*y,z) = R_2(x,z*^{-1}y)*y \tag{4.43}$$

$$\left(y*^{-1}R_1(x,z)\right)*x = (y*R_2(x,z))*^{-1}z \tag{4.44}$$

$$R_2(x,y) = R_1(y,x*y) \tag{4.45}$$

$$R_1(x,y)*R_2(x,y) = R_2(y,x*y). \tag{4.46}$$

Let's have an oriented singular knot and apply the quandle coloring. In the previous section, quandle coloring had been explained. Until now, we have had information under over-crossing. When we have a singular knot, some crossings do not have this information. So, when we apply the quandle coloring, we need two maps.

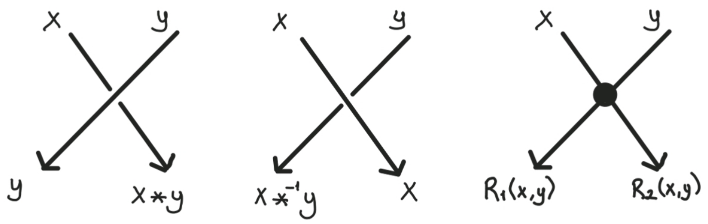


Figure 4.13. Quandle coloring at a positive, negative, and singular crossing respectively

There are three types of crossings in figure 4.13. In positive and negative crossing, one should use quandle coloring. But the other crossing is a singular crossing; and therefore we use two maps, respectively, R_1 and R_2. When we color each arc of the knot diagram, we see that the axioms give invariant singquandle coloring under Reidemeister moves.

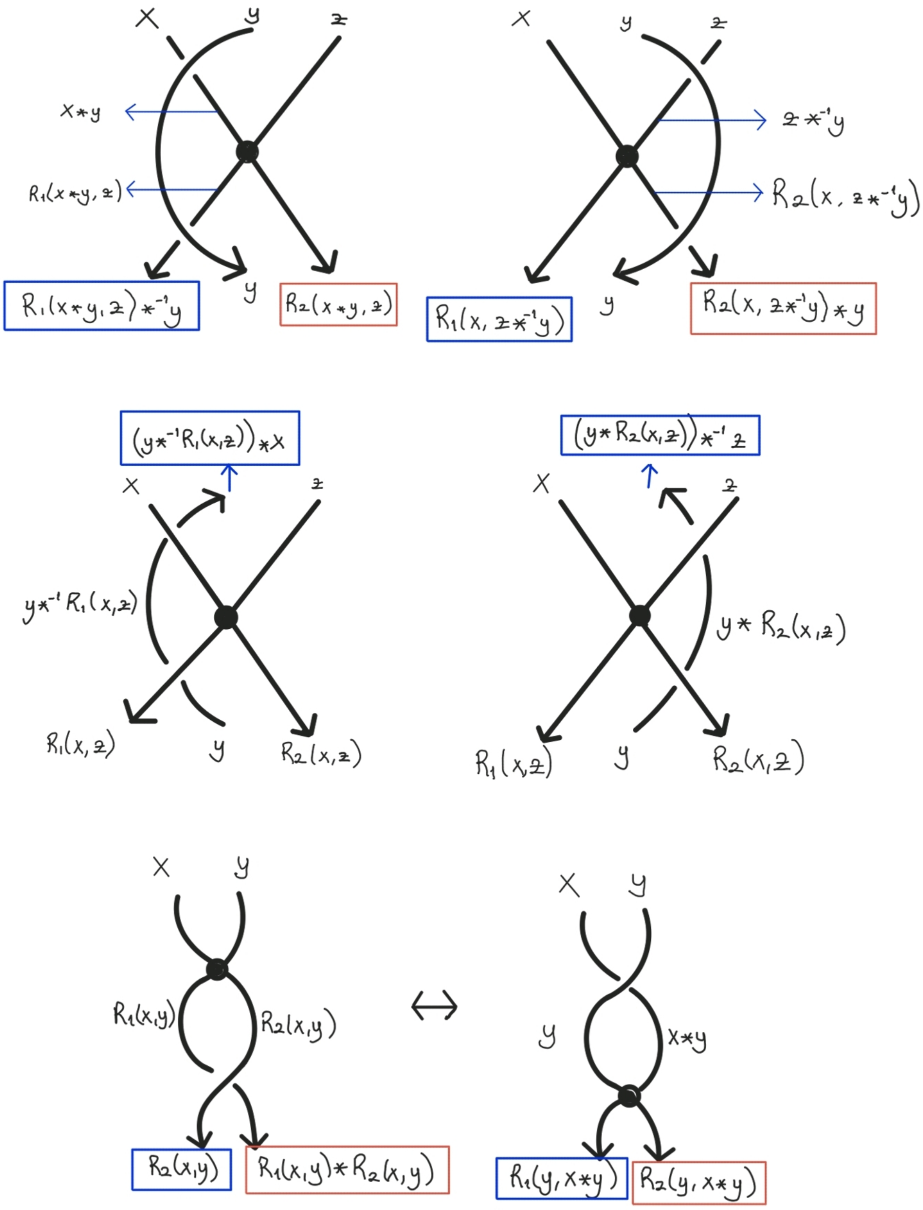


Figure 4.14. Singquandle Reidemeister moves

4.4. Bondles

In the previous section, we explained the quandle and singquandle structures. In this section, we will define the bondle structure. Although singquandles were defined for singular knots with singular crossings, we will analyze these crossings in a new way. This

new interpretation will allow us to develop the singquandle structure further and ultimately define the bondle structure. In the following chapter, we will apply the bondle structure to the analysis of protein structures.
There are two different bondles that are depending on the chosen orientation which can seen in 4.15.

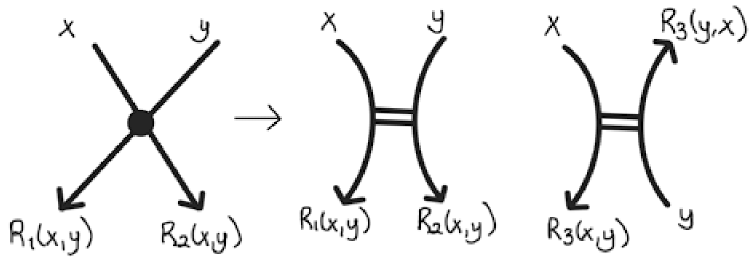


Figure 4.15. Parallel and anti-parallel strands

Despite the addition of two more functions, which is $R_3(x,y)$ and $R_4(x,y)$, doing the act of rotating by 180 degrees results in a switch of the roles of x and y, which also changes the roles of R3 and R4. Therefore, the result change $R_3(x,y) = R_3(y,x)$. This will be used to eliminate $R_4(x,y)$ from all that follows relations.

Definition 25 An orientated bondle is a quandle that has an operation denoted by $*$, and also has three functions, $R_1(x,y)$, $R_2(x,y)$, and $R_3(x,y)$, which are used for determining choices. These functions must satisfy the following conditions:

$$R_3\left(y, x *^{-1} z\right) = R_3(y * z, x) *^{-1} z, \tag{4.47}$$

$$R_3(x, y * z) = R_3\left(x *^{-1} z, y\right) * z, \tag{4.48}$$

$$\left(z *^{-1} R_3(x,y)\right) * x = \left(z *^{-1} y\right) * R_3(y,x), \tag{4.49}$$

$$R_3(x,y) *^{-1} y = R_3\left(x *^{-1} R_3(y,x), y\right). \tag{4.50}$$

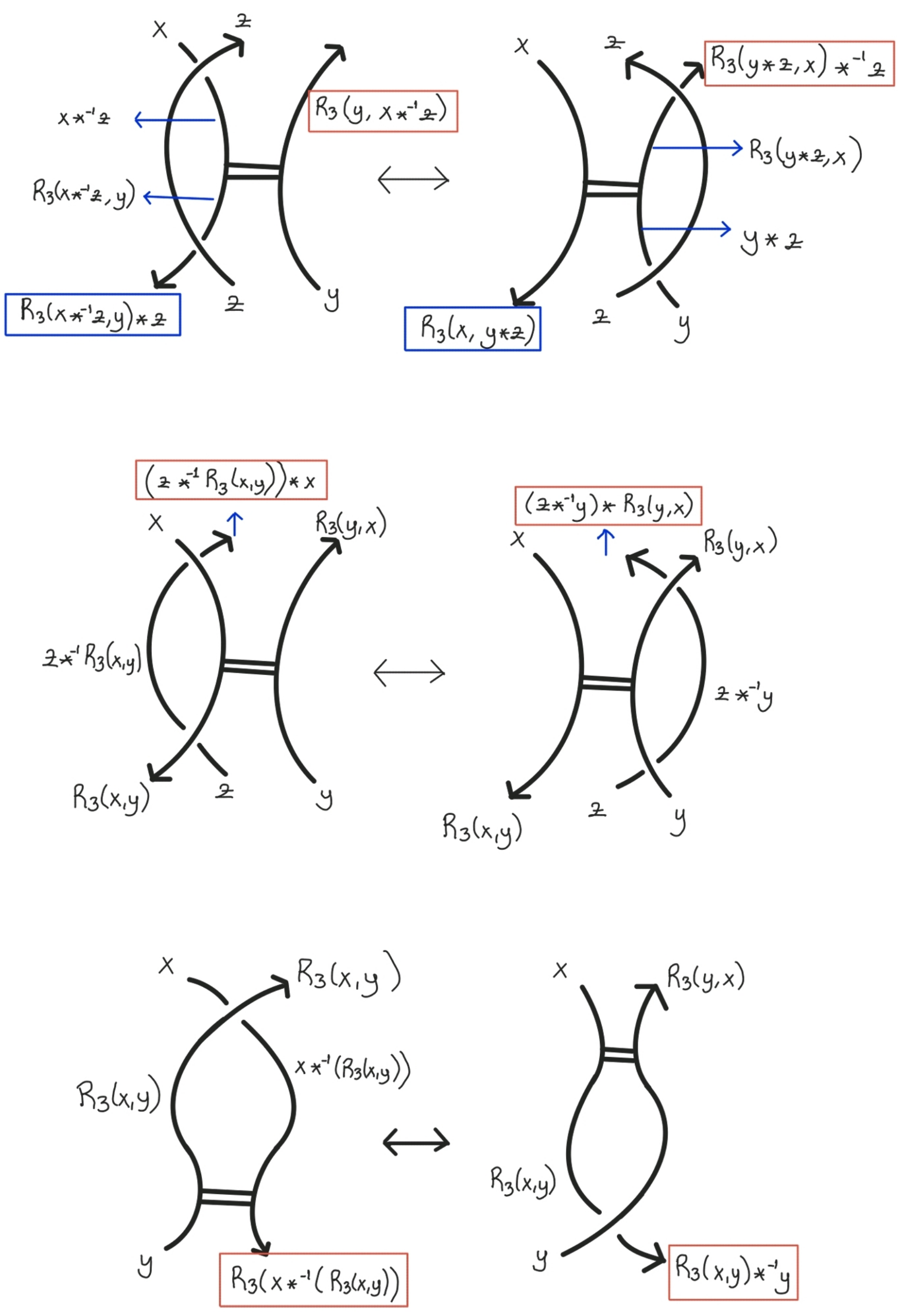


Figure 4.16. Bondle Reidemeister Moves

CHAPTER 5

CIRCUIT TOPOLOGY

The aim of this chapter is to use the algebraic structures of quandles to study circuit topology. Firstly, the chapter provides motivation for understanding the circuit topology and the structure of proteins. Then it explains the relation with bondle and finally, it defines the invariant of circuit topology. In this section, we will use these articles (Ceniceros et al., 2021), (Mashaghi et al., 2014), (Golovnev and Mashaghi, 2020) and (Adams et al., 2020).

5.1. A Motivation

Circuit topology refers to a mathematical method used to classify the organization of electrical connections. Researchers are now investigating the connection between circuit topology and the well-accepted field of knot theory. Although circuit topology focuses on interactions, knot theorists' methods might enhance its effectiveness. Knot theorists have created many coloring structures that might potentially be expanded to categorize topological circuits. A new quandle coloring method has been recently devised for protein analysis, with the potential to be modified to determine the range of potential circuit topologies. The objective of this thesis is to utilize the algebraic structures of quandles for the analysis of circuit topology. Using knot theory techniques, we create an unchanging characteristic that enables us to categorize and differentiate chain configurations within the context of circuit architecture.

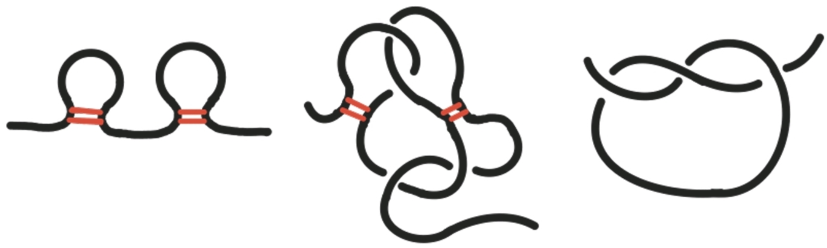


Figure 5.1. These represent circuit topology, generalized circuit topology, and knot theory respectively

Proteins and nucleic acids, which are folded linear molecule chains, play an important part in cell processes and the transmission of biological information. They frequently fold to function to accomplish their purpose, offering important information for the development of proteins. Several techniques have examined the geometric and chemical characteristics of folded proteins and genomic DNA. However, the topological qualities were given less attention due to the absence of a beneficial conceptual framework. The application of knot theory to the study of proteins and nucleic acids has produced successful outcomes. However, the effect on protein science has been restricted because almost all of the discovered proteins belong to a single topological class, namely the unknot.

The knot theory approach fails to effectively classify proteins due to its disregard for intra-molecular interactions or contacts, which drive the process of folding in molecular chains and provide an essential part of their function. The inclusion of intra-chain interaction is considered, and the prevalence of knots and links significantly improves. Hence, it is important to develop a new topology framework that covers interactions inside a chain and can effectively categorize the folding topology of biomolecular chains, namely proteins.

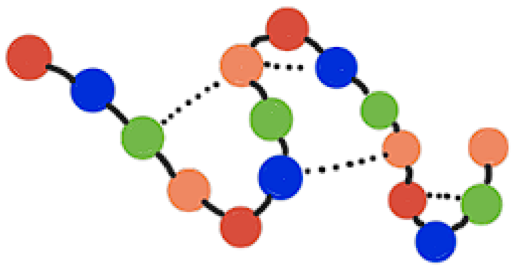


Figure 5.2. Structure of a protein

5.2. Coloring invariant for circuit topology

In this part, we examine the invariant of circuit topology. The topological equivalence of two circuit topologies can be determined by comparing the number of distinct colorings for each topology, according to the chosen bondle. We want to determine a set of colorings for a circuit topology X using an identified bondle B.

In circuit topology, the topological arrangements of the loops are defined by examining a linear polymer chain with N contacts and M loops. In this part, the contact sites are labelled as $Cs = Cs_1, Cs_2, \ldots, Cs_N$ and loops as $L = L_1, L_2, \ldots,$ and L_M.

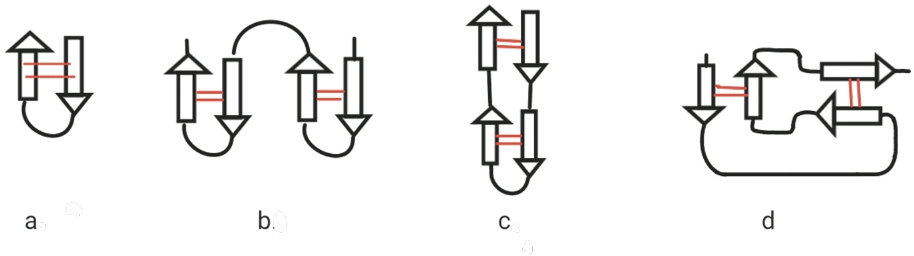


Figure 5.3. Structure and topology figuration general, series, parallel and cross relation

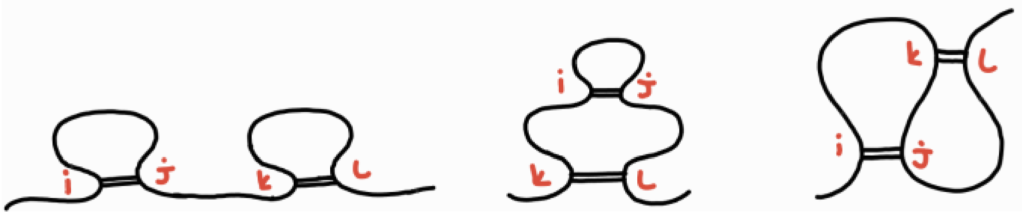


Figure 5.4. Series, parallel, and cross contacts

The circuit topology relations:

Series: $L_1\mathbf{S}L_2 \Leftrightarrow \left[Cs_i, Cs_j\right] \cap [Cs_k, Cs_l] = \varnothing$.

Parallel: $L_1\mathbf{P}L_2 \Leftrightarrow \left[Cs_i, Cs_j\right] \subset [Cs_k, Cs_l]$.

cross: $L_1\mathbf{X}L_2 \Leftrightarrow \left[Cs_i, Cs_j\right] \cap [Cs_k, Cs_l] \notin \left\{\varnothing, \left[Cs_i, Cs_j\right], \left[Cs_k, Cs_j\right]\right\}$.

Definition 26 $Col_B(X)$ is defined as the set of colorings bondle B in the circuit topology X.

Definition 27 Let's assume a circuit topology X and a bondle B. Then, circuit topology X has the bondle counting invariant by coloring bondle B and is defined by

$$\phi_B(X) = |Col_B(X)|. \tag{5.1}$$

$\phi_B(X) = |Col_B(X)|$.

Here we see two different structures for intra-chain contacts. The first is h-contact, and the second is S-contacts.

Figure 5.5. h-contact and s-contact

Next, we will verify the connections using h-contact. With this arrangement, we get the following coloring equations:

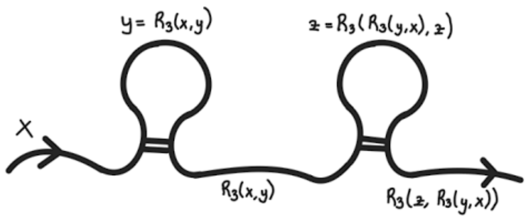


Figure 5.6. Two h-contacts for series arrangement, S

Analyze the S arrangement, and we get the following equations:

$$y = (R_3(x,y)) \tag{5.2}$$

$$z = R_3\left(R_3(y,x), z\right) \tag{5.3}$$

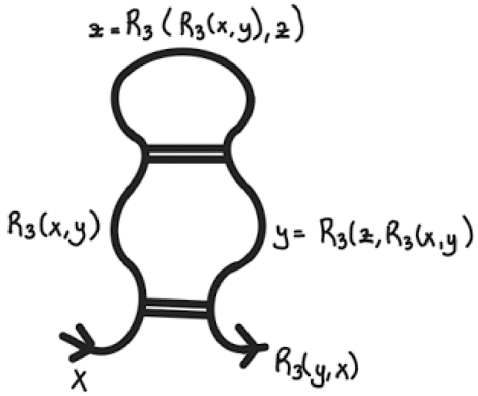


Figure 5.7. Two h-contacts for parallel arrangement, P

Analyze the P arrangement, and we get these equations:

$$z = R_3\,(R_3(x,y),z) \tag{5.4}$$

$$y = R_3\,(z, R_3(x,y)) \tag{5.5}$$

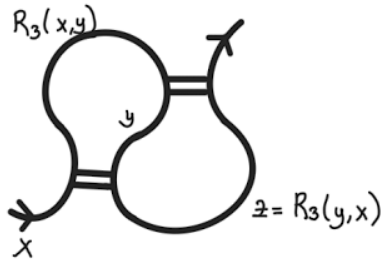


Figure 5.8. Two h-contacts for cross arrangement, C

Analyze the x arrangement, and we get these equations:

$$y = R_3\,(R_3(x,y),z) \tag{5.6}$$

$$z = (R_3(y,x)) \tag{5.7}$$

We find coloring equations for each arrangement. Now we use these equations for a specific bondle example.

Example 5.1 Let oriented bondle B, defined by $(B,*,R_1,R_2,R_3)$ and with $B=\mathbb{Z}_{15}$. It has operations defined by $x*y = 4x+12y = x*^{-1}y$, $R_1(x,y) = 10+14x+12y$, $R_2(x,y) = 10+3x+8y$, $R_3(x,y) = 10+10x+10x^2+11y$.

We find coloring equations for series arrangement. This equation gives this form,

$$y = R_3(x,y) = 10+10x+10x^2+11y, \tag{5.8}$$

$$z = R_3\,(R_3(y,x),z) = 10x^2+10xy+10y^2+10xy^2+5y^3+10y^4+11z. \tag{5.9}$$

Calculate by Mathematica gave 375 solutions. Therefore, $\Phi_B(S) = 375$.

For parallel arrangement. This equation becomes this form,

$$z = R_3\left(R_3(x,y),z\right) = 10x^2+5x^3+10x^4+10xy+10x^2y+10y^2+11z, \tag{5.10}$$

$$y = R_3\left(z,R_3(x,y)\right) = 5x+5x^2+y+10z+10z^2. \tag{5.11}$$

Calculate by Mathematica and Maple gave 750 solutions. Therefore, $\Phi_B(P) = 750$.

Lastly, for cross arrangement. This equation gives this form,

$$y = R_3\left(R_3(x,y),z\right) = 10x^2+5x^3+10x^4+10xy+10x^2y+10y^2+11z \tag{5.12}$$

$$z = R_3(y,x) = 10+11x+10y+10y^2. \tag{5.13}$$

In the first, changing the second equation gives the following equation to be solved in $\mathbb{Z}_{15}$. :

$$10+10x^2+11x+9y+10\left(10x^2+10x+11y+10\right)^2+5y^2 = 0. \tag{5.14}$$

The calculations by Mathematica gave 10 solutions. Since z can be written by x and y, we give the possible values of the pairs (x,y). We list the 10 solutions: $(0,10),(2,2),(3,13)$
$,(5,5),(6,1),(8,8),(9,4),(11,11),(12,7)$, and $(14,14)$. The system equation has 10 solutions. Hence, $\Phi_B(X) = 10$.

Example 5.2 Let $(B,*,R_1,R_2,R_3)$ be the oriented bondle with $B = \mathbb{Z}_{15}$ and operations defined by

$$x*y = 4x+12y = x*^{-1}y, \tag{5.15}$$

$$R_1(x,y) = 10+14x+12y, \tag{5.16}$$

$$R_2(x,y) = 10+3x+8y, \tag{5.17}$$

$$R_3(x,y) = 10+10x+10x^2+11y. \tag{5.18}$$

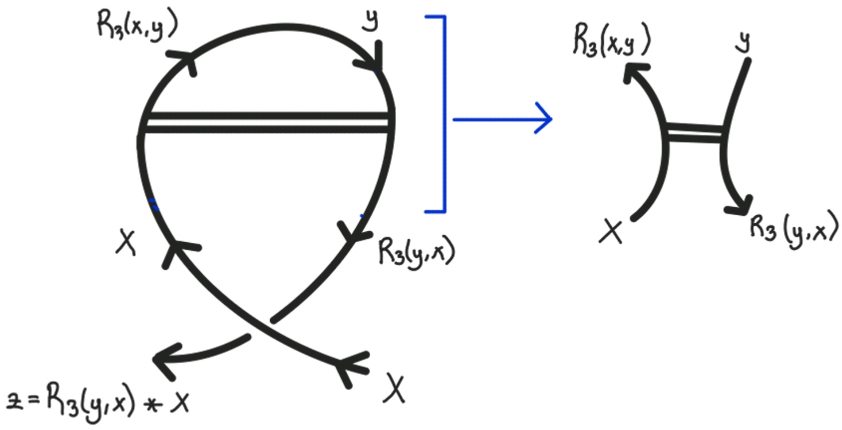


Figure 5.9. Oriented Bondle

This system of equations has 75 solutions. Hence, $\phi_B(X) = 75$.

CHAPTER 6

VIRTUAL KNOTS

A virtual knot theory, a generalization of classical knot theory discovered by L.H. Kauffman in 1996. In this section, firstly we will define virtual knot. Then we will give a virtual quandle and virtual singquandle. In this section, we will use the articles (Kauffman, 2006), (Kauffman and Manturov, 2004).

The chapter will begin with the definition of the virtual knot.

Definition 28 *A virtual link diagram* is a four-valent graph on a plane. At each crossing, it is either classical (it has information over crossing or under crossing) or virtual (it has an extra structure by a circle).

There is a knot in three-dimensional space. And we take projection this knot. When we take the projection of the knot, some crossing information can be unknown. So, this crossing shows with a circle.

We allow a new type of crossing, denoted by a 4-valent vertex with a small circle around it in 6.1.

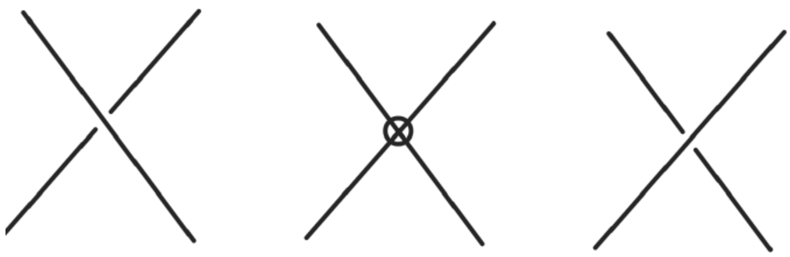


Figure 6.1. Crossings and virtual crossings

We will explain the purpose of this attribute by the use of axioms that extend the classical Reidemeister moves in 6.2

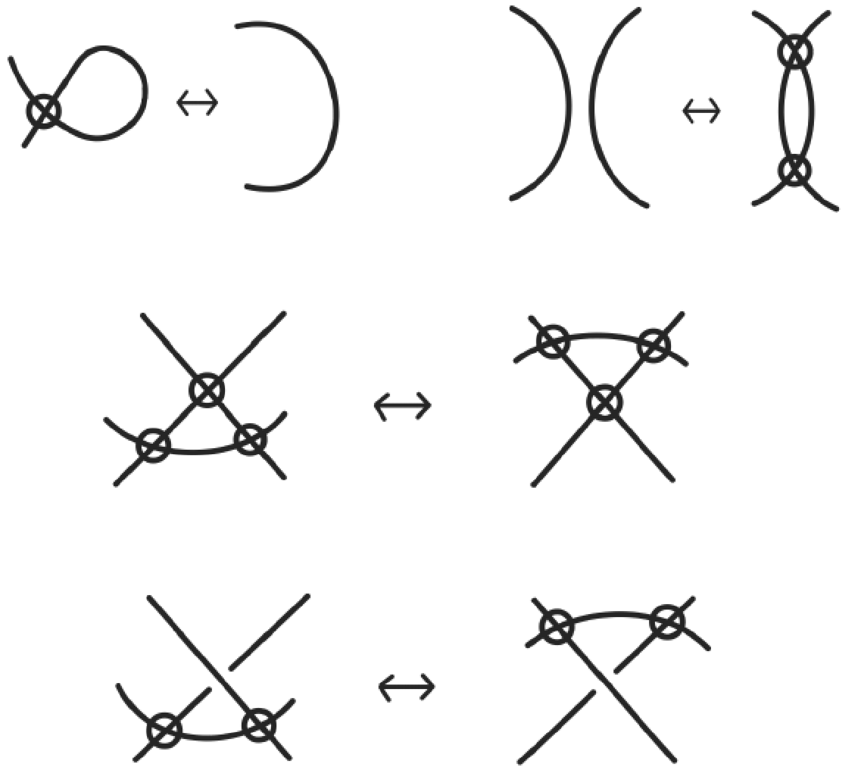


Figure 6.2. Virtual Reidemeister moves

6.1. A Motivation

This section gives explanations for two sources of motivation. The first topic is the study of knots in higher genus thickened surfaces. The second method involves the utilization of knot theory in the analysis of Gauss codes and Gauss diagrams, both of which are exclusively focused on combinatorial aspects.

6.1.1. A Review of Surfaces

We have illustrated the process of creating a diagram on the surface of a torus in figure 6.3. There is a trefoil knot on the surface of the torus. Subsequently, the virtual intersections are perceived as vestiges of the torus's projection onto the plane. The knots depicted on the surface of the torus T correspond to knots on the three-dimensional manifold T times I, where I represents the unit interval. If S_g denotes a surface of genus-g, then diagrams on S_g that undergo the standard Reidemeister movements can be used to describe knot theory on the surface $S_g \times I$.

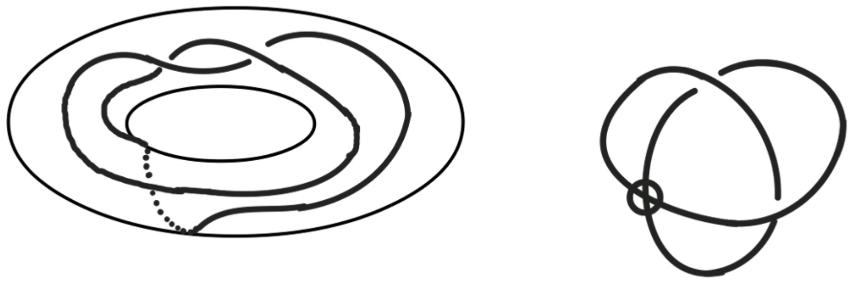


Figure 6.3. Virtual knot on the torus and its projection on the plane

6.1.2. A Review of Gauss codes

The utilization of Gauss codes, offers additional motivation for the representation of knots and links. The Gauss code is a series of crossing labels that are repeated twice to indicate a path along the diagram, beginning and ending at a given position. When we mention numerous link components, we are referring to a series of labels that are divided by the partition symbol "/" to indicate the component circuits of the code. Every label is repeated. In order to indicate whether a crossing is situated below (undercrossing) or above (overcrossing), it is possible to include the symbols O and U in the labels assigned to the crossings.

Example 6.1 *We have two examples: trefoil and unknot.*

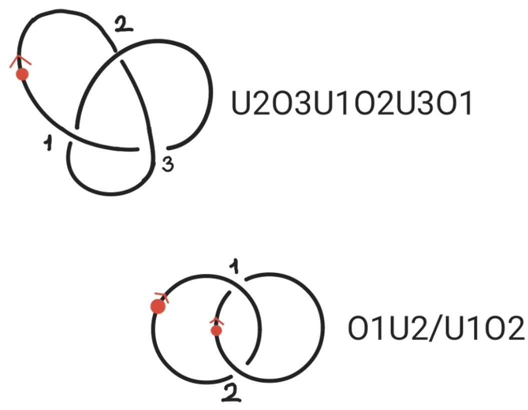


Figure 6.4. Trefoil and Hopf link

We have a virtual crossing in a diagram. We want to color it with elements of quandle X. When we color a virtual crossing, we have a new map, v.

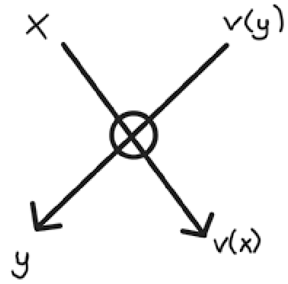


Figure 6.5. Virtual quandle coloring at a virtual crossing

We color each arc of knot diagram; we see that the axioms giving as invariant coloring under Reidemeister moves.

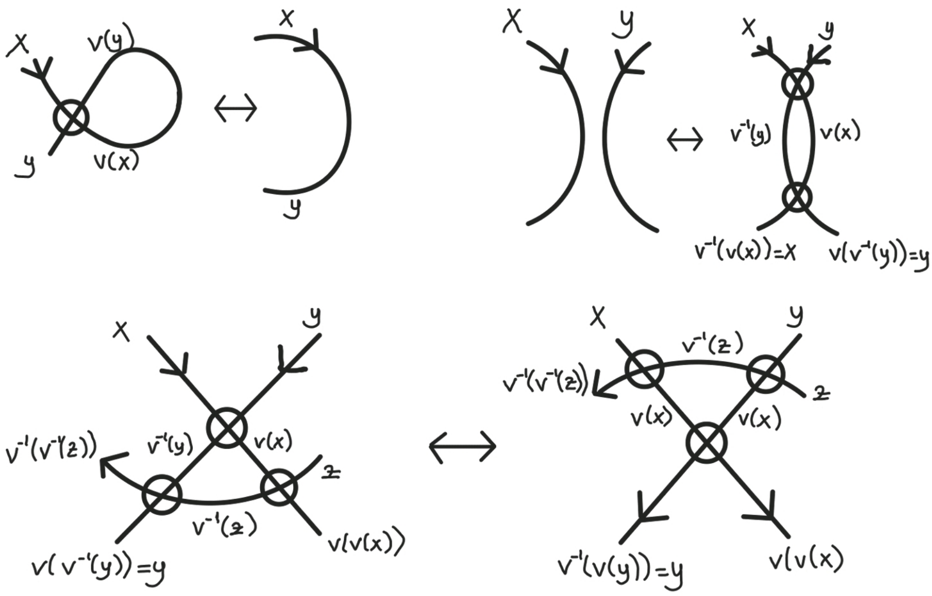


Figure 6.6. Virtual Reidemeister moves

6.2. Virtual Singquandles

We combine singquandle in the previous section. We will introduce whose axioms are motivated of Reidemeister of virtual singquandle.

Definition 29 Let $(X,*)$ be a *quandle*. Let R_1 and R_2 be two maps from $X \times X$ to X and v is an unary operation. The quintet $(X,*,R_1,R_2,v)$ is called an oriented *virtual singquandle* if the following axioms are satisfied for all $x,y,z \in X$:

$$R_1\left(x *^{-1} y, z\right) * y = \quad R_1(x, z * y) \tag{6.1}$$

$$R_2\left(x *^{-1} y, z\right) = \quad R_2(x, z * y) *^{-1} y \tag{6.2}$$

$$\left(y *^{-1} R_1(x,z)\right) * x = \quad (y * R_2(x,z)) *^{-1} z \tag{6.3}$$

$$R_2(x,y) = \quad R_1(y, x * y) \tag{6.4}$$

$$R_1(x,y) * R_2(x,y) = \quad R_2(y, x * y). \tag{6.5}$$

$$v^{-1}\left(R_1(v(x), z\right) = \quad R_1(x, v^{-1}(z)) \tag{6.6}$$

$$R_2\left(v(x), z\right) = \quad v(R_2(x, v^{-1}(z)) \tag{6.7}$$

We see axioms in the singquandle section. We analyze the virtual knot with the singquandle, we get two axioms from the Reidemester moves.

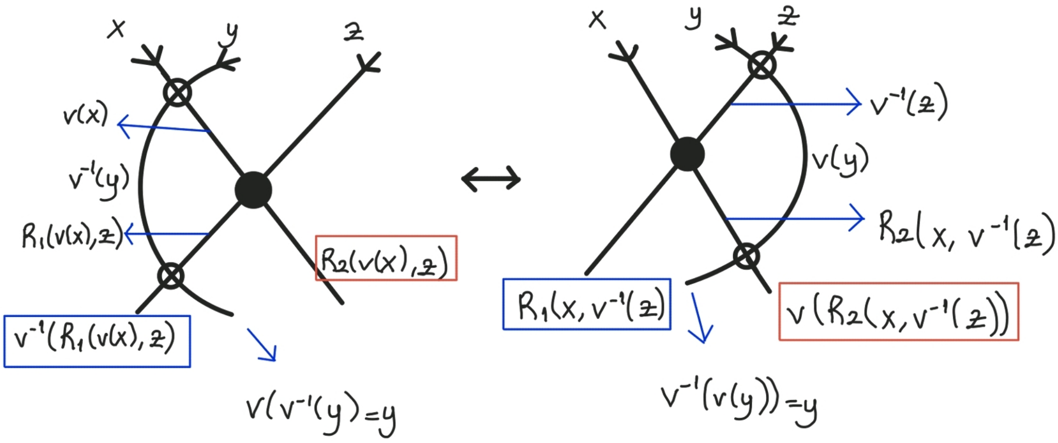


Figure 6.7. Virtual Singquandle Reidemeister move

We color all arcs of the knot diagram with operation v,$*$ and map R_1, R_2. We see that the axioms give invariant coloring under Reidemeister moves.

Definition 30 Let K be a oriented virtual singular knot diagram, *the fundamental virtual singquandle of K* is the set of equivalence classes of finite length which consider words in a set of generators by the edges of the labels in the graph, under the equivalence relation generated by all axioms and the relations at the crossings.

Theorem 13 *The fundamental virtual singquandle is a knot invariant.*

Proof Let K be an oriented knot diagram colored appropriate to the axioms. Then we get the finite-length set of equivalence classes. Since all axioms are obtained from Reidemeister moves, these axioms guarantee that these equivalence classes are invariant under the Reidemeister moves. Hence, the fundamental virtual singquandle of K is a knot invariant. □

Here we have an example.

Example 6.2 We have a trefoil knot with three different types of crossing: singular crossing, virtual crossing, and classical crossing. First, we label each arc with a generator and color it with axioms. We find a fundamental virtual singquandle.

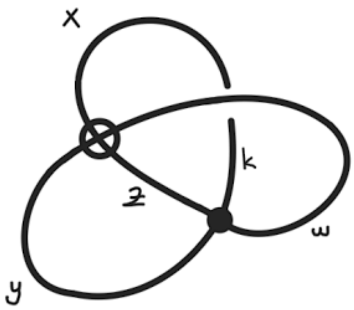


Figure 6.8. Three types crossing: classical crossings, virtual crossing, and singular crossing

$$\mathcal{VSQ}(K) = \langle x, y, z, k, w, \mid z = v(x), w = v(y), k = R_2(y * z), \tag{6.8}$$

$$w = R_1(y * z), x = k *^{-1} w\rangle. \tag{6.9}$$

We rewrite equation

$$\mathcal{VSQ}(K) = \langle x, y, k \mid k = R_2(y * v(x)), v(y) = R_1(y * v(x)), \tag{6.10}$$

$$x = R_2(y * v(x)) *^{-1} R_1(y * (v(x)))\rangle. \tag{6.11}$$

CHAPTER 7

CONCLUSION

In this thesis, we determined algebraic structures for knots. Colorability is the main idea behind algebraic structures. So firstly, we gave some definitions and theorems for colorability, saying that it is a knot invariant. Tricolorable and fox-n coloring was satisfying in this classification, but sometimes it is not enough. Then we explained the algebraic structures that are improved when we classify knots. These structures are quandle, singquandle, and bondle. These structures offer a comprehensive understanding of both classical and singular knots. In this part of this thesis, we define singquandle because of proteins. In biology, protein structure is analyzed in many different ways. Circuit topology is one of these methods. In this topology, we use knot theory. The protein structure is similar to that of singular knots. So, we define a singquandle. Then we used a bond for each singular crossing and defined bondle. The main idea of bondle in knot theory is to apply to proteins. So, we found out how many different ways it can be colored with bondle. Finally, we defined virtual knots and quandle virtual knots. We also defined a new structure a virtual singquandle. As a result, algebraic structures offer a comprehensive classification of knots.

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